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On the Coulomb-Mohr Failure Criterion

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Coulomb's criterion for the shear fracture of a brittle material is that total shearing resistance is the sum of the cohesive shear strength (independent of direction) and the product of the effective normal stress and the coefficient of internal friction (a constant independent of normal stress). Mohr generalized this criterion by extending it to a three-dimensional state of stress, and by allowing for a variable coefficient. The coefficients of internal and external (sliding) friction are not the same in general. Both tend to decrease with increasing normal stress, and their relative magnitudes may determine if failure occurs by new shear fracturing or by slip on pre-existing cohesionless surfaces like joints in rocks.

In the literature of rock deformation there is confusion about the concept of internal friction upon which the Coulomb-Mohr failure criterion is based. Although a correct account of this criterion has been given recently by *Jaeger and Cook* [1969], errors are still being promulgated in this journal, and its readers may find that a brief historical preface is helpful to this discussion.

Coulomb's [1776] problem was the shear fracture in a prism of isotropic material under uniaxial compression σ_1 (compressive stresses counted positive). He wrote down the equations for the shear stress τ and normal stress σ on a plane inclined at an angle, say θ, to the loading direction. He assumed that 'la cohésion se mesure par la résistance que les corps solides opposent à la désunion directe de leur parties', and 'je suppose ici que l' adhérence oppose une egalé résistance, soit que la force soit dirigée parallèlment ou perpendiculairement au plan de rupture.' He then solved for the value of θ for which the uniaxial stress (compressive breaking strength) would be a minimum, and he found, of course, that $\theta = 45°$.

During the following two centuries, writers of authority have erroneously stated that Coulomb proceeded no further. For example, in the first edition of his widely known book on faulting *Anderson* [1942] ascribed the notion of internal friction to Navier. *Jaeger* [1962] repeated this mistake. In one edition of his great book, *The Earth*, *Jeffreys* [1952] in turn credits Anderson with this concept!

In fact, however, Coulomb went on to say 'je supposerai ici que la résistance dûe au frottement [sic] est proportionelle a la pression comme l'a trouvé M. Amontons.' He then added the cohesive shear strength, which is now commonly denoted as τ_0, to the frictional shearing resistance $n\sigma$, where n is the coefficient of internal friction, and obtained the equation

$$\tau = \tau_0 + n\sigma.$$

Coulomb then differentiated the uniaxial compressive stress with respect to θ to find the minimum value, and showed that

$$\theta_f = \pm 45° \mp \phi/2,$$

where ϕ = the angle of internal friction, $n = \tan \phi$, and θ_f = the angle between a potential shear fracture and σ_1.

The physics of the Coulomb equation is obscure. Adding the 2 terms is like adding apples and oranges because, until cohesion is broken, no sliding surface exists in the intact body. Nevertheless, for whatever it may be worth, the concept of internal friction is clearly due to Coulomb. *Navier* [1833] himself so stated, and today he would no doubt be surprised by the credit implied by those who speak of the 'Coulomb-Navier equation.'

A century later *Mohr* [1900] examined the older failure criteria, and finding that none conformed well to the empirical data then available to him, he postulated that 'at the limit the shear stress on a glide plane reaches a maximum value depending on the normal pressure and the character of the material.' This criterion was intended to apply to both shear fracture strength in the brittle state and yield stress in the ductile state of a material. Mohr's most important contribution is his recognition that material properties are themselves functions of the state of stress. Of course, he also analyzed the problem in three dimensions and used his graphical construction published earlier [*Mohr*, 1882] to derive his famous failure envelope in shear and normal stress space. 'At each point in a body in which the elastic limit [yield stress] or the breaking limit is exceeded, two glide planes form: they intersect each other in the σ_2 axis of the point and make two similar angles with the direction of each of the principal stresses σ_1 and σ_3.' Implicit is the assumption that the intermediate principal stress σ_2 has no influence on failure. *Handin et al.* [1967], *Mogi* [1967], and others have shown that this assumption is not correct, but that the effects on strength are small relative to those of the extreme principal stresses σ_1 and σ_3.

Thus the Coulomb-Mohr criterion is merely the Coulomb condition generalized to handle the triaxial stress state and variable material properties. If the Mohr envelope is linear, it is described by Coulomb's equation. If the material has no cohesive strength, the envelope passes through the origin of $\tau - \sigma$ space; the material also has no tensile strength; and

$$\tau = \sigma \tan \phi,$$

which is just *Rankine*'s [1857] condition for the failure of a loose granular aggregate like sand.

Now what is internal friction? In dry sand it is of the nature of external (or sliding) friction because the coefficient of internal friction in Rankine's equation is about equal to the tangent of the angle of repose. In cohesive material, however, internal friction is a fictitious quantity that can not be measured directly. As *Maurer* [1965] emphasizes, it is not the post-fracture friction associated with sliding on new surfaces in broken material. Neither is it equivalent to the frictional resistance to slip on artificial rock surfaces. Indeed the coefficient of internal friction should be regarded as no more than the slope of the Mohr envelope for intact material, that is

$$n = (\tau - \tau_0)/\sigma.$$

This quantity n can be regarded as a 'physical constant' for a given state of the material; however, the material properties are themselves functions of at least the state of stress, the temperature, and the strain rate. With increasing pressure and/or temperature, and with decreasing strain rate, rocks tend to pass from the brittle to the ductile state; and the value of n decreases to approach zero in the limit. Even at constant temperature and strain rate, the Mohr envelope is always a curve that is concave toward the σ-axis, and that approaches a horizontal asymptote if only the normal stress is high enough. In the fully ductile (or 'plastic') state n becomes nearly zero as it is in ductile metals.

The distinction between internal and external (or sliding) friction is important. The mechanics of large masses of rock, treated as engineering structures to support foundations, slopes, and underground openings, involves the frictional properties of the natural fractures (or joints), of which few rocks are free. To predict the response of the rock mass to static or dynamic loading one must know the mechanical properties both of the fracture systems and of the intact rock.

To investigate the preference for shear fracturing or for slip on pre-existing rock surfaces, *Handin and Stearns* [1964] conducted triaxial compression tests on 1.90 by 5.0-cm, copper-jacketed cylinders of dolomite, limestone, and sandstone in the apparatus described by *Handin and Hager* [1957]. Intact specimens proved to be statistically homogeneous and isotropic in that breaking strengths were reproducible and independent of load orientation. The strengths are linear functions of confining pressure to at least 1 kb in the range of 0.25 to 2 kb investigated.

Tests were also made on cylinders that had been cut with a diamond saw at 5° increments from 30° to 70° to the longitudinal axis, that is to σ_1. The surfaces were lightly polished merely to make them all as nearly alike as possible. From the measured values of σ_1 and of σ_3, which is equal to the confining pressure, one can calculate the normal stress σ and the shear stress τ on the cut of inclination θ, at the onset of slip. The coefficient of static friction μ is then taken as the ratio of τ to σ. These experiments are similar to *Jaeger*'s [1959] except that a separate run was made at each 0.25-kb increment of confining pressure from 0.25 to 2 kb because the process of slip modifies the nature of the surface. They are also similar to those of *Byerlee* [1967*a*] on granite except that his maximum friction was reached after 1 mm of slip rather than at the onset.

On a cut of given inclination, the value of μ decreases with increasing normal stress as *Maurer* [1965] and *Byerlee* [1968] have also observed. The values are listed in Table 1 for sliding on 45° cuts in 4 rocks at 8 different confining pressures. The reproducibility of a measurement is about 5%. The values are also plotted as functions of normal stress (Figure 1). In general all these values, ranging from about 0.4 to 0.8, are consistent with those measured by *Jaeger* [1959], *Maurer* [1965], *Byerlee* [1967*a*, 1968], and others (see *Jaeger and Cook* [1969], Table 3.3.1).

In Tennessee sandstone, the constituents of which are brittle under the test conditions imposed here, the decrease in μ is relatively small, 11% over the normal stress range of about 1 to 6 kb. *Byerlee* [1967*b*] has suggested that at low normal stresses surfaces slide by lifting over asperities, whereas at higher stresses the asperities must be broken through. If the material is in the brittle state, then in the latter stage the shear stress for sliding should increase linearly with normal stress as the Coulomb criterion predicts; that is, μ should remain constant. The behavior of this sandstone does not differ widely from this idealization, since the rate of change of μ is high at low normal stress and very small at high normal stress.

TABLE 1. Coefficients of Sliding Friction as Functions of Confining Pressure on 45° Saw Cuts in 4 Rocks

Confining Pressure σ_3 kb	Blair Dolomite	Knox Dolomite	Solenhofen Limestone	Tennessee Sandstone
0.25	0.55	0.60	0.67	0.76
0.50	0.52	0.58	0.66	0.72
0.75	0.51	0.55	0.64	0.71
1.00	0.49	0.52	0.62	0.70
1.25	0.47	0.50	0.61	0.70
1.50	0.45	0.49	0.60	0.69
1.75	0.42	0.48	0.57	0.69
2.00	0.40	0.47	0.55	0.68

In the dolomites the decreases in μ with increasing normal stress from 0.5 to 3.5 kb are large, 27% for Blair and 22% for Knox. The changes of μ are about the same, and they are nearly constant. The only ready explanation for these reductions of the coefficients of friction is that these rocks are passing through the brittle-ductile transition and are behaving as do 'plastic' metals in which a variable μ is well known. A large reduction (18%) is also observed in Solenhofen limestone, and here the

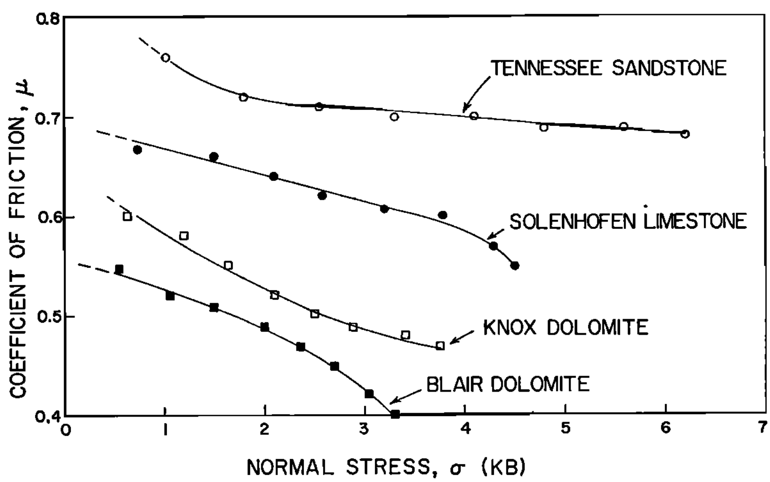

Fig. 1. Coefficients of sliding friction on 45° saw cuts in four rocks as functions of normal stress.

rate of change of μ definitely increases above a normal stress of 4 kb. The corresponding mean pressure is about 3.5 kb, which is well above the 2.7-kb transition pressure reported by *Handin et al.* [1967].

In Table 2 are listed the cohesive strengths and coefficients of internal friction of the intact rocks, together with representative values of the coefficients of sliding friction. From these data let us superimpose a 'sliding line' on the Mohr diagram for shear fracture, and then focus attention on the particular stress circle for $\sigma_3 = 1$ kb (Figure 2). Since the cuts are cohesionless, the sliding line passes through the origin, and its slope is equal to the angle of sliding friction, the angle whose tangent is the coefficient μ from Table 2. The 2 pairs of coordinates at the intersections of the line and the stress circle satisfy the condition for sliding that the shearing resistance τ_s equals the product of the coefficient of friction and the normal stress σ_s. That is to say, sliding on these 2 planes is just as likely as fracture, and sliding is favored on all planes lying between them.

TABLE 2. Cohesive Strengths and Coefficients of Internal and Sliding Friction of Some Rock under 1 kb Confining Pressure

Rock	Cohesive Strength τ_0 kb	Coefficient of Internal Friction n	Coefficient of Sliding Friction μ
Blair Dolomite	0.45	1.00	0.40
Lueders Limestone	0.15	0.53	0.60
Solenhofen Limestone	1.05	0.53	0.62
Tennessee Sandstone	0.50	0.84	0.70

Because one of these is at a low angle to σ_1 and would intersect the ends of the specimen, it was not investigated, and the sliding lines are extrapolated linearly through the origin to simplify the diagrams. Slip on the other plane, inclined at an angle θ_s of 30° or more to σ_1, can be compared with fracture.

Consider Blair dolomite with cohesive

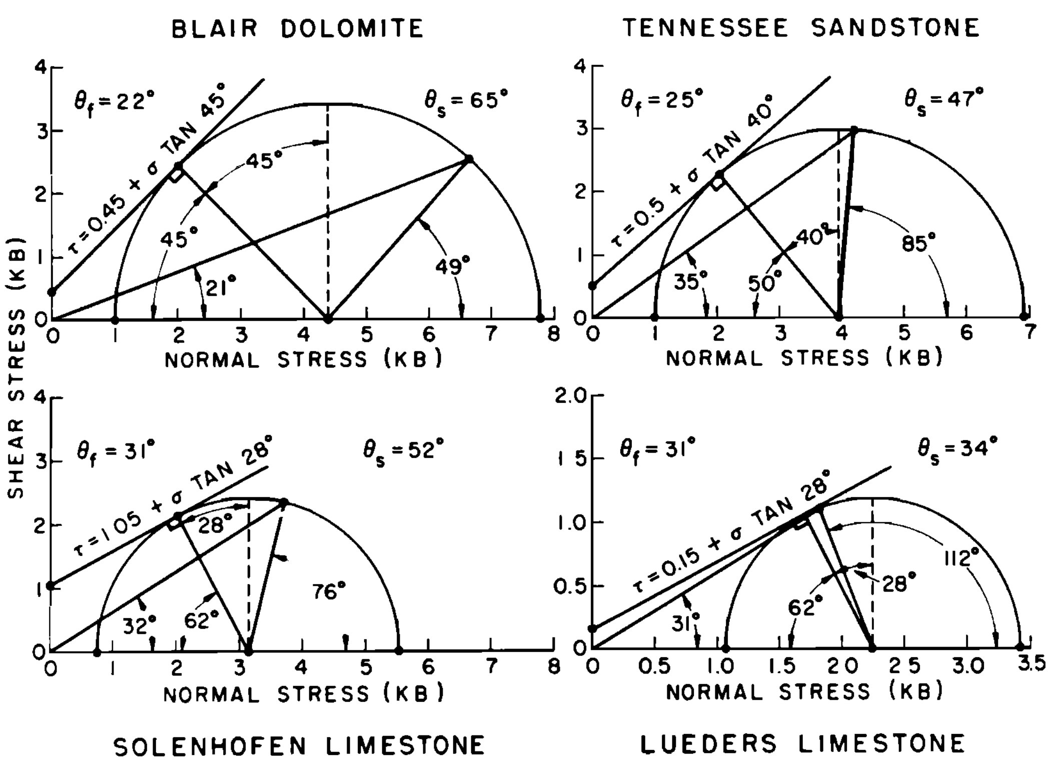

Fig. 2. Mohr envelopes and sliding lines for some rocks.

strength of 0.45 kb and with angle ϕ of 45°. For the normal stresses on high-angle cuts, μ would be about 0.4 at this confining pressure. The contrast between the angle of internal friction, 45°, and the angle of sliding friction, 21°, is the largest encountered. One predicts that slip on a 65° cut is as likely as fracture. It is observed that sliding always occurs at 60° and less. Fracturing alone occurs at about 20° in 70° cylinders in accord with the Coulomb condition. Either sliding, fracturing, or both, is seen in specimens with 65° cuts (Figure 3). The traces of the cut and the fracture on the circular section of the specimen (their 'strikes') are nearly always parallel. Dip directions are inconsistent. In the central cylinder the 22° fracture intersects and offsets the cut, and slip on the cut offsets the fault as well.

Tennessee sandstone has a τ_0 of 0.50 kb, a ϕ of 40°, and the highest angle of sliding friction of ~35°. One predicts that fracture at 25° is preferred over slip on cuts of 50° or more, and this is just what is observed. For Solenhofen limestone the angle of sliding friction, 32°, exceeds the angle of internal friction, 28°, and the 1.05-kb cohesive strength is very high. Fracturing at 31° rather than sliding occurs for cuts of 55° and more. For Lueders limestone the 31° sliding friction angle (measured on a 45° cut at a single confining pressure of 1 kb) also exceeds the internal friction angle of 28°, and the cohesive strength of only 0.15 kb is so low that the sliding line almost fails to intersect the stress circle at all. Slip should become barely possible at 34°. Displacement on a 30° cut does occur; but, since this surface lies very near the 31° plane of potential failure, sliding and faulting are not distinguishable.

These examples serve to demonstrate that the Coulomb-Mohr criterion can be useful even though its basis, the concept of internal friction, is unsatisfying. This criterion has certainly been widely and successfully applied to engineering problems involving both soil and rock

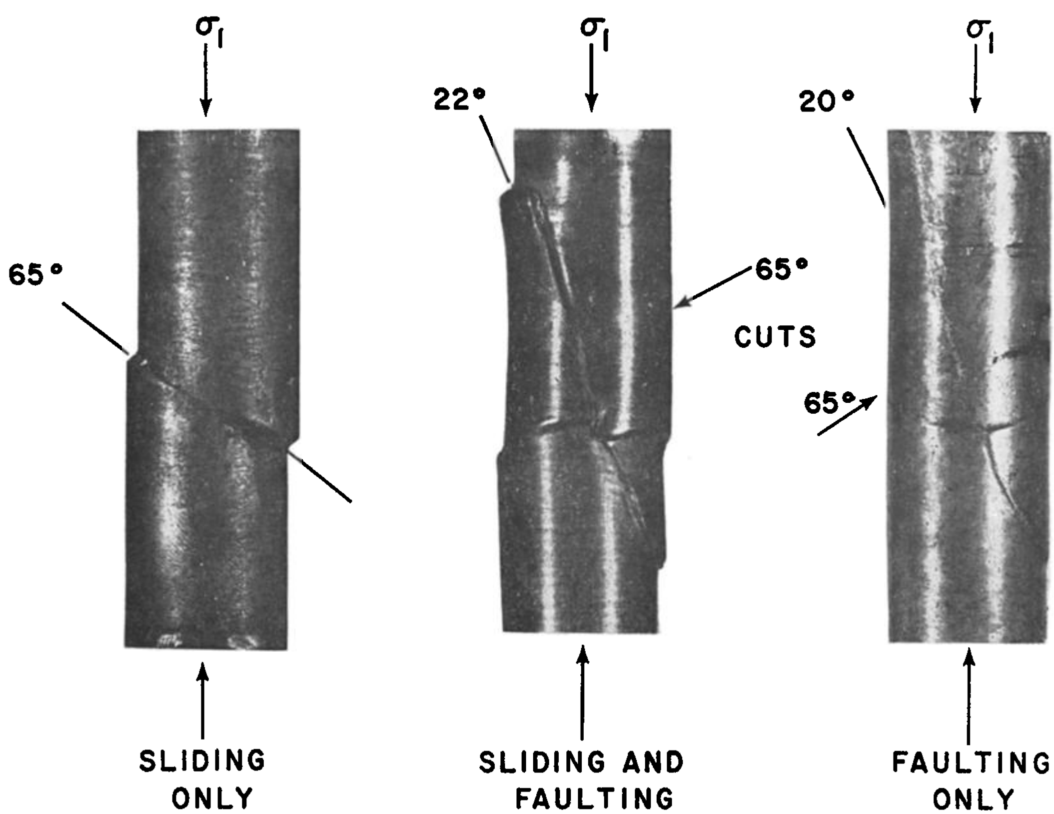

Fig. 3. Blair dolomite specimens with 65° saw cuts.

mechanics of the shallow crust, and doubtless its use will continue until an adequate mechanistic theory of fracture is developed.

Note that slip on a pre-existing cut (or fracture) may be a poor criterion for estimating principal stress directions. In the Blair dolomite, for example, a given orientation of σ_1 can be associated with slip on any cohesionless surface lying between about 5° and 65°. *McKenzie* [1969] has emphasized the fact that fault-plane solutions for shallow-focus earthquakes poorly determine principal stress directions simply because such events occur in already broken, not intact, rock.

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