Fluid-Structure Interaction

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1. Introduction

Fluid-structure interaction (FSI) refers to the coupling and interaction between any moving or deformable continuum solid with surrounding fluid. These interactions find applications in many engineering design systems, like aerospace engineering (aircraft design) or civil engineering (bridges). Another important application of FSI is studying the blood flow through arteries. These are used to design artificial heart valves for analysis of aneurysms.

The reason for coupling between the solid and the fluid is that as the solid deforms, it changes shape which in turn causes the geometry of the fluid to change, thereby affecting its dynamics. This effect will not be pronounced if the deformation of the structure is very small and the temporal variation is very slow. In such a case, each system can be treated independently and we would be concerned only with the resultant stresses in the solid parts.

My research is concerned with simulating earthquakes across a large timescale, and studying the nucleation and recurrence intervals of repeating earthquakes. I am currently focused on the Parkfield region in California, which has a history of well documented repeating earthquakes. Many seismic studies across Parkfield have identified bodies at seismogenic depths with velocity that could be caused by high fluid pressure. This fluid activity has a known effect of reducing the frictional strength of the fault, thereby accelerating the earthquake nucleation. The onset and timing of earthquakes are therefore directly controlled by variation in fluid pressure. Despite there being numerous studies on the effect of static fluid pressure (Miller, 1996; Roeloffs, 2000), there has not been a lot of study conducted on the coupling of fluid and the surrounding rocks, and how it affects the onset of earthquake nucleation or its size. Through this study, I hope to learn the physics of fluid-structure interaction, and apply it to the study of earthquakes.

2. Model setup

Fluid structure interaction encompasses a wide range of problems, for various geometries of solid and fluid, like a solid contained inside a moving fluid, or a fluid is contained in a thin membrane solid. Here we will be dealing with elastoacoustic problem, which concerns the motion of an elastic structure in contact with a compressible fluid. Specifically, we will look at free harmonic motions of a coupled solid-fluid system, also known as the normal modes of vibration or natural frequencies of vibration. When these natural frequencies coincide with forced frequencies, we have resonance that is detrimental to the structure. Therefore, in structure design, it is important to study the free harmonic motion. The frequencies obtained from the free harmonic motion are referred to as eigenfrequencies. My model setup is similar to Bermudez et al., 2008, described in Fig. 1. It is a 2D model with isotropic, homogeneous, linear elastic solid (e.g., steel) surrounding an ideal barotropic fluid.

We have used the following assumptions in formulating the problem:

1. The displacements are small, therefore the response of the structure is linear.
2. Fluid is homogeneous with constant density and viscous effects are ignored.
3. The displacement is harmonic.
4. No external force is acting on the system.

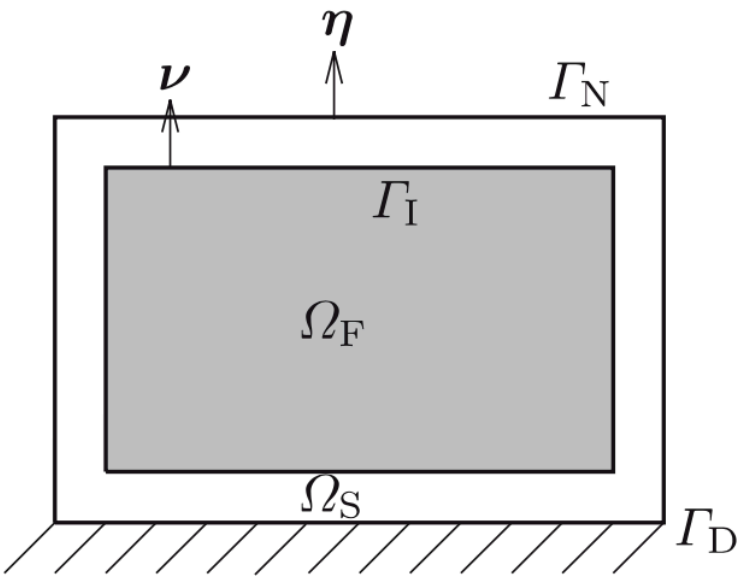


Fig. 1. Two-dimensional model setup for the elastoacoustic problem (from Bermudez et al., 2008).

All the variables are defined in their standard definition of Sobolev Spaces. The domain definitions are as follows:

$$\Omega_F \; is \; the \; fluid \; domain.$$
$$\Omega_S \; is \; the \; solid \; domain.$$
$$\Gamma_I \; is \; the \; interface \; of \; solid \; and \; fluid.$$
$$\Gamma_N \; is \; the \; top \; free \; surface.$$
$$\Gamma_D \; is \; the \; rigid \; boundary \; at \; the \; bottom.$$

The following variables would be used in subsequent formulations:

$$\boldsymbol{u}^F : the \; displacement \; vector \; in \; fluid \; domain.$$
$$p : acoustic \; pressure \; in \; the \; fluid \; domain.$$
$$\rho_F : the \; fluid \; density.$$
$$\boldsymbol{u}^S : the \; displacement \; vector \; in \; the \; solid \; domain.$$
$$\rho_S : the \; solid \; density.$$
$$\lambda_S \; and \; \mu_S : Lame \; parameters.$$
$$\boldsymbol{\varepsilon}(\boldsymbol{u}^S) = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T) - \; strain \; tensor.$$
$$\boldsymbol{\sigma}(\boldsymbol{u}^S) = \; \boldsymbol{C} : \boldsymbol{\varepsilon}(\boldsymbol{u}^S) - \; stress \; tensor, \boldsymbol{C} \; is \; the \; stiffness \; tensor.$$

3. Governing equations

We begin from the equation of motion in the solid and fluid domains. In the absence of external force, the acceleration of the mass in the solid domain is given by the divergence of stress tensor.

$$\rho_S \ddot{\boldsymbol{u}}^S = \nabla . \boldsymbol{\sigma} \tag{1}$$

In the fluid domain, the acceleration is given by the pressure gradient in the fluid. Also, by definition, the acoustic pressure is the lame's 1st parameter times the divergence of displacement. These two equations can be written as:

$$\rho_F \ddot{\boldsymbol{u}}^F = \; -\nabla p \tag{2}$$
$$p = \; -\nabla . \boldsymbol{u}^F \tag{3}$$

Replacing $\ddot{\boldsymbol{u}}$ with $- \; \omega^2 \boldsymbol{u}$ for a harmonic regime, we can rewrite the above three equations as:

$$\nabla p - \omega^2 \rho_F \boldsymbol{u}^F = \mathbf{0};\ in\ \Omega_F \tag{4}$$

$$p + c^2 \rho_F \nabla . \boldsymbol{u}^F = 0;\ in\ \Omega_F \tag{5}$$

$$\nabla . \boldsymbol{\sigma} + \ \omega^2 \rho_S \boldsymbol{u}^S = \mathbf{0};\ in\ \Omega_S \tag{6}$$

The interface coupling conditions will give us two equations based on the physical constraints. The first equation is the called the slippery condition, which is the condition for the interface to be frictionless. In such a case, the normal component of surface velocity of the solid must coincide with the normal component of the acoustic velocity of the fluid. The second condition is the kinetic condition, or the action-reaction principle for the stresses acting at the interface. This condition states that the pressure exerted by the fluid must be equal to the force acting due to the solid stresses on the interface. These two interface conditions are the essence of fluid structure interaction since they describe the coupling between fluid and the solid. They can be expressed as follows:

$$\boldsymbol{u}^S . \boldsymbol{\nu} - \boldsymbol{u}^F . \boldsymbol{\nu} = 0;\ on\ \Gamma_I \tag{7}$$

$$\boldsymbol{\sigma} . \boldsymbol{\nu} - p\, \boldsymbol{\nu} = 0;\ on\ \Gamma_I \tag{8}$$

where $\boldsymbol{\nu}$ is the normal vector to the interface.

In our model domain (Fig. 1), the top surface Γ_N is a free surface with normal vector $\boldsymbol{\eta}$, and the bottom surface Γ_D is a rigid surface clamped to the boundary. The respective equations for the boundaries are:

$$\boldsymbol{\sigma} . \boldsymbol{\eta} = 0; on\ \Gamma_N \tag{9}$$

$$\boldsymbol{u}^S = \mathbf{0}; on\ \Gamma_D \tag{10}$$

We will describe two different formulations of partial differential equations (PDE) based on the above equations of motion in a coupled system, similar to the approach described in Bermudez et al., 2008.

The first PDE is called the pressure-displacement formulation for the elastoacoustic problem, where the unknowns are the pressure field in the fluid domain, and the displacement field in the solid domain. Therefore, we have to eliminate the fluid displacement from the PDEs above. We can eliminate $\boldsymbol{u}^F$ using equations (4) and (5) to obtain the pressure-displacement formulation, hereafter referred to as **PDE1:**

$$\nabla^2 p + \frac{\omega^2}{c^2} p = 0; \quad in\ \Omega_F;$$
$$\nabla . \boldsymbol{\sigma} + \omega^2 \rho_S \boldsymbol{u}^S = 0; \quad in\ \Omega_S;$$
$$\boldsymbol{\sigma}\boldsymbol{\nu} + p\boldsymbol{\nu} = \mathbf{0}; \quad on\ \Gamma_I;$$
$$\frac{1}{\rho_F} \frac{\partial p}{\partial \boldsymbol{\nu}} - \omega^2 \boldsymbol{u}^S . \boldsymbol{\nu} = 0; \quad on\ \Gamma_I;$$
$$\boldsymbol{\sigma} . \boldsymbol{\eta} = 0; \quad on\ \Gamma_N;$$
$$\boldsymbol{u}^S = \mathbf{0}; \quad on\ \Gamma_D$$

The above PDE results in a non-symmetric weak form, when assembling into matrix representation for finite element analysis. This means that the eigenvalues could be complex in principle. It is shown in Bermudez et al., 2008 that the eigenvalues do turn out to be real numbers.

We define another set of PDE known as the pure displacement formulation for the elastoacoustic problem, where the unknowns in both the solid and the fluid domain are displacements fields. We can eliminate *p* from equations (4) and (5) to obtain the pure displacement formulation. This will be called PDE2:

$$c^2 \rho_F \nabla(\nabla . \boldsymbol{u}^F) + \omega^2 \rho_F \boldsymbol{u}^F = \mathbf{0}; \quad in\ \Omega_F;$$
$$\nabla . \boldsymbol{\sigma} + \omega^2 \rho_S \boldsymbol{u}^S = 0; \quad in\ \Omega_S;$$
$$\boldsymbol{\sigma}\boldsymbol{\nu} + \rho_F c^2 (\nabla . \boldsymbol{u})\, \boldsymbol{\nu} = \mathbf{0}; \quad on\ \Gamma_I;$$
$$\boldsymbol{u}^F . \boldsymbol{\nu} - \boldsymbol{u}^S . \boldsymbol{\nu} = 0; \quad on\ \Gamma_I;$$
$$\boldsymbol{\sigma} . \boldsymbol{\eta} = 0; \quad on\ \Gamma_N;$$
$$\boldsymbol{u}^S = \mathbf{0}; \quad on\ \Gamma_D$$

We derive the weak form of the above two sets of PDEs, PDE1 and PDE2, which is essential for the finite element analysis. Finite element analysis shines over other computational methods when there is dissimilar material properties or complex geometry. The idea behind finite element method is to divide the domain into a number of smaller elements, and numerically compute the dependent variables at each element. The simplest form is to divide the domain into smaller triangular elements, with 3 nodes as the vertices of triangles for each element. At each element, the partial differential equations are approximated by simpler equations. In order to do this, we define a test function over the domain and compute the weighted integral of the trial function (the dependent variable to be computed), where the weights are the test function. Then we minimize the residual error caused by the trial functions and finding the best fit.

The weak form is obtained by multiplying the partial differential equation with a test function, and using Green's formula (Integration by parts) to rearrange the integral equation. We can obtain a single weak form equation for the coupled system with some algebraic manipulations. For the PDE1, we define a test function q in the fluid domain, and a test function $\boldsymbol{v}^S$ in the solid domain, where q has the dimensions of pressure and $\boldsymbol{v}^S$ has the dimensions of velocity. The weak problem (WP1) corresponding to the PDE1 is given by

$$\int_{\Omega_S} \boldsymbol{\sigma}:\boldsymbol{\varepsilon} + \int_{\Omega_F} \frac{1}{\rho_F} \nabla p.\nabla q - \int_{\Gamma_I} p\boldsymbol{v^S}.\boldsymbol{\nu} \;=\; \omega^2 \left(\int_{\Omega_S} \rho_S \boldsymbol{u}^S.\boldsymbol{v}^S + \int_{\Omega_F} \frac{1}{\rho_F c^2} pq + \int_{\Gamma_I} q\boldsymbol{u^S}.\boldsymbol{\nu}) \right)$$

For the weak formulation of PDE2, we multiply equations of motion in the fluid and solid domain by test functions $\boldsymbol{v}^F$ and $\boldsymbol{v}^S$, where each of the test functions has the dimensions of velocity. Applying Green's formula, we obtain the weak problem 2 (WP2):

$$\int_{\Omega_S} \boldsymbol{\sigma}:\boldsymbol{\varepsilon} + \int_{\Omega_F} \rho_F c^2 (\nabla.\boldsymbol{u}^F)(\nabla.\boldsymbol{v}^F) \;=\; \omega^2 \left(\int_{\Omega_S} \rho_S \boldsymbol{u}^S.\boldsymbol{v}^S + \int_{\Omega_F} \rho_F \boldsymbol{u}^F.\boldsymbol{v}^F \right)$$

$$\boldsymbol{v}^S.\boldsymbol{\nu} = \boldsymbol{v}^F.\boldsymbol{\nu} \qquad on\ \boldsymbol{\Gamma_I}\ .$$

4. Results

We specify the physical parameters of the model for simulation. The outer solid container in figure 1 is steel, and the fluid is water. The parameters are as follows:

$$\rho_s = 7850 \frac{kg}{m^3}$$
$$\rho_F = 1000 \frac{kg}{m^3}$$
$$E = 200 \times 10^9 \, Pa$$
$$c = 1430 \frac{m}{s}$$

where E is the Young's modulus, c is the acoustic wave speed in water. The domain size considered is shown is Fig. 2. We run the simulation in Comsol Multiphysics solving for eigenfrequencies of the system. Comsol solves the system using PDE1 as the formulation, and corresponding WP1 for the finite elements. But it solves for to equations simultaneously rather than considering them as a single equation. We use a simple triangular mesh with 'finer grid' as defined in Comsol.

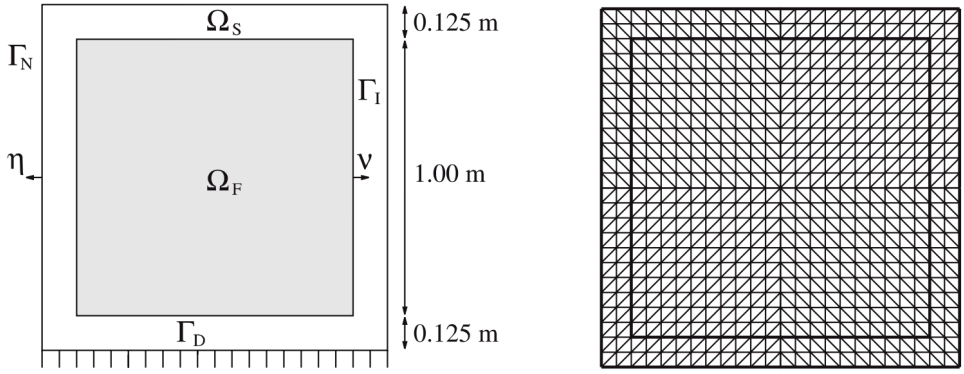
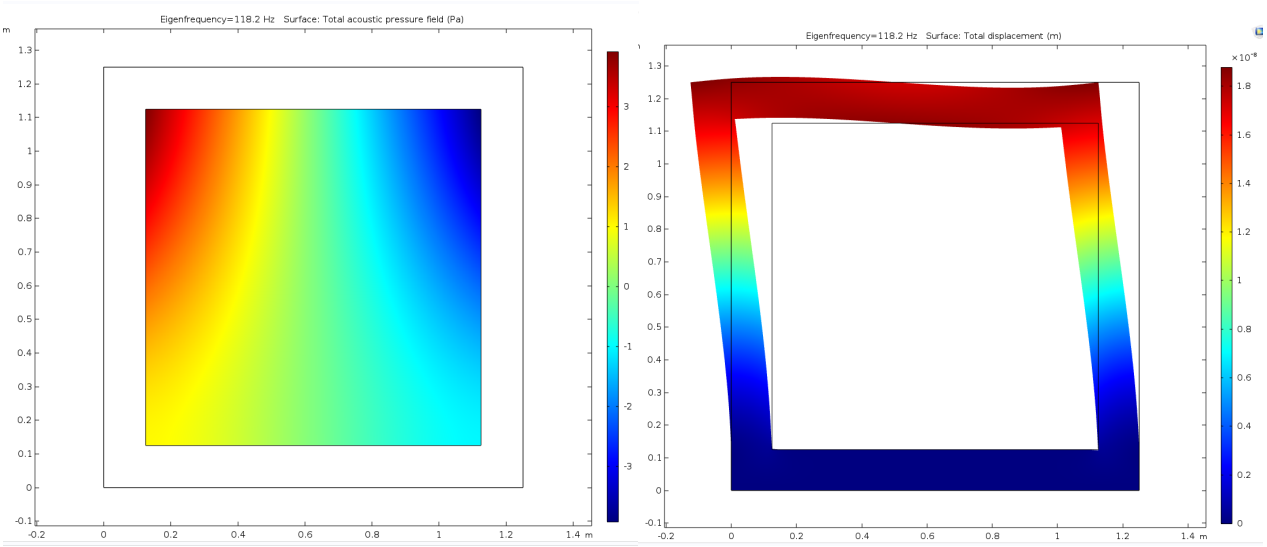


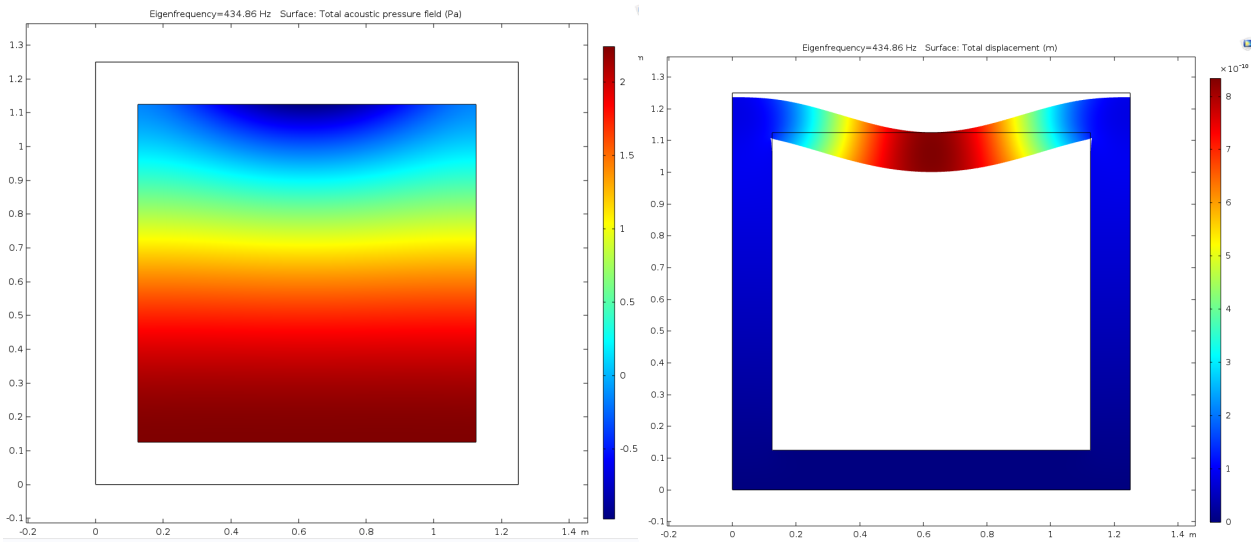
Fig. 2. The domain size and domain discretization is shown here for the simulation. From Bermudez et al., 2008.

The first five normal modes of vibrations for the eigenfrequencies are shown in Fig. 3. For each of these frequencies, we show the pressure field in the acoustic fluid domain, and the displacement field in the solid domain. These normal modes are extremely important to study during the design of a structure. An example of such a design would be the design of a nuclear powerplant, and from the perspective of earthquake engineering, it would be important to study the resonant frequency in which the structure vibrates to minimize the potential damage caused during an earthquake.

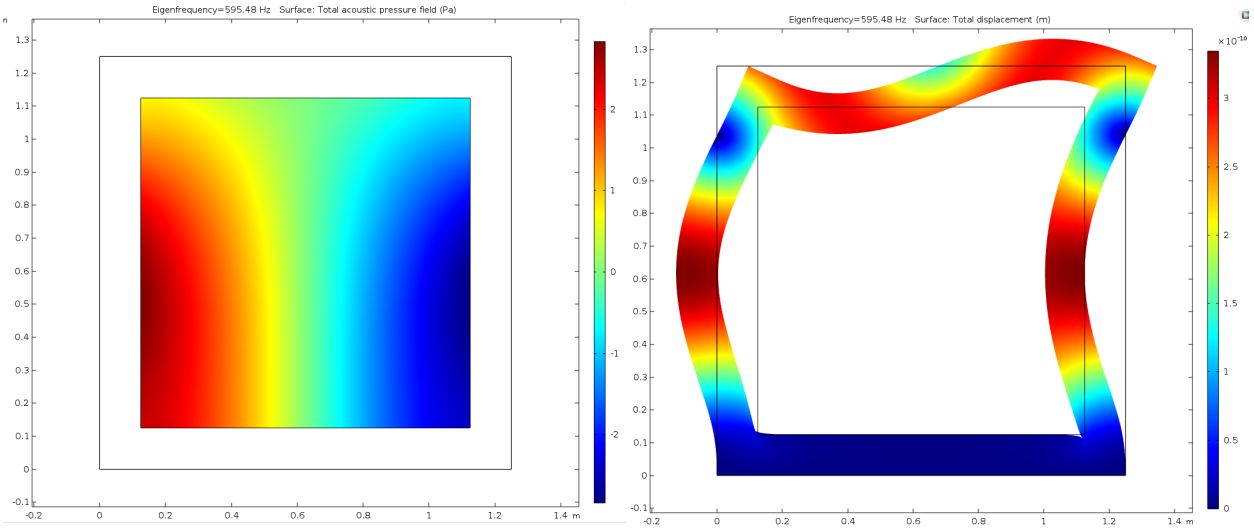
(a) Eigenfrequency = 118.2 Hz



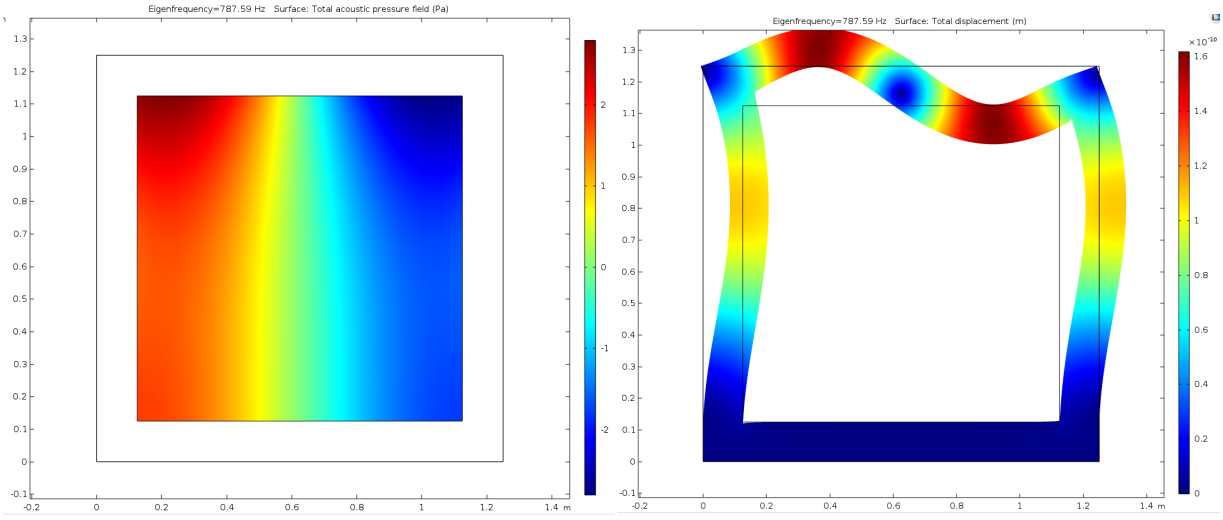
(b) Eigenfrequency = 434.86 Hz



(c) Eigenfrequency = 595.48 Hz



(d) Eigenfrequency = 787.59 Hz



(e) Eigenfrequency = 812.99 Hz

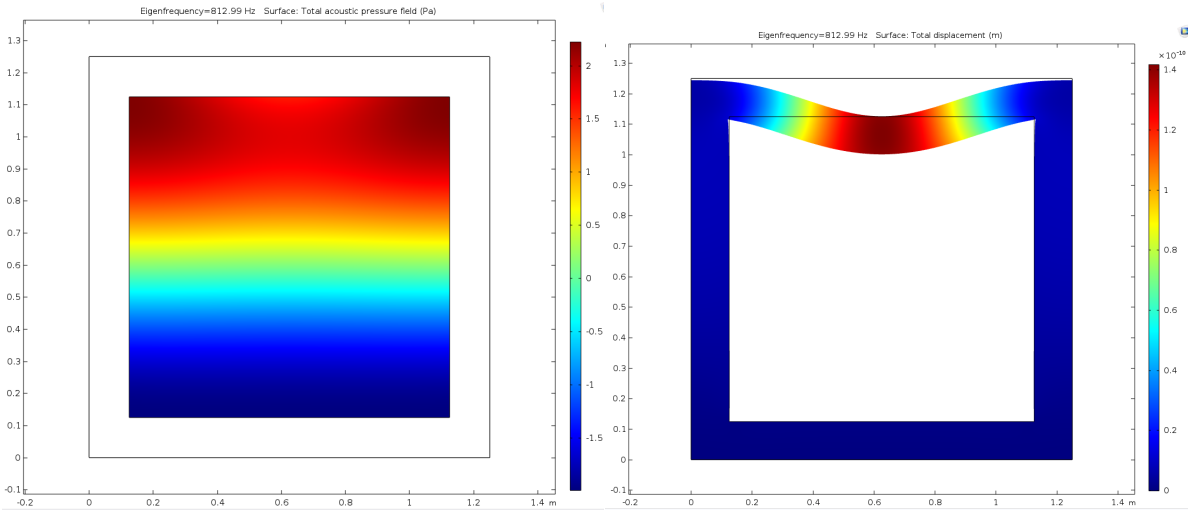


Fig. 3. The acoustic pressure field in Pa (left) and the solid displacement field in m (right) are shown for the first five natural eigenfrequencies. Simulation was done using COMSOL Multiphysics using a finite element approach.

5. Discussion and Conclusions

We have derived the partial differential equations in two different forms for the fluid-structure interaction elastoacoustic problem. We have discussed the weak form of the equations, and simulated the eigenfrequencies for first five modes for the described model. Fluid-srtucture interaction is a vast topic, and through this study, I have understood the mathematical formulations and the numerical tools used to study the deformations of a coupled system. My future work would entail the study of acoustic and elastic coupling for the interior of the earth at seismogenic depths to better understand the role of fluids in the onset of earthquakes.

References:

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