Module 4
2D Fourier Transform

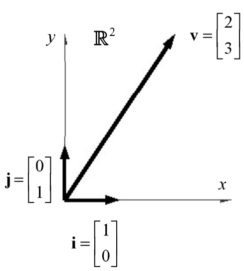
- **Fourier Transform Basics**
- **Sinusoidal Image**
- **2D Discrete Fourier Transform**
- **Meaning of Image Frequencies**
- **Sampling Theorem**
- **Some Important 2D DFT Pairs**

Joseph Fourier

Yesterday was my 21st birthday, at that age Newton *and* Pascal *had already acquired many claims to immortality.* - *In a letter by Fourier*

- Fourier was a math professor. He studied heat conduction. His PhD advisor was Lagrange.
- His most important contribution was development of the idea that an arbitrary function could be written as an infinite sum (series) of sines and cosines.
- Ironically, this work was bitterly criticized by both Laplace and Lagrange at the time!

Vectors in $\mathbb{R}^2$



- The natural basis: $\{\mathbf{i},\mathbf{j}\}$
- Representation: write an arbitrary vector $\mathbf{v}$ as a linear combination of the basis.
- **Question:** how do we do this?
- **Answer:** use the **dot product** (inner product).
- For two vectors $\mathbf{v}=\begin{bmatrix}v_1 & v_2\end{bmatrix}^T$ and $\mathbf{w}=\begin{bmatrix}w_1 & w_2\end{bmatrix}^T$, the dot product is often written using angle brackets:

$$\langle \mathbf{v}\,,\mathbf{w}\rangle = v_1w_1+v_2w_2$$

- Recall that the dot product is a **scalar**.

- Any arbitrary $\mathbf{v}\in\mathbb{R}^2$ can be written as $\mathbf{v}=\langle\mathbf{v},\mathbf{i}\rangle\mathbf{i}+\langle\mathbf{v},\mathbf{j}\rangle\mathbf{j}$.

- For example, on the last page we had $\mathbf{v}=\begin{bmatrix}2 & 3\end{bmatrix}^T$. Using the dot product,

$$\begin{aligned}\mathbf{v}&=\langle\mathbf{v},\mathbf{i}\rangle\mathbf{i}+\langle\mathbf{v},\mathbf{j}\rangle\mathbf{j}\\ &=\left\langle\begin{bmatrix}2\\3\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix}\right\rangle\begin{bmatrix}1\\0\end{bmatrix}+\left\langle\begin{bmatrix}2\\3\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\rangle\begin{bmatrix}0\\1\end{bmatrix}\\ &=(2\cdot1+3\cdot0)\begin{bmatrix}1\\0\end{bmatrix}+(2\cdot0+3\cdot1)\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}2\\3\end{bmatrix}\quad\checkmark\end{aligned}$$

- This is a powerful way to think about things because it works for **any** orthonormal basis.
- **Fact:** another orthonormal basis for $\mathbb{R}^2$ is given by the vectors

$$\mathbf{e}_1=\begin{bmatrix}\frac{1}{\sqrt2} & \frac{1}{\sqrt2}\end{bmatrix}^T \qquad \mathbf{e}_2=\begin{bmatrix}\frac{-1}{\sqrt2} & \frac{1}{\sqrt2}\end{bmatrix}^T$$

- For $\mathbf{v}=\begin{bmatrix}2 & 3\end{bmatrix}^T$, you can easily check that

$$\langle\mathbf{v},\mathbf{e}_1\rangle\mathbf{e}_1+\langle\mathbf{v},\mathbf{e}_2\rangle\mathbf{e}_2=\left\langle\begin{bmatrix}2\\3\end{bmatrix},\begin{bmatrix}\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}\end{bmatrix}\right\rangle\begin{bmatrix}\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}\end{bmatrix}+\left\langle\begin{bmatrix}2\\3\end{bmatrix},\begin{bmatrix}\frac{-1}{\sqrt2}\\\frac{1}{\sqrt2}\end{bmatrix}\right\rangle\begin{bmatrix}\frac{-1}{\sqrt2}\\\frac{1}{\sqrt2}\end{bmatrix}=\begin{bmatrix}2\\3\end{bmatrix}\quad\checkmark$$

Some Important Things About the Dot Product

- In higher dimensional spaces like $\mathbb{R}^{100}$, it becomes convenient to use "Capital Sigma" notation to write the dot product:

$$\langle\mathbf{v},\mathbf{w}\rangle=v_1w_1+v_2w_2+\cdots+v_{100}w_{100}=\sum_{k=1}^{100}v_kw_k$$

- But the whole procedure still works exactly like before in $\mathbb{R}^2$.
- For an orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\ldots,\mathbf{e}_{100}\}$, the representation of $\mathbf{v}$ is

$$\mathbf{v}=\langle\mathbf{v},\mathbf{e}_1\rangle\mathbf{e}_1+\langle\mathbf{v},\mathbf{e}_2\rangle\mathbf{e}_2+\cdots+\langle\mathbf{v},\mathbf{e}_{100}\rangle\mathbf{e}_{100}=\sum_{k=1}^{100}\langle\mathbf{v},\mathbf{e}_k\rangle\mathbf{e}_k$$

- The operation of computing the dot products $\langle\mathbf{v},\mathbf{e}_k\rangle$ defines a **transform**. It gives the representation (or coordinates) of $\mathbf{v}$ relative to the basis $\{\mathbf{e}_1,\mathbf{e}_2,\ldots,\mathbf{e}_{100}\}$.
- The operation of writing $\mathbf{v}$ as sum of the basis vectors defines an **inverse transform**.

Important Things About Dot Products

- One thing you may not have been taught about the dot product is: if the vectors have **complex** entries, then you must **conjugate** one of the vectors in the dot product.
- In engineering, we always conjugate the **second** vector.
- The dot product then becomes

$$\langle\mathbf{v},\mathbf{w}\rangle=\sum_k v_kw_k^*$$

- To write a vector $\mathbf{v}$ as a sum of the basis, you still add up the dot products times the basis vectors just like before:

$$\mathbf{v}=\sum_k\langle\mathbf{v},\mathbf{e}_k\rangle\mathbf{e}_k$$

- **Note:** sometimes in math and physics, it is the **first** vector that gets conjugated in the dot product. But in engineering, we **always** conjugate the **second** vector.

Infinite Dimensional Vector Spaces

- These ideas extend easily to infinite dimensional vector spaces.
- In your undergraduate Signals & Systems course, you dealt with discrete-time signals like $x[n]$.
- You can consider these signals to be vectors in $\mathbb{C}^\infty$ or $\mathbb{R}^\infty$.
- The dot product in these spaces works just like you think it should:

$$\langle x[n], y[n] \rangle = \sum_{n=-\infty}^{\infty} x[n] y^*[n].$$

- For an orthonormal basis $\{\ldots, \mathbf{e}_{-1}, \mathbf{e}_0, \mathbf{e}_1, \ldots\}$, you can write $x[n]$ as

$$x[n] = \sum_{k=-\infty}^{\infty} \langle x[n], \mathbf{e}_k \rangle \mathbf{e}_k$$

Natural Basis in $\mathbb{C}^\infty$ or $\mathbb{R}^\infty$

- Think of the signals $\delta[n]$, $\delta[n-1]$, $\delta[n+1]$, and $\delta[n-k]$ from your undergraduate Signals & systems course.
- Each one of them is "turned on" in exactly one place where it is equal to one.
- For vector spaces of discrete-time signals $x[n]$, the set of **translates** $\delta[n-k]$ of the Kronecker delta for $k \in \mathbb{Z}$ plays the **same role** that the basis $\{\mathbf{i}, \mathbf{j}\}$ plays in $\mathbb{R}^2$; i.e., it is the **natural basis**.
- The dot product of $x[n]$ with the basis vector $\delta[n-k]$ is the number

$$\langle x[n], \delta[n-k] \rangle = \sum_{n=-\infty}^{\infty} x[n]\delta[n-k] = x[k].$$

- The signal $x[n]$ can be written as a sum of the basis just like before:

$$\begin{aligned} x[n] &= \cdots + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + \cdots \\ &= \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] = \sum_{k=-\infty}^{\infty} \langle x[n], \delta[n-k] \rangle \delta[n-k]. \end{aligned}$$

Uncountably Infinite Dimensions

- We can let the number of dimensions grow even larger, so that it is equal to the number of real numbers in $\mathbb{R}$.
- The vectors in these spaces are exactly the continuous-time signals $x(t)$ from your undergraduate Signals & Systems course.
- Everything still works just like it did before, but now you have to use "Capital S" adding instead of "Capital Sigma" adding.
- The dot product in these spaces is given by

$$\langle x(t), y(t) \rangle = \int_{-\infty}^{\infty} x(t) y^*(t)\, dt.$$

- For an orthonormal basis $\{\mathbf{e}_\tau\}_{\tau \in \mathbb{R}}$, you can write $x(t)$ just like before:

$$x(t) = \int_{-\infty}^{\infty} \langle x(\tau), \mathbf{e}_\tau \rangle \mathbf{e}_\tau \, d\tau$$

Uncountably Infinite Dimensions

- The natural basis is a little bit harder to talk about in this case, because **distribution theory** is required to treat the Dirac delta $\delta(t)$ rigorously.
- Here, "distribution theory" means the theory of generalized functions. It has nothing to do with "probability distributions."
- We won't go into the details (take ECE 5213, DSP, for that).
- Nevertheless, it should make intuitive sense to you at this point that the natural basis is given by the translates of the Dirac delta:

$$\{\delta(t-\tau)\}_{\tau \in \mathbb{R}}$$

- The dot product of $x(t)$ with a basis function is the number

$$\langle x(t), \delta(t-\tau) \rangle = \int_{-\infty}^{\infty} x(t)\delta(t-\tau)\, dt = x(\tau).$$

- You write $x(t)$ as a sum of the basis just like before, by adding up the dot products times the basis vectors:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\, d\tau.$$

Change of Basis

- Recall from calculus that changing coordinate systems can sometimes turn a hard problem into an easier one.
  - For example, when computing a solid of revolution, switching from rectangular coordinates to cylindrical coordinates often makes the problem easier.
- Similarly, when working on signals or images, it is sometimes very useful to change from the natural basis to some other basis.
- This can let you look at the signal in a different way where it's easier to see different information that might be important for a particular problem.
- It can also turn some "hard" problems into easy ones. For example, an appropriate change of basis turns convolution into multiplication (more on this later).

Spectral Basis for Discrete-Time Signals

- **Fact:** the set of signals $\{e^{j2\pi fn}\}$ for $f \in (-\frac{1}{2}, \frac{1}{2}]$ is an orthonormal basis for the vector space of signals $x[n]$ that you dealt with in your undergraduate Signals & Systems course.
- Because the basis vectors (basis signals) have **complex** entries, it is important for you to remember to conjugate when you take dot products!
- How do we write $x[n]$ as a sum of this basis?
  1. Take the dot product of $x[n]$ with each of the basis vectors. This gives us a (complex) number for each basis vector.
  2. Add up the dot products times the basis vectors to get $x[n]$.
- Since there are an infinite number of basis vectors, we will have a lot of dot products to keep track of… one for each $f \in (-\frac{1}{2}, \frac{1}{2}]$.
- Notice that this defines a **function**. The domain is $f \in (-\frac{1}{2}, \frac{1}{2}]$ and the range is complex numbers (the dot products).
- We will call it $X(e^{j2\pi f})$ and use it to keep track of the dot products.

1D Discrete-Time Fourier Transform

- **Step 1:** take the dot product of $x[n]$ with a basis vector (don't forget to conjugate the entries of the second vector!):

$$X(e^{j2\pi f}) = \left\langle x[n], e^{j2\pi fn} \right\rangle = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi fn}$$

- This is called the **discrete-time Fourier transform (DTFT).**
- **Step 2:** add up the dot products times the basis functions to get $x[n]$. Because the number of basis functions is uncountable, we have to use "Capital S" adding:

$$x[n] = \int_{-1/2}^{1/2} \left\langle x[n], e^{j2\pi fn} \right\rangle e^{j2\pi fn}\, df = \int_{-1/2}^{1/2} X(e^{j2\pi f}) e^{j2\pi fn}\, df$$

- This is called the **inverse DTFT**.

Why Use the DTFT?

- Looking at the graph of $X(e^{j2\pi f})$ can make it easy to see certain things that are hard to see from the graph of $x[n]$.
  - For example, if $x[n]$ is a digital audio signal, changing to the spectral basis and looking at the graph of $X(e^{j2\pi f})$ may make it easy to see if there is too much bass or not enough midrange.
- Let H be a discrete-time LTI system with impulse response $h[n]$ and frequency response $H(e^{j2\pi f})$.
- For an arbitrary input signal $x[n]$, the output signal is given by the convolution $y[n] = x[n] * h[n]$ which can take a lot of work to compute.
- But if $x[n]$ is an **eigenfunction** of the system, then the output is easy to compute. It is given by $y[n] = \lambda\, x[n]$, where λ is an eigenvalue that is complex-valued in general.
- For an arbitrary input signal $x[n]$, the output will be easy to compute if we can write $x[n]$ as a sum of eigenfunctions. Each term of the sum will just get multiplied by a complex eigenvalue.

- **Fact:** the DTFT basis signals are all eigenfunctions of **any** discrete-time LTI system. The eigenvalues are given by the system frequency response.
- **Proof:** let the system input be $x[n] = e^{j2\pi fn}$. The output is given by

$$\begin{aligned} y[n] &= x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k] \\ &= \sum_{k=-\infty}^{\infty} e^{j2\pi f(n-k)} h[k] \\ &= e^{j2\pi fn}\left[\sum_{k=-\infty}^{\infty} h[k] e^{-j2\pi fk}\right] = H(e^{j2\pi f})x[n]. \qquad \blacksquare \end{aligned}$$

- Use the DTFT to write an arbitrary input $x[n]$ as a sum of eigenfunctions. The system output is given by

$$\begin{aligned} y[n] &= H\{x[n]\} = H\left\{\int_{-1/2}^{1/2} X(e^{j2\pi f}) e^{j2\pi fn}\, df\right\} \\ &= \int_{-1/2}^{1/2} X(e^{j2\pi f}) H\left\{e^{j2\pi fn}\right\} df \\ &= \int_{-1/2}^{1/2} X(e^{j2\pi f}) H(e^{j2\pi f}) e^{j2\pi fn}\, df = \int_{-1/2}^{1/2} Y(e^{j2\pi f}) e^{j2\pi fn}\, df. \end{aligned}$$

- In other words, for an arbitrary input signal $x[n]$, the output of the LTI system H is given by

$$y[n] = \mathrm{IDTFT}\left\{X(e^{j2\pi f})H(e^{j2\pi f})\right\}.$$

- If we represent the input signal using the natural basis, then convolution is required to compute the system output.
- But if we use the DTFT to change to the spectral basis, then the output is given by multiplication.

1D Fourier Transform

- **Fact:** the set of signals $\left\{e^{j2\pi ft}\right\}$ for $f \in \mathbb{R}$ is an orthonormal basis for the vector space of signals $x(t)$ that you dealt with in your undergraduate Signals & Systems course.
- Let's write $x(t)$ as a sum of this basis.
- **Step 1:** take the dot product of $x(t)$ with a basis vector (don't forget to conjugate the entries of the second vector!):

$$X(f) = \left\langle x(t), e^{j2\pi ft} \right\rangle = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft}\, dt$$

- This is called the **Fourier transform (FT).**
- **Step 2:** add up the dot products times the basis functions to get $x(t)$:

$$x(t) = \int_{-\infty}^{\infty} \left\langle x(t), e^{j2\pi ft} \right\rangle e^{j2\pi ft}\, df = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft}\, df$$

- This is called the **inverse Fourier Transform (IFT)**.

Orthogonal Basis

- Going back to $\mathbb{R}^2$, consider the basis

$$\mathbf{e}_1 = \begin{bmatrix} 3 & 0 \end{bmatrix}^T \qquad \mathbf{e}_2 = \begin{bmatrix} 0 & 3 \end{bmatrix}^T$$

- This basis is **orthogonal**, but it is **not orthonormal** since each basis vector has length **three** instead of one.
- If we try to write the vector $\mathbf{v} = \begin{bmatrix} 2 & 3 \end{bmatrix}^T$ as a sum of this basis using our method, we get

$$\begin{aligned} \left\langle \mathbf{v}, \mathbf{e}_1 \right\rangle \mathbf{e}_1 + \left\langle \mathbf{v}, \mathbf{e}_2 \right\rangle \mathbf{e}_2 &= \left\langle \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix} \right\rangle \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \left\langle \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right\rangle \begin{bmatrix} 0 \\ 3 \end{bmatrix} \\ &= (2\cdot 3 + 3\cdot 0)\begin{bmatrix} 3 \\ 0 \end{bmatrix} + (2\cdot 0 + 3\cdot 3)\begin{bmatrix} 0 \\ 3 \end{bmatrix} \\ &= 6\begin{bmatrix} 3 \\ 0 \end{bmatrix} + 9\begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 18 \\ 27 \end{bmatrix} = 9\begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad ✗ \end{aligned}$$

- We get the **wrong** answer.
  - Because the basis was not orthonormal, the dot products were all too big by a factor of 3.
  - We multiplied these dot products that were all too big times basis vectors that were all too long by a factor of 3.
- All together, our answer is **too big** by a factor of 3×3 = 9, which is the length of a basis vector squared.
- When the basis is orthogonal, but **not** orthonormal, our method needs some "fixing up."
  - We can divide by the squared length of a basis vector when we compute the dot products (fix up on the **transform**),
  - or we can divide by the squared length of a basis vector when we add up the dot products times the basis vectors (fix up on the **inverse transform**).
  - or we can divide by the length of a basis vector in both places (fix up on both the transform and the inverse transform).

- You will often see the DTFT basis written in terms of radian frequency instead of Hertzian frequency.
- The basis is then $\{e^{j\omega n}\}$ for $\omega \in (-\pi, \pi]$.
- This basis is **orthogonal**, but it is **not** orthonormal.
  - Using the Riemann-Lebesgue lemma of distribution theory, one can show that the length of each basis vector is $\sqrt{2\pi}$, not one (take ECE 5213, DSP, for the details).
  - Because of this, we have to divide by 2π somewhere to make our method work.
- In engineering, the most common convention is to divide by 2π on the inverse transform. This gives the alternate DTFT/IDTFT definition

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} \, d\omega$$

which is found in many textbooks.

- Likewise, you will often see the FT basis written in terms of radian frequency instead of Hertzian frequency.
- The basis is then given by $\{e^{j\omega t}\}$ for $\omega \in \mathbb{R}$.
- This basis is again **orthogonal** but **not** orthonormal. Each basis vector has length $\sqrt{2\pi}$ instead of one.
- This gives the alternate FT/IFT definition

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} \, d\omega$$

1D Discrete Fourier Transform

- Consider a finite-length discrete-time signal $x[n]$ defined for $0 \le n \le N\text{-}1$.
- Such signals can be considered as vectors in $\mathbb{R}^N$ or $\mathbb{C}^N$.
- Since the space is N-dimensional, there are N vectors in a basis.
- The spectral (or Fourier) basis is given by

$$\left\{ e^{j\frac{2\pi k}{N} n} \right\}_{0 \le k \le N-1}, \quad 0 \le n \le N-1$$

- The **discrete Fourier transform** or **DFT** is defined (as always) by taking the dot product of the signal with a basis vector:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k}{N} n}, \quad 0 \le k \le N-1$$

- The **inverse DFT (IDFT)** is then obtained by adding up the dot products times the basis vectors and dividing out the squared length of a basis vector:

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi k}{N} n}, \quad 0 \le n \le N-1$$

- It is customary to write the DFT and IDFT using the shorthand notation $W_N = e^{-j2\pi/N}$.
- NOTE: W_N is a complex **number**… a **constant** (for any fixed choice of N).
- The forward and reverse transforms then become

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \quad 0 \le k \le N-1$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}, \quad 0 \le n \le N-1$$

- Notice that the conjugation operation is **built in** to the symbol W_N.
  - This makes the "–" sign in the exponent appear on the inverse transform instead of the forward transform when the "W_N" notation is used.
- **Note:** it's important that the DFT math starts with $n = 0$ and $k = 0$.
- In a Matlab DFT array, you have to remember that the first element has array index 1, but it's for $k = 0$.

*n*D Sinusoids – Continuous Case

- The 1D cosine: $\cos(2\pi f t)$.
- Instead of the scalar t, in nD we have a position **vector** $\mathbf{x} = [x_1 \; x_2 \; \dots \; x_n]^T$.
- Instead of the scalar f, we have a frequency **vector** $\mathbf{u} = [u_1 \; u_2 \; \dots \; u_n]^T$.
- But the argument of $\cos$ must still be **scalar**.
- It is the dot product of the frequency vector and the position vector:

$$\mathbf{u}^T \mathbf{x} = u_1 x_1 + u_2 x_2 + \cdots + u_n x_n$$

- The nD cosine: $\cos(2\pi \mathbf{u}^T \mathbf{x})$.
- The nD sine: $\sin(2\pi \mathbf{u}^T \mathbf{x})$.
- The nD complex exponential: $e^{j2\pi \mathbf{u}^T \mathbf{x}} = \cos(2\pi \mathbf{u}^T \mathbf{x}) + j\sin(2\pi \mathbf{u}^T \mathbf{x})$.
- Note: if $n = 1$, these reduce to the usual 1D definitions.
- We'll develop intuition about them later; for now we focus on the math.

nD Fourier Transform

- For the nD continuous case, the spectral basis is $\left\{e^{j2\pi\mathbf{u}^T\mathbf{x}}\right\}_{\mathbf{u}\in\mathbb{R}^n}$.
- Let $s(\mathbf{x})$ be an nD signal.
- For the FT, you simply take the dot product of $s(\mathbf{x})$ with a basis signal (as always) – don't forget to conjugate the entries of the second vector:

$$S(\mathbf{u}) = \int_{\mathbb{R}^n} s(\mathbf{x}) e^{-j2\pi\mathbf{u}^T\mathbf{x}} d\mathbf{x}$$

- You can "write out" the nD integral like this:

$$S(\mathbf{u}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} s(\mathbf{x}) e^{-j2\pi(u_1x_1+u_2x_2+\cdots+u_nx_n)} dx_1 dx_2 \cdots dx_n$$

- The nD inverse Fourier Transform is just what you would expect:

$$s(\mathbf{x}) = \int_{\mathbb{R}^n} S(\mathbf{u}) e^{j2\pi\mathbf{u}^T\mathbf{x}} d\mathbf{u} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} S(\mathbf{u})\, e^{j2\pi(u_1x_1+u_2x_2+\cdots+u_nx_n)} du_1 du_2 \cdots du_n$$

nD Fourier Transform

- If you want to use a radian frequency vector $\boldsymbol{\omega} = 2\pi\mathbf{u}$, then you have to divide out a "2π" for **each** dimension.
- In engineering, we usually do it on the inverse transform.
- So in terms of radian frequency, the continuous nD FT and IFT become:

$$S(\boldsymbol{\omega}) = \int_{\mathbb{R}^n} s(\mathbf{x}) e^{-j\boldsymbol{\omega}^T\mathbf{x}} d\mathbf{x}$$

$$s(\mathbf{x}) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} S(\boldsymbol{\omega}) e^{j\boldsymbol{\omega}^T\mathbf{x}} d\boldsymbol{\omega}$$

Important Notation Changes

- For the rest of the module, we will write (m,n) for image spatial coordinates. For discussing Fourier transforms, this almost always means (*column*, *row*).
  - Up to now we have thought of (m,n) as (*row*, col), consistent with Matlab and usual matrix "row-col" addressing.
  - But for the FT we think of 2D functions $f(x,y)$, where x is horizontal.
- We will also usually think of the frequency in cycles per image (cpi) instead of cycles per sample (pixel), since this is easier to interpret visually.
- For the case of 2D digital images, we will write the 2D frequency vector as $\mathbf{u} = [u \quad v]^T$, where u = "horizontal frequency" and v = "vertical frequency."

Sinusoidal Images

- We shall make frequent discussion in this module of image **frequency content**.
- The image having the **simplest** frequency content is the **sinusoidal image.**

Sinusoidal Images

- A **discrete sine image $\mathbf{I}$** is given by

$$\mathrm{I}(m,n) = \sin\left[2\pi\left(\tfrac{u}{M}m + \tfrac{v}{N}n\right)\right]$$

  and a **discrete cosine image** is given by

$$\mathrm{I}(m,n) = \cos\left[2\pi\left(\tfrac{u}{M}m + \tfrac{v}{N}n\right)\right]$$

  for $0 \le m \le M\text{-}1,\ 0 \le n \le N\text{-}1$
- Here, m is the column and n is the row.

Important Note

- For the mathematics of the 2D Discrete Fourier Transform (DFT) to work out, it is **very important** that
  - The first row in the image is row ZERO, **not** row one!
  - The first column is column ZERO, **not** column one!
- If the image is in a Matlab array, you must remember that the first row has index 1, but in terms of the mathematics, it is row 0!
- Similarly, in a Matlab array, the first column has index 1, but in terms of the mathematics it is column 0!

Allowable Values for the Frequencies

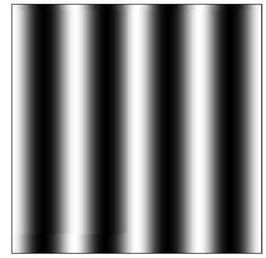
- The horizontal frequency u and vertical frequency v can be any **real** numbers.
- However, for constructing the 2D DFT basis for an $M \times N$ digital image, we will only need
  - u to be integers between 0 and M-1.
  - v to be integers between 0 and N-1.
- For this reason, u and v will often show up as integers.

Cosine Images

- If $M = N = 256$, $u = 4$, and $v = 0$, then the image is

$$\mathrm{I}(m,n) = \cos\,[2\pi\,\tfrac{4}{256}\,m]$$

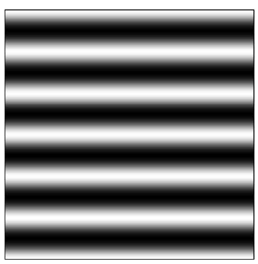
- Since the image does not depend on n, **every row is the same**.
- Every row is a cosine with 4 cycles per image:



- If $M = N = 256$, $u = 0$, and $v = 6$, then the image is

$$\mathrm{I}(m,n) = \cos\,[2\pi\,\tfrac{6}{256}\,n]$$

- Since the image does not depend on m, **every column is the same**.
- Every column is a cosine with 6 cycles per image:



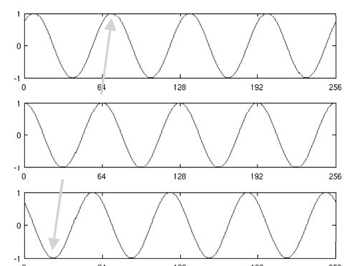
- For a moment, think of the 1D signal $\cos(2\pi ft)$.
- The graph of $\cos(2\pi ft - \phi)$ is the same, but shifted right by an amount $\phi/(2\pi f)$ on the t axis, since

$$\cos(2\pi ft - \phi) = \cos\left[2\pi f\left(t - \frac{\phi}{2\pi f}\right)\right].$$

$$\cos\left[\frac{2\pi 4}{256}t - \frac{\pi}{4}\right]$$

$$\cos\left[\frac{2\pi 4}{256}t\right]$$

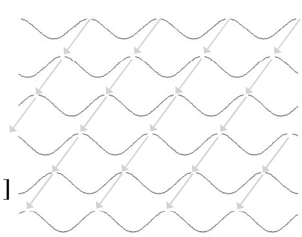
$$\cos\left[\frac{2\pi 4}{256}t + \frac{\pi}{4}\right]$$



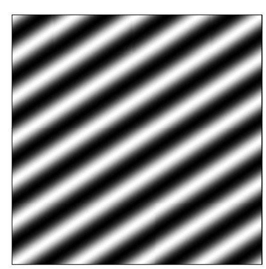
- Now, let $M = N = 256$, $u = 4$, and $v = 6$. The image is

$$\mathrm{I}(m,n) = \cos\,[2\pi(\tfrac{4}{256}\,m + \tfrac{6}{256}\,n)]$$

- On the first row, n=0, so $\mathrm{I}(m,n) = \cos\,[2\pi\,\tfrac{4}{256}\,m]$
- On the 2nd row, n=1, so $\mathrm{I}(m,n) = \cos\,[2\pi\tfrac{4}{256}m + \tfrac{2\pi 6}{256}]$
- On the 3rd row, n=2, so $\mathrm{I}(m,n) = \cos\,[2\pi\tfrac{4}{256}m + \tfrac{2\pi 12}{256}]$
- On the kth row, n=k-1, so $\mathrm{I}(m,n) = \cos\,[2\pi\tfrac{4}{256}m + \tfrac{2\pi 6(k-1)}{256}]$
- They are all cosines with horizontal frequency 4 cpi.
- But each row is shifted by $\phi = \tfrac{2\pi 6}{256}$ compared to the row above.

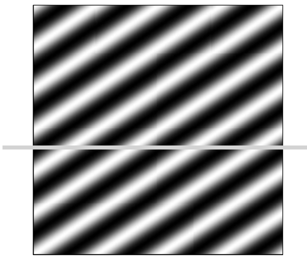


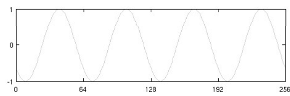
- As an image, this looks like: ($M = N = 256$, $u = 4$, $v = 6$)



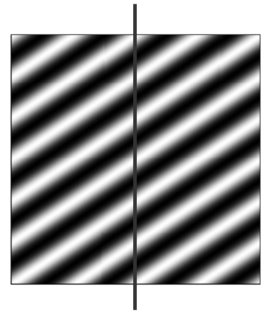
- Horizontally, every row goes through u=4 cycles across the image (but they start at different phase offsets).
- Vertically, every column goes through v=6 cycles down the image (but starting at different phase offsets).

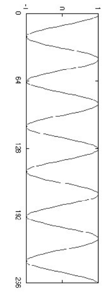
- If you take any horizontal slice through the image, you get a 1D cosine with frequency u=4 cpi.





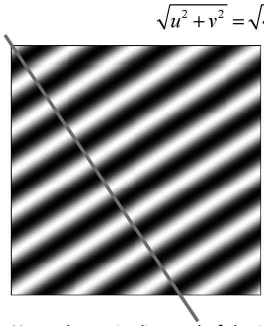
- If you take any vertical slice through the image, you get a 1D cosine with frequency v=6 cpi.

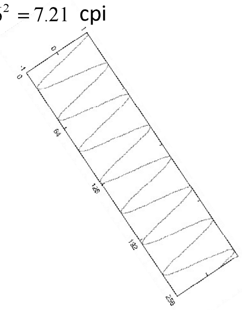




- If you take a slice through the image in a direction normal to the wavefronts, you get a 1D cosine with frequency

$$\sqrt{u^2+v^2} = \sqrt{4^2+6^2} = 7.21 \text{ cpi}$$

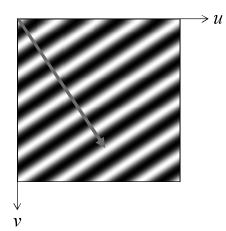




Note: the main diagonal of the image is longer than one "image." It has a length of $\sqrt{1^2+1^2} = 1.41$ images. So along a line the length of the main diagonal, we expect to see about $1.41 \times 7.21 = 10.17$ cycles.

Interpreting the 2D Frequency

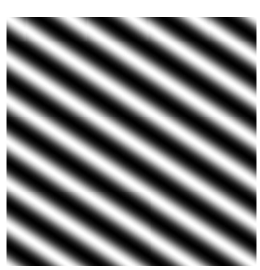
- The 2D frequency vector $[u\ v]^T$ has a direction $\theta = \tan^{-1}(v/u)$.
- This is usually called the **frequency orientation**.
- If you draw u and v axes that agree with the sense of the image coordinates, then the frequency vector points in a direction normal to the wavefronts; it points in the direction of "propagation."
- The magnitude of the frequency vector is
  $$\Omega = \sqrt{u^2+v^2}$$
- This is called the **radial frequency** or **magnitude frequency**. It is also often denoted by R.
- In the direction θ, the frequency of oscillation is Ω cpi.



More on Cosine Images

- If u = -4 and v = -6, we get the same image over again since cosine is **even**.
- However, the relative signs of u and v matter.
- If u = -4 and v = 6, it changes the orientation and we get a different image.

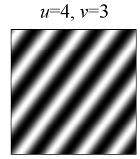
$u = -4;\ v = 6$



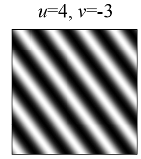
Sine Images

- Everything works pretty much the same way for sine images.
- Except: since sine is **odd**, if you negate both u and v it multiplies the image by **-**1.
- Also, for a sine image
  $$\mathrm{I}(m,n) = \sin\left[2\pi\left(\tfrac{u}{M}m + \tfrac{v}{N}n\right)\right]$$
  the pixel at m=0, n=0 is gray instead of white.

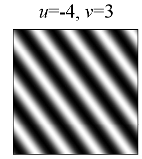
u=4, v=3



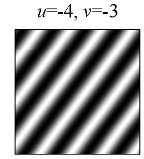
u=4, v=-3



u=-4, v=3



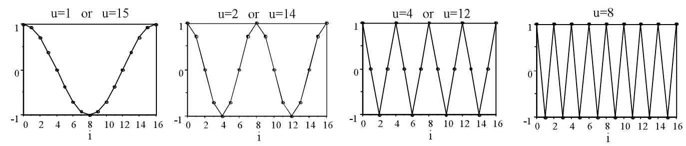
u=-4, v=-3



Spatial Frequencies

Digital Sinusoid Example

- Let $M = 16$, $v = 0$: $\mathrm{I}(m) = \cos(2\pi um/16)$: a cosine wave oriented in m-direction with frequency u. One row:



- Note that $\mathrm{I}(m) = \cos(2\pi um/16) = \cos[2\pi(16\text{-}u)m/16]$.
- Thus the **highest frequency wave occurs at $u = M/2$** (M is even here). **This will be important later.**

Explanation

- Why does $\cos(2\pi um/16) = \cos[2\pi(16\text{-}u)m/16]$?
- The reason is that adding any integer multiple of 2π to the argument of cos does not change the value.

$$\begin{aligned}\cos\left[\frac{2\pi(M-u)}{M}m\right] &= \cos\left[\frac{2\pi M}{M}m - \frac{2\pi u}{M}m\right] \\ &= \cos\left[-\frac{2\pi u}{M}m + 2\pi m\right] \\ &= \cos\left[-\frac{2\pi u}{M}m\right] = \cos\left[\frac{2\pi u}{M}m\right].\end{aligned}$$

Complex Exponential Image

- We'll use **complex exponential functions** to define the **Discrete Fourier Transform**.
- Define the **2-D complex exponential:**

$$\exp\left[-j2\pi\left(\frac{u}{M}m + \frac{v}{N}n\right)\right] \text{ for } 0 \le m \le M-1,\ 0 \le n \le N-1$$

- Here, we put the "-" sign in front of the $j2\pi$ for compatibility with the W_N notation.

- Intuitively, the complex exponential image can be understood in terms of sine and cosine images, since

$$\begin{aligned}\exp\left[-j2\pi\left(\frac{u}{M}m + \frac{v}{N}n\right)\right] &= \cos\left[2\pi\left(\frac{u}{M}m + \frac{v}{N}n\right)\right] - j\sin\left[2\pi\left(\frac{u}{M}m + \frac{v}{N}n\right)\right] \\ &= \cos\left[2\pi\left(\frac{u}{M}m + \frac{v}{N}n\right)\right] + j\sin\left[2\pi\left(\frac{-u}{M}m + \frac{-v}{N}n\right)\right].\end{aligned}$$

- Also note that the complex exponential image is **separable**:

$$e^{-j2\pi\left(\frac{u}{M}m + \frac{v}{N}n\right)} = e^{-j2\pi\frac{u}{M}m} e^{-j2\pi\frac{v}{N}n}$$

Properties of Complex Exponential

- Recall from page 4.23:

$$W_N = \exp\left(-j\frac{2\pi}{N}\right)$$

(N = image dimension).

- Hence, for a square image with $M=N$,

$$\exp\left[-j2\pi\left(\frac{u}{N}m + \frac{v}{N}n\right)\right] = W_N^{(um + vn)} = W_N^{um} W_N^{vn}$$

- More generally, when the image is **not** square, we have

$$\exp\left[-j2\pi\left(\frac{u}{M}m+\frac{v}{N}n\right)\right]=\exp\left[-j2\pi\frac{u}{M}m\right]\exp\left[-j2\pi\frac{v}{N}n\right]$$
$$=W_M^{um}W_N^{vn}$$

- The magnitude and phase of the complex exponential image are

$$\left|\exp\left[-j2\pi\left(\frac{u}{M}m+\frac{v}{N}n\right)\right]\right|=1$$
$$\angle\exp\left[-j2\pi\left(\frac{u}{M}m+\frac{v}{N}n\right)\right]=-2\pi\left(\frac{u}{M}m+\frac{v}{N}n\right)$$

Complex Exponential Image

- From **Euler's identity**:

$$W_M=\cos\left(\frac{2\pi}{M}\right)-j\sin\left(\frac{2\pi}{M}\right)$$

and

$$W_M^{um}=\cos\left(2\pi\frac{u}{M}m\right)-j\sin\left(2\pi\frac{u}{M}m\right)$$

- The u and v **powers** of W_M and W_N index the frequencies of the horizontal and vertical sinusoids.

Simple Properties of W_M and W_N

$$\cos\left[2\pi\left(\frac{u}{M}m+\frac{v}{N}n\right)\right]=\frac{1}{2}\left(W_M^{um}W_N^{vn}+W_M^{-um}W_N^{-vn}\right)$$

$$\sin\left[2\pi\left(\frac{u}{M}m+\frac{v}{N}n\right)\right]=j\frac{1}{2}\left(W_M^{um}W_N^{vn}-W_M^{-um}W_N^{-vn}\right)$$

$$\left|W_M^{um}\right|=\left\{\cos^2\left(2\pi\frac{u}{M}m\right)+\sin^2\left(2\pi\frac{u}{M}m\right)\right\}=1$$

$$\angle W_M^{um}=\tan^{-1}\left\{\sin\left(2\pi\frac{u}{M}m\right)\Big/\cos\left(2\pi\frac{u}{M}m\right)\right\}=2\pi\frac{u}{M}m$$

Comments

- It's possible to develop frequency domain concepts without complex numbers - but the math is much lengthier.
- Using $W_M^{um}W_N^{vn}$ to represent a **frequency component** oscillating horizontally at u cpi and vertically at v cpi simplifies things considerably.
- It is useful to think of $W_M^{um}W_N^{vn}$ as a **representation** of a **direction and frequency of oscillation**.

Max/Min Values for u and v

- The complex exponential

$$W_M^{um}=\exp\left(\frac{-j2\pi}{M}um\right)$$

is a sinusoid with frequency indexed by exponent u.

- **Minimum physical frequencies**: $u = kM,\ k \in \mathbb{Z}$

$$W_M^{0m}=W_M^{kMm}=1\quad\forall\ m,k\in\mathbb{Z}$$

- **Maximum physical frequencies**: $u = (k+1/2)M$ (period 2)

$$W_M^{(kM+M/2)m}=1\cdot W_M^{(M/2)m}=(-1)^m\qquad(M\ \text{even})$$

2D DISCRETE FOURIER TRANSFORM

- Any $M \times N$ image $\mathbf{I}$ can be uniquely written as a **weighted sum** of MN **complex exponential** basis images:

$$\mathrm{I}(m,n)=\frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}\tilde{\mathrm{I}}(u,v)W_M^{-um}W_N^{-vn}\qquad\text{(IDFT)}$$

- This is called the **2D Inverse Discrete Fourier Transform** or **IDFT.**
- The integers $0 \le u \le M\text{-}1$ and $0 \le v \le N\text{-}1$ index the basis images and give their frequencies in cpi.
- The weights $\tilde{\mathbf{I}}$ are the **2D DFT coefficients** of $\mathbf{I}$.

Forward Transform (DFT)

- The forward transform is given by the dot product of the image $\mathbf{I}$ with a basis vector:

$$\tilde{\mathrm{I}}(u,v) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} \mathrm{I}(m,n)\ W_M^{um} W_N^{vn} \qquad \text{(DFT)}$$

- Remember that (m, n) are **space indices**, while (u,v) are **spatial frequency indices**.

2D DFT as an Array

- The 2D DFT is an array with the **same dimensions** $(M \times N)$ as the image $\mathbf{I}$:

$$\tilde{\mathbf{I}} = \left[\tilde{\mathrm{I}}(u,v);\ 0 \le u \le M-1,\ 0 \le v \le N-1\right]$$

- You can think of it as an image (complex-valued).
- Note that the DFT is a **linear transformation**, so it commutes with linear combinations:

$$\mathrm{DFT}\left[a_1\mathbf{I}_1 + a_2\mathbf{I}_2 + \cdots + a_L\mathbf{I}_L\right] = a_1\tilde{\mathbf{I}}_1 + a_2\tilde{\mathbf{I}}_2 + \cdots + a_L\tilde{\mathbf{I}}_L$$

DFT Array Properties

- The DFT of an image $\mathbf{I}$ is generally **complex:**

$$\tilde{\mathbf{I}} = \tilde{\mathbf{I}}_{\mathrm{Real}} + j\,\tilde{\mathbf{I}}_{\mathrm{Imag}}$$

where

$$\tilde{\mathrm{I}}_{\mathrm{Real}}(u,v) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} \mathrm{I}(m,n)\cos\left[2\pi\left(\frac{u}{M}m + \frac{v}{N}n\right)\right]$$

$$\tilde{\mathrm{I}}_{\mathrm{Imag}}(u,v) = -\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} \mathrm{I}(m,n)\sin\left[2\pi\left(\frac{u}{M}m + \frac{v}{N}n\right)\right]$$

Magnitude and Phase of the DFT

- It is usually more intuitive to look at the **magnitude** and **phase** of the DFT rather than the real and imaginary parts**:**

$$\left|\tilde{\mathbf{I}}\right| = \left[\left|\tilde{\mathrm{I}}(u,v)\right|; 0 \le u \le M-1,\ 0 \le v \le N-1\right]$$

$$\angle\tilde{\mathbf{I}} = \left[\angle\tilde{\mathrm{I}}(u,v); 0 \le u \le M-1,\ 0 \le v \le N-1\right]$$

where

$$\left|\tilde{\mathrm{I}}(u,v)\right| = \sqrt{\tilde{\mathrm{I}}_{\mathrm{Real}}^2(u,v) + \tilde{\mathrm{I}}_{\mathrm{Imag}}^2(u,v)}$$

$$\angle\tilde{\mathrm{I}}(u,v) = \tan^{-1}\left[\tilde{\mathrm{I}}_{\mathrm{Imag}}(u,v) / \tilde{\mathrm{I}}_{\mathrm{Real}}(u,v)\right]$$

- In polar form, the 2D DFT is

$$\tilde{\mathrm{I}}(u,v) = \left|\tilde{\mathrm{I}}(u,v)\right| \exp\left(j\angle\tilde{\mathrm{I}}(u,v)\right)$$

Symmetry of the DFT

- For a real image $\mathbf{I}$, the DFT is **conjugate symmetric**:

$$\tilde{\mathrm{I}}(M-u, N-v) = \tilde{\mathrm{I}}^*(u,v)\ ;\ 0 \le u \le M-1,\ 0 \le v \le N-1$$

Proof:

$$\begin{aligned}\tilde{\mathrm{I}}(M-u, N-v) &= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} \mathrm{I}(m,n)\ W_M^{(M-u)m} W_N^{(N-v)n} \\ &= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} \mathrm{I}(m,n)\ W_M^{Mm} W_M^{-um} W_N^{Nn} W_N^{-vn} \\ &= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} \mathrm{I}(m,n)\ \left[W_M^{um} W_N^{vn}\right]^* = \mathrm{I}^*(u,v)\end{aligned}$$

More Symmetry Properties

- The symmetry of the DFT array for a real $\mathbf{I}$ implies that it is **redundant**. If you know half of the coefficients, you can figure out the rest from symmetry.
- We also have

$$\tilde{\mathrm{I}}_{\mathrm{Real}}(M-u, N-v) = \tilde{\mathrm{I}}_{\mathrm{Real}}(u,v)$$
$$\tilde{\mathrm{I}}_{\mathrm{Imag}}(M-u, N-v) = -\tilde{\mathrm{I}}_{\mathrm{Imag}}(u,v)$$

and

$$\left|\tilde{\mathrm{I}}(M-u, N-v)\right| = \left|\tilde{\mathrm{I}}(u,v)\right|$$
$$\angle\tilde{\mathrm{I}}(M-u, N-v) = -\angle\tilde{\mathrm{I}}(u,v)$$

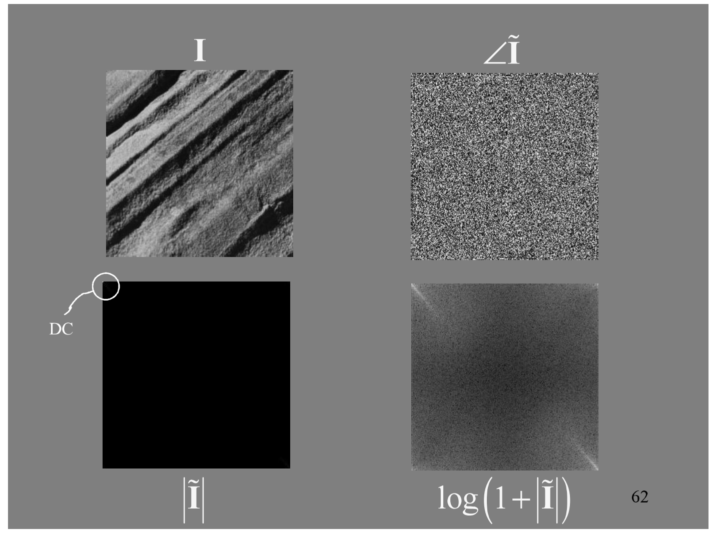
for $0 \le u \le M$-1, $0 \le v \le N$-1

Viewing the DFT

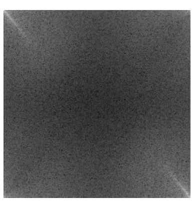
- To view the DFT, we usually display the **magnitude and phase as images**.
- In most cases, the phase is very difficult to interpret visually.
- Hence, it is common to look at the magnitude **only**.
- Logarithmic compression and a full-scale stretch are almost always applied to the magnitude for display:

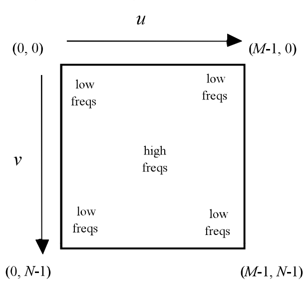
$$\log\left[1+\left|\tilde{\mathrm{I}}(u,v)\right|\right]$$

- This is the **log-magnitude spectrum** of the image $\mathbf{I}$.



- Note that the coefficients of the highest physical frequencies are **located near the center** of the DFT array: near $(u,v) = (M/2, N/2)$.





Periodicity of the DFT

- In the computer, the DFT coefficients are an array with a **finite** size the same as the image ($M \times N$):

$$\tilde{\mathbf{I}} = \left[\tilde{\mathrm{I}}(u,v)\right] \quad 0 \le u \le M\text{-}1, 0 \le v \le N\text{-}1$$

- However, for values of u,v **outside** this range, the mathematics of the DFT is **periodic**.
- Conceptually, the DFT coefficient array is thus **infinite in size and periodic** with

$$\tilde{\mathrm{I}}(u+pM, v+qN) = \tilde{\mathrm{I}}(u,v) \quad \forall\, p,q \in \mathbb{Z}.$$

- Mathematically, the image $\mathbf{I}$ can be recovered by applying the IDFT equation to **any one period** of $\tilde{\mathbf{I}}$.

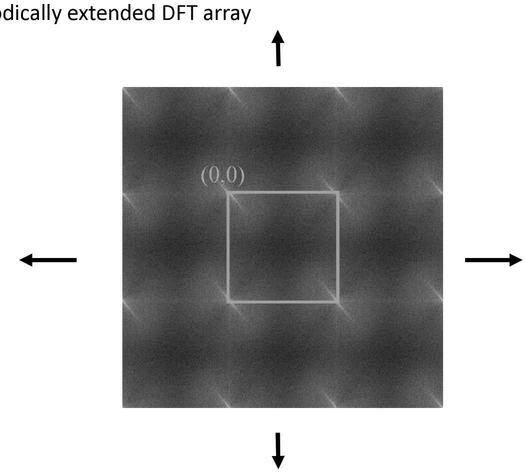
Proof of DFT Periodicity

- Let u,v be in the range $0 \le u \le M\text{-}1$ and $0 \le v \le N\text{-}1$.
- Let p,q be any integers. Then

$$\begin{aligned}
\tilde{\mathrm{I}}(u+pM, v+qN) &= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} \mathrm{I}(m,n)\; W_M^{(u+pM)m} W_N^{(v+qN)n} \\
&= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} \mathrm{I}(m,n)\; W_M^{um} W_N^{vn} W_M^{pMm} W_N^{qNn} \\
&= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} \mathrm{I}(m,n)\; W_M^{um} W_N^{vn} = \tilde{\mathrm{I}}(u,v)
\end{aligned}$$

This follows because $W_M^{qMm} = e^{-j2\pi\frac{pMm}{M}} = e^{-j2\pi(\text{integer})} = 1.$

Periodically extended DFT array



Periodic Extension of Image

- Mathematically, the DFT basis functions $W_M^{-um}\,W_N^{-vn}$ are defined for all integers m, n. Conceptually, they are **infinite in size and periodic**.
- When we write $\mathbf{I}$ as a sum of these basis functions

$$\mathrm{I}(m,n)=\frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}\tilde{\mathrm{I}}(u,v)W_M^{-um}W_N^{-vn},$$

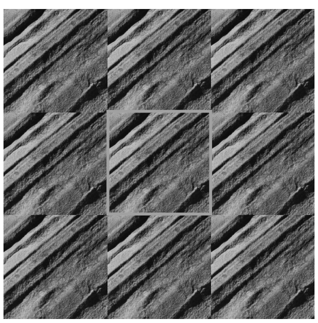
it defines an image that is also **infinite in size and periodic**.

- The DFT does not understand the concept of a finite-sized image.
- To the mathematics of the DFT, **every** image is **infinite in size and periodic** and **every** image has a DFT coefficient array that is also **infinite in size and periodic**.

Proof that the Image is Periodic

- Let m,n be in the range $0 \le m \le M\text{-}1$ and $0 \le n \le N\text{-}1$.
- Let p,q be any integers. Then

$$\begin{aligned}\mathrm{I}(m+pM,n+qM)&=\frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}\tilde{\mathrm{I}}(u,v)W_M^{-u(m+pM)}W_N^{-v(n+qN)}\\&=\frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}\tilde{\mathrm{I}}(u,v)W_M^{-um}W_N^{-vn}W_M^{-upM}W_N^{-vqN}\\&=\frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}\tilde{\mathrm{I}}(u,v)W_M^{-Mm}W_N^{-Nn}=\mathrm{I}(m,n)\end{aligned}$$



Periodically extended image

Displaying the Centered DFT

- For interpretation, it's usually desirable to display the log-magnitude spectrum with all of the low frequencies together in the center (instead of at the corners – see page 4.63).
- This can be accomplished using the frequency shift property of the DFT if we multiply the image $\mathbf{I}$ times an alternating image at the Nyquist rate:

$$(\text{-}1)^{m+n}\mathrm{I}(m,n)$$

- Taking the DFT of this modified image then produces the **centered log-magnitude spectrum**.
- For images, this is how the DFT is normally viewed.

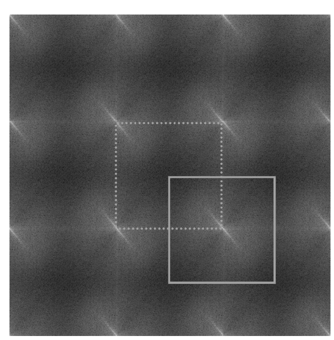
Centering the DFT

- How it works: note that

$$(-1)^{m+n}=(-1)^m(-1)^n=W_M^{-mM/2}W_N^{-nN/2},$$

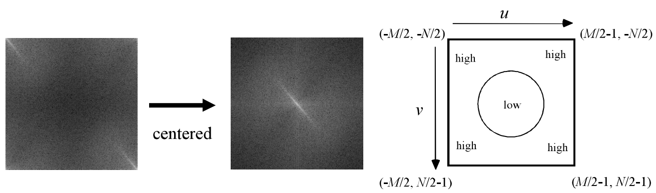
so

$$\begin{aligned}\mathrm{DFT}\Big[(-1)^{m+n}\mathrm{I}(m,n)\Big]&=\sum_{i=0}^{M-1}\sum_{j=0}^{N-1}\mathrm{I}(m,n)(-1)^{m+n}\;W_M^{um}W_N^{vn}\\&=\sum_{i=0}^{M-1}\sum_{j=0}^{N-1}\mathrm{I}(m,n)W_M^{um}W_N^{vn}W_M^{-(M/2)m}W_N^{-(N/2)n}\\&=\sum_{i=0}^{M-1}\sum_{j=0}^{N-1}\mathrm{I}(m,n)W_M^{[u-(M/2)]m}W_N^{[v-(N/2)]n}\\&=\tilde{\mathrm{I}}\big[u-(M/2),v-(N/2)\big]\end{aligned}$$

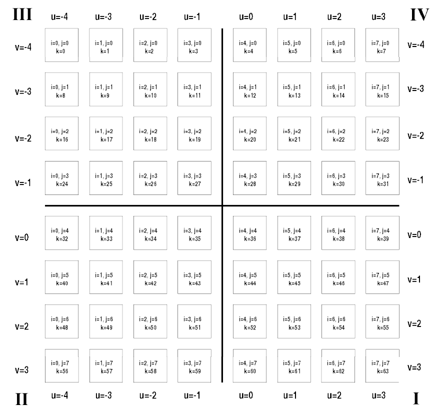


Shifted (centered) DFT from periodic extension

Centered DFT



Understanding the Centered DFT



- Suppose the image $\mathbf{I}$ is $8{\times}8$.
- This picture shows the frequencies u,v for every coefficient in the centered 2D DFT array.
- i,j are the col/row indices for a C array. For Matlab, add 1.
- k is the offset from the base of the array in memory. For a 1D C array, it is the index. For a 1D Matlab array, add 1.

- We think of the 2D frequency plane as being divided into four **quadrants**:
  - Quadrant I: $u{\geq}0$, $v{\geq}0$
  - Quadrant II: $u{<}0$, $v{\geq}0$
  - Quadrant III: $u{<}0$, $v{<}0$
  - Quadrant IV: $u{\geq}0$, $v{<}0$

Understanding the Centered DFT

- Matlab provides builtin functions fft2 and ifft2 to compute the 2D DFT equations on pages 4.54-4.55.
- Matlab also provides a builtin function fftshift to center the DFT as shown on pages 4.70-4.73.
  - If the length of the DFT array is even, then applying fftshift a second time will undo the shift.
  - Matlab also provides a function ifftshift to undo the shift. If the length of the DFT array is odd, then you *must* use ifftshift.
- The Matlab FFT implementation calls an open source package named FFTW3.
- If you use another language, you should also install FFTW3 and call it. There is a link to it on the course web site.

Example

- Let $\mathbf{X}_1$ be a cosine image with $M{=}N{=}64$ and $u{=}v{=}8$:

$$\mathbf{X}_1(m,n)=\cos\left[\tfrac{2\pi}{64}\left(8m+8n\right)\right]=\tfrac{1}{2}\mathrm{W}_{64}^{-(-8m\,-\,8n)}+\tfrac{1}{2}\mathrm{W}_{64}^{-(8m\,+\,8n)}$$

- Note that $MN{=}4096$.
- Following the pattern on page 4.74, the DC coefficient of the DFT array ($u{=}v{=}0$) will be at (row,col) = (33,33) in a Matlab array or (32,32) in a C array.
- The real part of the DFT array $\tilde{\mathbf{X}}_1$ will be equal to 4096/2 = 2048 at (u,v) = $(-8,-8)$ and $(8,8)$.
  - This is (row,col) = (33-8, 33-8) = (25,25) and (33+8, 33+8) = (41,41) in a Matlab array.
  - Or (row,col) = (24,24) and (40,40) in a C array.
  - Everywhere else, the real part of the DFT array will be zero.
- Because cosine is **real and even**, the imaginary part of the DFT array is all zero.

Example

- Let $\mathbf{X}_2$ be a sine image with $M{=}N{=}64$, $u = -4$, and $v = 7$:

$$\mathbf{X}_2(m,n)=\sin\left[\tfrac{2\pi}{64}\left(-4m+7n\right)\right]=\tfrac{-j}{2}\mathrm{W}_{64}^{-(-4m\,+\,7n)}+\tfrac{j}{2}\mathrm{W}_{64}^{-(4m\,-\,7n)}$$

- We have again that $MN{=}4096$.
- Because sine is **pure imaginary and odd**, the real part of the DFT array is all zero.
- The imaginary part of the DFT array $\tilde{\mathbf{X}}_2$ will be equal to -4096/2 = -2048 at $(u,v) = (-4,7)$.
  - This is (row,col) = (33+7, 33-4) = (40,29) in a Matlab array.
  - Or (row,col) = (39,28) in a C array.
- The imaginary part will be 4096/2 = 2048 at $(u,v) = (4,-7)$.
  - This is (row,col) = (33-7, 33+4) = (26,37) in a Matlab array.
  - Or (row,col) = (25,36) in a C array.
- Everywhere else, the imaginary part will be zero.

Example

- Let $\mathbf{X}_3 = \mathbf{X}_1 + \mathbf{X}_2$.
- Then $\tilde{\mathbf{X}}_3 = \tilde{\mathbf{X}}_1 + \tilde{\mathbf{X}}_2$.
- Because $\tilde{\mathbf{X}}_1$ is real and $\tilde{\mathbf{X}}_2$ is pure imaginary, we have

$$\mathrm{Re}\left\{\tilde{\mathbf{X}}_3\right\}=\tilde{\mathbf{X}}_1 \qquad j\,\mathrm{Im}\left\{\tilde{\mathbf{X}}_3\right\}=\tilde{\mathbf{X}}_2$$

$$\tilde{\mathbf{X}}_3=\mathrm{Re}\left\{\tilde{\mathbf{X}}_3\right\}+j\,\mathrm{Im}\left\{\tilde{\mathbf{X}}_3\right\}=\tilde{\mathbf{X}}_1+\tilde{\mathbf{X}}_2$$

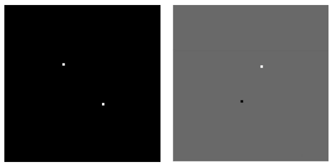
- In simple words:
  - The real part of $\tilde{\mathbf{X}}_3$ is equal to $\tilde{\mathbf{X}}_1$.
  - The imaginary part of $\tilde{\mathbf{X}}_3$ (times j) is equal to $\tilde{\mathbf{X}}_2$.
- Matlab code to implement this example:

```
[COLS,ROWS] = meshgrid(0:63,0:63);
X1 = cos(2*pi/64 * (8*COLS + 8*ROWS));
X2 = sin(2*pi/64 * (-4*COLS + 7*ROWS));
X3 = X1 + X2;
X3tilde = fftshift(fft2(X3));
```

```
>> imshow(255*((X1+1)/2),[0 255])
>> imshow(255*((X2+1)/2),[0 255])
>> imshow(255*((X3+2)/4),[0 255])
```

```
>>real(X3tilde(25,25))

ans =
   2.0480e+03

>>real(X3tilde(41,41))

ans =
   2.0480e+03

>>imag(X3tilde(26,37))

ans =
        2048

>>imag(X3tilde(40,29))

ans =
       -2048
```

```
>> imshow(real(X3tilde),[0 2048])
>> imshow(imag(X3tilde),[-2048 2048])
```



Example in C

- On the course web site, there is a program "DoFFT.c" that implements this example in C.
- You can find it under the "code" link on the web site.
- It contains a function fft2d that calls FFTW3 to compute the 2D DFT and then centers it.
  - Although only the forward transform is used in this example, fft2d can do both forward and inverse transforms.
  - There is a parameter "direction." If direction=1, then it does the forward transform. If direction=-1, then it does the inverse transform.
- The comment block at the top of the program explains how to compile it and link it to the FFTW3 library.
- You need to install FFTW3 **before** you compile the program.

Complex Exponential Matrix

- We define a special matrix of complex numbers:

$$\mathbf{W}_N = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^1 & W_N^2 & W_N^3 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & W_N^6 & \cdots & W_N^{2(N-1)} \\ 1 & W_N^3 & W_N^6 & W_N^9 & \cdots & W_N^{3(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & W_N^{3(N-1)} & \cdots & W_N^{(N-1)^2} \end{bmatrix}$$

- Note that it is a **symmetric** matrix

Inverse Matrix

$$\mathbf{W}_N^{-1} = \frac{1}{N}\mathbf{W}_N^* = \frac{1}{N}\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^{-1} & W_N^{-2} & W_N^{-3} & \cdots & W_N^{1-N} \\ 1 & W_N^{-2} & W_N^{-4} & W_N^{-6} & \cdots & W_N^{-2(N-1)} \\ 1 & W_N^{-3} & W_N^{-6} & W_N^{-9} & \cdots & W_N^{-3(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{1-N} & W_N^{-2(N-1)} & W_N^{-3(N-1)} & \cdots & W_N^{-(N-1)^2} \end{bmatrix}$$

Matrix-Product Form of DFT

- For a square image $\mathbf{I}$, we can then concisely write the DFT and IDFT as **matrix products**:

$$\tilde{\mathbf{I}} = \mathbf{W}_N \mathbf{I} \mathbf{W}_N \qquad \text{(DFT)}$$

$$\mathbf{I} = \frac{1}{N^2}\mathbf{W}_N^* \tilde{\mathbf{I}} \mathbf{W}_N^* \qquad \text{(IDFT)}$$

- This form is often useful for analysis and proofs and for programming a computer implementation.

Computation of the DFT

- Fast DFT/IDFT algorithms are collectively referred to as the **Fast Fourier Transform** (FFT).
- Matlab provides routines fft2, ifft2, and fftshift (for centering).
- For C, C++, and other languages, the FFTW package is strongly recommended.
- The speedup is the same as in 1D: from $\mathcal{O}(M^2N^2)$ to $\mathcal{O}[MN\log(MN)]$.

Interpreting Spatial Frequencies in the Image

- It's easy to lose track of the meaning of the DFT and the notion of **frequency content** in all the math.
- By examining the DFT or **spectrum** of an image (especially its magnitude), we can often deduce much about the image.

Qualitative Properties of the DFT

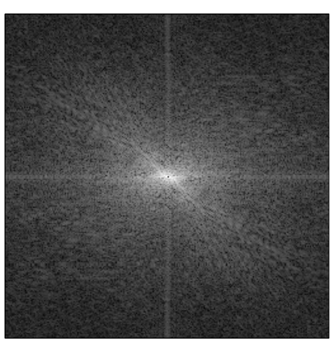
- We may regard the DFT magnitude as an **image of frequency content**.
- Bright regions in the DFT magnitude "image" correspond to frequencies having **large magnitudes** in the actual image.
- It is intuitive to think of image frequency content in terms of **granularity** (distribution of radial frequencies) and **orientation** (distribution of frequency vector directions).

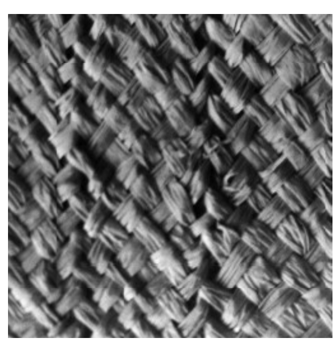
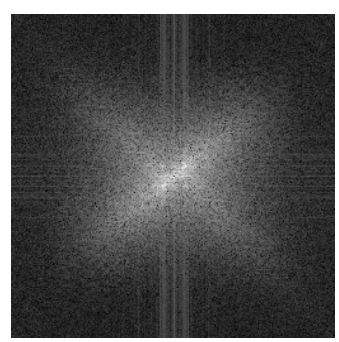
Image Granularity

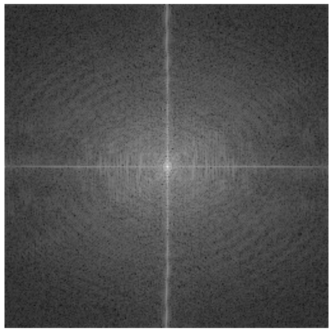
- DFT coefficients near the origin correspond to large, **smooth structure in the image -** often background.
- Since most images are non-negative, the DFT usually shows a strong peak at $(u, v) = (0, 0)$.
- Low frequency = long wavelength = large-scale structure in the image.
- High frequency = short wavelength = small-scale structure in the image.

Image Directionality

- Large DFT coefficients along certain orientations correspond to **directional** structure in the image.
- When there are strong, oriented structures in the DFT, strongly directional structures will be visible in the image.
- Visible structure in the centered log-magnitude spectrum is **orthogonal** to the structure in the image.
- This is because the direction of the frequency vector $[u\ v]^T$ is normal to the wavefronts in the image (see page 4.40).



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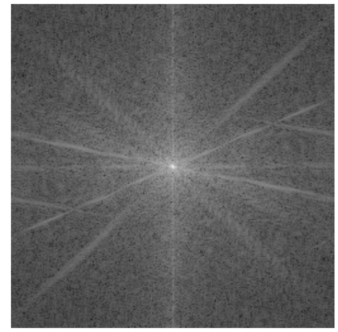

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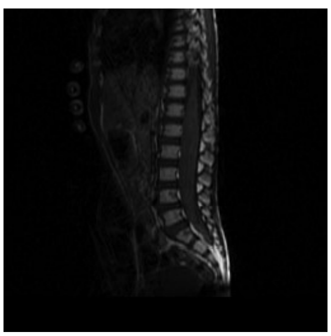

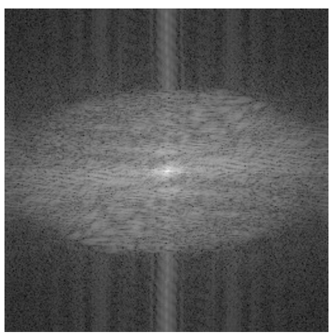

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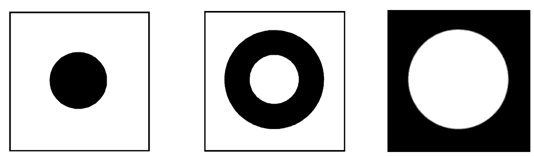
Granularity Selective Masking

- Define **zero-one masks** (black = 1)



low-frequency mask   mid-frequency mask   high-frequency mask

- **Masking** (multiplying) a DFT with these will produce IDFT images retaining only low, middle, or high radial frequencies.

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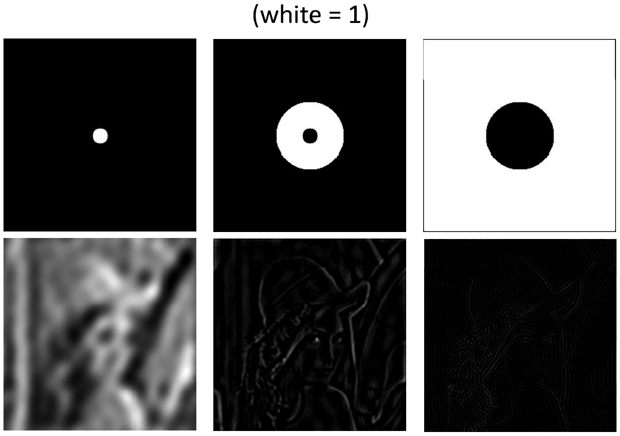
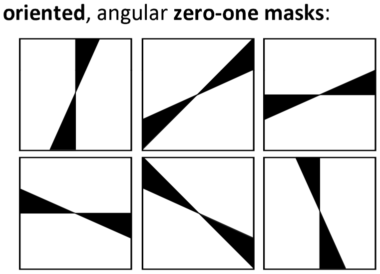
Granularity Selective Masking

(white = 1)



Orientation Selective Masking

- Define **oriented**, angular **zero-one masks**:



- Multiplying these times the DFT and inverting will retain **only** the **selected** orientations.

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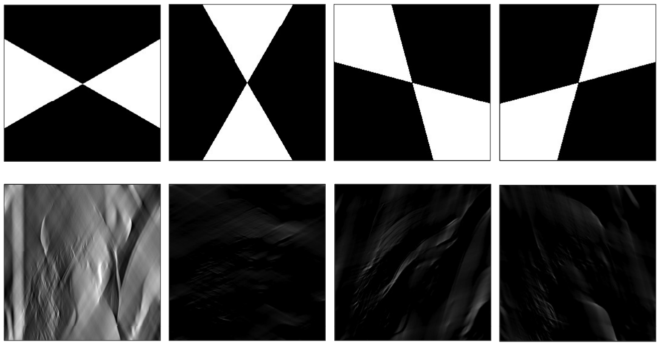
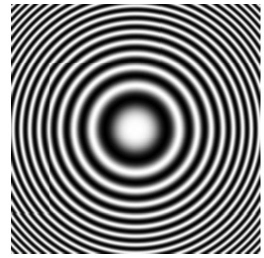
Orientation Selective Masking

(white = 1)

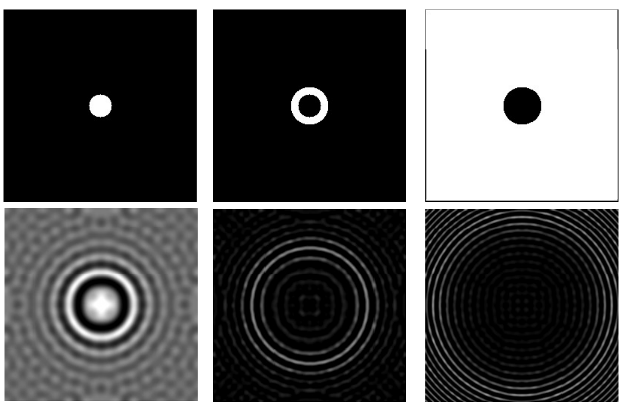


Chirp Image



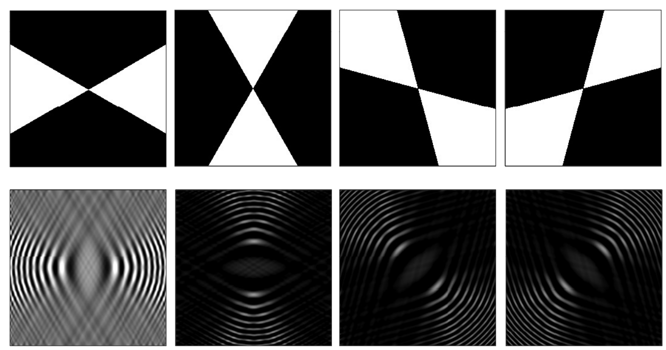
Granularity Selective Masking

(white = 1)



Orientation Selective Masking

(white = 1)



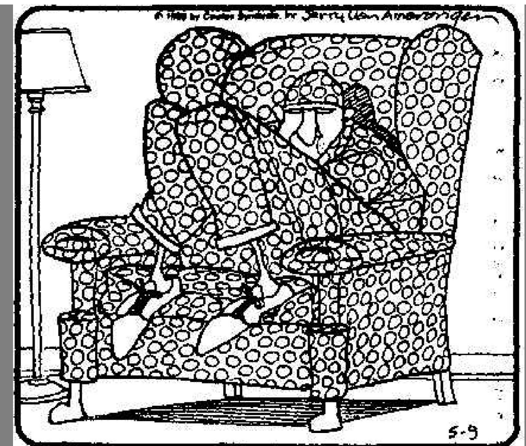
Granularity Selective Masking

```
fidx = fopen('/home/joebob/Images/lena.bin','r');
[x,junk] = fread(fidx,[256,256],'uchar');
x = x';
[COLS,ROWS] = meshgrid(0:255,0:255);
Lcutoff = 10;
LMask = (sqrt((ROWS-128).^ 2 + (COLS-128).^ 2) <= Lcutoff);
imshow(255*LMask, [0 255]);
imwrite(255*LMask,'LMask.tif','tif');
xLow = ifft2(fftshift(fftshift(fft2(x)) .* LMask));
imshow(real(xLow), [0 255]);
imwrite(uint8(real(xLow)),'LenaLow.tif','tif');

Hcutoff = 45;
HMask = (sqrt((ROWS-128).^ 2 + (COLS-128).^ 2) >= Hcutoff);
xHi = ifft2(fftshift(fftshift(fft2(x)) .* HMask));
imshow(real(xHi), [0 255]);
imwrite(uint8(real(xHi)),'LenaHi.tif','tif');

MMask = (1 - HMask) - LMask;
imshow(255*MMask, [0 255]);
imwrite(255*MMask,'MMask.tif','tif');
xMid = ifft2(fftshift(fftshift(fft2(x)) .* MMask));
imshow(real(xMid), [0 255]);
imwrite(uint8(real(xMid)),'LenaMid.tif','tif');
```


Orientation Selective Masking

```
fidx = fopen('/home/joebob/Images/lena.bin','r');
[x,junk] = fread(fidx,[256,256],'uchar');
x = x';
[COLS,ROWS] = meshgrid(0:255,0:255);

Cp = 3*pi/4;            % center angle (plus)
Cm = Cp+pi;       % center angle (opposite)
W = pi/6;         % anglular half-bandwidth
U = COLS-128;
V = ROWS-128;
Theta = atan2(V,U);
Mask = ((Theta >= Cp-W) .* (Theta <= Cp+W)) + ...
       ((Theta+2*pi >= Cp-W) .* (Theta+2*pi <= Cp+W)) + ...
       ((Theta-2*pi >= Cp-W) .* (Theta-2*pi <= Cp+W)) + ...
       ((Theta >= Cm-W) .* (Theta <= Cm+W)) + ...
       ((Theta+2*pi >= Cm-W) .* (Theta+2*pi <= Cm+W)) + ...
       ((Theta-2*pi >= Cm-W) .* (Theta-2*pi <= Cm+W));
imshow(255*Mask, [0 255]);
imwrite(255*Mask,'Mask.tif','tif');
y = ifft2(fftshift(fftshift(fft2(x)) .* Mask));
%
imshow(real(y), [0 255]);
title('Orientation Masked Image');
imwrite(uint8(real(y)),'y.tif','tif');
```

When the monster came, Lola, like the peppered moth and the arctic hare, *remained motionless and undetected.* Harold, of course, was immediately devoured.



You get the feeling Bob's not going out and grabbing life by the throat anytime soon.

DISCRETE-SPACE FOURIER TRANSFORM (DSFT)

- We have been looking at the 2D DFT. It's the 2D version of the 1D DFT on page 4.22.
- In 1D, we usually use n for the time variable (or index) and k for the frequency variable (which you can think of as indexing the basis signals).
- In a general ND formulation, you could use $n = (n_1\ n_2\ \dots\ n_N)$ for the "time" index and $k = (k_1\ k_2\ \dots\ k_N)$ for the frequency index.
- For the 2D case, we used (m,n) for the space index and (u,v) for the frequency index. We did this to minimize the use of subscripts. This is a ***widely*** used convention.
- Now we are going to look at the 2D version of the 1D DTFT on page 4.13. It is called the 2D **Discrete-Space Fourier Transform** (DSFT).
- For the 2D DSFT, we will use (m,n) for the space variables and (U,V) for the frequency variables.

THE DISCRETE-SPACE FOURIER TRANSFORM (DSFT)

- Let the image $\mathbf{I}$ be defined on *all* of $\mathbb{Z}^2$.
- The 2D **Discrete-Space Fourier Transform** is given by:

$$\tilde{\mathrm{I}}_D(U,\ V) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \mathrm{I}(m,n) e^{-j2\pi(Um+Vn)} \qquad \text{(DSFT)}$$

$$\mathrm{I}(m,n) = \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{\mathrm{I}}_D(U,\ V)\, e^{j2\pi(Um+Vn)}\, dU\, dV \qquad \text{(IDSFT)}$$

Discrete-Space Fourier Transform

- The units of U and V are cycles per pixel.
  - We could alternatively use units of radians per pixel. The limits of integration in the IDSFT would then be $-\pi$ to π and a factor of $1/(2\pi)^2$ would be required on the IDSFT.
- The **DSFT is periodic** with period ONE along each axis.
- The DSFT is **continuous** in spatial frequency.
- In most practical cases, the image $\mathbf{I}$ has a **finite** size.
- For (m,n) **outside** the range $0 \le m \le M\text{-}1$, $0 \le n \le N\text{-}1$, we set $\mathrm{I}(m,n) = 0$ for the DSFT.

Relating the DSFT and DFT

- In the theoretical case, it's possible to have an image $\mathbf{I}$ of infinite size. Such an image has a DSFT $\tilde{\mathbf{I}}_D$, but it does not have a DFT $\tilde{\mathbf{I}}$.
- Any practical image $\mathbf{I}$ is finite in size. It has a DFT $\tilde{\mathbf{I}}$. By zero padding the image to all of $\mathbb{Z}^2$, we can also give it a DSFT $\tilde{\mathbf{I}}_D$.
- For a finite sized image, there is a relationship between the DSFT and the DFT. The DFT is given by **samples** of the DSFT:

$$\tilde{\mathrm{I}}(u,v) = \tilde{\mathrm{I}}_D(U,\ V)\Big|_{U=\frac{u}{M},\, V=\frac{v}{N}}$$

$$;u=0,\dots,M-1, \quad v=0,\dots,N-1$$

DSFT

- It is more usual to introduce the DSFT first, followed by the DFT.
- However, our emphasis is on **algorithms** and **applications**, which generally use the **DFT**.
- We will find the **DSFT** to be a **valuable analytic tool**.

Sampling

- We will now study the relationships between the DFT/DSFT and the continuous 2D Fourier transform of the **original, unsampled optical image** that exists inside the camera.
- The digital image $\mathbf{I}$ is a sampled version of the **continuous optical intensity image** $\mathrm{I}_C(x, y)$ that is focused through the lens system to become incident upon a sensor located on the camera focal plane.

Notation and Units

- For the nD continuous Fourier transform on page 4.25, we used a vector $\mathbf{x}$ for the spatial variable and a vector $\mathbf{u}$ for the frequency variable.
- For 2D continuous optical images, we will use $\mathbf{x} = (x, y)$ to avoid subscripts. Thus, we will write $\mathrm{I}_C(x, y)$.
- The spatial coordinates (x, y) can be expressed in mm, inches, or any other appropriate unit of length.
- Often, it is convenient to express them in units given by the horizontal and vertical detector spacings on the focal plane, i.e., in units that are given by the physical size of a pixel in the camera.
- Again to avoid subscripts, we will write the 2D continuous Fourier transform frequency variable as $\mathbf{u} = (\Omega, \Lambda)$, where Ω = horizontal frequency and Λ = vertical frequency.

Continuous Fourier Transform

- The continuous image $\mathrm{I}_C(x, y)$ has a **Continuous Fourier Transform (CFT)** $\tilde{\mathrm{I}}_C(\Omega, \Lambda)$ where (x,y) are space coordinates and (Ω, Λ) are Hertzian spatial frequencies:

$$\tilde{\mathrm{I}}_C(\Omega, \Lambda) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{I}_C(x, y) e^{-j2\pi(\Omega x + \Lambda y)}\, dx\, dy \qquad \text{(CFT)}$$

$$\mathrm{I}_C(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{\mathrm{I}}_C(\Omega, \Lambda) e^{j2\pi(\Omega x + \Lambda y)}\, d\Omega\, d\Lambda \qquad \text{(ICFT)}$$

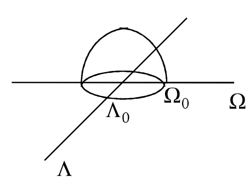
Continuous Fourier Transform

- The CFT is **not periodic** in general.
- The horizontal frequency Ω and vertical frequency Λ are continuous and expressed in cycles per unit of length, i.e. per unit of (x,y).
- The unit of length must be specified. For convenience, we will usually assume it is the height/width of a pixel on the detector.

Relating the CFT and DSFT/DFT

- Assume $\tilde{\mathrm{I}}_C(\Omega, \Lambda)$ is **bandlimited**, or **zero** outside a certain range of frequencies:

$$\tilde{\mathrm{I}}_C(\Omega, \Lambda) = 0 \text{ for } |\Omega| > \Omega_0, |\Lambda| > \Lambda_0$$



On Bandlimitedness

- **Any** real-world image is **approximately bandlimited:** its CFT becomes vanishingly small for large Ω, Λ.
- If it were not so the image would contain infinite energy since (Parseval)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\mathrm{I_C}(x,y)\right|^2 dx\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\tilde{\mathrm{I}}_{\mathrm{C}}(\Omega,\ \Lambda)\right|^2 d\Omega\, d\Lambda$$

- Ex: a Gaussian image has a Gaussian CFT which is not bandlimited. But the integrals converge because the CFT vanishes as $\Omega,\Lambda \to \infty$.

Image Sampling

- The detector creates $\mathrm{I}(m,n)$ by sampling the optical image with spacings X, Y in the x- & y-directions:

$$\mathrm{I}(m,n) = \mathrm{I_C}(mX, nY)\ ;\ m = 0, \ldots, M-1, \quad n = 0, \ldots, N-1$$

- The DSFT and CFT are related by:

$$\tilde{\mathrm{I}}_D(U,\ V) = \frac{1}{\mathrm{XY}}\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\tilde{\mathrm{I}}_{\mathrm{C}}\left(\Omega - \frac{p}{\mathrm{X}},\ \Lambda - \frac{q}{\mathrm{Y}}\right)\Bigg|_{(\Omega,\Lambda)=\left(\frac{U}{\mathrm{X}},\frac{V}{\mathrm{Y}}\right)}$$

$$= \frac{1}{\mathrm{XY}}\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\tilde{\mathrm{I}}_{\mathrm{C}}\left[\frac{1}{\mathrm{X}}(U-p),\ \frac{1}{\mathrm{Y}}(V-q)\right]$$

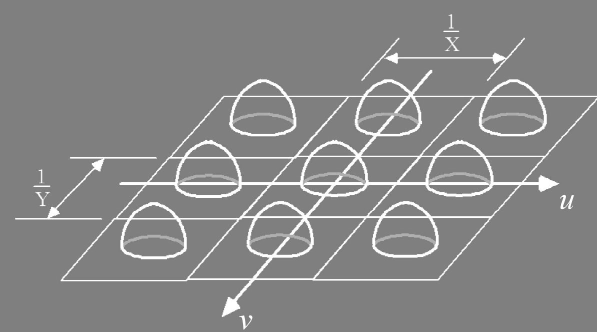
Relating DFT to CFT

- We have:

$$\tilde{\mathrm{I}}(u,v) = \tilde{\mathrm{I}}_D(U,\ V)\Big|_{U=\frac{u}{M},\ V=\frac{v}{N}}$$

$$= \frac{1}{\mathrm{XY}}\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\tilde{\mathrm{I}}_{\mathrm{C}}\left[\frac{1}{\mathrm{X}}\left(\frac{u}{M}-p\right),\ \frac{1}{\mathrm{Y}}\left(\frac{v}{N}-q\right)\right]$$

- A sum of shifted versions of the CFT. It is periodic in the u- and v-directions with periods $1/\mathrm{X}$ and $1/\mathrm{Y}$, respectively.



Unit-Period Case

- We can always set $\mathrm{X} = \mathrm{Y} = 1$.
- This simply says that we measure length and area in units that are given by the physical size of a pixel on the detector array (usually on the order of microns).
- Then

$$\tilde{\mathrm{I}}(u,v) = \sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\tilde{\mathrm{I}}_{\mathrm{C}}\left[\left(\frac{u}{M}-p\right),\ \left(\frac{v}{N}-q\right)\right]$$

Sampling Theorem

- If $\Omega_0 > \dfrac{1}{2\mathrm{X}}$ **or** $\Lambda_0 > \dfrac{1}{2\mathrm{Y}}$, the replicas of the CFT will **overlap** (sum up), **distorting them**. This is called **aliasing**.
- The digital image can be severely **distorted**.
- To avoid aliasing, the **sampling frequencies** $1/\mathrm{X}$ and $1/\mathrm{Y}$ must be at least **twice** the highest frequencies Ω_0 and Λ_0 in the continuous image.

Comments on Sampling

- This is a **mathematical** reason why images must be sampled sufficiently densely. If violated, image distortion can be **visually severe**.
- If the Sampling Theorem is **satisfied**, then the DFT is periodic (sampled) replicas of the CFT.

Aliased Chirp Image

- A **chirp image**

$$I_C(x,y) = A\cos[\varphi(x,y)] = A\cos\left(ax^2 + by^2\right)$$

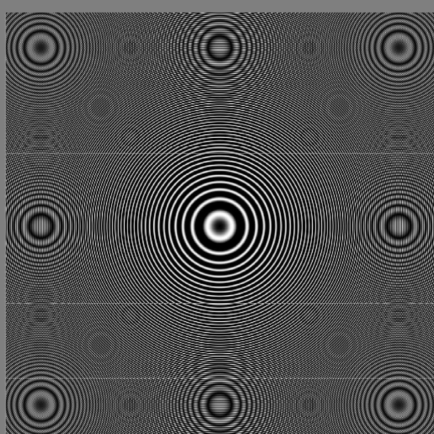
has **instantaneous spatial frequencies**

$$(\Omega_{\text{inst}},\ \Lambda_{\text{inst}}) = \left(\varphi_x(x,y),\ \varphi_y(x,y)\right) = (2ax,\ 2by)$$

which **increase linearly** away from the origin.

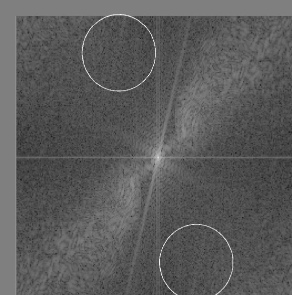
- For such an image, the origin $(x,y) = (0,0)$ is usually taken at the center of the image.

Sampled Chirp



Aliased Image

Sand Dune Image



Centered DFT Showing Aliasing

IMPORTANT 2-D FUNCTIONS AND THEIR DFTS

- It is worthwhile to examine the DFTs of some **specific images**. This is usually hard to do explicitly by hand for the DFT/DSFT.
- So we'll give some simple ones.
- Then state some others as **CFT transform pairs.**

Constant Image

- Let $I(m,n) = c$ for $0 \le m \le M$-1, $0 \le n \le N$-1
- Then

$$\tilde{I}(u,v) = c\,MN\,\delta(u,v)$$

where

$$\delta(u,v) = \begin{cases} 1\ ;\ u = v = 0 \\ 0\ ;\ \text{else} \end{cases} = \text{unit impulse}$$

2-D Unit Pulse Image

- Let $\mathrm{I}(m,n) = c\delta(m,n)$
- Then

$$\tilde{\mathrm{I}}(u,v) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} c\delta(m,n)\ W_M^{um}W_N^{vn}$$
$$= cW_N^0 W_M^0 = c \quad \text{(constant DFT)}$$

Cosine Wave Image

- Let

$$\mathrm{I}(m,n) = d\cos\left[2\pi\left(\frac{b}{M}m + \frac{c}{N}n\right)\right] = \left(\frac{d}{2}\right)\left[W_M^{bm}W_N^{cn} + W_M^{-bm}W_N^{-cn}\right]$$

- Then

$$\tilde{\mathrm{I}}(u,v) = \left(\frac{d}{2}\right)\sum_{m=0}^{N-1}\sum_{n=0}^{M-1}\left[W_M^{bm}W_N^{cn} + W_M^{-bm}W_N^{-cn}\right] W_M^{um}W_N^{vn}$$
$$= \left(\frac{d}{2}\right)\sum_{m=0}^{N-1}\sum_{n=0}^{M-1}\left[W_M^{(u+b)m}W_N^{(v+c)n} + W_M^{(u-b)m}W_N^{(v-c)n}\right]$$
$$= \left(\frac{d}{2}\right)MN\left[\delta(u-b, v-c) + \delta(u+b, v+c)\right]$$

Sine Wave Image

- Likewise, if

$$\mathrm{I}(m,n) = d\sin\left[2\pi\left(\frac{b}{M}m + \frac{c}{N}n\right)\right]$$

then

$$\tilde{\mathrm{I}}(u,v) = j\left(\frac{d}{2}\right)MN\left[\delta(u+b, v+c) - \delta(u-b, v-c)\right]$$

- The Fourier representations of pure sinusoids are concentrated **single frequencies**

DFT AS A SAMPLED CFT

- Now some CFT pairs that are either **difficult to express** or are too lengthy to do by hand as DFT (or even as DSFT) pairs.
- If **sampled adequately**, then
  - the DSFT is given by appropriately scaled periodic repetitions of the CFT.
  - the DFT is given by samples of the DSFT.

Rectangle Function

- Let

$$\mathrm{I_C}(x,y) = c\cdot\mathrm{rect}\left(\frac{x}{\mathrm{A}}\right)\cdot\mathrm{rect}\left(\frac{y}{\mathrm{B}}\right) = \begin{cases} c\ ;\ |x|\le\frac{\mathrm{A}}{2},\ |y|\le\frac{\mathrm{B}}{2} \\ 0\ ;\ \text{else}\end{cases}$$

- Then

$$\tilde{\mathrm{I}}_\mathrm{C}(\Omega,\Lambda) = c\cdot\mathrm{A}\cdot\mathrm{B}\cdot\mathrm{sinc}(\mathrm{A}\Omega)\cdot\mathrm{sinc}(\mathrm{B}\Lambda)$$

where

$$\mathrm{rect}(x) = \begin{cases} 1\ ;\ |x|\le\frac{1}{2} \\ 0\ ;\ \text{else}\end{cases} \quad \text{and} \quad \mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

Sinc Function

- Let

$$\mathrm{I_C}(x,y) = c\cdot\mathrm{sinc}(ax)\cdot\mathrm{sinc}(by)$$

then

$$\tilde{\mathrm{I}}_\mathrm{C}(\Omega,\Lambda) = \frac{c}{ab}\cdot\mathrm{rect}\left(\frac{\Omega}{a}\right)\cdot\mathrm{rect}\left(\frac{\Lambda}{b}\right) = \begin{cases}\frac{c}{ab};\ |\Omega|\le\frac{a}{2},\ |\Lambda|\le\frac{b}{2} \\ 0\ ;\ \text{else}\end{cases}$$

Gaussian Function

- Let
$$\mathrm{I_C}(x,y) = \exp\left[\frac{-\left(x^2+y^2\right)}{\sigma^2}\right]$$
then
$$\tilde{\mathrm{I}}_{\mathrm{C}}(\Omega,\ \Lambda) = \pi\sigma^2 \exp\left[-\pi^2\sigma^2\left(\Omega^2+\Lambda^2\right)\right]$$
- The Fourier transform of a Gaussian is also Gaussian – an **unusual property**.
- Note: since $\tilde{\mathrm{I}}_{\mathrm{C}}(\Omega, \Lambda)$ is **not** bandlimited, the digital image **will** be aliased.

Comments

- We now have a basic understanding of **frequency-domain concepts** in 2D.
- Let's put them to use in **linear filtering applications**… onward to **Module 5**.