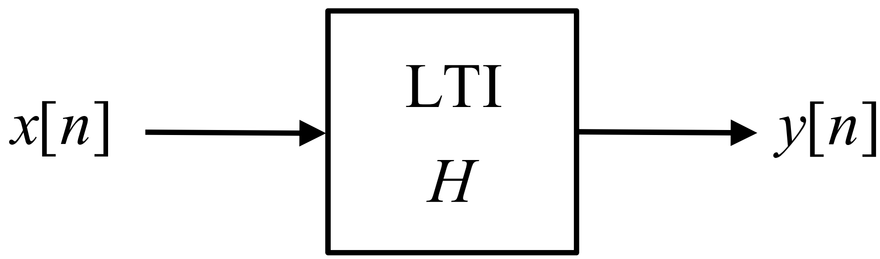
Module 5
Linear Image Filtering

- **Wraparound and Linear Convolution**
- **Linear Image Filters**
- **Linear Image Denoising**
- **Linear Image Restoration (Deconvolution)**

<u>REVIEW OF 1D LTI SYSTEMS</u>



- Operator notation: $y[n] = H\{x[n]\}$

- In English, this is read: "$y[n]$ is the output of the system H when $x[n]$ is the input."

THE MOST IMPORTANT PROPERTIES

- **Impulse response:** when the input is $\delta[n]$, the output is $h[n]$.

$$h[n] = H\left\{\delta[n]\right\}$$

- **Homogeneity:** if $y[n] = H\{x[n]\}$ and c is a constant, then

$$H\left\{cx[n]\right\} = cH\left\{x[n]\right\} = cy[n].$$

  - In other words, the action of the system **commutes** with multiplication by constants.

- **Superposition:** if $y_1[n] = H\{x_1[n]\}$ and $y_2[n] = H\{x_2[n]\}$, then

$$H\left\{x_1[n] + x_2[n]\right\} = H\left\{x_1[n]\right\} + H\left\{x_2[n]\right\} = y_1[n] + y_2[n].$$

  - In other words, the action of the system **commutes** with sums.

IMPORTANT LTI PROPERTIES

- Together, homogeneity and superposition are called **linearity**.
- Together, they imply that the action of a **linear** system commutes with linear combinations:

$$H\{c_1x_1[n]+c_2x_2[n]\}=c_1y_1[n]+c_2y_2[n].$$

- **Translation Invariance:** if $y[n]=H\{x[n]\}$ and n_0 is an integer constant, then

$$H\{x[n-n_0]\}=y[n-n_0].$$

  - In other words, the action of the system **commutes** with (time) shifts.
  - Also called **time invariance** or **shift invariance**.

1D Linear Convolution

- As we have seen, any 1D discrete-time signal $x[n]$ can be written as a linear combination of the translates of $\delta[n]$:

$$x[n] = \cdots + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + \cdots$$

- Here, it is **important** to realize that $x[-1]$, $x[0]$, $x[1]$, etc., are constants; they are **numbers**.
- So, if $x[n]$ is the input to an LTI system H, the output is

$$\begin{aligned} y[n] &= H\left\{x[n]\right\} \\ &= H\left\{\cdots + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + \cdots\right\} \\ &= \cdots + x[-1]H\left\{\delta[n+1]\right\} + x[0]H\left\{\delta[n]\right\} + x[1]H\left\{\delta[n-1]\right\} + \cdots \\ &= \cdots + x[-1]h[n+1] + x[0]h[n] + x[1]h[n-1] + \cdots \end{aligned}$$

Σ Notation

- To save time and paper, we can write this **exact same thing** using “capital Sigma do-loops”:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

$$\begin{aligned} y[n] &= H\{x[n]\} \\ &= H\left\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right\} \\ &= \sum_{k=-\infty}^{\infty} x[k]H\{\delta[n-k]\} \\ &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \end{aligned}$$

- This is called **linear convolution**; written $y[n] = x[n] * h[n]$.

Interpretation

- For each n, the output signal $y[n]$ is a number.
- This number is given by the dot product of the input $x[n]$ with a **flipped-and-shifted version** of the impulse response:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \langle x[k], h[-k-(-n)] \rangle$$

- Another way to think of it:
  - Let the input be $x[n] = 2\delta[n] + 3\delta[n\text{-}1] + 4\delta[n\text{-}2]$.
  - We can think of this as a sum of three input signals.
  - For $2\delta[n]$, the output is $2h[n]$.
  - For $3\delta[n\text{-}1]$, the output is $3h[n\text{-}1]$.
  - For $4\delta[n\text{-}2]$, the output is $4h[n\text{-}2]$.
  - The total output is the sum of these: that's **convolution**.

More About 1D Linear Convolution

- Continuous-time version:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\theta)h(t-\theta)d\theta.$$

- Computing 1D linear convolution in the transform domain:

$$Y\left(e^{j\omega}\right) = X\left(e^{j\omega}\right)H\left(e^{j\omega}\right)$$
$$Y(z) = X(z)H(z)$$

discrete time

$$Y(\Omega) = X(\Omega)H(\Omega)$$
$$Y(s) = X(s)H(s)$$

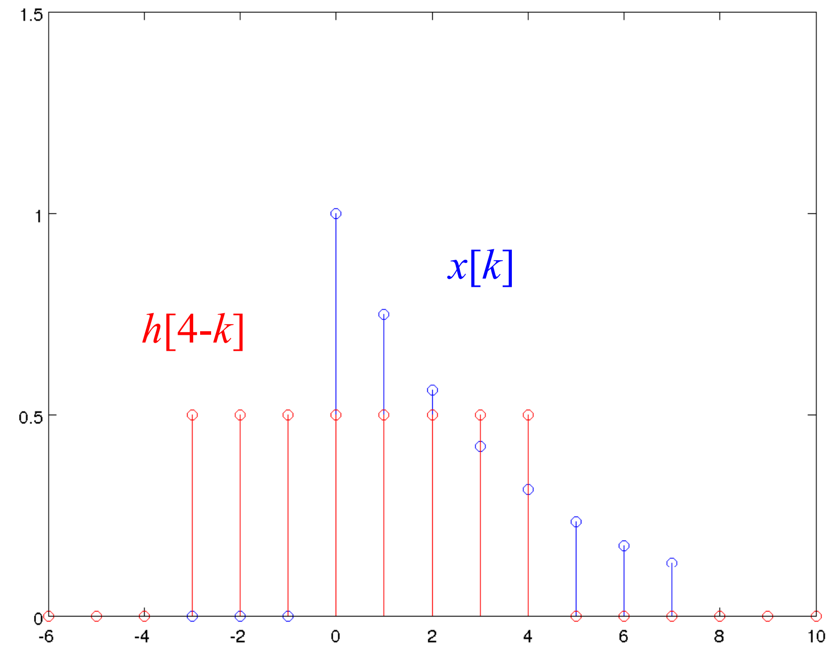
continuous time

An Important Idea

- Suppose $x[n] = (3/4)^n$, $0 \le n \le 7$, and zero otherwise.
- Let $h[n] = 1/2$, $0 \le n \le 7$, and zero otherwise.
- Then, according to the convolution formula,

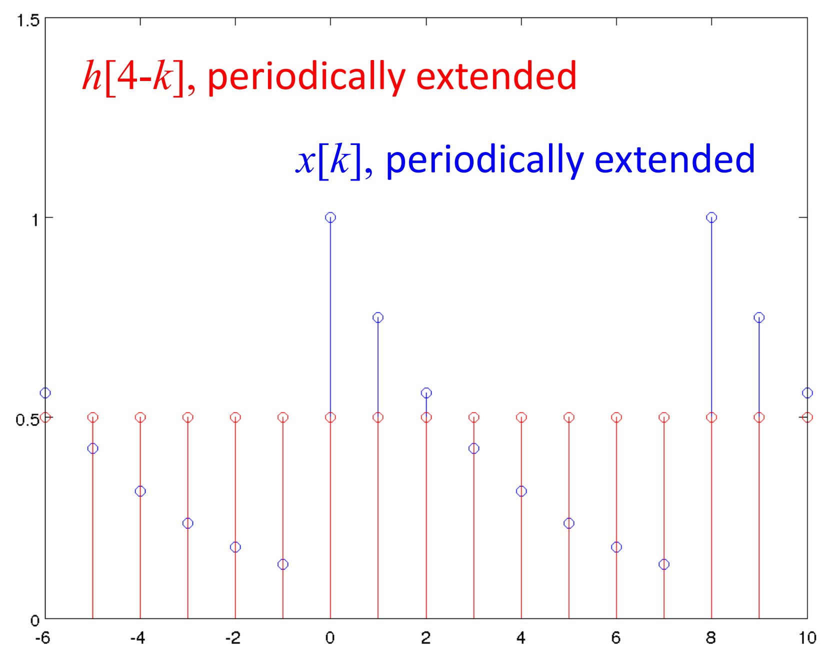
$$y[4] = \sum_{k=-\infty}^{\infty} x[k]h[4-k]$$

- Notice that the product $x[k]h[4\text{-}k]$ is zero for $k < 0$ because $x[k]$ is zero there.
- Similarly, the product is zero for $k > 4$ because $h[4\text{-}k]$ is zero there.
- In the linear convolution sum, we get **zero** for the product in places k where one of the signals "hangs over."



- Now suppose we try to compute this same convolution by multiplying the 8-point DFT's $X[k]$ and $H[k]$.
- Recall that, to the DFT, a finite-length signal is **one period** of a **periodic** signal.
- So the picture will be different this time!

- Because the signals are now **periodically extended**, there will no longer be zeros in the sum at places where one of the signals "hangs over."
- The number we get for $y[4]$ this way will **not** be the same as what we got by linear convolution on the last page.
- It is something **different** – it is called **wraparound convolution**.
- More on this in a minute…



nD Convolution

- In nD, a discrete LTI system H has an impulse response $h[\mathbf{p}]$, where $\mathbf{p} \in \mathbb{Z}^n$.
- For an nD input $x[\mathbf{p}]$, the output $y[\mathbf{p}]$ is given by the nD discrete linear convolution

$$y[\mathbf{p}] = x[\mathbf{p}] * h[\mathbf{p}] = \sum_{\mathbf{q} \in \mathbb{Z}^n} x[\mathbf{q}]h[\mathbf{p}-\mathbf{q}].$$

- Note that this has the same form as the 1D version.
- For continuous space with $\mathbf{v},\mathbf{w} \in \mathbb{R}^n$, the linear convolution is given by

$$y(\mathbf{v}) = x(\mathbf{v}) * h(\mathbf{v}) = \int_{\mathbb{R}^n} x(\mathbf{w})h(\mathbf{v}-\mathbf{w})d\mathbf{w}.$$

The 2D Case

- Now let's go back to our usual notation and write out the formulas from the last page for the special case of 2D.
- Let $\mathrm{H}(m, n)$ be the impulse response of a discrete 2D LTI system.
- Let the input be $\mathrm{I}(m, n)$.
- The output is given by the 2D discrete linear convolution

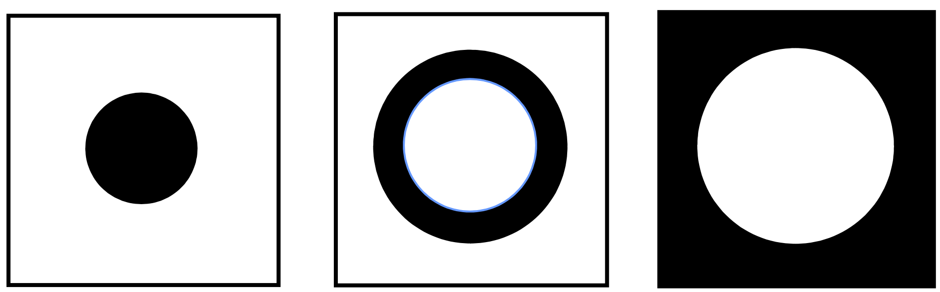
$$\mathrm{J}(m,n) = \mathrm{I}(m,n) * \mathrm{H}(m,n) = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \mathrm{I}(p,q)\mathrm{H}(m-p, n-q).$$

- For continuous space, the 2D linear convolution is given by

$$\mathrm{J}_C(x,y) = \mathrm{I}_C(x,y) * \mathrm{H}_C(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{I}_C(\alpha,\beta)\mathrm{H}_C(x-\alpha, y-\beta)\,d\alpha\, d\beta.$$

2D Wraparound Convolution

- Modifying the DFT of an image changes its appearance. For example, multiplying a DFT by a **zero-one mask** predictably modifies image appearance:



- But is this linear convolution?

Multiplying 2D DFTs

- What happens when two arbitrary 2D DFTs are multiplied together pointwise?

$$\tilde{\mathbf{J}} = \tilde{\mathbf{I}}_1 \otimes \tilde{\mathbf{I}}_2 \quad \text{or} \quad \tilde{\mathbf{J}} = \tilde{\mathbf{I}}_1 \Delta \tilde{\mathbf{I}}_2$$

- The answer has profound consequences in image processing.
- Pointwise division can be treated as multiplication by the reciprocal of $\tilde{\mathbf{I}}_2$, but special handling is needed if $\tilde{\mathbf{I}}_2$ contains values that are zero or nearly zero.

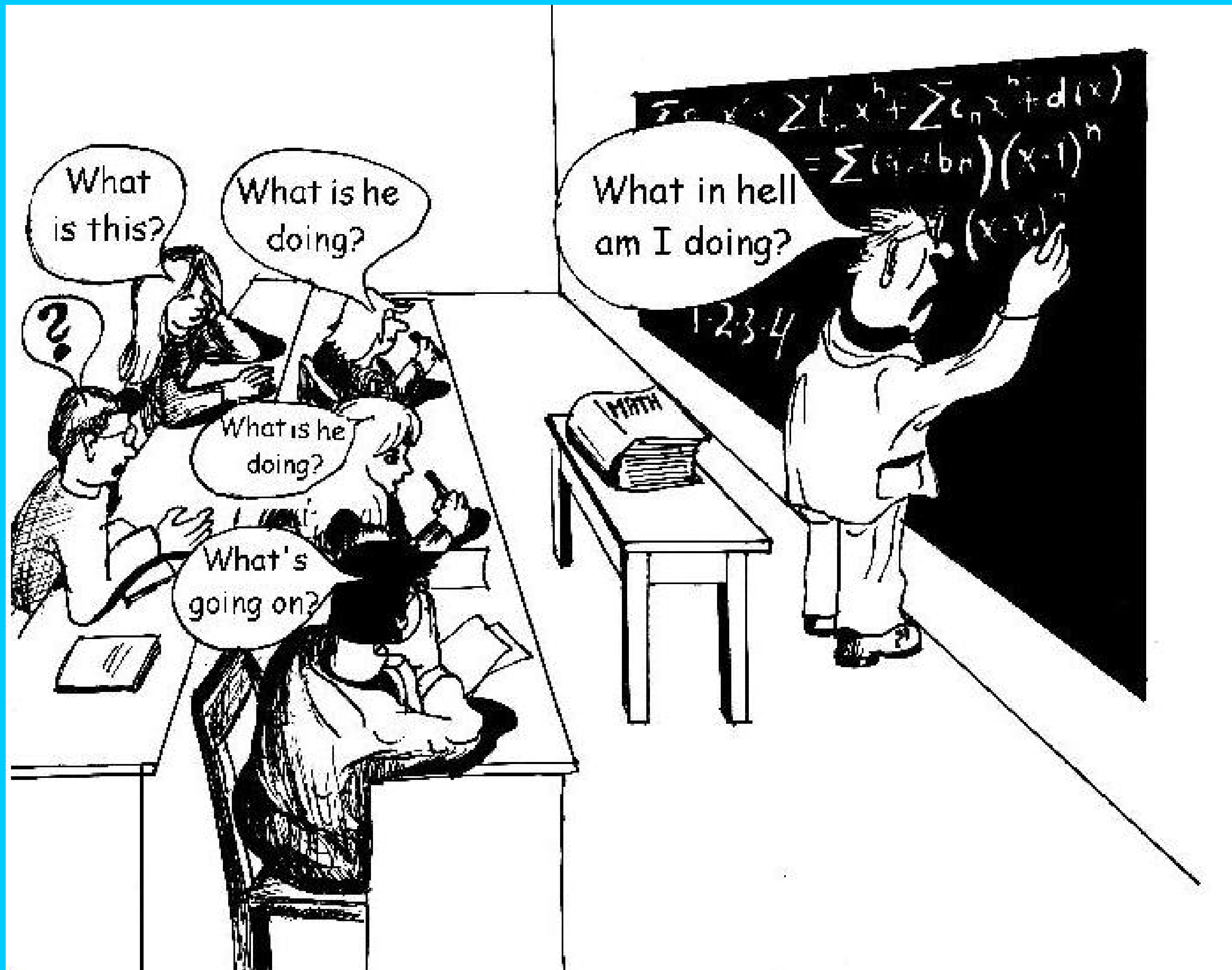
Multiplying DFTs

- Consider the **product**

$$\tilde{\mathbf{J}} = \tilde{\mathbf{I}}_1 \otimes \tilde{\mathbf{I}}_2$$

- This has inverse DFT

$$\begin{aligned}
\mathrm{J}(m,n) &= \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}\tilde{\mathrm{J}}(u,v)W_M^{-um}W_N^{-vn} \\
&= \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}\tilde{\mathrm{I}}_1(u,v)\tilde{\mathrm{I}}_2(u,v)W_M^{-um}W_N^{-vn} \\
&= \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}\left\{\sum_{p=0}^{M-1}\sum_{q=0}^{N-1}\mathrm{I}_1(p,q)W_M^{up}W_N^{vq}\right\}\left\{\sum_{r=0}^{M-1}\sum_{s=0}^{N-1}\mathrm{I}_2(r,s)W_M^{ur}W_N^{vs}\right\}W_M^{-um}W_N^{-vn}
\end{aligned}$$


What is this?
What is he doing?
What in hell am I doing?
MATH
What is he doing?
What's going on?

$$=\frac{1}{MN}\sum_{p=0}^{M-1}\sum_{q=0}^{N-1}\mathrm{I}_1(p,q)\sum_{r=0}^{M-1}\sum_{s=0}^{N-1}\mathrm{I}_2(r,s)\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} W_M^{u(p+r-m)}W_N^{v(q+s-n)}$$

★ $$=\frac{1}{MN}\sum_{p=0}^{M-1}\sum_{q=0}^{N-1}\mathrm{I}_1(p,q)\sum_{r=0}^{M-1}\sum_{s=0}^{N-1}\mathrm{I}_2(r,s)\cdot MN\cdot\delta(p+r-m,q+s-n)$$

★★ $$=\sum_{p=0}^{M-1}\sum_{q=0}^{N-1}\mathrm{I}_1(p,q)\mathrm{I}_2\left[(m-p)_M,\ (n-q)_N\right]$$

Note : $(p)_N = p \bmod N$

$$=\sum_{r=0}^{M-1}\sum_{s=0}^{N-1}\mathrm{I}_1\left[(m-r)_M,\ (n-s)_N\right]\mathrm{I}_2(r,s)$$

$$=\mathrm{I}_1(m,n)\circledast\mathrm{I}_2(m,n)$$

$\mathbf{I}_1\circledast\mathbf{I}_2$ is the **wraparound convolution** of $\mathbf{I}_1$ with $\mathbf{I}_2$

The steps ★ and ★★ are explained on the next 3 pages.

- To understand ★ on page 5.17, let

😐 $= \displaystyle\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} W_M^{u(p+r-m)} W_N^{v(q+s-n)} = \underbrace{\sum_{u=0}^{M-1} W_M^{u(p+r-m)}}_{☺} \underbrace{\sum_{v=0}^{N-1} W_N^{v(q+s-n)}}_{☹}$

- If $p = m\text{-}r$ and $q = n\text{-}s$, then $p{+}r\text{-}m = q{+}s\text{-}n = 0$, so

😐 $= \displaystyle\sum_{u=0}^{M-1} W_M^0 \sum_{v=0}^{N-1} W_N^0 = \sum_{u=0}^{M-1} 1 \sum_{v=0}^{N-1} 1 = MN$

- A useful sum formula: $\displaystyle\sum_{n=A}^{B} a^n = \frac{a^A - a^{B+1}}{1-a}$, provided $a \neq 1$.

- Now, if $p \neq m\text{-}r$, then $p{+}r\text{-}m \neq 0$, so with $A = 0,\ B = N-1,$ and $a = W_M^{p+r-m}$,

☺ $= \displaystyle\sum_{u=0}^{M-1} W_M^{u(p+r-m)} = \frac{W_M^0 - W_M^{M(p+r-m)}}{1 - W_N^{p+r-m}}$

$$= \frac{1-e^{j2\pi M(p+r-m)/M}}{1-e^{j2\pi(p+r-m)/M}} = \frac{1-e^{j2\pi(\text{integer})}}{1-e^{j2\pi(\text{not integer})}}$$

$$= \frac{1-1}{1-(\text{not }1)} = \frac{0}{\text{not zero}} = 0.$$

- Understanding ★ …

- Similarly, if $q \neq n$-s, then q+s-n $\neq 0$, so ☹ $= 0$.

- All together,

$$😐 = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} W_M^{u(p+r-m)} W_N^{v(q+s-n)} = \begin{cases} MN, & p = m - r \text{ and } q = n - s, \\ 0, & \text{otherwise,} \end{cases}$$

$$= MN\delta(p + r - m, q + s - n),$$

which establishes the equality ★ on page 5.17.

- To understand ★★ on page 5.17, observe that

$$\sum_{r=0}^{M-1}\sum_{s=0}^{N-1} \mathrm{I}_2(r,s)\cdot MN\cdot\delta(p+r-m,\ q+s-n)$$
$$=MN\sum_{r=0}^{M-1}\sum_{s=0}^{N-1} \mathrm{I}_2(r,s)\,\delta[r-(m-p),s-(n-q)]$$
$$=MN\,\mathrm{I}_2(m-p,n-q).$$

- Now, $0 \le m,p \le M-1$. So $(1-M) \le m-p \le (M-1)$.
- Likewise, $0 \le n,q \le N-1$. So $(1-N) \le n-q \le (N-1)$.
- In other words, $m-p$ can be outside the range 0 to $M-1$ and $n-q$ can be outside the range 0 to $N-1$, i.e., **outside** the bounds of the original $\mathbf{I}_2$ image.
- But here $\mathbf{I}_2$ must be interpreted as the IDFT of $\tilde{\mathbf{I}}_2$, which is the **periodic extension** of the original $\mathbf{I}_2$ image.
- Thus, pixels outside the range $[0, M\text{-}1]$, $[0, N\text{-}1]$ must be taken from the periodic extension. In terms of the original $\mathbf{I}_2$ image, this is conveniently written using modular arithmetic as $MN\mathrm{I}_2[(p-m)_M,\ (q-n)_N]$.

Wraparound Convolution

- The summation

$$\mathrm{J}(m,n)=\mathrm{I}_1(m,n)\circledast \mathrm{I}_2(m,n)$$
$$=\sum_{p=0}^{M-1}\sum_{q=0}^{N-1}\mathrm{I}_1(p,q)\,\mathrm{I}_2\Big[\big(m-p\big)_M\,,\ \big(n-q\big)_N\Big]$$

  is also sometimes called **cyclic convolution**, **circular convolution**, or **periodic convolution**.
- It is an inner product between one sequence and a (doubly) reversed, shifted, and **periodically extended** version of the other.
- Mathematically, the periodic extension is accounted for by writing the indices of $\mathbf{I}_2$ **modulo-$\mathbf{M,N}$**.

Understanding Wraparound Convolution

- Consider **hypothetical images** $\mathbf{I}_1$ and $\mathbf{I}_2$

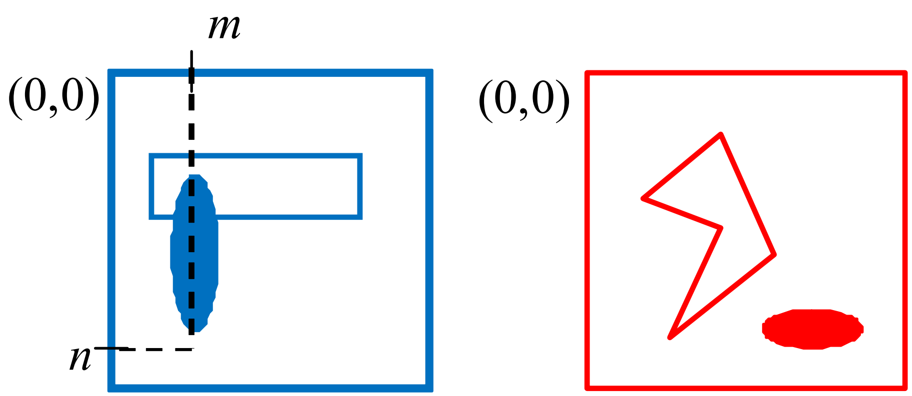
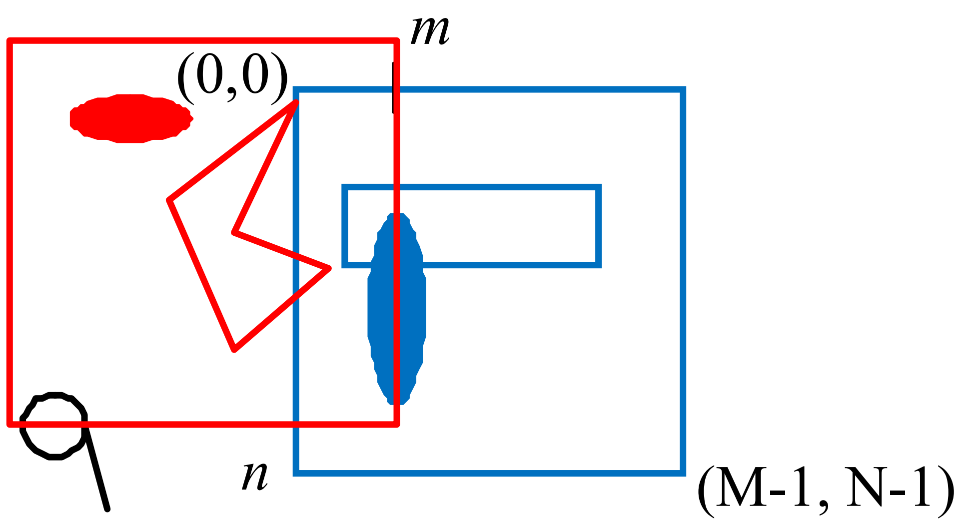


Image $\mathbf{I}_1$ Image $\mathbf{I}_2$

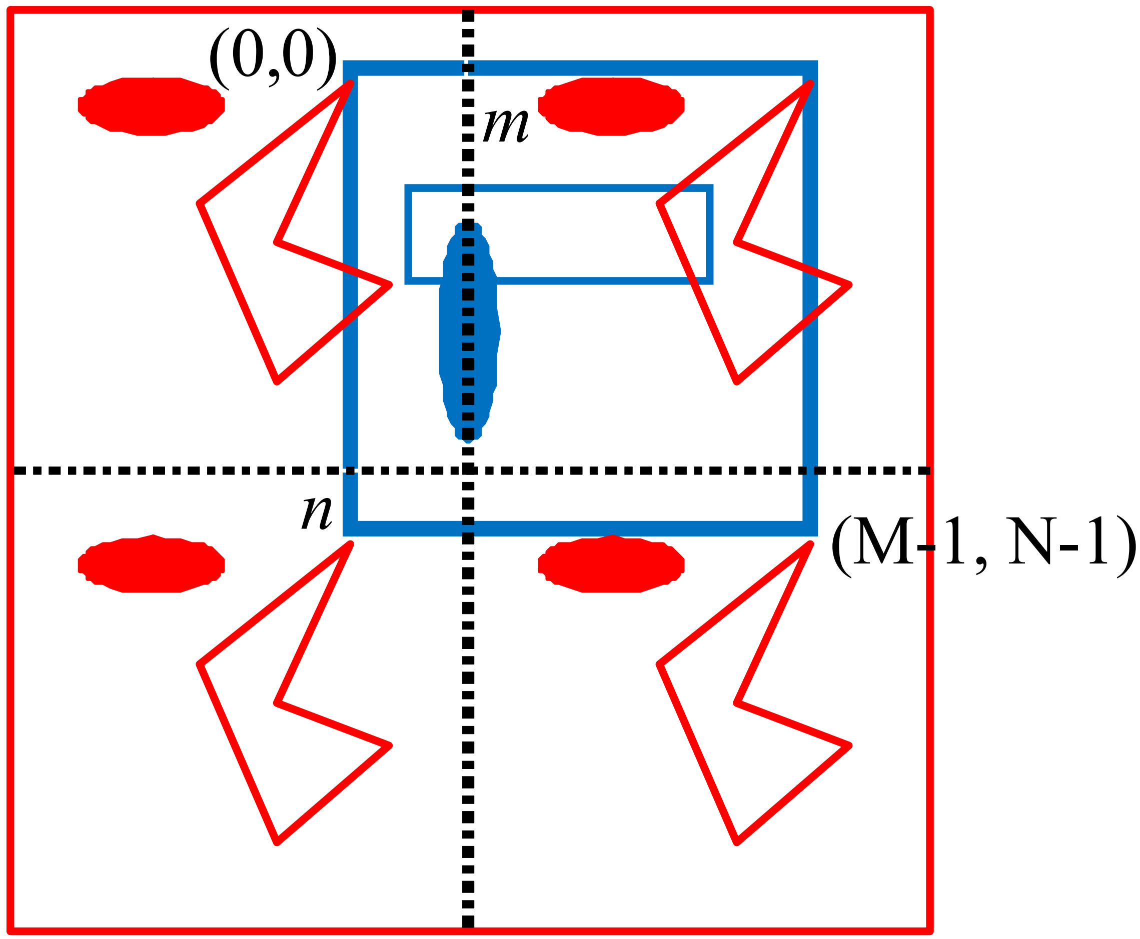
for which we wish to compute the **cyclic convolution** at (m, n) in the spatial domain (without DFTs).

- Without wraparound:



$\mathbf{I}_2$ Doubly-reversed and shifted

- For linear convolution, we would now compute the dot product of these two.
- But for wraparound convolution, the modulo arithmetic will periodically extend $\mathbf{I}_2$.



Overlay of periodic extension of shifted $\mathbf{I}_2$

Summation occurs over $0 < p < \text{M-1}, 0 < \text{q} < \text{N-1}$

i.e., over the blue $\mathbf{I}_1$ image.

Computation of Wraparound Convolution

- Direct computation of

$$
\begin{aligned}
\mathrm{J}(m,n) &= \mathrm{I}_1(m,n) \circledast \mathrm{I}_2(m,n) \\
&= \sum_{p=0}^{M-1} \sum_{q=0}^{N-1} \mathrm{I}_1(p,q)\,\mathrm{I}_2\left[(m-p)_M,\ (n-q)_N\right]
\end{aligned}
$$

  is simple but expensive.

- For an MxN image:
  - for **each** of MN pixels in **J**: **MN additions and MN multiplies**.
  - or (MN)(MN) multiply-add operations in total.
  - for M=N=512, this is $\mathbf{2^{26} = 6.9 \times 10^{10}}$

DFT Computation of Wraparound Convolution

- Because of **FFT**, computing $\circledast$ in the DFT domain is much faster, provided that $\mathrm{M,N}$ = powers of 2.
- Simply put,

$$\mathbf{J} = \mathbf{I}_1 \circledast \mathbf{I}_2 = \mathrm{IFFT}_{\mathrm{M \times N}} \left[\mathrm{FFT}_{\mathrm{M \times N}} \left[\mathbf{I}_1 \right] \otimes \mathrm{FFT}_{\mathrm{M \times N}} \left[\mathbf{I}_2 \right] \right]$$

- Computing an $(\mathrm{M \times N})$ FFT is $\mathcal{O}[\mathrm{MN} \cdot \log(\mathrm{MN})]$, so computation of $\circledast$ is as well.
- We will now discover that $\circledast$ must be modified in order to make it useful in most applications.

2D LINEAR CONVOLUTION

- Wraparound convolution is a consequence of the **periodic DFT**.
- If two **DSFTs** are multiplied together:

$$\tilde{J}_D(U,V) = \tilde{I}_{D1}(U,V)\,\tilde{I}_{D2}(U,V)$$

then useful **linear convolution** results:

$$J(m,n) = I_1(m,n) * I_2(m,n)$$

- **Wraparound convolution** is an **artifact** of sampling the DSFT – which causes **spatial periodicity**.

About Linear Convolution

- Most of circuit theory, optics, and analog filter theory is based on **linear convolution**.

- And … (linear) digital filter theory is based on the concept of **discrete linear convolution**.

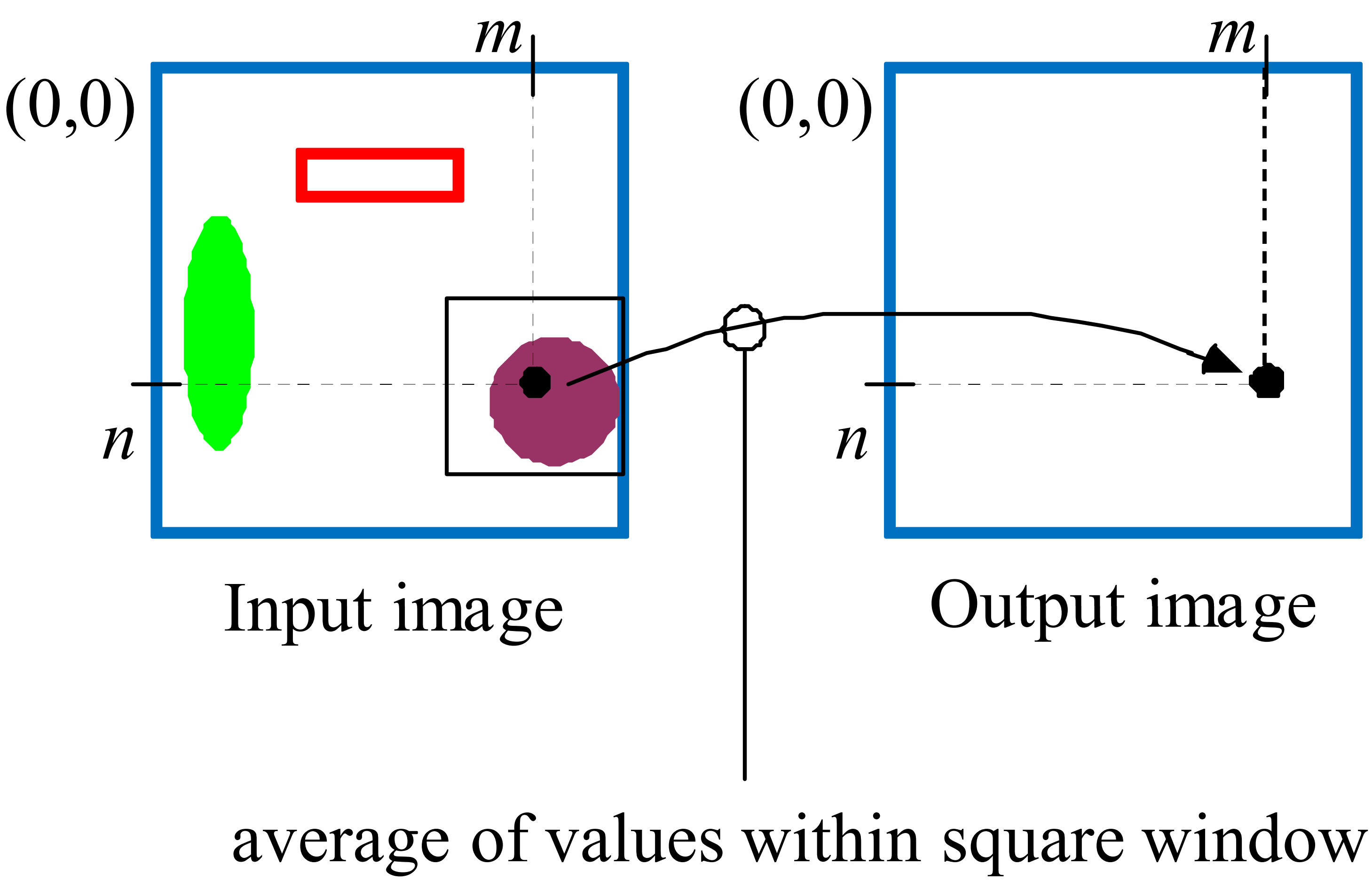
- Fortunately, **wraparound convolution** can be used to compute **linear convolution**.

Undesirability of Wraparound

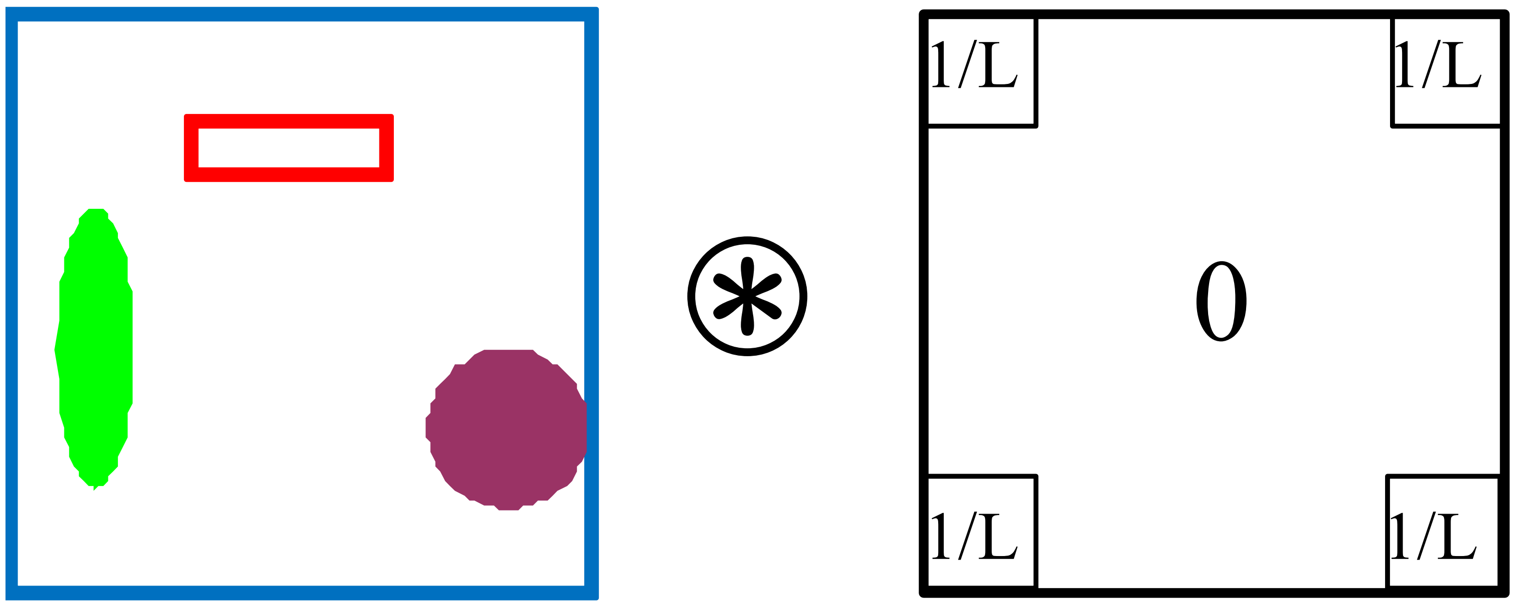
- A very simple type of linear convolution is the **local average operation** (or averaging filter).

- Each image pixel is replaced by the **average** of its neighbors within a window:

Depiction of Average Filtering



Computation of Average Filtering

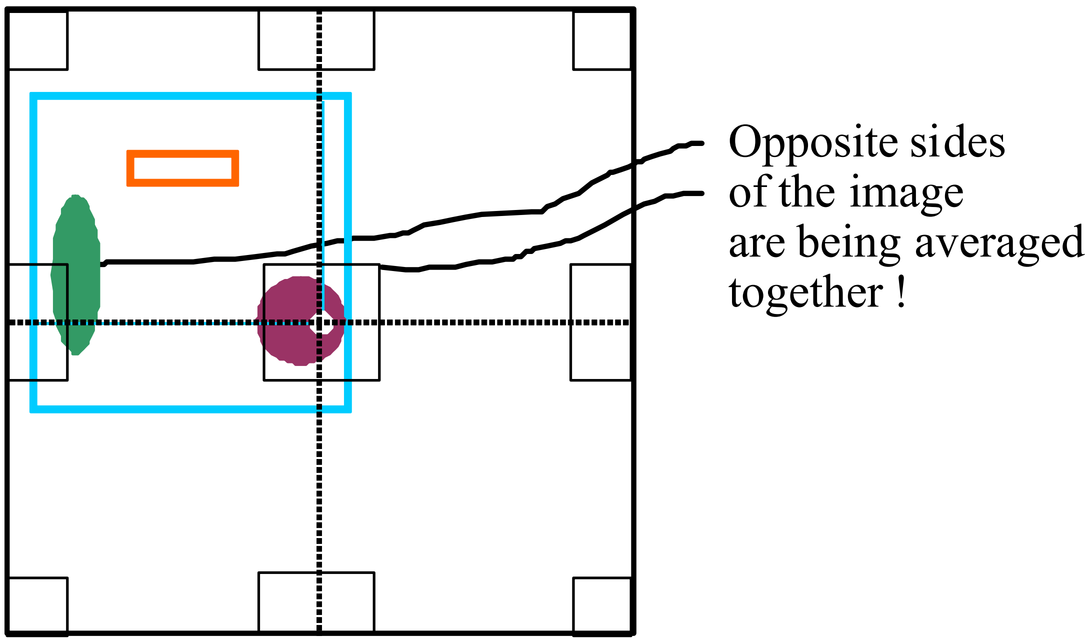


Input image

Image of square

The average filter operation may be expressed (at most points) as the wraparound convolution of the image with an image of a square with intensity $1/L$, where $L = \#$ pixels in the square

Wraparound Effect



- Near the image borders, however, wraparound effects occur.
- Usually, it is desirable to average **only neighboring pixels** ...
- The effect is **much worse** if the filter is large.
- If the filter is **small**, then this can sometimes be fixed by trimming off the borders of the averaged image.

Linear Convolution by Zero Padding

- Adapting wraparound convolution to do linear convolution is **conceptually simple**.
- Accomplished by **padding** the two image arrays with **zero values**.
- Then, the periodic extension wraps in only zero values.
- **Typically**, both image arrays are doubled in size, both vertically and horizontally:

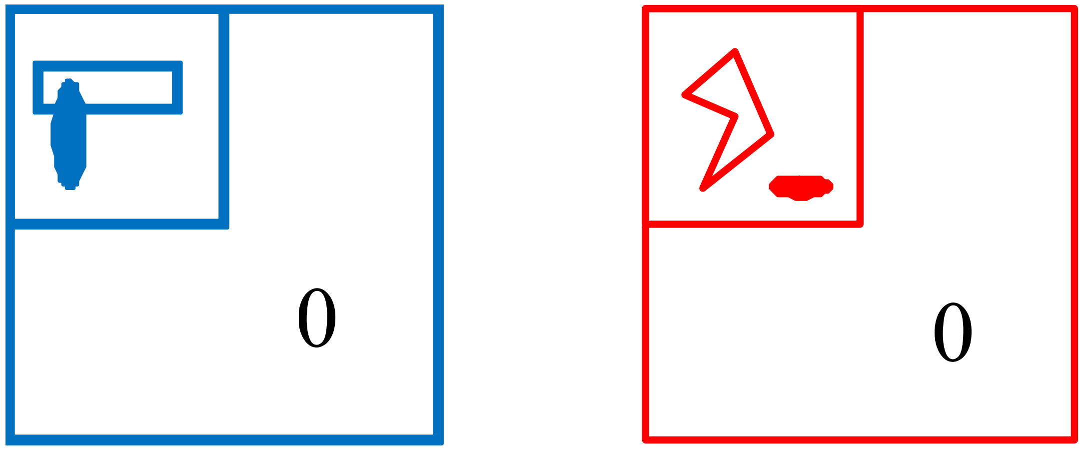


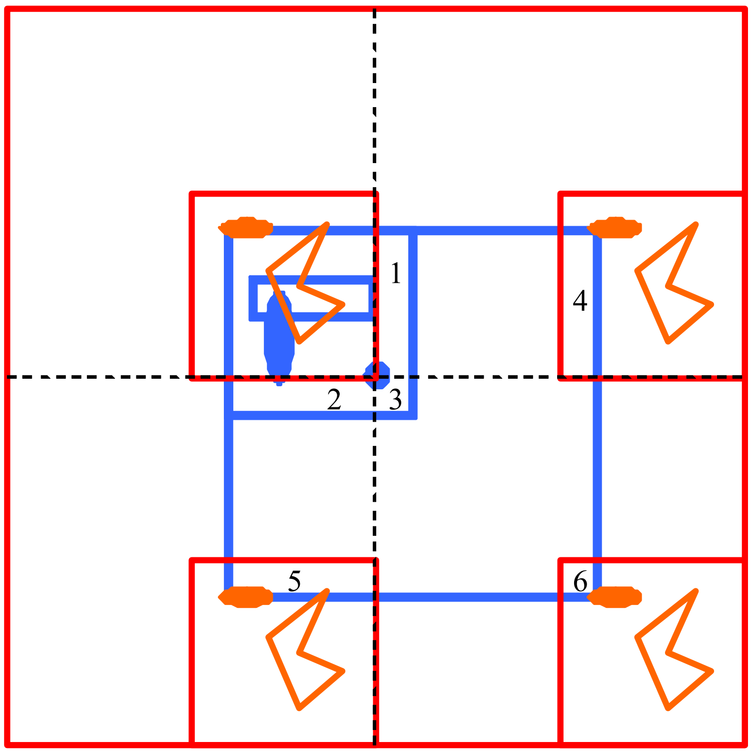
Image $\mathbf{I}_1$
(zero padded)

Image $\mathbf{I}_2$
(zero padded)

$2\text{M} \times 2\text{N}$ zero padded images

- **Wraparound eliminated**, since the zero padded, periodically extended, twice-reversed, and twice-shifted $\mathbf{I}_2$ image will now wrap in only zero pixels over the original $\mathbf{I}_1$ image.
- Can be seen by looking at the overlaps when computing the convolution at a point (m, n):

Wraparound Convolution of Zero Padded Images



Linear convolution by zero padding

- Remember, the wraparound convolution sum runs only over the **blue** square: $(0 \le p \le 2M\text{-}1,\ 0 \le q \le 2N\text{-}1)$.

- In the areas 1, 2, 3 the wraparound effect is eliminated because the zero-padded $\mathbf{I}_2$ image (red) is zero.
- In the areas 4, 5, 6 the wraparound effect is eliminated because the zero-padded $\mathbf{I}_1$ image (blue) is zero.
- This makes the result the same as performing linear convolution – as we did on p. 5.23.

2D Linear Convolution by DFT

- Let $\mathbf{I}_1$ and $\mathbf{I}_2$ be $\mathrm{M} \times \mathrm{N}$ digital images.
- We desire the **linear** convolution $\mathbf{J} = \mathbf{I}_1 * \mathbf{I}_2$.
- Let $\mathbf{I}_1'$ and $\mathbf{I}_2'$ be $2\mathrm{M} \times 2\mathrm{N}$ zero padded versions of $\mathbf{I}_1$ and $\mathbf{I}_2$.
- Compute:
  $$\mathbf{J}' = \mathbf{I}_1' \circledast \mathbf{I}_2' = \mathrm{IFFT}_{2\mathrm{M}\times 2\mathrm{N}}\left\{\mathrm{FFT}_{2\mathrm{M}\times 2\mathrm{N}}\left[\mathbf{I}_1'\right] \otimes \mathrm{FFT}_{2\mathrm{M}\times 2\mathrm{N}}\left[\mathbf{I}_2'\right]\right\}$$
- Then $\mathbf{J}'$ is the $2\mathrm{M} \times 2\mathrm{N}$ **wraparound** convolution of $\mathbf{I}_1'$ and $\mathbf{I}_2'$.
- It contains the desired **linear** convolution $\mathbf{J} = \mathbf{I}_1 * \mathbf{I}_2$.
- But, we usually want the filtered image:
  - to be the same size as the original,
  - to not be shifted relative to the original.
- So, which part of $\mathbf{J}'$ should we keep?
- The answer depends on the details of the application…

Which part of $\mathbf{J}'$ Should We Keep?

- It will take some work to answer this question – because the answer depends on exactly what we are trying to do.
- Some textbooks (e.g., Gonzalez and Woods) give complicated rules for this. Invariably, the rules depend on specific assumptions about what you are trying to do.
- For us, it will be better to develop an understanding of what is going on from the **fundamentals**.
- Then you will be able to figure out the right answer no matter what the particular situation.
- As usual, it will be helpful to look at the 1D case first…

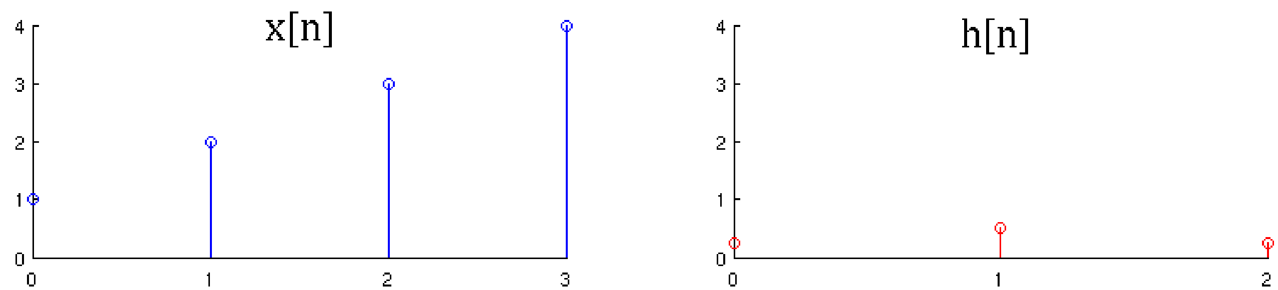
Finite-Length 1D Linear Convolution

- Let H be a 1D LTI causal 3-point weighted average filter with

$$h[n] = \tfrac{1}{4}\delta[n] + \tfrac{1}{2}\delta[n-1] + \tfrac{1}{4}\delta[n-2].$$

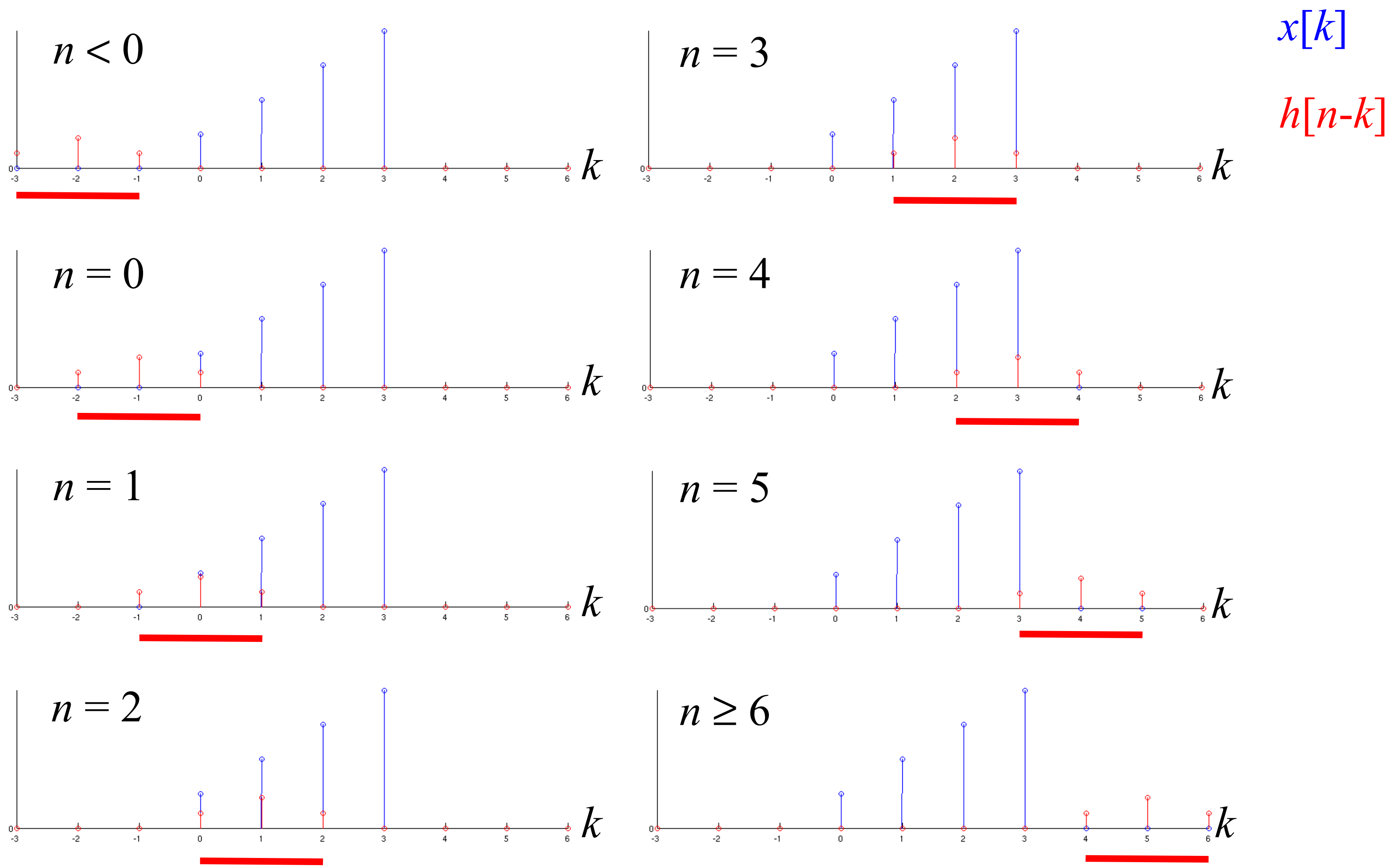
- Let the input be the 4-point signal

$$x[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2] + 4\delta[n-3].$$



- Let N_1 = length($x[n]$) = 4 and N_2 = length($h[n]$) = 3.
- The system output is the 1D linear convolution

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k].$$



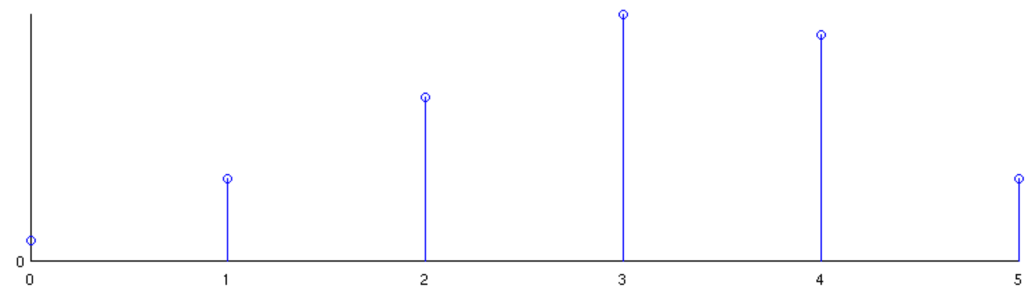
- There is nonzero overlap for $n = 0, 1, 2, 3$; i.e., once for each sample in $x[n]$, and for $n = 4, 5$; i.e., once for each sample in $h[n]$ **but** the last.
- After that there is **no** overlap.
- So the length of the convolution is $N_1 + N_2 - 1 = 6$.

- More precisely, computing $y[n]$ means computing the dot product of $x[k]$ and $h[n\text{-}k]$ for each n.
- Consider values of n starting at the far left ($-\infty$) and going right:
  - for $n < 0$, the rightmost sample of $h[n\text{-}k]$ does not yet reach the leftmost sample of $x[k]$. So the dot product is **zero**.
  - starting at $n = 0$, the rightmost sample of $h[n\text{-}k]$ starts to overlap with the graph of $x[k]$, so we get **nonzero** in general.
  - this situation continues for N_1 values of n, as the rightmost sample of $h[n\text{-}k]$ progresses to overlap each sample in $x[k]$. Thus, in general we get a nonzero dot product for $0 \le n \le N_1\text{-}1$; that is, for exactly N_1 values of n.
  - then, at $n = N_1$, the rightmost sample of $h[n\text{-}k]$ hangs over the right edge of the graph of $x[k]$. But we still get a nonzero dot product in general.
  - at $n = N_1+1$, the rightmost **two** samples of $h[n\text{-}k]$ hang over the right edge of the graph of $x[k]$, but the dot product is still nonzero in general.
  - this situation continues until **all but one sample** of $h[n\text{-}k]$ hang over. After that, the graph of $h[n\text{-}k]$ is entirely past the graph of $x[k]$ and the dot product is again zero.
- So, counting this up, we see that in general the dot product can be nonzero one time for each sample in $x[k]$ and one time for each sample in $h[n\text{-}k]$ **except the last one**… because once the last one hangs over there is no overlap.

- So, the convolution of two finite-length sequences with lengths N_1 and N_2 has a length that is given by $N_1 + N_2 - 1$.

- Matlab:

```
>> xn = [1 2 3 4];
>> hn = [0.25 0.5 0.25];
>> yn = conv(xn,hn);
>> length(yn)
ans = 6
>> stem([0:5],yn);
```



- To re-compute this same example using the 1D DFT and circular convolution, we need to zero pad both sequences to a length of at least $N_1 + N_2 - 1 = 6$:

```
>> xprime = [xn zeros(1,2)];
>> hprime = [hn zeros(1,3)];
>> yprime = ifft(fft(xprime).*fft(hprime));
>> max(abs(yn-yprime))
ans = 2.2204e-16
>> % output signal is the same as before
```

Notes on this 1D Example

- You can zero pad to a length $> N_1 + N_2 - 1$. Especially in 2D, the DFT may run faster if the padded length is a power of 2.
  - if you do this, the linear convolution still has length $N_1 + N_2 - 1$. It is contained in the first (leftmost) $N_1 + N_2 - 1$ samples of the result sequence.
- Interpreting $y[n]$ as the weighted 3-point average of $x[n]$:
  - There are **edge effects** on both ends of $y[n]$: zeros are averaged in where the graph of $h[n\text{-}k]$ hangs over the graph of $x[k]$ ($n = 0, 1, 4, 5$).
  - Because the filter is **causal**, it is **not** a centered average. For example, $y[4] = 0.25x[2] + 0.5x[3] + 0.25x[4]$.
  - In other words, there is **delay** (the filter introduces nonzero phase).

Notes 1D Example…

- Often, the edge effects are not a concern in 1D applications.
  - the impulse response $h[n]$ is often short compared to the signal $x[n]$.
  - for example, applying a 13-point average to one minute of digital audio at 44 kHz: $x[n]$ has length 2.64×10^6. But the edge effects impact only the first 7 samples and the last 7.
- A reasonable time delay is also okay in many 1D applications:
  - audio CD player, MP3 player
- A reasonable time delay **may** be of no concern in some image/video applications:
  - DVD/Blu ray player, cable set top box, youtube…

Extending these Ideas to 2D

- Let $\mathbf{I}_1$ be a $\mathrm{M}_1 \times \mathrm{N}_1$ digital image.
- Let $\mathbf{I}_2$ be a $\mathrm{M}_2 \times \mathrm{N}_2$ digital image.
- As before, we desire the **linear** convolution $\mathbf{J} = \mathbf{I}_1 * \mathbf{I}_2$.
- Extending the ideas we just saw in 1D, we can zero pad both images to a size of $\mathrm{M} \times \mathrm{N}$, where $\mathrm{M} = \mathrm{M}_1 + \mathrm{M}_2 - 1$ and $\mathrm{N} = \mathrm{N}_1 + \mathrm{N}_2 - 1$.
- Call the zero padded images $\mathbf{I}_1'$ and $\mathbf{I}_2'$.
- Then, the 2D linear convolution is given by

$$\mathbf{J} = \mathbf{I}_1' \circledast \mathbf{I}_2' = \mathrm{IFFT}_{\mathrm{M}\times\mathrm{N}}\left\{\mathrm{FFT}_{\mathrm{M}\times\mathrm{N}}\left[\mathbf{I}_1'\right] \otimes \mathrm{FFT}_{\mathrm{M}\times\mathrm{N}}\left[\mathbf{I}_2'\right]\right\}.$$

- It has size $\mathrm{M} \times \mathrm{N}$, i.e., $(\mathrm{M}_1 + \mathrm{M}_2 - 1) \times (\mathrm{N}_1 + \mathrm{N}_2 - 1)$.

- In general, the 2D FFT is faster if the number of rows and the number of columns are both powers of 2.
- So there is a tradeoff between using the minimum size FFTs that will eliminate the wraparound effect and using larger FFTs where the number of rows/cols are powers of 2.
- This tradeoff depends on the details of the particular FFT implementation and can be complicated to analyze.
- In rapid prototyping and in research/exploratory work, it is common to skip the analysis and zero pad both images up to the next power of 2, both horizontally and vertically.
- As before, call the zero padded size $\mathrm{M} \times \mathrm{N}$, which are now the next powers of 2 that are $\geq (\mathrm{M}_1 + \mathrm{M}_2 - 1)$ and $(\mathrm{N}_1 + \mathrm{N}_2 - 1)$.
- We then get $\mathbf{J}' = \mathbf{I}'_1 \circledast \mathbf{I}'_2 = \mathrm{IFFT}_{\mathrm{M}\times\mathrm{N}}\left\{\mathrm{FFT}_{\mathrm{M}\times\mathrm{N}}\left[\mathbf{I}'_1\right] \otimes \mathrm{FFT}_{\mathrm{M}\times\mathrm{N}}\left[\mathbf{I}'_2\right]\right\}$.
- The linear convolution $\mathbf{J} = \mathbf{I}_1 * \mathbf{I}_2$ still has size

$$(\mathrm{M}_1 + \mathrm{M}_2 - 1) \times (\mathrm{N}_1 + \mathrm{N}_2 - 1).$$

- It is contained in the top $\mathrm{N}_1 + \mathrm{N}_2 - 1$ rows and left $\mathrm{M}_1 + \mathrm{M}_2 - 1$ columns of the image $\mathbf{J}'$.

Some Notes on Zero Padding

- So why did it say on page 5.36 to zero pad both images to size $2M \times 2N$?
  - If the original images $\mathbf{I}_1$ and $\mathbf{I}_2$ are the same size (i.e., if both are $M \times N$), then the **minimum** FFT size to eliminate the wraparound effect is $(2M\text{-}1) \times (2N\text{-}1)$.
  - **Often**, the original sizes M and N **are** powers of 2. In this case, $2M \times 2N$ is the smallest FFT size that **both** eliminates the wraparound effect **and** gives the "power of 2" speedup in the FFT computations.
- If the original sizes M and N are **not** powers of 2, then it will generally be slightly faster to zero pad to a size of $(2M\text{-}1) \times (2N\text{-}1)$ instead of $2M \times 2N$.

Practical Considerations

- When you convolve two images that each have 8-bit pixels, the result image generally has pixels that are **outside** the range [0, 255].
  - For example, if $\mathbf{I}_1$ and $\mathbf{I}_2$ are both 256×256 and are both constant images with $\mathbf{I}_1(m, n) = \mathbf{I}_2(m, n) = 255$, then the maximum pixel value in $\mathbf{J} = \mathbf{I}_1 * \mathbf{I}_2$ will be $256^2 \cdot 255^2 = 4.2615 \times 10^9$!
  - So you have to do a full-scale contrast stretch before you can look at the result image (or do practically **anything** useful with it).
- When computing the linear (or circular) convolution of two **real** images using FFTs, numerical roundoff errors may cause the result image to have a small but nonzero imaginary part.
  - If $\mathbf{I}_1$ and $\mathbf{I}_2$ are both real, then so is the convolution. You should discard any nonzero imaginary part.

2D Linear Convolution by FFT: Example 1

- Use FFTs to compute the linear convolution of:

$\mathbf{I}_1$
256×256 Lena

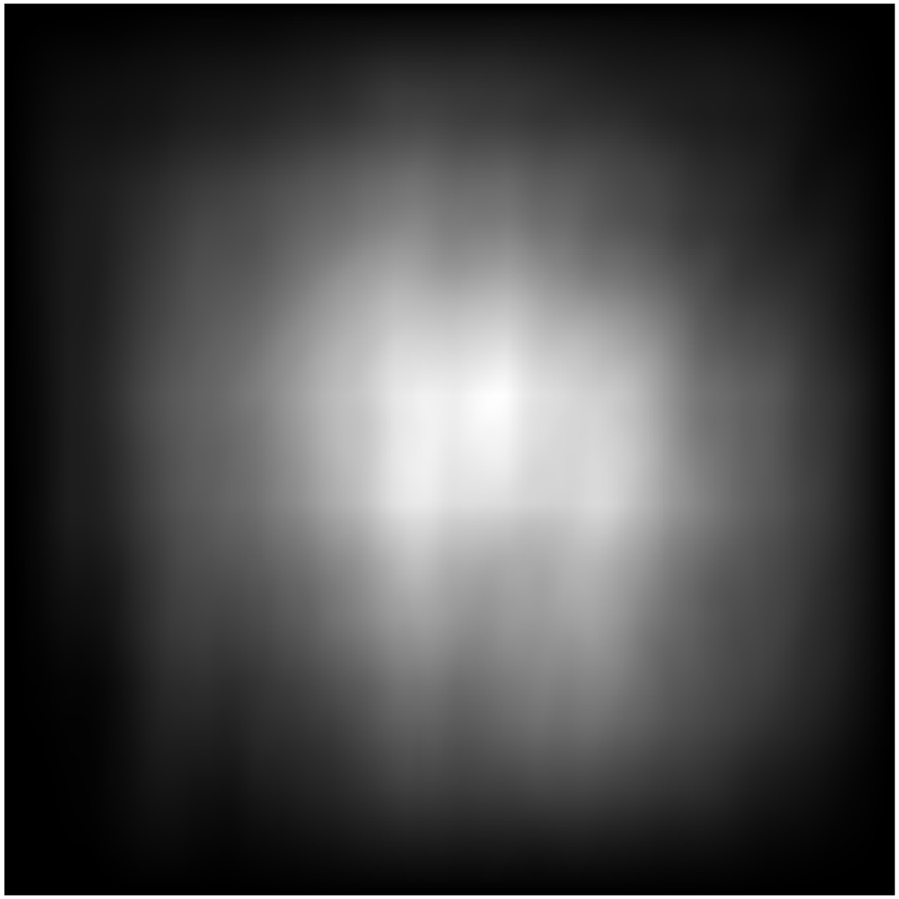
$\mathbf{I}_2$
200×200 Peppers

- Zero pad both images to size 256 + 200 – 1 = 455 rows/cols.

- Matlab:

```
[Lena,junk] = fread(fidLena,[256,256],'uchar');
[Peppers,junk] = fread(fidPeppers,[256,256],'uchar');
Lena = Lena'; Peppers = Peppers';
Peppers200 = Peppers(1:200,1:200);
Padsize = 256 + 200 - 1;
ZPLena = zeros(Padsize,Padsize);
ZPLena(1:256,1:256) = Lena;
ZPPeppers200 = zeros(Padsize,Padsize);
ZPPeppers200(1:200,1:200) = Peppers200;
J = real(ifft2(fft2(ZPLena) .* fft2(ZPPeppers200)));
JJ = stretch(J);
```

Result:



- Check:

```
J2 = conv2(Lena,Peppers200);
JJ2 = stretch(J2);
% exact same result
```

Comments on Example 1:

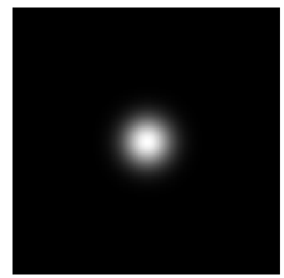
- The result is correct…
- But not very useful!
- In most cases, we aren't really interested in convolving two "typical" images directly.
- Rather, what we usually **are** interested in doing with linear convolution is applying a 2D LTI filter that will modify the image in a predictable way.
- This implies that we must **design** the filter:
  - design $\mathbf{I}_2$ to be the filter impulse response, or
  - design $\tilde{\mathbf{I}}_2$ to be the filter frequency response.

2D Linear Convolution by FFT: Example 2

- Use FFTs to apply a 128×128 LTI Gaussian low pass filter.
- Impulse resp: $H(r) = e^{-r^2/(2\sigma^2)}$, where $r = \sqrt{(m-64)^2 + (n-64)^2}$.

$\mathbf{I}_1$

256×256 Lena



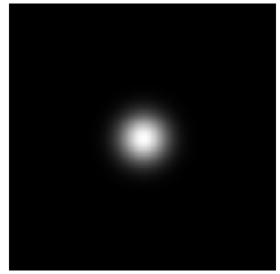
$\mathbf{I}_2 = \mathbf{H}$

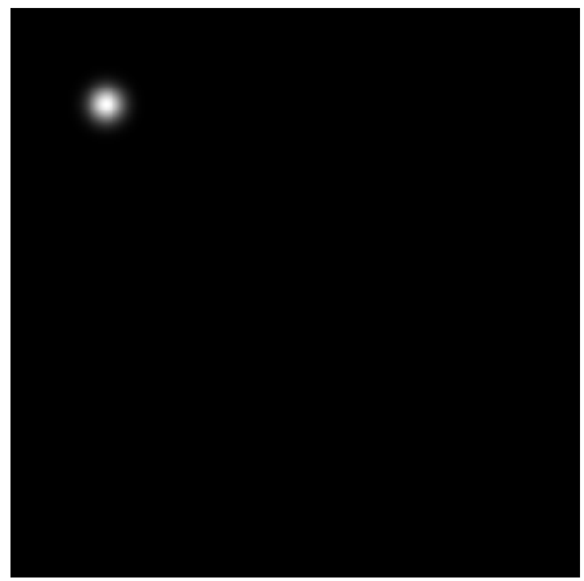
128×128 Gaussian

$\sigma = 8$

- Zero pad both images to size 256 + 128 – 1 = 383 rows/cols.

```
[COLS,ROWS] = meshgrid(0:127,0:127);
COLS = COLS - 64;
ROWS = ROWS - 64;
R = sqrt(ROWS .^ 2 + COLS .^ 2);
sigma = 8;
h = exp(-R.^2/(2*sigma^2));

Padsize = 256 + 128 - 1;
ZPLena = zeros(Padsize,Padsize);
ZPLena(1:256,1:256) = Lena;
ZPh = zeros(Padsize,Padsize);
ZPh(1:128,1:128) = h;
```





```
y1 = real(ifft2(fft2(ZPLena) .* fft2(ZPh)));
yy1 = stretch(y1);
```

- Usually, we want the result to be the same size as the original image, so:

```
y3 = y1(1:256,1:256);
yy3 = stretch(y3);
```

Comments on Example 2:

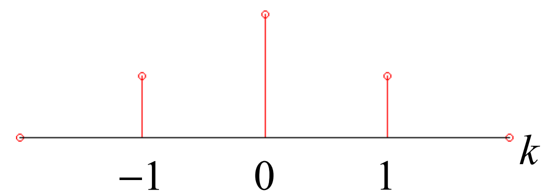
- This result is more useful than the one in Example 1!
  - In Example 2, we **did** get a low pass filtered Lena!
- But there are two main problems:
  - the filtered Lena image (i.e., the convolution result) is not the same size as the original.
  - since the main lobe of $\mathbf{H}$ is not centered around pixel (0,0), the filter introduces a nontrivial phase shift.
  - In other words, the convolution result has a **spatial shift** relative to the original image.
- One approach to fixing these two problems is to simply "undo" the spatial shift and crop the result to the size of the original image.
- To understand this, we once again go back to 1D…

1D Linear Convolution Again

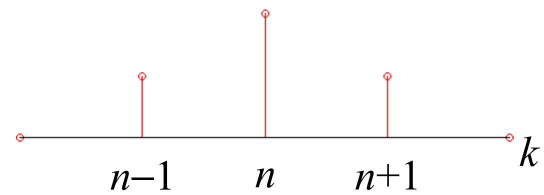
- Back on page 5.38, we had the 1D LTI causal 3-point weighted average filter with
$$h[n] = \tfrac{1}{4}\delta[n] + \tfrac{1}{2}\delta[n-1] + \tfrac{1}{4}\delta[n-2].$$
- By shifting $h[n]$ in time, we can turn this into a non-causal 3-point weighted average that **is centered**, so that it has zero phase and introduces no phase shift:
$$g[n] = h[n+1] = \tfrac{1}{4}\delta[n+1] + \tfrac{1}{2}\delta[n] + \tfrac{1}{4}\delta[n-1].$$
- Because $g[n]$ is real and even, it's DTFT is also **real** and even.
  - This means that the phase $\angle G(e^{j\omega})$ is **identically zero**.
  - So the filter G is not causal, but it introduces no phase shift between the input signal and the output signal.

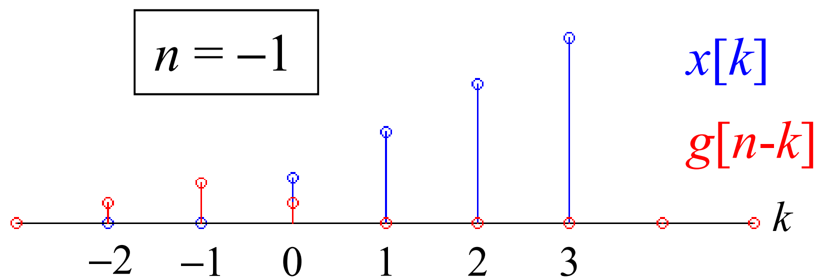
- Recall: $x[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2] + 4\delta[n-3]$.

Graph of $g[k]$



Graph of $g[n–k]$

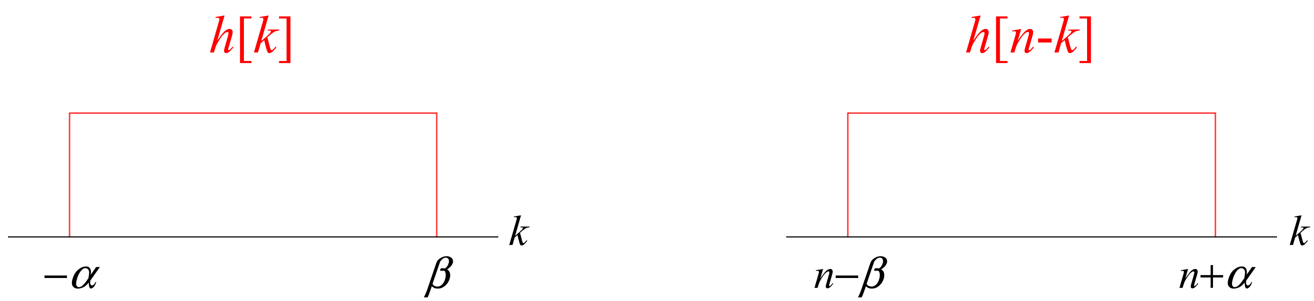




- Now, the first nonzero overlap occurs when $n = -1$ instead of $n = 0$.
- The linear convolution $y[n] = x[n] * g[n]$ still has length $N_1 + N_2 - 1 = 6$.
- But now this corresponds to times $n = -1$ to 4 **instead** of n=0 to 5.
- In Matlab, **everything is still exactly the same as before**.

- So even though everything is exactly the same in Matlab, we have to remember that this time the first array element of `yn=conv(xn,hn)` is for $n = -1$, not n=0!
- The length 4 weighted 3-point average signal corresponding to $x[n]$ is obtained by taking `yn` for n=0 to 3 **only**.
- In Matlab, this is `yn(2:5)`.
- Why?
  - The convolution $y[n]$ has length 6 and goes from $n = -1$ to n=4.
  - So, in Matlab, the first element of `yn` is for $n = -1$, the second element is for n=0, and so on…
- Notice that `yn(2:5)` is the same size as $x[n]$ and is not shifted relative to $x[n]$.

- More generally, suppose we have a 1D LTI filter H with an impulse response $h[n]$ that is nonzero from $n = -\alpha$ to $n = +\beta$:



- If $x[n]$ starts at n=0, then the **first** nonzero overlap in the linear convolution $y[n] = x[n] * h[n]$ will occur at n = $-\alpha$.
- So the α+1'st nonzero sample of the convolution will be the one that corresponds to n=0.
- Thus, in the Matlab array `yn=conv(xn,hn)`, it is the element `yn(α+1)` that corresponds to n=0.
- To obtain an output sequence the same length as the input that is **not** shifted, we keep length($x[n]$) samples from `yn` starting at index `α+1` .
- In the example on pages 5.56-5.57, $\alpha = 1$ and length($x[n]$) = 4, so we kept `yn(2:5)`.

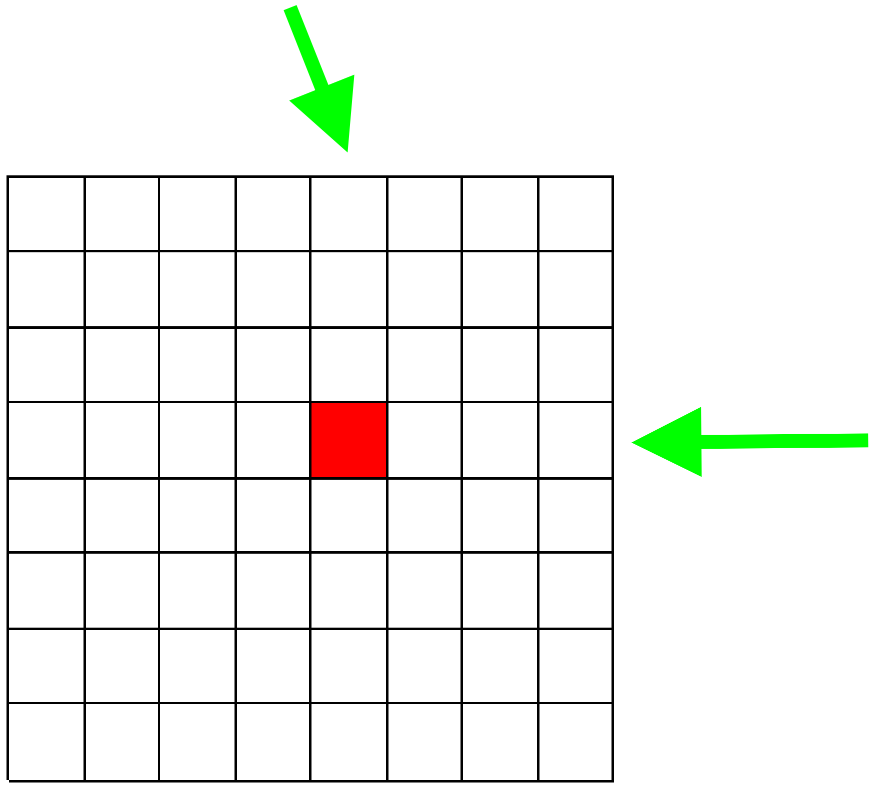
Extension to 2D

- Let $\mathbf{I}$ be a $\mathrm{M}_1 \times \mathrm{N}_1$ digital image.
- We will design a $\mathrm{M}_2 \times \mathrm{N}_2$ digital impulse response $\mathbf{H}$.
  - This time, the spatial origin (0,0) will **not** be located in the upper left corner of the $\mathbf{H}$ image.
  - Let (m, n) indicate the **true** col/row spatial coordinates in the mathematical formula for the impulse response function $\mathbf{H}$.
  - Let (p, q) indicate zero-based col/row indexing in the $\mathbf{H}$ image.
  - Let (r, s) indicate Matlab row/col indexing in the $\mathbf{H}$ image.
- We desire the **linear** convolution $\mathbf{J} = \mathbf{I} * \mathbf{H}$.
- In computing the image pixels $\mathrm{H}(p, q)$, suppose we place the mathematical origin $(m,n) = (0,0)$ at $(p, q) = (p_0, q_0)$.
- Then we have horizontal offset $\alpha_x = p_0$ and vertical offset $\alpha_y = q_0$.
- The upper left pixel of the $\mathbf{H}$ image will be for $(m, n) = (-p_0, -q_0)$.
- The first pixel to **keep** will be $(m, n) = (0,0)$, $(p, q) = (p_0, q_0)$, and Matlab row/col $(r, s) = (q_0+1, p_0+1)$.

- For example, suppose $\mathrm{M}_2 = \mathrm{N}_2 = 8$ and $(p_0, q_0) = (4,3)$:

$p=4$, $m=0$, Matlab column $s=5$.

the $\mathbf{H}$ image



$q=3$, $n=0$, Matlab row $r=4$.

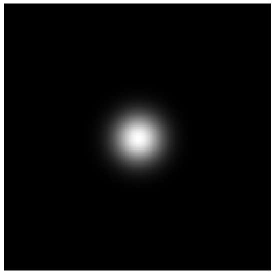
- The first pixel of the convolution to keep will be the one at $(p, q) = (4,3)$, or Matlab row/col $(r, s) = (4,5)$.

2D Linear Convolution by FFT: Example 3

- Use FFTs to apply a 128×128 LTI Gaussian low pass filter.
- The result must be the same size as the original image and must not be shifted relative to the original.
- We place the origin $(m, n) = (0,0)$ at $(p, q) = (64,64)$.
- Impulse response:

$$\mathrm{H}(m,n) = \exp\left[-\frac{m^2+n^2}{2\sigma^2}\right]$$

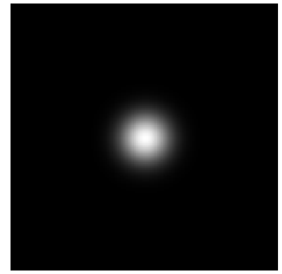
$$= \exp\left[-\frac{(p-64)^2+(q-64)^2}{2\sigma^2}\right]$$



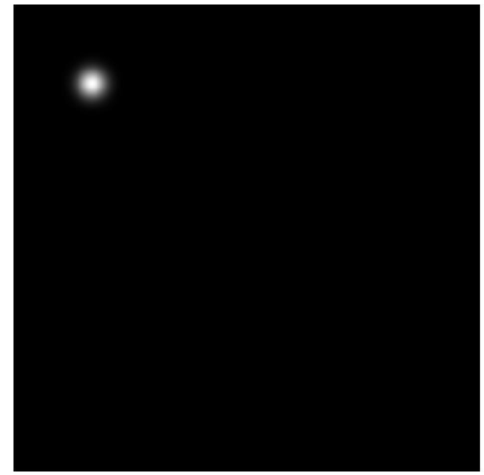
$\mathrm{H}(p, q)$
128×128 Gaussian
$\sigma = 8$

- We have $(p_0, q_0) = (64,64)$. So the first pixel to keep is $(p, q) = (64,64)$, which is Matlab row/col $(r, s) = (65,65)$.

```
[COLS,ROWS] = meshgrid(0:127,0:127);
sigma = 8;
h = exp(-((ROWS-64).^ 2 + (COLS-64).^ 2)/(2*sigma^2));
```



```
Padsize = 256 + 128 - 1;
ZPLena = zeros(Padsize,Padsize);
ZPLena(1:256,1:256) = Lena;
ZPh = zeros(Padsize,Padsize);
ZPh(1:128,1:128) = h;
```



```
y1 = real(ifft2(fft2(ZPLena) .* fft2(ZPh)));
y3 = stretch(y1(65:320,65:320));
```

A Common Approach

- For a $\mathrm{M} \times \mathrm{N}$ digital image $\mathbf{I}$, it is common to also design the impulse response $\mathbf{H}$ to be $\mathrm{M} \times \mathrm{N}$.
- Assuming M and N are even, we place the spatial origin $(m, n) = (0,0)$ at $(p, q) = (\mathrm{M}/2,\mathrm{N}/2)$, which is Matlab row/col $(r, s) = (\mathrm{N}/2 + 1, \mathrm{M}/2 + 1)$.
- This gives offsets $p_0 = \mathrm{M}/2$ and $q_0 = \mathrm{N}/2$.
- Usually, $\mathbf{I}$ and $\mathbf{H}$ are both zero padded to size $2\mathrm{M} \times 2\mathrm{N}$ to obtain the zero padded images $\mathbf{I}'$ and $\mathbf{H}'$.
- Then,

$$\mathbf{J}' = \mathbf{I}' \circledast \mathbf{H}' = \mathrm{IFFT}_{2\mathrm{M}\times 2\mathrm{N}}\left\{\mathrm{FFT}_{2\mathrm{M}\times 2\mathrm{N}}\left[\mathbf{I}'\right] \otimes \mathrm{FFT}_{2\mathrm{M}\times 2\mathrm{N}}\left[\mathbf{H}'\right]\right\}.$$

- The desired $\mathrm{M} \times \mathrm{N}$ image $\mathbf{J}$ is then taken from $\mathbf{J}'$ starting with pixel $(p, q) = (\mathrm{M}/2,\mathrm{N}/2)$.
- In Matlab row/col coordinates, this is

$$\mathbf{J} = \mathbf{J}'\left(\tfrac{\mathrm{N}}{2} + 1 : \tfrac{3\mathrm{N}}{2}, \tfrac{\mathrm{M}}{2} + 1 : \tfrac{3\mathrm{M}}{2}\right).$$

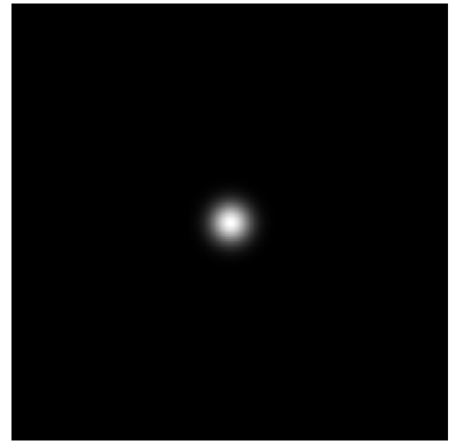
2D Linear Convolution by FFT: Example 4

- Input image $\mathbf{I}$ is 256×256 Lena.
- Use FFTs to apply a 256×256 Gaussian low pass LTI filter with space constant $\sigma = 8$.
  - We have $\mathrm{M} = \mathrm{N} = 256$.
  - For $\mathbf{H}$, we place the origin $(m, n) = (0,0)$ at $(p, q) = (128,128)$. This is Matlab row/col $(r, s) = (129,129)$.
  - This gives horizontal/vertical offsets $p_0 = q_0 = 128$.
  - With $\sigma = 8$, we compute the $\mathbf{H}$ image according to

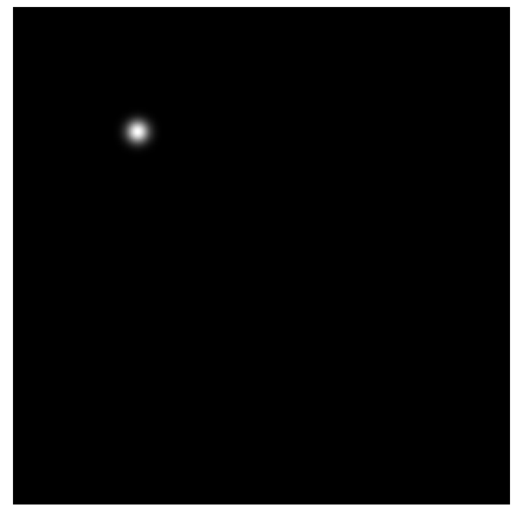
$$\mathrm{H}(p,q) = \exp\left[-\frac{m^2+n^2}{2\sigma^2}\right] = \exp\left[-\frac{(p-128)^2+(q-128)^2}{2\sigma^2}\right]$$

  - For Matlab, remember that (row, col) $= (r, s) = (q+1, p+1)$.
  - we zero pad both images to size $2\mathrm{M} \times 2\mathrm{N} = 512 \times 512$.
  - The desired 256×256 image $\mathbf{J}$ is then taken from $\mathbf{J}'$ starting with pixel $(p, q) = (128,128)$; for Matlab this is $(r, s) = (129,129)$.

```
[COLS,ROWS] = meshgrid(0:255,0:255);
sigma = 8;
h = exp(-((ROWS-128).^ 2 + (COLS-128).^ 2)/(2*sigma^2));
```



```
Padsize = 512;
ZPLena = zeros(Padsize,Padsize);
ZPLena(1:256,1:256) = Lena;
ZPh = zeros(Padsize,Padsize);
ZPh(1:256,1:256) = h;
```



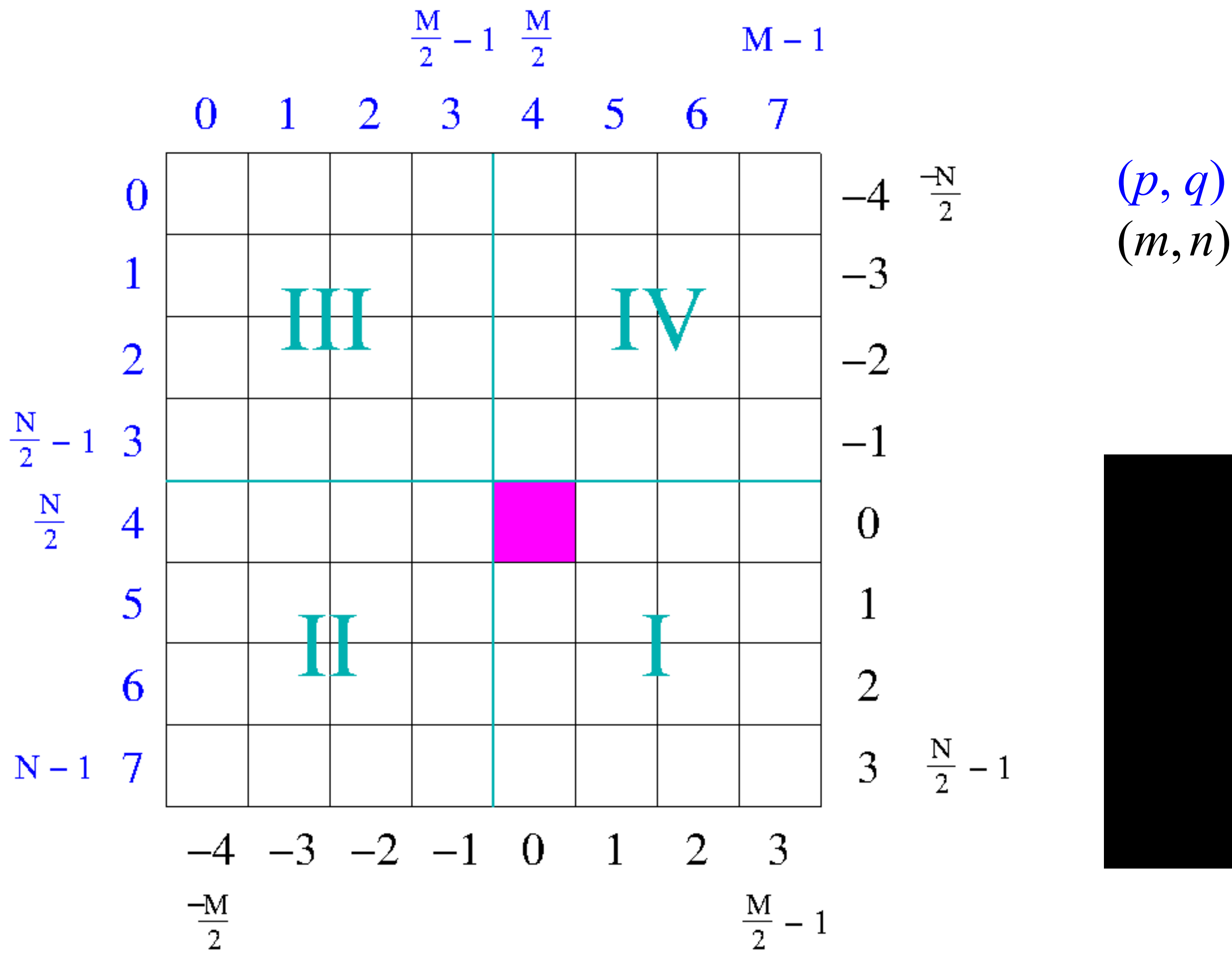
```
y = real(ifft2(fft2(ZPLena) .* fft2(ZPh)));
yy = stretch(y(129:384,129:384));
```

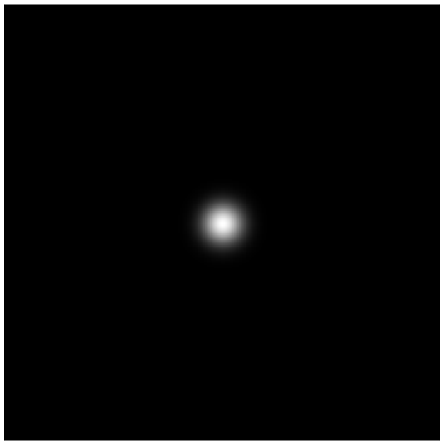
Zero Phase Impulse Response

- Another way to prevent the output image from being shifted relative to the original is to design a true zero phase impulse response directly.
- There are two main requirements for this:
  1. The impulse response must be real and even:
     - $H(-m, -n) = H(m, n)$
     - $\mathrm{Im}[H(m, n)] = 0$
  2. The spatial origin $(m, n) = (0,0)$ must be placed on the upper left pixel $(p, q) = (0,0)$ of the impulse response image $\mathbf{H}$.
- Because the DFT implies that the image is periodic, this means that the main lobe of the impulse response will be split across the four corners of the image $\mathbf{H}$.
- As we will see, this creates some issues when we zero pad the $\mathbf{H}$ image to implement linear convolution with FFTs.

- To understand how this works, let’s think about it in terms of Example 4 on pages 5.64-5.65.
- Input image $\mathbf{I}$ is 256×256. So M = #cols = 256 and N = #rows = 256.
- We want to use FFTs to apply a 256×256 Gaussian low pass LTI filter with space constant $\sigma = 8$.
- Recall: we have **three** coordinate systems:
  - (p, q): zero based col/row in the $\mathbf{H}$ image.
  - (m, n): “true” col/row that we use in the math to compute the $\mathbf{H}$ image.
  - (r, s): Matlab row/col: $(r, s) = (q+1, p+1)$.
- To design a true zero-phase impulse response image $\mathbf{H}$, we need for the (m, n) coordinate system and the (p, q) coordinate system to “line up.”
- This means that the offsets are $p_0 = q_0 = 0$.
- It will be useful to think of the $\mathbf{H}$ image as being divided into quadrants.
- Let’s start by looking at a small $\mathbf{H}$ image with $\mathrm{M} = \mathrm{N} = 8$.

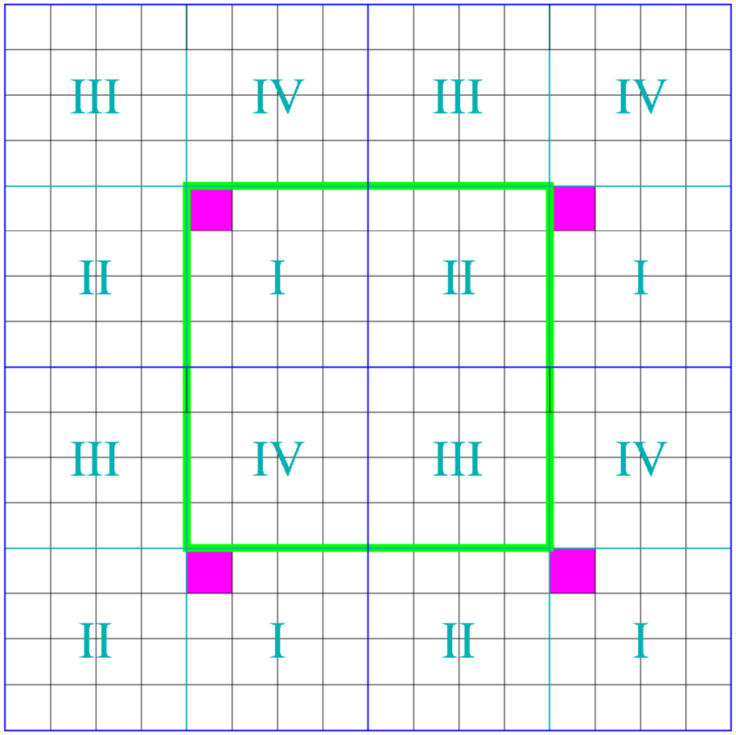
- Here is the division of the $\mathbf{H}$ image into quadrants I-IV, with the coordinate systems as they were in Example 4:

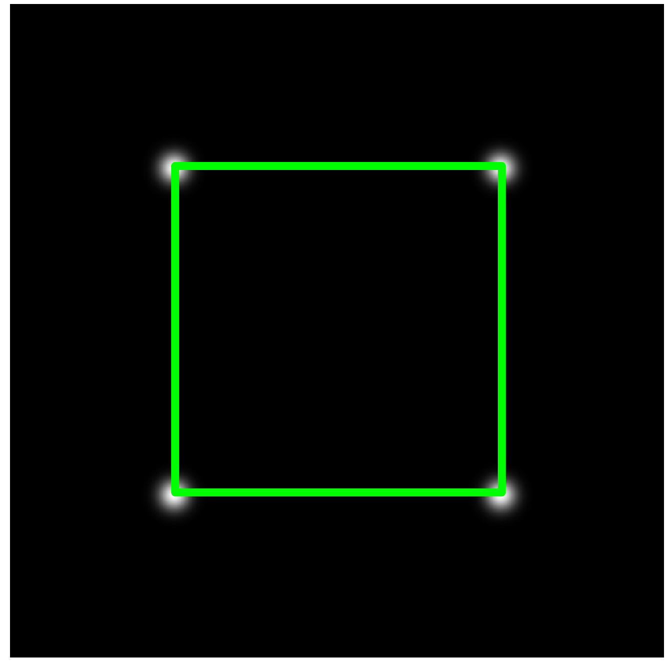




- The DFT implies that this is one period of a periodic image.

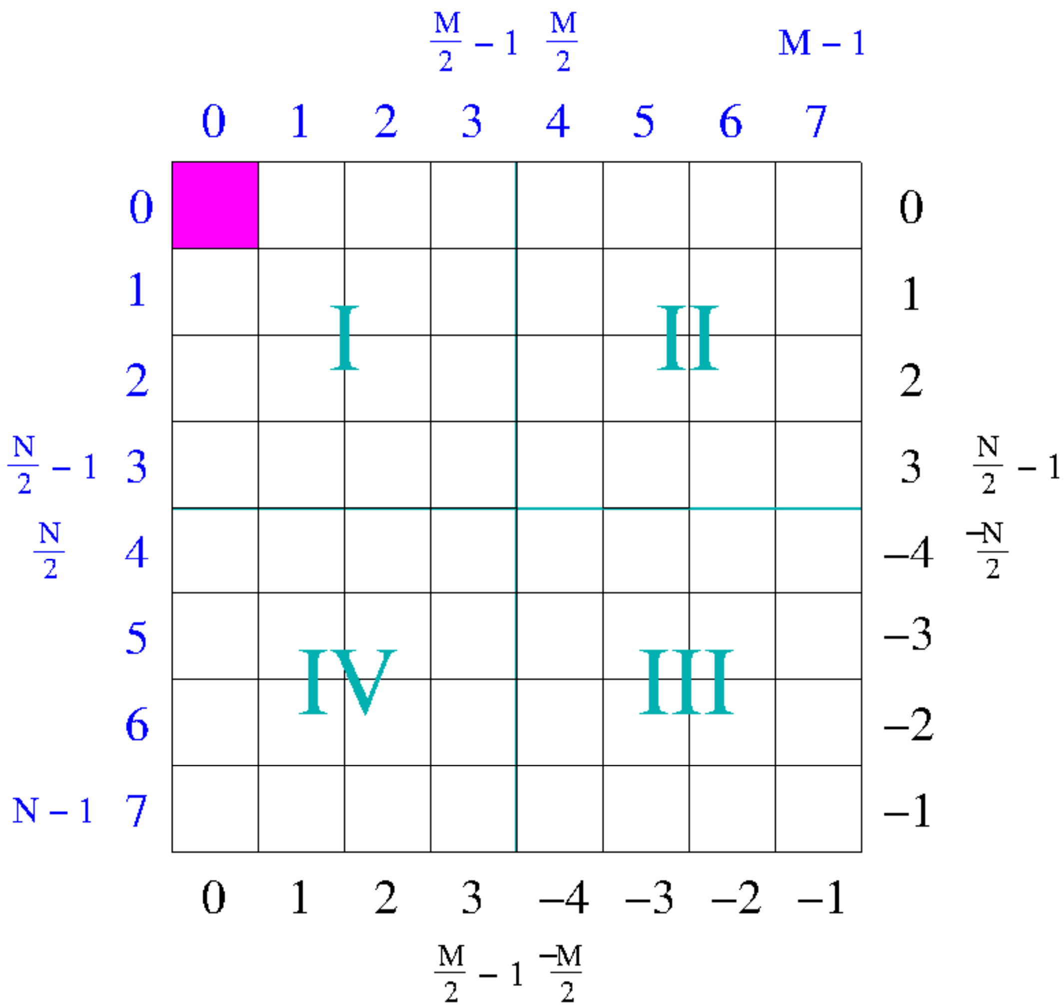
- Here is the periodic extension:





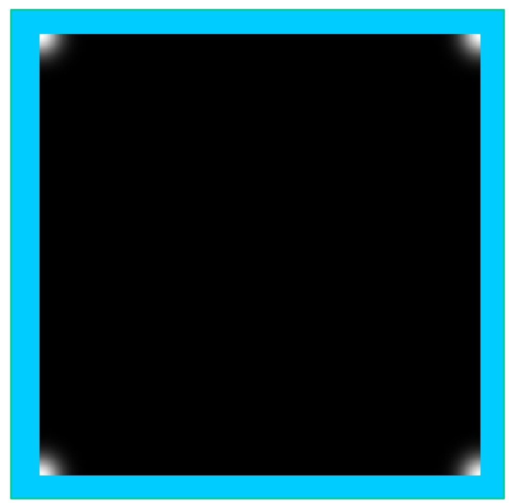
- The part in the green box is a zero phase image, with $(m, n) = (0{,}0)$ in the upper left corner.
- Notice that this splits the main lobe of the impulse response across the four corners of the image in the green box.
- To get this image, we have to permute the quadrants around…

- Here is the zero phase image in the green box on the last page:



(p, q)

(m, n)



- You can write a program to design this directly.
- Or you can do the design as in Example 4, and then shift the quadrants around:
  - Can write a program to do this directly.
  - Or take the FFT, multiply by $(-1)^{u+v}$, then invert.
  - In Matlab, you can call fftshift on the **H image**.

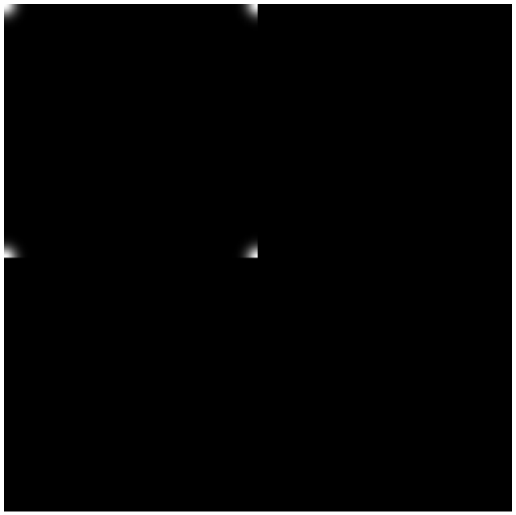
```
[COLS,ROWS] = meshgrid(0:255,0:255);
sigma = 8;
h = exp(-((ROWS-128).^ 2 + (COLS-128).^2)/(2*sigma^2));
h2 = fftshift(h);
```

- If you take the FFT, multiply it pointwise with the FFT of the image, and invert, then you will get circular convolution **without any phase shift**.

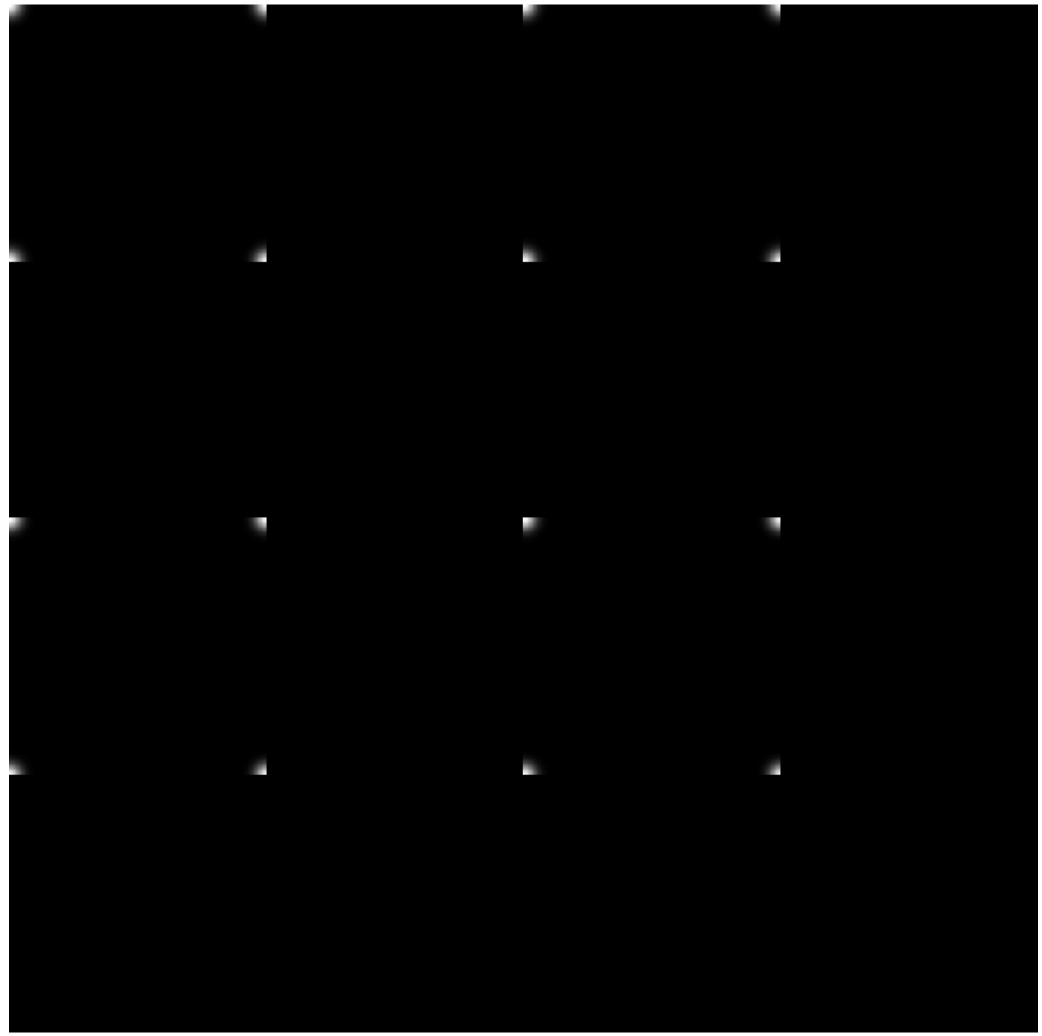
```
ifft2(fft2(Lena).*fft2(h2));
```

- However, our previous zero padding scheme will not work with the zero phase impulse response.

zero padded



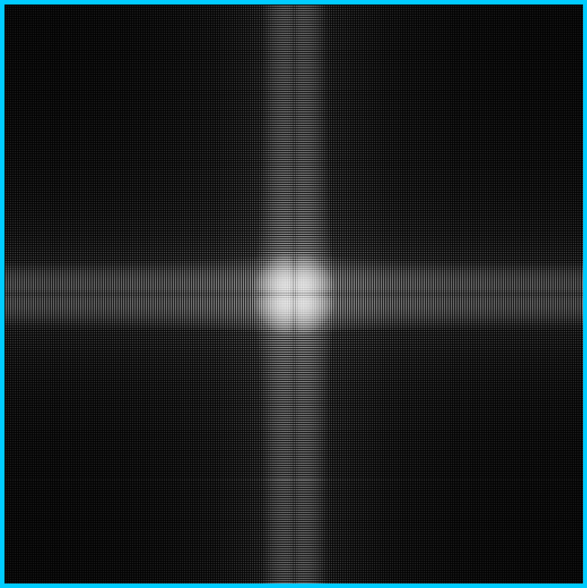
periodic extension



- The quadrants of the main lobe don’t fit together correctly.
- This is **not** the filter that we set out to design.
- Will give **incorrect** results.

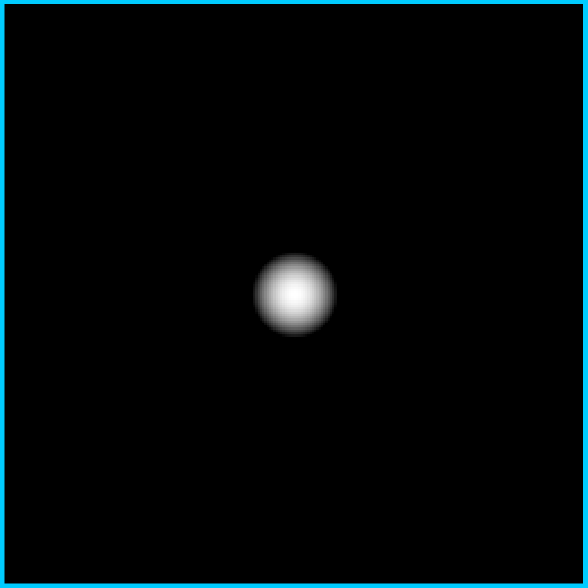
Incorrect filtered image

Incorrect frequency response

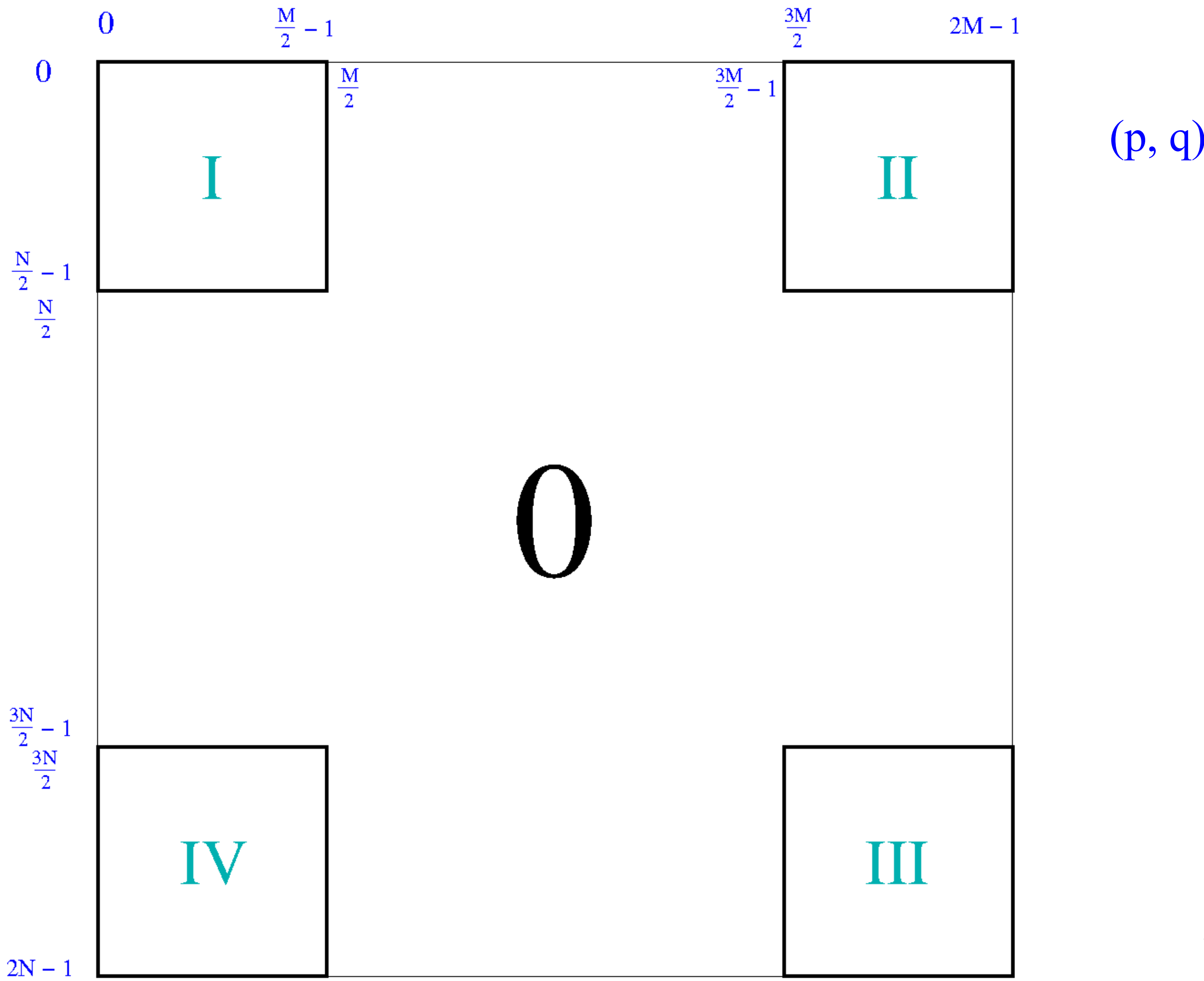


Correct filtered image

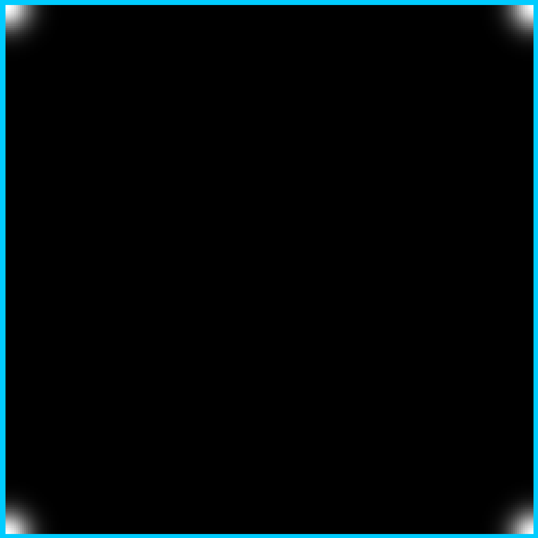
Correct frequency response



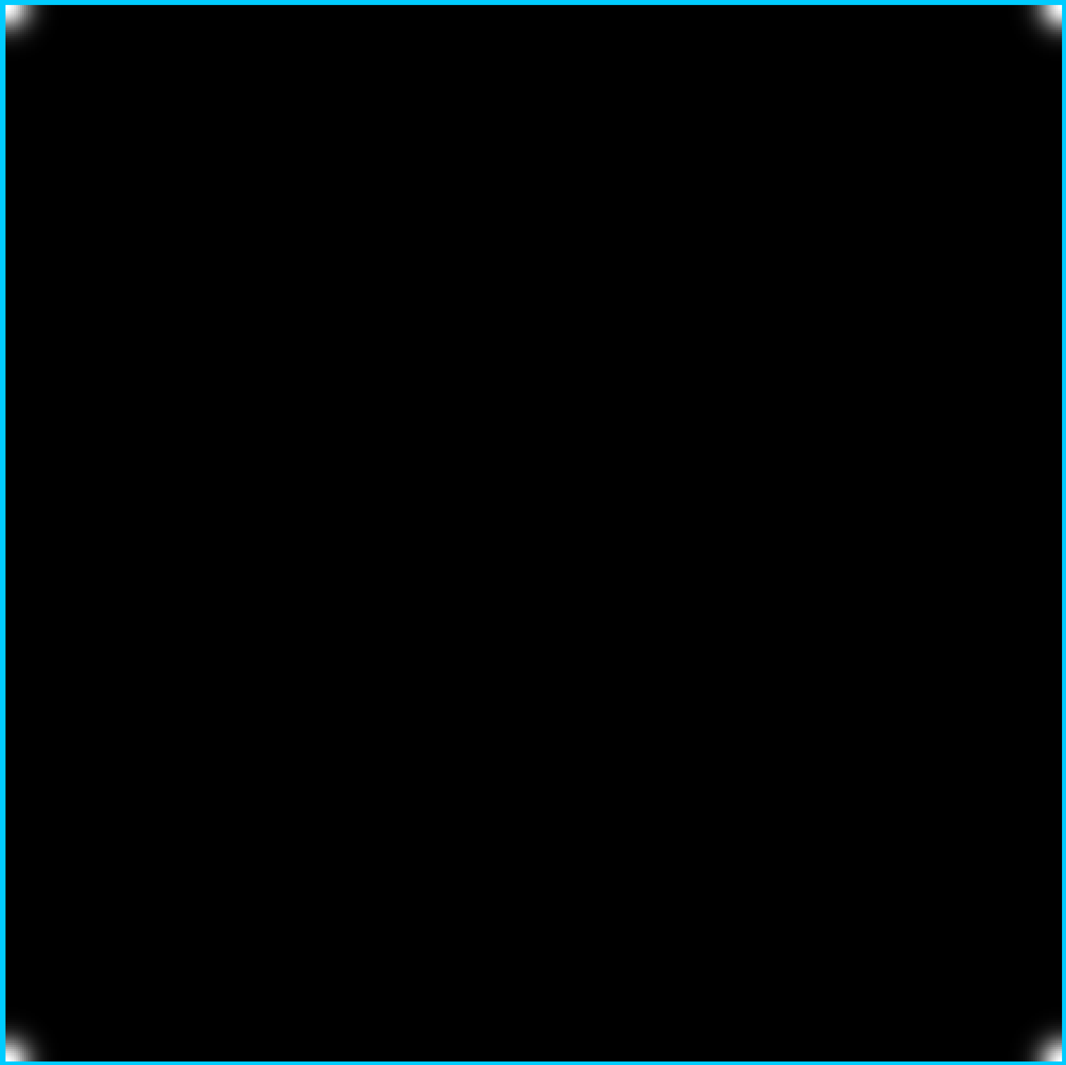
- To get the correct zero padded filter image, you must place the four quadrants of the zero phase impulse response in a $2M \times 2N$ zero image:



zero phase
impulse
response



correctly zero padded



2D Linear Convolution by FFT: Example 5

- Use FFTs to apply a 256×256 true zero phase Gaussian low pass LTI filter with space constant $\sigma = 8$.

```
ZPLena = zeros(512,512);
ZPLena(1:256,1:256) = Lena;
[COLS,ROWS] = meshgrid(0:255,0:255);
h = exp(-((ROWS-128).^ 2 + (COLS-128).^ 2)/(2*sigma^2));
h2 = fftshift(h);
h2ZP = zeros(512,512);
h2ZP(1:128,1:128) = h2(1:128,1:128);
h2ZP(1:128,385:512) = h2(1:128,129:256);
h2ZP(385:512,1:128) = h2(129:256,1:128);
h2ZP(385:512,385:512) = h2(129:256,129:256);
yz = ifft2(fft2(ZPLena).*fft2(h2ZP));
yz = yz(1:256,1:256);
```

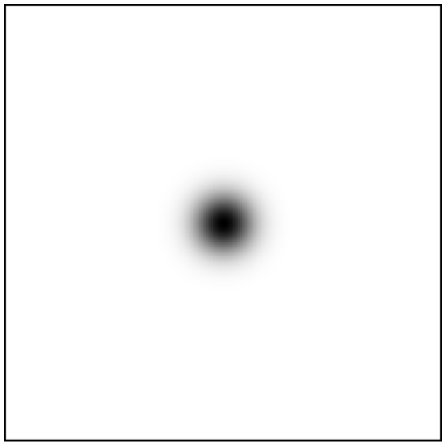
Frequency Response Design

- Often, it is desirable to design the frequency response $\tilde{\mathbf{H}}$ directly (instead of the impulse response $\mathbf{H}$).
- Normally, the frequency response will be designed at a size of $\mathrm{M} \times \mathrm{N}$ (same size as the image) so that the frequencies will be the same as those in the DFT of the image.
- The usual way of doing this is to fill in the $\mathrm{M} \times \mathrm{N}$ DFT array $\tilde{\mathbf{H}}$ from a designed formula for the frequency response (this is called **frequency sampling**).
- However, to implement linear convolution by multiplication of DFTs, what we need is a $2\mathrm{M} \times 2\mathrm{N}$ DFT array that can be pointwise multiplied with the DFT of the zero padded image.
- There is no simple analytical way to get this from the designed $\mathrm{M} \times \mathrm{N}$ DFT array $\tilde{\mathbf{H}}$.

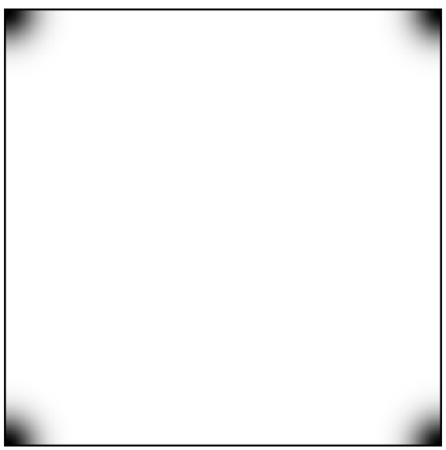
- Common procedure for direct design of the frequency response:
  1. Fill in the $\mathrm{M} \times \mathrm{N}$ DFT array $\tilde{\mathbf{H}}$ from the design equation.
     - Usually done with the **centered** DFT.
     - Then call fftshift to “un-center” it.
  2. Take the inverse FFT to get the zero phase $\mathrm{M} \times \mathrm{N}$ impulse response $\mathbf{H}$.
  3. Then **either**:
     - Proceed as in Example 5, **or**
     - Shift the impulse response to place the origin $(m,n) = (0,0)$ in the center at $(p,q) = (\mathrm{M}/2, \mathrm{N}/2)$ and proceed as in Example 4.
       – In Matlab, the shifting can be done by calling fftshift.

2D Linear Convolution by FFT: Example 6

- Use frequency sampling to apply a high pass Gaussian LTI filter to the 256×256 Lena image.
- Hi pass Gaussian design equation: $\tilde{\mathrm{H}}_{\mathrm{D}}(U,V) = 1 - e^{-(U^2+V^2)/2\sigma^2}$



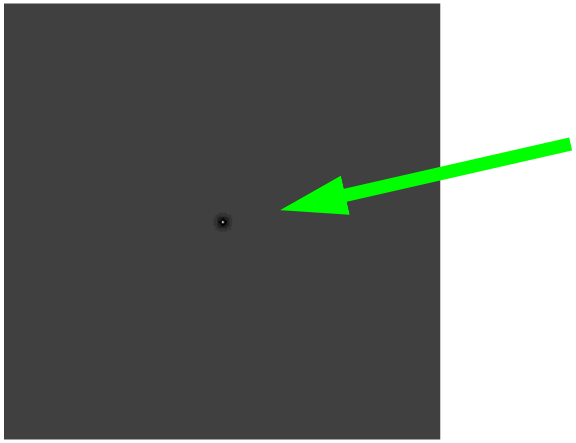
```
% Follow the steps on p. 5.78
ZPLena = zeros(512,512);
ZPLena(1:256,1:256) = Lena;
% Fill in the centered DFT array for the filter from design eqn:
[COLS,ROWS] = meshgrid(0:255,0:255);
sigma = 12;
H = 1 - exp(-((ROWS-128).^ 2 + (COLS-128).^ 2)/(2*sigma^2));
```



```
% call fftshift to un-center the filter DFT array:
Hshift = fftshift(H);
```

- Use frequency sampling…

```
% ifft to get zero-phase impulse resp:
hzero = ifft2(Hshift);
disp = stretch(log(1 + stretch(hzero)));

% shift the zero-phase impulse response
% to proceed as in Example 4:
h = fftshift(hzero);
```

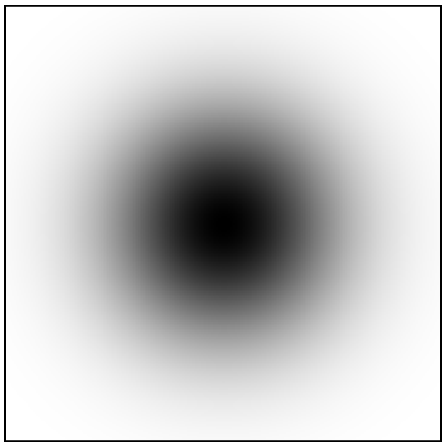


The interesting part is here; may be small and hard to see on your monitor!

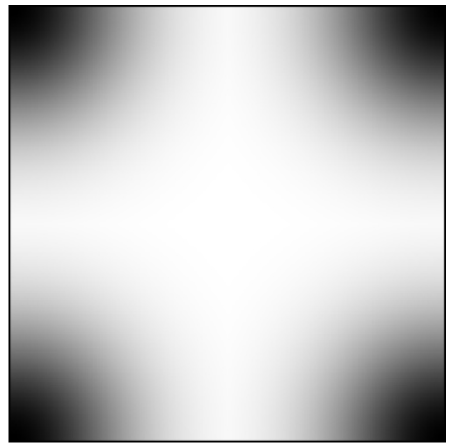
```
% compute filtered result as in Example 4:
ZPh = zeros(512,512);
ZPh(1:256,1:256) = h;

y = real(ifft2(fft2(ZPLena) .* fft2(ZPh)));
yy = stretch(y(129:384,129:384));
```

- Run again with $\sigma = 48$:

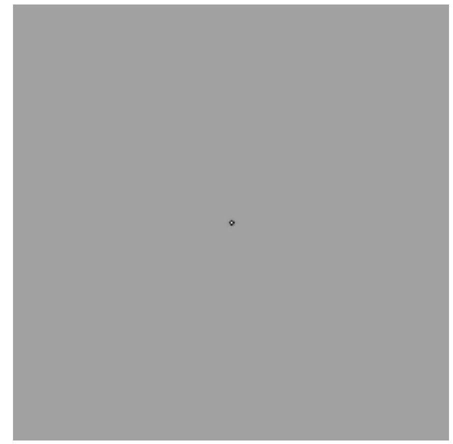
centered freq resp



un-centered freq resp



impulse resp

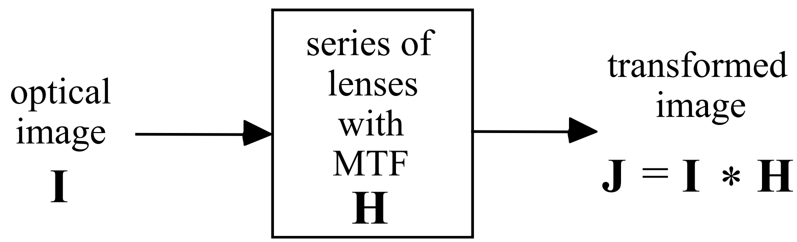


output

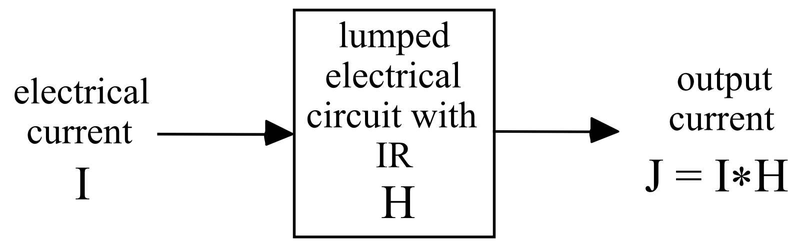
original

LINEAR IMAGE FILTERING

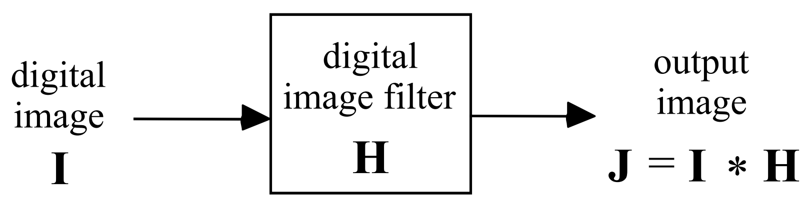
- A process that transforms a signal or image $\mathbf{I}$ by linear convolution is a type of **linear system.**



MTF = modulation transfer function



IR = impulse response



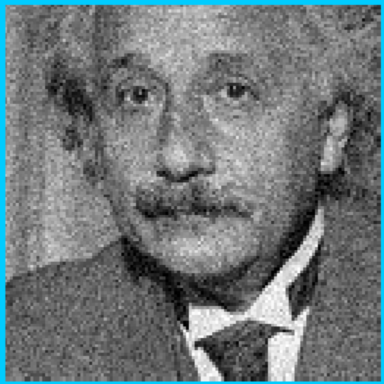
Of interest to us

Goals of Linear Image Filtering

- Process sampled, quantized images to **transform** them into
  - images of **better quality** (by some criteria)
  - images with certain features **enhanced**
  - images with certain features **de-emphasized** or **eradicated**

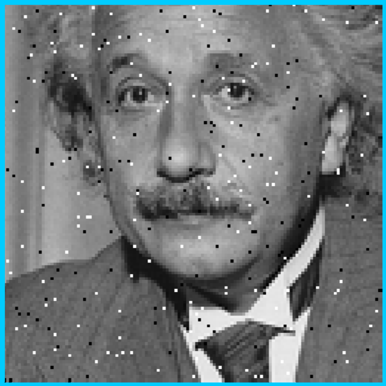
Some Specific Goals

- **smoothing** - remove noise from bit errors, transmission, etc
- **deblurring** - increase **sharpness** of blurred images
- **sharpening** - emphasize significant features, such as **edges**
- **combinations** of these

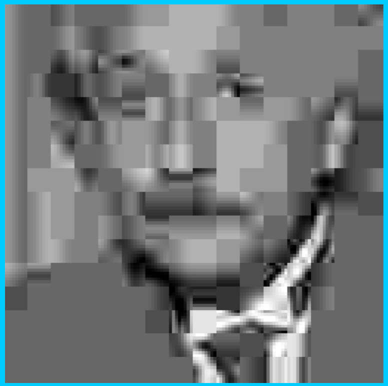


Gaussian white noise

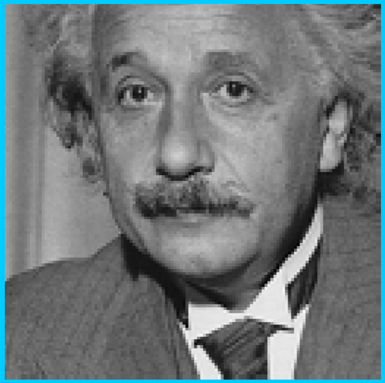
blur



impulse noise



JPEG compression



Albert

Variety of Image Distortions

A Tough One!

<u>Characterizing Linear Filters</u>

- Any **linear** digital image filter can be characterized in one of two **equivalent** ways:

  (1) The filter **impulse response** $\mathbf{H} = \left[\mathrm{H}(m,n)\right]$

  (2) The filter **frequency response** $\tilde{\mathbf{H}} = \left[\tilde{\mathrm{H}}(u,v)\right]$

- These are a DFT pair:

$$\tilde{\mathbf{H}} = \mathrm{DFT}\left[\mathrm{H}\right]$$

$$\mathbf{H} = \mathrm{IDFT}\left[\tilde{\mathrm{H}}\right]$$

Frequency Response

- The **frequency response** describes how the system affects each frequency in an image that is passed through the system.
- Since

$$\tilde{\mathrm{H}}(u,v)=\left|\tilde{\mathrm{H}}(u,v)\right|\exp\left\{j\angle\tilde{\mathrm{H}}(u,v)\right\}$$

an image frequency component at $(u, v) = (\mathrm{a}, \mathrm{b})$ is amplified or attenuated by the amount $\left|\tilde{\mathrm{H}}(\mathrm{a},\mathrm{b})\right|$ and shifted by the amount $\angle\tilde{\mathrm{H}}(\mathrm{a},\mathrm{b})$

Frequency Response Example

- Let the **input** to a system $\mathbf{H}$ be a cosine image:

$$\mathrm{I}(m,n)=\cos\left[2\pi\left(\frac{b}{M}m+\frac{c}{N}n\right)\right]=\frac{1}{2}\left(W_M^{bm}W_N^{cn}+W_M^{-bm}W_N^{-cn}\right)$$

- If $\mathbf{H}$ is real, the **output** is

$$\mathrm{J}(m,n)=\mathrm{H}(m,n)*\mathrm{I}(m,n)=\frac{1}{2}\sum_{p=0}^{M-1}\sum_{q=0}^{N-1}\mathrm{H}(p,q)\left[W_M^{-b(m-p)}W_N^{-c(n-q)}+W_M^{b(m-p)}W_N^{c(n-q)}\right]$$

$$=\frac{1}{2}W_M^{-bm}W_N^{-cn}\sum_{p=0}^{M-1}\sum_{q=0}^{N-1}\mathrm{H}(p,q)W_M^{bp}W_N^{cq}+\frac{1}{2}W_M^{bm}W_N^{cn}\sum_{p=0}^{M-1}\sum_{q=0}^{N-1}\mathrm{H}(p,q)W_M^{-bp}W_N^{-cq}$$

$$=\frac{1}{2}W_M^{-bm}W_N^{-cn}\tilde{\mathrm{H}}(b,c)+W_M^{bm}W_N^{cn}\tilde{\mathrm{H}}(-b,-c)\;=\;\underline{\underline{\left|\tilde{\mathrm{H}}(b,c)\right|\cos\left[2\pi\left(\frac{b}{M}m+\frac{c}{N}n\right)+\angle\tilde{\mathrm{H}}(b,c)\right]}}$$

Impulse Response

- The response of system $\mathbf{H}$ to the **unit impulse**

$$\delta(m,n)=\begin{cases} 1; & m=n=0 \\ 0; & \text{else} \end{cases}$$

- An effective way to model responses since every input image is a **weighted sum of unit pulses**

$$\mathrm{I}(m,n)=\sum_{p=0}^{M-1}\sum_{q=0}^{N-1}\mathrm{I}\left(m-p,n-q\right)\delta(p,q)=\sum_{p=0}^{M-1}\sum_{q=0}^{N-1}\mathrm{I}\left(p,q\right)\delta(m-p,n-q)$$

Linear Filter Design

- Often a filter is to be designed according to **frequency-domain** specifications.

- **Models of linear distortion** in the **continuous domain** lead to **linear digital solutions**.

Sampled Analog Specification

- Given an **analog** or **continuous-space** spec:

$$\mathrm{H_C}(\mathrm{x}, \mathrm{y}) \overset{\mathcal{F}}{\Leftrightarrow} \tilde{\mathrm{H}}_\mathrm{C}(\Omega, \Lambda)$$

- Impulse Invariance $(\mathrm{X}=\mathrm{Y}=1)$:

$$\mathrm{H}(m,n) = \mathrm{H_C}(m,n)\ ; \quad \begin{array}{c} -\frac{\mathrm{M}}{2} \le m \le \frac{\mathrm{M}}{2} - 1 \\ -\frac{\mathrm{N}}{2} \le n \le \frac{\mathrm{N}}{2} - 1 \end{array} \qquad (1)$$

- DFT frequency sampling:

$$\tilde{\mathrm{H}}(u,v) = \tilde{\mathrm{H}}_C\left(\frac{u}{M}, \frac{v}{N}\right) \qquad (2)$$

Low-Pass, Band-Pass, and High-Pass Filters

- The terms low-pass, band-pass, and high-pass are **qualitative descriptions** of a system's frequency response.

- "**Low-pass**" - attenuates all but the "lower" frequencies.
- "**Band-pass**" - attenuates all but an intermediate range of "middle" frequencies.
- "**High-pass**" - attenuates all but the "higher" frequencies.

- We have seen examples of these: the **zero-one** frequency masking results.

Generic Uses of Filter Types

- **Low-pass filters** are typically used to
  - smooth noise
  - blur image details to emphasize gross features

- **High-pass filters** are typically used to
  - enhance image details and contrast
  \- **remove** image blur

- **Bandpass filters** are usually **special-purpose**

Example Low-Pass Filter

- The **Gaussian filter** with frequency response

$$\tilde{H}_C(\Omega, \Lambda) = \exp\left[-2(\pi\sigma)^2(\Omega^2 + \Lambda^2)\right]$$

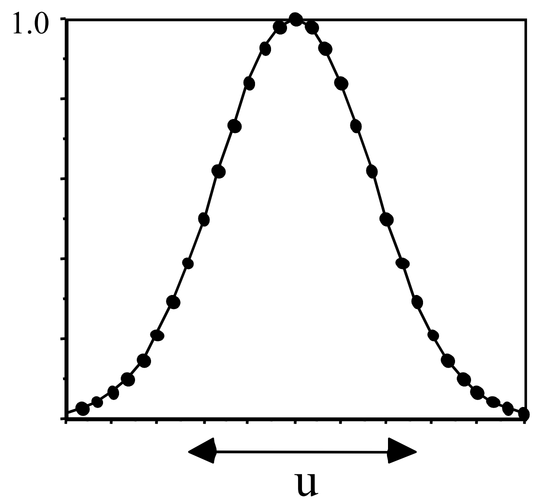
  hence

$$\tilde{H}(u, v) = \exp\left\{-2\pi^2\sigma^2\left[\left(\frac{u}{N}\right)^2 + \left(\frac{v}{M}\right)^2\right]\right\}$$

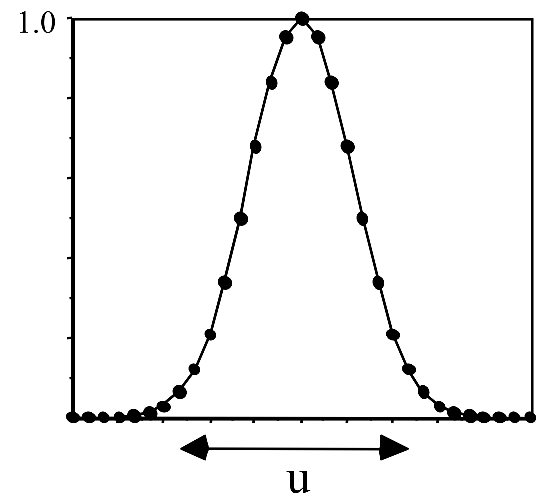
  which quickly falls at larger frequencies.

- The Gaussian is an **important** low-pass filter.

Gaussian Filter Profile



$N = 32, \sigma = 1$



$N = 32, \sigma = 1.5$

Plots show one row of DFT array $(v = 0)$

Example Band-Pass Filter

- Can define a BP filter as the **difference of two LPFs** identical except for a scaling factor.
- A common choice in image processing is the **difference-of-gaussians** (DOG) filter:

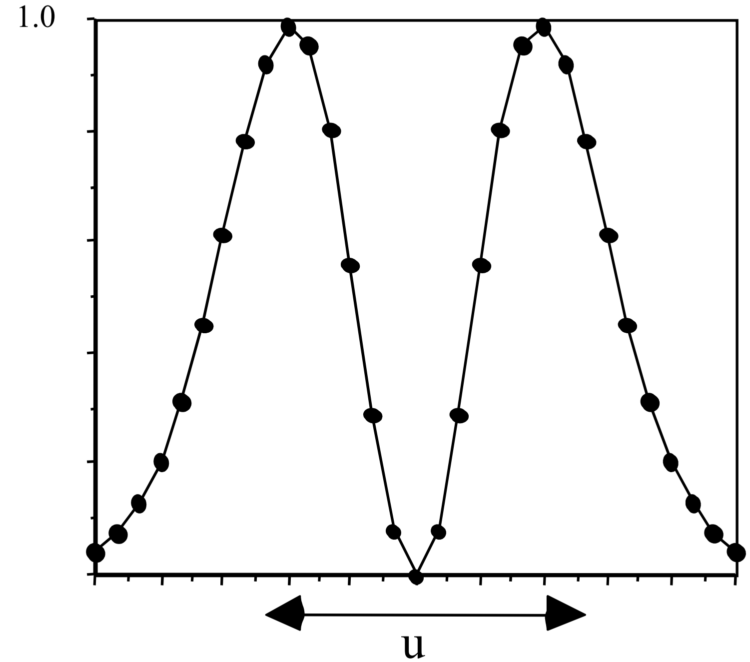
$$\tilde{H}_C(\Omega, \Lambda) = \exp\left[-2(\pi\sigma)^2\left(\Omega^2 + \Lambda^2\right)\right] - \exp\left[-2(K\pi\sigma)^2\left(\Omega^2 + \Lambda^2\right)\right]$$

hence

$$\tilde{H}(u, v) = \exp\left\{-2(\pi\sigma)^2\left[\left(\frac{u}{N}\right)^2 + \left(\frac{v}{M}\right)^2\right]\right\} - \exp\left\{-2(K\pi\sigma)^2\left[\left(\frac{u}{N}\right)^2 + \left(\frac{v}{M}\right)^2\right]\right\}$$

- **Typically, $\mathbf{K \approx 1.5}$**.

DOG Filter Profile



N=32

- DOG filters are very useful for image analysis – and in human visual modelling.

Example High-Pass Filter

- The **Laplacian** filter is also important

$$\tilde{H}_C(\Omega, \Lambda) = A\left(\Omega^2 + \Lambda^2\right)$$

  hence

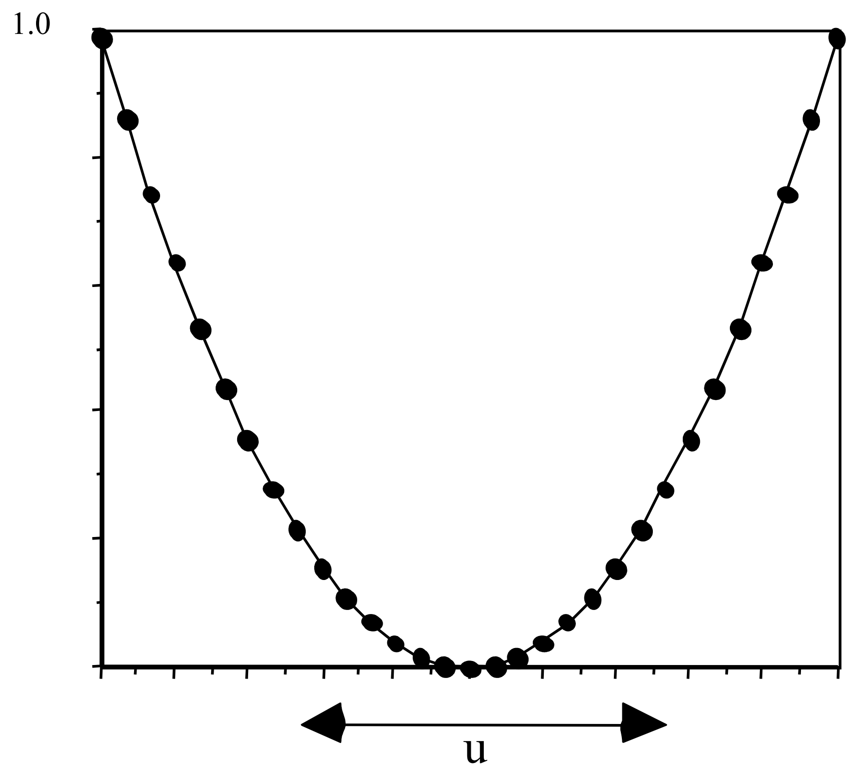
$$\tilde{H}(u, v) = A\left[\left(\frac{u}{N}\right)^2 + \left(\frac{v}{M}\right)^2\right]$$

  although this is a **severely truncated** approximation! Best used **in combination** with another filter (later).

- An approximation to the Fourier transform of the **continuous Laplacian:**

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Laplacian Profile



A = 4.5, N = 32

LINEAR IMAGE DENOISING

- **Linear image denoising** means a process that smooths noise **without** destroying the image information.
- The noise is usually modeled as **additive** or **multiplicative**.
- We consider **additive** noise now.
- **Multiplicative** noise is better handled by **homomorphic filtering** which uses a **nonlinear** preprocessing step.

Additive White Noise Model

- Model **additive white noise** as an image $\mathbf{N}$ with highly chaotic, unpredictable elements.
- Can be **thermal circuit noise**, **channel noise**, **sensor noise**, etc.
- Noise may effect the **continuous image** before sampling:

$$\underbrace{J_C(x,y)}_{\text{observed}} = \underbrace{I_C(x,y)}_{\text{original}} + \underbrace{N_C(x,y)}_{\text{white noise}}$$

Zero-Mean White Noise

- The white noise is **zero-mean** if the limit of the average of P arbitrary noise image realizations vanishes as $\mathrm{P} \rightarrow \infty$:

$$\frac{1}{\mathrm{P}} \sum_{\mathrm{p}=1}^{\mathrm{P}} \mathrm{N}_{\mathrm{C,p}}(\mathrm{x}, \mathrm{y}) \rightarrow 0 \text{ for all } (\mathrm{x},\mathrm{y}) \text{ as } \mathrm{P} \rightarrow \infty$$

- **On average**, the noise falls around the value zero.*

*Strictly speaking, the noise is also "mean-ergodic."

<u>Spectrum of White Noise</u>

- The **noise energy spectrum** is

$$\tilde{N}_C(\Omega, \Lambda) = \mathcal{F}\left\{N_C(x, y)\right\}$$

- If the **noise is white**, then, on average, the **energy spectrum will be flat** (flat spectrum = 'white'):

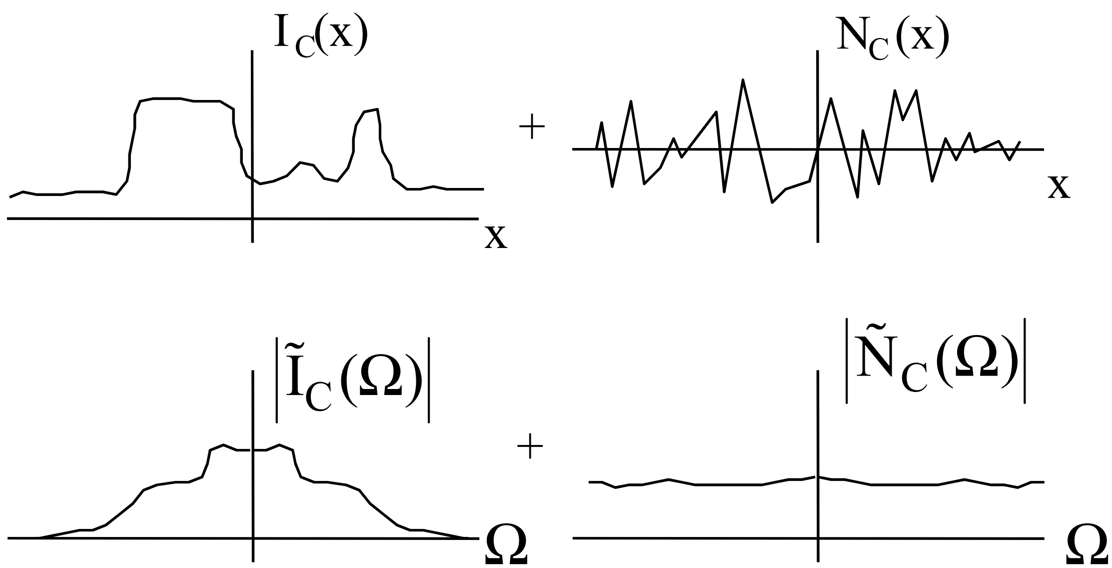
$$\frac{1}{P}\sum_{p=1}^{P}\left|\tilde{N}_{C,p}(\Omega, \Lambda)\right| \to \eta \text{ for all } (\Omega, \Lambda) \text{ as } P \to \infty$$

- **<u>Note</u>**: η^2 is called the **noise power**.

White Noise Model

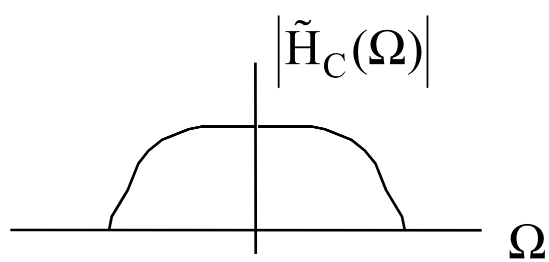
- White noise is an **approximate** model of additive **broadband** noise:

$$\underbrace{J_C(x,y)}_{\text{observed}} = \underbrace{I_C(x,y)}_{\text{original}} + \underbrace{N_C(x,y)}_{\text{broadband noise}}$$



<u>Linear Denoising</u>

- **<u>Objective</u>**: **Remove** as much of the high-frequency noise as possible while **preserving** as much of the image spectrum as possible.
- Generally accomplished by a LPF of fairly wide bandwidth (images are fairly wideband):



Digital White Noise

- We make a similar model for **digital zero-mean additive white noise**:

$$\underbrace{\mathbf{J}}_{\text{observed}} = \underbrace{\mathbf{I}}_{\text{original}} + \underbrace{\mathbf{N}}_{\text{noise}}$$

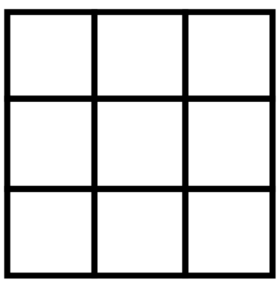
- **On average**, the elements of **N** will be zero.
- The DFT of the noisy image is the sum of the DFTs of the original image and the noise image:

$$\underbrace{\tilde{\mathbf{J}}}_{\text{observed}} = \underbrace{\tilde{\mathbf{I}}}_{\text{original}} + \underbrace{\tilde{\mathbf{N}}}_{\text{noise}}$$

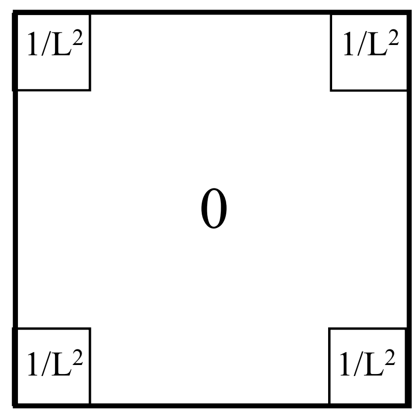
- **On average** the noise DFT will contain a **broad band** of frequencies.

Denoising - Average Filter

- To smooth an image: replace each pixel in a noisy image by the **average** of its $L \times L$ neighbors:



3x3 window



zero phase impulse response

Average Filter Rationale

- Averaging elements **reduces the noise mean towards zero**.
- Also reduces the noise variance and the noise power η^2 (actually, they are equal).
- The **window size** is usually an intermediate value to balance the tradeoff between noise smoothing and image smoothing.
- **Typical** average filter window sizes: $L \times L = 3 \times 3$, 5×5 ,..., 15×15 (lots of smoothing), e.g. for a 512×512 image.

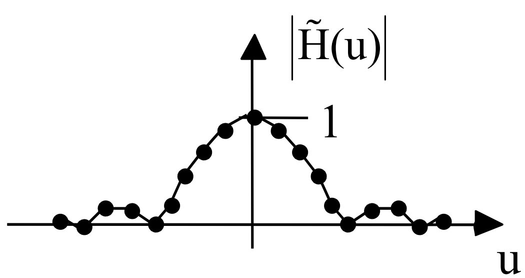
Average Filter Rationale

- **Linear filtering the image** (with zero-padding assumed hereafter)

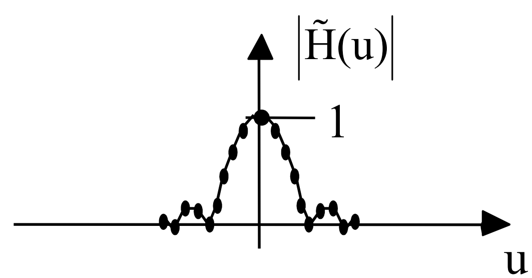
$$\mathbf{K} = \mathbf{H} * \mathbf{J} = \mathbf{H} * \mathbf{I} + \mathbf{H} * \mathbf{N}$$

$$\tilde{\mathbf{K}} = \tilde{\mathbf{H}} \otimes \tilde{\mathbf{J}} = \tilde{\mathbf{H}} \otimes \tilde{\mathbf{I}} + \tilde{\mathbf{H}} \otimes \tilde{\mathbf{N}}$$

will affect image / noise spectra in the same way:



smaller L



larger L

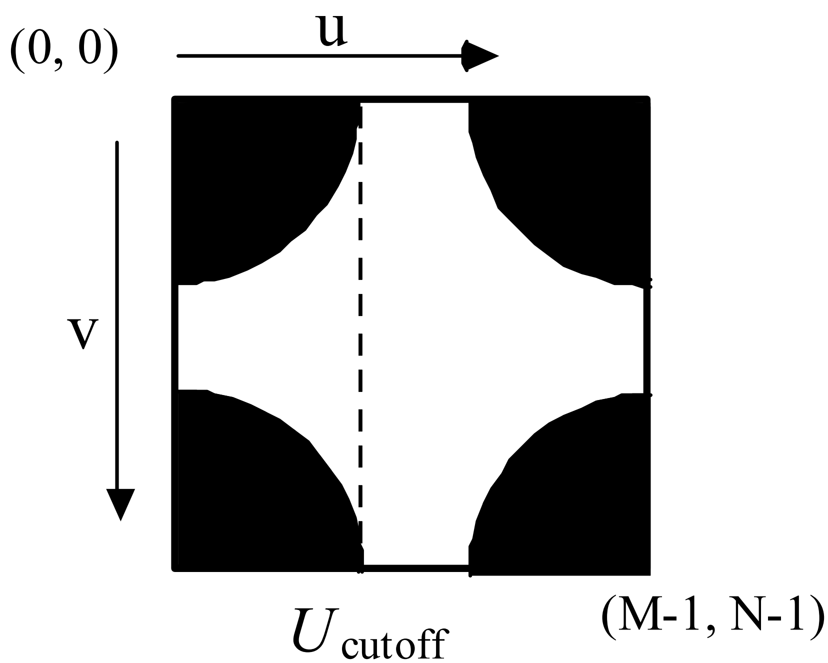
Denoising – Ideal Low-Pass Filter

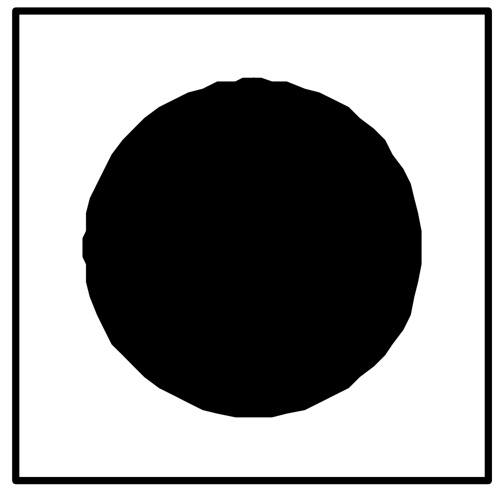
- Also possible to use an **ideal low-pass filter** by designing in the DFT domain:

$$\tilde{\mathrm{H}}(u,v) = \begin{cases} 1\,;\ \text{if } \sqrt{u^2+v^2} \le U_{\text{cutoff}} \\ 0\,;\ \text{otherwise} \end{cases}$$

- Possibly useful if its possible to estimate the highest important **radial** frequency U_{cutoff} in the original image.

Ideal LPF





centered DFT

Denoising - Gaussian Filter

- The **isotropic** Gaussian filter is an **effective** smoother:

$$\tilde{\mathrm{H}}(u,v) \;=\; \exp\left[-2\pi^2\sigma^2\left(\frac{u^2+v^2}{N^2}\right)\right]$$

- It **gives more weight to "closer" neighbors**.
- DFT design: Set the **half-peak bandwidth** to $\mathrm{U}_{\mathrm{cutoff}}$ by solving for σ:

$$\exp\left[-2\pi^2\sigma^2\left(\frac{U_{\mathrm{cutoff}}^2}{N^2}\right)\right] \;=\; \frac{1}{2}$$

$$\Rightarrow \quad \sigma = \left(\frac{N}{\pi U_{\mathrm{cutoff}}}\right)\sqrt{\log\sqrt{2}} \approx 0.19\left(\frac{N}{U_{\mathrm{cutoff}}}\right)$$

Measuring the Success of the Filter

- The observed image: $\underbrace{\mathbf{J}}_{\text{observed}} = \underbrace{\mathbf{I}}_{\text{original}} + \underbrace{\mathbf{N}}_{\text{noise}}$

- The filtered (denoised) image: $\mathbf{K} = \mathbf{H} * \mathbf{J}$
- There are two main **figures of merit** that have traditionally been used to measure or quantify the success of the denoising algorithm:
  - Mean squared error (MSE):

$$MSE(K) = \frac{\sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \left[K(m,n) - I(m,n) \right]^2}{MN}$$

Better Performance
= **lower** MSE

Measuring Denoising Success

– Improvement in signal-to-noise ratio (ISNR):

$$ISNR(\mathbf{K}) = 10\log_{10}\frac{MSE(\mathbf{J})}{MSE(\mathbf{K})} \quad \text{dB}$$

Better Performance = **higher** ISNR

- Both MSE and ISNR are **widely** used to quantify denoising performance.
- Most of our existing theory of optimal signal processing is based on them.
- But they do have limitations.

Limitations of MSE and ISNR

- Both are based on MSE, which does not always agree well with **human visual perception**.
  - In other words, the result that has better MSE (or better ISNR) may sometimes look **worse**.
- Example:
  - Let $\mathbf{J}$ be a *Lena* image with terrible noise.
  - Suppose that some fantastic denoising algorithm produces an output image $\mathbf{K}$ that is **exactly** the original *Lena* image, but shifted right by one column.
  - Even though the result looks **fantastic**, the MSE and ISNR will both be **terrible**.
- This has motivated recent work to develop new **perceptually based** figures of merit for images like the **structural similarity index** (SSIM).

LINEAR IMAGE DEBLURRING

- Often an image that is obtained digitally has already been corrupted by a linear process.
  - in other words, the optical image was corrupted **prior** to being captured by the camera.
- This may be due to **motion blur**, blurring due to **defocusing**, etc.
- We can model such an observed image as the result of a **linear convolution**:

$$\underbrace{J_C(x, y)}_{\text{observed}} = \underbrace{G_C(x, y)}_{\text{linear distortion}} * \underbrace{I_C(x, y)}_{\text{original}}$$

so

$$\underbrace{\tilde{J}_C(\Omega, \Lambda)}_{\text{observed}} = \underbrace{\tilde{G}_C(\Omega, \Lambda)}_{\text{linear distortion}} \cdot \underbrace{\tilde{I}_C(\Omega, \Lambda)}_{\text{original}}$$

Digital Blur Function

- The **sampled image** will then be of the form (assuming sufficient sampling rate)

$$\mathbf{J} = \mathbf{G} * \mathbf{I}$$

hence

$$\tilde{\mathbf{J}} = \tilde{\mathbf{G}} \otimes \tilde{\mathbf{I}}$$

- The distortion $\mathbf{G}$ is **almost always low-pass** (blurring).
- Our goal is to use digital filtering to reduce blur – a VERY hard problem!

Deblur - Inverse Filter

- Often it is possible to make an **estimate** of the distortion $\mathbf{G}$.
- This may be possible by examining the **physics** of the situation.
- For example, **motion blur** (relative camera movement) is usually along **one direction**. If this can be determined, then a filter can be designed.
- The MTF of a camera can often be determined – and hence, a **digital deblur** filter designed.

Deconvolution

- Reversing the linear blur $\mathbf{G}$ is **deconvolution**. It is done using the **inverse filter** of the distortion:

$$\tilde{\mathrm{G}}_{\mathrm{inverse}}(u,v) = \frac{1}{\tilde{\mathrm{G}}(u,v)}$$

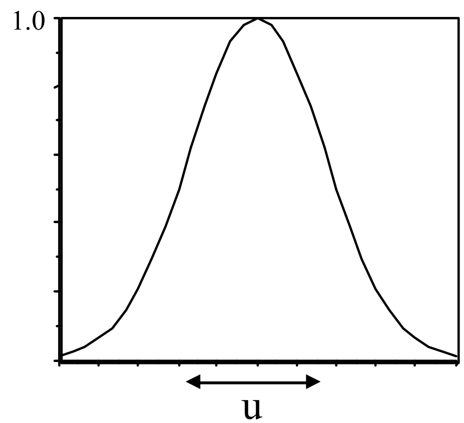
provided that $\tilde{\mathrm{G}}(u,v) \neq 0$ for any (u,v).

- Then the restored image is:

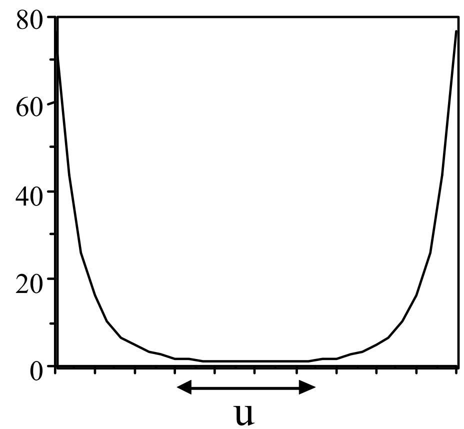
$$\tilde{\mathbf{K}} = \tilde{\mathbf{G}}_{\mathrm{inverse}} \otimes \tilde{\mathbf{G}} \otimes \tilde{\mathbf{I}} = \tilde{\mathbf{I}} \quad !$$

Blur Estimation

- An **estimate** of blur $\mathbf{G}$ might be obtainable.
- The inverse of a **low-pass** blur is **high-pass**:



Gaussian distortion



Inverse filter

- At high frequencies the designer must be **careful**!
- **Note**: The inverse takes value 1.0 at $(u, v) = (0,0)$

Deblur - Missing Frequencies

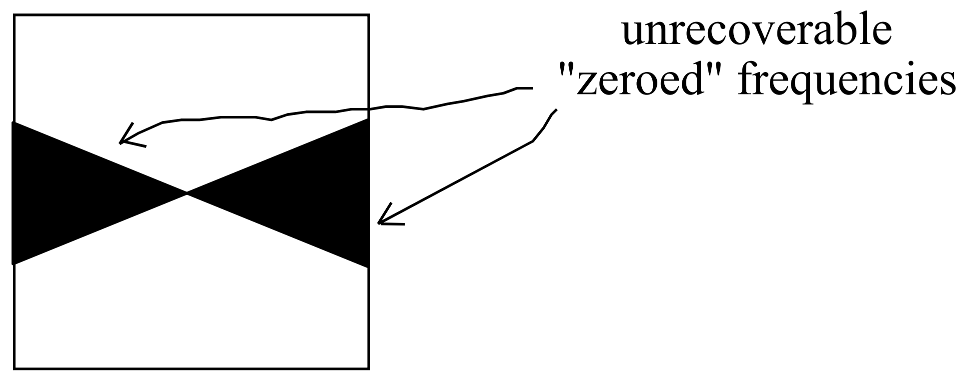
- Unfortunately, things are not always so "ideal" in the real world.
- Sometimes the blur frequency response takes **zero value(s)**.
- If

$$\tilde{G}(u_0, v_0) = 0 \text{ for some } (u_0, v_0), \text{ then } \tilde{G}_{\text{inverse}}(u_0, v_0) = \infty$$

which is **meaningless**.

Zeroed Frequencies

- The reality: any frequencies that are **zeroed** by a linear distortion are **unrecoverable in practice** (at least by linear means) - lost forever!
- The best that can be done is to reverse the distortion at the **non-zero values**.
- Sometimes much of the frequency plane is lost. Some optical systems remove a **large** angular spread of frequencies:



Pseudo-Inverse Filter

- The **pseudo-inverse filter** is defined by

$$\tilde{G}_{\text{p-inverse}}(u,v) = \begin{cases} 1/\tilde{G}(u,v) & ;\ \text{if } \tilde{G}(u,v) \neq 0 \\ 0 & ;\ \text{if } \tilde{G}(u,v) = 0 \end{cases}$$

- Thus **no attempt** is made to recover lost frequencies.
- The pseudo-inverse is set to **zero** in the known region of missing frequencies – a conservative approach.
- In this way spurious (noise) frequencies will be **eradicated**.

Deblur in the Presence of Noise

- A **worse case** is when the image $\mathbf{I}$ is distorted both by linear blur $\mathbf{G}$ and additive noise $\mathbf{N}$:

$$\mathbf{J} = \mathbf{G} * \mathbf{I} + \mathbf{N}$$

- This may occur, e.g., if an image is linearly distorted then sent over a noisy channel.
- The DFT:

$$\tilde{\mathbf{J}} = \tilde{\mathbf{G}} \otimes \tilde{\mathbf{I}} + \tilde{\mathbf{N}}$$

Filtering a Blurred, Noisy Image

- Filtering with a linear filter $\mathbf{H}$ will produce the result

$$\mathbf{K} = \mathbf{H} * \mathbf{J} = \mathbf{H} * \mathbf{G} * \mathbf{I} + \mathbf{H} * \mathbf{N}$$

or

$$\tilde{\mathbf{K}} = \tilde{\mathbf{H}} \otimes \tilde{\mathbf{J}} = \tilde{\mathbf{H}} \otimes \tilde{\mathbf{G}} \otimes \tilde{\mathbf{I}} + \tilde{\mathbf{H}} \otimes \tilde{\mathbf{N}}$$

- The problem is that **neither** a low-pass filter (to smooth noise, but won't correct the blur) nor a high-pass filter (the inverse filter, which will **amplify** the noise) will work.

Failure of Inverse Filter

- If the inverse filter were used, then

$$\mathbf{K} = \mathbf{G}_{\text{inverse}} * \mathbf{J} = \mathbf{I} + \mathbf{G}_{\text{inverse}} * \mathbf{N}$$

or

$$\tilde{\mathbf{K}} = \tilde{\mathbf{G}}_{\text{inverse}} \otimes \tilde{\mathbf{J}} = \tilde{\mathbf{I}} + \tilde{\mathbf{G}}_{\text{inverse}} \otimes \tilde{\mathbf{N}}$$

- In this case the blur is corrected, **but** the restored image has **horribly amplified high-frequency noise** added to it.

Wiener Filter

- The **Wiener filter** (after Norbert Wiener) or minimum-mean-square-error (MMSE) filter is a "best" linear approach.
- The Wiener filter for **blur $\mathbf{G}$** and **white noise $\mathbf{N}$** is

$$\tilde{\mathrm{G}}_{\mathrm{Wiener}}(u,v) = \frac{\tilde{\mathrm{G}}^{*}(u,v)}{\left|\tilde{\mathrm{G}}(u,v)\right|^{2} + \eta^{2}}$$

- Often the noise power η is **unknown** or **unobtainable**. The designer will usually experiment with heuristic values for η.
- In fact, better **visual results** may often be obtained by using such heuristic values for η in the Wiener filter.

Wiener Filter Rationale

- We won't derive the Wiener filter here. But:
- If $\eta = 0$ (**no noise**), the Wiener filter reduces to the **inverse filter**:

$$\tilde{G}_{\text{Wiener}}(u,v) = \frac{\tilde{G}^*(u,v)}{\left|\tilde{G}(u,v)\right|^2} = \frac{1}{\tilde{G}(u,v)}$$

which is **highly desirable**.

Wiener Filter Rationale

- If $\tilde{G}(u,v) = 1 \text{ for all } (u,v)$ (**no blur**) the Wiener filter reduces to:

$$\tilde{G}_{\text{Wiener}}(u,v) = \frac{1}{1+\eta^2}$$

  which does **nothing except scale the variance** so that the MSE is minimized.

- So, the Wiener filter is not useful unless there is **blur**.

Pseudo-Wiener Filter

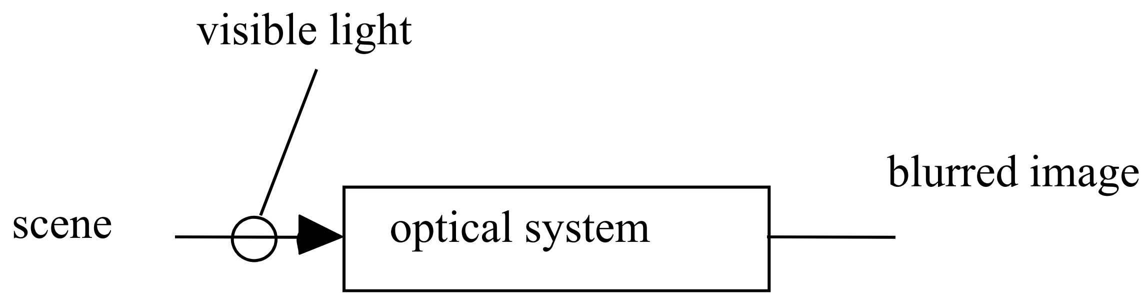
- Obviously, if there are **frequencies** zeroed by the linear distortion $\mathbf{G}$ then it is best to define a **pseudo-Wiener filter**:

$$\tilde{\mathrm{G}}_{\mathrm{Wiener}}(u,v) = \begin{cases} \dfrac{\tilde{\mathrm{G}}^{*}(u,v)}{\left|\tilde{\mathrm{G}}(u,v)\right|^{2} + \eta^{2}} & ; \text{if } \tilde{\mathrm{G}}(u,v) \neq 0 \\ 0 & ; \text{if } \tilde{\mathrm{G}}(u,v) = 0 \end{cases}$$

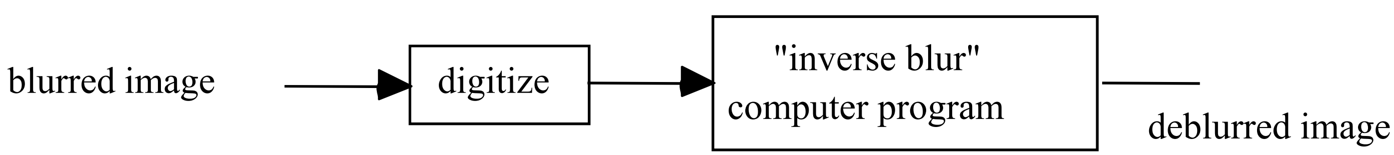
- Noise in the "missing region" of frequencies will be **eradicated**.

APPLICATION EXAMPLE: OPTICAL SERIAL SECTIONING

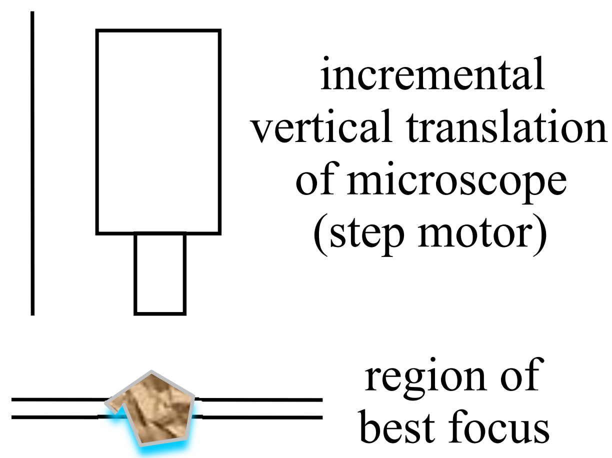
- **Optical** systems often **blur** images:



one possible solution:

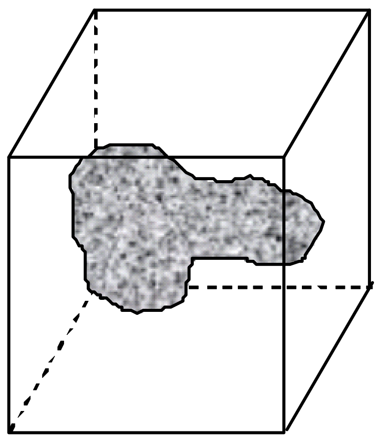


Optical Sectioning Microscopy

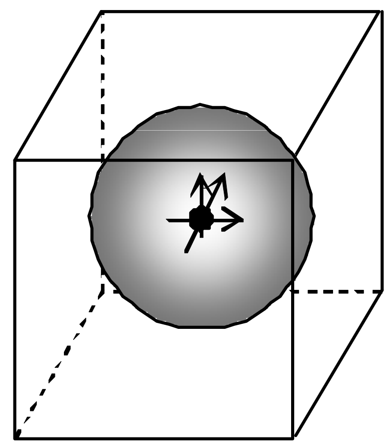


- A very **narrow-depth of field** microscope. One image taken at **each** focusing plane, giving a **sequence** of 2-D images - or **3-D image of optical density**.

3-D Image of Optical Density



3-D image of optical density

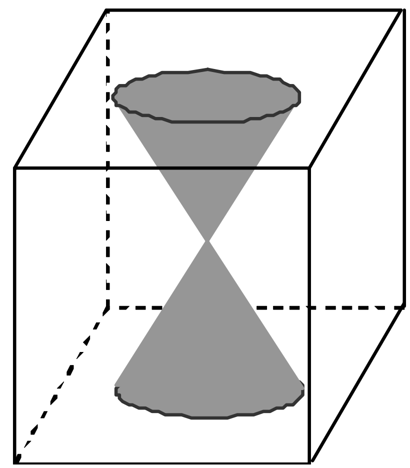


Magnitude of 3-D DFT

3-D Optical System Analysis

- In this system **three** effects occur:
  (1) A **linear low-pass distortion** $\mathbf{G}$.
  (2) A large **biconic region of frequencies** aligned along the **optical axis** is **zeroed**.
  (3) Approximately **additive white noise**.
- Items (1) and (2) shown using **principles of geometric optics**. Item (3) shown **empirically**.

<u>3-D Biconic Spread of Lost Frequencies</u>



Region of zeroed 3-D frequencies

- Note that DC $(u, v, w) = (0, 0, 0)$ is zeroed also. Hence the **background level** (AOD) **is lost**.
- There is **no linear filtering way** to recover this biconic region of frequencies.

3-D Restoration

- So: the 3-D images are **blurred**, have a large 3-D region of **missing frequencies**, and are corrupted by low-level **white noise** added.
- The **processed results** show the efficacy of
    - pseudo-inverse filtering
    - pseudo-Wiener filtering

  applied to two optical sectioned 3-D images:
    - a pollen grain
    - a pancreas Islet of Langerhans (collection of cells)

Shroud of Turin Image

An intensely enhanced, denoised, deblurred, etc etc etc and **debated** image

Comments

- Next we shall dispense with **frequency-domain concepts** and **linear filtering tools**.
- We shall now study **non-linear filtering methods**… onward to **Module 6**.