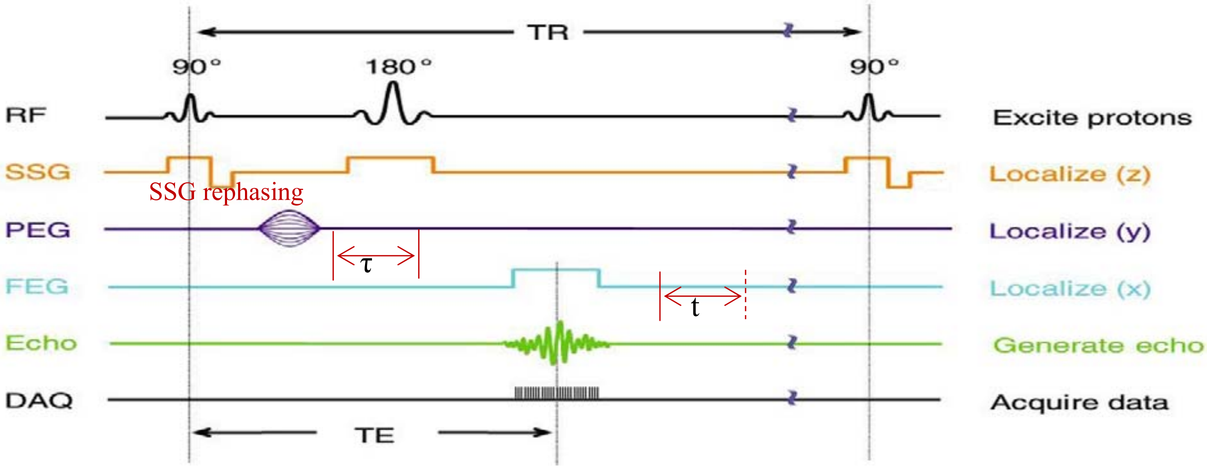
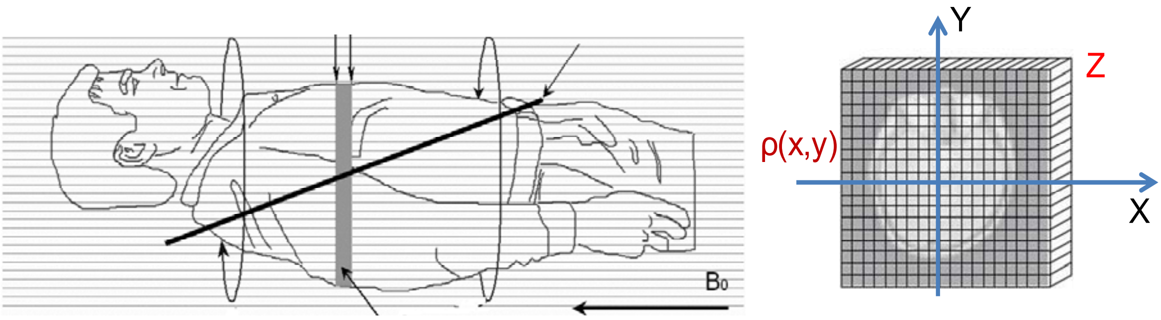
Homework #12:

Magnetic field gradients could be used to localize the NMR signals in x-, y-, and z-directions in MRI. The following figure demonstrates an example of spin-echo pulse sequence and timing of MR imaging.



After a SSG and RF pulse have been applied, an excited slice in z-direction is identified. The desired proton density function of a small voxel at location (x,y) on the excited slice, which we hope to display as a grayscale image, can be defined as ρ(x,y), shown as follows.



(a) The phase encode gradient (PEG), with a gradient of G_y is turned on for a brief period of time τ along y-direction, please determine the corresponding phase shift.
(b) If the signal is sampled at time t after turning the frequency encode gradient (FEG) on, what is the phase shift induced on the spins at location (x,y)? Note that the corresponding gradient is G_x,
(c) Then what is total phase shift Φ(x,y) for a voxel at location (x,y)?
(d) Assume that the complex signal emitted by a small voxel at location (*x*,*y*) can be expressed as:

$$\rho(x, y)e^{i\Phi(x,y)}dxdy$$

And, we can define

$$k_x = -\gamma G_x t$$
$$k_y = -\gamma G_y \tau$$

Please determine the signal detected by the RF coil $S(k_x,k_y)$ in K-space, considering that the RF coil equally sums contributions of protons (spins) from all voxels.

(e) Please solve for 2D MR image ρ(x,y) based on the signal detected by the RF coil (What mathematical method should be used to solve for ρ(x,y)?).

Solution:

(a) At first, phase shift in y direction ΔΦ(y) after phase encoding finishing:

$$\Delta\Phi(y) = \Phi(y) - \Phi_0 = y\gamma G_y \tau$$

(b) Then, phase shift in x direction $\Phi(x)_t$ at time ***t*** after turning the *x* gradient (Gx) during frequency encoding:

$$\Phi(x)_t = \omega(x)t = [\omega_0 + \gamma G_x x] \times t = \omega_0 t + \gamma G_x x t$$

(c) Thus the total phase shift $\Phi(x,y)_t$ for a voxel at time ***t*** for a voxel location (x,y) is:

$$\begin{aligned}\Phi(x,y)_t &= \Phi(x)_t + \Delta\Phi(y) \\ &= \omega_0 t + x\gamma G_x t + y\gamma G_y \tau\end{aligned}$$

(d) The complex signal emitted by a small voxel at location (*x*,*y*) can be expressed with total phase shift $\Phi(x,y)_t$ as:

$$\rho(x,y)e^{i\Phi(x,y)_t}dxdy = \rho(x,y)e^{i(\omega_0 t + x\gamma G_x t + y\gamma G_y \tau)}dxdy$$

Because of the RF coil equally sums contributions of protons (spins) from all voxels, the signal detected by the coil should be the sum of them:

$$\begin{aligned}S'(\gamma G_x t, \gamma G_y \tau) &= \iint_{Object} \rho(x,y)e^{i[\omega_0 t + x\gamma G_x t + y\gamma G_y \tau]}dxdy \\ &= e^{i\omega_0 t}\iint_{Object} \rho(x,y)e^{i[x\gamma G_x t + y\gamma G_y \tau]}dxdy\end{aligned}$$

The factor $e^{(i\omega_0 t)}$ is just a simple modulation factor representing the Larmor precession of the protons (spins). This carrier frequency can be moved outside the integral.

Therefore:

$$S(\gamma G_x t, \gamma G_y \tau) = \iint\limits_{Object} \rho(x, y) e^{i[x\gamma G_x t + y\gamma G_y \tau]} dxdy$$

we can define:

$$\begin{aligned} k_x &= -\gamma G_x t \\ k_y &= -\gamma G_y \tau \end{aligned}$$

Final:

$$S(k_x, k_y) = \iint\limits_{Object} \rho(x, y) e^{-i[k_x x + k_y y]} dxdy$$

(e) Once filled, the k-space matrix contain positionally dependent variations along the k_y and k_x directions respectively.

$$S(k_x, k_y) = \sum_{k_y(-\max)}^{k_y(\max)} S(k_x)_{k_y}$$

Then a 2D inverse Fourier transform applies to produce the spatial domain representation.

$$\rho(x, y) = FT^{-1}\left[S(k_x, k_y)\right]$$