Poisson Distribution and X-ray Quantum Noise

Recall the earlier discussions, that the quantum noise in radiological imaging obeys Poisson distribution:

$$P_K = \frac{N^K e^{-N}}{K!} \qquad (2.1)$$

where: P_K is probability, in a given time interval, of emitting K photons from an x-ray source, and N is the average number of photons emitted during that interval.

Here: K! is the factorial of non-negative integer as:

$$K! = K\times(K-1)\times(K-2)\ldots\times 1 = K\times(K-1)! = K\times(K-1)\times(K-2)! \qquad (2.2)$$

$$0! \equiv 1$$

In the following let us show that in Poisson process, the variance is given as

$$\sigma^2 = N.$$

❖ The variance σ^2 can be determined as:

$$\sigma^2 = \sum_{K=0}^{\infty}(K-N)^2 \times P_K = \sum_{K=0}^{\infty}(K^2 - 2KN + N^2)\times P_K$$

$$= \sum_{K=0}^{\infty} K^2 \times P_K - 2N \times \sum_{K=0}^{\infty} K \times P_K + N^2 \times \sum_{K=0}^{\infty} P_K$$

$$= \sum_{K=0}^{\infty} K^2 \times P_K - 2N \times N + N^2 = \sum_{K=0}^{\infty} K^2 \times P_K - N^2 \qquad (2.3)$$

Note definitions in statistics:

$$\sigma^2 = \sum_{K=0}^{\infty}(K-N)^2 \times P_K$$

$$N = \sum_{K=0}^{\infty} K \times P_K$$

$$\sum_{K=0}^{\infty} P_K = 1$$

Recall Eq (2.1) :

$$\sum_{K=0}^{\infty} K^2 \times P_K = \sum_{K=0}^{\infty} K^2 \times \frac{N^K e^{-N}}{K!} \qquad (2.4)$$

Recall Eq. (2.2), Eq. (2.4) become:

$$\sum_{K=0}^{\infty} K^2 \times P_K = \sum_{K=0}^{\infty} K^2 \times \frac{N^K e^{-N}}{K!} = \sum_{K=1}^{\infty} K^2 \times \frac{N^K e^{-N}}{K \times (K-1)!} = \sum_{K=1}^{\infty} K \times \frac{N^K e^{-N}}{(K-1)!}$$

$$= \sum_{K=1}^{\infty} (K-1+1) \times \frac{N^K e^{-N}}{(K-1)!} = \sum_{K=2}^{\infty} (K-1) \times \frac{N^K e^{-N}}{(K-1)!} + \sum_{K=1}^{\infty} \frac{N^K e^{-N}}{(K-1)!}$$

$$= \sum_{K=2}^{\infty} (K-1) \times \frac{N^K e^{-N}}{(K-1) \times (K-2)!} + \sum_{K=1}^{\infty} \frac{N^K e^{-N}}{(K-1)!}$$

$$= \sum_{K=2}^{\infty} \frac{N^K e^{-N}}{(K-2)!} + \sum_{K=1}^{\infty} \frac{N^K e^{-N}}{(K-1)!} \quad = A + B \qquad (2.5)$$

Note:

$$K^2 \times \frac{N^K e^{-N}}{K!} \bigg|_{K=0} = 0$$

$$(K-1) \times \frac{N^K e^{-N}}{(K-1)!} \bigg|_{K=1} = 0$$

From Eq. (2.5):

$$B = \sum_{K=1}^{\infty} \frac{N^K e^{-N}}{(K-1)!} = \sum_{K=1}^{\infty} \frac{N \times N^{K-1} \times e^{-N}}{(K-1)!} = N \times e^{-N} \sum_{K=1}^{\infty} \frac{N^{K-1}}{(K-1)!}$$

Let $L = K - 1$

$$B = \sum_{K=1}^{\infty} \frac{N^K e^{-N}}{(K-1)!} = N \times e^{-N} \times \sum_{L=0}^{\infty} \frac{N^L}{L!}$$

Note: $\sum_{L=0}^{\infty} \frac{N^L}{L!} = e^N$

$$B = \sum_{K=1}^{\infty} \frac{N^K e^{-N}}{(K-1)!} = N \times e^{-N} \times e^N = N \qquad (2.6)$$

Also from Eq. (2.5), we have:

$$A = \sum_{K=2}^{\infty} \frac{N^K e^{-N}}{(K-2)!} = \sum_{K=2}^{\infty} \frac{N^2 \times N^{K-2} \times e^{-N}}{(K-2)!} = N^2 \times e^{-N} \times \sum_{K=2}^{\infty} \frac{N^{K-2}}{(K-2)!}$$

Let $M = K - 2$

$$A = \sum_{K=2}^{\infty} \frac{N^K e^{-N}}{(K-2)!} = N^2 \times e^{-N} \times \sum_{M=0}^{\infty} \frac{N^M}{M!}$$

Note: $\sum_{M=0}^{\infty} \frac{N^M}{M!} = e^N$

$$A = \sum_{K=2}^{\infty} \frac{N^K e^{-N}}{(K-2)!} = N^2 \times e^{-N} \times e^N = N^2 \qquad (2.7)$$

Then Eq (2.5) becomes:

$$\sum_{K=0}^{\infty} K^2 \times P_K = \sum_{K=2}^{\infty} \frac{N^K e^{-N}}{(K-2)!} + \sum_{K=1}^{\infty} \frac{N^K e^{-N}}{(K-1)!} = A + B = N^2 + N \quad (2.8)$$

Recall Eq. (2.3), the variance σ^2

$$\sigma^2 = \sum_{K=0}^{\infty} (K-N)^2 P_K = \sum_{K=0}^{\infty} K^2 \times P_K - N^2 = N^2 + N - N^2 = N \quad (2.9)$$

Finally, we have shown that in Poisson process, the variance is given as $\sigma^2 = N$. Therefore, the x-ray quantum noise is described as:

Variance: $\sigma^2 = N$

Standard deviation $\sigma = \sqrt{\sigma^2} = \sqrt{N}$

References:

A. Macovski, *Medical Imaging Systems*, Prentice Hall, New Jersey, 1983

H. H. Barrett and W. Swindell, *Radiological Imaging*, Vol.1, Academic Press, 1981

Homework:

S-1. The quantum noise in x-ray imaging obeys Poisson distribution:

$$P_K = \frac{N^K e^{-N}}{K!}$$

where: P_K is probability, in a given time interval, of emitting K photons from an x-ray source, N is the average number of photons emitted during that interval, and K! is the factorial of non-negative integer.

Please prove that in Poisson process, the variance is given as: $\sigma^2 = N$.

Hint: the following may be helpful:

$$\sigma^2 = \sum_{K=0}^{\infty} (K-N)^2 \times P_K$$

$$\sum_{K=0}^{\infty} K^2 \times P_K = N^2 + N$$

$$N = \sum_{K=0}^{\infty} K \times P_K$$