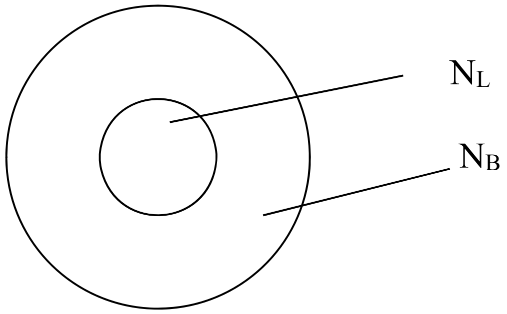
Home Work #2

Questions 1 and 2 refer to the figure which represents an x-ray image of a uniform, disk-shaped lesion with an average of N_L photons, where N_L is just slightly smaller than the average background count level, N_B photons.



1. What is the lesion's signal-to-noise ratio (SNR)?

Solution:

$$SNR = \frac{N_B - N_L}{N_B}\sqrt{N_B}$$

2. If the same object is imaged for twice as long (twice the photons), how is the new SNR related to the old SNR?

Solution:

$$SNR = C \times \sqrt{N}$$

$$\because N_{new} = 2N_{old}$$

$$SNR_{new} = C \times \sqrt{N_{new}} = C \times \sqrt{2N_{old}} = \sqrt{2} \times SNR_{old}$$

3. Assume an imaging system has a point spread function (PSF) given as follows:

$$PSF(x,y) = \frac{1}{2\pi} \times \frac{e^{-|x|}}{1+y^2}$$

What is the 2D modulation transfer function, $MTF(u,v)$ of this system?

[The flowing integral may help]

$$\int_{-\infty}^{\infty} \frac{e^{-i2\pi vy}}{1+y^2} dy = \pi e^{-|2\pi v|}$$

Solution:

$$MTF(u,v) = F_{2D}\{PSF(x,y)\}$$

$$= \int \int_{-\infty}^{\infty} PSF(x,y) e^{-i2\pi(ux+vy)} dxdy$$

$$= \int \int_{-\infty}^{\infty} \frac{1}{2\pi} \times \frac{e^{-|x|}}{1+y^2} e^{-i2\pi(ux+vy)} dxdy$$

$$= \frac{1}{2\pi} \times \int \int_{-\infty}^{\infty} e^{-|x|} \times \left(\frac{1}{1+y^2}\right) e^{-i2\pi(ux+vy)} dxdy$$

$$= \frac{1}{2\pi} \times \left[\int_{-\infty}^{\infty} e^{-|x|} e^{-i2\pi ux} dx\right]\left[\int_{-\infty}^{\infty} \frac{e^{-i2\pi vy}}{1+y^2} dy\right]$$

Because:

$$\int_{-\infty}^{\infty} e^{-|x|} e^{-i2\pi ux} dx = \int_{-\infty}^{0} e^{x} e^{-i2\pi ux} dx + \int_{0}^{\infty} e^{-x} e^{-i2\pi ux} dx$$

$$= \int_{-\infty}^{0} e^{(1-i2\pi u)x} dx + \int_{0}^{\infty} e^{(-1-i2\pi u)x} dx$$

$$= \frac{1}{1-i2\pi u} e^{(1-i2\pi u)x}\Big|_{-\infty}^{0} + \frac{1}{-1-i2\pi u} e^{(-1-i2\pi u)x}\Big|_{0}^{\infty}$$

$$= \frac{1}{1-i2\pi u}[1-0] + \frac{1}{-1-i2\pi u}[0-1]$$

$$= \frac{1}{1-i2\pi u} + \frac{1}{1+i2\pi u}$$

$$= \frac{2}{1+(2\pi u)^2}$$

$$\int_{-\infty}^{\infty} \frac{e^{-i2\pi vy}}{1+y^2} dy = \pi e^{-2\pi|v|}$$

Therefore,

$$MTF(u,v) = \frac{1}{2\pi} \times \frac{2\pi e^{-2\pi|v|}}{1+(2\pi u)^2}$$

$$= \frac{e^{-2\pi|v|}}{1+(2\pi u)^2}$$

4. Consider the 1-D ***rect*** function:

$$rect(x) = \begin{cases} 1 \ \text{ for } |x| \le \dfrac{1}{2} \\ 0 \ \text{ for } |x| > \dfrac{1}{2} \end{cases}$$

What is its Fourier transform?

Solution:

We have:

$$F_{1D}\{rect(x)\} = \int_{-\infty}^{\infty} rect(x) \times e^{-j2\pi ux} dx$$

$$= \int_{-1/2}^{1/2} e^{-j2\pi ux} dx = -\frac{1}{j2\pi u} e^{-j2\pi ux}\Big|_{-1/2}^{1/2}$$

$$= \frac{1}{\pi u} \frac{e^{j\pi u} - e^{-j\pi u}}{2j} = \frac{\sin(\pi u)}{\pi u} = \mathrm{sinc}(\pi u)$$

5. Detectors of some medical imaging systems can be modeled as ***rect*** functions of different sizes and locations. Compute the Fourier transform of the following scaled and translated ***rect*** function:

$$f(x,y) = rect(\frac{x-x_0}{\Delta x_0}) \times rect(\frac{y-y_0}{\Delta y_0})$$

$$\text{where}: \mathrm{x}_0, \mathrm{y}_0, \Delta\mathrm{x}_0 \text{ and } \Delta\mathrm{y}_0 \text{ are constant.}$$

Solution:

We can use the shift, magnification and separability property of the Fourier transform.

$$F\left[f(x-a, y-b)\right] = F(u,v)e^{-i2\pi(ua+vb)}$$

$$F\left[f(ax)\right] = \frac{1}{|a|}F(\frac{u}{a})$$

The Fourier transform of the ***rect*** function is the ***sinc*** function,

Therefore, we can get:

$$F_{2D}\left[f(x,y)\right] = F_{2D}\left[rect(\frac{x-x_0}{\Delta x_0}) \times rect(\frac{y-y_0}{\Delta y_0})\right]$$

$$= F_{1D}\left[rect\left(\frac{x}{\Delta x_0} - \frac{x_0}{\Delta x_0}\right)\right] \times F_{1D}\left[rect\left(\frac{y}{\Delta y_0} - \frac{y_0}{\Delta y_0}\right)\right]$$

$$= F_{1D}\left[rect\left(\frac{x}{\Delta x_0}\right)\right] \times e^{-i2\pi u\frac{x_0}{\Delta x_0}} F_{1D}\left[rect\left(\frac{y}{\Delta y_0}\right)\right] \times e^{-i2\pi v\frac{y_0}{\Delta y_0}}$$

Finally:

$$
\begin{aligned}
&F_{2D}\left[f(x,y)\right] \\
&= \Delta x_0 \times \frac{\sin(\pi\Delta x_0 u)}{\pi\Delta x_0 u} \times e^{-i2\pi\Delta x_0 u \frac{x_0}{\Delta x_0}} \times \Delta y_0 \times \frac{\sin(\pi\Delta y_0 v)}{\pi\Delta y_0 v} \times e^{-i2\pi\Delta y_0 v \frac{y_0}{\Delta y_0}} \\
&= \Delta x_0 \times \frac{\sin(\pi\Delta x_0 u)}{\pi\Delta x_0 u} \times \Delta y_0 \times \frac{\sin(\pi\Delta y_0 v)}{\pi\Delta y_0 v} \times e^{-i2\pi(\frac{\Delta x_0 u x_0}{\Delta x_0} + \frac{\Delta y_0 v y_0}{\Delta y_0})} \\
&= \frac{\sin(\pi\Delta x_0 u)}{\pi u} \times \frac{\sin(\pi\Delta y_0 v)}{\pi v} \times e^{-i2\pi(u x_0 + v y_0)}
\end{aligned}
$$

6. Please prove the following:

$$\delta(kx) * \delta(\frac{x}{k}) = \delta(x)$$

where $*$ is convolution operation, and k is a constant.

Solution:

We know that:

$$\delta(ax) * \delta(bx) = \delta(abx)$$

Therefore:

$$\delta(kx) * \delta(\frac{x}{k}) = \delta(k \times \frac{1}{k} \times x) = \delta(x)$$