ANSWERS

Name:

Please answer *question 1* and *any other* 5 out of the remaining 9 questions. You can answer in any order you wish. Please hand in all the exam questions and answers at the end of the class.

Circle the questions in table 1 that you have answered ! Time allocated 50 min.

Question	Marks earned	Out of
Q1	MUST DO	5
Q2		5
Q3		5
Q4		5
Q5		5
Q6		5
Q7		5
Q8		5
Q9		5
Q10		5
Total	Qu 1 + Best 5	30

Table 1: Mark allocation

1. Proof $E(S^2) = \sigma^2$
2. a) $\hat{p} = \frac{y}{n}$ b) $E(\hat{p}) = p$ c) Yes d) $\frac{pq}{n}$
3. a) $c = 4$ b) $f_1(x) = 2x$, $\mu_x = \frac{2}{3}$ c) $f_2(y) = 2y$, $\mu_y = \frac{2}{3}$ d) $E(xy) = \frac{4}{9}$ e) $cov(x,y) = 0$
4. a) $\bar{Y} = 3$ b) $(0.75, 5.25)$ c) $(1.84, 4.16)$ d) $(1.82, 4.18)$
5. a) $c = 1$ b) $1 - e^{-y}$ c) $F(2.6) = 0.9257$ d) $F(0) = 0$, $F(\infty) = 1$ e) 0.3611
6. Proof
7. a) $L \sim N$ b) $E(L) = 0$ c) $V(L) = 14$
8. a) $\left.\frac{dm}{dt}\right|_{t=0} = \nu$ b) $\nu^2 + 2\nu - \nu^2 = 2\nu$
9. a) – e) TRUE f) $x = 4$
10. a) NR proof b) IV ℓ'' c)(1) distance.

3

1. Prove that S^2 is an unbiased estimator of σ^2. That is prove that

$$E(S^2) = \sigma^2$$

where

$$S^2 = \frac{\sum_i Y_i^2 - n\overline{Y}^2}{n-1}.$$

This is the estimator for the population variance.
Hint: $\sigma_Y^2 = E(Y^2) - E(Y)^2$

$$\sigma_Y^2 = E(Y^2) - E(Y)^2$$

$$E(Y^2) = \sigma_Y^2 + \mu_Y^2$$

$$E(\overline{Y}^2) = \sigma_{\overline{Y}}^2 + \mu_{\overline{Y}}^2 = \frac{\sigma_Y^2}{n} + \mu_Y^2$$

$$(n-1)S^2 = \sum Y_i^2 - n\overline{Y}^2$$

$$(n-1)E(S^2) = \sum E(Y_i^2) - nE(\overline{Y}^2)$$

$$= \sum(\sigma_Y^2 + \mu_Y^2) - n\left(\frac{\sigma_Y^2}{n} + \mu_Y^2\right)$$

$$= n\sigma_Y^2 + \cancel{n\mu_Y^2} - \sigma_Y^2 - \cancel{n\mu_Y^2}$$

$$= \sigma_Y^2(n-1)$$

$$E(S^2) = \sigma_Y^2$$

$\therefore$ unbiased

2. Suppose that a Binomial experiment was performed so that in n trials y successes were observed.

$$P(y) = \binom{n}{y} p^y (1-p)^{n-y} \quad 0 \le y \le n$$

~~$\mu_Y = \beta,\ \sigma_Y^2 = \beta^2$~~, $\mu_Y = np \quad \sigma_Y^2 = npq$

(a) Find $\hat{p}_{mle}$ the maximum likelihood estimator of p. (Show working)

(b) Find $E(\hat{p}_{mle})$? (Show working)

(c) Is $\hat{p}_{mle}$ UNBIASED?

(d) Find $V(\hat{p}_{mle})$ (Show working)

a) $\ell(p) = \log\binom{n}{y} + y\log p + (n-y)\log(1-p)$

$$\frac{\partial \ell}{\partial p} = \frac{y}{p} - \frac{(n-y)}{1-p}$$

$$\frac{\partial \ell}{\partial p} = 0 \Rightarrow 0 = \frac{y}{\hat{p}} - \frac{(n-y)}{1-\hat{p}}$$

$$0 = y(1-\hat{p}) - \hat{p}(n-y)$$

$$0 = y - \cancel{y\hat{p}} - \hat{p}n + \cancel{\hat{p}y}$$

$$0 = y - \hat{p}n$$

$$\hat{p} = \frac{y}{n}$$

b) $E(\hat{p}) = E\left(\frac{y}{n}\right) = \frac{1}{n}E(y) = \frac{1}{n}np = p$ unbiased

c) Yes unbiased

d) $V(\hat{p}) = V\left(\frac{y}{n}\right) = \frac{1}{n^2}V(y) = \frac{npq}{n^2} = \frac{pq}{n}$

3. Let X and Y have the joint density

$$f(x,y) = \begin{cases} cxy & if\ 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & otherwise \end{cases}$$

(a) Find the value of c that makes $f(x,y)$ a probability density function.

(b) Find the marginal density $f_1(x)$ and μ_x.

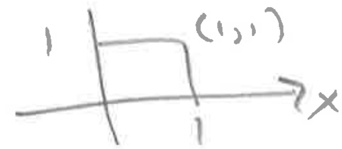
(c) Find the marginal density $f_2(y)$ and μ_y.

(d) Find $E(XY)$.

(e) Using

$$cov(X,Y) = E(XY) - \mu_x \mu_y$$

find the covariance.



(ai) $\iint\limits_{\text{AREA}} cxy\,dx\,dy = 1$

$\int_0^1 cxy\,dx = c\frac{1}{2}x^2y\Big|_0^1 = c\frac{1}{2}y$

$\int_0^1 c\frac{1}{2}y\,dy = \frac{1}{4}cy^2\Big|_0^1 = \frac{1}{4}c$

$\frac{1}{4}c = 1 \quad c = 4$

b) $f_1(x) = \int_0^1 f(x,y)\,dy = \int_{x=0}^{x=1} 4xy\,dy = \left[2xy^2\right]_0^1 = 2x$

$\mu_x = \int_0^1 x f_1(x)\,dx = \int_0^1 2x^2\,dx = \frac{2}{3}x^3\Big|_0^1 = \frac{2}{3}$

4. If $Y = \{2, 2, 3, 5\}$ is a sample from a Normal distribution with unknown mean and variance. Find the following:

(a) A point estimate for μ

(b) A 95% ci for μ

(c) A 80% ci for μ

(d) A 95% ci for μ when it was discovered that the population standard deviation was 1.2

> A $100(1-\alpha)$ % confidence interval for μ is:
>
> $$\overline{Y} \pm t_{\alpha/2} s/\sqrt{n}$$
> $$\overline{Y} \pm Z_{\alpha/2} \sigma/\sqrt{n}$$

R output

```
> y=c(2,2,3,5)
> sd(y)
[1] 1.414214
> qt(1-0.05/2,3)
[1] 3.182446
> qt(1-0.05/2,4)
[1] 2.776445
> qt(1-0.2/2,3)
[1] 1.637744
> qt(1-0.1/2,4)
[1] 2.131847
>
> qnorm(1-0.05/2,0,1)
[1] 1.959964
> qnorm(1-0.05/2,1,1)
[1] 2.959964
> qnorm(1-0.2/2,0,1)
[1] 1.281552
> qnorm(1-0.1/2,0,1)
[1] 1.644854
>
```

a) $\overline{Y} = \frac{2+2+3+5}{4} = \frac{12}{4} = 3$

b) $\overline{Y} \pm t_{0.05/2} \frac{s}{\sqrt{n}} = 3 \pm 3.182446 \times \frac{1.414214}{\sqrt{4}}$

$= (0.7497, 5.2503)$

c) $(1.8419, 4.1581)$

d) $\overline{Y} \pm Z_{\alpha/2} \frac{\sigma}{\sqrt{n}} = 3 \pm 1.9599 \times \frac{1.2}{\sqrt{4}}$

$(1.824, 4.1756)$

(Please show working here:)

5. (Taken from MS pg. 233 5.117) Let c be a constant and consider the density function for the random variable Y:

$$f(y) = \begin{cases} ce^{-y} & \text{if } y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Show $c = 1$.

(b) Show that the cumulative distribution function is $F(y) = 1 - e^{-y}$ Hint: $F(y) = P(Y \leq y) = \int_{-\infty}^{y} f(t)dt$

(c) Compute $F(2.6)$

(d) Show that $F(0) = 0$ and $F(\infty) = 1$.

(e) Compute $P(1 \leq Y \leq 5)$.

a) $\int_{-\infty}^{\infty} f(y)\,dy = 1$

$\Rightarrow \int_{0}^{\infty} ce^{-y}\,dy = 1$

$c\int_{0}^{\infty} e^{-y}\,dy = 1$

$-e^{-y}\Big|_{0}^{\infty} c = 1$

$(-e^{-\infty} + e^{0})c = 1$

$1 \cdot c = 1$

$c = 1$

b) $F(y) = \int_{0}^{y} e^{-t}\,dt = \left[-e^{-t}\right]_{0}^{y}$

$= -e^{-y} + 1$

$= 1 - e^{-y}$

c) $F(2.6) = 1 - e^{-2.6}$

$= 0.9257$

d) $F(0) = 1 - e^{-0} = 1 - 1 = 0$

$F(\infty) = 1 - e^{-\infty} = 1$

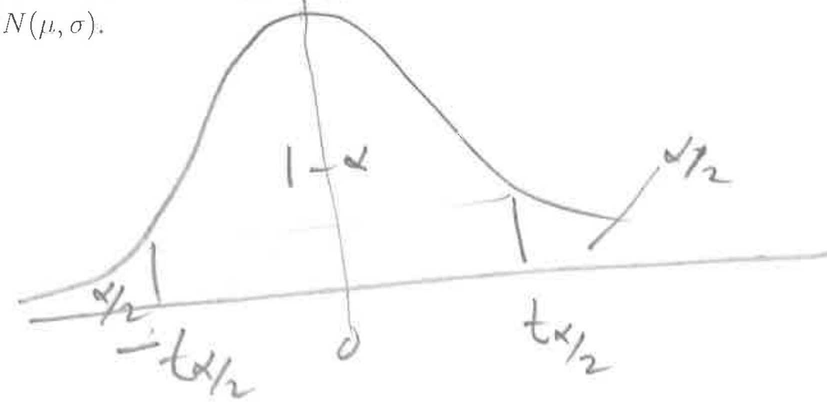
e) $P(1 \leq Y \leq 5) = F(5) - F(1)$

$= 1 - e^{-5} - (1 - e^{-1})$

$= e^{-1} - e^{-5}$

$= 0.3611$

6. Using the pivotal statistic $T = \frac{\overline{Y}-\mu}{s/\sqrt{n}}$ derive the $(1-\alpha)100\%$ confidence interval for the population mean μ. Suppose that $y_1, \ldots, y_n$ is a sample of size n taken from a very large population where $Y_i \overset{iid}{\sim} N(\mu, \sigma)$.



$$P(-t_{\alpha/2} \le T \le t_{\alpha/2}) = 1-\alpha$$

$$P\left(-t_{\alpha/2} \le \frac{\overline{Y}-\mu}{S/\sqrt{n}} \le t_{\alpha/2}\right) = 1-\alpha$$

$$P\left(\overline{Y} - t_{\alpha/2}\frac{S}{\sqrt{n}} \le \mu \le \overline{Y} + t_{\alpha/2}\frac{S}{\sqrt{n}}\right) = 1-\alpha$$

$\therefore$ $100(1-\alpha)\%$ c.i for μ is

$$\overline{Y} \mp t_{\alpha/2}\frac{S}{\sqrt{n}}$$

7. If $L = Y_1 + 2Y_2 - 3Y_3$ and $Y_i \overset{iid}{\sim} N(\mu = 1, \sigma^2 = 1)$

(a) Find the distribution of L.

$L \sim N$

(b) Find $E(L)$

$E(L) = E(Y_1) + 2E(Y_2) - 3E(Y_3)$

$= 1 + 2 \times 1 - 3 \times 1$

$= 0$

(c) Find $V(L)$

$V(L) = V(Y_1) + 4V(Y_2) + 9V(Y_3)$

$= 1 + 4 + 9$

$= 14$

8. The moment generator for a chi square density with random variable Y is

$$M_Y(t) = (1-2t)^{-\nu/2}$$

Using the first and second derivative of $M_Y(t)$ and $\sigma_Y^2 = E(Y^2) - E(Y)^2$, prove the following:

(a) $\mu_Y = \nu$

let $u = 1-2t$

$M_Y(t) = u^{-\nu/2}$

$\frac{dM}{dt} = \frac{dM}{du} \cdot \frac{du}{dt}$

$= -\nu/2 \, u^{-\nu/2 - 1} \cdot (-2)$

$= \nu u^{-\nu/2-1}$

so $\left.\frac{dM}{dt}\right|_{t=0} = \nu 1^{-\nu/2-1} = \underline{\nu}$

(b) $\sigma_Y^2 = 2\nu$

$\frac{d^2M}{dt^2} = \frac{d}{dt}\frac{dM}{dt} = \frac{d}{dt} \nu u^{-\nu/2-1} = \frac{d}{du} \nu u^{-\nu/2-1} \cdot (-2)$

$= \nu(-\nu/2 - 1) u^{-\nu/2-2} (-2)$

$\left.\frac{d^2M}{dt^2}\right|_{t=0} = (\nu^2 + 2\nu) \cdot 1^{-\nu/2-2}$

$= \nu^2 + 2\nu$

$\sigma_Y^2 = E(Y^2) - \mu_1^2$

$= \nu^2 + 2\nu - \nu^2$

$= \underline{2\nu}$

9. In order to find the maximum likelihood estimates of a density one needs to locate the roots of $l'(\theta)$ where θ is the parameter of interest. To investigate this we will look at the general problem of finding the roots of any function $f(x)$. Below is the R code for the function `mynewt()`. There are four lines marked # A, # B # C and # D respectfully.

```
mynewt=function(x0,delta=0.001,f,fdash){
d=1000
i=0
x=c()
y=c()
x[1]=x0
y[1]=f(x[1])
while(d > delta & i<20){ # A
i=i+1
x[i+1]=x[i]-f(x[i])/fdash(x[i]) # B
y[i+1]=f(x[i+1])
d=abs(y[i+1]) # C
}
windows()
curve(f(x),xlim=range(c(range(x),-range(x))),xaxt="n", main="Newton-Raphson Algorithm")
points(x,y,col="Red",pch=19,cex=1.5)
axis(1,x,round(x,2),las=2)
abline(h=0,col="Red")

segments(x[1:(i-1)],y[1:(i-1)],x[2:i],rep(0,i-1),col="Blue",lwd=2) # D
segments(x[2:i],rep(0,i-1),x[2:i],y[2:i],lwd=0.5,col="Pink")

list(x=x,y=y)
}
```

The following applies to the code above.

(a) In line A explain why there are two conditions. to obtain accuracy and stop hang ups

(b) In line A explain what each condition does. d> delta checks accuracy to be more than delta and iterations must not be 20 or more

(c) In line B explain what fdash is. NR step

(d) In line C explain what d means geometrically – i.e. if you were to plot it. Vertical distance to functi

(e) In line D the blue segments are subsets of lines. These are called tangents - TRUE or FALSE?

(f) Suppose the following was called from within R after the function had been sent to the workspace:

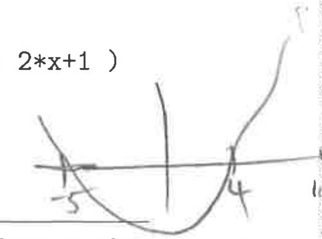
```
mynewt(x0=10,delta=0.0001,f=function(x) x^2 +x -20,fdash=function(x) 2*x+1 )
```

What root would the algorithm find?

$(x+5)(x-4)=0$

$X=-5$ or $X=4$

$X = 4$



10. Figure 1 shows a function $y = f(x)$ with at least one root. The Newton Raphson algorithm is to be run in order to find the root. A tangent is drawn to the curve at $(x_n, f(x_n))$ the x intercept of the tangent is at x_{n+1} – if the algorithm continues the root will be more closely approximated. The following is the update algorithm:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad (1)$$

$$f'(x_n) = \frac{f(x_n)}{x_n - x_{n+1}}$$

$$x_n - x_{n+1} = \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

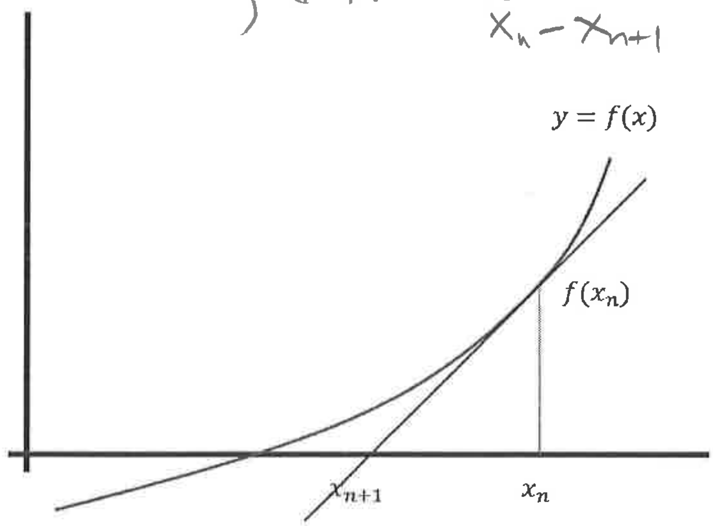


Figure 1: Newton-Raphson Alogorithm to find roots

(a) Derive the update formula from the picture in Figure 1.

(b) Below is the code for obtaining max. lik. estimates. Suppose L=likelihood, l is the log likelihood and dashes are derivatives wrt the parameter. In line A, `fdash` is

- (i) l
- (ii) L
- (iii) l'
- (iv) l''

(c) In line B, d is:

- (i) The distance between the x axis at $(x_{n+1}, 0)$ and the point (x_{n+1}, y_{n+1})
- (ii) delta
- (iii) x0
- (iv) A number < 0

```
myNRML=function(x0,delta=0.001,llik,xrange,parameter="param"){
h=delta/100
f=function(x) (llik(x+h)-llik(x))/h
fdash=function(x) (f(x+h)-f(x))/h # LINE A
d=1000
i=0
x=c()
y=c()
x[1]=x0
y[1]=f(x[1])
while(d > delta & i<100){
i=i+1
x[i+1]=x[i]-f(x[i])/fdash(x[i])
y[i+1]=f(x[i+1])
d=abs(y[i+1]) # LINE B
}
windows()
layout(matrix(1:2,nr=1,nc=2,byrow=TRUE),width=c(1,2))
curve(llik(x), xlim=xrange,xlab=parameter,ylab="log Lik",main="Log Lik")
curve(f(x),xlim=xrange,xaxt="n", xlab=parameter,ylab="derivative",
main=  "Newton-Raphson Algorithm \n on the derivative")
points(x,y,col="Red",pch=19,cex=1.5)
axis(1,x,round(x,2),las=2)
abline(h=0,col="Red")

segments(x[1:(i-1)],y[1:(i-1)],x[2:i],rep(0,i-1),col="Blue",lwd=2)
segments(x[2:i],rep(0,i-1),x[2:i],y[2:i],lwd=0.5,col="Green")

list(x=x,y=y)
}
```