Exam 4/14/2017 MATH 4753

Qu 1

a) $(n-1)s^2 = \sum Y_i^2 - n\bar{Y}^2$ ①

$E(Y^2) = \sigma_Y^2 + \mu^2$

$E(\bar{Y}^2) = \sigma_{\bar{Y}}^2 + \mu_{\bar{Y}}^2 = \frac{\sigma^2}{n} + \mu^2$

$\therefore E((n-1)s^2) = E(\sum Y_i^2 - n\bar{Y}^2)$

$= \sum_{i=1}^{n} E(Y_i^2) - nE(\bar{Y}^2)$

$= \sum_i (\sigma_Y^2 + \mu^2) - n(\frac{\sigma^2}{n} + \mu^2)$ ④

$= n\sigma_Y^2 + \cancel{n\mu^2} - \sigma^2 - \cancel{n\mu^2}$

$E[(n-1)s^2] = (n-1)\sigma_Y^2$

$(n-1)E(s^2) = (n-1)\sigma_Y^2$

$E(s^2) = \sigma_Y^2 = \sigma^2$

b) $s_b^2 = \frac{\sum Y_i^2 - n\bar{Y}^2}{n}$

$ns_b^2 = \sum Y_i^2 - n\bar{Y}^2$ ②

$nE(s_b^2) = E(\sum Y_i^2 - n\bar{Y}^2)$

$= (n-1)\sigma_Y^2$ by a)

$E(s_b^2) = \frac{(n-1)}{n}\sigma_Y^2$

c) $\frac{E(s^2)}{E(s_b^2)} = \frac{\sigma^2}{\frac{(n-1)}{n}\sigma^2} = \frac{n}{n-1} > 1$

$\therefore E(s^2) > E(s_b^2)$ TRUE ①

Qu 2.

a) $\ell(p) = \log\binom{n}{y} + y\log p + (n-y)\log(1-p)$

$$\frac{\partial \ell}{\partial p} = \frac{y}{p} - \frac{(n-y)}{1-p}$$

$$\frac{\partial \ell}{\partial p} = 0 \quad \therefore \quad 0 = \frac{y}{\hat{p}} - \frac{(n-y)}{1-\hat{p}}$$

$$0 = y(1-\hat{p}) - \hat{p}(n-y)$$

$$= y - y\hat{p} - \hat{p}n + \hat{p}y$$

$$0 = y - \hat{p}n$$

$$\hat{p}n = y$$

$$\underline{\hat{p} = \frac{y}{n}} \quad 1$$

b) $E(\frac{y}{n}) = \frac{E(y)}{n} = \frac{np}{n} = p$ 1

c) Since $E(\hat{p}) = p$ unbiased 1

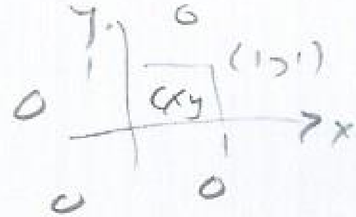
d). $V(\hat{p}) = V(\frac{y}{n}) = \frac{1}{n^2}V(Y) = \frac{1}{n^2}npq = \frac{pq}{n}$ 1

e) $P(Y \le 5) = P(0) + P(1) + P(2) + P(3) + P(4) + P(5)$

$P(y) = \binom{10}{y} 0.5^y (0.5)^{10-y} = \binom{10}{y} 0.5^{10}$ 1

Q3.

a) $$\int_{y=0}^{y=1}\int_{x=0}^{x=1} cxy \, dx\,dy = 1 =$$



$$\int_{x=0}^{x=1} cxy \, dx = c\frac{x^2}{2}y\Big]_{x=0}^{x=1} = \frac{cy}{2}$$

a) $c=4$
b) $\mu_x = \frac{2}{3}$, $f_1(x) = 2x$
c) $\mu_y = \frac{2}{3}$, $f_2(y) = 2y$
d) 0.

$$\int_{y=0}^{y=1} \frac{cy}{2}\,dy = \frac{cy^2}{4}\Big]_0^1 = \frac{c}{4}$$

1

$$\frac{c}{4} = 1 \qquad c = \underline{4}$$

b) $$f_1(x) = \int f(x,y)\,dy = \int_{y=0}^{y=1} 4xy\,dy = 4x\frac{y^2}{2}\Big|_{y=0}^{y=1}$$

1

$$= 2x$$

$$\mu_x = \int x\,2x = \frac{2x^3}{3}\Big|_0^1 = \frac{2}{3}$$

c) $$f_2(y) = \int f(x,y)\,dx = \int_{x=0}^{x=1} 4xy\,dx = 2y$$

1

$$\mu_y = \frac{2}{3}$$

d) $$E(XY) = \int_{y=0}^{y=1}\int_{x=0}^{x=1} XY\,4xy\,dx\,dy = \int_{y=0}^{y=1}\int_{x=0}^{x=1} 4x^2y^2\,dx\,dy$$

e) $$\mathrm{Cov}(X,Y) = \frac{4}{9} - \frac{2}{3}\cdot\frac{2}{3} = \underline{0}$$

1

1

$$\text{Inner} = \Big[\frac{4}{3}x^3y^2\Big]_{x=0}^{x=1} = \frac{4}{3}y^2$$

$$\text{outer} = \int_{y=0}^{y=1} \frac{4}{3}y^2\,dy = \frac{4}{9}\,y^3\Big|_{y=0}^{y=1} = \frac{4}{9}$$

Qu 4.

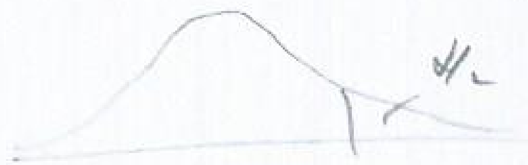
a) $\bar{Y} = \frac{12}{4} = 3.$ ✓

b) $\bar{Y} \pm t_{\alpha/2}\, s/\sqrt{n}$. 1

$3 \pm 3.182446 * 1.414214/\sqrt{4}$

c). $3 \pm 1.637744 * 1.414214/\sqrt{4}$ 1

d) $3 \pm 1.959964 * 1.2/\sqrt{4}$ 1



$t_{\alpha/2} = qt(1-\alpha/2, 3)$

95% ci $\Rightarrow qt(1-0.05/2, 3)$

$= 3.182446$

80% ci $\Rightarrow qt(1-0.2/2, 3)$

$= 1.637744$

Qu5

a) $\int_0^\infty ce^{-y}\,dy = 1$

$\left[-ce^{-y}\right]_0^\infty = 1$

$-c\left(e^{-\infty} - e^{0}\right) = 1$

$-c(-1) = 1$

$c = 1$ 2

b) $F(y) = \int_0^y f(t)\,dt = \left[-e^{-t}\right]_0^y = -e^{-y} + e^{0}$

$= 1 - e^{-y}$ 1

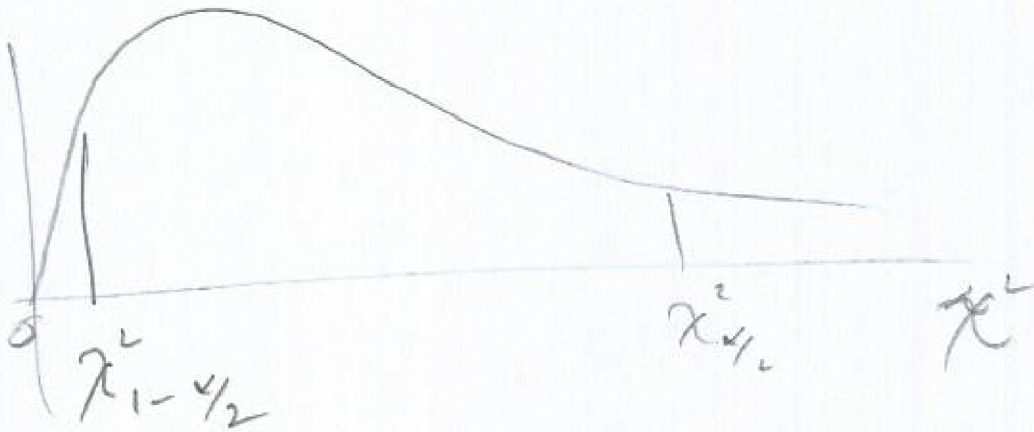
c) $F(0) = 1 - e^{-0} = 1 - 1 = 0$, $F(\infty) = 1 - e^{-\infty} = 1$. 1

d) $P(1 \le Y \le 5) = F(5) - F(1)$

$= 1 - e^{-5} - (1 - e^{-1})$ 1

$= e^{-1} - e^{-5}$

Qu 6.



2

$$P\left(\chi^2_{1-\alpha/2} \le \chi^2 \le \chi^2_{\alpha/2} \right) = 1-\alpha$$

1

$$\chi^2_{1-\alpha/2} \le \frac{(n-1)s^2}{\sigma^2} \le \chi^2_{\alpha/2}$$

$$\sigma^2 \chi^2_{1-\alpha/2} \le (n-1)s^2$$

$$\sigma^2 \le \frac{(n-1)s^2}{\chi^2_{1-\alpha/2}}$$

2.

$$\sigma^2 \ge \frac{(n-1)s^2}{\chi^2_{\alpha/2}}$$

$$\therefore \frac{(n-1)s^2}{\chi^2_{1-\alpha/2}} \le \sigma^2 \le \frac{(n-1)s^2}{\chi^2_{\alpha/2}}$$ 13 9

$100(1-\alpha)\%$ ci fo σ^2.

Qu 7

a) $L \sim N$ ①

b) $E(L) = 2E(Y_1) - 3E(Y_2) + 2E(Y_3)$ ②

$= 2 - 3 + 2$

$= 1$

c) $V(L) = 4V(Y_1) + 9V(Y_2) + 4V(Y_3)$ ②

$= 4 \times 3 + 9 \times 3 + 4 \times 3$

$= 17 \times 3$

$= \underline{51}$