Derivation of

$$\hat\beta_0 \sim N\left(\beta, \frac{\sigma^2}{n}\left(\frac{\sum x_i^2}{SS_{xx}}\right)\right)$$

$$\overline{Y} = \hat\beta_0 + \hat\beta_1 \overline{X}$$

$$\hat\beta_0 = \overline{Y} - \hat\beta_1 \overline{X}$$

$$= \sum \frac{Y_i}{n} - \frac{\sum (x_i - \overline{x}) Y_i \overline{X}}{SS_{xx}}$$

$$= \sum \left[\frac{1}{n} - \frac{(x_i - \overline{x})\overline{x}}{SS_{xx}}\right] Y_i$$

$$= \sum c_i Y_i \;;\; c_i = \left[\frac{1}{n} - \frac{(x_i - \overline{x})\overline{x}}{SS_{xx}}\right] \quad \text{—— } (*)$$

$\therefore \hat\beta_0 \sim N$ since a linear sum of iid N r.vs is N

$$E(\hat\beta_0) = E\left[\sum c_i Y_i\right] = \sum_{i=1}^{n} c_i E(Y_i) = \sum_{i=1}^{n} c_i (\beta_0 + \beta_1 x_i) \quad \text{— } (**)$$

$(**)$ must be simplified

$$\sum c_i \beta_0 = \sum_{i=1}^{n} c_i \beta_0 = \beta_0 \sum c_i = \beta_0 \sum_i \left[\frac{1}{n} - \frac{(x_i - \overline{x})\overline{x}}{SS_{xx}}\right] = \beta_0 [1 - 0]$$

$$\sum_i c_i \beta_1 x_i = \beta_1 \sum c_i x_i = \beta_1 \sum \left[\frac{x_i}{n} - \frac{(x_i - \overline{x}) x_i \overline{x}}{SS_{xx}}\right] = \beta_1 \left[\overline{x} - \overline{x}\cancel{\frac{\sum (x_i - \overline{x}) x_i}{SS_{xx}}}\right]$$

$$= \beta_1 [\overline{x} - \overline{x}] = 0$$

$$\hat\beta_0 \sim N(\mu = \beta_1,)$$

$$V(\hat\beta_0) = V\left(\sum_i c_i Y_i\right) = \sum_i c_i^2 V(Y_i) = \sigma^2 \sum_i c_i^2 = \sigma^2 \sum \left[\frac{1}{n} - \frac{(x_i - \overline{x})\overline{x}}{SS_{xx}}\right]^2$$

$$= \sigma^2 \sum \left[\frac{1}{n^2} - 2\frac{1}{n}\frac{(x_i - \overline{x})\overline{x}}{SS_{xx}} + \frac{(x_i - \overline{x})^2\overline{x}^2}{SS_{xx}^2}\right] = \sigma^2\left[\frac{1}{n} + \cancel{\frac{\sum (x_i - \overline{x})^2}{SS_{xx}^2}}\,\overline{x}^2\right]$$

$$= \sigma^2\left[\frac{1}{n} + \frac{\overline{x}^2}{SS_{xx}}\right]$$

$$\sigma^2\left[\frac{SS_{xx} + n\overline{x}^2}{n SS_{xx}}\right]$$

now $SS_{xx} = \sum x_i^2 - n\overline{x}^2$

so

$$\sigma^2\left[\frac{\sum x_i^2 - n\overline{x}^2 + n\overline{x}^2}{n SS_{xx}}\right]$$

$$= \frac{\sigma^2}{n}\left[\frac{\sum x^2}{SS_{xx}}\right]$$

$$\hat\beta_0 \sim N\left(\beta_0, \frac{\sigma^2}{n}\left[\frac{\sum x^2}{SS_{xx}}\right]\right)$$

For Marissa and others

$$SS_{xx} = \sum (x_i - \bar{x})(x_i - \bar{x})$$

$$= \sum (x_i - \bar{x}) x_i - \sum (x_i - \bar{x}) \bar{x}$$

$$= \sum (x_i - \bar{x}) x_i - \bar{x} \sum (x_i - \bar{x}) = 0$$

$$= \sum (x_i - \bar{x}) x_i .$$

$$\boxed{SS_{xx} = \sum_{i=1}^{n} (x_i - \bar{x}) x_i}$$