ANSWERS

Department of Mathematics (OU)

Name:

Please answer *question 1* and *any other* 7 out of the remaining 9 questions. You can answer in any order you wish. Please hand in all the exam questions and answers at the end of the class.

Circle the questions in table 1 that you have answered ! Time allocated 1hr 15 min.

Question	Marks earned	Out of
Q1	MUST DO	10
Q2		10
Q3		10
Q4		10
Q5		10
Q6		10
Q7		10
Q8		10
Q9		10
Q10		10
Total	Best 8	80

Table 1: Mark allocation

If $\hat{\theta}$ is an unbiased estimator of θ then

$$E(\hat{\theta}) = \theta$$

1. In this question you will prove that S^2 is an unbiased estimator of σ^2 by answering the sub questions a)-e) . That is you will prove:

$$E(S^2) = \sigma^2$$

where

$$S^2 = \frac{\sum_{i=1}^{n} Y_i^2 - n\overline{Y}^2}{n-1}.$$

This is the estimator for the population variance.
Hint: $\sigma_Y^2 = E(Y^2) - E(Y)^2$

(a) Show that $E(Y^2) = \sigma_Y^2 + E(Y)^2$ **Hint:** You can use anything given above.

(b) Show that $\sigma_{\overline{Y}}^2 = \frac{\sigma_Y^2}{n}$

(c) Show that

$$E(S^2) = \frac{\sum_{i=1}^{n} E(Y_i^2) - nE(\overline{Y}^2)}{n-1}$$

make sure you say what properties of expectation you use.

(d) Show that $E(S^2) = \sigma^2$ by using well defined algebraic steps. Hint: $E(\overline{Y}^2) = \sigma_{\overline{Y}}^2 + E(\overline{Y})^2$

(e) If S^2 had an n instead of $n-1$ in the denominator what would the value of c be in $E(S^2) = c\sigma^2$ - show all working?

a) Given $\sigma_Y^2 = E(Y^2) - E(Y)^2 \therefore E(Y^2) = \sigma_Y^2 + E(Y)^2$

b) $\sigma_{\bar{Y}}^2 = V(\bar{Y}) = V\left(\frac{\sum Y_i}{n}\right) = \frac{1}{n^2}\sum V(Y_i) = \frac{n\sigma_Y^2}{n^2} = \frac{\sigma_Y^2}{n}$

c) $E(S^2) = E\left[\frac{\sum_{i=1}^{n} Y_i^2 - n\bar{Y}^2}{n-1}\right] = \frac{1}{n-1} E\left[\sum_{i=1}^{n} Y_i^2 - n\bar{Y}^2\right]$ — $E(cX) = cE(X)$

$= \frac{1}{n-1}\left[\sum_{i=1}^{n} E(Y_i^2) - nE(\bar{Y}^2)\right]$ — $E(g_1(x) + g_2(x)) = E(g_1) + E(g_2)$

d) $E(S^2) = \frac{1}{n-1}\left[\sum(\sigma_Y^2 + \mu^2) - n(\sigma_{\bar{Y}}^2 + \mu^2)\right]$

$= \frac{1}{n-1}\left[n\sigma_Y^2 + n\mu^2 - n\left(\frac{\sigma_Y^2}{n} + \mu^2\right)\right]$

$= \frac{1}{n-1}\left[n\sigma_Y^2 + n\mu^2 - \sigma_Y^2 - n\mu^2\right] = \frac{1}{\cancel{n-1}}\left[\sigma_Y^2(\cancel{n-1})\right] = \sigma_Y^2$

e) $E(S^{*2}) = E\left(\frac{n-1}{n}S^2\right) = \frac{n-1}{n}E(S^2) = \frac{n-1}{n}\sigma^2$

So $c = \frac{n-1}{n}$

2. Let $y_1, y_2, \ldots, y_n$ be a random sample of n observations from a Poisson distribution with probability function

$$p(y) = \begin{cases} \frac{e^{-\lambda}\lambda^y}{y!} & y = 0, 1, 2, \ldots \\ 0 & otherwise \end{cases}$$

$\mu_Y = \lambda$, $\sigma_Y^2 = \lambda$

(a) Write down the expression for the joint density $p(\mathbf{y})$. Show ALL working!

(b) Write down the expression for the Likelihood $L(\lambda)$. This will not require much working!

(c) Write down the simplified expression for the log likelihood $l(\lambda)$. Show all working!

(d) Find the maximum likelihood estimator. You will need to show all working.

(e) Is the maximum likelihood estimator unbiased? Show why or why not with working.

a) $p(y) = \frac{e^{-\lambda}\lambda^{y_1}}{y_1!} \cdot \frac{e^{-\lambda}\lambda^{y_2}}{y_2!} \cdots \frac{e^{-\lambda}\lambda^{y_n}}{y_n!} = \frac{e^{-n\lambda}\lambda^{\Sigma y_i}}{y_1!\ldots y_n!}$

b) $L(\lambda) = \frac{e^{-n\lambda}\lambda^{\Sigma y_i}}{y_1!\ldots y_n!}$

c) $l(\lambda) = \log\left(\frac{e^{-n\lambda}\lambda^{\Sigma y_i}}{y_1!\ldots y_n!}\right) = -n\lambda + \Sigma y_i \log\lambda - \log(y_1!\ldots y_n!)$

d) $\frac{dl}{d\lambda} = -n + \frac{\Sigma y_i}{\lambda} - 0$

$0 = -n + \frac{\Sigma y_i}{\hat{\lambda}}$

$\hat{\lambda} = \frac{\Sigma y_i}{n} = \bar{y}$

e) $E(\hat{\lambda}) = E(\bar{y}) = E(y) = \mu_y = \lambda$

$E(\hat{\lambda}) = \lambda$ $\therefore$ unbiased.

3. Let $y_1, y_2, \ldots, y_n$ be a random sample of n observations on a random variable Y where $f(y)$ is an exponential density function:

$$f(y) = \begin{cases} \frac{e^{-y/\beta}}{\beta} & y \geq 0 \\ 0 & otherwise \end{cases}$$

$\mu_Y = \beta$, $\sigma_Y^2 = \beta^2$,

(a) Find $\hat{\beta}_{mle}$ the maximum likelihood estimator of β. (Show working)

(b) Find $E(\hat{\beta}_{mle})$? (Show working)

(c) Is $\hat{\beta}_{mle}$ UNBIASED?

(d) Find $V(\hat{\beta}_{mle})$ (Show working)

(e) Find the numerical value of $f(-3.5)$

(a) $f(\underline{y}) = \frac{e^{-y_1/\beta}}{\beta} \cdot \frac{e^{-y_2/\beta}}{\beta} \cdots \frac{e^{-y_n/\beta}}{\beta}$

$= \frac{e^{-\frac{\sum y_i}{\beta}}}{\beta^n}$

$\ell(\beta) = -\frac{\sum y_i}{\beta} - n \log \beta$

$\ell'(\beta) = \frac{\sum y_i}{\beta^2} - \frac{n}{\beta}$

$\ell'(\beta) = 0 \Rightarrow 0 = \frac{\sum y_i}{\hat{\beta}^2} - \frac{n}{\hat{\beta}}$

$0 = \sum y_i - n\hat{\beta}$

$\hat{\beta} = \overline{Y}$

(b) $E(\hat{\beta}) = E(\overline{Y}) = E(Y) = \beta$

So $E(\hat{\beta}) = \beta$ hence unbiased

c) From b) unbiased

d) $V(\hat{\beta}) = V(\overline{Y}) = \frac{\sigma_Y^2}{n}$

$= \frac{\beta^2}{n}$

e) $f(-3.5) = 0$

4. If $x = \{8, 9, 10\}$ is a sample from a Normal distribution with unknown mean and variance. Find the following by using a calculator (unless told otherwise) and information supplied:

(a) A point estimate for μ

(b) A 95% ci for μ

(c) A point estimate for σ^2

(d) A 90% ci for σ^2

(e) Using any output available, find a 70% ci for μ

$$\overline{Y} \pm t_{\alpha/2} s/\sqrt{n}$$
$$((n-1)s^2/\chi^2_{\alpha/2}, (n-1)s^2/\chi^2_{1-\alpha/2})$$

R output

```
> x=8:10
> summary(x)
   Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
    8.0     8.5     9.0     9.0     9.5    10.0
> t.test(x,conf.level=0.70)

One Sample t-test

data:  x
t = 15.5885, df = 2, p-value = 0.00409
alternative hypothesis: true mean is not equal to 0
70 percent confidence interval:
 8.199673 9.800327
sample estimates:
mean of x
        9

> qt(1-0.05/2,2)
[1] 4.302653
> qt(1-0.01/2,2)
[1] 9.924843
> qt(1-0.1/2,2)
[1] 2.919986
> qt(1-0.05/2,3)
[1] 3.182446
> qchisq(0.10/2,2)
[1] 0.1025866
> qchisq(1-0.10/2,2)
```

a) $\hat{\mu} = 9$

b) $\overline{Y} \pm t_{\alpha/2} s/\sqrt{n}$

$9 \pm 4.3 \; 1/\sqrt{3}$

$(6.5, 11.5)$

c) $s^2 = 1 = \hat{\sigma}^2$

d) $\frac{(n-1)s^2}{\chi^2_{\alpha/2}}, \frac{(n-1)s^2}{\chi^2_{1-\alpha/2}}$

$= \frac{2.1^2}{5.99}, \frac{2.1^2}{0.103}$

$= (0.3, 19.5)$

e) $(8.2, 9.8)$ from t.test

```
[1] 5.991465
> qchisq(0.10/2,3)
[1] 0.3518463
> qchisq(1-0.10/2,3)
[1] 7.814728
> var(x)
[1] 1
```

(Please show working here:)

5. Find $cov(X,Y)$ for the following bivariate distribution by answering the questions below. You might need

$$cov(X,Y) = E(XY) - \mu_X \mu_Y.$$

Please note: ALL working MUST be shown!

$$f(x,y) = \begin{cases} 2x & \text{if } 0 \le x \le 1; 0 \le y \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$E(g(X,Y)) = \int_Y \int_X g(x,y) f(x,y)\,dx\,dy \qquad (1)$$

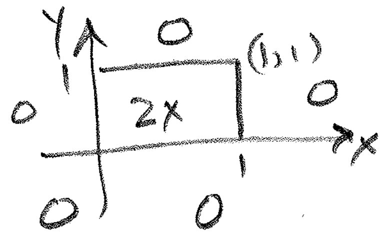
(a) Find $E(XY)$ – show all working

(b) Find $f_1(X)$ — show all working

(c) Find $f_2(Y)$ — show all working

(d) Find μ_x and μ_y – show all working

(e) Find $Cov(X,Y)$ – all working please.



(a) $E(XY) = \int_{y=0}^{y=1}\int_{x=0}^{x=1} xy\,2x\,dx\,dy = \int\int 2x^2y\,dx\,dy = \int_{y=0}^{y=1} \left[\frac{2}{3}x^3y\right]_{x=0}^{x=1} dy$

$= \frac{2}{3}\int_{y=0}^{y=1} y\,dy = \frac{2}{3}\left[\frac{1}{2}y^2\right]_{y=0}^{y=1} = \frac{1}{3}$

(b) $f_1(x) = \int_Y f(x,y)\,dy = \int_0^1 2x\,dy = 2x$

(c) $f_2(y) = \int_X f(x,y)\,dx = \int_0^1 2x\,dx = [x^2]_0^1 = 1$

(d) $\mu_x = E(X) = \int_X x f_1(x)\,dx = \int 2x^2\,dx = \frac{2}{3}x^3\Big|_0^1 = \frac{2}{3}$

$\mu_y = E(Y) = \int_Y y f_2(y)\,dy = \int_0^1 y \cdot 1\,dy = \frac{1}{2}y^2\Big|_0^1 = \frac{1}{2}$

(e) $Cov(X,Y) = E(XY) - \mu_x\mu_y = \frac{1}{3} - \frac{2}{3}\cdot\frac{1}{2} = \frac{1}{3} - \frac{1}{3} = 0$

(cont. covariance calculation here)

6. Suppose that $y_1, \ldots, y_n$ is a sample of size n taken from a very large population where $Y_i \overset{iid}{\sim} N(\mu, \sigma)$ - using this sample and the pivotal statistic $T = \frac{\overline{Y}-\mu}{s/\sqrt{n}}$ derive the $(1-\alpha)100\%$ confidence interval for the population mean i.e. μ, namely

$$\boxed{\overline{Y} \pm t_{\alpha/2} s/\sqrt{n}}$$

by answering the following questions:

(a) Draw a sketch of the t distribution with areas $1-\alpha$, $\alpha/2$ and quantiles $t_{\alpha/2}, -t_{\alpha/2}$

(b) Make a probability statement from the sketch by filling in the blank (the argument)

$$P(\ldots \leq T \leq \ldots) = 1 - \alpha$$

(c) Substitute the pivotal statistic into the equation.

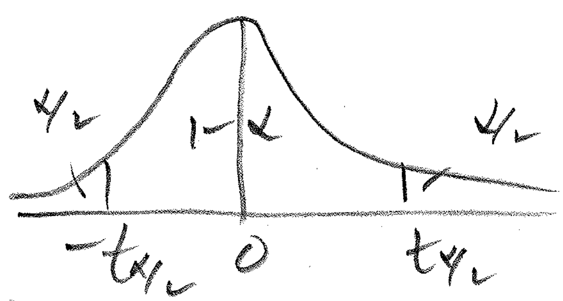
(d) Rearrange the argument and make μ the subject to get

$$P(\ldots \leq \mu \leq \ldots) = 1 - \alpha$$

fill in the gaps after showing all working.

(e) Now give the correct interpretation of the interval.

(a)



b) $P(-t_{\alpha/2} \leq T \leq t_{\alpha/2}) = 1-\alpha$

c) $P\left(-t_{\alpha/2} \leq \frac{\overline{Y}-\mu}{s/\sqrt{n}} \leq t_{\alpha/2}\right) = 1-\alpha$

d) Argument:

$\mu \leq \overline{Y} + t_{\alpha/2}\frac{s}{\sqrt{n}}$

$\mu \geq \overline{Y} - t_{\alpha/2}\frac{s}{\sqrt{n}}$

So $P\left(\overline{Y} - t_{\alpha/2}\frac{s}{\sqrt{n}} \leq \mu \leq \overline{Y} + t_{\alpha/2}\frac{s}{\sqrt{n}}\right) = 1-\alpha$

e) With $100(1-\alpha)\%$ Confidence the True and underlying mean will lie in the interval

$\overline{Y} \pm t_{\alpha/2}\frac{s}{\sqrt{n}}$.

OR frequency interpretation

7. If $L = 2Y_1 - 3Y_2 - Y_3$ and $Y_i \overset{iid}{\sim} N(\mu = 1, \sigma^2 = 2)$

(a) Find the distribution of L.

(b) Find $E(L)$

(c) Find $V(L)$

(d) Find $E(2L)$

(e) Find $V(2L+1)$

(a) $L \sim N$

(b) $E(L) = 2E(Y_1) - 3E(Y_2) - 1E(Y_3)$

$= 2\times 1 - 3\times 1 - 1\times 1$

$= -2$

(c) $V(L) = 2^2 V(Y_1) + (-3)^2 V(Y_2) + (-1)^2 V(Y_3)$

$= 4\times 2 + 9\times 2 + 1\times 2$

$= 8 + 18 + 2$

$= 28$

(d) $E(2L) = 2E(L) = 2\times(-2) = -4$

(e) $V(2L+1) = 2^2 V(L) = 4\times 28 = 112$

8. The moment generator for a chi-sq density is

$$M_Y(t) = (1 - 2t)^{-\nu/2}$$

$$\frac{d^k}{dt^k} M_Y(t)\bigg|_{t=0} = \mu_k' \qquad (2)$$

Using the first and second derivative of $M_Y(t)$ and $\sigma_Y^2 = E(Y^2) - E(Y)^2$, where $Y \sim Chisq(\nu)$ answer the following:

(a) Find $\frac{d}{dt} M_Y(t)$

(b) Find $\frac{d^2}{dt^2} M_Y(t)$

(c) Using one of the derivatives above prove $\mu_Y = \nu$

(d) Using the derivatives above prove $\sigma_Y^2 = 2\nu$

(e) True/False: The variance of the chisq will always be twice the mean.

(a) $M_Y(t) = u^{-\nu/2}$, $u = 1-2t$

$$\frac{dM}{dt} = \frac{dM}{du} \cdot \frac{du}{dt} = -\nu/2\, u^{-\nu/2 - 1} \cdot -2 = \nu(1-2t)^{-\nu/2 - 1}$$

(b) $$\frac{d}{dt}\frac{dM}{dt} = \nu(-\nu/2 - 1)(1-2t)^{-\nu/2 - 1 - 1} \cdot (-2)$$

$$= 2\nu(\nu/2 + 1)(1-2t)^{-\nu/2 - 2}$$

(c) $$\mu_1' = \mu = \nu(1-2t)^{-\nu/2 - 1}\Big|_{t=0} = \nu(1-0)^{-\nu/2 - 1} = \nu$$

(d) $$\mu_2' = E(Y^2) = 2\nu(\nu/2 + 1)(1-2t)^{-\nu/2 - 2}\Big|_{t=0} = 2\nu(\nu/2 + 1)(1-0)^{-\nu/2 - 2} = \nu^2 + 2\nu$$

$$\sigma_Y^2 = E(Y^2) - E(Y)^2 = \nu^2 + 2\nu - \nu^2 = 2\nu$$

(e) TRUE.

9. In order to find the maximum likelihood estimates of a density one needs to locate the roots of $l'(\theta)$ where θ is the parameter of interest. To investigate this we will look at the general problem of finding the roots of any function $f(x)$. Below is the R code for the function mynewt(). There are four lines marked # A, # B # C and # D respectfully.

```
mynewt=function(x0,delta=0.001,f,fdash){ #A
d=1000
i=0
x=c()
y=c()
x[1]=x0
y[1]=f(x[1])
while(d > delta & i<20){ # B
i=i+1
x[i+1]=x[i]-f(x[i])/fdash(x[i]) # C
y[i+1]=f(x[i+1])
d=abs(y[i+1])
}
windows()
curve(f(x),xlim=range(c(range(x),-range(x))),xaxt="n", main="Newton-Raphson Algorithm")
points(x,y,col="Red",pch=19,cex=1.5)
axis(1,x,round(x,2),las=2)
abline(h=0,col="Red")

segments(x[1:(i-1)],y[1:(i-1)],x[2:i],rep(0,i-1),col="Blue",lwd=2)
segments(x[2:i],rep(0,i-1),x[2:i],y[2:i],lwd=0.5,col="Pink")

list(x=x,y=y) #D
}
```

The following applies to the code above.

(a) In Line A what is the default value of delta 0.001

(b) In line B explain why there are two conditions. to stop hanging up and stop with desired accuracy

(c) In line C explain what fdash is. Slope of tangent

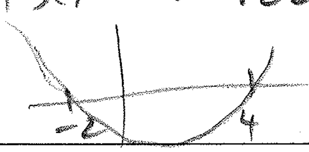
(d) In line D what will the last value of y be in absolute value when compared to delta? ≤ delta.

(e) Suppose the following was called from within R after the function had been sent to the workspace:

```
mynewt(x0=-10,delta=0.0001,f=function(x) x^2-2*x - 8,fdash=function(x) 2*x-2 )
```

What root would the algorithm find? $f = (x-4)(x+2)$ roots $4, -2$

-2



10. Figure 1 shows a function $y = f(x)$ with at least one root. The Newton Raphson algorithm is to be run inorder to find the root. A tangent is drawn to the curve at $(x_n, f(x_n))$ the x intercept of the tangent is at x_{n+1} – if the algorithm continues the root will be more closely approximated. The following is the update algorithm:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad (3)$$

(a) $f'(x_n) = \dfrac{f(x_n)}{x_n - x_{n+1}}$

(b) $f'(x_n)(x_n - x_{n+1}) = f(x_n)$

$x_n - x_{n+1} = \dfrac{f(x_n)}{f'(x_n)}$

$x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

(c) TRUE

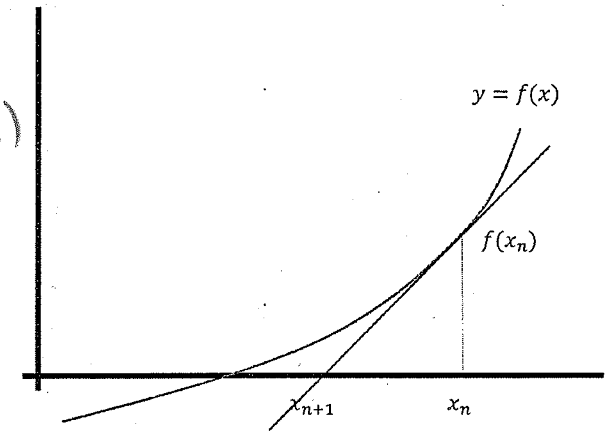


Figure 1: Newton-Raphson Alogorithm to find roots

(d)

(a) Derive the equation for the slope of the tangent in Figure 1, i.e $f'(x_n) =$?.

(b) Using the slope equation solve for x_{n+1}

(c) TRUE/FALSE?. x_{n+1} is closer to the root than x_n.

(d) Below is the code for obtaining max. lik. estimates. Suppose L=likelihood, l is the log likelihood and dashes are derivatives wrt the parameter. In line A f approximates:

  (i) L
  (ii) ~~l~~
  (iii) l'
  (iv) l''

(e)

(e) In line B fdash approximates:

  (i) L
  (ii) l
  (iii) ~~l'~~
  (iv) l''

```
myNRML=function(x0,delta=0.001,llik,xrange,parameter="param"){
h=delta/100
f=function(x) (llik(x+h)-llik(x))/h # A
fdash=function(x) (f(x+h)-f(x))/h # B
d=1000
i=0
x=c()
y=c()
x[1]=x0
y[1]=f(x[1])
while(d > delta & i<100){
i=i+1
x[i+1]=x[i]-f(x[i])/fdash(x[i])
y[i+1]=f(x[i+1])
d=abs(y[i+1])
}
windows()
layout(matrix(1:2,nr=1,nc=2,byrow=TRUE),width=c(1,2))
curve(llik(x), xlim=xrange,xlab=parameter,ylab="log Lik",main="Log Lik")
curve(f(x),xlim=xrange,xaxt="n", xlab=parameter,ylab="derivative",
main=  "Newton-Raphson Algorithm \n on the derivative")
points(x,y,col="Red",pch=19,cex=1.5)
axis(1,x,round(x,2),las=2)
abline(h=0,col="Red")

segments(x[1:(i-1)],y[1:(i-1)],x[2:i],rep(0,i-1),col="Blue",lwd=2)
segments(x[2:i],rep(0,i-1),x[2:i],y[2:i],lwd=0.5,col="Green")

list(x=x,y=y)
}
```