On Polarization Properties of Light, Gaussian Beams and Spin-Orbit Interaction of Light

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Topics of discussion

Polarization properties of light

Gaussian Beam

Polarization properties of light

- Jones Formalism
- Stokes-Muller formalism

Jones Vector

Electric field of *fully polarized* EM wave propagating along z-axis is given by

$$\boldsymbol{E}(\boldsymbol{r},t) = \begin{bmatrix} A_x(\boldsymbol{r})e^{i\delta_x} \\ A_y(\boldsymbol{r})e^{i\delta_y} \\ 0 \end{bmatrix} e^{-i(kz-\omega t)}$$

Define normalized **Jones vector** *s.t.* $\boldsymbol{J}^*\boldsymbol{J} = 1$ as

$$\boldsymbol{J}(\boldsymbol{r},t) = \frac{1}{\sqrt{A_x^2 + A_y^2}} \begin{bmatrix} A_x(\boldsymbol{r})e^{i\delta_x} \\ A_y(\boldsymbol{r})e^{i\delta_y} \end{bmatrix}$$

Note that *intensity*, $I = A_x^2 + A_y^2 = \boldsymbol{J}^*\boldsymbol{J}$

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Jones vector of usual polarization state

Polarization state	$\boldsymbol{J}$
$\vert H\rangle$	$\begin{bmatrix}1\\0\end{bmatrix}$
$\vert V\rangle$	$\begin{bmatrix}0\\1\end{bmatrix}$
$\vert P\rangle$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$
$\vert M\rangle$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$
$\vert L\rangle$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}$
$\vert R\rangle$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-i\end{bmatrix}$

Jones Matrix & evolution of Jones vector

Jones matrix for an optical element be $\boldsymbol{M}$ *s.t.*

$$\boldsymbol{M} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$$

If a polarized light of Jones vector $\boldsymbol{J}_{in}$ passes through that optical element then the Jones vector of output light is given by

$$\boldsymbol{J}_{out} = \boldsymbol{M}\ \boldsymbol{J}_{in}$$

- Composition rule: $\boldsymbol{M} = \boldsymbol{M}_1\ \boldsymbol{M}_2 \ldots \boldsymbol{M}_n$
- Frame rotation by θ: $\boldsymbol{M}_\theta = R(-\theta)\ \boldsymbol{M}\ R(\theta)$ where $R(\theta)$ is *passive rotation matrix.*

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Jones matrix of usual optical element

Optical element	$\boldsymbol{M}$
Free space	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
x-Polariser	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
Right circular polariser	$\frac{1}{2}\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$
Linear di-attenuator	$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$
Half-wave plate with fast axis horizontal	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Quarter-wave plate with fast axis horizontal	$\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$
General phase retarder	$\begin{bmatrix} e^{i\phi_x} & 0 \\ 0 & e^{i\phi_y} \end{bmatrix}$

Polarization properties of light

- Jones Formalism
- Stokes-Muller formalism

Coherency matrix

Coherency matrix, $\boldsymbol{C}$ defined as

$$\boldsymbol{C} = \underbrace{\left\langle \boldsymbol{E} \otimes \boldsymbol{E}^{\dagger} \right\rangle = \left\langle \boldsymbol{E}\boldsymbol{E}^{\dagger} \right\rangle = \begin{bmatrix} \langle E_x E_x^* \rangle & \left\langle E_x E_y^* \right\rangle \\ \langle E_y E_x^* \rangle & \left\langle E_y E_y^* \right\rangle \end{bmatrix}}_{\text{only for fully polarized light}} = \begin{bmatrix} c_{xx} & c_{xy} \\ c_{yx} & c_{yy} \end{bmatrix}$$

- $\boldsymbol{C} = \boldsymbol{C}^{\dagger}$ (Hermitian).
- *Time averaged intensity*, $I = \mathrm{Tr}(\boldsymbol{C})$
-

Let basis set,

$$\left\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\boldsymbol{V_0}}, \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{\boldsymbol{V_1}}, \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\boldsymbol{V_2}}, \underbrace{\begin{bmatrix} 0 & i \\ -i & 1 \end{bmatrix}}_{\boldsymbol{V_3}} \right\} \; s.t. \; \boldsymbol{C} = \frac{1}{2} \sum_{i=0}^{3} S_i \boldsymbol{V}_i$$

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Stokes vector

$$\boldsymbol{C} = \frac{1}{2}\sum_{i=0}^{3} S_i \boldsymbol{V_i} \longrightarrow \begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = \boldsymbol{S} \ (\textit{Stokes vector})$$

Measurement of Stokes parameters by passing the light through

1. isotropic medium, $I_0 = S_0$
2. x-polariser, $I_1 = \frac{1}{2}(S_0 + S_1)$
3. 45°-polariser, $I_2 = \frac{1}{2}(S_0 + S_2)$
4. circular polariser, $I_3 = \frac{1}{2}(S_0 + S_3)$

Stokes vector

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Stokes vector of usual polarization state

Polarization state	$\boldsymbol{C}$	$\boldsymbol{S}$
$\|H\rangle$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}^T$
$\|V\rangle$	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix}^T$
$\|P\rangle$	$\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}^T$
$\|M\rangle$	$\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix}^T$
$\|L\rangle$	$\frac{1}{2}\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}^T$
$\|R\rangle$	$\frac{1}{2}\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & -1 \end{bmatrix}^T$
Un-polarized	$\frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T$

Muller Matrix & evolution of Stokes vector

Muller matrix for an optical element $\mathfrak{M}$ *s.t.*

$$\mathfrak{M} = \begin{bmatrix} \mu_{11} & \cdots & \mu_{14} \\ \vdots & \ddots & \vdots \\ \mu_{41} & \cdots & \mu_{44} \end{bmatrix}$$

Evolution of Stokes vector as, $\boldsymbol{S}_{out} = \mathfrak{M}\ \boldsymbol{S}_{in}$

- Composition rule: $\mathfrak{M} = \mathfrak{M}_1\ \mathfrak{M}_2 \ldots \mathfrak{M}_n$
- Frame rotation by θ: $\mathfrak{M}_\theta = T^{-1}(\theta)\ \mathfrak{M}\ T(\theta)$ where

$$T(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\theta & \sin 2\theta & 0 \\ 0 & -\sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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Poincare sphere representation

For elliptically polarized light, Stokes vector

$$\boldsymbol{S} = S_0 \begin{bmatrix} 1 \\ \cos 2\chi \cos 2\psi \\ \cos 2\chi \sin 2\psi \\ \sin 2\chi \end{bmatrix} \longrightarrow \underbrace{S_0(\cos 2\chi \cos 2\psi, \cos 2\chi \sin 2\psi, \sin 2\psi)}_{\text{On Poincare sphere of radius } S_0}$$

where azimuth (ψ) and ellipticity (χ) is defined.

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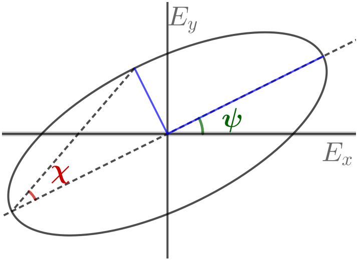
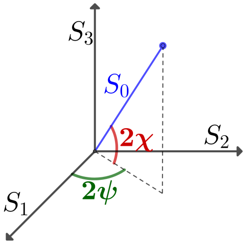
Polarization ellipse



Poincare sphere



Poincare sphere representation

For un-polarized light,

$$\boldsymbol{S} = S_0 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow \underbrace{(0,0,0)}_{\text{At Origin}}$$

For partially polarized light,

$$\boldsymbol{S} = \begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} \longrightarrow \underbrace{(S_1, S_2, S_3)}_{\substack{\text{Inside sphere } s.t. \\ 0 < S_1^2 + S_2^2 + S_3^2 < S_0^2}}$$

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Gaussian Beam and properties

- Paraxial wave
- Gaussian beam solution and properties
- Modes of Gaussian beam

Paraxial wave

Maxwell's wave equation:

$$\nabla^2 \boldsymbol{E}(\boldsymbol{r}, t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\boldsymbol{E}(\boldsymbol{r}, t) = 0$$

Paraxial beam propagating predominantly in z-direction,

$$\boldsymbol{E}(x, y, z, t) = \boldsymbol{\psi}(x, y, z)e^{i(\omega t - kz)}$$

and taking slowly varying amplitude approx. *i.e.*

$$\left|\frac{\partial^2 \boldsymbol{\psi}}{\partial z^2}\right| \ll k\left|\frac{\partial \boldsymbol{\psi}}{\partial z}\right| \ll k^2 \left|\boldsymbol{\psi}\right|$$

Paraxial wave equation:

$$\frac{\partial^2 \boldsymbol{\psi}}{\partial x^2} + \frac{\partial^2 \boldsymbol{\psi}}{\partial y^2} - 2ik\frac{\partial \boldsymbol{\psi}}{\partial r} = 0$$

One of the solutions is Gaussian beam.

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Gaussian Beam

- Paraxial wave
- Gaussian beam solution and properties
- Modes of Gaussian beam

Gaussian beam solution

$$\text{Ansatz: } \psi(\boldsymbol{r}, z) = A \exp\left[-i\left(p(z) + \frac{kr^2}{2q(z)}\right)\right]$$

$$\psi(\boldsymbol{r}, z) = A\left(\frac{w_0}{w(z)}\right)\exp\left(\tan^{-1}\left(\frac{z}{z_0}\right) - i\frac{kr^2}{2R(z)} - \frac{r^2}{w^2(z)}\right)$$

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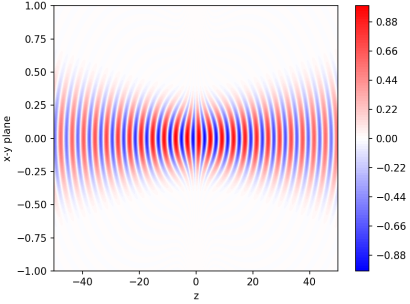
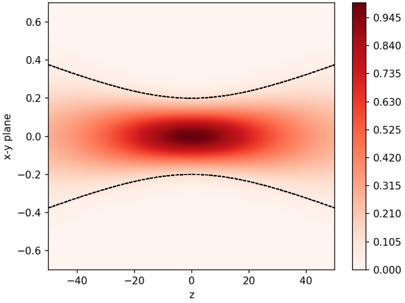
Electric field



Intensity



Gaussian beam properties

$$\psi(\boldsymbol{r}, z) = A\underbrace{\left(\frac{w_0}{w(z)}\right)}_{\text{term I}}\exp\left(i\tan^{-1}\left(\frac{z}{z_0}\right) - i\frac{kr^2}{2R(z)} - \frac{r^2}{w^2(z)}\right)$$

Term I related to **spreading of beam**.

$w \rightarrow$ Physical radius

$w_0 \rightarrow$ Beam waist

$$w(z) = w_0\sqrt{1 + \left(\frac{z}{z_0}\right)^2}$$

$z_0 \rightarrow$ Rayleigh length

$$z_0 = \frac{\pi w_0^2}{\lambda}$$

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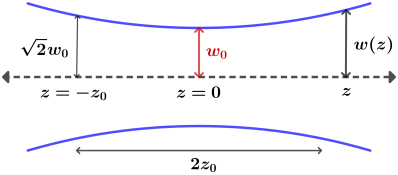

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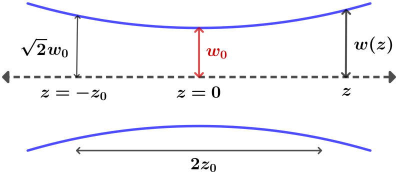


Gaussian beam properties

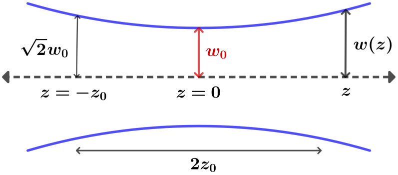
$$\psi(\boldsymbol{r}, z) = A\underbrace{\left(\frac{w_0}{w(z)}\right)}_{\text{term I}} \exp\left(i\tan^{-1}\left(\frac{z}{z_0}\right) - i\frac{kr^2}{2R(z)} - \frac{r^2}{w^2(z)}\right)$$

Term I related to **spreading of beam**.

$w \rightarrow$ Physical radius

$w_0 \rightarrow$ Beam waist

$$w(z) = w_0\sqrt{1 + \left(\frac{z}{z_0}\right)^2}$$



$z_0 \rightarrow$ Rayleigh length

$$z_0 = \frac{\pi w_0^2}{\lambda}$$

Gaussian beam properties

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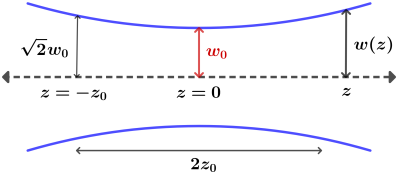

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Term II related to **Gouy phase** (ϕ_G).

$$\phi_G = \tan^{-1}\left(\frac{z}{z_0}\right)$$

Gaussian beam properties

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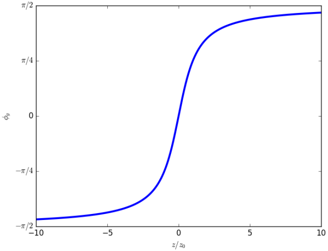

Gaussian beam properties

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Term III related to **radius of curvature** (R) of beam wave-front.

$$R(z) = z\left[1 + \left(\frac{z_0}{z}\right)^2\right]$$

Gaussian beam properties

$$\psi(\boldsymbol{r}, z) = A\left(\frac{w_0}{w(z)}\right)\exp\left(i\tan^{-1}\left(\frac{z}{z_0}\right)\underbrace{-i\frac{kr^2}{2R(z)}}_{\text{term III}}-\frac{r^2}{w^2(z)}\right)$$

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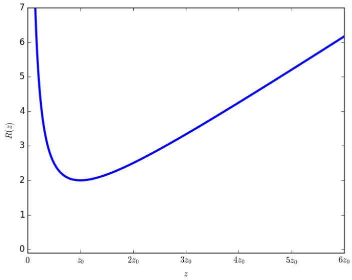

Gaussian beam properties

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Term IV related to **Gaussian intensity profile**.

$$I(r, z) \sim \exp\left(-\frac{2r^2}{w^2(z)}\right)$$

Gaussian beam properties

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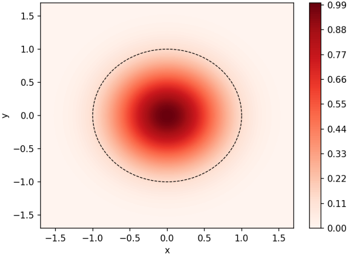

q parameter and beam tracing

$$\psi(\boldsymbol{r}, z) = A \exp\left[-i\left(p(z) + \frac{kr^2}{2\,q(z)}\right)\right]$$

$q(z) \longrightarrow$ characteristic of a beam if λ known.

$$q(z) = z + i\,z_0$$

$$\frac{1}{q(z)} = \frac{1}{R(z)} - i\,\frac{\lambda}{\pi\,w^2(z)}$$

$$q_{in} \longrightarrow \underbrace{\boxed{\begin{bmatrix} A & B \\ C & D \end{bmatrix}}}_{\text{Optical element}} \longrightarrow q_{out} = \frac{Aq_{in} + B}{Cq_{in} + D}$$

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Gaussian Beam and properties

- Paraxial wave
- Gaussian beam solution and properties
- Modes of Gaussian beam

Hermite-Gaussian mode

$$\psi_{m,n}(\boldsymbol{r}, z) = A \left(\frac{w_0}{w(z)}\right) H_m \left(\frac{\sqrt{2}x}{w(z)}\right) H_n \left(\frac{\sqrt{2}y}{w(z)}\right) \exp\left(-\frac{r^2}{w^2(z)}\right) \cdot \exp\left(i\,(m+n+1)\tan^{-1}\left(\frac{z}{z_0}\right) - i\frac{kr^2}{2R(z)}\right)$$

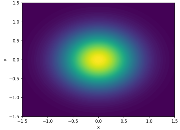
$m, n = 0 \;\Rightarrow\; \psi =$ Gaussian

Hermite-Gaussian mode

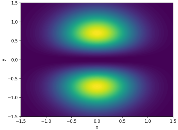
$$\psi_{m,n}(\boldsymbol{r}, z) = A\left(\frac{w_0}{w(z)}\right) H_m\left(\frac{\sqrt{2}x}{w(z)}\right) H_n\left(\frac{\sqrt{2}y}{w(z)}\right) \exp\left(-\frac{r^2}{w^2(z)}\right) \cdot$$
$$\exp\left(i\,(m+n+1)\tan^{-1}\left(\frac{z}{z_0}\right) - i\frac{kr^2}{2R(z)}\right)$$

$m, n = 0 \;\Rightarrow\; \psi =$ Gaussian

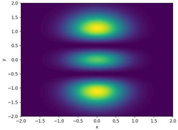
Hermite-Gaussian Intensity profile



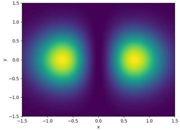
(a) TEM_{00}



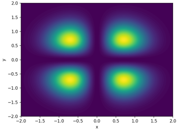
(b) TEM_{01}



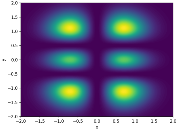
(c) TEM_{02}



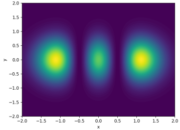
(d) TEM_{10}



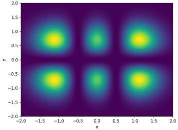
(e) TEM_{11}



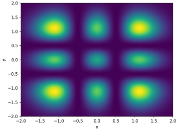
(f) TEM_{12}



(g) TEM_{20}



(h) TEM_{21}



(i) TEM_{22}

Laguerre-Gaussian mode

$$\psi_{p,l}(r,\phi,z) = A\,\frac{w_0}{w(z)}\left[\frac{r\sqrt{2}}{w(z)}\right]^{|l|} L_p^{|l|}\left(\frac{2r^2}{w^2(z)}\right)\exp\left(-\frac{r^2}{w^2(z)}\right)\cdot$$
$$\exp\left(-il\phi + i(2p+l+1)\tan^{-1}\left(\frac{z}{z_0}\right) - i\frac{kr^2}{2R(z)}\right)$$

$l, p = 0 \;\Rightarrow\; \psi =$ Gaussian

$\exp(-il\phi) \longrightarrow$ Helical phase
(carries OAM)

Laguerre-Gaussian mode

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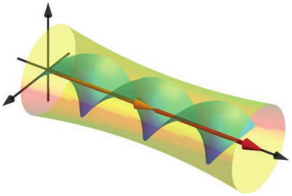
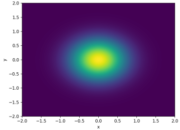
$$\exp\left(-il\phi + i(2p+l+1)\tan^{-1}\left(\frac{z}{z_0}\right) - i\frac{kr^2}{2R(z)}\right)$$

Helical phase front

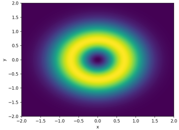


$\exp(-il\phi) \longrightarrow$ Helical phase (carries OAM)

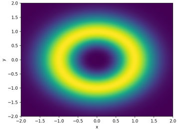
Laguerre-Gaussian Intensity profile



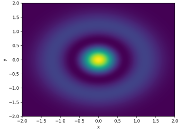
(a) $p = 0, |l| = 0$



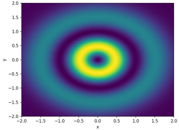
(b) $p = 0, |l| = 1$



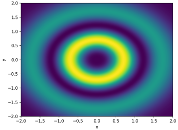
(c) $p = 0, |l| = 2$



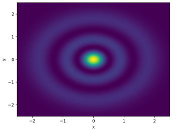
(d) $p = 1, |l| = 0$



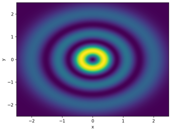
(e) $p = 1, |l| = 1$



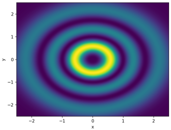
(f) $p = 1, |l| = 2$



(g) $p = 2, |l| = 0$

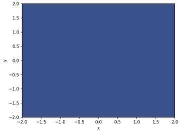


(h) $p = 2, |l| = 1$



(i) $p = 2, |l| = 2$

Laguerre-Gaussian Phase profile



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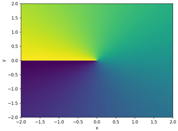
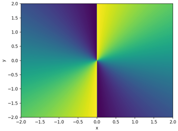
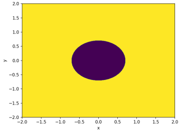


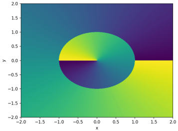
(b) $p = 0, l = 1$



(c) $p = 0, l = 2$



(d) $p = 1, l = 0$



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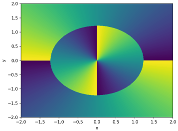

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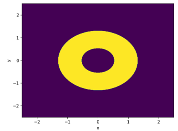
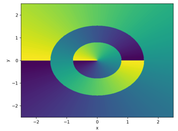
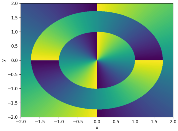


(g) $p = 2, l = 0$



(h) $p = 2, l = 1$



(i) $p = 2, l = 2$