Summer Project

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21MS179

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1.2.1 Jones Vector

Vector form of electric field of fully polarized EM wave propagating along z-axis is given by

$$\mathbf{E}(\mathbf{x},t) = \begin{bmatrix} E_x(\mathbf{x},t) \\ E_y(\mathbf{x},t) \\ E_z(\mathbf{x},t) \end{bmatrix} = \begin{bmatrix} A_x(\mathbf{x})e^{-i(kz-\omega t-\delta_x)} \\ A_y(\mathbf{x})e^{-i(kz-\omega t-\delta_y)} \\ 0 \end{bmatrix} = \begin{bmatrix} A_x(\mathbf{x})e^{i\delta_x} \\ A_y(\mathbf{x})e^{i\delta_y} \\ 0 \end{bmatrix} e^{-i(kz-\omega t)} \tag{1.1}$$

We define normalized[1] *Jones vector* as

$$\mathbf{J}(\mathbf{x},t) = \frac{1}{\sqrt{A_x^2 + A_y^2}} \begin{bmatrix} A_x(\mathbf{x})e^{i\delta_x} \\ A_y(\mathbf{x})e^{i\delta_y} \end{bmatrix} \tag{1.2}$$

Such examples of usual polarization states are given below [3],

Polarization state	$\mathbf{J}$
$\|H\rangle$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$
$\|V\rangle$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
$\|P\rangle$	$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
$\|M\rangle$	$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$
$\|L\rangle$	$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$
$\|R\rangle$	$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}$

Table 1: Jones vector of usual polarization state

Some properties of Jones vector are

1. The intensity of the EM wave is given by

$$I = \frac{1}{2}c\epsilon_0(A_x^2 + A_y^2) = \frac{1}{2}c\epsilon_0(E^*E) \tag{1.3}$$

2. For general elliptically polarized light we can measure the azimuth (α) ellipticity (ϵ) of the polarization ellipse by comparing Jones vector $\mathbf{J}$ with [1]

$$\begin{bmatrix} \cos\alpha\cos\epsilon - i\sin\alpha\sin\epsilon \\ \sin\alpha\cos\epsilon - i\cos\alpha\sin\epsilon \end{bmatrix}$$

[1] normalized as $\mathbf{J}\,\mathbf{J}^* = 1$

1.2.2 Jones Matrix & evolution of Jones vector

Jones matrix is a 2×2 matrix assigned for a particular optical element. Let $\mathbf{M}$ be the Jones matrix for an optical element *s.t.*

$$\mathbf{M} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$$

then if a polarized light of Jones vector $\mathbf{J}_{in}$ passes through that optical element then the Jones vector of output light is given by

$$\mathbf{J}_{out} = \mathbf{M}\,\mathbf{J}_{in} \tag{1.4}$$

$$\Rightarrow \mathbf{E}_{out} = \mathbf{M}\,\mathbf{E}_{in} \tag{1.5}$$

To determine m_{ij} in $\mathbf{M}$,

1. Pass x-polarized light and determine $\mathbf{J}_{out}$, then

$$\mathbf{J}_{out} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} m_{11} \\ m_{21} \end{bmatrix}$$

2. Pass y-polarized light and determine $\mathbf{J}_{out}$, then

$$\mathbf{J}_{out} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} m_{12} \\ m_{22} \end{bmatrix}$$

Such examples of usual Jones matrix [2] are given below,[3]

Optical element	$\mathbf{M}$
Free space	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
x-Polariser	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
y-Polariser	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$
Right circular polariser	$\frac{1}{2}\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$
Left circular polariser	$\frac{1}{2}\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}$
Linear di-attenuator	$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$
Half-wave plate with fast axis horizontal	$e^{-i\pi/2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Quarter-wave plate with fast axis horizontal	$e^{-i\pi/4}\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$
General phase retarder	$\begin{bmatrix} e^{i\phi_x} & 0 \\ 0 & e^{i\phi_y} \end{bmatrix}$

Table 2: Jones matrix related to usual optical element

[2]For polariser the Jones matrix $\mathbf{M} = \mathbf{J}\,\mathbf{J}^*$ where $\mathbf{J}$ is normalized Jones vector corresponding polarization state *s.t.* $\mathbf{J}_{out} = \mathbf{MJ} = (\mathbf{JJ}^*)\mathbf{J} = \mathbf{J}(\mathbf{J}^*\mathbf{J}) = \mathbf{J}$

Some properties of Jones matrix are

1. Resultant Jones matrix for composition of n optical elements is given by
$$\mathbf{M} = \mathbf{M}_1\, \mathbf{M}_2 \ldots \mathbf{M}_n \tag{1.6}$$

2. For an optical element when its optical axis aligned at an angle θ *w.r.t.* x-axis then resultant Jones matrix for this rotated optical element is given by
$$\mathbf{M}_\theta = R(-\theta)\, \mathbf{M}\, R(\theta) \tag{1.7}$$
where $R(\theta)$ is passive rotation matrix *s.t.*
$$R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{1.8}$$

1.2.3 Drawback of Jones formalism

Main drawback of Jones formalism is that its application is restricted in fully polarized light. This formalism cannot explain the partially polarized or unpolished light which we frequently observe in practical use.

1.3 Stokes-Muller formalism

1.3.1 Coherency matrix

Coherency matrix of a EM wave is defined as [1]
$$\mathbf{C} = \left\langle \mathbf{E} \otimes \mathbf{E}^\dagger \right\rangle = \left\langle \mathbf{E}\mathbf{E}^\dagger \right\rangle = \begin{bmatrix} \langle E_x E_x^* \rangle & \langle E_x E_y^* \rangle \\ \langle E_y E_x^* \rangle & \langle E_y E_y^* \rangle \end{bmatrix} = \begin{bmatrix} c_{xx} & c_{xy} \\ c_{yx} & c_{yy} \end{bmatrix} \tag{1.9}$$
where $\otimes$ denotes Kronecker product, $\langle\cdot\rangle$ denotes the temporal avg of the corresponding quantity and $\delta = \delta_y - \delta_x$.

Examples of coherency matrix of usual polarization states are given below [4],

Polarization state	$\mathbf{J}$	$\mathbf{C}$
$\lvert H\rangle$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
$\lvert V\rangle$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$
$\lvert P\rangle$	$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$
$\lvert M\rangle$	$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$
$\lvert L\rangle$	$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}$
$\lvert R\rangle$	$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$
Un-polarized	–	$\frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Table 3: Coherency matrix of usual polarization state

Some properties of coherency matrix are

1. It is a hermitian matrix *i.e.* $\mathbf{C} = \mathbf{C}^\dagger$
2. Trace and determinant off the matrix are non-negative[3] *i.e.* $\text{tr}(\mathbf{C}) > 0$ & $\det(\mathbf{C}) \geq 0$.
3. $\text{Tr}(\mathbf{C}) = \langle E_x E_x^* \rangle + \left\langle E_y E_y^* \right\rangle$ is the time averaged intensity of input light.
4. let the polarized light (of electric field $\mathbf{E}_{in}$ & coherency matrix $\mathbf{C}_{in}$) passes through an optical element (of Jones matrix $\mathbf{M}$) then let output electric field be $\mathbf{E}_{out}$ by the equation 1.5, then output coherency matrix $\mathbf{C}_{out}$ is given by

$$\begin{aligned} \mathbf{C}_{out} = \left\langle \mathbf{E}_{out} \mathbf{E}_{out}^\dagger \right\rangle &= \left\langle (\mathbf{M}\mathbf{E}_{in}) (\mathbf{M}\mathbf{E}_{in})^\dagger \right\rangle \\ &= \left\langle (\mathbf{M}\mathbf{E}_{in}) \left(\mathbf{E}_{in}^\dagger \mathbf{M}^\dagger\right) \right\rangle \\ &= \mathbf{M} \left\langle \mathbf{E}_{in} \mathbf{E}_{in}^\dagger \right\rangle \mathbf{M}^\dagger \\ &= \mathbf{M}\, \mathbf{C}_{in}\, \mathbf{M}^\dagger \end{aligned} \tag{1.10}$$

1.3.2 Stokes parameters and Stokes vector

Now we see that coherency matrix $\mathbf{C}$ of any polarization state in table 3 can be written in the linear combination of the 4 basis given below [5]

$$\beta = \left\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\boldsymbol{V_0}}, \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{\boldsymbol{V_1}}, \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\boldsymbol{V_2}}, \underbrace{\begin{bmatrix} 0 & i \\ -i & 1 \end{bmatrix}}_{\boldsymbol{V_3}} \right\} \tag{1.11}$$

Now we can write any coherency matrix $\mathbf{C}$ as

$$\mathbf{C} = \frac{1}{2} \sum_{i=0}^{3} S_i \boldsymbol{V_i} \tag{1.12}$$

Note that all V_i's are Hermitian, so obviously is $\mathbf{C}$.

We call $\{S_0, S_1, S_2, S_3\}$ as a *Stokes parameter* and the values of S_i's are experimentally measurable.

A *Stokes vector* $\mathbf{S}$ is defined as[4]

$$\mathbf{S} = \begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} \tag{1.13}$$

Examples of Stokes vector for different polarization states are given below

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[3] Proofs to be done

[4] for intensity normalised Stokes vector, $\mathbf{S} = \begin{bmatrix} 1 & s_1 & s_2 & s_3 \end{bmatrix}$ where $s_i = S_i/S_0$

Polarization state	C	S
$\|H\rangle$	$\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$	$\begin{bmatrix}1 & 1 & 0 & 0\end{bmatrix}^T$
$\|V\rangle$	$\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$	$\begin{bmatrix}1 & -1 & 0 & 0\end{bmatrix}^T$
$\|P\rangle$	$\frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$	$\begin{bmatrix}1 & 0 & 1 & 0\end{bmatrix}^T$
$\|M\rangle$	$\frac{1}{2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$	$\begin{bmatrix}1 & 0 & -1 & 0\end{bmatrix}^T$
$\|L\rangle$	$\frac{1}{2}\begin{bmatrix}1 & -i\\ i & 1\end{bmatrix}$	$\begin{bmatrix}1 & 0 & 0 & 1\end{bmatrix}^T$
$\|R\rangle$	$\frac{1}{2}\begin{bmatrix}1 & i\\ -i & 1\end{bmatrix}$	$\begin{bmatrix}1 & 0 & 0 & -1\end{bmatrix}^T$
Un-polarized	$\frac{1}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$	$\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}^T$

Table 4: Stokes vector of usual polarization state

Note that all Jones vectors has Stokes vectors but converse need not to be true.

Now we see from the equation 1.12

$$\begin{bmatrix}\langle E_x E_x^*\rangle & \left\langle E_x E_y^*\right\rangle \\ \langle E_y E_x^*\rangle & \left\langle E_y E_y^*\right\rangle\end{bmatrix} = \mathbf{C} = \frac{1}{2}\sum_{i=0}^{3} S_i \boldsymbol{V_i} = \frac{1}{2}\begin{bmatrix} S_0 + S_1 & S_2 + iS_3 \\ S_2 - iS_3 & S_0 - S_1\end{bmatrix} \tag{1.14}$$

From there we can write

$$\boldsymbol{S} = \begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = \begin{bmatrix} \langle E_x E_x^*\rangle + \left\langle E_x E_y^*\right\rangle \\ \langle E_x E_x^*\rangle - \left\langle E_x E_y^*\right\rangle \\ \langle E_y E_x^*\rangle + \left\langle E_x E_y^*\right\rangle \\ i\left(\langle E_y E_x^*\rangle - \left\langle E_x E_y^*\right\rangle\right) \end{bmatrix} \tag{1.15}$$

Now for a polarized light,

$$\mathbf{C} = \begin{bmatrix} \langle A_x^2\rangle & \left\langle A_x A_y e^{-i\delta}\right\rangle \\ \left\langle A_x A_y e^{i\delta}\right\rangle & \langle A_y^2\rangle \end{bmatrix} \text{ and } \boldsymbol{S} = \begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = \begin{bmatrix} \left\langle A_x^2 + A_y^2\right\rangle \\ \left\langle A_x^2 - A_y^2\right\rangle \\ \langle 2A_x A_y \cos\delta\rangle \\ \langle 2A_x A_y \sin\delta\rangle \end{bmatrix} \tag{1.16}$$

1.3.3 Measurement of Stokes parameters

To measure the 4 Stokes parameter of EM wave associated with, we have to do 4 steps experiment. In each case, we pass the light through various optical elements and measure the (time-averaged) intensity [6],

Step I Pass the light through homogenous isotropic medium (or, free space) and measure the intensity. From table 2 and eq. 1.10, we get,

$$\begin{aligned}\mathbf{C}_{out} &= \mathbf{M}\,\mathbf{C}_{in}\,\mathbf{M}^{\dagger} \\ &= \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\frac{1}{2}\begin{bmatrix}S_0+S_1 & S_2+iS_3\\S_2-iS_3 & S_0-S_1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix}S_0+S_1 & 0\\0 & S_0-S_1\end{bmatrix}\end{aligned} \tag{1.17}$$

So the measured intensity will be

$$I_0 = \mathrm{tr}(\mathbf{C}_{out}) = S_0 \tag{1.18}$$

Step II Pass the light through x-polariser and measure the intensity. From table 2 and eq. 1.10, we get,

$$\begin{aligned}\mathbf{C}_{out} &= \mathbf{M}\,\mathbf{C}_{in}\,\mathbf{M}^{\dagger} \\ &= \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\frac{1}{2}\begin{bmatrix}S_0+S_1 & S_2+iS_3\\S_2-iS_3 & S_0-S_1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix}S_0+S_1 & 0\\0 & 0\end{bmatrix}\end{aligned} \tag{1.19}$$

So the measured intensity will be

$$I_1 = \mathrm{tr}(\mathbf{C}_{out}) = \frac{1}{2}(S_0+S_1) \tag{1.20}$$

Step III Pass the light through the polariser with transmission axis is at 45° and measure the intensity. Then from eq. 1.7, **M** for this polariser will be

$$\mathbf{M} = R(-45^\circ)\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}R(45^\circ) = \frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} \tag{1.21}$$

From eq. 1.10, we get,

$$\begin{aligned}\mathbf{C}_{out} &= \mathbf{M}\,\mathbf{C}_{in}\,\mathbf{M}^{\dagger} \\ &= \mathbf{M}\,\mathbf{C}_{in}\,\mathbf{M}^{\dagger} \\ &= \frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\frac{1}{2}\begin{bmatrix}S_0+S_1 & S_2+iS_3\\S_2-iS_3 & S_0-S_1\end{bmatrix}\frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} \\ &= \frac{1}{4}\begin{bmatrix}S_0+S_2 & S_0+S_2\\S_0+S_2 & S_0+S_2\end{bmatrix}\end{aligned} \tag{1.22}$$

So the measured intensity will be

$$I_1 = \mathrm{tr}(\mathbf{C}_{out}) = \frac{1}{2}(S_0+S_2) \tag{1.23}$$

Step IV Pass the light through right circular polariser and measure the intensity. From table 2 and eq. 1.10, we get,

$$\begin{aligned}\mathbf{C}_{out} &= \mathbf{M}\,\mathbf{C}_{in}\,\mathbf{M}^{\dagger} \\ &= \frac{1}{2}\begin{bmatrix}1 & i\\-i & 1\end{bmatrix}\frac{1}{2}\begin{bmatrix}S_0+S_1 & S_2+iS_3\\S_2-iS_3 & S_0-S_1\end{bmatrix}\frac{1}{2}\begin{bmatrix}1 & i\\-i & 1\end{bmatrix}\end{aligned} \tag{1.24}$$

So the measured intensity will be

$$I_1 = \mathrm{tr}(\mathbf{C}_{out}) = \frac{1}{2}(S_0 + S_3) \tag{1.25}$$

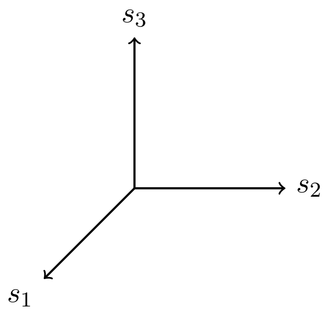
From the equations 1.18, 1.20, 1.23 and 1.25, we can get the values of all S_i's.

1.3.4 Poincare sphere representation

For total intensity normalised Stokes vector is $\mathbf{S} = \begin{bmatrix} 1 & s_1 & s_2 & s_3 \end{bmatrix}$ where $s_i = S_i/S_0$. Observe that $\mathbf{S}$ is a 3-dimensional quantity. Therefore we can write,

$$\begin{bmatrix} 1 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} \rightarrow \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}$$

Poincare sphere representation is a coordinate system to define the state of polarization of light where the mutually orthogonal coordinate axes are $\{s_1, s_2, s_3\}$.



Example of special cases are

Case I For fully polarized light

$$s_1 = \frac{A_x^2 - A_y^2}{A_x^2 + A_y^2} \tag{1.26}$$

$$s_2 = \frac{2A_yA_y \cos\delta}{A_x^2 + A_y^2} \tag{1.27}$$

$$s_3 = \frac{2A_yA_y \sin\delta}{A_x^2 + A_y^2} \tag{1.28}$$

from there we can see

$$s_1^2 + s_2^2 + s_3^2 = 1 \tag{1.29}$$

which implies that fully polarized has the locus at any point in the sphere of radius 1 in Poincare sphere representation.

Case II For fully un-polarized light

$$s_1 = s_2 = s_3 = 0 \tag{1.30}$$

which implies that fully un-polarized has the locus at any the centre $(0, 0, 0)$ in the sphere of radius 1 in Poincare sphere representation.

1.3.5 Degree of Polarization

Degree of Polarization is the measure of polarization of light.
We define

- Total degree of polarization, $DOP = \sqrt{s_1^2 + s_2^2 + s_3^2}$
- Degree of linear polarization $= \sqrt{s_1^2 + s_2^2}$
- Degree of circular polarization $= \sqrt{s_1^2 + s_2^2 + s_3^2}$

For any mixed polarization state we can decompose the Stokes vector into fullu polarized and un polarized components,

$$\begin{bmatrix} 1 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \underbrace{\begin{bmatrix} \sqrt{s_1^2 + s_2^2 + s_3^2} \\ s_1 \\ s_2 \\ s_3 \end{bmatrix}}_{\text{fully polarized, } DOP=1} + \underbrace{\begin{bmatrix} 1 - \sqrt{s_1^2 + s_2^2 + s_3^2} \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\text{un-polarized}} \tag{1.31}$$

1.3.6 Muller Matrix & evolution of Stokes vector

Similar to the Jones matrix, *Muller matrix* is a 4×4 matrix assigned for a particular optical element. Let $\mathfrak{M}$ be the Muller matrix for an optical element *s.t.*

$$\mathfrak{M} = \begin{bmatrix} \mu_{11} & \cdots & \mu_{14} \\ \vdots & \ddots & \vdots \\ \mu_{41} & \cdots & \mu_{44} \end{bmatrix}$$

then if a light of Stokes vector $\mathbf{S}_{in}$ passes through that optical element, then the Stokes vector of output light is given by

$$\mathbf{S}_{out} = \mathfrak{M}\,\mathbf{S}_{in} \tag{1.32}$$

Some properties of Jones matrix are

1. Resultant Muller matrix for composition of n optical elements is given by

$$\mathfrak{M} = \mathfrak{M}_1\,\mathfrak{M}_2 \ldots \mathfrak{M} \tag{1.33}$$

2. When th optical element is aligned at an angle θ *w.r.t.* x-axis then resultant Muller matrix (similar to Jones matrix) for this rotated optical element is given by

$$\mathbf{M}_\theta = T^(- 1)(\theta)\,\mathbf{M}\,T(\theta) \tag{1.34}$$

where $T(\theta)$ is passive rotation matrix in Poincare sphere representation *w.r.t* s_3 axis, *s.t.*

$$T(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\theta & \sin 2\theta & 0 \\ 0 & -\sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1.35}$$

Note that, in eq. 1.35, if we write

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\theta & \sin 2\theta & 0 \\ 0 & -\sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} \cos 2\theta & \sin 2\theta & 0 \\ -\sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.36}$$

we see that it is proper rotation matrix of rotation angle 2θ in Poincare sphere *w.r.t* s_3 axis. And as we know that rotation of θ of electric field results in rotation of 2θ in azimuth angle of Stokes vector in Poincare sphere.

1.3.7 Relationship between Jones & Stokes-Muller formalism

Let, $\mathbf{J}$ be jones vector, $\mathbf{M}$ be the Jones matrix, $\mathbf{S}$ be the Stokes vector and $\mathfrak{M}$ be the Muller matrix *s.t.* equations 1.4 and 1.32 is satisfied.

Let us define *coherency vector* of 1.9 as

$$\mathbf{L} = \begin{bmatrix} c_{xx} & c_{xy} & c_{yx} & c_{yy} \end{bmatrix}^T \tag{1.37}$$

and *Wolf matrix* $\mathbf{W}$ as

$$\mathbf{L}_{out} = \mathbf{W}\,\mathbf{L}_{in} \tag{1.38}$$

then the relation between Jones and Wolf matrix[5] is

$$\mathbf{W} = \mathbf{J} \otimes \mathbf{J}^* \tag{1.39}$$

Now from equations 1.9 and 1.15, one can write

$$\mathbf{S} = \mathbf{A}\,\mathbf{L} \tag{1.40}$$

where,

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & -i & i & 0 \end{bmatrix} \tag{1.41}$$

then the relation between Jones and Muller matrix[6] is

$$\mathfrak{M} = \mathbf{A}\left(\mathbf{J} \otimes \mathbf{J}^*\right)\mathbf{A}^{-1} \tag{1.42}$$

Note that this relationship is only possible in both ways, if the light is fully polarized light as all Jones vectors has Stokes vectors but converse need not to be true.

[5] proof to be done

[6] proof to be done

1.4 More on Elliptically polarized light

1.4.1 Jones vector of elliptically polarized light

In this section we will discuss the generalized polarization ellipse of an EM wave. Let our electric field vector of EM wave is given by

$$\mathbf{E} = \begin{bmatrix} E_x \\ E_y \end{bmatrix} = \begin{bmatrix} a_1 \cos(\tau + \delta_1) \\ a_2 \cos(\tau + \delta_2) \end{bmatrix} \text{ where } \tau = kz - \omega \text{ and } ta_1, a_2 \geq 0 \tag{1.43}$$

by eliminating τ we get,

$$\frac{1}{a_1^2} E_x^2 + \frac{1}{a_2^2} E_y^2 - \frac{2\cos\delta}{a_1 a_2} E_x E_y = \sin^2(\delta) \tag{1.44}$$

where $\delta = \delta_2 - \delta_1$. The eq. 1.44 is equation of circle when $a_1 = a_2$, otherwise, of ellipse[2].

Now we do the change of basis $\{E_x, E_y\} \longmapsto \{E_\xi, E_\eta\}$ (See fig. 1) *s.t.* electric field in $\{E_\xi, E_\eta\}$ basis be

$$\begin{aligned} \mathbf{E} \rightarrow & \mathbf{F} \\ \mathbf{F} = & \begin{bmatrix} E_\xi \\ F_\chi \end{bmatrix} = \begin{bmatrix} a\cos(\tau + \delta_0) \\ \pm b \cos(\tau + \delta_0) \end{bmatrix} \text{ where } a \geq b \geq 0 \end{aligned} \tag{1.45}$$

which is parametric form of canonical ellipse[7] in $\{E_\xi, E_\eta\}$ basis.

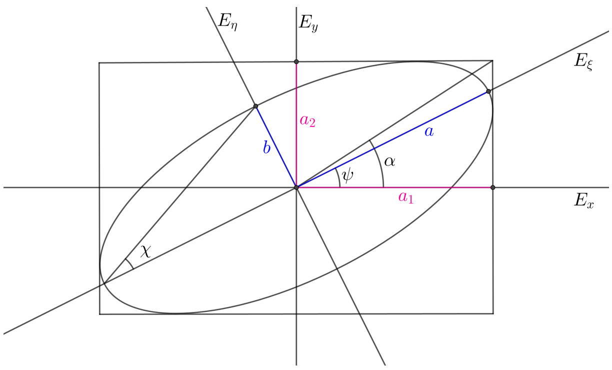


Figure 1: Polarization ellipse

Let ψ be the azimuth angle of the ellipse then

$$\mathbf{F} = R(\psi)\,\mathbf{E} \tag{1.46}$$

$$\Rightarrow \begin{bmatrix} E_\xi \\ F_\chi \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{bmatrix} \begin{bmatrix} E_x \\ F_y \end{bmatrix} \tag{1.47}$$

$$\Rightarrow \begin{bmatrix} a\cos(\tau + \delta_0) \\ \pm b\cos(\tau + \delta_0) \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{bmatrix} \begin{bmatrix} a_1 \cos(\tau + \delta_1) \\ a_2 \cos(\tau + \delta_2) \end{bmatrix} \tag{1.48}$$

[7] $\pm$ before b denotes the handedness of the rotation of electric field vector in transverse plane.

We want value of a, b, After some tedious calculation [2], we reach to some important results, given below

$$a^2 + b^2 = a_1^2 + a_2^2 \tag{1.49}$$

$$\pm\, ab = a_1 a_2 \sin\delta \tag{1.50}$$

$$\tan 2\chi := \pm\frac{b}{a} \text{ where } \chi \in [-\frac{\pi}{4}, \frac{\pi}{4}] \tag{1.51}$$

$$\tan 2\alpha := \frac{a_2}{a_1} \text{ where } \alpha \in [0, \frac{\pi}{2}] \tag{1.52}$$

$$\tan 2\psi = \tan 2\alpha \sin\delta \tag{1.53}$$

$$\tan 2\chi = \sin 2\alpha \sin\delta \tag{1.54}$$

where ψ is the *azimuth* and χ is *ellipticity* of the polarization ellipse.

To see the handedness of the rotation of electric field vector in transverse plane,

Case I For right-handed polarization, $\sin\delta > 0$, then from equations 1.50, and 1.51, we can say

$$\tan 2\chi \geq 0 \Rightarrow \chi \in \left(0, \frac{\pi}{4}\right]$$

Case II Similarly for left-handed polarization, $\sin\delta < 0$, then from equations 1.50, and 1.51, we can say

$$\tan 2\chi \leq 0 \Rightarrow \chi \in \left[\frac{\pi}{4}, 0\right)$$

Now lets calculate the Jones vector of elliptical polarisation,

1.4.2 Stokes vector and corresponding Poincare representation

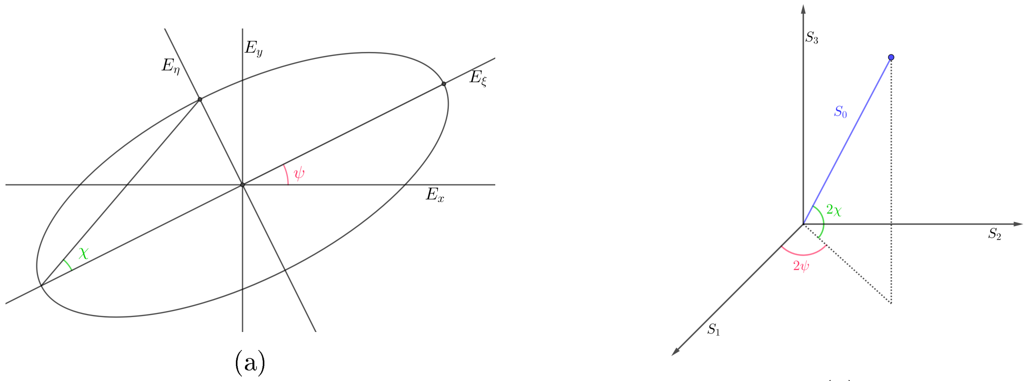
From the eq. 1.16, we can write for our case,

$$\boldsymbol{S} = \begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = \begin{bmatrix} a_1^2 + a_2^2 \\ a_1^2 - a_2^2 \\ 2a_1 a_2 \cos\delta \\ 2a_1 a_2 \sin\delta \end{bmatrix} = S_0 \begin{bmatrix} 1 \\ \cos 2\chi \cos 2\psi \\ \cos 2\chi \sin 2\psi \\ \sin 2\chi \end{bmatrix} \tag{1.55}$$

So in Poincare sphere representation with axes S_1, S_2, S_3, the required vector is

$$S_0 \begin{bmatrix} 1 \\ \cos 2\chi \cos 2\psi \\ \cos 2\chi \sin 2\psi \\ \sin 2\chi \end{bmatrix} \longrightarrow (S_0 \cos 2\chi \cos 2\psi, S_0 \cos 2\chi \sin 2\psi, S_0 \sin 2\chi) \tag{1.56}$$

The evolution of azimuth (ψ) and ellipticity (χ) of the polarization state in Poincare representation is shown in the figure 2.



(a)

(b)

Figure 2: polarisation ellipse and corresponding Poincare representation

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2 GAUSSIAN BEAM

2.1 Introduction

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2.2 Parallax wave equaion and solutions

2.2.1 Scalar wave solution (without polrisation)

2.2.2 Vector wave solution (with polrisation)

2.3 Gaussian Beam properties

2.4 Differenrt modes of Gaussian beams

2.5 Relationship between 1st order LG & HG beam

3 SPIN-ORBIT INTERACTION

3.1 Introduction

soi

3.2 Angular momentum of Light

3.3 Orbital Angular Momentum (OAM)

3.3.1 Intrinsic vs Extrinsic OAM

3.3.2 OAM of LG Beam

3.4 Spin Angular Momentum (SAM)

3.5 Spin orbit energy

3.6 Geometric phase of light

3.6.1 Spin redirection Berry phase

3.6.2 Pancharatnam-Berry Phase

3.6.3 LG-HG Mode transformation

3.7 Types of SOI

3.8 SOI in inhomogeneous anisotropic medium