Summer Project 2023

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21MS179

Contents

Listings

1 GAUSSIAN and ITS FOURIER TRANSFORM

1.1 Standard normal

Standard normal curve is

$$f(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$$

```python
import matplotlib.pyplot as plt
import numpy as np
plt.style.use("classic")

def f(t):
    return np.exp(-(t)**2/2)/(np.sqrt(2*np.pi))

xv = np.linspace(-7,7,1000)
yv = f(xv)

plt.plot(xv, yv, lw=1)
plt.xlabel("$t$")
plt.ylabel("$f(t)$")
plt.grid(True)
```

Listing 1: Standard Normal curve

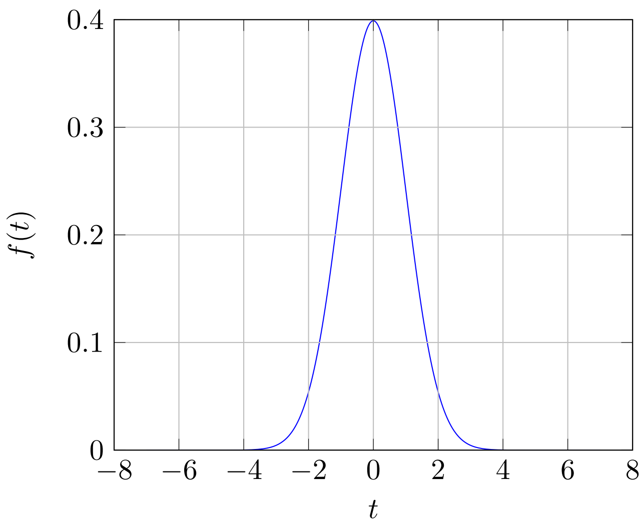


Figure 1: Standard Normal curve

1.2 Fourier transform of Standard normal

If $f(t) = \int_{-\infty}^{\infty} g(\omega) e^{i\omega t} d\omega$, then fourier transform of that is $g(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$. Here, $f(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$

$$
\begin{aligned}
g(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\cos(\omega t)dt - i\frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\sin(\omega t)dt
\end{aligned}
$$

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import quad

def f(t):
    return np.exp(-(t)**2/2)/(np.sqrt(2*np.pi))

def ft(y):
    int_re = lambda t: f(t)*np.cos(y*t)
    int_im = lambda t: f(t)*np.sin(y*t)
    g_re = quad(int_re,-np.inf,np.inf)[0]/(2*np.pi)
    g_im = quad(int_im,-np.inf,np.inf)[0]/(2*np.pi)
    return g_re - 1j*g_im
g = np.frompyfunc(ft, 1, 1)

xv = np.linspace(-7,7,1000)
yv = np.abs(g(xv))
plt.plot(xv, yv, lw=1)
plt.xlabel("$\omega$")
plt.ylabel("$abs(g(\omega))$")
plt.grid(True)
```

Listing 2: Fourier transform of Standard normal

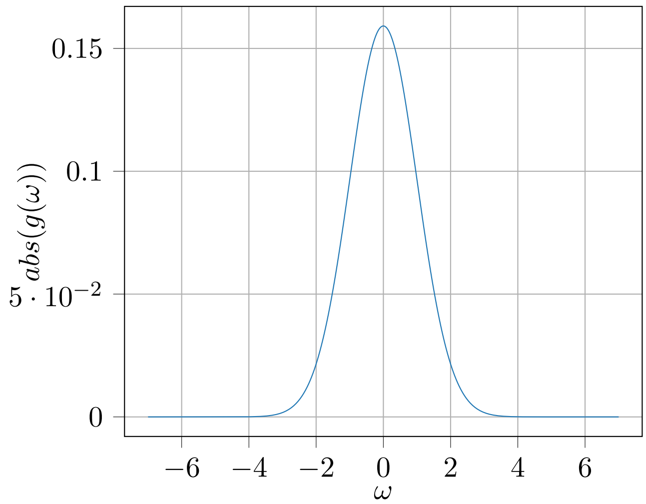


Figure 2: Fourier transform of Standard Normal

Mathematically,

$$
\begin{aligned}
g(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt = \frac{1}{2\pi}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-t^2/2}e^{-i\omega t}dt \\
&= \frac{1}{2\pi}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-t^2/2-i\omega t}dt = \frac{1}{2\pi}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(t^2/2+i\omega t)}dt \\
&= \frac{1}{2\pi}\frac{1}{\sqrt{2\pi}}e^{-\omega^2/2}\int_{-\infty}^{\infty} e^{-(t/\sqrt{2}+i\omega/\sqrt{2})^2}dt \\
&= \frac{1}{2\pi}\frac{1}{\sqrt{\pi}}e^{-\omega^2/2}\int_{-\infty}^{\infty} e^{-\zeta^2}d\zeta \;\; [\text{ substitute } \zeta = t/\sqrt{2}+i\omega/\sqrt{2}\,] \\
&= \frac{1}{2\pi}e^{-\omega^2/2} \;\; [\text{ as } \int_{-\infty}^{\infty} e^{-\zeta^2}d\zeta = \sqrt{\pi} \;\;]
\end{aligned}
$$

2 GAUSSIAN BEAMS

2.1 Solutions of the paraxial wave equation

$$E(r,z,t) = \psi(r,z)e^{i\omega t}$$
$$\psi(r,z) = A\frac{w_0}{w(z)}\exp\left[\frac{-r^2}{w^2(z)}\right]\exp\left[i\left(kz - \arctan(\frac{z}{z_0}) + \frac{kr^2}{2R(z)}\right)\right]$$

where,

$$w(z) = w_0\sqrt{1+z^2/z_02}$$
$$z_0 = \pi w_0^2/\lambda$$
$$R(z) = z + \frac{z_0^2}{z}$$

2.2 Intensity profile

Intensity of Gaussian beam is given by,

$$I(x,y,z) = \frac{c\epsilon}{2}|A|^2\left[\frac{w_0}{w(z)}\right]^2\exp\left(-2(x^2+y^2)/w^2(z)\right)$$

```
import numpy as np
import matplotlib.pyplot as plt

# define intensity at z=0, z_0=1
R=lambda r: np.exp(-2*r**2)
a1=np.linspace(-1.7,1.7,200)
xv,yv=np.meshgrid(a1,a1)
zv=R(np.sqrt(xv**2+yv**2))
plt.contourf(xv,yv,zv,levels=100,cmap='viridis')
plt.xlabel("x")
plt.ylabel("y")
plt.colorbar()

theta=np.linspace(0,2*np.pi,500)
x=np.cos(theta)
y=np.sin(theta)
plt.plot( x, y, "--", lw=0.8)
```

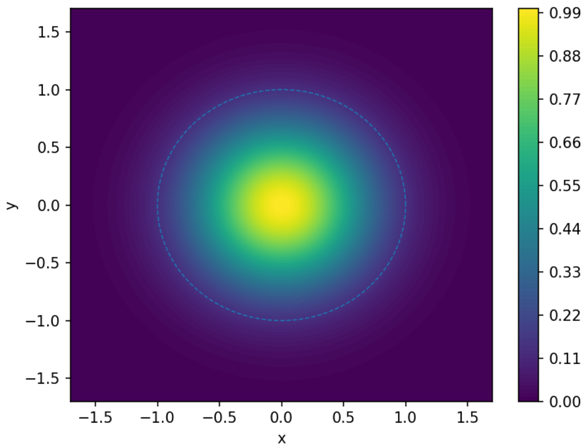


Figure 3: Intensity variation in a cross section $(z = 0, z_0 = 1, w_0 = 1)$

```python
import numpy as np
import matplotlib.pyplot as plt

# define intensity at z_0=1
R=lambda x,z: np.exp(-2*x**2/(1+z**2))/(1+z**2)
a1=np.linspace(-1.5,1.5,500)
a2=np.linspace(-3,3,500)
xv,zv=np.meshgrid(a2,a1)
I=R(xv,zv)
plt.contourf(zv,xv,I,levels=100,cmap='viridis')
plt.colorbar()
plt.xlabel("z")
plt.ylabel("x-y plane")

w=lambda z1: np.sqrt(1+z1**2)
plt.plot(zv,w(zv),"--",lw=0.7)
plt.plot(zv,-w(zv),"--",lw=0.7)
```

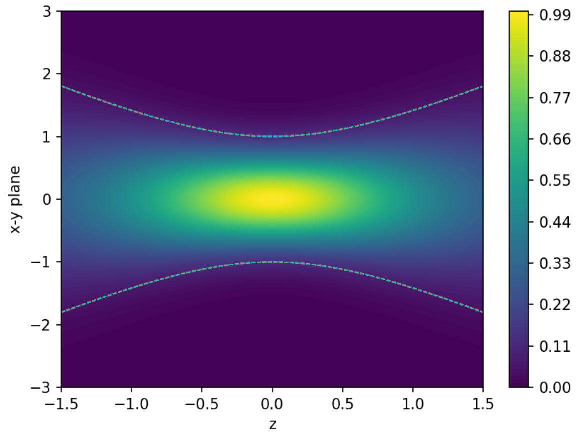


Figure 4: Intensity variation along z $(z_0 = 1, w_0 = 1)$

3 HERMITE–GAUSSIAN BEAMS

3.1 Solutions of the paraxial wave equation

$$
\begin{aligned}
E(x,y,z,t) =& \psi_(x,y,z)e^{i\omega t} \\
\psi(x,y,z) =& A\frac{w_0}{w(z)} H_m\left(\sqrt{2}x/w(z)\right) H_n\left(\sqrt{2}y/w(z)\right) \exp\left[\frac{-(x^2+y^2)}{w^2(z)}\right] \\
& \exp\left[i\left(kz-(m+n+1)\arctan(\frac{z}{z_0})+\frac{k(x^2+y^2)}{2R(z)}\right)\right]
\end{aligned}
$$

where,

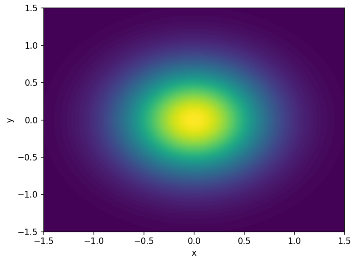
$$
\begin{aligned}
w(z) &= w_0\sqrt{1+z^2/z_0 2} \\
z_0 &= \pi w_0^2/\lambda \\
R(z) &= z+\frac{z_0^2}{z} \\
H_n &= \text{n-th order Hermite polynomial}
\end{aligned}
$$

3.2 Intensity profile

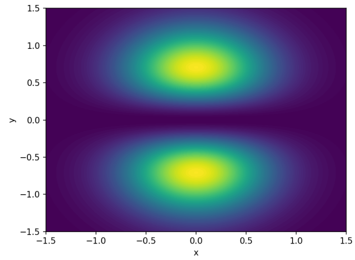
Intensity of Hermite-Gaussian beam is given by,

$$
I_{m,n}(x,y,z) = \frac{c\epsilon}{2}|A|^2\left[H_m(\sqrt{2}x/w(z))\right]^2\left[H_n(\sqrt{2}y/w(z))\right]^2\exp\left(-2(x^2+y^2)/w^2(z)\right)
$$

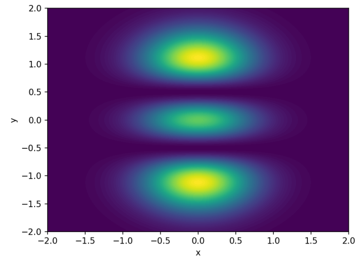
```python
import numpy as np
import matplotlib.pyplot as plt
from scipy import special

H=lambda n: special.hermite(n, monic=False)

m=int(input("m="))
n=int(input("n="))

# define intensity at z=0, z_0=1
intensity=lambda m,n,x,y: (H(m)(np.sqrt(2)*x)*H(n)(np.sqrt(2)*y))**2*np.exp(-2*(
    x**2+y**2))

l=2
a1=np.linspace(-l,l,200)
xv,yv=np.meshgrid(a1,a1)
zv=intensity(m,n,xv,yv)
plt.contourf(xv,yv,zv,levels=100,cmap='viridis')
plt.xlabel("x")
plt.ylabel("y")
#plt.colorbar()
```



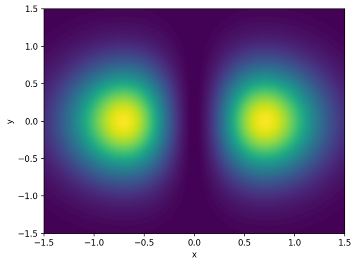
(a) TEM_{00}



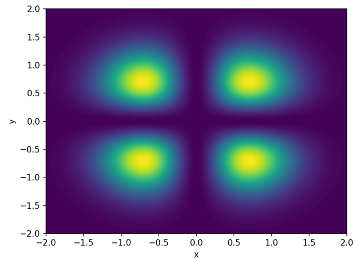
(b) TEM_{01}



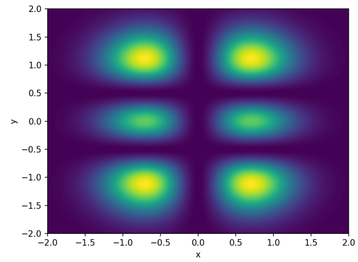
(c) TEM_{02}



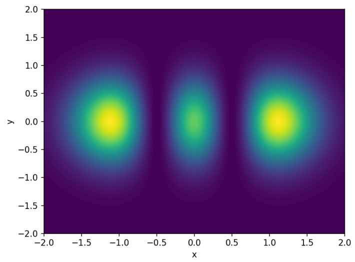
(d) TEM_{10}



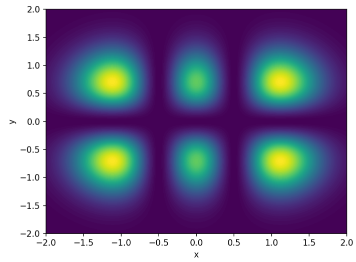
(e) TEM_{11}



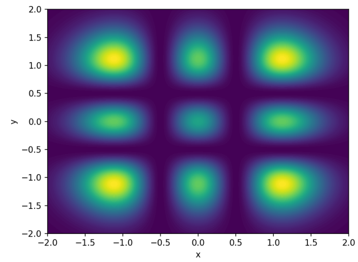
(f) TEM_{12}



(g) TEM_{20}



(h) TEM_{21}



(i) TEM_{22}

Figure 5: Intensity variation for different TEM in a cross section $(z = 0, z_0 = 1, w_0 = 1)$

4 LAGUERRE–GAUSSIAN BEAMS

4.1 Solutions of the paraxial wave equation

$$
\begin{aligned}
E(r,z,t) =& \psi_{p,l}(r,z)e^{i\omega t} \\
\psi_{p,l}(r,z) =& A\frac{w_0}{w(z)}\left[\frac{r\sqrt{2}}{w(z)}\right]^{|l|} L_p^{|l|}\left(\frac{2r^2}{w^2(z)}\right)\exp\left[\frac{-r^2}{w^2(z)}\right] \\
& \exp\left[i\left(l\phi-(2p+l+1)\arctan(\frac{z}{z_0})+\frac{kr^2}{2R(z)}\right)\right]
\end{aligned}
$$

where,

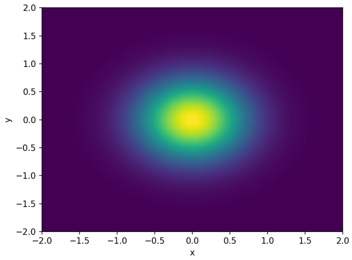
$$
\begin{aligned}
w(z) &= w_0\sqrt{1+z^2/z_02} \\
z_0 &= \pi w_0^2/\lambda \\
R(z) &= z+\frac{z_0^2}{z} \\
L_p^{|l|} &= \text{Associated Laguerre polynomial}
\end{aligned}
$$

4.2 Intensity profile

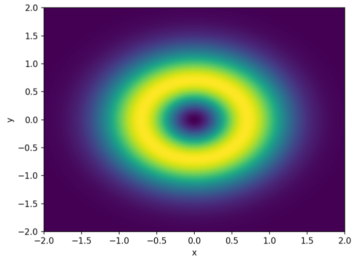
Intensity of Hermite-Gaussian beam is given by,

$$
I_{p,l}(r,z) = \frac{c\epsilon}{2}|A|^2\left[\frac{w_0}{w(z)}\right]^2\left[\frac{r\sqrt{2}}{w(z)}\right]^{2|l|}\left[L_p^{|l|}\left(\frac{2r^2}{w^2(z)}\right)\right]^2\exp\left(-2r^2/w^2(z)\right)
$$

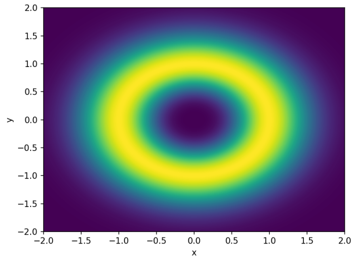
```
import numpy as np
import matplotlib.pyplot as plt
from scipy import special

L= lambda x,p,l: special.assoc_laguerre(x,p,l)

p=int(input("p="))
l=int(input("l="))
\
# define intensity at z=0, z_0=1
Intensity=lambda p,l,x,y: ((np.sqrt(2*(x**2+y**2)))**(2*np.abs(l)))*(L(2*(x**2+y
    **2),p,np.abs(l)))**2*np.exp(-2*(x**2+y**2))

a=2
a1=np.linspace(-a,a,200)
xv,yv=np.meshgrid(a1,a1)
zv=Intensity(p,l,xv,yv)
plt.contourf(xv,yv,zv,levels=300,cmap='viridis')
plt.xlabel("x")
plt.ylabel("y")
#plt.colorbar()
```



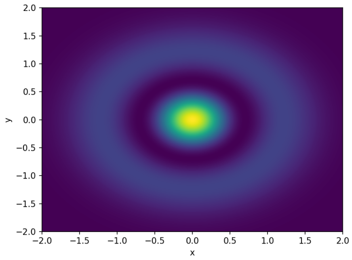
(a) $p = 0, |l| = 0$



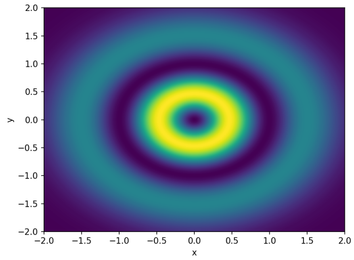
(b) $p = 0, |l| = 1$



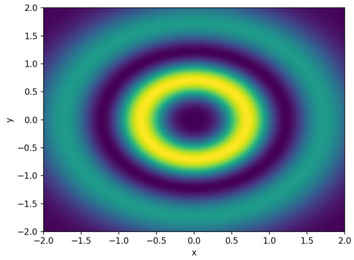
(c) $p = 0, |l| = 2$



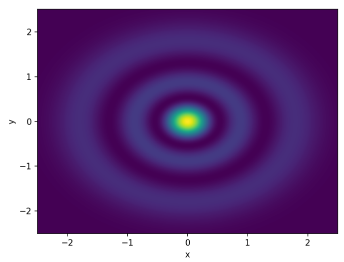
(d) $p = 1, |l| = 0$



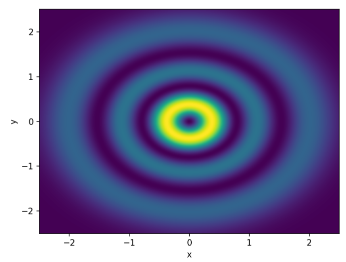
(e) $p = 1, |l| = 1$



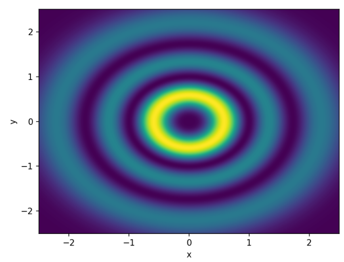
(f) $p = 1, |l| = 2$



(g) $p = 2, |l| = 0$



(h) $p = 2, |l| = 1$



(i) $p = 2, |l| = 2$

Figure 6: Intensity variation for different modes in a cross section $(z = 0, z_0 = 1, w_0 = 1)$

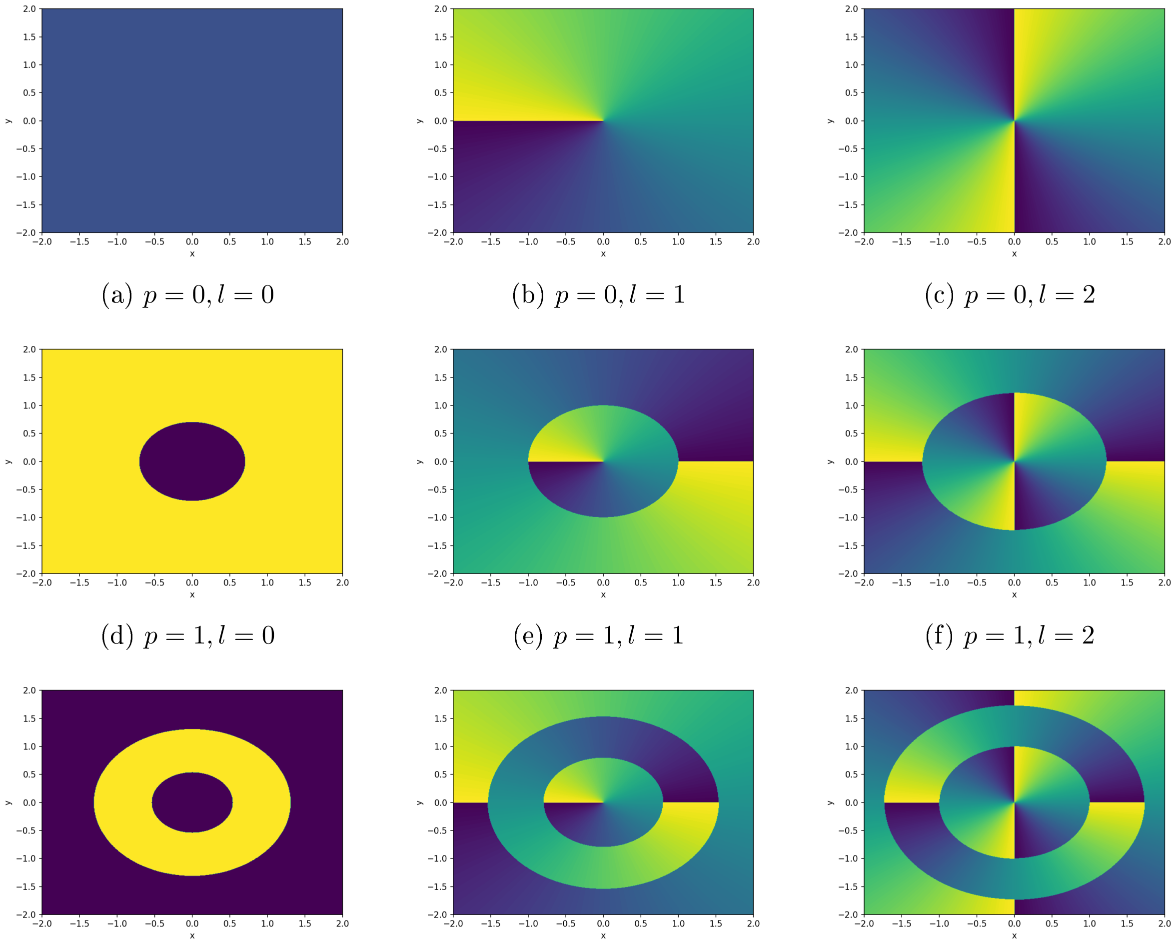
4.3 Phase plot

Phase difference of Hermite-Gaussian beam at $t = 0$ is given by,

$$\text{Phase} = \arg(\psi_{p,l}(r, z))$$

```
import numpy as np
import matplotlib.pyplot as plt
from scipy import special

p=int(input("p="))
l=int(input("l="))

# define phi of range 0 to 2*pi
def taninv(x, y):
    return np.angle(x+1j*y)
phi_angle = np.vectorize(taninv)

```

```
L= lambda x,p,l: special.assoc_laguerre(x,p,l)

# define phase function on z=0 plane, z_0=1
def R(p,l,r,phi):
    return np.angle(((np.sqrt(2*(r**2)))**(np.abs(l)))*(L(2*(r**2),p,np.abs(l)))
    *np.exp(1j*(l*phi)))
phase=np.vectorize(R)

a=2
a1=np.linspace(-a,a,500)
xv,yv=np.meshgrid(a1,a1)
zv=phase(p,l,np.sqrt(xv**2+yv**2),phi_angle(xv,yv))
plt.contourf(xv,yv,zv,levels=200,cmap='viridis')
plt.xlabel("x")
plt.ylabel("y")
#plt.colorbar()
```



(a) $p = 0, l = 0$ (b) $p = 0, l = 1$ (c) $p = 0, l = 2$

(d) $p = 1, l = 0$ (e) $p = 1, l = 1$ (f) $p = 1, l = 2$

(g) $p = 2, l = 0$ (h) $p = 2, l = 1$ (i) $p = 2, l = 2$

Figure 7: Phase variation for different modes in a cross section $(z = 0, z_0 = 1, w_0 = 1)$