Summer Project

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21MS179

" Nobody ever figures out what life is all about, and it doesn't matter. Explore the world. Nearly everything is really interesting if you go into it deeply enough. "

RICHARD P. FEYNMAN

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1 POLARIZATION

1.1 Introduction

We know, in EM wave, the electric field and magnetic field oscillating perpendicularly in the transverse plane *w.r.t.* the propagation direction. *Polarization* is the property of an EM wave, which deals with the temporal and spatial variation of the orientation of field vector (mainly, electric field) of the EM wave. Here we mainly discuss Jones formalism, Stokes-Muller formalism and finally apply those thing in elliptically polarized light.

1.2 Jones formalism

1.2.1 Jones Vector

Vector form of electric field of fully polarized EM wave propagating along z-axis is given by

$$\boldsymbol{E}(\boldsymbol{x},t) = \begin{bmatrix} E_x(\boldsymbol{x},t) \\ E_y(\boldsymbol{x},t) \\ E_z(\boldsymbol{x},t) \end{bmatrix} = \begin{bmatrix} A_x(\boldsymbol{x})e^{-i(kz-\omega t-\delta_x)} \\ A_y(\boldsymbol{x})e^{-i(kz-\omega t-\delta_y)} \\ 0 \end{bmatrix} = \begin{bmatrix} A_x(\boldsymbol{x})e^{i\delta_x} \\ A_y(\boldsymbol{x})e^{i\delta_y} \\ 0 \end{bmatrix} e^{-i(kz-\omega t)} \tag{1.1}$$

We define normalized[1] *Jones vector* as

$$\boldsymbol{J}(\boldsymbol{x},t) = \frac{1}{\sqrt{A_x^2 + A_y^2}} \begin{bmatrix} A_x(\boldsymbol{x})e^{i\delta_x} \\ A_y(\boldsymbol{x})e^{i\delta_y} \end{bmatrix} \tag{1.2}$$

Such examples of usual polarization states are given below [3],

Polarization state	$\boldsymbol{J}$
$|H\rangle$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$
$|V\rangle$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
$|P\rangle$	$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
$|M\rangle$	$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$
$|L\rangle$	$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$
$|R\rangle$	$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}$

Table 1: Jones vector of usual polarization state

Some properties of Jones vector are

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[1]normalized as $\boldsymbol{J}\,\boldsymbol{J}^* = 1$

1. The intensity of the EM wave is given by

$$I = \frac{1}{2}c\epsilon_0(A_x^2 + A_y^2) = \frac{1}{2}c\epsilon_0(E^*E) \tag{1.3}$$

2. For general elliptically polarized light we can measure the azimuth (α) ellipticity (ϵ) of the polarization ellipse by comparing Jones vector $\boldsymbol{J}$ with [1]

$$\begin{bmatrix} \cos\alpha\cos\epsilon - i\sin\alpha\sin\epsilon \\ \sin\alpha\cos\epsilon + i\cos\alpha\sin\epsilon \end{bmatrix}$$

1.2.2 Jones Matrix & evolution of Jones vector

Jones matrix is a 2×2 matrix assigned for a particular optical element. Let $\boldsymbol{M}$ be the Jones matrix for an optical element *s.t.*

$$\boldsymbol{M} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$$

then if a polarized light of Jones vector $\boldsymbol{J}_{in}$ passes through that optical element then the Jones vector of output light is given by

$$\boldsymbol{J}_{out} = \boldsymbol{M}\,\boldsymbol{J}_{in} \tag{1.4}$$
$$\Rightarrow \boldsymbol{E}_{out} = \boldsymbol{M}\,\boldsymbol{E}_{in} \tag{1.5}$$

To determine m_{ij} in $\boldsymbol{M}$,

1. Pass x-polarized light and determine $\boldsymbol{J}_{out}$, then

$$\boldsymbol{J}_{out} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} m_{11} \\ m_{21} \end{bmatrix}$$

2. Pass y-polarized light and determine $\boldsymbol{J}_{out}$, then

$$\boldsymbol{J}_{out} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} m_{12} \\ m_{22} \end{bmatrix}$$

Such examples of usual Jones matrix [2] are given below,[3]

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[2] For polariser the Jones matrix $\boldsymbol{M} = \boldsymbol{J}\,\boldsymbol{J}^*$ where $\boldsymbol{J}$ is normalized Jones vector corresponding polarization state *s.t.* $\boldsymbol{J}_{out} = \boldsymbol{M}\boldsymbol{J} = (\boldsymbol{J}\boldsymbol{J}^*)\boldsymbol{J} = \boldsymbol{J}(\boldsymbol{J}^*\boldsymbol{J}) = \boldsymbol{J}$

Optical element	M
Free space	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
x-Polariser	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
y-Polariser	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$
Right circular polariser	$\frac{1}{2}\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$
Left circular polariser	$\frac{1}{2}\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}$
Linear di-attenuator	$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$
Half-wave plate with fast axis horizontal	$e^{-i\pi/2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Quarter-wave plate with fast axis horizontal	$e^{-i\pi/4}\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$
General phase retarder	$\begin{bmatrix} e^{i\phi_x} & 0 \\ 0 & e^{i\phi_y} \end{bmatrix}$

Table 2: Jones matrix related to usual optical element

Some properties of Jones matrix are

1. Resultant Jones matrix for composition of n optical elements is given by

$$M = M_1 \, M_2 \ldots M_n \qquad (1.6)$$

2. For an optical element when its optical axis aligned at an angle θ *w.r.t.* x-axis then resultant Jones matrix for this rotated optical element is given by

$$M_\theta = R(-\theta) \, M \, R(\theta) \qquad (1.7)$$

where $R(\theta)$ is passive rotation matrix *s.t.*

$$R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \qquad (1.8)$$

1.2.3 Drawback of Jones formalism

Main drawback of Jones formalism is that its application is restricted in fully polarized light. This formalism cannot explain the partially polarized or unpolished light which we frequently observe in practical use.

1.3 Stokes-Muller formalism

1.3.1 Coherency matrix

Coherency matrix of a EM wave is defined as [1]

$$\boldsymbol{C} = \left\langle \boldsymbol{E} \otimes \boldsymbol{E}^\dagger \right\rangle = \left\langle \boldsymbol{E}\boldsymbol{E}^\dagger \right\rangle = \begin{bmatrix} \langle E_x E_x^* \rangle & \langle E_x E_y^* \rangle \\ \langle E_y E_x^* \rangle & \langle E_y E_y^* \rangle \end{bmatrix} = \begin{bmatrix} c_{xx} & c_{xy} \\ c_{yx} & c_{yy} \end{bmatrix} \tag{1.9}$$

where $\otimes$ denotes Kronecker product, $\langle \cdot \rangle$ denotes the temporal avg of the corresponding quantity and $\delta = \delta_y - \delta_x$.

Examples of coherency matrix of usual polarization states are given below [4],

Polarization state	$\boldsymbol{J}$	$\boldsymbol{C}$
$\lvert H \rangle$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
$\lvert V \rangle$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$
$\lvert P \rangle$	$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$
$\lvert M \rangle$	$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$
$\lvert L \rangle$	$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}$
$\lvert R \rangle$	$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$
Un-polarized	$-$	$\frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Table 3: Coherency matrix of usual polarization state

Some properties of coherency matrix are

1. It is a hermitian matrix *i.e.* $\boldsymbol{C} = \boldsymbol{C}^\dagger$

2. Trace and determinant off the matrix are non-negative[3] *i.e.* $\mathrm{tr}(\boldsymbol{C}) > 0$ & $\det(\boldsymbol{C}) \geq 0$.

3. $\mathrm{Tr}(\boldsymbol{C}) = \langle E_x E_x^* \rangle + \langle E_y E_y^* \rangle$ is the time averaged intensity of input light.

4. let the polarized light (of electric field $\boldsymbol{E}_{in}$ & coherency matrix $\boldsymbol{C}_{in}$) passes through an optical element (of Jones matrix $\boldsymbol{M}$) then let output electric field be $\boldsymbol{E}_{out}$ by the equation 1.5, then output coherency matrix $\boldsymbol{C}_{out}$ is given by

$$\begin{aligned} \boldsymbol{C}_{out} = \left\langle \boldsymbol{E}_{out}\boldsymbol{E}_{out}^\dagger \right\rangle &= \left\langle (\boldsymbol{M}\boldsymbol{E}_{in})(\boldsymbol{M}\boldsymbol{E}_{in})^\dagger \right\rangle \\ &= \left\langle (\boldsymbol{M}\boldsymbol{E}_{in})\left(\boldsymbol{E}_{in}^\dagger \boldsymbol{M}^\dagger\right) \right\rangle \\ &= \boldsymbol{M}\left\langle \boldsymbol{E}_{in}\boldsymbol{E}_{in}^\dagger \right\rangle \boldsymbol{M}^\dagger \\ &= \boldsymbol{M}\,\boldsymbol{C}_{in}\,\boldsymbol{M}^\dagger \end{aligned} \tag{1.10}$$

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[3]Proofs to be done

1.3.2 Stokes parameters and Stokes vector

Now we see that coherency matrix $\boldsymbol{C}$ of any polarization state in table 3 can be written in the linear combination of the 4 basis given below [5]

$$\beta = \left\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\boldsymbol{V_0}}, \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{\boldsymbol{V_1}}, \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\boldsymbol{V_2}}, \underbrace{\begin{bmatrix} 0 & i \\ -i & 1 \end{bmatrix}}_{\boldsymbol{V_3}} \right\} \tag{1.11}$$

Now we can write any coherency matrix $\boldsymbol{C}$ as

$$\boldsymbol{C} = \frac{1}{2} \sum_{i=0}^{3} S_i \boldsymbol{V_i} \tag{1.12}$$

Note that all V_i's are Hermitian, so obviously is $\boldsymbol{C}$.

We call $\{S_0, S_1, S_2, S_3\}$ as a *Stokes parameter* and the values of S_i's are experimentally measurable.

A *Stokes vector* $\boldsymbol{S}$ is defined as[4]

$$\boldsymbol{S} = \begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} \tag{1.13}$$

Examples of Stokes vector for different polarization states are given below

Polarization state	$\boldsymbol{C}$	$\boldsymbol{S}$
$\lvert H \rangle$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}^T$
$\lvert V \rangle$	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix}^T$
$\lvert P \rangle$	$\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}^T$
$\lvert M \rangle$	$\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix}^T$
$\lvert L \rangle$	$\frac{1}{2}\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}^T$
$\lvert R \rangle$	$\frac{1}{2}\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & -1 \end{bmatrix}^T$
Un-polarized	$\frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T$

Table 4: Stokes vector of usual polarization state

Note that all Jones vectors has Stokes vectors but converse need not to be true.

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[4]for intensity normalised Stokes vector, $\boldsymbol{s} = \begin{bmatrix} 1 & s_1 & s_2 & s_3 \end{bmatrix}$ where $s_i = S_i/S_0$

Now we see from the equation 1.12

$$\begin{bmatrix} \langle E_x E_x^* \rangle & \langle E_x E_y^* \rangle \\ \langle E_y E_x^* \rangle & \langle E_y E_y^* \rangle \end{bmatrix} = \boldsymbol{C} = \frac{1}{2} \sum_{i=0}^{3} S_i \boldsymbol{V_i} = \frac{1}{2} \begin{bmatrix} S_0 + S_1 & S_2 + iS_3 \\ S_2 - iS_3 & S_0 - S_1 \end{bmatrix} \tag{1.14}$$

From there we can write

$$\boldsymbol{S} = \begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = \begin{bmatrix} \langle E_x E_x^* \rangle + \langle E_x E_y^* \rangle \\ \langle E_x E_x^* \rangle - \langle E_x E_y^* \rangle \\ \langle E_y E_x^* \rangle + \langle E_x E_y^* \rangle \\ i\left(\langle E_y E_x^* \rangle - \langle E_x E_y^* \rangle \right) \end{bmatrix} \tag{1.15}$$

Now for a polarized light,

$$\boldsymbol{C} = \begin{bmatrix} \langle A_x^2 \rangle & \langle A_x A_y e^{-i\delta} \rangle \\ \langle A_x A_y e^{i\delta} \rangle & \langle A_y^2 \rangle \end{bmatrix} \text{ and } \boldsymbol{S} = \begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = \begin{bmatrix} \langle A_x^2 + A_y^2 \rangle \\ \langle A_x^2 - A_y^2 \rangle \\ \langle 2A_x A_y \cos \delta \rangle \\ \langle 2A_x A_y \sin \delta \rangle \end{bmatrix} \tag{1.16}$$

1.3.3 Measurement of Stokes parameters

To measure the 4 Stokes parameter of EM wave associated with, we have to do 4 steps experiment. In each case, we pass the light through various optical elements and measure the (time-averaged) intensity [6],

Step I Pass the light through homogenous isotropic medium (or, free space) and measure the intensity. From table 2 and eq. 1.10, we get,

$$\begin{aligned} \boldsymbol{C}_{out} &= \boldsymbol{M} \, \boldsymbol{C}_{in} \, \boldsymbol{M}^\dagger \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} S_0 + S_1 & S_2 + iS_3 \\ S_2 - iS_3 & S_0 - S_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} S_0 + S_1 & 0 \\ 0 & S_0 - S_1 \end{bmatrix} \end{aligned} \tag{1.17}$$

So the measured intensity will be

$$I_0 = \text{tr}(\boldsymbol{C}_{out}) = S_0 \tag{1.18}$$

Step II Pass the light through x-polariser and measure the intensity. From table 2 and eq. 1.10, we get,

$$\begin{aligned} \boldsymbol{C}_{out} &= \boldsymbol{M} \, \boldsymbol{C}_{in} \, \boldsymbol{M}^\dagger \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \frac{1}{2} \begin{bmatrix} S_0 + S_1 & S_2 + iS_3 \\ S_2 - iS_3 & S_0 - S_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} S_0 + S_1 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned} \tag{1.19}$$

So the measured intensity will be

$$I_1 = \text{tr}(\boldsymbol{C}_{out}) = \frac{1}{2}(S_0 + S_1) \tag{1.20}$$

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Step III Pass the light through the polariser with transmission axis is at 45° and measure the intensity. Then from eq. 1.7, $\boldsymbol{M}$ for this polariser will be

$$\boldsymbol{M} = R(-45°) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} R(45°) = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{1.21}$$

From eq. 1.10, we get,

$$\begin{aligned} \boldsymbol{C}_{out} &= \boldsymbol{M}\,\boldsymbol{C}_{in}\,\boldsymbol{M}^\dagger \\ &= \boldsymbol{M}\,\boldsymbol{C}_{in}\,\boldsymbol{M}^\dagger \\ &= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\frac{1}{2}\begin{bmatrix} S_0 + S_1 & S_2 + iS_3 \\ S_2 - iS_3 & S_0 - S_1 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{4}\begin{bmatrix} S_0 + S_2 & S_0 + S_2 \\ S_0 + S_2 & S_0 + S_2 \end{bmatrix} \end{aligned} \tag{1.22}$$

So the measured intensity will be

$$I_1 = \mathrm{tr}(\boldsymbol{C}_{out}) = \frac{1}{2}(S_0 + S_2) \tag{1.23}$$

Step IV Pass the light through right circular polariser and measure the intensity. From table 2 and eq. 1.10, we get,

$$\begin{aligned} \boldsymbol{C}_{out} &= \boldsymbol{M}\,\boldsymbol{C}_{in}\,\boldsymbol{M}^\dagger \\ &= \frac{1}{2}\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}\frac{1}{2}\begin{bmatrix} S_0 + S_1 & S_2 + iS_3 \\ S_2 - iS_3 & S_0 - S_1 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} \end{aligned} \tag{1.24}$$

So the measured intensity will be

$$I_1 = \mathrm{tr}(\boldsymbol{C}_{out}) = \frac{1}{2}(S_0 + S_3) \tag{1.25}$$

From the equations 1.18, 1.20, 1.23 and 1.25, we can get the values of all S_i's.

1.3.4   Poincare and sphere representation

AM For total intensity normalised Stokes vector is $\boldsymbol{s} = \begin{bmatrix} 1 & s_1 & s_2 & s_3 \end{bmatrix}^T$ where $s_i = S_i/S_0$. Observe that $\boldsymbol{s}$ is a 3-dimensional quantity. Therefore we can write,

$$\begin{bmatrix} 1 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} \rightarrow \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}$$

Poincare sphere representation is a coordinate system to define the state of polarization of light where the mutually orthogonal coordinate axes are $\{s_1, s_2, s_3\}$.

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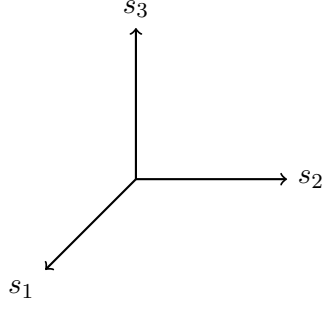
Example of special cases are

Case I For fully polarized light

$$s_1 = \frac{A_x^2 - A_y^2}{A_x^2 + A_y^2} \tag{1.26}$$

$$s_2 = \frac{2A_y A_y \cos\delta}{A_x^2 + A_y^2} \tag{1.27}$$

$$s_3 = \frac{2A_y A_y \sin\delta}{A_x^2 + A_y^2} \tag{1.28}$$

from there we can see

$$s_1^2 + s_2^2 + s_3^2 = 1 \tag{1.29}$$

which implies that fully polarized has the locus at any point in the sphere of radius 1 in Poincare sphere representation.

Case II For fully un-polarized light

$$s_1 = s_2 = s_3 = 0 \tag{1.30}$$

which implies that fully un-polarized has the locus at any the centre $(0, 0, 0)$ in the sphere of radius 1 in Poincare sphere representation.

1.3.5 Degree of Polarization

Degree of Polarization is the measure of polarization of light.
We define

- Total degree of polarization, $DOP = \sqrt{s_1^2 + s_2^2 + s_3^2}$

- Degree of linear polarization $= \sqrt{s_1^2 + s_2^2}$

- Degree of circular polarization $= \sqrt{s_1^2 + s_2^2 + s_3^2}$

For any mixed polarization state we can decompose the Stokes vector into fully polarized and un-polarized components,

$$\begin{bmatrix} 1 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \underbrace{\begin{bmatrix} \sqrt{s_1^2 + s_2^2 + s_3^2} \\ s_1 \\ s_2 \\ s_3 \end{bmatrix}}_{\text{fully polarized, } DOP=1} + \underbrace{\begin{bmatrix} 1 - \sqrt{s_1^2 + s_2^2 + s_3^2} \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\text{un-polarized}} \tag{1.31}$$

1.3.6 Muller Matrix & evolution of Stokes vector

Similar to the Jones matrix, *Muller matrix* is a 4×4 matrix assigned for a particular optical element. Let $\mathfrak{M}$ be the Muller matrix for an optical element *s.t.*

$$\mathfrak{M} = \begin{bmatrix} \mu_{11} & \cdots & \mu_{14} \\ \vdots & \ddots & \vdots \\ \mu_{41} & \cdots & \mu_{44} \end{bmatrix}$$

then if a light of Stokes vector $\boldsymbol{S}_{in}$ passes through that optical element, then the Stokes vector of output light is given by

$$\boldsymbol{S}_{out} = \mathfrak{M} \, \boldsymbol{S}_{in} \tag{1.32}$$

Some properties of Jones matrix are

1. Resultant Muller matrix for composition of n optical elements is given by

$$\mathfrak{M} = \mathfrak{M}_1 \, \mathfrak{M}_2 \ldots \mathfrak{M} \tag{1.33}$$

2. When th optical element is aligned at an angle θ *w.r.t.* x-axis then resultant Muller matrix (similar to Jones matrix) for this rotated optical element is given by

$$\mathfrak{M}_\theta = T^{-1}(\theta) \, \mathfrak{M} \, T(\theta) \tag{1.34}$$

where $T(\theta)$ is passive rotation matrix in Poincare sphere representation *w.r.t* s_3 axis, *s.t.*

$$T(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\theta & \sin 2\theta & 0 \\ 0 & -\sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1.35}$$

Note that, in eq. 1.35, if we write

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\theta & \sin 2\theta & 0 \\ 0 & -\sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} \cos 2\theta & \sin 2\theta & 0 \\ -\sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.36}$$

we see that it is proper rotation matrix of rotation angle 2θ in Poincare sphere *w.r.t* s_3 axis. And as we know that rotation of θ of electric field results in rotation of 2θ in azimuth angle of Stokes vector in Poincare sphere.

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1.3.7 Relationship between Jones & Stokes-Muller formalism

Let, $\boldsymbol{J}$ be jones vector, $\boldsymbol{M}$ be the Jones matrix, $\boldsymbol{S}$ be the Stokes vector and $\mathfrak{M}$ be the Muller matrix *s.t.* equations 1.4 and 1.32 is satisfied.

Let us define *coherency vector* of 1.9 as

$$\boldsymbol{L} = \begin{bmatrix} c_{xx} & c_{xy} & c_{yx} & c_{yy} \end{bmatrix}^T \tag{1.37}$$

and *Wolf matrix* $\boldsymbol{W}$ as

$$\boldsymbol{L}_{out} = \boldsymbol{W}\,\boldsymbol{L}_{in} \tag{1.38}$$

then the relation between Jones and Wolf matrix[5] is

$$\boldsymbol{W} = \boldsymbol{M} \otimes \boldsymbol{M}^* \tag{1.39}$$

Now from equations 1.9 and 1.15, one can write

$$\boldsymbol{S} = \boldsymbol{A}\,\boldsymbol{L} \tag{1.40}$$

where,

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & -i & i & 0 \end{bmatrix} \tag{1.41}$$

then the relation between Jones and Muller matrix[6] is

$$\mathfrak{M} = \boldsymbol{A}\,(\boldsymbol{M} \otimes \boldsymbol{M}^*)\,\boldsymbol{A}^{-1} \tag{1.42}$$

Note that this relationship is only possible in both ways, if the light is fully polarized light as all Jones vectors has Stokes vectors but converse need not to be true.

1.4 More on Elliptically polarized light

1.4.1 Jones vector of elliptically polarized light

In this section we will discuss the generalized polarization ellipse of an EM wave. Let our electric field vector of EM wave is given by

$$\boldsymbol{E} = \begin{bmatrix} E_x \\ E_y \end{bmatrix} = \begin{bmatrix} a_1 \cos(\tau + \delta_1) \\ a_2 \cos(\tau + \delta_2) \end{bmatrix} \text{ where } \tau = kz - \omega \text{ and } ta_1, a_2 \geq 0 \tag{1.43}$$

by eliminating τ we get,

$$\frac{1}{a_1^2}E_x^2 + \frac{1}{a_2^2}E_y^2 - \frac{2\cos\delta}{a_1 a_2}E_x E_y = \sin^2(\delta) \tag{1.44}$$

where $\delta = \delta_2 - \delta_1$. The eq. 1.44 is equation of circle when $a_1 = a_2$, otherwise, of ellipse[2].

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[5] proof to be done

[6] proof to be done

Now we do the change of basis $\{E_x, E_y\} \longmapsto \{E_\xi, E_\eta\}$ (See fig. 1) *s.t.* electric field in $\{E_\xi, E_\eta\}$ basis be

$$\boldsymbol{E} \to \boldsymbol{F}$$

$$\boldsymbol{F} = \begin{bmatrix} E_\xi \\ E_\eta \end{bmatrix} = \begin{bmatrix} a\cos(\tau + \delta_0) \\ \pm b\cos(\tau + \delta_0) \end{bmatrix} \quad \text{where } a \geq b \geq 0 \tag{1.45}$$

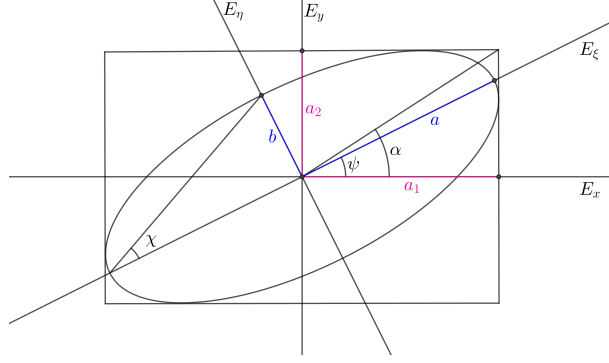which is parametric form of canonical ellipse[7] in $\{E_\xi, E_\eta\}$ basis.

Figure 1: Polarization ellipse

Let ψ be the azimuth angle of the ellipse then

$$\boldsymbol{F} = R(\psi)\,\boldsymbol{E} \tag{1.46}$$

$$\Rightarrow \begin{bmatrix} E_\xi \\ E_\eta \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{bmatrix} \begin{bmatrix} E_x \\ F_y \end{bmatrix} \tag{1.47}$$

$$\Rightarrow \begin{bmatrix} a\cos(\tau + \delta_0) \\ \pm b\cos(\tau + \delta_0) \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{bmatrix} \begin{bmatrix} a_1\cos(\tau + \delta_1) \\ a_2\cos(\tau + \delta_2) \end{bmatrix} \tag{1.48}$$

We want value of a, b, After some tedious calculation [2], we reach to some important results, given below

$$a^2 + b^2 = a_1^2 + a_2^2 \tag{1.49}$$

$$\pm\, ab = a_1 a_2 \sin\delta \tag{1.50}$$

$$\tan\chi := \pm\frac{b}{a} \quad \text{where } \chi \in [-\frac{\pi}{4}, \frac{\pi}{4}] \tag{1.51}$$

$$\tan\alpha := \frac{a_2}{a_1} \quad \text{where } \alpha \in [0, \frac{\pi}{2}] \tag{1.52}$$

$$\tan 2\psi = \tan 2\alpha \cos\delta \tag{1.53}$$

$$\sin 2\chi = \sin 2\alpha \sin\delta \tag{1.54}$$

where ψ is the *azimuth* and χ is *ellipticity* of the polarization ellipse.

To see the handedness of the rotation of electric field vector in transverse plane,

Case I For right-handed polarization, $\sin\delta > 0$, then from equations 1.50, and 1.51, we can say

$$\tan\chi \geq 0 \Rightarrow \chi \in \left(0, \frac{\pi}{4}\right]$$

---

[7] $\pm$ before b denotes the handedness of the rotation of electric field vector in transverse plane.

Case II Similarly for left-handed polarization, $\sin\delta < 0$, then from equations 1.50, and 1.51, we can say

$$\tan\chi \leq 0 \Rightarrow \chi \in \left[-\frac{\pi}{4}, 0\right)$$

Now the Jones vector of elliptical polarization in the form of ellipticity and azimuth[8] will be,

$$\boldsymbol{J} = \begin{bmatrix} \cos\psi\cos\chi - i\sin\psi\sin\chi \\ \sin\psi\cos\chi + i\cos\psi\sin\chi \end{bmatrix} \tag{1.55}$$

1.4.2   Stokes vector and corresponding Poincare representation

From the eq. 1.16, we can write for our case[9],

$$\boldsymbol{S} = \begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = \begin{bmatrix} a_1^2 + a_2^2 \\ a_1^2 - a_2^2 \\ 2a_1 a_2 \cos\delta \\ 2a_1 a_2 \sin\delta \end{bmatrix} = S_0 \begin{bmatrix} 1 \\ \cos 2\chi \cos 2\psi \\ \cos 2\chi \sin 2\psi \\ \sin 2\chi \end{bmatrix} \tag{1.56}$$

So in Poincare sphere representation with axes $\{S_1, S_2, S_3\}$, the required vector is

$$S_0 \begin{bmatrix} 1 \\ \cos 2\chi \cos 2\psi \\ \cos 2\chi \sin 2\psi \\ \sin 2\chi \end{bmatrix} \longrightarrow (S_0 \cos 2\chi \cos 2\psi, S_0 \cos 2\chi \sin 2\psi, S_0 \sin 2\chi) \tag{1.57}$$

The evolution of azimuth (ψ) and ellipticity (χ) of the polarization state in Poincare representation is shown in the figure 2.
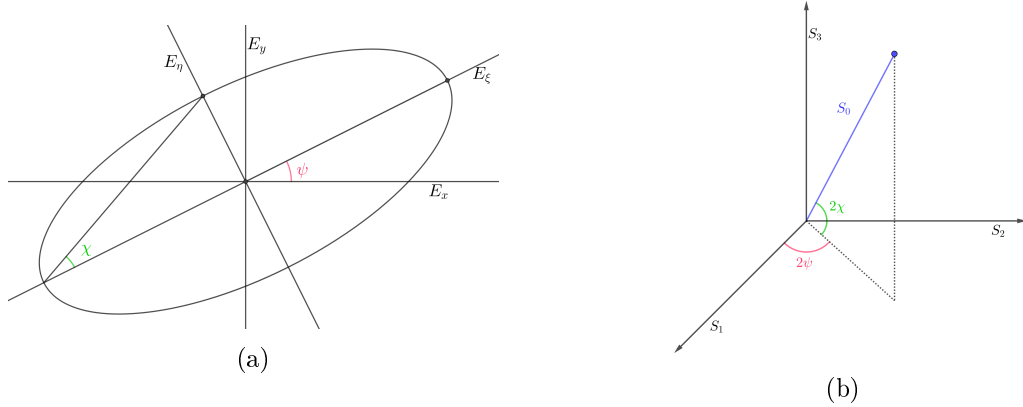

Figure 2: polarization ellipse and corresponding Poincare representation

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[8]proof to be done

[9]proof to be done

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2 GAUSSIAN BEAM

2.1 Introduction

2.2 Paraxial wave equation and solutions

From Maxwell's 3-D wave equation for electric field in vacuum,

$$\nabla^2 \boldsymbol{E}(\boldsymbol{r}, t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\boldsymbol{E}(\boldsymbol{r}, t) = 0 \tag{2.1}$$

Here we will consider the scalar form of the equation.

To find the scalar solution, let our first ansatz be,

$$E(x, y, z, t) = F(x, y, z)e^{i\omega t} \tag{2.2}$$

Putting it in eq. 2.1, we get time -independent *Helmholtz equation i.e.*

$$\nabla^2 F(x, y, z) + k^2 F(x, y, z) = 0 \text{ where } k^2 = \frac{\omega^2}{c^2} \tag{2.3}$$

For light to travel in z-direction, our 2nd ansatz be,

$$F(\boldsymbol{r}) = \psi(x, y, z)e^{-ikz} \tag{2.4}$$

Considering the slowly varying envelope approximation[1] that,

$$\lambda^2 \left|\frac{\partial^2\psi}{\partial z^2}\right| << \lambda \left|\frac{\partial\psi}{\partial z}\right| << |\psi| \tag{2.5}$$

and putting 2.4 in eq. 2.3 gives *Paraxial wave equation*,

$$\nabla_T^2 \psi - 2ik\frac{\partial\psi}{\partial r} = 0 \tag{2.6}$$

where transverse Laplacian, $\nabla_T = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ or in cylindrical coordinate, $\nabla_T = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial \phi^2}$.

2.2.1 Scalar wave solution

We will solve the paraxial wave equation in cylindrical coordinate $\{r, \phi, z\}$.[8] To get a solution for which intensity is of Gaussian like and radially symmetric (*i.e.* no variation with ϕ), our first ansatz be,[9][8]

$$\psi(\boldsymbol{r}, z) = A\exp\left[-i\left(p(z) + \frac{kr^2}{2q(z)}\right)\right] = A\underbrace{\exp\left[-ip(z)\right]}_{\text{first term}}\underbrace{\exp\left[-i\frac{kr^2}{2q(z)}\right]}_{\text{second term}} \tag{2.7}$$

where second term is related to Gaussian intensity and first term is additional phase factor. Putting this, in eq, 2.6, we get

$$\left[\frac{k^2}{q^2}\left(\frac{dq}{dz} - 1\right)r^2 - 2k\left(\frac{dp}{dz} + \frac{i}{q}\right)\right]\psi = 0 \tag{2.8}$$

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To satisfy this, for all r, we get

$$\frac{dq}{dz} - 1 = 0 \tag{2.9}$$

$$\frac{dp}{dz} + \frac{i}{q} = 0 \tag{2.10}$$

Lets first calculate eq. 2.9.

$$\frac{dq}{dz} - 1 = 0 \Rightarrow q(z) = z + q_0 \tag{2.11}$$

putting this in the second term of expression 2.7 at $z = 0$,

$$\exp\left[-i\frac{kr^2}{2q(0)}\right] = \exp\left[-i\frac{kr^2}{2q_0}\right] \tag{2.12}$$

which is a phase factor does not give Gaussian intensity. So to get Gaussian intensity, q_0 must be imaginary. Let $j_0 = iz_0$, then

$$\boxed{q(z) = z + iz_0} \tag{2.13}$$

Now, the second term of expression 2.7 at $z = 0$ be,

$$\exp\left[-\frac{kr^2}{2z_0}\right] = \exp\left[-\frac{r^2}{w_0^2}\right] \tag{2.14}$$

where

$$w_0^2 = \frac{2z_0}{k} = \frac{\lambda z_0}{\pi} \Rightarrow \boxed{z_0 = \frac{\pi w_0^2}{\lambda}} \tag{2.15}$$

We call z_0 *confocal parameter* or *Rayleigh range* of the beam.

Now from expression 2.13, we calculate $1/q(z)$.

$$\frac{1}{q(z)} = \frac{1}{z + iz_0} = \frac{z}{z^2 + z_0^2} - i\frac{z_0}{z^2 + z_0^2} \tag{2.16}$$

Then the second term of expression 2.7 be,

$$\exp\left[-i\frac{kr^2}{2q(z)}\right] = \underbrace{\exp\left[-\frac{kr^2 z_0}{2(z^2 + z_0^2)}\right]}_{\text{term A}} \underbrace{\exp\left[-i\frac{kr^2 z}{2(z^2 + z_0^2)}\right]}_{\text{term B}} \tag{2.17}$$

Write term A of 2.17 as

$$\exp\left[-\frac{kr^2 z_0}{2(z^2 + z_0^2)}\right] = \exp\left[-\frac{r^2}{w^2(z)}\right] \tag{2.18}$$

which is Gaussian, where

$$\boxed{w^2(z) = w_0^2\left[1 + \left(\frac{z}{z_0}\right)^2\right]} \tag{2.19}$$

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We call $w(z)$ *physical radius/half-width* of the beam.

Write term B of 2.17 as

$$\exp\left[-i\frac{kr^2 z}{2(z^2 + z_0^2)}\right] = \exp\left[-i\frac{kr^2}{2R(z)}\right] \tag{2.20}$$

where

$$\boxed{R(z) = z\left[1 + \left(\frac{z_0}{z}\right)^2\right]} \tag{2.21}$$

We know for spherical wave,

$$E(\boldsymbol{r}, t) \sim \frac{1}{r}e^{i(\omega t - kr)} \tag{2.22}$$
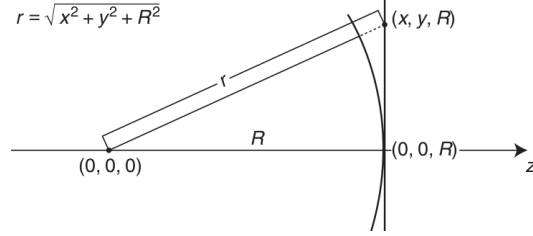
Figure 3: Radius of curvature of spherical wavefront (Ref. [7])

Let R be the radius of curvature of spherical wavefront. Now for any point $\boldsymbol{r} = (x, y, R)$ on $z = R$ plane, r will be

$$r = \sqrt{x^2 + y^2 + R^2} \tag{2.23}$$

For collimated beam, we restrict radius of curvature measurement near $\boldsymbol{r} = (0, 0, R)$, so

$$r = \sqrt{x^2 + y^2 + R^2} \approx R + \frac{x^2 + y^2}{2R} \tag{2.24}$$

Now from 2.22,

$$E(\boldsymbol{r}, t) \sim \frac{1}{r}e^{i\omega t}e^{-ikr}e^{-ik\frac{x^2+y^2}{2R}} \tag{2.25}$$

comparing with 2.2

$$\psi(x, y, z) \sim e^{-ik\frac{x^2+y^2}{2R}} \tag{2.26}$$

comparing 2.26 in the above expression with term B of 2.17, we conclude that $R(z)$ in 2.21 is *radius of curvature* of wavefront near $r = 0$ of collimated beam in far field.

Putting 2.19 and 2.21 in eq. 2.16,

$$\boxed{\frac{1}{q(z)} = \frac{1}{R(z)} - i\,\frac{\lambda}{\pi w^2(z)}} \tag{2.27}$$

Now we simplify the first term in 2.7. Putting 2.13 in 2.10 and by solving the differential equation, we get,

$$\frac{dp}{dz} = \frac{-i}{q} = \frac{-i}{z + iz_0} \Rightarrow i\, p(z) = \ln\left[1 - i\frac{z}{z_0}\right] \tag{2.28}$$

As we can write

$$1 - i\frac{z}{z_0} = \sqrt{1 + \left(\frac{z}{z_0}\right)^2}\, \exp\left[-i\,\tan^{-1}\left(\frac{z}{z_0}\right)\right]$$

putting this in the above expression of $i\, p(z)$, our final expression will be

$$i\, p(z) = \frac{1}{2}\ln\left[1 + \left(\frac{z}{z_0}\right)^2\right] - i\,\tan^{-1}\left(\frac{z}{z_0}\right) \tag{2.29}$$

Finally putting 2.27 and 2.29 in 2.7, we get,[9][8]

$$\psi(\boldsymbol{r}, z) = A\exp\left[-i\left(p(z) + \frac{kr^2}{2q(z)}\right)\right]$$

$$= A\exp\left[-\frac{1}{2}\ln\left[1 + \left(\frac{z}{z_0}\right)^2\right] + i\,\tan^{-1}\left(\frac{z}{z_0}\right) - i\frac{kr^2}{2}\left(\frac{1}{R(z)} - i\frac{\lambda}{\pi w^2(z)}\right)\right]$$

$$= \frac{A}{\sqrt{1 + \left(\frac{z}{z_0}\right)^2}}\exp\left(i\,\tan^{-1}\left(\frac{z}{z_0}\right)\right)\exp\left(-i\frac{kr^2}{2R(z)}\right)\exp\left(-\frac{r^2}{w^2(z)}\right)$$

$$\Rightarrow \psi(\boldsymbol{r}, z) = A\underbrace{\left(\frac{w_0}{w(z)}\right)}_{\text{term I}}\underbrace{\exp\left(i\,\tan^{-1}\left(\frac{z}{z_0}\right)\right)}_{\text{term II}}\underbrace{\exp\left(-i\frac{kr^2}{2R(z)}\right)}_{\text{term III}}\underbrace{\exp\left(-\frac{r^2}{w^2(z)}\right)}_{\text{term IV}} \tag{2.30}$$

In that expression,

1. Term I $\longrightarrow$ related to spreading of beam along propagation in z.

2. Term II $\longrightarrow$ related to *Gouy phase*.

3. Term III $\longrightarrow$ gives radius of curvature of beam wave front.

4. Term IV $\longrightarrow$ gives radially symmetric Gaussian intensity profile.
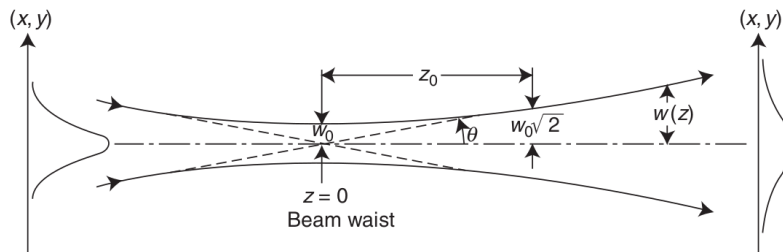
2.3   Characteristics of Gaussian Beam

Figure 4: A beam profile (Ref. [7])

Some characteristics of Gaussian beam are

1. *Intensity* of Gaussian beam in any transverse plain is,

$$I(r, z) = \frac{1}{2}\epsilon_0 c |E^* E| = \frac{1}{2}\epsilon_0 c |\psi^* \psi|$$

$$= \frac{1}{2}\epsilon_0 c |A|^2 \left(\frac{w_0}{w(z)}\right)^2 \exp\left(-\frac{2r^2}{w^2(z)}\right) \tag{2.31}$$

(a) Intensity variation in a cross section

(b) Intensity variation along z-axis
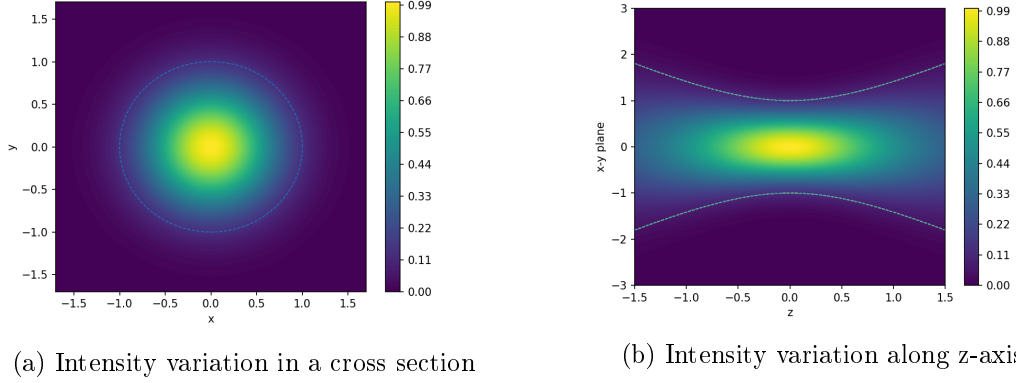
Figure 5: Gaussian intensity profile for $z_0 = 1, w_0 = 1$

2. *Rate of energy* of Gaussian beam passes through any transverse plain is given by

$$W = \iint_{-\infty}^{\infty} dx\, dy\, I(x, y, z)$$

$$= \frac{1}{2}\epsilon_0 c |A|^2 \left(\frac{w_0}{w(z)}\right)^2 \iint_{-\infty}^{\infty} dx\, dy\, \exp\left(-\frac{2(x^2 + y^2)}{w^2(z)}\right)$$

$$= \frac{1}{2}\epsilon_0 c |A|^2 \left(\frac{w_0}{w(z)}\right)^2 \int_{-\infty}^{\infty} dx\, \exp\left(-\frac{2x^2}{w^2(z)}\right) \int_{-\infty}^{\infty} dy\, \exp\left(-\frac{2y^2}{w^2(z)}\right)$$

$$= \frac{1}{2}\epsilon_0 c |A|^2 \left(\frac{w_0}{w(z)}\right)^2 (\sqrt{\pi}w(z))^2 = \frac{1}{2}\epsilon_0 c |A|^2 w_0^2 \tag{2.32}$$

which is constant throughout the propagation along z-axis.

3. *Radius of curvature* $R(z)$ of the wavefront is given by

$$R(z) = z\left[1 + \left(\frac{z_0}{z}\right)^2\right] \tag{2.33}$$

For $z = 0$ $R \to \infty$

For $z >> z_0$, then $R \approx z$

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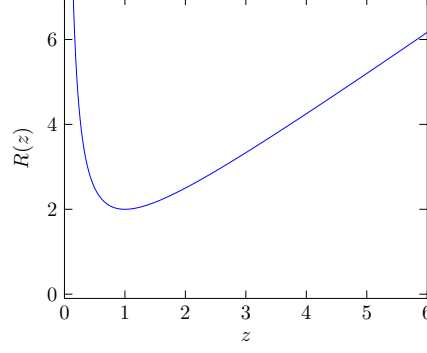
Figure 6: Variation of radius of curvature with z ($z_0 = 1$)

4. *Beam half-width* (see fig. 4) is given by

$$w(z) = w_0 \sqrt{1 + \left(\frac{z}{z_0}\right)^2} \tag{2.34}$$

For $z = 0$, then $w = w_0$ (*Beam waist*)

For $z >> z_0$, then $w(z) = w_0 \frac{z}{z_0}$

Diffraction angle at far field is given by

$$2\theta = 2\frac{dw}{dz} = 2\frac{w_0}{z_0} = \frac{2\lambda}{\pi w_0} \tag{2.35}$$

Effective area of the beam in a cross section is $\frac{1}{2}\pi w^2(z)$

5. *Gouy phase* represents the difference in phase shift of a Gaussian beam *w.r.t.* a plane wave of the same wavelength near $r = 0$.[13] Gouy phase of a Gaussian beam is given by

$$\phi_g(z) = \tan^{-1}\left(\frac{z}{z_0}\right) \tag{2.36}$$

The Gouy phase vary form $-\pi/2$ to $\pi/2$ continuously as z goes from $-\infty$ to ∞, shown in fig. 7

Figure 7: Variation of Gouy phase with z

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6. Inside Rayleigh length (z_0), the laser beam is highly collimated and intensity is also very high. So gain medium is kept in between $z = -z_0$ and z_0 to get maximum stimulated emission from gain medium.

2.4 Beam Tracing and Resonator Stability

2.5 Different modes of Gaussian beams

Here we will discuss mainly two types of higher order Gaussian beams *i.e.*

1. Hermite-Gaussian (HG) beam

2. Laguerre-Gaussian (LG) beam

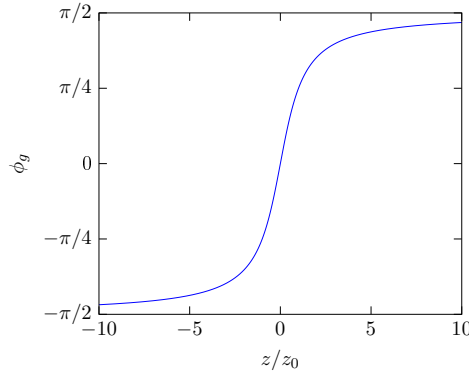### 2.5.1 Hermite-Gaussian beam

In the expression 2.30, we get the radially symmetric Gaussian beam solution. But we now seek higher order solution of Gaussian beam which is rectangular symmetric.

Lets take the ansatz as,

$$\psi(\boldsymbol{r}, z) = A\, g\left(\frac{x}{w(z)}\right) h\left(\frac{y}{w(z)}\right) \exp\left[-i\left(p(z) + \frac{kr^2}{2q(z)}\right)\right] \tag{2.37}$$

Putting this in Paraxial wave equation 2.6, and solving the differential equation,[7] we get,

$$\psi_{m,n}(\boldsymbol{r}, z) = A\frac{w_0}{w(z)} H_m\left(\frac{\sqrt{2}x}{w(z)}\right) H_n\left(\frac{\sqrt{2}y}{w(z)}\right) \exp\left(-\frac{r^2}{w^2(z)}\right) \cdot$$
$$\exp\left(i\,(m+n+1)\tan^{-1}\left(\frac{z}{z_0}\right) - i\frac{kr^2}{2R(z)}\right) \tag{2.38}$$

where H_i is *i th order Hermite polynomial* and other symbols are as usual. See for $m = 0 = n$ we recover the Gaussian solution of 2.30 which we call as *zero order* HG beam. For different values of m an n, we will get diferent type of higher order HG beam, these are called *transverse electromagnetic mode* of order (m, n) or, TEM_{mn}.

Some characteristics of HG beams are given below,

1. *Intensity* of HG beam is given by

$$I_{m,n}(x, y, z) = \frac{c\epsilon}{2}|A|^2 \left[H_m\left(\frac{\sqrt{2}x}{w(z)}\right)\right]^2 \left[H_n\left(\frac{\sqrt{2}y}{w(z)}\right)\right]^2 \exp\left(\frac{2(x^2 + y^2)}{w^2(z)}\right) \tag{2.39}$$

Due to the number of zeros equals the order of Hermite polynomial, we will see m number of horizontal and n number of vertical node in intensity profile of the TEM_{mn} beam. See figure 8.

2. *Rate of energy* of HG beam passes through any transverse plain is given by

$$W = \iint_{-\infty}^{\infty} dx\, dy\, I(x, y, z)$$

$$= \frac{1}{2}\epsilon_0 c |A|^2 \left(\frac{w_0}{w(z)}\right)^2 \int_{-\infty}^{\infty} dx\, \exp\left(-\frac{2x^2}{w^2(z)}\right) \left[H_m\left(\frac{\sqrt{2}x}{w(z)}\right)\right]^2$$

$$\int_{-\infty}^{\infty} dy\, \exp\left(-\frac{2y^2}{w^2(z)}\right) \left[H_n\left(\frac{\sqrt{2}y}{w(z)}\right)\right]^2 \tag{2.40}$$

the integration terms of 2.40 are in the form of

$$\int_{-\infty}^{\infty} H_l \exp\left(-a\xi^2\right) d\xi = \int_{-\infty}^{\infty} \left(\sum_{k=0}^{l} c_k \xi_k\right) \exp\left(-a\xi^2\right) d\xi = \sum_{k=0}^{l} c_k \int_{-\infty}^{\infty} \xi_k \exp\left(-a\xi^2\right) d\xi$$

as the values of $\int_{-\infty}^{\infty} \xi_k \exp\left(-a\xi^2\right) d\xi$ is fixed[10] and for finite value of l, W is finite and constant throughout the propagation along z-axis.

3. *Radius of curvature* of HG beam is same as simple Gaussian beam for all modes.

4. *Gouy phase* for different order HG beam is given by

$$\phi_g(\eta, z) = \eta \tan^{-1}\left(\frac{z}{z_0}\right) \text{ where } \eta = m + n + 1 \tag{2.41}$$
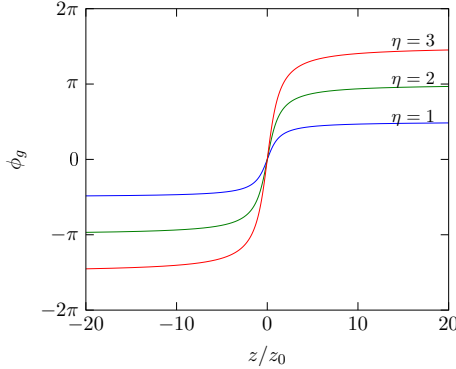
Figure 9: Variation of Gouy phase with z for HG beam

2.5.2 Laguerre-Gaussian beam

In the expression 2.30, we get the radially symmetric Gaussian beam solution. But we now seek higher order solution of Gaussian beam which is not radially symmetric i.e. vary with ϕ.

Lets take the ansatz as,

$$\psi(\boldsymbol{r}, z) = A\, g\left(\frac{y}{w(z)}\right) \exp\left[-i\left(p(z) + \frac{kr^2}{2q(z)} + l\phi\right)\right] \tag{2.42}$$

---

[10]as n_{th} moment of a random variable of Gaussian distribution has a fixed value for a particular integer values of n [15]

(a) TEM_{00} (b) TEM_{01} (c) TEM_{02}

(d) TEM_{10} (e) TEM_{11} (f) TEM_{12}

(g) TEM_{20} (h) TEM_{21} (i) TEM_{22}
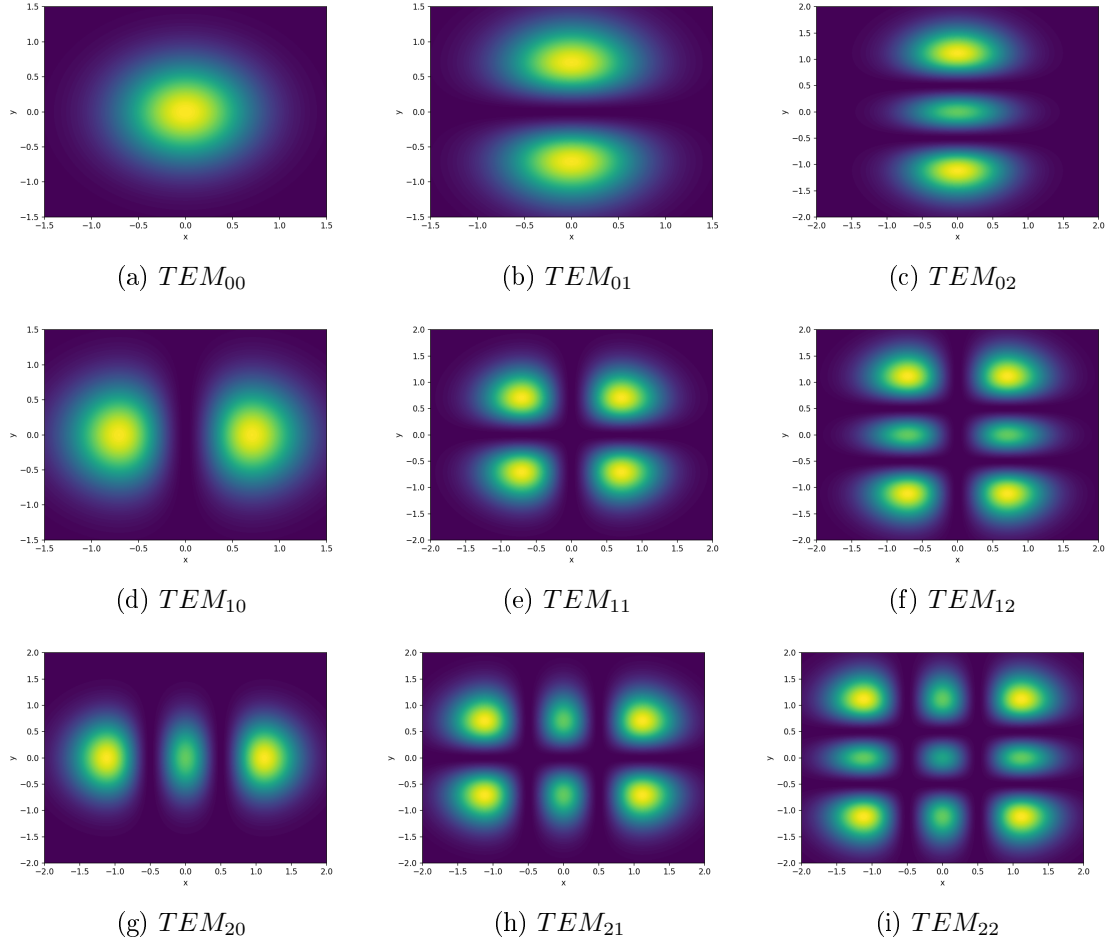
Figure 8: Intensity variation for different TEM in a cross section ($z = 0, z_0 = 1, w_0 = 1$)

Putting this in Paraxial wave equation 2.6, and solving the differential equation,[16][9] we get,

$$\psi_{p,l}(\boldsymbol{r}, z) = A \frac{w_0}{w(z)} \left[\frac{r\sqrt{2}}{w(z)} \right]^{|l|} L_p^{|l|} \left(\frac{2r^2}{w^2(z)} \right) \exp\left(-\frac{r^2}{w^2(z)} \right) \cdot$$
$$\exp\left(-il\phi + i(2p + l + 1)\arctan\left(\frac{z}{z_0} \right) - i\frac{kr^2}{2R(z)} \right) \quad (2.43)$$

where l is *vortex quantum number*,takes integer value, $L_p^{|l|}$ is *associated Laguerre polynomial* and other terms are as usual.

Some characteristics of HG beams are given below,

1. *Intensity* of LG beam is given by

$$I_{p,l}(r, z) = \frac{c\epsilon}{2} |A|^2 \left[\frac{w_0}{w(z)} \right]^2 \left[\frac{r\sqrt{2}}{w(z)} \right]^{2|l|} \left[L_p^{|l|} \left(\frac{2r^2}{w^2(z)} \right) \right]^2 \exp\left(-\frac{2r^2}{w^2(z)} \right) \quad (2.44)$$

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(a) $p = 0, |l| = 0$ (b) $p = 0, |l| = 1$ (c) $p = 0, |l| = 2$

(d) $p = 1, |l| = 0$ (e) $p = 1, |l| = 1$ (f) $p = 1, |l| = 2$

(g) $p = 2, |l| = 0$ (h) $p = 2, |l| = 1$ (i) $p = 2, |l| = 2$
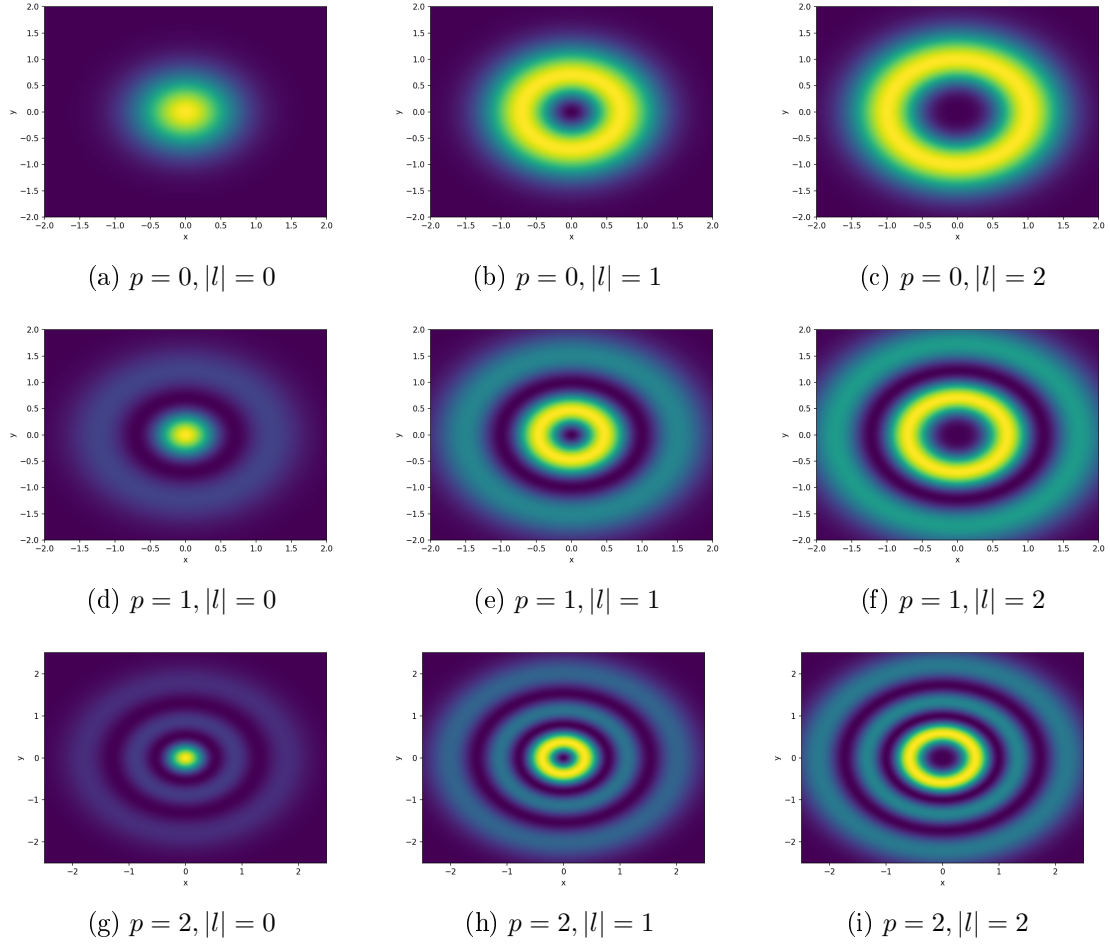
Figure 10: Intensity variation for different modes in a cross section ($z = 0, z_0 = 1, w_0 = 1$)

See intensity plot for corresponding LG beam in figure 10. For $|l| \neq 0$ the intensity of centre is zero and the value of p denotes the number of radial nodes as $L_p^{|l|}$ has p number of zeros.

2. *Rate of energy* of HG beam passes through any transverse plain is given by

$$
\begin{aligned}
W &= \iint_{-\infty}^{\infty} dx \, dy \, I(x, y, z) \\
&= \frac{1}{2} \epsilon_0 c |A|^2 \left(\frac{w_0}{w(z)} \right)^2 \int_{-\infty}^{\infty} dx \, \left(\frac{2r^2}{w^2(z)} \right)^{|l|} \left[L_p^{|l|} \left(\frac{2r^2}{w^2(z)} \right) \right]^2 \exp\left(-\frac{2r^2}{w^2(z)} \right)
\end{aligned}
\tag{2.45}
$$

By same argument as HG beam, we can conclude W is finite and constant throughout the propagation along z-axis

3. *Phase* of the LG beam, unlike Gaussian beam, not only depends on the r and z, but also on *phi*. Phase of LG beam is given by

$$
\Phi_{LG}(r, \phi, z) = -il\phi + i(2p + l + 1) \arctan\left(\frac{z}{z_0} \right) - i \frac{kr^2}{2R(z)}
\tag{2.46}
$$

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(a) $p = 0, l = 0$ (b) $p = 0, l = 1$ (c) $p = 0, l = 2$

(d) $p = 1, l = 0$ (e) $p = 1, l = 1$ (f) $p = 1, l = 2$

(g) $p = 2, l = 0$ (h) $p = 2, l = 1$ (i) $p = 2, l = 2$
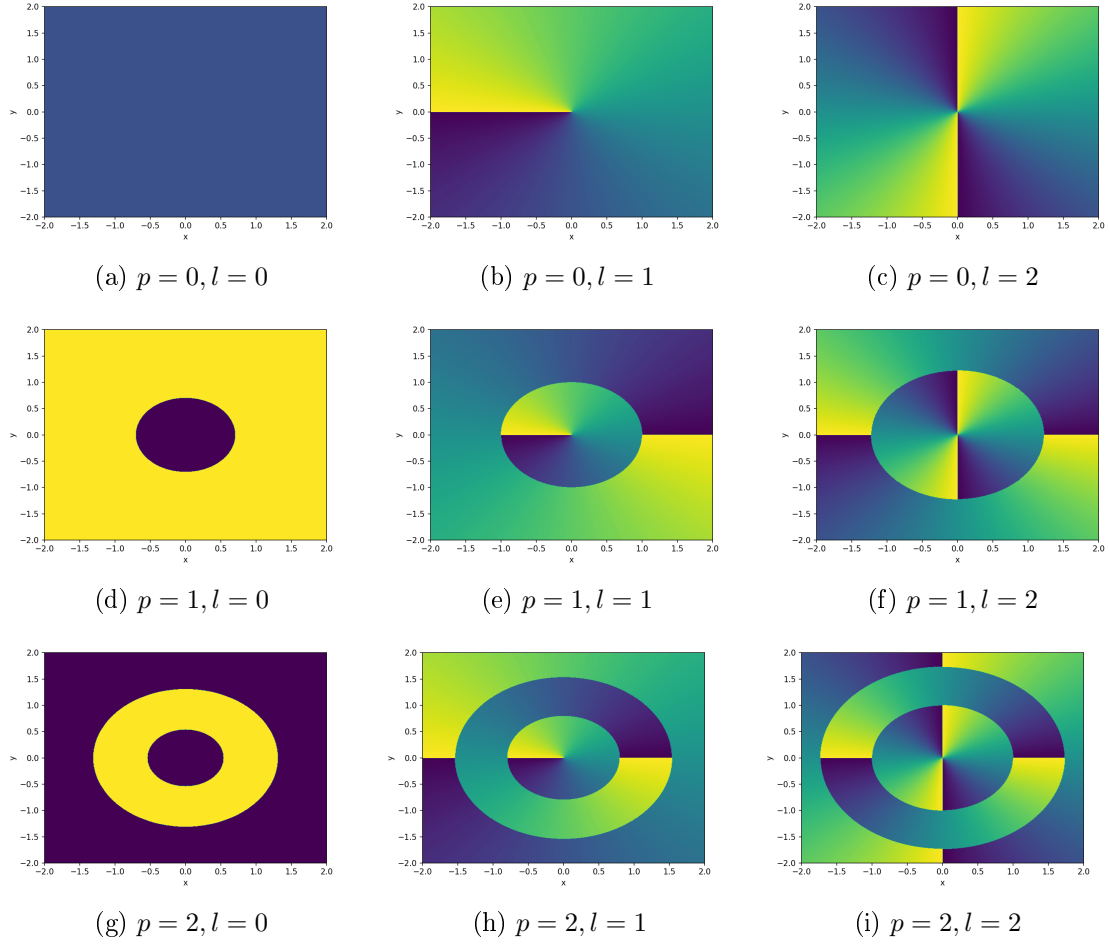
Figure 11: Phase variation for different modes in a cross section ($z = 0, z_0 = 1, w_0 = 1$)

Phase plot for different order of LG beam at $z = 0$ is given in figure 11.

4. As we will see later section, unlike the HG beam, LG bream carry orbital angular momentum due to its phase variation *w.r.t.* ϕ which results helical phase-front of the beam.

2.6   Relationship between LG & HG beam

[18]

2.7   Maxwell-Gaussian beam (with polarization)

[17][13][12]

Let us consider electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{E}$ of a EM wave propagating in

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z-direction, we write, in Cartesian coordinate[11],

$$\boldsymbol{E}(\boldsymbol{r}, t) = \boldsymbol{E_0} \, e^{i(kz - \omega t)} = \begin{bmatrix} E_{0x}(\boldsymbol{r}) \\ E_{0y}(\boldsymbol{r}) \\ E_{0y}(\boldsymbol{r}) \end{bmatrix} e^{i(kz - \omega t)} \tag{2.47}$$

$$\boldsymbol{B} = \boldsymbol{B_0} \, e^{i(kz - \omega t)} = \begin{bmatrix} B_{0x}(\boldsymbol{r}) \\ B_{0y}(\boldsymbol{r}) \\ B_{0y}(\boldsymbol{r}) \end{bmatrix} e^{i(kz - \omega t)} \tag{2.48}$$

As $\boldsymbol{E}$ and $\boldsymbol{B}$ satisfies Maxwell's equation in free space[11], and using $\omega/k = c$, we get,

$$\nabla \cdot \boldsymbol{E} = 0 \tag{2.49}$$

$$\Rightarrow ik E_{0z} + \nabla \cdot \boldsymbol{E_0} = 0 \tag{2.50}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{2.51}$$

$$\Rightarrow ik B_{0z} + \nabla \cdot \boldsymbol{B_0} = 0 \tag{2.52}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{2.53}$$

$$\Rightarrow ik\hat{z} \times \boldsymbol{E_0} + \nabla \times \boldsymbol{E_0} = ikc\boldsymbol{B_0} \tag{2.54}$$

$$\nabla \times \boldsymbol{B} = \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} \tag{2.55}$$

$$\Rightarrow ik\hat{z} \times \boldsymbol{B_0} + \nabla \times \boldsymbol{B_0} = -ik\frac{1}{c}\boldsymbol{E_0} \tag{2.56}$$

Assuming paraxial approximation 2.6, we get paraxial wave equation in vector form,

$$\nabla_T^2 \boldsymbol{E_0} - 2ik\frac{\partial \boldsymbol{E_0}}{\partial r} = 0 \tag{2.57}$$

$$\nabla_T^2 \boldsymbol{B_0} - 2ik\frac{\partial \boldsymbol{B_0}}{\partial r} = 0 \tag{2.58}$$

Here each component of $\boldsymbol{E_0}$ and $\boldsymbol{B_0}$ will satisfy the paraxial equation.

Now considering slowly varying envelop approximation 2.5 for each component of $\boldsymbol{E_0}$ and $\boldsymbol{B_0}$, from 2.50 and 2.52, we get,

$$E_{0z} = \frac{i}{k}\left(\frac{\partial E_{0x}}{\partial x} + \frac{\partial E_{0y}}{\partial y}\right) \tag{2.59}$$

$$B_{0z} = \frac{i}{k}\left(\frac{\partial B_{0x}}{\partial x} + \frac{\partial B_{0y}}{\partial y}\right) \tag{2.60}$$

[10]

___

[11]For cylindrical coordinate, refer [14].

3 MOMENTUM OF LIGHT

3.1 Introduction

We know EM wave carries energy as well as momentum. Momentum can be two types, linear and angular momentum. The *momentum density* of EM wave is momentum per unit volume.

3.2 Linear and angular momentum of light

If linear momentum density is $\boldsymbol{p}$ and linear momentum is $\boldsymbol{P}$ then,

$$\boldsymbol{p} = \frac{1}{c^2}\boldsymbol{S} = \epsilon_0 \boldsymbol{E} \times \boldsymbol{B} \tag{3.1}$$

$$\boldsymbol{P} = \int \boldsymbol{p}\, d\tau \tag{3.2}$$

Now let the monochromatic field with angular frequency ω, then

$$\boldsymbol{\mathcal{E}}(\boldsymbol{r}, t) = \mathrm{Re}\big\{\boldsymbol{E}(\boldsymbol{r})e^{-i\omega t}\big\} = \frac{1}{2}(\boldsymbol{E}e^{-i\omega t} + \boldsymbol{E}^* e^{i\omega t}) \tag{3.3}$$

$$\boldsymbol{\mathcal{B}}(\boldsymbol{r}, t) = \mathrm{Re}\big\{\boldsymbol{B}(\boldsymbol{r})e^{-i\omega t}\big\} = \frac{1}{2}(\boldsymbol{B}e^{-i\omega t} + \boldsymbol{B}^* e^{i\omega t}) \tag{3.4}$$

From Maxwell-Faraday equation in free space, we know that, [22]

$$\nabla \times \boldsymbol{\mathcal{E}} = -\frac{\partial}{\partial t}\boldsymbol{\mathcal{B}} \tag{3.5}$$

$$\Rightarrow \nabla \times \boldsymbol{E} = i\omega \boldsymbol{B} \tag{3.6}$$

The total energy is given by

$$W = \int d\tau\, \frac{I}{c} = \frac{\epsilon_0}{2} \int d\tau\, \boldsymbol{E}^* \cdot \boldsymbol{E} \tag{3.7}$$

Now time-averaged Poynting vector is given by, [21][22]

$$\langle \boldsymbol{S} \rangle = \frac{1}{\mu_0} \langle \boldsymbol{\mathcal{E}} \times \boldsymbol{\mathcal{B}} \rangle = \frac{1}{\mu_0}\, \frac{1}{2}\, \mathrm{Re}\{\boldsymbol{E}^* \times \boldsymbol{B}\} \tag{3.8}$$

From eq. 3.1 and 3.2 and using 3.6 and 3.8, time-averaged linear momentum is given by

$$\boldsymbol{P} = \int d\tau\, \langle \boldsymbol{S} \rangle \tag{3.9}$$

$$= \frac{\epsilon_0}{2} \int d\tau\, \mathrm{Re}\{\boldsymbol{E}^* \times \boldsymbol{B}\}$$

$$= \frac{\epsilon_0}{2i\omega} \int d\tau\, \mathrm{Re}\{\boldsymbol{E}^* \times (\nabla \times \boldsymbol{E})\}$$

$$= \frac{\epsilon_0}{2i\omega}\, \mathrm{Re}\left\{\int d\tau\, \boldsymbol{E}^* \times (\nabla \times \boldsymbol{E})\right\} \tag{3.10}$$

On partial integration and using the transversality[12] of E, considering the magnitude of $\boldsymbol{E}$ decreasing very rapidly when $r \to 0$, [19] we get[13]

$$\boldsymbol{P} = \frac{\epsilon_0}{2i\omega} \int d\tau \left[\sum_{\xi=x,y,z} E_\xi^* \nabla E_\xi \right] \tag{3.11}$$

which is quantum mechanical equivalent for linear momentum of a particle.

Let angular momentum density is $\boldsymbol{j}$ and angular momentum is $\boldsymbol{J}$ then,

$$\boldsymbol{j} = \frac{1}{c^2} \boldsymbol{r} \times \boldsymbol{S} = \epsilon_0 \boldsymbol{r} \times (\boldsymbol{E} \times \boldsymbol{B}) \tag{3.12}$$

$$\boldsymbol{J} = \int \boldsymbol{j} \, d\tau \tag{3.13}$$

Similarly from eq. 3.12 and 3.13 and using 3.6 and 3.8, time-averaged angular momentum,

$$\boldsymbol{J} = \int d\tau \, \boldsymbol{r} \times \langle \boldsymbol{S} \rangle \tag{3.14}$$

$$= \frac{\epsilon_0}{2} \int d\tau \, \boldsymbol{r} \times \text{Re}\{\boldsymbol{E}^* \times \boldsymbol{B}\}$$

$$= \frac{\epsilon_0}{2i\omega} \int d\tau \, \text{Re}\{\boldsymbol{r} \times (\boldsymbol{E}^* \times (\nabla \times \boldsymbol{E}))\}$$

$$= \frac{\epsilon_0}{2i\omega} \text{Re}\left\{ \int d\tau \, \boldsymbol{r} \times (\boldsymbol{E}^* \times (\nabla \times \boldsymbol{E})) \right\} \tag{3.15}$$

Now similarly on partial integration and using the transversality[14] of E, considering the magnitude of $\boldsymbol{E}$ decreasing very rapidly when $r \to 0$, [19][20] we get[15]

$$\boldsymbol{J} = \frac{\epsilon_0}{2i\omega} \int d\tau \left[\sum_{\xi=x,y,z} E_\xi^* (\boldsymbol{r} \times \nabla) E_\xi \right] + \frac{\epsilon_0}{2i\omega} \int d\tau \boldsymbol{E}^* \times \boldsymbol{E} \tag{3.16}$$

Now according to [23][19], within the paraxial approximation to get time-averaged energy, linear momentum and angular momentum per unit length *i.e.* $\mathcal{W}$, $\mathcal{P}$ and $\mathcal{J}$ respectively, along the beam propagating in z-direction, we have to integrate throughout the transverse xy-plane. So,

$$\mathcal{W}_z = \frac{\epsilon_0}{2i\omega} \iint dx \, dy \, \boldsymbol{E}^* \cdot \boldsymbol{E} \tag{3.17}$$

$$\mathcal{P}_z = \frac{\epsilon_0}{2i\omega} \iint dx \, dy \, \langle \boldsymbol{E} \times (\nabla \times \boldsymbol{E}) \rangle_z \tag{3.18}$$

$$\mathcal{J}_z = \frac{\epsilon_0}{2i\omega} \iint dx \, dy \, [\boldsymbol{r} \times \langle \boldsymbol{E} \times (\nabla \times \boldsymbol{E}) \rangle]_z \tag{3.19}$$

---

[12]what is it?
[13]have to see derivation in Chapter I in ref. [24]
[14]what is it?
[15]have to see derivation in Chapter I in ref. [24]

3.3 More on angular momentum

Let the electric field of the paraxial EM wave is [1]

$$\boldsymbol{E}(x, y, z) = \boldsymbol{F}(x, y, z) \, e^{-ikz} \tag{3.20}$$

s.t. $\boldsymbol{F}$ is the slowly varying spatial envelop, satisfies the paraxial wave equation 2.6 *i.e.*

$$\nabla_T^2 \boldsymbol{F} - 2ik\frac{\partial \boldsymbol{F}}{\partial r} = 0$$

also assumed that the beam waist w_0 (transverse dimension of the beam) is assumed to be much smaller than the diffraction length, $l_d = kw_0^2$ and z- component of $\boldsymbol{F}$ is smaller than transverse component by a factor of w_0/l_d. [25][19][1]

From previous eq. 3.19, z-component of $\mathcal{J}$ for 3.20 is given by

$$\mathcal{J}_z(z) = \underbrace{\frac{\epsilon_0}{2i\omega} \iint dx \, dy \, \left[F_\xi^* \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right) F_\xi \right]_{\xi=x,y}}_{\text{1st term}} + \underbrace{\frac{\epsilon_0}{2i\omega} \iint dx \, dy \, (F_x^* F_y + F_y^* F_x)}_{\text{2nd term}} \tag{3.21}$$

Note that, in 3.21, we can separately identify two terms. According to [1], the first is associated with transverse distribution of electric field (amplitude and phase) as $x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ is equivalent to $\frac{\partial}{\partial \phi}$ in cylindrical coordinate and the second term to the polarization of electric field. Thus it is concluded,[19] within the paraxial approximation, that

1. first term is the orbital angular momentum (OAM) per unit length in z of the EM wave,

$$\mathcal{L} = \frac{\epsilon_0}{2i\omega} \iint dx \, dy \, \left[F_\xi^* \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right) F_\xi \right]_{\xi=x,y} \tag{3.22}$$

   which is quantum mechanical equivalent of z-component of the angular momentum of a particle.

2. second term is the spin angular momentum (SAM) per unit length in z,

$$\mathcal{S} = \frac{\epsilon_0}{2i\omega} \iint dx \, dy \, (F_x^* F_y + F_y^* F_x) \tag{3.23}$$

3.3.1 Orbital and spin Angular Momentum

Let envelop $\boldsymbol{F}$ corresponding to the EM wave 3.20 for a vortex beam (*e.g.* LG beam) be,

$$\boldsymbol{F}(r, \phi, t) = u(r) \, \exp(-il\phi) \, \hat{\boldsymbol{p}}(t) \tag{3.24}$$

where $e^{il\phi}$ refers to helical phase front, and $\hat{\boldsymbol{p}} = p_x\hat{x} + p_y\hat{y} + p_z\hat{z}$ refers to unit polarization direction of electric field. Putting it in 3.22 considering $x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \equiv \frac{\partial}{\partial \phi}$, we get,

$$\mathcal{L} = \frac{\epsilon_0}{2i\omega} \iint r \, dr \, d\phi \, \left[F_\xi^* \frac{\partial}{\partial \phi} F_\xi \right]_{\xi=x,y}$$

$$= \frac{\epsilon_0}{2\omega} l \left(|p_x|^2 + |p_y|^2 \right) 2\pi \int r \, dr \, |u(r)|^2$$

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considering z-component of $\hat{\boldsymbol{p}}$ is smaller than transverse component by a factor of w_0/l_d, so $|p_x|^2 + |p_y|^2 \approx |p_x|^2 + |p_y|^2 + |p_z|^2 = |\hat{\boldsymbol{p}}|^2 = 1$, then

$$\mathcal{L} = \frac{\pi \epsilon_0 l}{\omega} \int r \, dr \, |u(r)|^2 \tag{3.25}$$

For the SAM part 3.23,

$$\begin{aligned} \mathcal{S} &= \frac{\epsilon_0}{2i\omega} \iint r \, dr \, d\phi \, (F_x^* F_y + F_y^* F_x) \\ &= \frac{\epsilon_0}{2i\omega} (p_x^* p_y - p_y^* p_x) 2\pi \int r \, dr \, |u(r)|^2 \end{aligned} \tag{3.26}$$

let us define a term *polarization helicity* σ as

$$\sigma = 2 \operatorname{Im}(p_x^* \, p_y) = \frac{1}{i}(p_x^* p_y - p_y^* p_x) \tag{3.27}$$

then

$$\mathcal{S} = \frac{\epsilon_0}{2\omega} \sigma \, 2\pi \int r \, dr \, |u(r)|^2 \tag{3.28}$$

The relation between polarization direction $\hat{\boldsymbol{p}}$ and Jones vector $\boldsymbol{J}$ is[16]

$$\hat{\boldsymbol{p}} = \boldsymbol{J} e^{i\omega t} \tag{3.29}$$

So

$$\sigma = \frac{1}{i}(p_x^* p_y - p_y^* p_x) = \frac{1}{i}(J_x^* J_y - J_y^* J_x) \tag{3.30}$$

Using table 1,

1. For LCP, $\sigma = \frac{1}{2i}(1i + 1i) = 1$

2. For RCP, $\sigma = \frac{1}{2i}(-1i - 1i) = -1$

3. for linearly polarized, $\sigma = \frac{1}{i}(1 - 1) = 0$, so no SAM.

Now for $\mathcal{W}$ in 3.17, we get,

$$\begin{aligned} \mathcal{W}_z &= \frac{\epsilon_0}{2} \iint r \, dr \, d\phi \, \boldsymbol{E}^* \cdot \boldsymbol{E} \\ &= \frac{\epsilon_0}{2} \iint r \, dr \, d\phi \, \boldsymbol{F}^* \cdot \boldsymbol{F} \\ &= \frac{\epsilon_0}{2} \left(|p_x|^2 + |p_y|^2 + |p_z|^2 \right) 2\pi \int r \, dr \, |u(r)|^2 \\ &= \pi \epsilon_0 \int r \, dr \, |u(r)|^2 \end{aligned} \tag{3.31}$$

---

[16]for fully polarized light only

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See from 3.25 and 3.31,

$$\frac{\mathcal{L}}{\mathcal{W}_z} = \frac{l}{\omega} = \frac{\text{OAM}}{\text{Total energy}} \tag{3.32}$$

from 3.28 and 3.31,

$$\frac{\mathcal{S}}{\mathcal{W}_z} = \frac{\sigma}{\omega} = \frac{\text{SAM}}{\text{Total energy}} \tag{3.33}$$

and from 3.32 and 3.33,

$$\frac{\mathcal{J}_z}{\mathcal{W}_z} = \frac{\mathcal{L} + \mathcal{S}}{\mathcal{W}_z} = \frac{l + \sigma}{\omega} = \frac{\text{Total AM}}{\text{Total energy}} \tag{3.34}$$

We know for a photon the energy associated with it is $\hbar\omega$, then from 3.32 and 3.33, we can say, the OAM associated with one photon is $l\hbar$ and SAM associated with one photon is $\sigma\hbar$

3.3.2 Intrinsic and Extrinsic nature of angular momentum

If we observe the expression of $\mathcal{J}_z$ in 3.19, it may depends on the choice of the axis (usually $\boldsymbol{r} = (0,0)$), from which we measure $\boldsymbol{r}$.[1][26] But if we shift the $\boldsymbol{r}$ *s.t.*

$$\boldsymbol{r} \longrightarrow \boldsymbol{r}' = \boldsymbol{r} + \boldsymbol{r_0} \tag{3.35}$$
$$(x, y) \longrightarrow (x', y') = (x, y) + (x_0, y_0) \tag{3.36}$$

then change of $\mathcal{J}_z$ will be

$$\mathcal{J}_z \longrightarrow \mathcal{J}'_z = \mathcal{J}_z + (\boldsymbol{r_0} \times \mathcal{P}) \cdot \hat{z} \tag{3.37}$$

The change in angular momentum,

$$\Delta\mathcal{J}_z = \mathcal{J}'_z - \mathcal{J}_z = (\boldsymbol{r_0} \times \mathcal{P}) \cdot \hat{z} \tag{3.38}$$
$$\Rightarrow \Delta\mathcal{J}_z = \frac{x_0\epsilon_0}{2} \iint dx\, dy\, \langle \boldsymbol{E} \times \boldsymbol{B} \rangle_y + \frac{y_0\epsilon_0}{2} \iint dx\, dy\, \langle \boldsymbol{E} \times \boldsymbol{B} \rangle_x \tag{3.39}$$

Now, we say the AM is *intrinsic*, when the AM does not depends on the choice of the reference axis, and *extrinsic* when it depends on the choice of axis.

If the AM is to be intrinsic, then for all (x_0, y_0)

$$\Delta\mathcal{J} = 0$$
$$\Rightarrow \iint dx\, dy\, \langle \boldsymbol{E} \times \boldsymbol{B} \rangle_y = 0 = \iint dx\, dy\, \langle \boldsymbol{E} \times \boldsymbol{B} \rangle_x \tag{3.40}$$

From the SAM part, we see $\mathcal{S}$ does not depend on the choice of the axis, so SAM is intrinsic. But for the OAM part, we see $\mathcal{L}$ depends on the choice of the axis, so OAM may be intrinsic or extrinsic determined by 3.40.

3.4 AM of Gaussian Beam

4 SPIN-ORBIT INTERACTION

4.1 Introduction

4.2 Spin-orbit energy

In quantum mechanics, spin-orbit interaction is a common phenomena. We can see it in the splitting of spectral lines into fine structure. For a simple case like H atom, a electron in orbit around the nucleus, the spin-orbit coupling results from the interaction between electron's spin magnetic moment and nucleus's orbital magnetic field, which we will discuss below. [1][27]

The spin magnetic moment of a electron is given by

$$\boldsymbol{\mu_s} = -\frac{e}{m_e c}\boldsymbol{S} \tag{4.1}$$

where e, m_e are absolute charge and mass of electron respectively and $\boldsymbol{S}$ is spin angular momentum of electron.

For our system, the electric field is centrally symmetric *i.e.*

$$\boldsymbol{E} = -\frac{1}{e}\frac{dU}{dr}\hat{\boldsymbol{r}} = -\frac{1}{er}\frac{dU}{dr}\boldsymbol{r} \tag{4.2}$$

where U is the potential energy,

$$U = \frac{1}{4\pi\epsilon_0}\frac{e^2}{r} \tag{4.3}$$

Now if electron rotates around the nucleus with instantaneous velocity $\boldsymbol{v}$, then from the rest frame of electron, the nucleus is rotating with $-\boldsymbol{v}$ around the electron thus creating a magnetic field in centre, which is[17]

$$\boldsymbol{B} = -\frac{1}{c}\boldsymbol{v} \times \boldsymbol{E} = -\frac{1}{m_e c}\boldsymbol{E} \times \boldsymbol{P} \tag{4.4}$$

where $\boldsymbol{P} = m_e\boldsymbol{v}$ is momentum.

Due to spin-orbit interaction, the corresponding interaction energy rises,

$$H_{so} = -\boldsymbol{\mu_s} \cdot \boldsymbol{B} = -\frac{e}{m_e c}\boldsymbol{S} \cdot \boldsymbol{B} = -\frac{e}{m_e^2 c^2}\boldsymbol{S} \cdot (\boldsymbol{E} \times \boldsymbol{P}) = \frac{1}{m_e^2 c^2 r}\frac{dU}{dr}\boldsymbol{S} \cdot (\boldsymbol{r} \times \boldsymbol{P}) \tag{4.5}$$

Taking orbital angular momentum $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{P}$, we get,

$$H_{so} = \frac{1}{m_e^2 c^2 r}\frac{dU}{dr}\boldsymbol{S} \cdot \boldsymbol{L} \tag{4.6}$$

We call H_{so} as *Spin-orbit energy.*[18]

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[17]the coefficient $1/c$ is in cgs, but $1/c^2$ in SI. [28]

[18]I have to check the Thomas precision.

4.3 Geometric phase of light

We know from the knowledge of waves, there is the phase factor corresponding to the EM wave due to optical path length difference, called *dynamical phase*. But there is another phase factor other than dynamical one, due to the geometry or topology of the evolution of the electromagnetic wave, we call it *Geometric phase*.[1] Geometric phases arises from intrinsic angular momentum and rotations of coordinates.[33] The two types of geometric phases, will be discussed, are

1. Spin-redirection Berry phase

2. Pancharatnam-Berry phase

4.3.1 Spin-redirection Berry phase

This *Berry phase* associated with adiabatic evolution of wave-vector $\boldsymbol{k}$. When the wave vector complete a adiabatic cycle, we will find a geometric phase arises from it. As an example, let a polarized light passes through a helical optic fibre with very low (negligible) birefringence and no torsional stress[19]. After one helical patch, the $\boldsymbol{k}$ wave-vector come to the same state, but gives rise of a helicity-dependant geometric phase[20] called *spin-redirection Berry phase*. [33]

For circularly polarized light, we can write Jones vector as

$$\boldsymbol{J} = \begin{bmatrix} 1 \\ i\sigma \end{bmatrix} \tag{4.7}$$

where σ is helicity (either $+1$ or -1).

Considering transversality of electric field, it will be tangent on the sphere in momentum space. And in one adiabatic cycle of wave-vector $w.r.t.$ k_z, the non-trivial parallel transport of the electric-field vectors takes place on the sphere.[33] So after transportation of a vector, the vector is rotated by an angle Θ [30] *i.e.* the frame is rotated by an angle Θ where

$$\Theta = 2\pi(1 - \cos\theta) \tag{4.8}$$

where θ is half-apex angle of the cone formed by sweeping the wave vector $\boldsymbol{k}$ (see fig 12b). Note that Θ is the solid angle at at the apex of the cone.

After one adiabatic cycle, electric field vector $\boldsymbol{J}$ transform into $\boldsymbol{J}'$

$$\boldsymbol{J} \longrightarrow \boldsymbol{J}' = \boldsymbol{J}\exp(i\Theta_G) \tag{4.9}$$

where Θ_G is the acquired helicity-dependant geometric phase $s.t.$

$$\Theta_G = \sigma\Theta \tag{4.10}$$

So,

$$|L\rangle \longrightarrow e^{i\Theta} |L\rangle \tag{4.11}$$

$$|R\rangle \longrightarrow e^{-i\Theta} |R\rangle \tag{4.12}$$

---

[19]Torsional stress produces circular birefringence by elasto-optic effect. [?]

[20]This is a *non-holonomic process*. [29]

(a) Evolution of polarization axis along helical optic fibre (ref. [32])

(b) Evolution of wave-vector $\boldsymbol{k}$ along helical optic fibre in momentum space
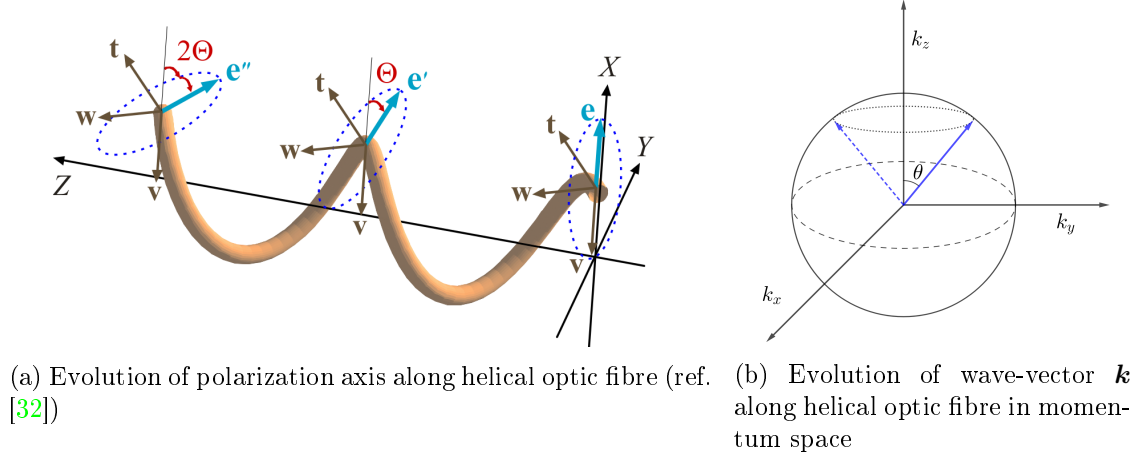
Figure 12: Geometric Berry phase arises along helical optic fibre

Now for x-polarized light, $|x_{in}\rangle$ and $|x_{out}\rangle$ will be

$$|x_{in}\rangle = \frac{1}{\sqrt{2}}\left(|L\rangle + |R\rangle\right) \tag{4.13}$$

$$|x_{out}\rangle = \frac{1}{\sqrt{2}}\left(e^{i\Theta}|L\rangle + e^{-i\Theta}|R\rangle\right) \tag{4.14}$$

So the x-polarized light is rotated by angle Θ, shown in fig 12a. Note that θ does not originate from intrinsic anisotropy and it is geometric one. We call it *spin-redirection Berry phase*.

4.3.2   Pancharatnam-Berry Phase

Unlike the previous one, the *Pancharatnam-Berry phase* arises after a cyclic evolution in Poincare sphere when the state of polarization changes keeping wave-vector $\boldsymbol{k}$ is fixed. Convenient example of observation of Pancharatnam-Berry phase is Michelson interferometer setup, (see fig. 13) in which one arm of the interferometer has two quarter wave-plates, one (QP1) is fixed and another one (QP2) is rotatable. [34]
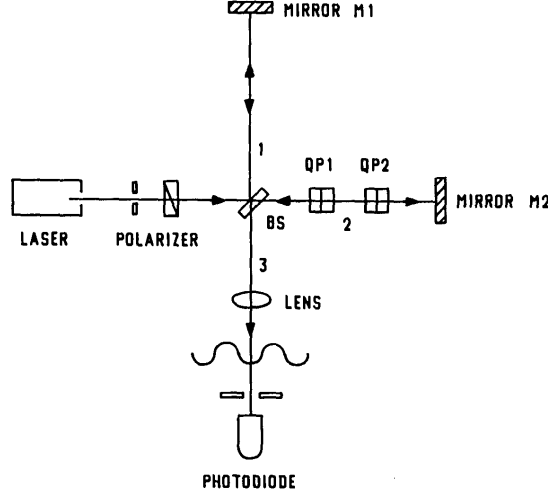
Figure 13: Schematic of Michelson interferometer setup for observation of Pancharatnam-Berry phase(ref. [34])

Fast axis of QP1 and QP2 are aligned at $\theta_1 = \pi/4$ and $\theta_2 = \beta$ $w.r.t.$ x-axis, respectively. The input light is x-polarized. Then the polarization state of output light at different part of arm 2 in Michelson interferometer is shown in fig 14, and corresponding Jones calculus shown in table 5.
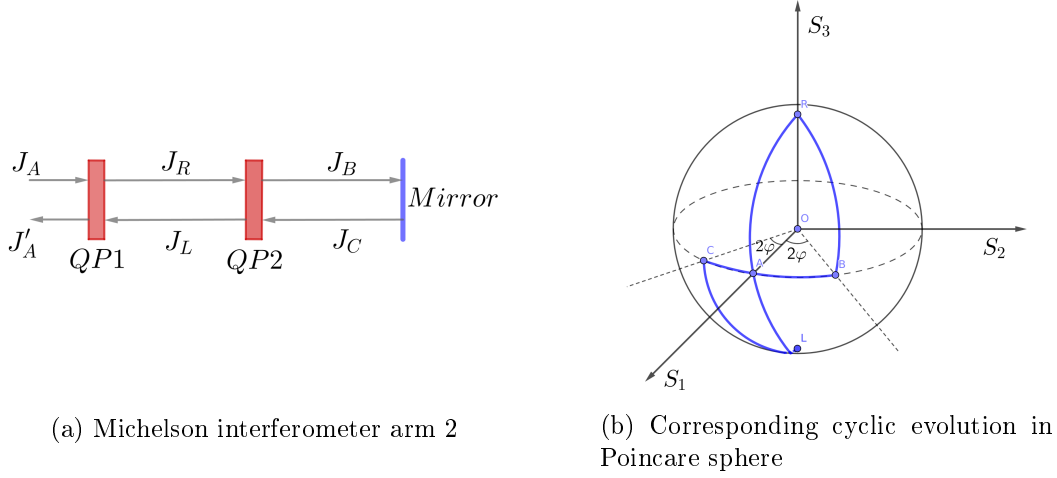

(a) Michelson interferometer arm 2

(b) Corresponding cyclic evolution in Poincare sphere

Figure 14: Evolution of polarization state in arm 2

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Poincare Sphere point (pol. state)	Polarization state (Jones vector, $\boldsymbol{J}$)
A ($|A\rangle$)	$\boldsymbol{J}_A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
R ($|R\rangle$)	$\boldsymbol{J}_R = R(-\pi/4)\,\boldsymbol{M}_{QP}\,R(\pi/4)\,\boldsymbol{J}_A = \dfrac{e^{i\phi'}}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $= \dfrac{e^{i\phi'}}{2}\begin{bmatrix} 1+i \\ 1-i \end{bmatrix} = \dfrac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}\exp(i\phi_1)$
B ($|B\rangle$)	$\boldsymbol{J}_B = R(-\beta)\,\boldsymbol{M}_{QP}\,R(\beta)\,\boldsymbol{J}_R$ $= e^{i\phi''}\begin{bmatrix} \cos^2(\beta)+i\sin^2(\beta) & (1-i)\sin(\beta)\cos(\beta) \\ (1-i)\sin(\beta)\cos(\beta) & \sin^2(\beta)+i\cos^2(\beta) \end{bmatrix}\dfrac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}\exp(i\phi_1)$ $= \begin{bmatrix} \cos(\beta+\pi/4) \\ \sin(\beta+\pi/4) \end{bmatrix}\exp(i\phi_2)\,\exp(-i\beta)$ $= \begin{bmatrix} \cos(\varphi) \\ \sin(\varphi) \end{bmatrix}\exp(i\phi_3)\exp(-i\varphi)\ \text{ where }\ \varphi=\beta+\pi/4$
C ($|C\rangle$)	$\boldsymbol{J}_C = \boldsymbol{M}_{Mirror}\,\boldsymbol{J}_B = e^{i\pi/2}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \cos(\varphi) \\ \sin(\varphi) \end{bmatrix}\exp(i\phi_3)\exp(-i\varphi)$ $= \begin{bmatrix} -\cos(\varphi) \\ \sin(\varphi) \end{bmatrix}\exp(i\phi_4)\exp(-i\varphi)$
L ($|L\rangle$)	$\boldsymbol{J}_L = \begin{bmatrix} 1 \\ i \end{bmatrix}\exp(i\phi_5)\exp(-i\,2\varphi)$
A ($|A'\rangle$)	$\boldsymbol{J}'_A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\exp(i\phi_6)\exp(-i\,2\varphi)$ $= \boldsymbol{J}_A\exp(i\phi_6)\exp(-i\,2\varphi)$

Table 5: Jones vector of polarization state

We see that the light with linear polarization state ($|B\rangle$) acquired a additional phase term *i.e.* $\exp(-i\,2\varphi)$ which only depends on orientation of QP2 *i.e.* $\varphi = \beta + \pi/4$ and does not depend on thickness and refractive index of the birefringent wave-plate, so this phase is purely geometric one. Other all ϕ's are dynamical phase factor.[1]
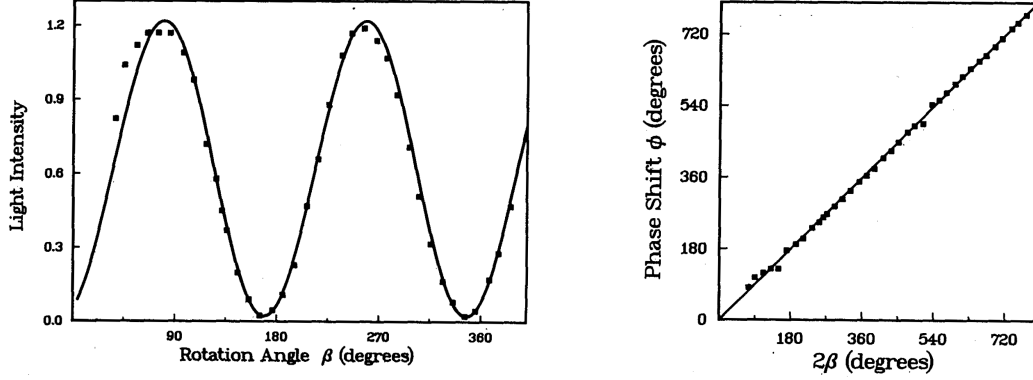
We see after one cyclic evolution in Poincare sphere in path ARBCLA,

$$|A\rangle \longrightarrow |A'\rangle = |A\rangle\exp(i\phi_6 - i\,2\varphi) \tag{4.15}$$

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So at photodiode the intensity variation *w.r.t.* β will be

$$
\begin{aligned}
I &= (\langle A| + \langle A'|)(|A\rangle + |A'\rangle) \\
&= \langle A|A\rangle \left(1 + \exp(-i\phi_6 + i2\varphi)\right)\left(1 + \exp(i\phi_6 - i2\varphi)\right) \\
&= \langle A|A\rangle \left(2 + 2\cos(2\varphi - \phi_6)\right) \\
\Rightarrow I(\beta) &= 2\langle A|A\rangle \left(1 - \sin(2\beta - \phi_6)\right)
\end{aligned}
\tag{4.16}
$$

Experimental verification by Chyba *et al* (ref. [34]) is given in figure 15.

(a) Measured intensity vs Rotation angle β of QP2, fitted with eq. **??**

(b) Measured phase shift vs rotation angle β, with solid line of $\phi = 2\beta$
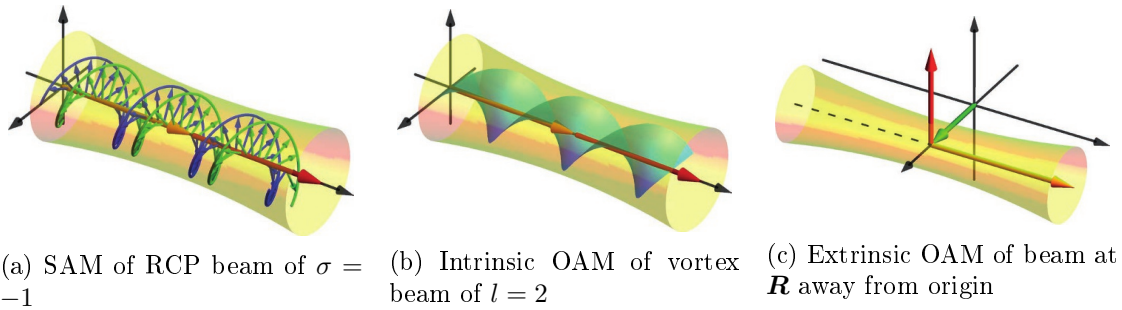
Figure 15: Measurement of the Pancharatnam phase by Chyba *et al* [34]

4.4 Types of SOI

We have discussed brief of angular momentum of light in previous chapter. The different types of angular momentum a EM beam carries are

1. Spin AM ($\boldsymbol{S}$)

2. Intrinsic orbital AM ($\boldsymbol{L}_{int}$)

3. Extrinsic orbital AM ($\boldsymbol{L}_{ext}$)

The diagrammatic representation is given in figure 16.

(a) SAM of RCP beam of $\sigma = -1$

(b) Intrinsic OAM of vortex beam of $l = 2$

(c) Extrinsic OAM of beam at $\boldsymbol{R}$ away from origin

Figure 16: Angular momentum of paraxial beam (ref. [33])

35

The inter-conversion between these different angular momentum in a process represents *spin-orbit interaction* of light.[33] The three type of interaction are

1. between $\boldsymbol{S}$ and $\boldsymbol{L}_{int}$

2. between $\boldsymbol{S}$ and $\boldsymbol{L}_{ext}$

3. between $\boldsymbol{L}_{int}$ and $\boldsymbol{L}_{ext}$

In the later section, we will see several manifestation of those interactions.

4.5   SOI in isotropic medium

[35]

4.6   SOI in inhomogeneous anisotropic medium

[36][20][33]

4.7   SOI in non-paraxial fields

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