Summer Project

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1 POLARIZATION

1.1 Introduction

We know, in EM wave, the electric field and magnetic field oscillating perpendicularly in the transverse plane *w.r.t.* the propagation direction. *Polarization* is the property of an EM wave, which deals with the temporal and spatial variation of the orientation of field vector (mainly, electric field) of the EM wave. Here we mainly discuss Jones formalism, Stokes-Muller formalism and finally apply those thing in elliptically polarized light.

1.2 Jones formalism

1.2.1 Jones Vector

Vector form of electric field of fully polarized EM wave propagating along z-axis is given by

$$\boldsymbol{E}(\boldsymbol{x},t) = \begin{bmatrix} E_x(\boldsymbol{x},t) \\ E_y(\boldsymbol{x},t) \\ E_z(\boldsymbol{x},t) \end{bmatrix} = \begin{bmatrix} A_x(\boldsymbol{x})e^{-i(kz-\omega t-\delta_x)} \\ A_y(\boldsymbol{x})e^{-i(kz-\omega t-\delta_y)} \\ 0 \end{bmatrix} = \begin{bmatrix} A_x(\boldsymbol{x})e^{i\delta_x} \\ A_y(\boldsymbol{x})e^{i\delta_y} \\ 0 \end{bmatrix} e^{-i(kz-\omega t)} \tag{1.1}$$

We define normalized[1] *Jones vector* as

$$\boldsymbol{J}(\boldsymbol{x},t) = \frac{1}{\sqrt{A_x^2 + A_y^2}} \begin{bmatrix} A_x(\boldsymbol{x})e^{i\delta_x} \\ A_y(\boldsymbol{x})e^{i\delta_y} \end{bmatrix} \tag{1.2}$$

Such examples of usual polarization states are given below [3],

Polarization state	$\boldsymbol{J}$
$\lvert H\rangle$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$
$\lvert V\rangle$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
$\lvert P\rangle$	$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
$\lvert M\rangle$	$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$
$\lvert L\rangle$	$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$
$\lvert R\rangle$	$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}$

Table 1: Jones vector of usual polarization state

Some properties of Jones vector are

[1] normalized as $\boldsymbol{J}\,\boldsymbol{J}^* = 1$

1. The intensity of the EM wave is given by

$$I = \frac{1}{2}c\epsilon_0(A_x^2 + A_y^2) = \frac{1}{2}c\epsilon_0(E^* E) \tag{1.3}$$

2. For general elliptically polarized light we can measure the azimuth (α) ellipticity (ϵ) of the polarization ellipse by comparing Jones vector $\boldsymbol{J}$ with [1]

$$\begin{bmatrix} \cos\alpha\cos\epsilon - i\sin\alpha\sin\epsilon \\ \sin\alpha\cos\epsilon + i\cos\alpha\sin\epsilon \end{bmatrix}$$

1.2.2 Jones Matrix & evolution of Jones vector

Jones matrix is a 2×2 matrix assigned for a particular optical element. Let $\boldsymbol{M}$ be the Jones matrix for an optical element *s.t.*

$$\boldsymbol{M} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$$

then if a polarized light of Jones vector $\boldsymbol{J}_{in}$ passes through that optical element then the Jones vector of output light is given by

$$\boldsymbol{J}_{out} = \boldsymbol{M}\ \boldsymbol{J}_{in} \tag{1.4}$$

$$\Rightarrow \boldsymbol{E}_{out} = \boldsymbol{M}\ \boldsymbol{E}_{in} \tag{1.5}$$

To determine m_{ij} in $\boldsymbol{M}$,

1. Pass x-polarized light and determine $\boldsymbol{J}_{out}$, then

$$\boldsymbol{J}_{out} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} m_{11} \\ m_{21} \end{bmatrix}$$

2. Pass y-polarized light and determine $\boldsymbol{J}_{out}$, then

$$\boldsymbol{J}_{out} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} m_{12} \\ m_{22} \end{bmatrix}$$

Such examples of usual Jones matrix [2] are given below,[3]

[2]For polariser the Jones matrix $\boldsymbol{M} = \boldsymbol{J}\ \boldsymbol{J}^*$ where $\boldsymbol{J}$ is normalized Jones vector corresponding polarization state *s.t.* $\boldsymbol{J}_{out} = \boldsymbol{M}\boldsymbol{J} = (\boldsymbol{J}\boldsymbol{J}^*)\boldsymbol{J} = \boldsymbol{J}(\boldsymbol{J}^*\boldsymbol{J}) = \boldsymbol{J}$

Optical element	$\boldsymbol{M}$
Free space	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
x-Polariser	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
y-Polariser	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$
Right circular polariser	$\frac{1}{2}\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$
Left circular polariser	$\frac{1}{2}\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}$
Linear di-attenuator	$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$
Half-wave plate with fast axis horizontal	$e^{-i\pi/2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Quarter-wave plate with fast axis horizontal	$e^{-i\pi/4}\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$
General phase retarder	$\begin{bmatrix} e^{i\phi_x} & 0 \\ 0 & e^{i\phi_y} \end{bmatrix}$

Table 2: Jones matrix related to usual optical element

Some properties of Jones matrix are

1. Resultant Jones matrix for composition of n optical elements is given by

$$\boldsymbol{M} = \boldsymbol{M}_1\ \boldsymbol{M}_2 \ldots \boldsymbol{M}_n \tag{1.6}$$

2. For an optical element when its optical axis aligned at an angle θ *w.r.t.* x-axis then resultant Jones matrix for this rotated optical element is given by

$$\boldsymbol{M}_\theta = R(-\theta)\ \boldsymbol{M}\ R(\theta) \tag{1.7}$$

where $R(\theta)$ is passive rotation matrix *s.t.*

$$R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{1.8}$$

1.2.3 Drawback of Jones formalism

Main drawback of Jones formalism is that its application is restricted in fully polarized light. This formalism cannot explain the partially polarized or unpolished light which we frequently observe in practical use.

1.3 Stokes-Muller formalism

1.3.1 Coherency matrix

Coherency matrix of a EM wave is defined as [1]

$$\boldsymbol{C} = \left\langle \boldsymbol{E} \otimes \boldsymbol{E}^\dagger \right\rangle = \left\langle \boldsymbol{E}\boldsymbol{E}^\dagger \right\rangle = \begin{bmatrix} \langle E_x E_x^* \rangle & \langle E_x E_y^* \rangle \\ \langle E_y E_x^* \rangle & \langle E_y E_y^* \rangle \end{bmatrix} = \begin{bmatrix} c_{xx} & c_{xy} \\ c_{yx} & c_{yy} \end{bmatrix} \tag{1.9}$$

where $\otimes$ denotes Kronecker product, $\langle \cdot \rangle$ denotes the temporal avg of the corresponding quantity and $\delta = \delta_y - \delta_x$.

Examples of coherency matrix of usual polarization states are given below [4],

Polarization state	$\boldsymbol{J}$	$\boldsymbol{C}$
$\lvert H\rangle$	$\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}1&0\\0&0\end{bmatrix}$
$\lvert V\rangle$	$\begin{bmatrix}0\\1\end{bmatrix}$	$\begin{bmatrix}0&0\\0&1\end{bmatrix}$
$\lvert P\rangle$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$	$\frac{1}{2}\begin{bmatrix}1&1\\1&1\end{bmatrix}$
$\lvert M\rangle$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$	$\frac{1}{2}\begin{bmatrix}1&-1\\-1&1\end{bmatrix}$
$\lvert L\rangle$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}$	$\frac{1}{2}\begin{bmatrix}1&-i\\i&1\end{bmatrix}$
$\lvert R\rangle$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-i\end{bmatrix}$	$\frac{1}{2}\begin{bmatrix}1&i\\-i&1\end{bmatrix}$
Un-polarized	–	$\frac{1}{2}\begin{bmatrix}1&0\\0&1\end{bmatrix}$

Table 3: Coherency matrix of usual polarization state

Some properties of coherency matrix are

1. It is a hermitian matrix *i.e.* $\boldsymbol{C} = \boldsymbol{C}^\dagger$
2. Trace and determinant off the matrix are non-negative[3] *i.e.* $\mathrm{tr}(\boldsymbol{C}) > 0$ & $\det(\boldsymbol{C}) \geq 0$.
3. $\mathrm{Tr}(\boldsymbol{C}) = \langle E_x E_x^* \rangle + \left\langle E_y E_y^* \right\rangle$ is the time averaged intensity of input light.
4. let the polarized light (of electric field $\boldsymbol{E}_{in}$ & coherency matrix $\boldsymbol{C}_{in}$) passes through an optical element (of Jones matrix $\boldsymbol{M}$) then let output electric field be $\boldsymbol{E}_{out}$ by the equation 1.5, then output coherency matrix $\boldsymbol{C}_{out}$ is given by

$$\begin{aligned} \boldsymbol{C}_{out} = \left\langle \boldsymbol{E}_{out}\boldsymbol{E}_{out}^\dagger \right\rangle &= \left\langle (\boldsymbol{M}\boldsymbol{E}_{in})\,(\boldsymbol{M}\boldsymbol{E}_{in})^\dagger \right\rangle \\ &= \left\langle (\boldsymbol{M}\boldsymbol{E}_{in}) \left(\boldsymbol{E}_{in}^\dagger \boldsymbol{M}^\dagger\right) \right\rangle \\ &= \boldsymbol{M} \left\langle \boldsymbol{E}_{in}\boldsymbol{E}_{in}^\dagger \right\rangle \boldsymbol{M}^\dagger \\ &= \boldsymbol{M}\,\boldsymbol{C}_{in}\,\boldsymbol{M}^\dagger \end{aligned} \tag{1.10}$$

[3] Proofs to be done

1.3.2 Stokes parameters and Stokes vector

Now we see that coherency matrix $\boldsymbol{C}$ of any polarization state in table 3 can be written in the linear combination of the 4 basis given below [5]

$$\beta = \left\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\boldsymbol{V_0}}, \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{\boldsymbol{V_1}}, \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\boldsymbol{V_2}}, \underbrace{\begin{bmatrix} 0 & i \\ -i & 1 \end{bmatrix}}_{\boldsymbol{V_3}} \right\} \tag{1.11}$$

Now we can write any coherency matrix $\boldsymbol{C}$ as

$$\boldsymbol{C} = \frac{1}{2}\sum_{i=0}^{3} S_i \boldsymbol{V_i} \tag{1.12}$$

Note that all V_i's are Hermitian, so obviously is $\boldsymbol{C}$.

We call $\{S_0, S_1, S_2, S_3\}$ as a *Stokes parameter* and the values of S_i's are experimentally measurable.

A *Stokes vector* $\boldsymbol{S}$ is defined as[4]

$$\boldsymbol{S} = \begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} \tag{1.13}$$

Examples of Stokes vector for different polarization states are given below

Polarization state	$\boldsymbol{C}$	$\boldsymbol{S}$
$\vert H\rangle$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}^T$
$\vert V\rangle$	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix}^T$
$\vert P\rangle$	$\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}^T$
$\vert M\rangle$	$\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix}^T$
$\vert L\rangle$	$\frac{1}{2}\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}^T$
$\vert R\rangle$	$\frac{1}{2}\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & -1 \end{bmatrix}^T$
Un-polarized	$\frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T$

Table 4: Stokes vector of usual polarization state

Note that all Jones vectors has Stokes vectors but converse need not to be true.

[4] for intensity normalised Stokes vector, $\boldsymbol{s} = \begin{bmatrix} 1 & s_1 & s_2 & s_3 \end{bmatrix}$ where $s_i = S_i/S_0$

Now we see from the equation 1.12

$$\begin{bmatrix} \langle E_x E_x^* \rangle & \langle E_x E_y^* \rangle \\ \langle E_y E_x^* \rangle & \langle E_y E_y^* \rangle \end{bmatrix} = \boldsymbol{C} = \frac{1}{2} \sum_{i=0}^{3} S_i \boldsymbol{V_i} = \frac{1}{2} \begin{bmatrix} S_0 + S_1 & S_2 + iS_3 \\ S_2 - iS_3 & S_0 - S_1 \end{bmatrix} \tag{1.14}$$

From there we can write

$$\boldsymbol{S} = \begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = \begin{bmatrix} \langle E_x E_x^* \rangle + \langle E_x E_y^* \rangle \\ \langle E_x E_x^* \rangle - \langle E_x E_y^* \rangle \\ \langle E_y E_x^* \rangle + \langle E_x E_y^* \rangle \\ i \left(\langle E_y E_x^* \rangle - \langle E_x E_y^* \rangle \right) \end{bmatrix} \tag{1.15}$$

Now for a polarized light,

$$\boldsymbol{C} = \begin{bmatrix} \langle A_x^2 \rangle & \langle A_x A_y e^{-i\delta} \rangle \\ \langle A_x A_y e^{i\delta} \rangle & \langle A_y^2 \rangle \end{bmatrix} \text{ and } \boldsymbol{S} = \begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = \begin{bmatrix} \langle A_x^2 + A_y^2 \rangle \\ \langle A_x^2 - A_y^2 \rangle \\ \langle 2A_x A_y \cos\delta \rangle \\ \langle 2A_x A_y \sin\delta \rangle \end{bmatrix} \tag{1.16}$$

1.3.3 Measurement of Stokes parameters

To measure the 4 Stokes parameter of EM wave associated with, we have to do 4 steps experiment. In each case, we pass the light through various optical elements and measure the (time-averaged) intensity [6],

Step I Pass the light through homogenous isotropic medium (or, free space) and measure the intensity. From table 2 and eq. 1.10, we get,

$$\begin{aligned} \boldsymbol{C}_{out} &= \boldsymbol{M}\, \boldsymbol{C}_{in}\, \boldsymbol{M}^\dagger \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} S_0 + S_1 & S_2 + iS_3 \\ S_2 - iS_3 & S_0 - S_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} S_0 + S_1 & 0 \\ 0 & S_0 - S_1 \end{bmatrix} \end{aligned} \tag{1.17}$$

So the measured intensity will be

$$I_0 = \mathrm{tr}(\boldsymbol{C}_{out}) = S_0 \tag{1.18}$$

Step II Pass the light through x-polariser and measure the intensity. From table 2 and eq. 1.10, we get,

$$\begin{aligned} \boldsymbol{C}_{out} &= \boldsymbol{M}\, \boldsymbol{C}_{in}\, \boldsymbol{M}^\dagger \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \frac{1}{2} \begin{bmatrix} S_0 + S_1 & S_2 + iS_3 \\ S_2 - iS_3 & S_0 - S_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} S_0 + S_1 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned} \tag{1.19}$$

So the measured intensity will be

$$I_1 = \mathrm{tr}(\boldsymbol{C}_{out}) = \frac{1}{2}(S_0 + S_1) \tag{1.20}$$

Step III Pass the light through the polariser with transmission axis is at 45° and measure the intensity. Then from eq. 1.7, $\boldsymbol{M}$ for this polariser will be

$$\boldsymbol{M} = R(-45^\circ)\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}R(45^\circ) = \frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} \tag{1.21}$$

From eq. 1.10, we get,

$$\begin{aligned}\boldsymbol{C}_{out} &= \boldsymbol{M}\,\boldsymbol{C}_{in}\,\boldsymbol{M}^\dagger\\ &= \boldsymbol{M}\,\boldsymbol{C}_{in}\,\boldsymbol{M}^\dagger\\ &= \frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\frac{1}{2}\begin{bmatrix}S_0+S_1 & S_2+iS_3\\S_2-iS_3 & S_0-S_1\end{bmatrix}\frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\\ &= \frac{1}{4}\begin{bmatrix}S_0+S_2 & S_0+S_2\\S_0+S_2 & S_0+S_2\end{bmatrix}\end{aligned} \tag{1.22}$$

So the measured intensity will be

$$I_1 = \mathrm{tr}(\boldsymbol{C}_{out}) = \frac{1}{2}(S_0+S_2) \tag{1.23}$$

Step IV Pass the light through right circular polariser and measure the intensity. From table 2 and eq. 1.10, we get,

$$\begin{aligned}\boldsymbol{C}_{out} &= \boldsymbol{M}\,\boldsymbol{C}_{in}\,\boldsymbol{M}^\dagger\\ &= \frac{1}{2}\begin{bmatrix}1 & i\\-i & 1\end{bmatrix}\frac{1}{2}\begin{bmatrix}S_0+S_1 & S_2+iS_3\\S_2-iS_3 & S_0-S_1\end{bmatrix}\frac{1}{2}\begin{bmatrix}1 & i\\-i & 1\end{bmatrix}\end{aligned} \tag{1.24}$$

So the measured intensity will be

$$I_1 = \mathrm{tr}(\boldsymbol{C}_{out}) = \frac{1}{2}(S_0+S_3) \tag{1.25}$$

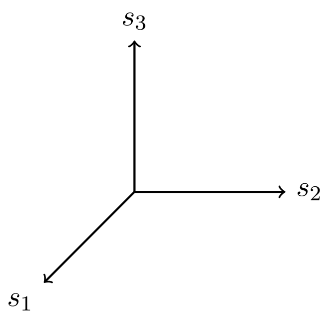
From the equations 1.18, 1.20, 1.23 and 1.25, we can get the values of all S_i's.

1.3.4 Poincare sphere representation

For total intensity normalised Stokes vector is $\boldsymbol{s} = \begin{bmatrix}1 & s_1 & s_2 & s_3\end{bmatrix}^T$ where $s_i = S_i/S_0$. Observe that $\boldsymbol{s}$ is a 3-dimensional quantity. Therefore we can write,

$$\begin{bmatrix}1\\s_1\\s_2\\s_3\end{bmatrix} \rightarrow \begin{bmatrix}s_1\\s_2\\s_3\end{bmatrix}$$

Poincare sphere representation is a coordinate system to define the state of polarization of light where the mutually orthogonal coordinate axes are $\{s_1, s_2, s_3\}$.



Example of special cases are

Case I For fully polarized light

$$s_1 = \frac{A_x^2 - A_y^2}{A_x^2 + A_y^2} \tag{1.26}$$

$$s_2 = \frac{2A_y A_y \cos\delta}{A_x^2 + A_y^2} \tag{1.27}$$

$$s_3 = \frac{2A_y A_y \sin\delta}{A_x^2 + A_y^2} \tag{1.28}$$

from there we can see

$$s_1^2 + s_2^2 + s_3^2 = 1 \tag{1.29}$$

which implies that fully polarized has the locus at any point in the sphere of radius 1 in Poincare sphere representation.

Case II For fully un-polarized light

$$s_1 = s_2 = s_3 = 0 \tag{1.30}$$

which implies that fully un-polarized has the locus at any the centre $(0, 0, 0)$ in the sphere of radius 1 in Poincare sphere representation.

1.3.5 Degree of Polarization

Degree of Polarization is the measure of polarization of light.
We define

- Total degree of polarization, $DOP = \sqrt{s_1^2 + s_2^2 + s_3^2}$
- Degree of linear polarization $= \sqrt{s_1^2 + s_2^2}$
- Degree of circular polarization $= \sqrt{s_1^2 + s_2^2 + s_3^2}$

For any mixed polarization state we can decompose the Stokes vector into fully polarized and un-polarized components,

$$
\begin{bmatrix} 1 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \underbrace{\begin{bmatrix} \sqrt{s_1^2 + s_2^2 + s_3^2} \\ s_1 \\ s_2 \\ s_3 \end{bmatrix}}_{\text{fully polarized, } DOP=1} + \underbrace{\begin{bmatrix} 1 - \sqrt{s_1^2 + s_2^2 + s_3^2} \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\text{un-polarized}} \tag{1.31}
$$

1.3.6 Muller Matrix & evolution of Stokes vector

Similar to the Jones matrix, *Muller matrix* is a 4×4 matrix assigned for a particular optical element. Let $\mathfrak{M}$ be the Muller matrix for an optical element *s.t.*

$$
\mathfrak{M} = \begin{bmatrix} \mu_{11} & \cdots & \mu_{14} \\ \vdots & \ddots & \vdots \\ \mu_{41} & \cdots & \mu_{44} \end{bmatrix}
$$

then if a light of Stokes vector $\boldsymbol{S}_{in}$ passes through that optical element, then the Stokes vector of output light is given by

$$
\boldsymbol{S}_{out} = \mathfrak{M}\, \boldsymbol{S}_{in} \tag{1.32}
$$

Some properties of Jones matrix are

1. Resultant Muller matrix for composition of n optical elements is given by

$$
\mathfrak{M} = \mathfrak{M}_1\, \mathfrak{M}_2 \ldots \mathfrak{M} \tag{1.33}
$$

2. When th optical element is aligned at an angle θ *w.r.t.* x-axis then resultant Muller matrix (similar to Jones matrix) for this rotated optical element is given by

$$
\mathfrak{M}_\theta = T^{-1}(\theta)\, \mathfrak{M}\, T(\theta) \tag{1.34}
$$

where $T(\theta)$ is passive rotation matrix in Poincare sphere representation *w.r.t* s_3 axis, *s.t.*

$$
T(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\theta & \sin 2\theta & 0 \\ 0 & -\sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1.35}
$$

Note that, in eq. 1.35, if we write

$$
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\theta & \sin 2\theta & 0 \\ 0 & -\sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} \cos 2\theta & \sin 2\theta & 0 \\ -\sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.36}
$$

we see that it is proper rotation matrix of rotation angle 2θ in Poincare sphere *w.r.t* s_3 axis. And as we know that rotation of θ of electric field results in rotation of 2θ in azimuth angle of Stokes vector in Poincare sphere.

1.3.7 Relationship between Jones & Stokes-Muller formalism

Let, $\boldsymbol{J}$ be jones vector, $\boldsymbol{M}$ be the Jones matrix, $\boldsymbol{S}$ be the Stokes vector and $\mathfrak{M}$ be the Muller matrix *s.t.* equations 1.4 and 1.32 is satisfied.

Let us define *coherency vector* of 1.9 as

$$\boldsymbol{L} = \begin{bmatrix} c_{xx} & c_{xy} & c_{yx} & c_{yy} \end{bmatrix}^T \tag{1.37}$$

and *Wolf matrix* $\boldsymbol{W}$ as

$$\boldsymbol{L}_{out} = \boldsymbol{W}\, \boldsymbol{L}_{in} \tag{1.38}$$

then the relation between Jones and Wolf matrix[5] is

$$\boldsymbol{W} = \boldsymbol{M} \otimes \boldsymbol{M}^* \tag{1.39}$$

Now from equations 1.9 and 1.15, one can write

$$\boldsymbol{S} = \boldsymbol{A}\, \boldsymbol{L} \tag{1.40}$$

where,

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & -i & i & 0 \end{bmatrix} \tag{1.41}$$

then the relation between Jones and Muller matrix[6] is

$$\mathfrak{M} = \boldsymbol{A}\,(\boldsymbol{M} \otimes \boldsymbol{M}^*)\,\boldsymbol{A}^{-1} \tag{1.42}$$

Note that this relationship is only possible in both ways, if the light is fully polarized light as all Jones vectors has Stokes vectors but converse need not to be true.

1.4 More on Elliptically polarized light

1.4.1 Jones vector of elliptically polarized light

In this section we will discuss the generalized polarization ellipse of an EM wave. Let our electric field vector of EM wave is given by

$$\boldsymbol{E} = \begin{bmatrix} E_x \\ E_y \end{bmatrix} = \begin{bmatrix} a_1 \cos(\tau + \delta_1) \\ a_2 \cos(\tau + \delta_2) \end{bmatrix} \text{ where } \tau = kz - \omega \text{ and } ta_1, a_2 \geq 0 \tag{1.43}$$

by eliminating τ we get,

$$\frac{1}{a_1^2} E_x^2 + \frac{1}{a_2^2} E_y^2 - \frac{2\cos\delta}{a_1 a_2} E_x E_y = \sin^2(\delta) \tag{1.44}$$

where $\delta = \delta_2 - \delta_1$. The eq. 1.44 is equation of circle when $a_1 = a_2$, otherwise, of ellipse[2].

[5] proof to be done
[6] proof to be done

Now we do the change of basis $\{E_x, E_y\} \longmapsto \{E_\xi, E_\eta\}$ (See fig. 1) *s.t.* electric field in $\{E_\xi, E_\eta\}$ basis be

$$\begin{aligned} \boldsymbol{E} &\to \boldsymbol{F} \\ \boldsymbol{F} &= \begin{bmatrix} E_\xi \\ E_\eta \end{bmatrix} = \begin{bmatrix} a\cos(\tau+\delta_0) \\ \pm b\cos(\tau+\delta_0) \end{bmatrix} \text{ where } a \geq b \geq 0 \end{aligned} \tag{1.45}$$

which is parametric form of canonical ellipse[7] in $\{E_\xi, E_\eta\}$ basis.

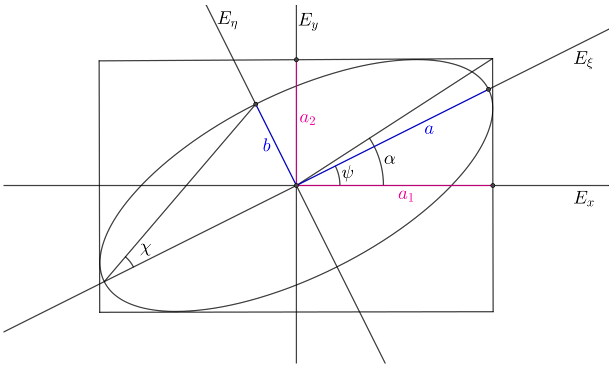


Figure 1: Polarization ellipse

Let ψ be the azimuth angle of the ellipse then

$$\boldsymbol{F} = R(\psi)\,\boldsymbol{E} \tag{1.46}$$

$$\Rightarrow \begin{bmatrix} E_\xi \\ E_\eta \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{bmatrix} \begin{bmatrix} E_x \\ F_y \end{bmatrix} \tag{1.47}$$

$$\Rightarrow \begin{bmatrix} a\cos(\tau+\delta_0) \\ \pm b\cos(\tau+\delta_0) \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{bmatrix} \begin{bmatrix} a_1\cos(\tau+\delta_1) \\ a_2\cos(\tau+\delta_2) \end{bmatrix} \tag{1.48}$$

We want value of a, b, After some tedious calculation [2], we reach to some important results, given below

$$a^2 + b^2 = a_1^2 + a_2^2 \tag{1.49}$$

$$\pm\, ab = a_1 a_2 \sin\delta \tag{1.50}$$

$$\tan\chi := \pm\frac{b}{a} \text{ where } \chi \in [-\frac{\pi}{4}, \frac{\pi}{4}] \tag{1.51}$$

$$\tan\alpha := \frac{a_2}{a_1} \text{ where } \alpha \in [0, \frac{\pi}{2}] \tag{1.52}$$

$$\tan 2\psi = \tan 2\alpha \cos\delta \tag{1.53}$$

$$\sin 2\chi = \sin 2\alpha \sin\delta \tag{1.54}$$

where ψ is the *azimuth* and χ is *ellipticity* of the polarization ellipse.

To see the handedness of the rotation of electric field vector in transverse plane,

Case I For right-handed polarization, $\sin\delta > 0$, then from equations 1.50, and 1.51, we can say

$$\tan\chi \geq 0 \Rightarrow \chi \in \left(0, \frac{\pi}{4}\right]$$

[7] $\pm$ before b denotes the handedness of the rotation of electric field vector in transverse plane.

Case II Similarly for left-handed polarization, $\sin\delta < 0$, then from equations 1.50, and 1.51, we can say

$$\tan\chi \leq 0 \Rightarrow \chi \in \left[-\frac{\pi}{4}, 0\right)$$

Now the Jones vector of elliptical polarization in the form of ellipticity and azimuth[8] will be,

$$\boldsymbol{J} = \begin{bmatrix} \cos\psi\cos\chi - i\sin\psi\sin\chi \\ \sin\psi\cos\chi + i\cos\psi\sin\chi \end{bmatrix} \tag{1.55}$$

1.4.2 Stokes vector and corresponding Poincare representation

From the eq. 1.16, we can write for our case[9],

$$\boldsymbol{S} = \begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = \begin{bmatrix} a_1^2 + a_2^2 \\ a_1^2 - a_2^2 \\ 2a_1a_2\cos\delta \\ 2a_1a_2\sin\delta \end{bmatrix} = S_0 \begin{bmatrix} 1 \\ \cos2\chi\cos2\psi \\ \cos2\chi\sin2\psi \\ \sin2\chi \end{bmatrix} \tag{1.56}$$

So in Poincare sphere representation with axes $\{S_1, S_2, S_3\}$, the required vector is

$$S_0 \begin{bmatrix} 1 \\ \cos2\chi\cos2\psi \\ \cos2\chi\sin2\psi \\ \sin2\chi \end{bmatrix} \longrightarrow (S_0\cos2\chi\cos2\psi, S_0\cos2\chi\sin2\psi, S_0\sin2\chi) \tag{1.57}$$

The evolution of azimuth (ψ) and ellipticity (χ) of the polarization state in Poincare representation is shown in the figure 2.

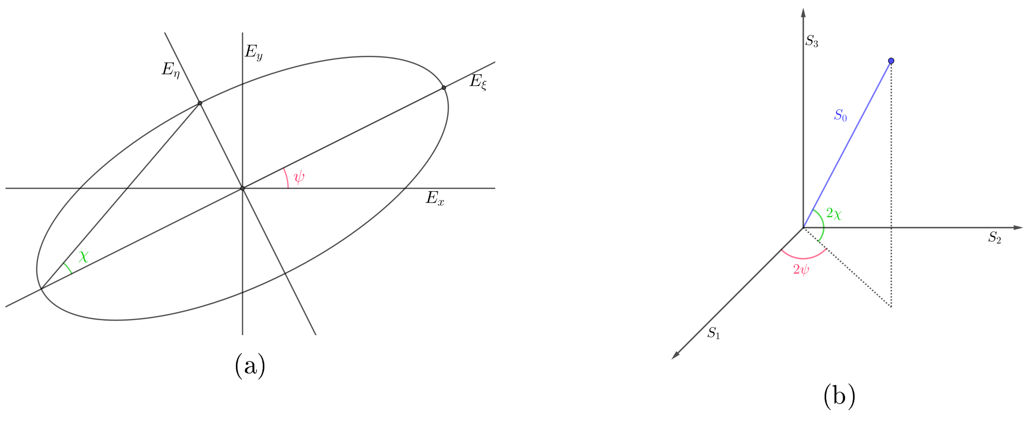


Figure 2: polarization ellipse and corresponding Poincare representation

[8] proof to be done
[9] proof to be done

2 GAUSSIAN BEAM

2.1 Introduction

2.2 Paraxial wave equation and solutions

From Maxwell's 3-D wave equation for electric field in vacuum,

$$\nabla^2 \boldsymbol{E}(\boldsymbol{r},t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\boldsymbol{E}(\boldsymbol{r},t) = 0 \tag{2.1}$$

Here we will consider the scalar form of the equation.

To find the scalar solution, let our first ansatz be,

$$E = F(x,y,z)e^{i\omega t} \tag{2.2}$$

Putting it in eq. 2.1, we get Helmholtz equation *i.e.*

$$\nabla^2 F(x,y,z) + k^2 f(x,y,z) = 0 \text{ where } k^2 = \frac{\omega^2}{c^2} \tag{2.3}$$

For light to travel in z-direction, our 2nd ansatz be,

$$F(\boldsymbol{r}) = \psi(x,y,z)e^{-ikz} \tag{2.4}$$

Considering the slowly varying envelope approximation[1] that,

$$\lambda^2 \left|\frac{\partial^2\psi}{\partial z^2}\right| << \lambda \left|\frac{\partial\psi}{\partial z}\right| << |\psi| \tag{2.5}$$

and putting 2.4 in eq. 2.3 gives *Paraxial wave equation*,

$$\nabla_T^2\psi - 2ik\frac{\partial\psi}{\partial r} = 0 \tag{2.6}$$

where transverse Laplacian, $\nabla_T = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ or in cylindrical coordinate, $\nabla_T = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}$.

2.2.1 Scalar wave solution (without polarization)

We will solve the paraxial wave equation in cylindrical coordinate $\{r,\phi,z\}$.[8] To get a solution for which intensity is of Gaussian like and radially symmetric (*i.e.* no variation with ϕ), our first ansatz be,[9][8]

$$\psi(\boldsymbol{r},z) = A\exp\left[-i\left(p(z) + \frac{kr^2}{2q(z)}\right)\right] = A\underbrace{\exp\left[-ip(z)\right]}_{\text{first term}}\underbrace{\exp\left[-i\frac{kr^2}{2q(z)}\right]}_{\text{second term}} \tag{2.7}$$

where second term is related to Gaussian intensity and first term is additional phase factor. Putting this, in eq, 2.6, we get

$$\left[\frac{k^2}{q^2}\left(\frac{dq}{dz} - 1\right)r^2 - 2k\left(\frac{dp}{dz} + \frac{i}{q}\right)\right]\psi = 0 \tag{2.8}$$

To satisfy this, for all r, we get

$$\frac{dq}{dz} - 1 = 0 \tag{2.9}$$

$$\frac{dp}{dz} + \frac{i}{q} = 0 \tag{2.10}$$

Lets first calculate eq. 2.9.

$$\frac{dq}{dz} - 1 = 0 \Rightarrow q(z) = z + q_0 \tag{2.11}$$

putting this in the second term of expression 2.7 at $z = 0$,

$$\exp\left[-i\frac{kr^2}{2q(0)}\right] = \exp\left[-i\frac{kr^2}{2q_0}\right] \tag{2.12}$$

which is a phase factor does not give Gaussian intensity. So to get Gaussian intensity, q_0 must be imaginary. Let $j_0 = iz_0$, then

$$\boxed{q(z) = z + iz_0} \tag{2.13}$$

Now, the second term of expression 2.7 at $z = 0$ be,

$$\exp\left[-\frac{kr^2}{2z_0}\right] = \exp\left[-\frac{r^2}{w_0^2}\right] \tag{2.14}$$

where

$$w_0^2 = \frac{2z_0}{k} = \frac{\lambda z_0}{\pi} \Rightarrow \boxed{z_0 = \frac{\pi w_0^2}{\lambda}} \tag{2.15}$$

We call z_0 *confocal parameter* or *Rayleigh range* of the beam.

Now from expression 2.13, we calculate $1/q(z)$.

$$\frac{1}{q(z)} = \frac{1}{z + iz_0} = \frac{z}{z^2 + z_0^2} - i\frac{z_0}{z^2 + z_0^2} \tag{2.16}$$

Then the second term of expression 2.7 be,

$$\exp\left[-i\frac{kr^2}{2q(z)}\right] = \underbrace{\exp\left[-\frac{kr^2 z_0}{2(z^2 + z_0^2)}\right]}_{\text{term A}} \underbrace{\exp\left[-i\frac{kr^2 z}{2(z^2 + z_0^2)}\right]}_{\text{term B}} \tag{2.17}$$

Write term A of 2.17 as

$$\exp\left[-\frac{kr^2 z_0}{2(z^2 + z_0^2)}\right] = \exp\left[-\frac{r^2}{w^2(z)}\right] \tag{2.18}$$

which is Gaussian, where

$$\boxed{w^2(z) = w_0^2\left[1 + \left(\frac{z}{z_0}\right)^2\right]} \tag{2.19}$$

We call $w(z)$ *physical radius/half-width* of the beam.

Write term B of 2.17 as

$$\exp\left[-i\frac{kr^2 z}{2(z^2+z_0^2)}\right] = \exp\left[-i\frac{kr^2}{2R(z)}\right] \tag{2.20}$$

where

$$\boxed{R(z) = z\left[1+\left(\frac{z_0}{z}\right)^2\right]} \tag{2.21}$$

We know for spherical wave,

$$E(\boldsymbol{r},t) \sim \frac{1}{r}e^{i(\omega t - kr)} \tag{2.22}$$

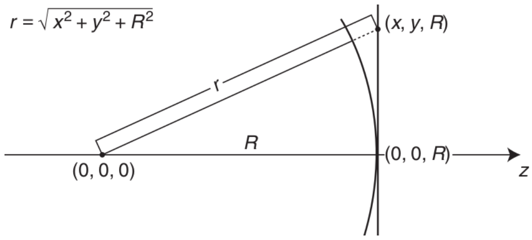


Figure 3: Radius of curvature of spherical wavefront (Ref. [7])

Let R be the radius of curvature of spherical wavefront. Now for any point $\boldsymbol{r} = (x, y, R)$ on $z = R$ plane, r will be

$$r = \sqrt{x^2+y^2+R^2} \tag{2.23}$$

For collimated beam, we restrict radius of curvature measurement near $\boldsymbol{r} = (0, 0, R)$, so

$$r = \sqrt{x^2+y^2+R^2} \approx R + \frac{x^2+y^2}{2R} \tag{2.24}$$

Now from 2.22,

$$E(\boldsymbol{r},t) \sim \frac{1}{r}e^{i\omega t}e^{-ikr}e^{-ik\frac{x^2+y^2}{2R}} \tag{2.25}$$

comparing with 2.2

$$\psi(x,y,z) \sim e^{-ik\frac{x^2+y^2}{2R}} \tag{2.26}$$

comparing 2.26 in the above expression with term B of 2.17, we conclude that $R(z)$ in 2.21 is *radius of curvature* of wavefront near $r = 0$ of collimated beam in far field.

Putting 2.19 and 2.21 in eq. 2.16,

$$\boxed{\frac{1}{q(z)} = \frac{1}{R(z)} - i\,\frac{\lambda}{\pi w^2(z)}} \tag{2.27}$$

Now we simplify the first term in 2.7. By solving the differential equation 2.10 and putting 2.13 we get,

$$\frac{dp}{dz} = \frac{-i}{q} = \frac{-i}{z + iz_0} \Rightarrow i\,p(z) = \ln\left[1 - i\frac{z}{z_0}\right] \tag{2.28}$$

As we can write

$$1 - i\frac{z}{z_0} = \sqrt{1 + \left(\frac{z}{z_0}\right)^2} \exp\left[-i\,\tan^{-1}\left(\frac{z}{z_0}\right)\right]$$

putting this in the above expression of $i\,p(z)$, our final expression will be

$$i\,p(z) = \frac{1}{2}\ln\left[1 + \left(\frac{z}{z_0}\right)^2\right] - i\,\tan^{-1}\left(\frac{z}{z_0}\right) \tag{2.29}$$

Finally putting 2.27 and 2.29 in 2.7, we get,[9][8]

$$\begin{aligned}
\psi(\boldsymbol{r}, z) &= A\exp\left[-i\left(p(z) + \frac{kr^2}{2q(z)}\right)\right] \\
&= A\exp\left[-\frac{1}{2}\ln\left[1 + \left(\frac{z}{z_0}\right)^2\right] + i\,\tan^{-1}\left(\frac{z}{z_0}\right) - i\frac{kr^2}{2}\left(\frac{1}{R(z)} - i\,\frac{\lambda}{\pi w^2(z)}\right)\right] \\
&= \frac{A}{\sqrt{1 + \left(\frac{z}{z_0}\right)^2}}\exp\left(i\,\tan^{-1}\left(\frac{z}{z_0}\right)\right)\exp\left(-i\frac{kr^2}{2R(z)}\right)\exp\left(-\frac{r^2}{w^2(z)}\right) \\
\Rightarrow \psi(\boldsymbol{r}, z) &= A\underbrace{\left(\frac{w_0}{w(z)}\right)}_{\text{term I}}\underbrace{\exp\left(i\,\tan^{-1}\left(\frac{z}{z_0}\right)\right)}_{\text{term II}}\underbrace{\exp\left(-i\frac{kr^2}{2R(z)}\right)}_{\text{term III}}\underbrace{\exp\left(-\frac{r^2}{w^2(z)}\right)}_{\text{term IV}}
\end{aligned} \tag{2.30}$$

In that expression,

1. Term I $\longrightarrow$ related to spreading of beam along propagation in z.
2. Term II $\longrightarrow$ related to *Gouy phase.*
3. Term III $\longrightarrow$ gives radius of curvature of beam wave front.
4. Term IV $\longrightarrow$ gives radially symmetric Gaussian intensity profile.

2.3 Characteristics of Gaussian Beam

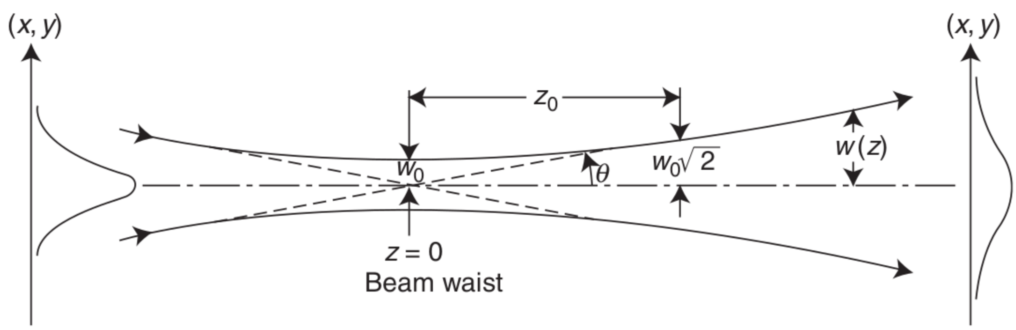
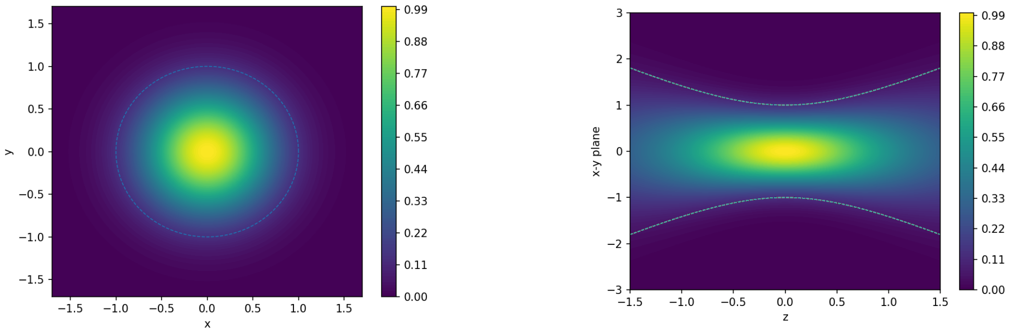


Figure 4: A beam profile (Ref. [7])

Some characteristics of Gaussian beam are

1. *Intensity* of Gaussian beam in any transverse plain is,

$$
\begin{aligned}
I(r,z) &= \frac{1}{2}\epsilon_0 c|E^*E| = \frac{1}{2}\epsilon_0 c|\psi^*\psi| \\
&= \frac{1}{2}\epsilon_0 c|A|^2\left(\frac{w_0}{w(z)}\right)^2 \exp\left(-\frac{2r^2}{w^2(z)}\right)
\end{aligned}
\tag{2.31}
$$



(a) Intensity variation in a cross section

(b) Intensity variation along z-axis

Figure 5: Gaussian intensity profile for $z_0 = 1, w_0 = 1$

2. *Rate of energy* of Gaussian beam passes through any transverse plain is given by

$$
\begin{aligned}
W &= \iint_{-\infty}^{\infty} dx\, dy\, I(x,y,z) \\
&= \frac{1}{2}\epsilon_0 c|A|^2\left(\frac{w_0}{w(z)}\right)^2 \iint_{-\infty}^{\infty} dx\, dy\, \exp\left(-\frac{2(x^2+y^2)}{w^2(z)}\right) \\
&= \frac{1}{2}\epsilon_0 c|A|^2\left(\frac{w_0}{w(z)}\right)^2 \int_{-\infty}^{\infty} dx\, \exp\left(-\frac{2x^2}{w^2(z)}\right) \int_{-\infty}^{\infty} dy\, \exp\left(-\frac{2y^2}{w^2(z)}\right) \\
&= \frac{1}{2}\epsilon_0 c|A|^2\left(\frac{w_0}{w(z)}\right)^2 (\sqrt{\pi}w(z))^2 = \frac{1}{2}\epsilon_0 c|A|^2 w_0^2
\end{aligned}
\tag{2.32}
$$

   which is constant throughout the propagation along z-axis.

3. *Radius of curvature* $R(z)$ of the wavefront is given by

$$
R(z) = z\left[1 + \left(\frac{z_0}{z}\right)^2\right]
\tag{2.33}
$$

   For $z = 0$ $R \to \infty$

   For $z >> z_0$, then $R \approx z$

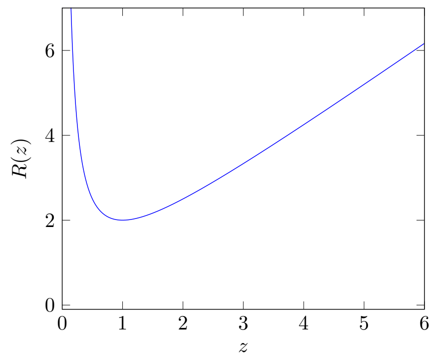


Figure 6: Variation of radius of curvature with z ($z_0 = 1$)

4. *Beam half-width* (see fig. 4) is given by

$$w^2(z) = w_0\sqrt{1 + \left(\frac{z}{z_0}\right)^2} \tag{2.34}$$

   For $z = 0$ $w = w_0$ (*Beam waist*)
   For $z >> z_0$, then $w(z) = w_0\frac{z}{z_0}$
   Diffraction angle at far field is given by

$$2\theta = 2\frac{dw}{dz} = 2\frac{w_0}{z_0} = \frac{2\lambda}{\pi w_0} \tag{2.35}$$

   Effective area of the beam in a cross section is $\dfrac{1}{2}\pi w^2(z)$

5. *Gouy phase* of a gaussian beam is given by

$$\phi_g(z) = \tan^{-1}\left(\frac{z}{z_0}\right) \tag{2.36}$$

   The Gouy phase vary form $-\pi/2$ to $\pi/2$ continuously as z goes from $-\infty$ to ∞, shown in fig. 7

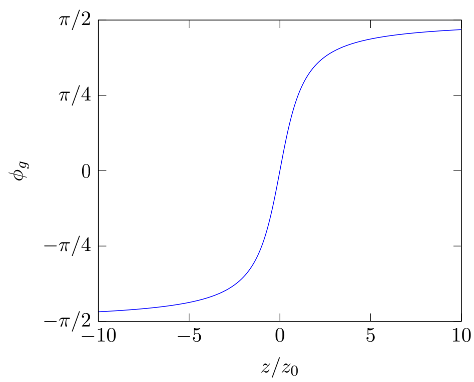


Figure 7: Variation of Gouy phase with z

6. Inside Rayleigh length (z_0), the laser beam is highly collimated and intensity is also very high. So gain medium is kept in between $z = -z_0$ and z_0 to get maximum stimulated emission from gain medium.

2.4 Different modes of Gaussian beams

Here we will discuss mainly two types of higher order Gaussian beams *i.e.*

1. Hermite-Gaussian (HG) beam
2. Laguerre-Gaussian (LG) beam

2.4.1 Hermite-Gaussian beam

In the expression 2.30, we get the radially symmetric Gaussian beam solution. But we now seek higher order solution of Gaussian beam which is rectangular symmetric.

Lets take the ansatz as,

$$\psi(\boldsymbol{r}, z) = A\, g\left(\frac{x}{w(z)}\right) h\left(\frac{y}{w(z)}\right) \exp\left[-i\left(p(z) + \frac{kr^2}{2q(z)}\right)\right] \tag{2.37}$$

Putting this in Paraxial wave equation 2.6, and solving the differential equation,[7] we get,

$$\begin{aligned}\psi_{m,n}(\boldsymbol{r}, z) = A\frac{w_0}{w(z)} H_m\left(\frac{\sqrt{2}x}{w(z)}\right) H_n\left(\frac{\sqrt{2}y}{w(z)}\right) \exp\left(-\frac{r^2}{w^2(z)}\right)\cdot \\ \exp\left(i\,(m+n+1)\tan^{-1}\left(\frac{z}{z_0}\right) - i\frac{kr^2}{2R(z)}\right)\end{aligned} \tag{2.38}$$

where H_i is *i th order Hermite polynomial*and other symbols are as usual. See for $m = 0 = n$ we recover the Gaussian solution of 2.30 which we call as *zero order* HG beam. For different values of m an n, we will get diferent type of higher order HG beam, these are called *transverse electromagnetic mode* of order (m, n), notionally, TEM_{mn}.

Some characteristics of HG beams are given below,

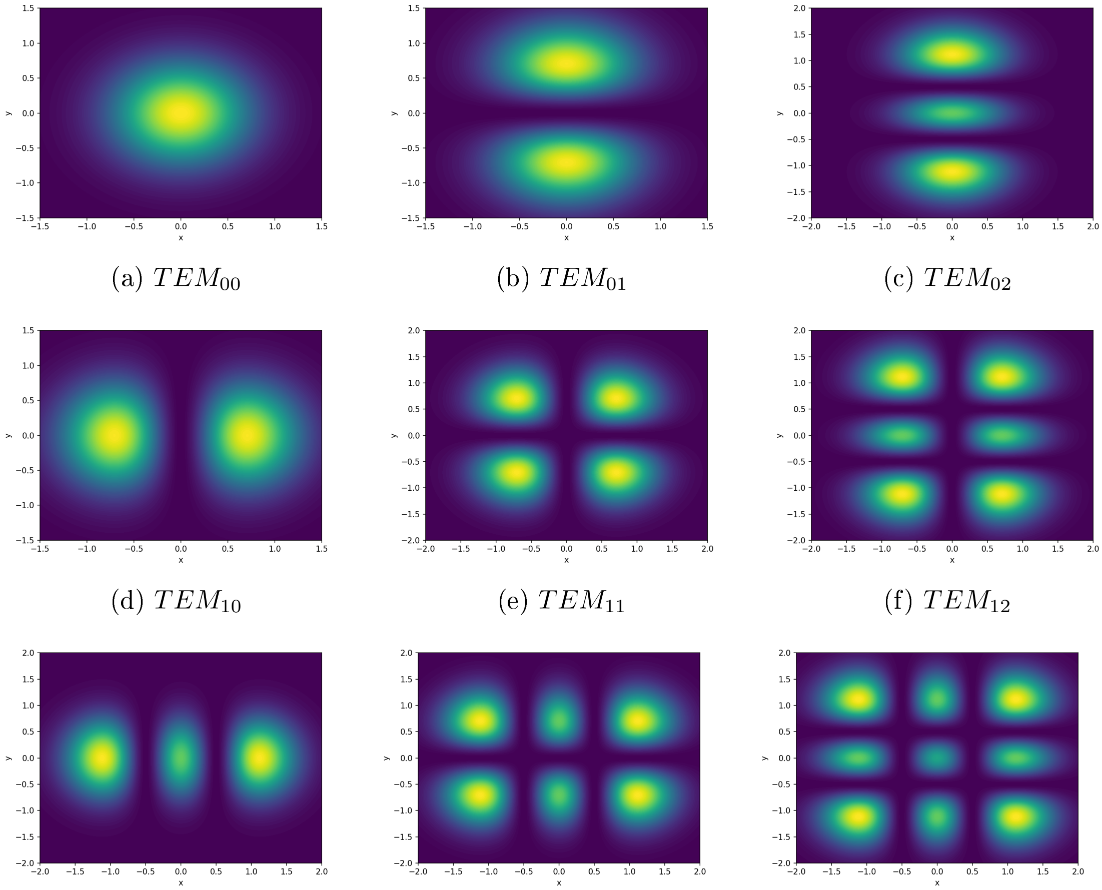
1. *Intensity* of HG beam is given by

$$I_{m,n}(x, y, z) = \frac{c\epsilon}{2}|A|^2 \left[H_m\left(\frac{\sqrt{2}x}{w(z)}\right)\right]^2 \left[H_n\left(\frac{\sqrt{2}y}{w(z)}\right)\right]^2 \exp\left(\frac{2(x^2+y^2)}{w^2(z)}\right) \tag{2.39}$$

Due to the number of zeros equals the order of Hermite polynomial, we will see m number of horizontal and n number of vertical node in intensity profile of the TEM_{mn} beam. See figure 8.

2. *Rate of energy* of HG beam passes through any transverse plain is given by

$$\begin{aligned}W &= \iint_{-\infty}^{\infty} dx\, dy\, I(x, y, z) \\ &= \frac{1}{2}\epsilon_0 c|A|^2 \left(\frac{w_0}{w(z)}\right)^2 \int_{-\infty}^{\infty} dx\, \exp\left(-\frac{2x^2}{w^2(z)}\right) \left[H_m\left(\frac{\sqrt{2}x}{w(z)}\right)\right]^2 \\ &\int_{-\infty}^{\infty} dy\, \exp\left(-\frac{2y^2}{w^2(z)}\right) \left[H_n\left(\frac{\sqrt{2}y}{w(z)}\right)\right]^2\end{aligned} \tag{2.40}$$



(a) TEM_{00} (b) TEM_{01} (c) TEM_{02}

(d) TEM_{10} (e) TEM_{11} (f) TEM_{12}

(g) TEM_{20} (h) TEM_{21} (i) TEM_{22}

Figure 8: Intensity variation for different TEM in a cross section $(z = 0, z_0 = 1, w_0 = 1)$

the integration terms of 2.40 are in the form of

$$\int_{-\infty}^{\infty} H_l \exp(-a\xi^2) d\xi = \int_{-\infty}^{\infty} \left(\sum_{k=0}^{l} c_k \xi_k \right) \exp(-a\xi^2) d\xi = \sum_{k=0}^{l} c_k \int_{-\infty}^{\infty} \xi_k \exp(-a\xi^2) d\xi$$

as the values of $\int_{-\infty}^{\infty} \xi_k \exp(-a\xi^2) d\xi$ is fixed[10] and for finite value of l, W is finite and constant throughout the propagation along z-axis.

3. *Radius of curvature* of HG beam is same as simple Gaussian beam for all modes.

4. *Gouy phase* for different order HG beam is given by

$$\phi_g(\eta, z) = \eta \tan^{-1}\left(\frac{z}{z_0}\right) \text{ where } \eta = m + n + 1 \tag{2.41}$$

[10]as n_{th} moment of a random variable of Gaussian distribution has a fixed value for a periarticular integer values of n [10]

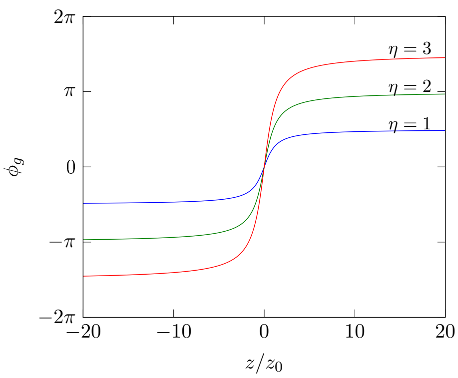


Figure 9: Variation of Gouy phase with z for HG beam

2.4.2 Laguerre-Gaussian beam

In the expression 2.30, we get the radially symmetric Gaussian beam solution. But we now seek higher order solution of Gaussian beam which is not radially symmetric i.e. vary with ϕ.

Lets take the ansatz as,

$$\psi(\boldsymbol{r}, z) = A\, g\left(\frac{y}{w(z)}\right) \exp\left[-i\left(p(z) + \frac{kr^2}{2q(z)} + l\phi\right)\right] \tag{2.42}$$

Putting this in Paraxial wave equation 2.6, and solving the differential equation,[11][9] we get,

$$\begin{aligned}\psi_{p,l}(\boldsymbol{r}, z) =& A\,\frac{w_0}{w(z)}\left[\frac{r\sqrt{2}}{w(z)}\right]^{|l|} L_p^{|l|}\left(\frac{2r^2}{w^2(z)}\right)\exp\left(-\frac{r^2}{w^2(z)}\right)\cdot \\ & \exp\left(-il\phi + i(2p+l+1)\arctan\left(\frac{z}{z_0}\right) - i\frac{kr^2}{2R(z)}\right)\end{aligned} \tag{2.43}$$

where $L_p^{|l|}$ is *Associated Laguerre polynomial* and other terms are as usual.

Some characteristics of HG beams are given below,

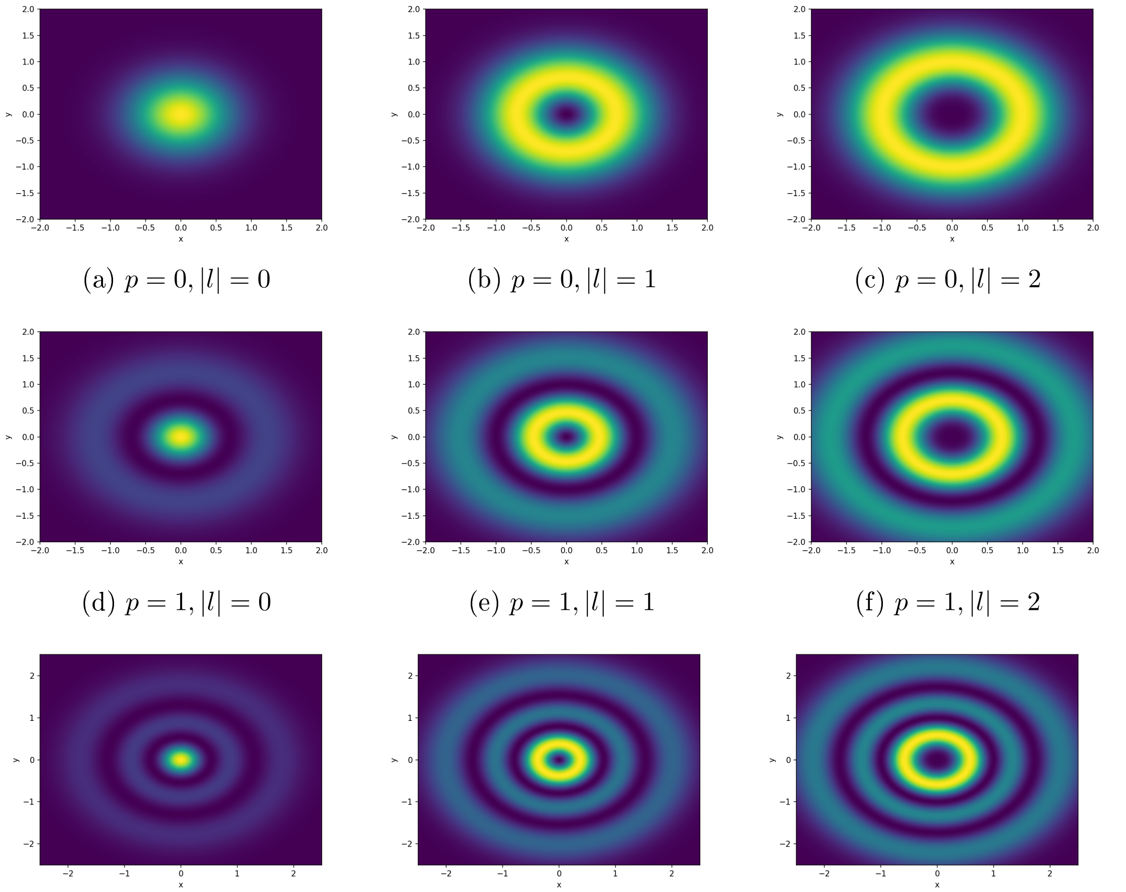
1. *Intensity* of LG beam is given by

$$I_{p,l}(r, z) = \frac{c\epsilon}{2}|A|^2\left[\frac{w_0}{w(z)}\right]^2\left[\frac{r\sqrt{2}}{w(z)}\right]^{2|l|}\left[L_p^{|l|}\left(\frac{2r^2}{w^2(z)}\right)\right]^2\exp\left(-\frac{2r^2}{w^2(z)}\right) \tag{2.44}$$

   See intensity plot for corresponding LG beam in figure 10. For $|l| \neq 0$ the intensity of centre is zero and the value of p denotes the number of radial nodes as $L_p^{|l|}$ has p number of zeros.

2. *Rate of energy* of HG beam passes through any transverse plain is given by

$$\begin{aligned}W &= \iint_{-\infty}^{\infty} dx\, dy\, I(x, y, z) \\ &= \frac{1}{2}\epsilon_0 c|A|^2\left(\frac{w_0}{w(z)}\right)^2\int_{-\infty}^{\infty} dx\,\left(\frac{2r^2}{w^2(z)}\right)^{|l|}\left[L_p^{|l|}\left(\frac{2r^2}{w^2(z)}\right)\right]^2\exp\left(-\frac{2r^2}{w^2(z)}\right)\end{aligned} \tag{2.45}$$



(a) $p = 0, |l| = 0$ (b) $p = 0, |l| = 1$ (c) $p = 0, |l| = 2$

(d) $p = 1, |l| = 0$ (e) $p = 1, |l| = 1$ (f) $p = 1, |l| = 2$

(g) $p = 2, |l| = 0$ (h) $p = 2, |l| = 1$ (i) $p = 2, |l| = 2$

Figure 10: Intensity variation for different modes in a cross section ($z = 0, z_0 = 1, w_0 = 1$)

By same argument as HG beam, we can conclude W is finite and constant throughout the propagation along z-axis

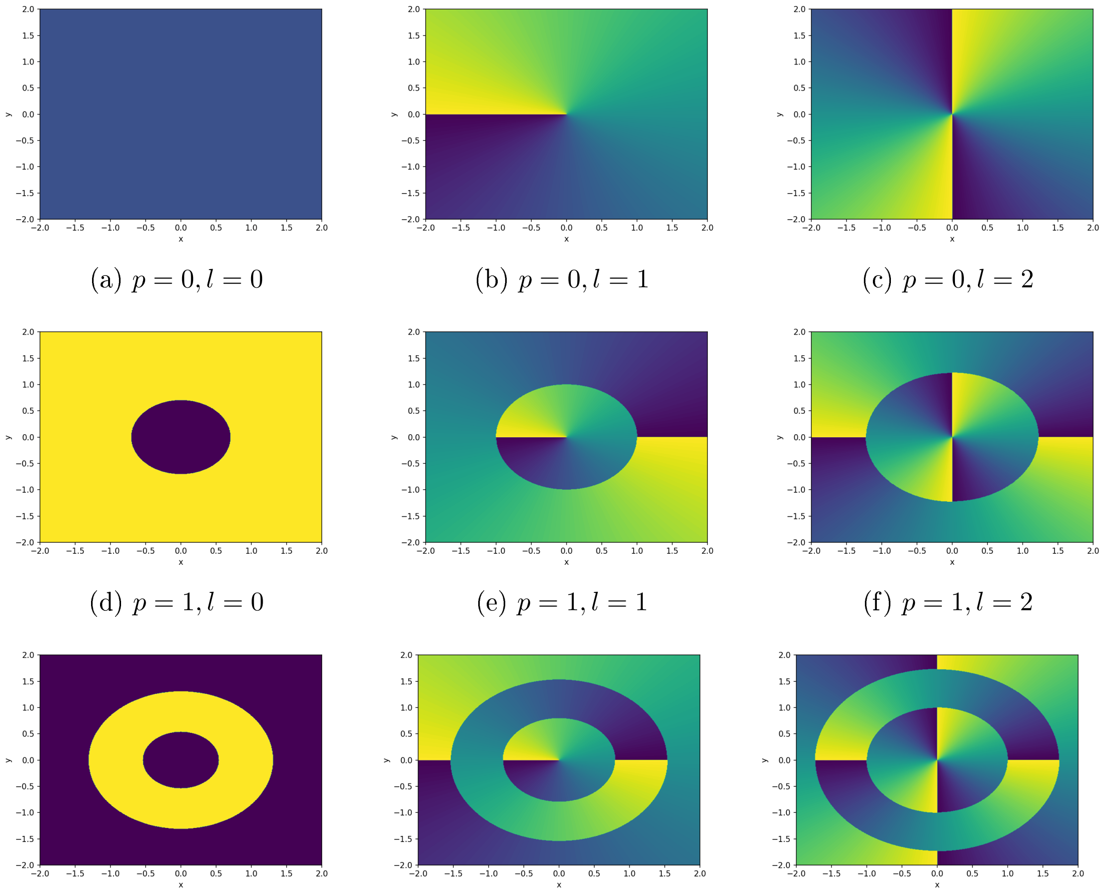
3. *Phase* of the LG beam, unlike Gaussian beam, not only depends on the r and z, but also on *phi*. Phase of LG beam is given by

$$\Phi_{LG}(r, \phi, z) = -il\phi + i(2p + l + 1) \arctan\left(\frac{z}{z_0}\right) - i\frac{kr^2}{2R(z)} \tag{2.46}$$

Phase plot for different order of LG beam at $z = 0$ is given in figure 11.

2.5 Relationship between LG & HG beam

[12]



(a) $p = 0, l = 0$ (b) $p = 0, l = 1$ (c) $p = 0, l = 2$

(d) $p = 1, l = 0$ (e) $p = 1, l = 1$ (f) $p = 1, l = 2$

(g) $p = 2, l = 0$ (h) $p = 2, l = 1$ (i) $p = 2, l = 2$

Figure 11: Phase variation for different modes in a cross section $(z = 0, z_0 = 1, w_0 = 1)$

3 SPIN-ORBIT INTERACTION

3.1 Introduction

3.2 Angular momentum of Light

3.3 Orbital Angular Momentum (OAM)

3.3.1 Intrinsic vs Extrinsic OAM

3.3.2 OAM of LG Beam

3.4 Spin Angular Momentum (SAM)

3.5 Spin orbit energy

3.6 Geometric phase of light

3.6.1 Spin redirection Berry phase

3.6.2 Pancharatnam-Berry Phase

3.6.3 LG-HG Mode transformation

3.7 Types of SOI

3.8 SOI in inhomogeneous anisotropic medium

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