Open Quantum Systems: Kraus Representation Theorem and Quantum Channels

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Abstract

This paper begins by discussing the general dynamics of an open system. We explore the theory of quantum measurements and elaborate on the intricacies of operator sum representation including the Kraus representation theorem. Finally, the main emphasis is given to the quantum channels including amplitude-damping, dephasing, bit-flip, phase-flip, and depolarizing channels.

1 Introduction

The study of open quantum systems constitutes a significant area of study within physics. While the dynamics of closed quantum systems, which are completely isolated from their environment, can be well understood but in reality perfectly closed systems are unattainable as environmental effects are inevitable. Even if these effects are small and often negligible, they can still influence the system's evolution. Consequently, it becomes necessary to model systems as open quantum systems to describe physical phenomena more accurately. Unlike closed systems, open quantum systems constantly interact with their surrounding environment, leading to a dynamic evolution that incorporates the influence of the environment. This paper will provide an introductory understanding of open quantum systems and how to formulate environmental effects, incorporating them into the system dynamics through a quantum channel.

This paper is organized as follows. In Section 2, we discuss the theoretical framework necessary for understanding the dynamics of open quantum systems, including their unitary

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evolution, and introduce the operator-sum representation. In Section 3, we provide a general introduction to quantum measurements, with an emphasis on orthogonal and generalized measurements. We then proceed to Section 4, where we discuss in-depth the operator-sum representation, and its general properties, and subsequently introduce the Kraus representation theorem. Following this, in Section 5, we provide a detailed discussion of various quantum channels, including amplitude-damping, dephasing, bit-flip, phase-flip, and depolarizing channels. Finally, we conclude our discussion in Section 6.

2 Open systems and their evolution

In closed quantum system, evolution of a density matrix $\rho(t_0)$ is performed by unitary operator $U(t,t_0) = e^{-i\boldsymbol{H}_S(t-t_0)}$ as $\rho(t) = U(t,t_0)\rho(t_0)U(t,t_0)^\dagger$, where $\boldsymbol{H}_S$ is the Hamiltonian of a closed system. An open system generally gets coupled with the environment, so the Hamiltonian of the system and its environment may not always be separable, which leads to a combined evolution of both. Let total Hamiltonian be

$$\boldsymbol{H} = \boldsymbol{H}_S \otimes \mathbb{1} + \mathbb{1} \otimes \boldsymbol{H}_B + \boldsymbol{H}_{int},$$

where $\boldsymbol{H}_S, \boldsymbol{H}_B$ be the Hamiltonian of system and bath respectively and $\boldsymbol{H}_{int}$ denotes the Hamiltonian of the interaction between the system and the bath.

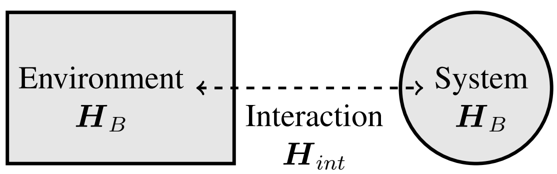


Figure 1: Interaction of open quantum system with environment.

The total Hilbert space $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_B$ (subscript 'S' & 'B' to be understood as system and bath respectively). Starting with a initial state $\rho(0) = \rho_S \otimes |0\rangle\langle 0|_B$, combined evolution is performed by propagator $U_{SB}(t)$ as $\rho(t) = U_{SB}(t)\rho(0)U_{SB}(t)^\dagger$. And only system evolution is thus given by,

$$\begin{aligned}
\rho_S(t) = \mathrm{Tr}_B[\rho(t)] &= \sum_{k\in\beta_B} \langle k|_B \, U_{SB}(\rho_S \otimes |0\rangle\langle 0|_B) U_{SB}^\dagger \, |k\rangle_B \\
&= \sum_{k\in\beta_B} \langle k|_B \, U_{SB} \, |0\rangle_B \, \rho_S \, \langle 0|_B \, U_{SB}^\dagger \, |k\rangle_B \\
&= \sum_{k\in\beta_B} M_k \rho_S M_k^\dagger
\end{aligned} \tag{2.1}$$

where β_S and β_B are orthonormal bases for $\mathcal{H}_S$ and $\mathcal{H}_B$ respectively and $M_k = \langle k| U_{SB} |0\rangle$ is called Kraus operator. This representation of $\rho_S(t)$ in eq. (2.1) is called *operator sum representation*.

Remark. 1. Operator sum representation of ρ_S with Kraus operator $\{M_k\}$ in eq. (2.1) depends on the choice of the basis β_B. So, it is not unique.

2. $\rho(t)$ is Hermitian and positive.

3. Unitarity of propagator $U_{SB}(t)$ implies

$$\begin{aligned}\mathrm{Tr}(\rho_S(0)) = \mathrm{Tr}(\rho_S(t)) &= \sum_{k\in\beta_B} \mathrm{Tr}\Big(M_k\rho_S(0)M_k^\dagger\Big) \\ &= \sum_{k\in\beta_B} \mathrm{Tr}\Big(M_k^\dagger M_k\rho_S(0)\Big) = \mathrm{Tr}\left(\sum_{k\in\beta_B} M_k^\dagger M_k\rho_S(0)\right),\end{aligned}$$

i.e., $\sum_{k\in\beta_B} M_k M_k^\dagger = \mathbb{1}$ (completeness relation).

4. Starting with a pure state, for unitary evolution, it remains pure.

3 An introduction to Quantum measurement

The postulates of quantum measurement say that any measurement in a closed system is described by a set of measurement operators $\{M_m\}$ corresponds to the measurement outcome $\{m\}$ such that $\sum_m M_m M_m^\dagger = \mathbb{1}$. For a open system, we define measurement operators $\{\mathsf{M}_m : \mathsf{M}_m(\rho) = M_m\rho M_m^\dagger\}$ corresponding to the measurement outcome $\{m\}$. Given the state ρ before measurement, the post measurement state will be $\mathsf{M}_m(\rho)/\,\mathrm{Tr}[\mathsf{M}_m(\rho)]$ with probability of the outcome m being $p(m) = \mathrm{Tr}[\mathsf{M}_m(\rho)]$.

According to Von Neumann, it is possible in principle to correlate the state of a microscopic quantum system with the value of a macroscopic classical variable, and we may take it for granted that we can perceive the value of the classical variable. So, to measure an observable of the system, we couple the system with a larger system (*i.e.* measuring pointer) from which we can perceive the corresponding macroscopic classical variable and that larger system is called *Ancillary* bath or environment.

We describe the measurement by modifying the Hamiltonian (of total system) by adding a coupling term between the observable $\boldsymbol{A}$ of the system and generator of translation $\boldsymbol{P}$ of the macroscopic classical variable ('x') corresponding to the pointer (or ancillary bath). The Hamiltonian is

$$\boldsymbol{H} = \boldsymbol{H}_S \otimes \mathbb{1} + \mathbb{1} \otimes \boldsymbol{H}_B + \lambda(t)\boldsymbol{A} \otimes \boldsymbol{P}, \tag{3.1}$$

where $\boldsymbol{H}_S$ and $\boldsymbol{H}_B$ are separate Hamiltonian of the system to be measured and the ancillary bath, respectively and λ is a coupling constant. Considering the time scale of the system and bath evolution being much larger than the measurement timescale (δt), we neglect $\boldsymbol{H}_S$, $\boldsymbol{H}_B$ over coupling term in eq. (3.1) and get $\boldsymbol{H} \simeq \lambda(t)\boldsymbol{A} \otimes \boldsymbol{P}$, and corresponding propagator is

$$U(\delta t) = \exp(-i\boldsymbol{H}\delta t) = \exp(-i\,\delta t\,\lambda\boldsymbol{A} \otimes \boldsymbol{P}) \tag{3.2}$$

Let $\beta_S = \{|i\rangle_S\}$ be the orthonormal eigenbasis for $\boldsymbol{A}$ with corresponding eigenvalue $\{a_i\}$ (discrete spectrum) and $\{|x\rangle_B\}$ be the orthogonal basis of x (continuous spectrum), that leads,

$$\begin{aligned} U(\delta t) &= \sum_{i\in\beta_S} \exp(-i\,\delta t\,\lambda\boldsymbol{A} \otimes \boldsymbol{P})\,|i\rangle\langle i|_S \otimes \int_{-\infty}^{\infty} dx\,|x\rangle\langle x|_B \\ &= \sum_i \int_{-\infty}^{\infty} dx\,|i\rangle\langle i|_S \otimes \exp(-i\,\delta t\,\lambda a_i\boldsymbol{P})\,|x\rangle\langle x|_B \\ &= \sum_i |i\rangle\langle i|_S \otimes \int_{-\infty}^{\infty} dx\,|x+\lambda a_i\delta t\rangle\langle x|_B \\ &= \sum_i |i\rangle\langle i|_S \otimes \int_{-\infty}^{\infty} dx\,|x\rangle\langle x-\lambda a_i\delta t|_B \end{aligned} \tag{3.3}$$

Starting with initial state $\sum_{j\in\beta_S} c_j\,|j\rangle_S \otimes |\psi\rangle_B$, after time δt, the state is

$$\begin{aligned} U(\delta t)\sum_{j\in\beta_S} c_j\,|j\rangle_S \otimes |\psi\rangle_B &= \sum_{i\in\beta_S} c_i\,|i\rangle_S \otimes \int_{-\infty}^{\infty} dx\,|x\rangle\,\langle x-\lambda a_i\delta t|\psi\rangle_B \\ &= \sum_{i\in\beta_S} c_i\,|i\rangle_S \otimes \int_{-\infty}^{\infty} dx\,\psi_B(x-\lambda a_i\delta t)\;|x\rangle_B \\ &= \sum_{i\in\beta_S} c_i\,|i\rangle_S \otimes |\phi_i\rangle_B\,, \end{aligned} \tag{3.4}$$

where

$$|\phi_i\rangle_B = \int_{-\infty}^{\infty} dx\,\psi_B(x-\lambda a_i\delta t)\,|x\rangle_B \Rightarrow \phi_{i\,B}(x) = \psi_B(x-\lambda a_i\delta t).$$

If the overlap between $|\phi_i\rangle_B$'s for different a_i's (*i.e.,* $\langle\phi_i|\phi_j\rangle$) is low, we can separately measure the pointer, and we will detect the eigenvalue a_i by measuring the change of the observable x of $|\phi_i\rangle_B$ by $\lambda a_i\delta t$ in the pointer, with probability $|c_i|^2$. In that case, the state of the system is prepared in the eigenstate $|i\rangle$. So, measuring $|\psi(x+\lambda a_i\delta t)\rangle_B$ in the pointer induces a measurement of the system in the orthogonal eigenbasis. This is an example of Von Neumann's model of orthogonal measurement. More abstractly, for a orthogonal measurement corresponding to N orthogonal projection operator of the system $\{M_i = |i\rangle\langle i|_S : i = 0, \ldots, N-1\}$, we introduce N orthonormal basis of pointer $\{|k\rangle_B : k = 0, \ldots, N-1\}$, then the unitary transformation (similar to eq. (3.3)) is

$$U(\delta t) = \sum_{i,k=0}^{N-1} M_i \otimes |k+i\rangle\langle k|_B\,, \tag{3.5}$$

where $|k+i\rangle \equiv |k+i \pmod N\rangle$ and $|k+i\rangle\langle k|_B$ denotes the jump in the pointer. An orthonormal basis for the pointer is taken to ensure that they do not overlap *i.e.,* $\langle k|k'\rangle_B = \delta_{k,k'}$.

Remark (*Generalized measurement*)**.** Unlike the previous one, in generalized measurement, we may not always project the system to an orthogonal eigenbasis $\{|i\rangle_S\}$ when measuring the pointer in different orthonormal basis, $\{|k'\rangle_B\}$. Now starting with a state of the system $|\psi\rangle_S$, when coupled with the state $|0\rangle_B$ in ancillary bath (pointer) gives $\rho(0) = |\psi\rangle\langle\psi|_S \otimes |0\rangle\langle 0|_B$ in $\mathcal{H}_S \otimes \mathcal{H}_B$. Define the propagator of the combined system as

$$\begin{aligned} U_{SB}(t) &= \sum_{k' \in \beta_B} M_{k'} \otimes |k'\rangle\, \langle 0|_B \\ \Rightarrow U_{SB}(t)\, |\psi\rangle_S \otimes |0\rangle_B &= \sum_{k' \in \beta_B} M_{k'}\, |\psi\rangle_S \otimes |k'\rangle_B \end{aligned} \tag{3.6}$$

where $\beta_B = \{|k'\rangle\}$ is orthonormal basis for $\mathcal{H}_B$. Observe that, $\{M_{k'}\, |\psi\rangle_S\}_{k' \in \beta_B}$ may not be orthogonal and (as in eq. (2.1)),

$$\rho_S(t) = \mathrm{Tr}_B[\rho(t)] = \sum_{k' \in \beta_B} M_{k'}\, |\psi\rangle\langle\psi|_S\, M_{k'}^{\dagger}.$$

Here measuring $|k'\rangle_B$ the state of the system is prepared in $M_{k'}\, |\psi\rangle_S$ with probability $\|M_k\, |\psi\rangle_S\|$. Generally, $\rho_S(t)$ is a mixed state. We can represent a mixed state of a system in $\mathcal{H}_S$ as a pure state of a bigger system (*i.e.* system + ancillary bath) in $\mathcal{H}_S \otimes \mathcal{H}_B$ as $\sum_{k \in \beta_B} M_k\, |\psi\rangle_S \otimes |k\rangle_B$ and by partial tracing over the bath we extract the state of the system.

4 Quantum operator and beyond

In the previous section, we describe the evolution of the quantum system through a unitary propagator, here we will discuss the quantum operator formalism. A *quantum operator*, defined from a vector space of density operators to another vector space, is a map acting on a density operator to give a new density operator. As an example, in eq. (2.1), we define quantum operator M acting on density operator $\rho_S(0)$, gives $\rho_S(t)$ as

$$\rho_S(t) = \mathsf{M}(\rho_S) = \sum_{k \in \beta_B} M_k \rho_S M_k^{\dagger}.$$

Observe that the quantum operator is written in operator sum representation. A question may arise: Can all quantum operators be written in the form of operator sum representation? We will soon see that if it follows some set of rules then it can have operator sum representation. Based on the physical realization of the evolution of a density operator, we claim that quantum operator M follows the following axioms:

Axiom 1. It says that for any ρ in Hilbert space $\mathcal{H}$, $0 \leq \mathrm{Tr}[\mathsf{M}(\rho)] \leq 1$. It is convenient as M will give a new density operator and trace of which, gives the total probability of measurement outcome of all constituent states of the ensemble.

Axiom 2. It says that the quantum operator is *convex linear map*. Suppose that $t = t_0$, an ensemble of states, ρ_i are prepared with probability p_i, then we expect the time evolved ensemble of states at $t = t_0 + \delta t$, will be $\mathsf{M}(\rho_\mathrm{i})$ with the same probability p_i.

Axiom 3. It says that the the quantum operator is *completely positive* (a map M defined in $\mathcal{H}$ Hilbert space, is 'completely positive' if $\mathsf{M} \otimes \mathbb{1}$ is positive in $\mathcal{H} \otimes \mathcal{H}'$ for any extension $\mathcal{H}'$ over $\mathcal{H}$). It is positive as the density operator is positive semi-definite. Moreover, it is completely positive, because starting with a density operator of the composite bipartite system (of subsystem and another ancillary system), applying $\mathsf{M} \otimes \mathbb{1}$, we still want to obtain a valid density operator.

Considering all three axioms for a quantum operator, we propose a necessary theorem regarding the operator sum representation.

Theorem 1. *The quantum operator* M*, defined from a vector space of density operators to another vector space, satisfies the axiom 1, axiom 2 and axiom 3, if and only if*

$$\mathsf{M}(\rho) = \sum_{i=1} M_i \rho M_i^\dagger$$

for some set of operators $\{M_i\}$ *defined from a Hilbert space to another Hilbert space and*

$$\sum_{i=1} M_i \rho M_i^\dagger \leq \mathbb{1}.$$

Remark. Some important points are as follows:

1. This theorem states only the existence of such operator sum, not the uniqueness.

2. Two operator-sum representations of the same quantum channel are related by a unitary change of basis for the Kraus operators. Let two different sets of Kraus operators $\{M_i\}$ and $\{N_j\}$ representing the same channel, they are related by the linear relation, $N_j = \sum_i v_{ji} M_i$, where $V = [v_{ji}]$ is some unitary matrix.

3. If the quantum operator is also trace-preserving, then completeness relation follows *i.e.,* $\sum_i M_i M_i^\dagger = \mathbb{1}$

4. For our discussion of a quantum channel, the domain Hilbert space and range Hilbert space are the same and finite-dimensional further, they are trace-preserving. The following Kraus representation theorem will elaborate on this.

Theorem 2 (Kraus representation theorem)**.** M *be a trace-preserving quantum operator defined in finite-dimensional Hilbert space $\mathcal{H}_S$ of dimensions d. Then any operator sum representation of the quantum channel has at most d^2 elements. For any $\rho \in \mathcal{H}_S$, the quantum operator $\mathsf{M}(\rho)$ can be written as*

$$\mathsf{M}(\rho) = \sum_{i=1}^{K} M_i \rho M_i^{\dagger}$$

where $K \leq d^2$ is called Kraus number and $\{M_k\}$ is the set of linear operators, called Kraus operator, that satisfy the completeness property, i.e.

$$\sum_{i=1}^{K} M_i M_i^{\dagger} = \mathbb{1}.$$

5 Quantum channels

We have discussed quantum operators and three axioms that a quantum operator follows. The quantum operator with trace-preserving, completely positive, convex linear property is called *quantum channel*. As it is trace preserving,

$$\sum_{i} M_i M_i^{\dagger} = \mathbb{1},$$

gives the completeness relation. Further, it follows Kraus representation theorem.

Our main focus is to describe the dynamics of a few quantum channels *viz*, amplitude-damping, phase-damping and decoherence channels via a two-level system, which gives a concise idea of how the environment affects the system. The general description is as follows, given a quantum channel with different Kraus operators $\{M_i\}_{i=1}^{N}$ acting on the system, we introduce an auxiliary bath having N orthonormal basis, and define the unitary propagator as given in eq. (3.6). Finally, partial tracing over the states of the auxiliary bath gives the dynamics of the system.

To describe the quantum channel, *Markov approximation* is taken into account. It says that the evolution of the state will only depend on the current state *i.e.,* the density operator $\rho_S(t + \delta t)$ will only depend on $\rho_S(t)$. So the history of the state of the system does not influence the evolution of the present state. But we know an open system is dissipative because information and energy can flow both ways from system to environment and vice versa, then the density operator $\rho_S(t+\delta t)$ not only depends on $\rho_S(t)$ but also depends on ρ_S at a time before t. In reality, Markov approximation is unreachable. Nonetheless, it is a good approximation to start considering the fact that the time scale of the evolution of the system is much larger than the time scale in which the environment completely forgot the information flows into it. For the case of a two-level system, the Markov approximation is taken into account. Let the initial state of the environment be $|0\rangle_B$. After each infinitesimal time increment during combined evolution, the evolved states of the environment (denoted as $|n\rangle_B$) reset themselves to the state $|0\rangle_B$. In other words, the environment forgets its previous state within that timescale. For example, if a quantum operator $U(\delta t)$ transforms $|e\rangle_S |0\rangle_B$ into $|g\rangle_S |n\rangle_B$, with $|n\rangle_B$ being some states of the bath, the subsequent evolution can be described as follows:

1. $|e\rangle_S |0\rangle_B$ evolves to $|g\rangle_S |n\rangle_B$ with probability p under $U(\delta t)$,
2. considering the Markov approximation, the environment state $|n\rangle_B$ is reset to $|0\rangle_B$ and $|g\rangle_S |n\rangle_B$ transforms into $|g\rangle_S |0\rangle_B$,
3. again the state $|g\rangle_S |0\rangle_B$ evolves under $U(\delta t)$ and reset its environment state to $|0\rangle_B$,
4. the process repeats.

The evolution in the quantum channels is given by trace-preserving completely positive linear operator M. For a density operator ρ_S the evolution is as follows:

$$\rho_S(t+\delta t) = \mathsf{M}(\delta t)\rho_S(t).$$

Considering the Markov approximation, the *Chapman-Kolmogorov equation* states that,

$$\mathsf{M}(N\delta t) = \mathsf{M}^N(\delta t) = \underbrace{\mathsf{M}(\delta t) \circ \mathsf{M}(\delta t) \circ \cdots \circ \mathsf{M}(\delta t)}_{N \text{ times}}$$

which implies

$$\rho_S(t+N\delta t) = \mathsf{M}(N\delta t)\rho_S(t) = \mathsf{M}(\delta t) \circ \cdots \circ \mathsf{M}(\delta t)\rho_S(t).$$

5.1 Amplitude-damping channel

In quantum mechanics, the concept of amplitude represents the probability of a state within the system. The amplitude-damping channel serves as a model to illustrate how the probability of certain states within the system decaying when it is connected to a bath due to the dissipation of energy. For instance, we can consider the spontaneous photon emission of an atomic two-level system as an example. Let, $|g\rangle$ & $|e\rangle$ be the ground and excited state of the system, respectively and $|0\rangle$ & $|1\rangle$ be the states of ancillary bath (or environment) such that $|1\rangle$ denotes photon is detected in the environment and $|0\rangle$ when not detected. The unitary evolution for small time δt is as follows:

$$\begin{aligned}
|g\rangle\,|0\rangle &\xrightarrow{U(\delta t)} |g\rangle\,|0\rangle \\
|e\rangle\,|0\rangle &\xrightarrow{U(\delta t)} \sqrt{1-p}\,|e\rangle\,|0\rangle + \sqrt{p}\,|g\rangle\,|1\rangle\,,
\end{aligned}$$

where probability $p = \Gamma\delta t$ and Γ is probability rate. Writing a possible unitary evolution operator U describing the above events in matrix form in the basis $\{|g\rangle\,|0\rangle\,, |g\rangle\,|1\rangle\,, |e\rangle\,|0\rangle\,, |e\rangle\,|1\rangle\}$ gives the following,

$$U(\delta t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{1-p} & \sqrt{p} & 0 \\ 0 & \sqrt{p} & \sqrt{1-p} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.1}$$

The corresponding Kraus operators are

$$\begin{aligned}
M_0 &= \langle 0|U|0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix} = |g\rangle\langle g| + \sqrt{1-p}\,|e\rangle\langle e|\,, \\
M_1 &= \langle 1|U|0\rangle = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix} = \sqrt{p}\,|g\rangle\langle e|\,.
\end{aligned}$$

It is clear that M_1 describes the photon emission as the environment detects it, so $|e\rangle$ decays to $|g\rangle$. And M_0 describes the evolution when there is no photon detected.

Starting with a single qubit density operator,

$$\rho_0 = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix}$$

with $\rho_{00} + \rho_{11} = 1$, the evolved state after δt is

$$\mathsf{M}(\rho_0) = \rho(\delta t) = M_0\rho_0 M_0^\dagger + M_1\rho_0 M_1^\dagger = \begin{pmatrix} \rho_{00} + p\rho_{11} & \sqrt{1-p}\rho_{01} \\ \sqrt{1-p}\rho_{10} & (1-p)\rho_{11} \end{pmatrix}. \tag{5.2}$$

Now, taking Markov approximation into account, after a finite time t, the evolved state is

$$\begin{aligned}
\rho(t) = \lim_{\delta t\to 0} \mathsf{M}^{t/\delta t}(\rho_0) &= \lim_{\delta t\to 0} \begin{pmatrix} \rho_{00} + (1-(1-\Gamma\delta t)^{t/\delta t})\rho_{11} & \sqrt{(1-\Gamma\delta t)^{t/\delta t}}\rho_{01} \\ \sqrt{(1-\Gamma\delta t)^{t/\delta t}}\rho_{10} & (1-\Gamma\delta t)^{t/\delta t}\rho_{11} \end{pmatrix} \\
&= \begin{pmatrix} \rho_{00} + (1-e^{-\Gamma t})\rho_{11} & e^{-\Gamma t/2}\rho_{01} \\ e^{-\Gamma t/2}\rho_{10} & e^{-\Gamma t}\rho_{11} \end{pmatrix}
\end{aligned} \tag{5.3}$$

which, in infinite time limit *i.e.,* $t \gg 1/\Gamma$, evolves into the following stationary state:

$$\rho_\infty = \rho(t \to \infty) = \begin{pmatrix} \rho_{00} + \rho_{11} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = |g\rangle\langle g| \,.$$

In terms of Bloch sphere representation, starting with

$$\rho_0 = \frac{1}{2}(\mathbb{1} + \vec{n}_0 \cdot \vec{\sigma}) = \frac{1}{2} \begin{pmatrix} 1 + n_z & n_x - i n_y \\ n_x + i n_y & 1 - n_z \end{pmatrix},$$

at time t,

$$\rho(t) = \frac{1}{2}\mathbb{1} + \frac{1}{2} \begin{pmatrix} n_x \, e^{-\Gamma t/2} \\ n_y \, e^{-\Gamma t/2} \\ 1 - (1 - n_z) e^{-\Gamma t} \end{pmatrix} \cdot \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix} \Rightarrow \vec{n}(t) = \begin{pmatrix} n_x \, e^{-\Gamma t/2} \\ n_y \, e^{-\Gamma t/2} \\ 1 - (1 - n_z) e^{-\Gamma t} \end{pmatrix}. \tag{5.4}$$

Further, the stationary state is given by

$$\rho_\infty = \frac{1}{2}\mathbb{1} + \frac{1}{2}\sigma_z \Rightarrow \vec{n}_\infty = (0, 0, 1). \tag{5.5}$$

That suggests that with time Bloch sphere will shrink towards the pole (0,0,1) and concentrate at the pole *i.e.,* the stationary state. So, over a long time, the system will relax into the ground state and the amplitude (or the probability) of the excited state will completely vanish.

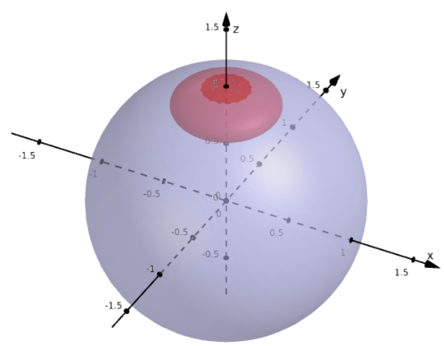


Figure 2: Bloch sphere representation of amplitude damping channel is shown here. The grey sphere is the initial Bloch sphere of unit polarization and the red closed surface denotes the evolved Bloch sphere for $e^{-\Gamma t/2} = 0.4$.

Now we can think of a more generalized amplitude-damping given by the following

Kraus operators:

$$
\begin{aligned}
M_0 &= \sqrt{\lambda}\left(|g\rangle\langle g| + \sqrt{1-p}\,|e\rangle\langle e|\right),\\
M_1 &= \sqrt{\lambda p}\;\, |g\rangle\langle e|,\\
M_2 &= \sqrt{1-\lambda}\left(\sqrt{1-p}\,|g\rangle\langle g| + |e\rangle\langle e|\right),\\
M_3 &= \sqrt{(1-\lambda)p}\;\, |e\rangle\langle g|
\end{aligned}
$$

where λ and p are in $[0,1]$. And the final stationary state will be

$$
\rho_\infty = \begin{pmatrix} \lambda & 0 \\ 0 & 1-\lambda \end{pmatrix} = \lambda\,|g\rangle\langle g| + (1-\lambda)\,|e\rangle\langle e| = \frac{1}{2}\mathbb{1} + \left(\frac{2\lambda-1}{2}\right)\sigma_z,
$$

and in Bloch sphere, $\vec{n} = (0,0,2\lambda-1)$. Here the Bloch sphere will shrink towards $(0,0,2\lambda-1)$ with time.

5.2 Dephasing channel

A dephasing (or phase damping) channel describes the evolution of a coherence state to a decoherence state by losing its phase information to the environment. For example, dust particles with two energy levels $\{|g\rangle, |e\rangle\}$ interact with photons by exchanging energy by collision. Point to note that the energy gap between the two levels of the dust particle is much larger than the exchange energy, so collision won't affect the system state but it certainly affects the state of the photon. Also the ground state $|g\rangle$ of the dust particle interacts differently with photon than the excited state $|e\rangle$. Starting with photon state $|0\rangle$, it evolves into $\{|0\rangle, |1\rangle, |2\rangle\}$ states depending on the state of the dust particle and collision probability $p = \Gamma\delta t$. The unitary evolution for small time δt is as follows;

$$
\begin{aligned}
|g\rangle\,|0\rangle &\xrightarrow{U(\delta t)} \sqrt{1-p}\,|g\rangle\,|0\rangle + \sqrt{p}\,|g\rangle\,|1\rangle\\
|e\rangle\,|0\rangle &\xrightarrow{U(\delta t)} \sqrt{1-p}\,|e\rangle\,|0\rangle + \sqrt{p}\,|e\rangle\,|2\rangle
\end{aligned}
$$

Similarly, a possible unitary evolution operator U in matrix form in the basis $\{|g\rangle|0\rangle, |g\rangle|1\rangle, |g\rangle|2\rangle, |e\rangle|0\rangle, |e\rangle|1\rangle, |e\rangle|2\rangle\}$ is the following,

$$
U(\delta t) = \begin{pmatrix}
\sqrt{1-p} & \sqrt{p} & 0 & 0 & 0 & 0\\
\sqrt{p} & \sqrt{1-p} & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & \sqrt{1-p} & 0 & \sqrt{p}\\
0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & \sqrt{p} & 0 & \sqrt{1-p}
\end{pmatrix}. \tag{5.6}
$$

The corresponding Kraus operators are

$$M_0 = \sqrt{1-p}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\ M_1 = \begin{pmatrix}\sqrt{p} & 0\\ 0 & 0\end{pmatrix},\ M_2 = \begin{pmatrix}0 & 0\\ 0 & \sqrt{p}\end{pmatrix}.$$

and the evolved state after δt is

$$\mathsf{M}(\rho_0) = \rho(\delta t) = \sum_{i=1}^{3} M_i \rho_0 M_i^\dagger = \begin{pmatrix}\rho_{00} & (1-p)\rho_{01}\\ (1-p)\rho_{10} & \rho_{11}\end{pmatrix}. \tag{5.7}$$

Taking Markov approximation into account, after a finite time t the state is,

$$\rho(t) = \lim_{\delta t \to 0} \mathsf{M}^{t/\delta t}(\rho_0) = \begin{pmatrix}\rho_{00} & e^{-\Gamma t}\rho_{01}\\ e^{-\Gamma t}\rho_{10} & \rho_{11}\end{pmatrix}. \tag{5.8}$$

In infinite time limit *i.e.,* $t \gg 1/\Gamma$, the state evolves into the following stationary state:

$$\rho(t \to \infty) = \begin{pmatrix}\rho_{00} & 0\\ 0 & \rho_{11}\end{pmatrix}$$

In terms of Bloch sphere representation, starting with single qubit $\rho_0 = \frac{1}{2}(\mathbb{1} + \vec{n}_0 \cdot \vec{\sigma})$, the evolved state at time t is the following:

$$\rho(t) = \frac{1}{2}\mathbb{1} + \frac{1}{2}\begin{pmatrix}n_x\, e^{-\Gamma t}\\ n_y\, e^{-\Gamma t}\\ n_z\end{pmatrix} \cdot \begin{pmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\end{pmatrix} \Rightarrow \vec{n}(t) = \begin{pmatrix}n_x\, e^{-\Gamma t}\\ n_y\, e^{-\Gamma t}\\ n_z\end{pmatrix}, \tag{5.9}$$

and the stationary state is given by

$$\rho_\infty = \frac{1}{2}\mathbb{1} + \frac{1}{2}n_z\sigma_z \Rightarrow \vec{n}_\infty = (0, 0, n_z). \tag{5.10}$$

That suggests that over a finite time, the Bloch sphere squeezes exponentially along x and y axes, keeping z coordinate intact and in the stationary state, it concentrates on the point $(0, 0, n_z)$. So, the coherence of the system is lost and the state of the system is approaching to complete decoherence state. We will see later that the dephasing channel is equivalent to the phase-flip channel.

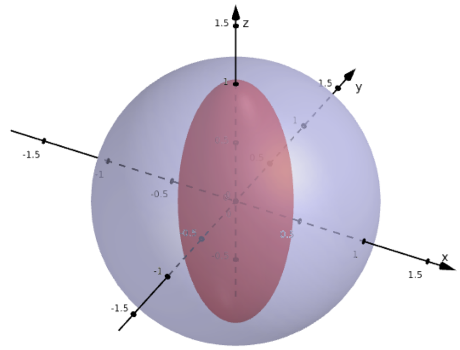


Figure 3: Bloch sphere representation of dephasing channel is shown here. The grey sphere is the initial Bloch sphere of unit polarization and the red ellipsoid denotes the evolved Bloch sphere for $e^{-\Gamma t/2} = 0.4$.

5.3 Bit-flip & phase-flip channel

Bit-flip and phase-flip are very important quantum noise channels. *Bit-flip* channel is a quantum channel where the transmitted qubit can be flipped during transmission due to environmental interaction with finite probability. This occurs in the following way:

$$\begin{aligned} &|0\rangle \longrightarrow |1\rangle\,, \quad |1\rangle \longrightarrow |0\rangle \\ \textit{i.e.,}\ &|\psi\rangle \longrightarrow \sigma_x |\psi\rangle\,. \end{aligned}$$

Let the bit-flip occur with probability p in single qubit $|\psi\rangle$ in the channel. This follows the unitary evolution as follows:

$$|\psi\rangle |0\rangle \xrightarrow{U} \sqrt{1-p}\,|\psi\rangle |0\rangle + \sqrt{p}(\sigma_x |\psi\rangle) |1\rangle$$

where $\{|0\rangle, |1\rangle\}$ orthogonal states of ancillary bath. The ancillary bath (or measurement pointer) keeps track of the evolution by the orthogonal states $\{|k\rangle : k = 0, 1, 2, \dots\}$. The corresponding Kraus operators are

$$M_0 = \sqrt{1-p}\,\mathbb{1} \text{ and } M_1 = \sqrt{p}\,\sigma_x$$

Starting with a single qubit state

$$\rho = \frac{1}{2}(\mathbb{1} + \vec{r}\cdot\vec{\sigma}) = \frac{1}{2}\begin{pmatrix} 1+r_z & r_x - ir_y \\ r_x + ir_y & 1-r_z \end{pmatrix},$$

the operator sum representation is as follows:

$$\begin{aligned} \mathsf{M}(\rho) &= (1-p)\rho + p\,\sigma_x\rho\sigma_x \\ &= \frac{1}{2}\begin{pmatrix} 1+(1-2p)r_z & r_x - i(1-2p)r_y \\ r_x + i(1-2p)r_y & 1-(1-2p)r_z \end{pmatrix} \end{aligned} \tag{5.11}$$

In terms of Bloch sphere representation

$$\mathsf{M}(\rho) = \frac{1}{2}\mathbb{1} + \frac{1}{2}\begin{pmatrix} r_x \\ r_y(1-2p) \\ r_z(1-2p) \end{pmatrix} \cdot \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix} \Rightarrow \vec{n} = \begin{pmatrix} r_x \\ r_y(1-2p) \\ r_z(1-2p) \end{pmatrix} \tag{5.12}$$

So the Bloch sphere squeezes along y and z axes by $(1-2p)$ times keeping x coordinate intact and converting into an ellipsoid.

Now we discuss the phase-flip channel. The *phase-flip* channel is described by the following way:

$$\begin{aligned} |0\rangle &\longrightarrow |0\rangle\,, \quad |1\rangle \longrightarrow -|1\rangle \\ \textit{i.e.,}\ |\psi\rangle &\longrightarrow \sigma_z|\psi\rangle \end{aligned}$$

Following the steps outlined previously, the operator representation is as follows:

$$\begin{aligned} \mathsf{M}(\rho) &= (1-p)\rho + p\,\sigma_z\rho\sigma_z \\ &= \frac{1}{2}\begin{pmatrix} 1+r_z & (1-2p)r_x - i(1-2p)r_y \\ (1-2p)r_x + i(1-2p)r_y & 1-r_z \end{pmatrix} \end{aligned} \tag{5.13}$$

and in terms of Bloch sphere representation

$$\mathsf{M}(\rho) = \frac{1}{2}\mathbb{1} + \frac{1}{2}\begin{pmatrix} r_x(1-2p) \\ r_y(1-2p) \\ r_z \end{pmatrix} \cdot \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix} \Rightarrow \vec{n} = \begin{pmatrix} r_x(1-2p) \\ r_y(1-2p) \\ r_z \end{pmatrix} \tag{5.14}$$

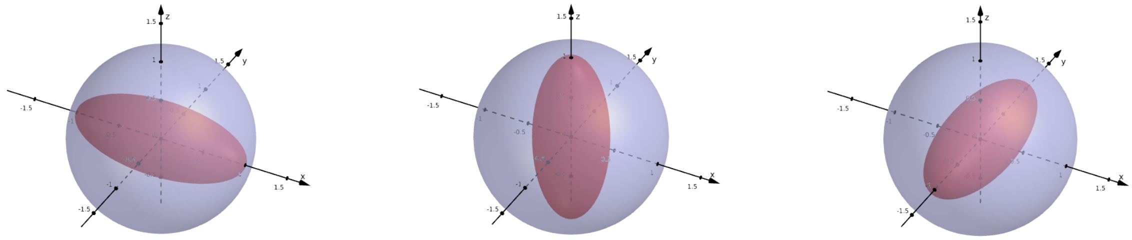
As we see before in the dephasing channel, the Bloch sphere representation for both channels is similar. So as mentioned before, the sphere is squeezed along x and y axes by $(1-2p)$ times keeping z coordinate intact and converted into an ellipsoid.

Further, *bit-phase flip* is described in the following way:

$$\begin{aligned} |0\rangle &\longrightarrow -i|1\rangle\,, \quad |1\rangle \longrightarrow i|0\rangle \\ \textit{i.e.,}\ |\psi\rangle &\longrightarrow i\sigma_x\sigma_z|\psi\rangle = \sigma_y|\psi\rangle \end{aligned}$$

Similarly, the Bloch representation will be

$$\mathsf{M}(\rho) = \frac{1}{2}\mathbb{1} + \frac{1}{2}\begin{pmatrix} r_x(1-2p) \\ r_y \\ r_z(1-2p) \end{pmatrix} \cdot \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix} \Rightarrow \vec{n} = \begin{pmatrix} r_x(1-2p) \\ r_y \\ r_z(1-2p) \end{pmatrix} \tag{5.15}$$



(a) Bit-flip channel (b) Phase-flip channel (c) Bit-phase flip channel

Figure 4: Bloch sphere representation of the channels are shown above. For each channel, the grey sphere is the initial Bloch sphere of unit polarization and the red ellipsoid denotes the evolved Bloch sphere for $(1-2p) = 0.4$.

Now take a special case where the error occurs in qubit in a particular way. The probability of occurrence of any error in the channel is p and each of the following errors has the same probability $\frac{p}{3}$,

$$
\begin{aligned}
|\psi\rangle &\to \sigma_x |\psi\rangle \text{ (Bit-flip)}\\
|\psi\rangle &\to \sigma_z |\psi\rangle \text{ (Phase-flip)}\\
|\psi\rangle &\to \sigma_y |\psi\rangle \text{ (Bit-phase flip).}
\end{aligned}
$$

The unitary evolution is as follows:

$$
|\psi\rangle|0\rangle \xrightarrow{U} \sqrt{1-p}\,|\psi\rangle|0\rangle + \sqrt{\frac{p}{3}}(\sigma_x|\psi\rangle)|1\rangle + \sqrt{\frac{p}{3}}(\sigma_y|\psi\rangle)|2\rangle + \sqrt{\frac{p}{3}}(\sigma_z|\psi\rangle)|3\rangle
$$

where $\{|k\rangle : k = 0,1,2,3\}$ are orthogonal states of ancillary bath. The corresponding Kraus operators are

$$
M_0 = \sqrt{1-p}\mathbb{1},\ M_1 = \sqrt{\frac{p}{3}}\sigma_x,\ M_2 = \sqrt{\frac{p}{3}}\sigma_y,\ M_3 = \sqrt{\frac{p}{3}}\sigma_z.
$$

Starting with single qubit $\rho = \frac{1}{2}(\mathbb{1} + \vec{r}\cdot\vec{\sigma})$, the operator sum representation is as follows:

$$
\begin{aligned}
\mathsf{M}(\rho) &= (1-p)\rho + \frac{p}{3}\sigma_x\rho\sigma_x + \frac{p}{3}\sigma_y\rho\sigma_y + \frac{p}{3}\sigma_z\rho\sigma_z\\
&= \frac{1}{2}\begin{pmatrix} 1 + (1-\frac{4}{3}p)r_z & (1-\frac{4}{3}p)r_x - i(1-\frac{4}{3}p)r_y \\ (1-\frac{4}{3}p)r_x + i(1-\frac{4}{3}p)r_y & 1-(1-\frac{4}{3}p)r_z \end{pmatrix}
\end{aligned} \tag{5.16}
$$

and in terms of Bloch sphere representation

$$
\mathsf{M}(\rho) = \frac{1}{2}\mathbb{1} + \frac{1}{2}\left(1-\frac{4}{3}p\right)\begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix}\cdot\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix} \Rightarrow \vec{n} = \left(1-\frac{4}{3}p\right)\begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix} \tag{5.17}
$$

that is to say, the Bloch sphere contracts uniformly in every direction. This is a specific example of a depolarizing channel, which will be discussed next.

5.4 Depolarizing channel

A depolarizing channel is a noise channel where the transmitted qubit may be depolarized (*i.e.,* qubit converted into completely mixed state $\mathbb{1}/2$) with a finite probability p.The depolarizing channel acts on a qubit state $\rho = \frac{1}{2}\mathbb{1} + \vec{r}\cdot\vec{\sigma}$ in the following way:

$$\begin{aligned}\mathsf{M}(\rho) &= p\frac{\mathbb{1}}{2} + (1-p)\rho \\ &= \frac{1}{2}\begin{pmatrix} 1+(1-p)r_z & (1-p)r_x - i(1-p)r_y \\ (1-p)r_x + i(1-p)r_y & 1-(1-p)r_z \end{pmatrix}.\end{aligned} \tag{5.18}$$

Similarly in terms of Bloch sphere representation,

$$\mathsf{M}(\rho) = \frac{1}{2}\mathbb{1} + \frac{1}{2}(1-p)\begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix}\cdot\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix} \Rightarrow \vec{n} = (1-p)\begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix} \tag{5.19}$$

As mentioned in the preceding example (see eq. (5.17)), the Bloch sphere contracts uniformly $(1-p)$ times in every direction. That is, in every direction the magnitude of the polarization vector, $\vec{r}$ decreases by the same amount, so the coherence state becomes a decoherence state.

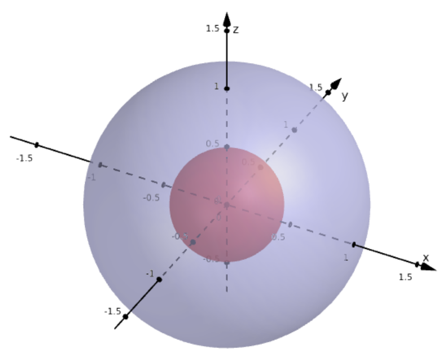


Figure 5: Bloch sphere representation of depolarizing channel is shown here. The grey sphere is the initial Bloch sphere of unit polarization and the red sphere denotes the evolved Bloch sphere for $(1-p) = 0.4$.

Remark (*Reversibility of quantum channel*)**.** It is essential to note that all the discussed quantum channels are irreversible. This irreversibility arises from the fact that if we have another quantum channel that reverses the effect of a particular quantum channel - meaning it increases the polarization (*i.e.,* $|\vec{r}|$) in a certain direction in the Bloch sphere that is decreased by the original channel - then starting with a state of unit polarization (*i.e.,* $|\vec{r}| = 1$) in that direction on the Bloch sphere, the evolved state would have a polarization greater than unity, which is physically nonsensical.

6 Conclusions

We have presented a comprehensive overview of the theoretical framework essential for understanding open quantum systems and their dynamics. Beginning with the discussion on the unitary evolution of open systems and quantum measurements, we examined open quantum systems through various types of quantum channels using a two-level system. We explored their properties and behaviours in detail. By providing a thorough discussion of these topics, we hope to provide readers with a foundation for further study in the field of open quantum systems and quantum information.

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References

[1] Nielsen, M. A., & Chuang, I. L. (2010). *Quantum Computation and Quantum Information: 10th Anniversary Edition.* Cambridge: Cambridge University Press.

[2] Cappellaro, P. (2012). *Quantum Theory of Radiation Interactions* [Lecture notes]. https://ocw.mit.edu/courses/22-51-quantum-theory-of-radiation-interactions-fall-2012/pages/lecture-notes/.

[3] Preskill, J.(2018). *Quantum Information* [Lecture notes]. http://theory.caltech.edu/~preskill/ph219/index.html.

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