

# From Unitary Evolution to Quantum Channels: Understanding Open Quantum Systems

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## Abstract

This paper begins with discussing the general dynamics of an open system. We explore the theory of quantum measurements and elaborate on the intricacy of operator sum representation including the Kraus representation theorem. Finally, the main emphasis is given to the quantum channels including amplitude-damping, dephasing, bit-flip, phase-flip, and depolarizing channels.

## 1 Introduction

The study of open quantum systems constitutes a significant area within physics. While the dynamics of closed quantum systems, which are completely isolated from their environment, can be well understood, they lack information about environmental interactions. In reality, perfectly closed systems are unattainable, as environmental effects are inevitable. Even if these effects are small and often negligible, they can still influence the system's behavior. Consequently, to describe physical phenomena more accurately, it becomes necessary to model systems as open quantum systems. Unlike closed systems, open quantum systems constantly interact with their surrounding environment, leading to a dynamic evolution that incorporates environmental influences.

This paper is organized as follows. In Section 2, we discuss the theoretical framework necessary for understanding the dynamics of open quantum systems, including their unitary evolution, and introduce the operator-sum representation. In Section 3, we provide a general

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introduction to quantum measurements, with an emphasis on orthogonal and generalized measurements. We then proceed to Section 4, where we discuss in-depth the operator-sum representation, its general properties, and subsequently introduce the Kraus representation theorem. Following this, in Section 5, we provide a detailed discussion of various quantum channels, including the amplitude-damping, dephasing, bit-flip, phase-flip, and depolarizing channels. Finally, we conclude our discussion in Section 6.

## 2 Open systems and their evolution

In closed quantum system, evolution of a density matrix  $\rho(t_0)$  is performed by unitary operator  $U(t, t_0) = e^{-i\mathbf{H}_S(t-t_0)}$  as  $\rho(t) = U(t, t_0)\rho(t_0)U(t, t_0)^\dagger$ , where  $\mathbf{H}_S$  is the Hamiltonian of the closed system. For an open system, it generally couples with the environment, so the Hamiltonian of the system and its environment may not always be separable, which leads to a combined evolution of both. Let total Hamiltonian be

$$\mathbf{H} = \mathbf{H}_S \otimes \mathbb{1} + \mathbb{1} \otimes \mathbf{H}_B + \mathbf{H}_{int}$$

where  $\mathbf{H}_S, \mathbf{H}_B$  be the individual Hamiltonian of system and bath respectively and  $\mathbf{H}_{int}$  denotes the Hamiltonian of the interaction between the system and bath.

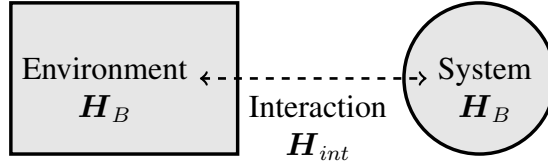


Figure 1: Interaction of open quantum system with environment.

The total Hilbert space  $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_B$  (subscript ‘S’ & ‘B’ to be understood as system and bath respectively). Starting with a initial state  $\rho(0) = \rho_S \otimes |0\rangle\langle 0|_B$ , combined evolution is performed by propagator  $U_{SB}(t)$  as  $\rho(t) = U_{SB}(t)\rho(0)U_{SB}(t)^\dagger$ . And only system evolution is thus given by,

$$\begin{aligned} \rho_S(t) &= \text{Tr}_B[\rho(t)] = \sum_{k \in \beta_B} \langle k|_B U_{SB}(\rho_S \otimes |0\rangle\langle 0|_B) U_{SB}^\dagger |k\rangle_B \\ &= \sum_{k \in \beta_B} \langle k|_B U_{SB} |0\rangle_B \rho_S \langle 0|_B U_{SB}^\dagger |k\rangle_B \\ &= \sum_{\substack{k \in \beta_B, \\ i, j, l, m \in \beta_S}} \langle i, k| U_{SB} |j, 0\rangle \langle j| \rho_S |l\rangle \langle l, 0| U_{SB}^\dagger |m, k\rangle \end{aligned}$$

$$= \sum_{k;i,j,l,m} M_k^{i,j} \rho_S^{j,l} (M_k^\dagger)^{l,m} = \sum_{k \in \beta_B} M_k \rho_S M_k^\dagger \quad (2.1)$$

where  $\beta_S$  and  $\beta_B$  are orthonormal bases for  $\mathcal{H}_S$  and  $\mathcal{H}_B$  respectively and  $M_k = \langle k | U_{SB} | 0 \rangle$  is called Kraus operator. This representation of  $\rho_S(t)$  in eq. (2.1) is called *operator sum representation*.

**Remark.** 1. Operator sum representation of  $\rho_S$  with Kraus operator  $\{M_k\}$  in eq. (2.1) depends on the choice of the basis  $\beta_B$ . So, it is not unique.

2.  $\rho(t)$  is Hermitian and positive.

3. If  $\rho_S(t)$  preserves its trace, then we get the *completeness relation*,

$$\begin{aligned} \text{Tr}(\rho_S(0)) &= \text{Tr}(\rho_S(t)) = \sum_{k \in \beta_B} \text{Tr}(M_k \rho_S(0) M_k^\dagger) \\ &= \sum_{k \in \beta_B} \text{Tr}(M_k^\dagger M_k \rho_S(0)) = \text{Tr}\left(\sum_{k \in \beta_B} M_k^\dagger M_k \rho_S(0)\right), \end{aligned}$$

$$\text{i.e., } \sum_{k \in \beta_B} M_k M_k^\dagger = \mathbb{1}.$$

4. Starting with a pure state, for unitary evolution, it remains pure.

### 3 An introduction to Quantum measurement

The postulates of quantum measurement say that any measurement in a closed system is described by a set of measurement operators  $\{M_m\}$  corresponding to the measurement outcome  $\{m\}$  such that  $\sum_m M_m M_m^\dagger = \mathbb{1}$ . For an open system, we define measurement operators  $\{M_m : M_m(\rho) = M_m \rho M_m^\dagger\}$  corresponding to the measurement outcome  $\{m\}$ . Given the state  $\rho$  before measurement, the post measurement state will be  $M_m(\rho) / \text{Tr}[M_m(\rho)]$  with probability of the outcome  $m$  being  $p(m) = \text{Tr}[M_m(\rho)]$ .

According to Von Neumann, it is possible in principle to correlate the state of a microscopic quantum system with the value of a macroscopic classical variable, and we may take it for granted that we can perceive the value of the classical variable. So, to measure an observable of the system, we couple the system with a larger system (*i.e.* measuring pointer) from which we can perceive the corresponding macroscopic classical variable and that larger system is called *Ancillary* bath or environment.

We describe the measurement by modifying the Hamiltonian (of total system) by adding a coupling term between the observable  $A$  of the system and generator of translation  $P$  of

the macroscopic classical variable (' $x$ ') corresponding to the pointer (or ancillary bath). The Hamiltonian is

$$\mathbf{H} = \mathbf{H}_S \otimes \mathbb{1} + \mathbb{1} \otimes \mathbf{H}_B + \lambda(t) \mathbf{A} \otimes \mathbf{P}, \quad (3.1)$$

where  $\mathbf{H}_S$  and  $\mathbf{H}_B$  are separate Hamiltonian of the system to be measured and the ancillary bath, respectively and  $\lambda$  is a coupling constant. Considering the time scale of the system and bath evolution being much larger than the measurement timescale ( $\delta t$ ), we neglect  $\mathbf{H}_S$ ,  $\mathbf{H}_B$  over coupling term in eq. (3.1) and get  $\mathbf{H} \simeq \lambda(t) \mathbf{A} \otimes \mathbf{P}$ , and corresponding propagator is

$$U(\delta t) = \exp(-i \mathbf{H} \delta t) = \exp(-i \delta t \lambda \mathbf{A} \otimes \mathbf{P}) \quad (3.2)$$

Let  $\beta_S = \{|i\rangle_S\}$  be the orthonormal eigenbasis for  $\mathbf{A}$  with corresponding eigenvalue  $\{a_i\}$  (discrete spectrum) and  $\{|x\rangle_B\}$  be the orthogonal basis of  $x$  (continuous spectrum), that leads,

$$\begin{aligned} U(\delta t) &= \sum_{i \in \beta_S} \exp(-i \delta t \lambda \mathbf{A} \otimes \mathbf{P}) |i\rangle_S \otimes \int_{-\infty}^{\infty} dx |x\rangle_B \langle x|_B \\ &= \sum_i \int_{-\infty}^{\infty} dx |i\rangle_S \otimes \exp(-i \delta t \lambda a_i \mathbf{P}) |x\rangle_B \langle x|_B \\ &= \sum_i |i\rangle_S \otimes \int_{-\infty}^{\infty} dx |x + \lambda a_i \delta t\rangle_B \langle x|_B \end{aligned} \quad (3.3)$$

Starting with initial state  $\sum_{j \in \beta_S} c_j |j\rangle_S \otimes |\psi\rangle_B$ , after time  $\delta t$ , the state is

$$\begin{aligned} U(\delta t) \sum_{j \in \beta_S} |j\rangle_S \otimes |\psi\rangle_B &= \sum_{i \in \beta_S} \int_{-\infty}^{\infty} dx c_i |i\rangle_S \otimes |x + \lambda a_i \delta t\rangle_B \langle x| \psi\rangle_B \\ &= \sum_{i \in \beta_S} \int_{-\infty}^{\infty} dx \psi_B(x - \lambda a_i \delta t) c_i |i\rangle_S \otimes |x\rangle_B \\ &= \sum_{i \in \beta_S} c_i |i\rangle_S \otimes |\phi_i\rangle_B, \end{aligned} \quad (3.4)$$

where

$$|\phi_i\rangle_B = \int_{-\infty}^{\infty} dx \psi_B(x - \lambda a_i \delta t) |x\rangle_B \Rightarrow \phi_B(x) = \psi_B(x - \lambda a_i \delta t).$$

If the overlap between  $|\phi_i\rangle_B$ 's for different  $a_i$ 's (i.e.,  $\langle \phi_i | \phi_j \rangle$ ) is low, we can separately measure the pointer, and we will detect the eigenvalue  $a_i$  by measuring the change of the observable  $x$  of  $|\phi_i\rangle_B$  by  $\lambda a_i \delta t$  in the pointer, with probability  $|c_i|^2$ . In that case, the state of the system is prepared in the eigenstate  $|i\rangle$ . So, measuring  $|\psi(x + \lambda a_i \delta t)\rangle_B$  in

the pointer induces a measurement of the system in the orthogonal eigenbasis. This is an example of Von Neumann's model of orthogonal measurement. More abstractly, for an orthogonal measurement corresponding to  $N$  orthogonal projection operator of the system  $\{M_i = |i\rangle\langle i|_S : i = 0, \dots, N-1\}$ , we introduce  $N$  orthonormal basis of pointer  $\{|k\rangle_B : k = 0, \dots, N-1\}$ , then the unitary transformation (similar to eq. (3.3)) is

$$U(\delta t) = \sum_{i,k=0}^{N-1} M_i \otimes |k+i\rangle\langle k|_B, \quad (3.5)$$

where  $|k+i\rangle \equiv |k+i \pmod{N}\rangle$  and  $|k+i\rangle\langle k|_B$  denotes the jump in the pointer. An orthonormal basis for the pointer is taken to ensure that they do not overlap *i.e.*,  $\langle k|k'\rangle_B = \delta_{k,k'}$ .

**Remark (Generalized measurement).** Unlike the previous one, in generalized measurement, we may not always project the system to an orthogonal eigenbasis  $\{|i\rangle_S\}$  when measuring the pointer in different orthonormal basis,  $\{|k'\rangle_B\}$ . Now starting with a state of the system  $|\psi\rangle_S$ , when coupled with the state  $|0\rangle_B$  in ancillary bath (pointer) gives  $\rho(0) = |\psi\rangle\langle\psi|_S \otimes |0\rangle\langle 0|_B$  in  $\mathcal{H}_S \otimes \mathcal{H}_B$ . Define the propagator of the combined system as

$$\begin{aligned} U_{SB}(t) &= \sum_{k' \in \beta_B} M_{k'} \otimes |k'\rangle\langle 0|_B \\ \Rightarrow U_{SB}(t) |\psi\rangle_S \otimes |0\rangle_B &= \sum_{k' \in \beta_B} M_{k'} |\psi\rangle_S \otimes |k'\rangle_B \end{aligned} \quad (3.6)$$

where  $\beta_B = \{|k'\rangle\}$  is orthonormal basis for  $\mathcal{H}_B$ . Observe that,  $\{M_{k'} |\psi\rangle_S\}_{k' \in \beta_B}$  may not be orthogonal and (as in eq. (2.1)),

$$\rho_S(t) = \text{Tr}_B[\rho(t)] = \sum_{k' \in \beta_B} M_{k'} |\psi\rangle\langle\psi|_S M_{k'}^\dagger.$$

Here measuring  $|k'\rangle_B$  the state of the system is prepared in  $M_{k'} |\psi\rangle_S$  with probability  $\|M_{k'} |\psi\rangle_S\|^2$ . Generally,  $\rho_S(t)$  is a mixed state. We can represent a mixed state of a system in  $\mathcal{H}_S$  as a pure state of a bigger system (*i.e.* system + ancillary bath) in  $\mathcal{H}_S \otimes \mathcal{H}_B$  as  $\sum_{k \in \beta_B} M_k |\psi\rangle_S \otimes |k\rangle_B$  and by partial tracing over the bath we extract the state of the system.

## 4 Quantum operator and beyond

In the previous section, we describe the evolution of the quantum system through a unitary propagator, here we will discuss quantum operator formalism. A *quantum operator*, defined from a vector space of density operators to another vector space, is a function acting

on an operator to give a new operator. As an example, in eq. (2.1), define quantum operator  $M$  acting on density operator  $\rho_S(0)$ , gives  $\rho_S(t)$  as

$$\rho_S(t) = M(\rho_S) = \sum_{k \in \beta_B} M_k \rho_S M_k^\dagger.$$

Observe that the quantum operator is written in operator sum representation. A question may arise: Can all quantum operators be written in the form of operator sum representation? We will soon see that if it follows some set of rules then it can have operator sum representation. Based on the physical realization of the evolution of a density operator, we claim that quantum operator  $M$  follows the following axioms:

*Axiom 1.* It says that for any  $\rho$  in Hilbert space  $\mathcal{H}$ ,  $0 \leq \text{Tr}[M(\rho)] \leq 1$ . It is convenient as  $M$  will give a new density operator and trace of which, gives the total probability of measurement outcome of all constituent states of the ensemble.

*Axiom 2.* It says that the quantum operator is *convex linear map*. Suppose that  $t = t_0$ , an ensemble of states,  $\rho_i$  are prepared with probability  $p_i$ , then we expect the time evolved ensemble of states at  $t = t_0 + \delta t$ , will be  $M(\rho)$  with the same probability  $p_i$ .

*Axiom 3.* It says that the quantum operator is *completely positive* (a map  $M$  defined in  $\mathcal{H}$  Hilbert space, is ‘completely positive’ if  $M \otimes \mathbb{1}$  is positive in  $\mathcal{H} \otimes \mathcal{H}'$  for any extension  $\mathcal{H}'$  over  $\mathcal{H}$ ). It is positive as the density operator is positive semi-definite. Moreover, it is completely positive, because starting with a density operator of the composite bipartite system (of subsystem and another ancillary system), applying  $M \otimes \mathbb{1}$ , we still want to obtain a valid density operator.

Considering all three axioms for a quantum operator, we propose a necessary theorem regarding the operator sum representation.

**Theorem 1.** *The quantum operator  $M$ , defined from a vector space of density operators to another vector space, satisfies the axiom 1, axiom 2 and axiom 3, if and only if*

$$M(\rho) = \sum_{i=1} M_i \rho M_i^\dagger$$

*for some set of operators  $\{M_i\}$  defined from a Hilbert space to another Hilbert space and*

$$\sum_{i=1} M_i \rho M_i^\dagger \leq \mathbb{1}$$

**Remark.** Some important points are as follows:

1. This theorem states only the existence of such operator sum, not the uniqueness.

2. Two operator-sum representations of the same quantum channel are related by a unitary change of basis for the Kraus operators. Let two different sets of Kraus operators  $\{M_i\}$  and  $\{N_j\}$  representing the same channel, they are related by  $N_j = \sum_i V_j^i M_i$ , where  $\{V_j^i\}$  are some unitary matrices.
3. If the quantum operator is also trace-preserving, then completeness relation follows *i.e.*,  $\sum_i M_i M_i^\dagger = \mathbb{1}$
4. For further discussion of a quantum channel, the domain Hilbert space and range Hilbert space are the same, moreover are finite-dimensional and trace-preserving. The following Kraus representation theorem will elaborate on this.

**Theorem 2** (Kraus representation theorem). *Let  $M$  be a trace-preserving quantum channel defined in finite-dimensional Hilbert space  $\mathcal{H}_S$  of dimensions  $d$ . Then any operator sum representation of the quantum channel has at most  $d^2$  elements. For any  $\rho \in \mathcal{H}_S$ , the quantum operator  $M(\rho)$  can be written as*

$$M(\rho) = \sum_{i=1}^K M_i \rho M_i^\dagger$$

where  $K \leq d^2$  is called *Kraus number* and  $\{M_k\}$  is the set of linear operators, called *Kraus operator*, that satisfy the completeness property, *i.e.*

$$\sum_{i=1}^K M_i M_i^\dagger = \mathbb{1}.$$

## 5 Quantum channels

We have discussed quantum operators and three axioms that a quantum operator follows. The quantum operator with trace-preserving, completely positive, convex linear property is called *quantum channel*. As it is trace preserving,

$$\sum_i M_i M_i^\dagger = \mathbb{1},$$

gives the completeness relation. Further, it follows Kraus representation theorem.

Our main focus is to describe the dynamics of a few quantum channels *viz.*, amplitude-damping, phase-damping and decoherence channels via two level system, which gives a concise idea of how the environment affects the system. The general description is as follows, given a quantum channel with different Kraus operators  $\{M_i\}_{i=1}^N$  acting on the system, we introduce an auxiliary bath having  $N$  orthonormal basis, and define the unitary propagator

as given in eq. (3.6). Finally, partial tracing over the states of the auxiliary bath gives the dynamics of the system.

To describe the quantum channel, *Markov approximation* is taken into account. It says that the evolution of the state will only depend on the current state *i.e.*, the density operator  $\rho_S(t + \delta t)$  will only depends on  $\rho_S(t)$ . So the past history of the state of the system does not influence the evolution of the present state. But we know an open system is dissipative because information and energy can flow both way from system to environment and vice versa, then the density operator  $\rho_S(t + \delta t)$  not only depends on  $\rho_S(t)$ , but also depends on  $\rho_S$  at time before  $t$ . In reality, Markov approximation is unreachable. Nonetheless it is a good approximation to start considering the fact that the time scale of the evolution of the system is much larger than the time scale in which environment completely forgot the information flows into it. For the case of two level system, the Markov approximation is taken into account such that, after each infinitesimal time increment for the combined evolution of the system and its environment, the states of the environment, denoted as  $|0\rangle_E, |1\rangle_E$ , corresponding to the states of the system,  $|g\rangle_S, |e\rangle_S$ , reset themselves. Here, the basis states  $|0\rangle_E$  and  $|1\rangle_E$  keep track of the evolution of  $|g\rangle_S$  and  $|e\rangle_S$  respectively. In other words, the environment forgets its past memory. As an example,

$$|e\rangle |0\rangle \xrightarrow[\text{Prob}=p]{U(\delta t)} |g\rangle |1\rangle \xrightarrow[|1\rangle \rightarrow |0\rangle]{\text{Reset basis of E}} |g\rangle |0\rangle \xrightarrow{U(\delta t)} \xrightarrow{\text{Reset}} \xrightarrow{U(\delta t)} \dots$$

The evolution in the quantum channels is given by trace-preserving completely positive linear operator  $M$ . For a density operator  $\rho_S$  the evolution is as follows,

$$\rho_S(t + \delta t) = M(\delta t)\rho_S(t).$$

The Chapman-Kolmogorov equation says that,

$$M(N\delta t) = M^N(\delta t) = \underbrace{M(\delta t) \circ \dots \circ M(\delta t)}_{N \text{ times}}$$

then

$$\rho_S(t + N\delta t) = M(N\delta t)\rho_S(t) = M(\delta t) \circ \dots \circ M(\delta t)\rho_S(t)$$

.

## 5.1 Amplitude-damping channel

In quantum mechanics, the concept of amplitude represents the probability of a state within the system. The amplitude-damping channel serves as a model to illustrate how the probability of certain states within the system decays when it is connected to a bath due to the dissipation of energy. For instance, we can consider the spontaneous photon emission of



an atomic two-level system as an example. Let,  $|g\rangle$  &  $|e\rangle$  be the ground and excited state of the system, respectively and  $|0\rangle$  &  $|1\rangle$  be the states of ancillary bath (or environment) such that  $|1\rangle$  denotes photon is detected in the environment and  $|0\rangle$  when not detected. The unitary evolution for small time  $\delta t$  is as follows:

$$\begin{aligned} |g\rangle |0\rangle &\xrightarrow{U(\delta t)} |g\rangle |0\rangle \\ |e\rangle |0\rangle &\xrightarrow{U(\delta t)} \sqrt{1-p} |e\rangle |0\rangle + \sqrt{p} |g\rangle |1\rangle, \end{aligned}$$

where probability  $p = \Gamma \delta t$  and  $\Gamma$  is probability rate. Writing a possible unitary evolution operator  $U$  describing the above events in matrix form in the basis  $\{|g\rangle |0\rangle, |g\rangle |1\rangle, |e\rangle |0\rangle, |e\rangle |1\rangle\}$  gives the following,

$$U(\delta t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{1-p} & \sqrt{p} & 0 \\ 0 & \sqrt{p} & \sqrt{1-p} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (5.1)$$

The corresponding Kraus operators are

$$\begin{aligned} M_0 &= \langle 0|U|0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix} = |g\rangle\langle g| + \sqrt{1-p} |e\rangle\langle e|, \\ M_1 &= \langle 1|U|0\rangle = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix} = \sqrt{p} |g\rangle\langle e|. \end{aligned}$$

It is clear that  $M_1$  describes the photon emission as the environment detects it, so  $|e\rangle$  decays to  $|g\rangle$ , and  $M_2$ . And  $M_0$  describes the evolution when there is no photon detected.

Starting with a single qubit density operator,

$$\rho_0 = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix}$$

with  $\rho_{00} + \rho_{11} = 1$ , the evolved state after  $\delta t$  is

$$M(\rho_0) = \rho(\delta t) = M_0 \rho_0 M_0^\dagger + M_1 \rho_0 M_1^\dagger = \begin{pmatrix} \rho_{00} + p\rho_{11} & \sqrt{1-p}\rho_{01} \\ \sqrt{1-p}\rho_{10} & (1-p)\rho_{11} \end{pmatrix}. \quad (5.2)$$

Now, taking Markov approximation into account, after a finite time  $t$ , the evolved state is

$$\begin{aligned} \rho(t) &= \lim_{\delta t \rightarrow 0} M^{t/\delta t}(\rho_0) = \lim_{\delta t \rightarrow 0} \begin{pmatrix} \rho_{00} + (1 - (1 - \Gamma \delta t)^{t/\delta t})\rho_{11} & \sqrt{(1 - \Gamma \delta t)^{t/\delta t}}\rho_{01} \\ \sqrt{(1 - \Gamma \delta t)^{t/\delta t}}\rho_{10} & (1 - \Gamma \delta t)^{t/\delta t}\rho_{11} \end{pmatrix} \\ &= \begin{pmatrix} \rho_{00} + (1 - e^{-\Gamma t})\rho_{11} & e^{-\Gamma t/2}\rho_{01} \\ e^{-\Gamma t/2}\rho_{10} & e^{-\Gamma t}\rho_{11} \end{pmatrix} \end{aligned} \quad (5.3)$$

which, in infinite time limit *i.e.*,  $t \gg 1/\Gamma$ , evolves into the following stationary state:

$$\rho_\infty = \rho(t \rightarrow \infty) = \begin{pmatrix} \rho_{00} + \rho_{11} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = |g\rangle\langle g|.$$

In terms of Bloch sphere representation, starting with

$$\rho_0 = \frac{1}{2}(\mathbb{1} + \vec{n}_0 \cdot \vec{\sigma}) = \frac{1}{2} \begin{pmatrix} 1 + n_z & n_x - in_y \\ n_x + in_y & 1 - n_z \end{pmatrix}$$

then at time  $t$ ,

$$\rho(t) = \frac{1}{2}\mathbb{1} + \frac{1}{2} \begin{pmatrix} n_x e^{-\Gamma t/2} \\ n_y e^{-\Gamma t/2} \\ 1 - (1 - n_z)e^{-\Gamma t} \end{pmatrix} \cdot \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix} \Rightarrow \vec{n}(t) = \begin{pmatrix} n_x e^{-\Gamma t/2} \\ n_y e^{-\Gamma t/2} \\ 1 - (1 - n_z)e^{-\Gamma t} \end{pmatrix}. \quad (5.4)$$

Further, the stationary state is given by

$$\rho_\infty = \frac{1}{2}\mathbb{1} + \frac{1}{2}\sigma_z \Rightarrow \vec{n}_\infty = (0, 0, 1). \quad (5.5)$$

That suggests that with time Bloch sphere will shrink towards the pole (0,0,1) and concentrate at the pole *i.e.*, the stationary state. So, over a long time, the system will relax into the ground state and the amplitude (or the probability) of the excited state will completely vanish.

Now we can think of a more generalized amplitude-damping given by the following Kraus operators:

$$\begin{aligned} M_0 &= \sqrt{\lambda} \left( |g\rangle\langle g| + \sqrt{1-p} |e\rangle\langle e| \right), \\ M_1 &= \sqrt{\lambda p} |g\rangle\langle e|, \\ M_2 &= \sqrt{1-\lambda} \left( \sqrt{1-p} |g\rangle\langle g| + |e\rangle\langle e| \right), \\ M_3 &= \sqrt{(1-\lambda)p} |e\rangle\langle g| \end{aligned}$$

where  $\lambda$  and  $p$  are in  $[0, 1]$ . And the final stationary state will be

$$\rho_\infty = \begin{pmatrix} \lambda & 0 \\ 0 & 1 - \lambda \end{pmatrix} = \lambda |g\rangle\langle g| + (1 - \lambda) |e\rangle\langle e| = \frac{1}{2}\mathbb{1} + \left( \frac{2\lambda - 1}{2} \right) \sigma_z,$$

and in Bloch sphere,  $\vec{n} = (0, 0, 2\lambda - 1)$ . Similarly here, the Bloch sphere will shrink towards  $(0, 0, 2\lambda - 1)$  with time.

## 5.2 Dephasing channel

A dephasing (or phase damping) channel describes the evolution of a coherence state into a decoherence state by losing its phase information to the environment. For example, dust particles with two energy levels  $\{|g\rangle, |e\rangle\}$  interact with photons by exchanging energy by collision. Point to note that the energy gap between the two levels of the dust particle is much larger than the exchange energy, so collision won't affect the system state but it certainly affects the state of the photon. Also the ground state  $|g\rangle$  of the dust particle interacts differently with photon than the excited state  $|e\rangle$ . Starting with photon state  $|0\rangle$ , it evolves into  $\{|0\rangle, |1\rangle, |2\rangle\}$  states depending on the state of the dust particle and collision probability  $p = \Gamma \delta t$ . The unitary evolution for small time  $\delta t$  is as follows;

$$\begin{aligned} |g\rangle |0\rangle &\xrightarrow{U(\delta t)} \sqrt{1-p} |g\rangle |0\rangle + \sqrt{p} |g\rangle |1\rangle \\ |e\rangle |0\rangle &\xrightarrow{U(\delta t)} \sqrt{1-p} |e\rangle |0\rangle + \sqrt{p} |e\rangle |2\rangle \end{aligned}$$

Similarly, a possible unitary evolution operator  $U$  in matrix form in the basis  $\{|g\rangle |0\rangle, |g\rangle |1\rangle, |g\rangle |2\rangle, |e\rangle |0\rangle, |e\rangle |1\rangle, |e\rangle |2\rangle\}$  is the following,

$$U(\delta t) = \begin{pmatrix} \sqrt{1-p} & \sqrt{p} & 0 & 0 & 0 & 0 \\ \sqrt{p} & \sqrt{1-p} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{1-p} & 0 & \sqrt{p} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{p} & 0 & \sqrt{1-p} \end{pmatrix}. \quad (5.6)$$

The corresponding Kraus operators are

$$M_0 = \sqrt{1-p} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, M_1 = \begin{pmatrix} \sqrt{p} & 0 \\ 0 & 0 \end{pmatrix}, M_2 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}.$$

and the evolved state after  $\delta t$  is

$$\mathbf{M}(\rho_0) = \rho(\delta t) = \sum_{i=1}^3 M_i \rho_0 M_i^\dagger = \begin{pmatrix} \rho_{00} & (1-p)\rho_{01} \\ (1-p)\rho_{10} & \rho_{11} \end{pmatrix}. \quad (5.7)$$

Taking Markov approximation into account, after a finite time  $t$  the state is,

$$\rho(t) = \lim_{\delta t \rightarrow 0} \mathbf{M}^{t/\delta t}(\rho_0) = \begin{pmatrix} \rho_{00} & e^{-\Gamma t} \rho_{01} \\ e^{-\Gamma t} \rho_{10} & \rho_{11} \end{pmatrix}. \quad (5.8)$$

In infinite time limit *i.e.*,  $t \gg 1/\Gamma$ , the state evolves into the following stationary state:

$$\rho(t \rightarrow \infty) = \begin{pmatrix} \rho_{00} & 0 \\ 0 & \rho_{11} \end{pmatrix}$$

In terms of Bloch sphere representation, starting with  $\rho_0 = \frac{1}{2}(\mathbb{1} + \vec{n}_0 \cdot \vec{\sigma})$ , the evolved state at time  $t$  is the following:

$$\rho(t) = \frac{1}{2}\mathbb{1} + \frac{1}{2} \begin{pmatrix} n_x e^{-\Gamma t} \\ n_y e^{-\Gamma t} \\ n_z \end{pmatrix} \cdot \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix} \Rightarrow \vec{n}(t) = \begin{pmatrix} n_x e^{-\Gamma t} \\ n_y e^{-\Gamma t} \\ n_z \end{pmatrix}, \quad (5.9)$$

and the stationary state is given by

$$\rho_\infty = \frac{1}{2}\mathbb{1} + \frac{1}{2}n_z\sigma_z \Rightarrow \vec{n}_\infty = (0, 0, n_z). \quad (5.10)$$

That suggests that over a finite time, the Bloch sphere squeezes exponentially along  $x$  and  $y$  axes, keeping  $z$  coordinate intact and in the stationary state, it concentrates on the point  $(0, 0, n_z)$ . So, the coherence of the system is lost and the state of the system is approaching to complete decoherence state. We will see later that the dephasing channel is equivalent to the phase-flip channel.

### 5.3 Bit-flip & phase-flip channel

Bit-flip and phase-flip are very important quantum noise channels. *Bit-flip channel* is a quantum channel where the transmitted qubit can be flipped during transmission due to environmental interaction with finite probability. This occurs in the following way:

$$\begin{aligned} |0\rangle &\longrightarrow |1\rangle, \quad |1\rangle \longrightarrow |0\rangle \\ \text{i.e., } |\psi\rangle &\longrightarrow \sigma_x |\psi\rangle. \end{aligned}$$

Let the bit-flip occur with probability  $p$  in single qubit  $|\psi\rangle$  in the channel. This follows the unitary evolution as follows:

$$|\psi\rangle |0\rangle \xrightarrow{U} \sqrt{1-p} |\psi\rangle |0\rangle + \sqrt{p} (\sigma_x |\psi\rangle) |1\rangle$$

where  $\{|0\rangle, |1\rangle\}$  orthogonal states of ancillary bath. The ancillary bath (or measurement pointer) keeps track of the evolution by the orthogonal states  $\{|k\rangle : k = 0, 1, 2, \dots\}$ . The corresponding Kraus operators are

$$M_0 = \sqrt{1-p} \mathbb{1} \text{ and } M_1 = \sqrt{p} \sigma_x$$

Starting with a single qubit state

$$\rho = \frac{1}{2}\mathbb{1} + \vec{r} \cdot \vec{\sigma} = \frac{1}{2} \begin{pmatrix} 1 + r_z & r_x - ir_y \\ r_x + ir_y & 1 - r_z \end{pmatrix},$$

the operator sum representation is as follows:

$$\begin{aligned} M(\rho) &= (1-p)\rho + p \sigma_x \rho \sigma_x \\ &= \frac{1}{2} \begin{pmatrix} 1 + (1-2p)r_z & r_x - i(1-2p)r_y \\ r_x + i(1-2p)r_y & 1 - (1-2p)r_z \end{pmatrix} \end{aligned} \quad (5.11)$$

In terms of Bloch sphere representation

$$M(\rho) = \frac{1}{2}\mathbb{1} + \frac{1}{2} \begin{pmatrix} r_x \\ r_y(1-2p) \\ r_z(1-2p) \end{pmatrix} \cdot \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix} \Rightarrow \vec{n} = \begin{pmatrix} r_x \\ r_y(1-2p) \\ r_z(1-2p) \end{pmatrix} \quad (5.12)$$

So the Bloch sphere squeezes along  $y$  and  $z$  axes  $(1-2p)$  times keeping  $x$  coordinate intact and converting into an ellipsoid.

Now we discuss the phase-flip channel. The *phase-flip channel* is described by the following way:

$$\begin{aligned} |0\rangle &\longrightarrow |0\rangle, \quad |1\rangle \longrightarrow -|1\rangle \\ \text{i.e., } |\psi\rangle &\longrightarrow \sigma_z |\psi\rangle \end{aligned}$$

Following the steps outlined previously, the operator representation is as follows:

$$\begin{aligned} M(\rho) &= (1-p)\rho + p \sigma_z \rho \sigma_z \\ &= \frac{1}{2} \begin{pmatrix} 1 + r_z & (1-2p)r_x - i(1-2p)r_y \\ (1-2p)r_x + i(1-2p)r_y & 1 - r_z \end{pmatrix} \end{aligned} \quad (5.13)$$

and in terms of Bloch sphere representation

$$M(\rho) = \frac{1}{2}\mathbb{1} + \frac{1}{2} \begin{pmatrix} r_x(1-2p) \\ r_y(1-2p) \\ r_z \end{pmatrix} \cdot \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix} \Rightarrow \vec{n} = \begin{pmatrix} r_x(1-2p) \\ r_y(1-2p) \\ r_z \end{pmatrix} \quad (5.14)$$

As we see before in the dephasing channel, the Bloch sphere representation for both channels is similar. So as mentioned before, the sphere is squeezed along  $x$  and  $y$  axes  $(1-2p)$  times keeping  $z$  coordinate intact and converted into an ellipsoid.

Further, both bit-phase flip is described in the following way:

$$\begin{aligned} |0\rangle &\longrightarrow i|1\rangle, \quad |1\rangle \longrightarrow -i|0\rangle \\ \text{i.e., } |\psi\rangle &\longrightarrow i\sigma_x \sigma_z |\psi\rangle = \sigma_y |\psi\rangle \end{aligned}$$

Similarly, the Bloch representation will be

$$M(\rho) = \frac{1}{2}\mathbb{1} + \frac{1}{2} \begin{pmatrix} r_x(1-2p) \\ r_y \\ (1-2p)r_z \end{pmatrix} \cdot \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix} \Rightarrow \vec{n} = \begin{pmatrix} r_x(1-2p) \\ r_y \\ r_z(1-2p) \end{pmatrix} \quad (5.15)$$

Now take a special case where the error occurs in qubit in a particular way. The probability of occurrence of the error in the channel is  $p$  and the probability of occurring one of these three errors is same,

$$\begin{aligned} |\psi\rangle &\rightarrow \sigma_x |\psi\rangle \text{ (Bit-flip )} \\ |\psi\rangle &\rightarrow \sigma_z |\psi\rangle \text{ (Phase-flip)} \\ |\psi\rangle &\rightarrow \sigma_y |\psi\rangle \text{ (Bit-phase flip).} \end{aligned}$$

The unitary evolution is as follows:

$$|\psi\rangle |0\rangle \xrightarrow{U} \sqrt{1-p} |\psi\rangle |0\rangle + \sqrt{\frac{p}{3}} (\sigma_x |\psi\rangle) |1\rangle + \sqrt{\frac{p}{3}} (\sigma_y |\psi\rangle) |2\rangle + \sqrt{\frac{p}{3}} (\sigma_z |\psi\rangle) |3\rangle$$

where  $\{|k\rangle : k = 0, 1, 2, 3\}$  orthogonal states of ancillary bath. The corresponding Kraus operators are

$$M_0 = \sqrt{1-p} \mathbb{1}, \quad M_1 = \sqrt{\frac{p}{3}} \sigma_x, \quad M_2 = \sqrt{\frac{p}{3}} \sigma_y, \quad M_3 = \sqrt{\frac{p}{3}} \sigma_z.$$

Starting with  $\rho$ , the operator sum representation is as follows:

$$\begin{aligned} M(\rho) &= (1-p)\rho + p \sigma_z \rho \sigma_z \\ &= \frac{1}{2} \begin{pmatrix} 1 + (1 - \frac{4}{3}p)r_z & (1 - \frac{4}{3}p)r_x - i(1 - \frac{4}{3}p)r_y \\ (1 - \frac{4}{3}p)r_x + i(1 - \frac{4}{3}p)r_y & 1 - (1 - \frac{4}{3}p)r_z \end{pmatrix} \end{aligned} \quad (5.16)$$

and in terms of Bloch sphere representation

$$M(\rho) = \frac{1}{2} \mathbb{1} + \frac{1}{2} \begin{pmatrix} 1 - \frac{4}{3}p \end{pmatrix} \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix} \cdot \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix} \Rightarrow \vec{n} = \begin{pmatrix} 1 - \frac{4}{3}p \end{pmatrix} \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix} \quad (5.17)$$

that is to say, the Bloch sphere contracts uniformly in every direction. This is a specific example of a depolarizing channel, which will be discussed next.

## 5.4 Depolarizing channel

A depolarizing channel is a noise channel where the transmitted qubit can be depolarized (*i.e.*, qubit converted into completely mixed state  $\mathbb{1}/2$ ) with a finite probability  $p$ . The depolarizing channel acts on a qubit state  $\rho = \frac{1}{2} \mathbb{1} + \vec{r} \cdot \vec{\sigma}$  in the following way:

$$\begin{aligned} M(\rho) &= p \frac{\mathbb{1}}{2} + (1-p)\rho \\ &= \frac{1}{2} \begin{pmatrix} 1 + (1-p)r_z & (1-p)r_x - i(1-p)r_y \\ (1-p)r_x + i(1-p)r_y & 1 - (1-p)r_z \end{pmatrix}. \end{aligned} \quad (5.18)$$

Similarly in terms of Bloch sphere representation,

$$M(\rho) = \frac{1}{2}\mathbb{1} + \frac{1}{2}(1-p) \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix} \cdot \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix} \Rightarrow \vec{n} = (1-p) \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix} \quad (5.19)$$

As mentioned in the preceding example (see eq. (5.17)), the Bloch sphere contracts uniformly  $(1-p)$  times in every direction. That is, in every direction the magnitude of the polarization vector,  $\vec{r}$  decrease by a same amount, so the coherence state becomes decoherence state.

**Remark (Reversibility of quantum channel).** It is worth noting that all the quantum channel discussed are irreversible. The irreversibility is coming from the fact that if we have a quantum channel which reverse the effect of the particular quantum channel, that is it increase polarization (*i.e.*,  $|\vec{r}|$ ) in certain direction, then starting with a state with unit polarization (*i.e.*,  $|\vec{r}| = 1$ ) in that direction on the Bloch sphere, the evolved state will be of polarization more than unity, which is absurd.

## 6 Conclusions

We have presented a comprehensive overview of the theoretical framework essential for understanding open quantum systems and their dynamics. Beginning with the discussion on the unitary evolution of open systems and quantum measurements, we proceeded to examine open quantum systems through various types of quantum channels using a two-level system. We explored their properties and behaviours in detail. By providing a thorough discussion of these topics, we hope to give readers a foundation for further study in the field of open quantum systems and quantum information.

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