Purity of $\mathcal{M}$-theory

P. Vanhove (SPhT/Saclay)

based on:

hep-th/0408171 with L. Anguelova and P.-A. Grassi

hep-th/0411167 with P.-A. Grassi

Outline of the Talk:

11d supergravity and beyond

$\mathcal{N}_{11} = 1$ supergravity theory in 11d describes the dof

$$g_{MN}, \qquad C_{MNP}, \qquad \Psi_M^A$$

the dynamics is described by the 2d actions [Cremmer, Julia, Scherk]

$$\begin{aligned} S_{CJS} &= \frac{1}{2\kappa_{(11)}^2}\int d^{11}x\left[\mathcal{R}_{(11)} + |G_4|^2\right] \\ &+ \frac{1}{2\kappa_{(11)}^2}\int d^{11}x + \bar{\Psi}_M\Gamma^{MNP}\left(D_N + \mathcal{T}_N\cdot G_4\right)\Psi_P \\ &+ \frac{1}{6}\int C_3\wedge G_4\wedge G_4 + \text{four fermi} \end{aligned}$$

This action is the strong coupling limit of the effective action for type IIa in the string frame [Townsend & Witten] and is the "effective" action for the Membrane [Bergshoeff, Sezgin, Townsend & Duff, Howe, Inami, Stelle] : κ-symmetry of the supermembrane forces the background field to satisfy the 11D supergravity equations-of-motion.

Supersymmetry requieres that the dimension zero torsion takes the form

$$T^c{}_{\alpha\beta} = (\mathcal{C}\Gamma^{r_1})_{\alpha\beta}X^c{}_{r_1} + (\mathcal{C}\Gamma^{r_1r_2})_{\alpha\beta}X^c{}_{r_1r_2} + (\mathcal{C}\Gamma^{r_1\cdots r_5})_{\alpha\beta}X^c{}_{r_1\cdots r_5}$$

5808 **1**+55+65 **429**+165+11 **4290**+462+330

$$T_{r\alpha}{}^{\beta} = \left(\Gamma_{[5]r}\cdot G_{[4]} \qquad -8\Gamma_{[3]}\cdot G_{[4]r}\right)_{\alpha}{}^{\beta}$$

Paul Howe showed that $X_{429} = 0 = X_{4290}$ implies 11d CJS action

Anything in **429** or **4290** is a (quantum) correction to the CJS action.

Quantum correction to $\mathcal{M}$-theory

$$\frac{1}{\ell_P^3}\int d^{11}x\,\left(\mathcal{R}^4+C_3\mathcal{R}^4\right)$$

The $\mathcal{R}^4$ corrections needs $X_{4290}\neq 0$

$$X_{4290}=(\alpha'_M)^3\,t_8^{a\,ts_1\cdots s_6}W^{s_1s_2}_{r_1r_2}W^{s_3s_4}_{r_3r_4}W^{s_5s_6}_{r_5t}$$

[Cedewall, Gran, Nielsen, Nilsson & Cedewall, Gran, Nilsson, Tsimpis]
[Peeters, Vanhove, Westerberg]

The analysis of [Howe, Tsimpis] shows that all the $\mathcal{R}^4$ term in 11d are related to the anomaly term

$$\int C_3\wedge\left(a_1\,\mathrm{tr}R^4+a_2\,(\mathrm{tr}R^2)^2\right)$$

Unicity of R^4 invariant ? is it a consequence of supersymmetry alone?

Should arise from $T_{M2}=\ell_P^{-3}$ Mtwobrane (two-loop world-volume?) corrections. But no perturbative approach for the membrane

$\Rightarrow$ the zero-mode approximation : The superparticle.

The superparticle perturbation

Aim: Formulate first quantized superparticle feynman rules for calculating g-loop amplitudes.

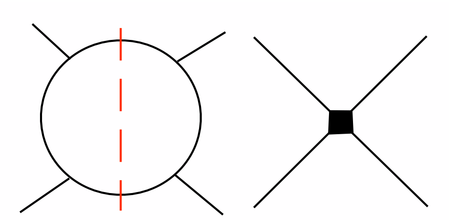
$$\mathcal{A}^{(g)}=\int\mathcal{D}S^A\,\mathcal{D}X\,\int_{r=1}^{c(g)+n}d\tau_r V(\tau_1)\cdots V(\tau_n)\;e^{-S_{part.}}$$

The superparticle perturbation

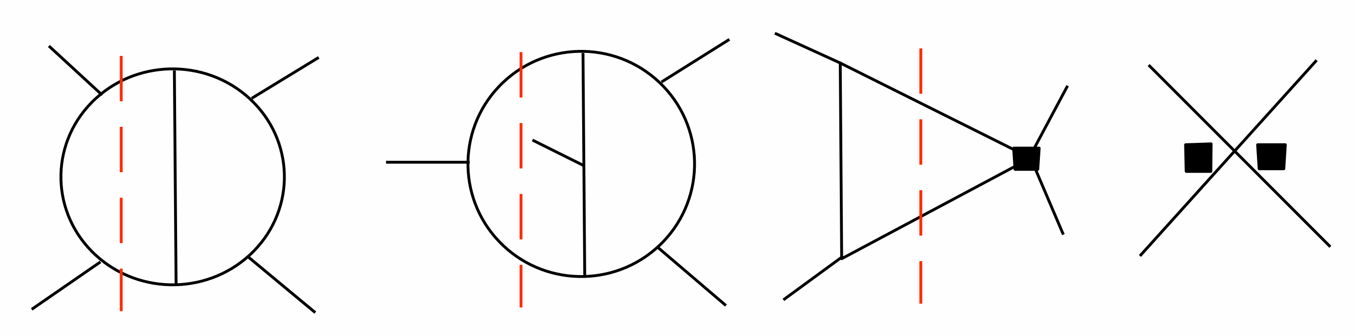
[Green, Gutperle, Vanhove & Green, Kwon, Vanhove]

- Use SO(9) light-cone vertices
- Constructs the diagram with the cutting technics of

[Dixon, Bern, Perelstein, Rozowsky & Green, Kwon, Vanhove]



$$\to \int d^9x\sqrt{G}\mathcal{V}\,\mathcal{R}^4\left(\frac{E_{\frac{3}{2}}(\Omega)}{\mathcal{V}^{\frac{3}{2}}}+\frac{4\pi}{3}\underbrace{(\Lambda\,\ell_P)^3}_{\to\frac{\pi}{2}}\right)$$



$$\to \quad \int d^9x\sqrt{G}\mathcal{V}\,\mathcal{D}^4\mathcal{R}^4\left(\mathcal{V}^{-\frac{5}{2}}\,E_{\frac{5}{2}}(\Omega,\bar{\Omega})+\frac{4}{\pi^2}\zeta(3)\zeta(4)\,V^{-4}\right)$$

Limitations:

- Possible to construct light-cone vertex operators and interaction Hamiltonian [Green, Gutperle, Kwon & Metsaev] but cannot compute $\int C_3\wedge\mathcal{R}^4$ because of the light-cone gauge $C_{+-I}=0$
- Cutting technics of [Dixon et al.] for gravity: very efficient upto 2-loop and 4 external states (All the necessary tree amplitudes are known).
  At 3-loop appear non 2-particule cut-constructible diagrams and not all the diagrams are known.
  At higher-loop the situation is worse, and does not give any way of computation $c(g)$ the number of moduli at g-loop order.

The Superparticle

$$S_{part.} = \int d\tau \left(\dot{X}^M P_M - e P_M P^M + i\dot{\bar{\Theta}}^A (\mathcal{C}\Gamma^M)_{AB} \Theta^B P_M \right)$$

Invariant under diffeomorphisms and κ-symmetry

$$\delta_\kappa \Theta = \Gamma \cdot P\kappa; \quad \delta x = \xi p - i\Theta\gamma\delta_\kappa\Theta; \quad \delta e = \dot{\xi} - 4i\dot{\Theta}\kappa; \quad \delta P = 0$$

[Siegel] added the term $\delta S = i\dot{\Theta}p - i\Psi\Gamma \cdot Pp + p^A \Lambda_{AB} p^B$ to get a superparticle theory with no 2nd class constraints, for which the following quantity is conserved

$$d_A = p_A + 2P_M\,(\mathcal{C}\Gamma^M\theta)_A; \qquad \{d_A, d_B\} = P_M\,(\mathcal{C}\Gamma^M)_{AB}$$

Gauge-fixing the action in the SO(9) semi-light-cone gauge ($a = 1, \cdots, 9$)

$$S_{g.f.} \quad = \quad \int d\tau \left(\dot{X}^M P_M - e\, P_M P^M + \dot{S}_a S_a \right)$$

gives the Pure Spinor action [Anguelova, Grassi, Vanhove & Berkovits, Marchioro]

$$\begin{aligned} S_{g.f.} &= \int d\tau \left(\dot{X}^M P_M - P_M P^M + \dot{\vartheta}^A p_A + \dot{\lambda}^A w_A \right) \\ Q &= \lambda^A\, d_A; \qquad Q^2 = P_M\, \lambda^A (\mathcal{C}\Gamma^M)_{AB} \lambda^B \end{aligned}$$

Notice: The BV-formalism for the superparticle needs an *infinite* towers of ghosts [Kallosh]. Here only a *single constrained ghost* is used. The infinite reducibility of the constraints appears when solving the pure spinor constraints:

$$\text{dof}(\lambda) = 32 - 11 + 32 - \cdots = 23$$

[Berkovits, Nekrasov & Movshev]

Pure Spinor Formalism: for the supermembrane

The κ-symmetry gauge fixed version of the supermembrane [Berkovits]

$$Q_{M2} = \lambda^A(\sigma^1, \sigma^2)\, \hat{d}_A$$

$Q^2_{M2}\phi = 0$ with $\phi = e^I, \theta^A, x^I, d_A$ give

$$\begin{aligned} (\lambda\Gamma^M\lambda) &= 0 \qquad (1) \\ (\lambda\Gamma^{MN}\lambda)\,\Pi_{I\,M} &= 0 \qquad (2) \\ \lambda^A\,\mathcal{C}_{AB}\nabla_I\lambda^B &= 0 \qquad (3) \end{aligned}$$

- For the superparticle (the zero-mode approximation) only (1) is needed.
- (2) and (3) do not reduce the number of components of λ

Pure Spinor Cohomology

The zero-momentum cohomology gives the antifield (BV-)formalism of 11d supergravity [Berkovits]

$$H^{\bullet}(Q|(1)) = \oplus_{p=0}^{7} H^{(p)}(Q|(1))$$

- $H^{(3)}$ contains the (linearized) physical fields for the 11d sugra

- $H^{(4)}$ contains the (linearized) anti-fields for the 11d sugra

- $H^{(p)}$ $p = 0, 1, 2, 5, 6, 7$ contains the (anti-)ghost and (anti-)ghost-for-ghost

We only consider physical states with positive λ-ghost-number n in the Q-cohomology.

$$QU^{(n)} = 0; \qquad \delta_{gauge} U^{(n)} = QU^{(n-1)}$$
$$U^{(n)} = \lambda^{A_1} \cdots \lambda^{A_n} U_{A_1 \cdots A_n}(x, \theta)$$

We consider wave-function of positive ghost number *only*.

<u>Problems:</u> One can find states with negative ghost numbers that trivializes the cohomology [Tonin et al.]

$$Y = \frac{v \cdot \Theta}{v \cdot \lambda} \text{ for any constant spinor } v_A; \qquad \{Q, Y\} = 1$$

What happens at $\lambda^A = 0$ do extra states arise ?

Vertex operators

Vertex operators are physical states in the pure spinor cohomology

We have λ-ghost number 1 and 3 vertex operators

$$
\begin{aligned}
V &= \int d\tau V^{(0)}; \\
V^{(0)} &= P^M(g_{MN}P^N + E_M{}^A d_A + \Omega_{MNR}(\lambda\Gamma^{MN}w)) \\
QV^{(0)} &= \partial_\tau U^{(1)}; \\
U^{(1)} &= \lambda^A\, E_{A\,M}\, P^M \\
U^{(3)} &= \lambda^A\lambda^B\lambda^C\, \Phi_{ABC}(x,\theta) \\
&= \cdots + (\lambda\theta)^3\,(g_{MN} + C_{MNP}) + (\lambda\theta)^3\,\theta\Psi_M + \cdots
\end{aligned}
$$

$V^{(0)}$ is the zero-mode approximation of the supermembrane v.op.

Superparticle Tree Amplitudes

The (zero-momentum) pure spinor cohomology has a single highest state $\dim(H^{(7)}) = 1$ and $H^{(p)} = 0$ for $p > 7$

[Berkovits & Cederwall, Nilsson, Tsimpis]

$$C^*(\theta, \lambda) = c^* \times (\lambda\Gamma^{M_1}\theta) \cdots (\lambda\Gamma^{M_7}\theta)\,(\theta\Gamma_{M_1\cdots M_7}\theta)$$

used to define the bracket [Chesterman & Berkovits]

$$\begin{aligned} 1 &= \langle (\lambda\Gamma^{M_1}\theta) \cdots (\lambda\Gamma^{M_7}\theta)\,(\theta\Gamma_{M_1\cdots M_7}\theta)\rangle \\ &= \int d^{23}\lambda\, d^{32}\theta\, \mathcal{T}_{[A_1\cdots A_{23}](B_1\cdots B_7)} \prod_{i=1}^{7} \partial_{\lambda^{B_i}} \prod_{j=1}^{23} \theta^{A_j} \\ &\times \quad (\lambda\Gamma^{M_1}\theta) \cdots (\lambda\Gamma^{M_7}\theta)\,(\theta\Gamma_{M_1\cdots M_7}\theta) \end{aligned}$$

A scalar measure of integration for the 23 complex components solution of $\lambda\Gamma^M\lambda = 0$

$$d\lambda^{A_1} \wedge \cdots \wedge d\lambda^{A_{23}} = [\mathcal{D}\lambda]_{+16}\, \mathcal{T}_{[A_1\cdots A_{23}](B_1\cdots B_7)} \prod_{i=1}^{7} \lambda^{B_i}$$

Field theory action [Anguelova, Grassi, Vanhove]

$$S_{PFT} = \langle U^{(3)} Q U^{(3)}\rangle + \langle U^{(3)} U^{(1)} U^{(3)}\rangle + \cdots$$

- gives all the CJS action including 3-point and 4-point terms.
- The superparticle perturbation does not need $U^{(4)}$
- Higher-point amplitudes are $\langle U^{(3)} U^{(1)} U^{(3)} \;\left(\int \mathcal{V}^0\right)^n\rangle$

Reduction to 10d : tree-level amplitudes

The Fierz identity in 11d

$$(\lambda\Gamma^{MN}\lambda)\,(\lambda\Gamma_N\lambda) \equiv 0 \begin{cases} (\lambda\Gamma^{11\,n}\lambda)\,(\lambda\Gamma_n\lambda) \equiv 0 \\ (\lambda\Gamma^{mn}\lambda)\,(\lambda\Gamma_n\lambda) + (\lambda\Gamma^{m\,11}\lambda)\,(\lambda\Gamma_{11}\lambda) \equiv 0 \end{cases}$$

Solving $\lambda^A = (\lambda_L^\alpha, \lambda_R^\alpha)$ and $\lambda\Gamma^{11}\lambda \neq 0$ gives

$$\lambda_L\gamma^m\lambda_L = 0 = \lambda_R\gamma^m\lambda_R; \qquad m = 0, \cdots, 9$$

$\lambda_{L,R}$ are 10d pure spinors with 11 complex components parametrizing

$$\mathbb{C}^* \times SO(10)/U(5)$$

- $Q_{string} = Q^{(1,0)} + Q^{(0,1)}$

$$\Rightarrow H^\bullet_{string} = H^\bullet(Q^{(1,0)}|p.s.) \otimes H^\bullet(Q^{(0,1)}|p.s.)$$

- For $\lambda\Gamma^{11}\lambda \neq 0$ the top element in the pure spinor cohomoly used to define the integration measure becomes

$$W_{11}^{(7)}; \qquad QW_{11}^{(7)} = \lambda\Gamma^{11}\lambda\,(W_{5,L}^{(3)}W_{5,R}^{(3)})$$

  ▷ $QU^{(3)} = (\lambda\Gamma^{11}\lambda)\,U^{(1,1)}$

  ▷ The reduction to 10D is given by [Grassi, Vanhove]

$$\langle U^{(3)}QU^{(3)}\rangle \to \langle U^{(1,1)}c_o^- Q_{string}U^{(1,1)}\rangle$$

  ▷ $c_o^- = \lambda\Gamma^{11}\theta$ which imposes the level-matching condition $b_o^- U = 0$ and arises from the PRO

$$Y = (\lambda\Gamma^{11}\theta)\,\delta(\lambda\Gamma^{11}\lambda)$$

Superparticle Loop Amplitudes

The prescription for multiloop superparticle computation in 11d [Grassi, Vanhove]

$$\mathcal{A}_N^g = \int [\mathcal{D}\lambda]_{+16}\, d^{32}\theta \prod_{i=1}^{g} \mathcal{D}N_i\, \mathrm{d}^{32}d_i \prod_{j=1}^{c(g)} \int dt_j\, b_B(t_j) \times$$

$$\times \prod_{k=c(g)+1}^{22\,g} Z_{B_k} \prod_{l=1}^{g} Z_{J_l} \prod_{m=1}^{23} Y_{C_m} \prod_{n=1}^{N} \int d\tau_n V_n^{(0)}(\tau_n)$$

- Picture Raising and Lowering operators

$$Z_B|^{+1} = [Q, \delta(B_{MN}\,(\lambda\Gamma^{MN} w))]$$

$$Y_C|^{-1} = C_A\theta^A\,\delta(C_A\lambda^A)$$

- The b-field is defined as

$$[Q, b_B] = Z_B\,T$$

- The λ-ghost number counting

$$16 - 16g + 23g - c(g) - 23 = 7(g-1) - c(g)$$

- Saturation of the 32 zero modes for the θ^A.
- Saturation of the $32 \times g$ zero modes for the d_A: non vanishing amplitude needs enough d zero modes from b_B

$$\frac{5}{3}\,c(g) + \frac{M}{3} + 2N \geq 9g$$

Supertring multiloop Amplitudes

Reducing to 10d with $\lambda^A = (\lambda^\alpha_L, \lambda^\alpha_R)$

$$
\begin{aligned}
[\mathcal{D}^{23}\lambda]_{+16} &= [\mathcal{D}^{11}\lambda_L]_{+8} \wedge [\mathcal{D}^{11}\lambda_R]_{+8} \wedge [\mathcal{D}\rho_\lambda]_{+0} \\
[\mathcal{D}^{23}w]_{-16} &= [\mathcal{D}^{11}w_L]_{-8} \wedge [\mathcal{D}^{11}w_R]_{-8} \wedge [\mathcal{D}\rho_w]_{+0}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}^g_{N,11d\to 10d} &= \int \mathcal{D}\rho_\lambda \prod_{i=1}^{g} \mathcal{D}(\rho_w)_i \int dX_{11} \prod_{i=1}^{g} d(P_{11})_i \, Z_{11}\, Y_{11} \prod_{i=1}^{g-1} b^i_{11} \\
&\times \left| \int [\mathcal{D}^{11}\lambda]_{+8}[\mathcal{D}^{11g}w]_{+8} \prod_{i=1}^{11g} Z_{B^i} \prod_{j=1}^{11} Y_C \right|^2 \int \mathcal{V} \cdots
\end{aligned}
$$

To be compared with Berkovits' superstring multiloop prescription

$$
\begin{aligned}
\mathcal{A}^g_N = \int & \Big| \mathcal{D}\lambda \, d^{32}\theta \prod_{i=1}^{g} \mathcal{D}N_i \, \mathrm{d}^{32} d_i \\
& \prod_{j=1}^{3(g-1)} \int dz_j \, (\mu | b_{B,L})(z_j) \prod_{k=3(g-1)+1}^{11\,g} Z_{B_k} \prod_{m=1}^{11} Y_{C_m} \Big|^2 \\
& \prod_{n=1}^{N} \int d^2 z_n V^{(0)}_n(z_n, \bar{z}_n)
\end{aligned}
$$

The extra pieces should corresponds to non-perturbative effects from D0-brane

Topological M-theory

The tree-level action is defined w.r.t. to the integration measure

$$1 = \int d\mu_n^{(p)}\, (\lambda\gamma^{m_1}\theta)\cdots(\lambda\gamma^{m_n}\theta)\, (\theta\gamma_{m_1\cdots m_n}\theta) = \int d\mu_n^{(p)}\, W_n^{(p)}$$

with $(n,p)=(5,3)$ in 10d and $(n,p)=(9,7)$ in 11d

We specify specific boundary conditions using [Grassi,Vanhove]

$$\delta_\theta^{(n)} = (\theta\gamma_{m_1\cdots m_n}\theta)\, \delta^{(n)}(y)\, dx^{m_1}\wedge\cdots\wedge dx^{m_n}$$

- $\delta_\theta^{(4)}$ in 11d tree-level action $\Rightarrow$ 7d Hamiltonian of [Shatashvili, Gerasimov]

$$\langle \delta_\theta^{(4)}\, U^{(3)} Q U^{(3)}\rangle + \langle \delta_\theta^4 U^{(3)} U^{(1)} U^{(3)}\rangle + \cdots = \int_{M_7} (CdC + \Psi\Gamma_{[5]}\Psi)$$

- $\delta_\theta^{(7)}$ in 10d open open-string field theory action $\Rightarrow$ 3d (super-)Chern-Simons

$$\langle \delta_\theta^{(7)}\, U^{(1)} Q_o U^{(1)}\rangle + \cdots \to \int d^3x\, (AdA + A^3)$$

- $\delta_\theta^{(7)} \times W_{5,R}$ un the closed string-field theory action $\Rightarrow$ the holomorphic Chern-Simons

$$W_{5,R} = (\lambda_R\gamma^m\theta_R)(\lambda_R\gamma^n\theta_R)(\lambda_R\gamma^p\theta_R)\, (\theta_R\gamma_{mnp}\theta_R)$$

$$\langle \underbrace{\delta_\theta^{(7)} W_{5,R}}_{\text{chiral measure}}\, U^{(1,0)} Q U^{(1,0)}\rangle + \cdots \to \int d^6x\, \bar\Omega \wedge (A\partial A + A^3)$$

Outlook / Conclusion

We presented a pure spinor formalism for computing multiloop amplitude for the superparticle in 11d.

Because the 11d generalized pure spinor satisfy

$$32\,\lambda_A\lambda_B = \frac{1}{2!}\,(\lambda\Gamma_{MN}\lambda)\,(\Gamma^{MN})_{AB} + \frac{1}{5!}\,(\lambda\Gamma_{M_1\cdots M_5}\lambda)\,(\Gamma^{M_1\cdots M_5})_{AB}$$

The pure spinor cohomology contains all the informations about the massless field of 11d.

Futur direction

- Definition of the sum over the various field theory diagrams (moduli space)
- Regularisation of the diagrams
- Unicity of the R^4 invariant : Unique answer in [Anguelova, Grassi, Vanhove].
- Topological M-theory and Hitchin's fonctional
  Exactness of the 7d action ? Membrane corrections ? Quantum corrections to Hitchin's functional?