Elective course unit in M.Sc. Computational Mechanics (ISE)

Effective Properties of Micro-Heterogeneous Materials

Project Description (Group 5)

Effective properties of a matrix with ellipsoid inclusion

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Contents

1. Abstract:

Analytical approximations are used to investigate the effective properties of the Micro heterogeneous material. In this paper homogenization methods are implemented on two phase material with a cylindrical inclusion with an ellipsoidal cross-section of infinite length embedded in a matrix material. Bounds for coefficient of elasticity tensor are calculated using classical homogenization methods like Voigt, Reuss, Dilute Distribution and Differential Scheme.

2. Fundamental Concepts with Respect to Homogenization

2.1 Introduction:

Heterogeneous materials are found in the nature as well as manmade Materials. Even though the macrostructure appears to be homogeneous the microstructure is inhomogeneous with defects in it. Defects are characterized as cracks, voids and inclusions. The effective properties of macroscopic materials are related to microscopic behavior exemplified by the interaction between constituents in the material. Microstructure of material under consideration is defined as the representative volume elements.

All microstructure shows inhomogeneous behavior even if they appear homogeneous in macrostructure. These in-homogeneities may exist in the form of Cracks, Voids or Inclusions in the crystal lattice. These in-homogeneities are usually referred as defects. Behavior of these defects impacts the macroscopic properties of the material.

2.2 Eshelby Result:

Eshelby(1957) developed a formulation to determine elasticity solution with single inclusion embedded in infinite matrix of material with uniform exterior loading. This result is basis of various approximation methods to obtain effective properties. Consider a single linearly elastic particle, at infinitesimal strains, embedded in an infinite matrix: If the shape of the particle is ellipsoidal, under uniform far field stress or strain, then the stress and strain in the particle are constant. Ellipsoidal shapes are qualitatively useful since the geometry can mimic a variety of microstructures.

For a particle in three dimensions Eigen strains in the inclusions are constant

$$\varepsilon_{kl}^{t} = \text{Constant,}$$

Then Eshelby result holds true that the stains inside the inclusion Ω are also constant. Fourth order Eshelby tensor is represented as $\mathbb{S}_{ijkl}$ and depends only on the Eigen strains.

$$\varepsilon = \mathbb{S}_{ijkl} * \varepsilon_{kl}^{t} = \text{Const in } \Omega$$

Similarly stresses are also constant inside the inclusion.

$$\sigma_{ij} = C_{ijmn}(\mathbb{S}_{mnkl} - I_{mnkl}) * \ \varepsilon^{t}{}_{kl} = \text{const in } \Omega$$

Where

$$I_{mnkl} = \frac{1}{2} * (\delta_{mk}\delta_{nl} + \delta_{ml}\delta_{nk})$$

is a fourth order identity tensor. The Eshelby tensor is symmetric in the first and second pair of indices, but in general it is not symmetric with regard to an exchange of these pairs:

$$\mathbb{S}_{ijkl} = \mathbb{S}_{jikl} = \mathbb{S}_{ijlk}, \ \mathbb{S}_{ijkl} \neq \mathbb{S}_{klij}$$

Material properties of the isotropic materials depend on the poison's ratio, Ratios of Principle axis and orientation of principle axis with respect to the Cartesian coordinate system.

Outside the inclusion Ω the stresses and strains are not constant; with increasing distance r from the inclusion they asymptotically decay according to ε_{ij} , $\sigma_{ij} \approx r^{-3}$ for $r \to \infty$, as in case of a center of dilatation. Starting from the general ellipsoid various special cases can be derived. For instance, the two-dimensional solution for an infinitely long cylinder of elliptic cross section in plane strain is obtained from the limit process $a_3 \to \infty$

In elliptical inclusion $a_3 \to \infty$

$$\mathbb{S}_{1111} = \frac{1}{2*(1-\nu)}\left[\frac{{a_2}^2 + 2{a_1}^2{a_2}^2}{(a_1+a_2)^2} + (1-2\nu)*\frac{a_2}{a_1+a_2}\right]$$

$$\mathbb{S}_{2222} = \frac{1}{2*(1-\nu)}\left[\frac{{a_1}^2 + 2{a_1}^2{a_2}^2}{(a_1+a_2)^2} + (1-2\nu)*\frac{a_1}{a_1+a_2}\right]$$

$$\mathbb{S}_{1122} = \frac{1}{2*(1-\nu)}\left[\frac{{a_2}^2}{(a_1+a_2)^2} - (1-2\nu)*\frac{a_2}{a_1+a_2}\right]$$

$$\mathbb{S}_{2211} = \frac{1}{2*(1-\nu)}\left[\frac{{a_1}^2}{(a_1+a_2)^2} - (1-2\nu)*\frac{a_1}{a_1+a_2}\right]$$

$$\mathbb{S}_{1212} = \frac{1}{2*(1-\nu)}\left[\frac{{a_1}^2 + {a_2}^2}{(a_1+a_2)^2} - \frac{(1-2\nu)}{2}\right]$$

$$\mathbb{S}_{1133} = \frac{\nu}{2*(1-\nu)}\left[\frac{2a_2}{a_1+a_2}\right]$$

$$\mathbb{S}_{2233} = \frac{\nu}{2*(1-\nu)}\left[\frac{2a_1}{a_1+a_2}\right]$$

$$\mathbb{S}_{1313} = \left[\frac{a_2}{2(a_1+a_2)}\right]$$

$$\mathbb{S}_{2323} = \left[\frac{a_1}{2(a_1+a_2)}\right]$$

$$\mathbb{S}_{3333} = 0$$

2.3 Isotropic material properties:

Effective Elastic properties of heterogeneous materials depend on the geometrical nature of mixture of microstructures and on the volume fraction of both inclusion and matrix. Each separate homogeneous region is characterized by its stiffness tensor, which describes properties of material. Stiffness tensor is denoted as $\mathbb{C}$ and it can be written in terms of bulk and shear constants given by:

$$\mathbb{C}^{s}_{ijkl} = \kappa_s \delta_{ij}\delta_{kl} + 2\mu_s(\delta_{ik}\delta_{jl} - \frac{1}{3}\delta_{ij}\delta_{kl}) \text{ , s = 1, 2}$$

For isotropic materials 4th order stiffness tensor can be represented as in matrix form

$$\mathbb{C}_{ijkl} = \begin{bmatrix} \kappa_s+4\mu_s/3 & \kappa_s-2\mu_s/3 & \kappa_s-2\mu_s/3 & 0 & 0 & 0 \\ \kappa_s-2\mu_s/3 & \kappa_s+4\mu_s/3 & \kappa_s-2\mu_s/3 & 0 & 0 & 0 \\ \kappa_s-2\mu_s/3 & \kappa_s-2\mu_s/3 & \kappa_s+4\mu_s/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_s & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_s & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu_s \end{bmatrix}$$

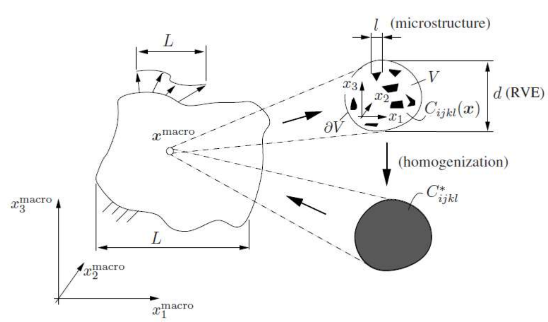
We set

$$\kappa_s = \frac{\mathrm{E}_s}{3(1-2\nu_s)}$$

$$\mu_s = \frac{\mathrm{E}_s}{2(1+2\nu_s)}$$

2.4 Representative volume Element:

Representative volume Element is a model of a sample of material used to determine corresponding macroscopic effective properties of the material. The effective properties are given as relations between the strains and stresses of the sample. These strains and stresses are measured from the sample's surface displacements and tractions, respectively. It is easily shown that these strains and stresses are actually the volume average of the corresponding microscopic variables within the samples.



2.5 Averaging:

A material point on the macroscopic level is related to volume V where as in microscopic level is related to fluctuating microscopic fields. The macroscopic stresses and strains can be characterized by volumetric average of the microscopic stresses and strains.

$$\langle \sigma_{ij} \rangle = \frac{1}{V}\int_V \sigma_{ij}(\mathrm{x})dV \ , \qquad \langle \varepsilon_{ij} \rangle = \frac{1}{V}\int_V \varepsilon_{ij}(\mathrm{x})dV$$

Average strain theorem:

$$\begin{aligned}
\langle \varepsilon \rangle_{\Omega} &= \frac{1}{2|\Omega|}\int_{\Omega}(\nabla \mathrm{u}+(\nabla \mathrm{u})^T)\,\mathrm{d}\,\Omega \\
&= \frac{1}{2|\Omega|}\left\{\int_{\Omega_1}(\nabla \mathrm{u}+(\nabla \mathrm{u})^T)\,\mathrm{d}\,\Omega+\int_{\Omega_2}(\nabla \mathrm{u}+(\nabla \mathrm{u})^T)\,\mathrm{d}\,\Omega\right\} \\
&= \frac{1}{2|\Omega|}\left\{\int_{\partial\Omega_1}(\mathrm{u}\otimes \mathrm{n}+n\otimes u)\,\mathrm{dA}+\int_{\Omega_2}(\mathrm{u}\otimes \mathrm{n}+n\otimes u)\,\mathrm{dA}\right\} \\
&= \frac{1}{2|\Omega|}\left\{\int_{\partial\Omega}((\varepsilon.\mathrm{x})\otimes \mathrm{n}+n\otimes(\varepsilon.\mathrm{x}))\,\mathrm{dA}+\int_{\partial\Omega_1\cap\partial\Omega_2}(\|\,\mathrm{u}\,\|\otimes \mathrm{n}+n\otimes\|\,u\,\|)\,\mathrm{dA}\right\} \\
&= \frac{1}{2|\Omega|}\left\{\int_{\partial\Omega}(\nabla(\varepsilon.\mathrm{x})+\nabla(\varepsilon.\mathrm{x})^T)\,\mathrm{d}\,\Omega+\int_{\partial\Omega_1\cap\partial\Omega_2}(\|\,\mathrm{u}\,\|\otimes \mathrm{n}+n\otimes\|\,u\,\|)\,\mathrm{dA}\right\} \\
&= \varepsilon+\frac{1}{2|\Omega|}\int_{\partial\Omega_1\cap\partial\Omega_2}(\|\,\mathrm{u}\,\|\otimes \mathrm{n}+n\otimes\|\,u\,\|)\,\mathrm{dA}
\end{aligned}$$

Average Stress Theorem:

$$\begin{aligned}
\langle \sigma \rangle_{\Omega} &= \frac{1}{|\Omega|}\int_{\Omega}(\nabla.(\sigma\otimes \mathrm{x})\,\mathrm{d}\,\Omega+\frac{1}{|\Omega|}\int_{\Omega}(\mathrm{f}\otimes \mathrm{x})\,\mathrm{d}\,\Omega \\
&= \frac{1}{|\Omega|}\int_{\partial\Omega}(\sigma\otimes \mathrm{x}).n\,\mathrm{dA}+\frac{1}{|\Omega|}\int_{\Omega}(\mathrm{f}\otimes \mathrm{x})\,\mathrm{d}\,\Omega \\
&= \sigma+\frac{1}{|\Omega|}\int_{\Omega}(\mathrm{f}\otimes \mathrm{x})\,\mathrm{d}\,\Omega
\end{aligned}$$

3. Analytical Approximations for Linear Elastic Problems

3.1 Voigt and Reuss Approximation:

In heterogeneous materials one of the micro-fields are assumed to be constant to apply as a boundary condition to apply on the micro heterogeneous material.

W. VOIGT (1889) derived an approximation to find the effective properties of the heterogeneous materials assuming strain as constant boundary condition.

From the boundary condition's of Voigt the strain is takes as constant in the volume V of matrix and influence tensor L = 1. From Average strain boundary condition the effective elasticity tensor is approximated by the average stiffness:

$$\mathbb{C}^*_{(\mathrm{Voigt})} = \langle \mathbb{C}:\mathrm{L}^{\varepsilon} \rangle = \sum_{\alpha=1}^{n} \mathrm{c}_{\alpha} \mathbb{C}_{\alpha}$$

$$\kappa^*_{(\mathrm{Voigt})} = \sum_{\alpha=1}^{n} \mathrm{c}_{\alpha} \kappa_{\alpha} \text{ and } \mu^*_{(\mathrm{Voigt})} = \sum_{\alpha=1}^{n} \mathrm{c}_{\alpha} \mu_{\alpha}$$

Analogously A. Reuss(1929) derived an analytical approximation assuming stress as constant boundary condition to obtain the effective properties of the heterogeneous material.

$$\mathbb{C}^*_{(\mathrm{Reuss})} = \langle \mathbb{C}^{-1}:\mathrm{L}^{\sigma} \rangle^{-1} = \langle \mathbb{C}^{-1} \rangle^{-1} = \left(\sum_{\alpha=1}^{n} \mathrm{c}_{\alpha} \mathbb{C}_{\alpha}^{-1} \right)^{-1}$$

$$\kappa^*_{(\mathrm{Reuss})} = \left(\sum_{\alpha=1}^{n} \frac{\mathrm{c}_{\alpha}}{\kappa_{\alpha}} \right)^{-1} \text{ and } \mu^*_{(\mathrm{Reuss})} = \left(\sum_{\alpha=1}^{n} \frac{\mathrm{c}_{\alpha}}{\mu_{\alpha}} \right)^{-1}$$

Voigt and Reuss method approximate the macroscopic material behavior to the isotropic even though physically the material is an anisotropic from the geometrical arrangement of the phases.

The approximation of the effective properties by Voigt and Reuss are exact only in one dimensional case. Voigt applies only when the materials are in parallel and Reuss when materials are in series.

- Assumption of constant strains leads to the violation of the equilibrium
- Assumption of the constant stress leads to the violation of compatibility of deformation

Despite of the deficiencies due to assumptions in by Voigt and Reuss define the exact Yield bounds for true effective elastic constants of heterogeneous materials

3.2 Dilute Distribution approach:

Dilute Distribution is a simplest solution to obtain effective properties of micro heterogeneous materials which satisfy local equilibrium and compatibility of deformation. Major back drop of this approach is assuming no interaction between the particles. Hence we consider material matrix with single inclusion.

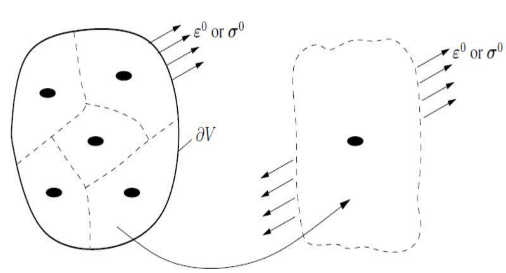


Fig: Dilute distribution approach

Figure describes the consideration of single inclusion in the material matrix discarding interactions between multiple inclusions subjected to uniform loading. Therefore characteristic inclusions considered have to be very small compared to the distance from the boundary of the RVE.

Localization tensor is given by:

$$L_I^{\varepsilon} = A_I^{\infty} = \left[I + \mathbb{S}_M : \mathbb{C}_M^{-1} : (\mathbb{C}_I - \mathbb{C}_M) \right]^{-1}$$

Effective elastic module is given by: $\mathbb{C}^* = \mathbb{C}_M + c_I(\mathbb{C}_I - \mathbb{C}_M) : \mathrm{L}_I^{\varepsilon}$

Effective Elasticity tensor by dilute distribution

$$\mathbb{C}^*_{(\mathrm{DD})} = \mathbb{C}_M + c_I(\mathbb{C}_I - \mathbb{C}_M) : \left[I + \mathbb{S}_M : \mathbb{C}_M^{-1} : (\mathbb{C}_I - \mathbb{C}_M) \right]^{-1}$$

3.3 Differential Scheme:

Differential Scheme homogenization is carried out on the embedding the entire volume fraction of each phase in infinitesimal steps. This process can be associated with the actual manufacturing of the heterogeneous materials by incorporating defects in homogeneous materials as a step wise process. Effective properties obtained using the Differential Scheme is exact

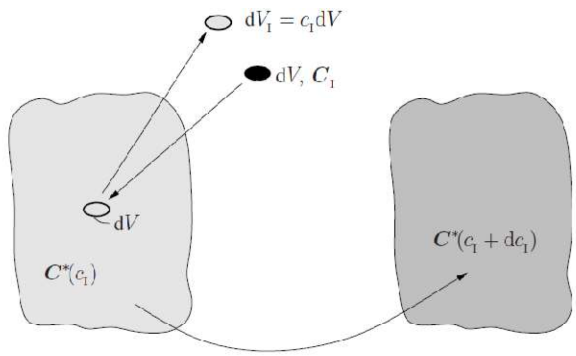


Fig: Differential Scheme

As described from figure, an infinitesimal volume dV of the inclusion having $\mathbb{C}_I$ as mechanical properties is embedded in the matrix having effective properties $\mathbb{C}_M$ of the matrix depends on volume fraction of the inclusion c^i and hence should be computed at each step. Starting from the balance of volume

$$(\mathrm{c}_I^n + \mathrm{dc}_I^{n+1})\,\mathrm{V} = \mathrm{c}_I^n V - c_I^n dV + dV$$

$$\frac{dV}{V} = \frac{dc_I}{1 - c_I}$$

Effective elasticity tensor of differential scheme approach can be derived and $\mathbb{C}_M$ is replaced by $\mathbb{C}^*(\mathrm{c}_I^n)$.

The localization tensor is given by,

$$L_{(\mathrm{DS})} = [\mathrm{I} + \mathbb{S}_M : \mathbb{C}_M^{-1} : (\mathbb{C}_I - \mathbb{C}_M)]^{-1}$$

We set

$$\mathbb{C}^*(c_I^{n+1}) = \mathbb{C}^*(c_I^n) + (\mathrm{d}\,\mathbb{C}^*)^{n+1}$$

We obtain the relation for effective modulus

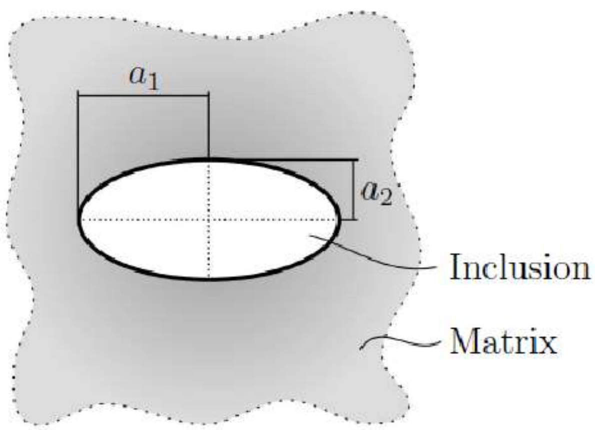
$$\mathbb{C}^*(c_I^n) + d(\mathbb{C}^*)^{n+1} = \mathbb{C}^*(c_I^n) + \frac{dV}{V}[\mathbb{C}_I - \mathbb{C}^*(c_I^n)] : \mathrm{L}_{(\mathrm{DS})}$$

And finally we get $\dfrac{d(\mathbb{C}^*)^{n+1}}{d(c_I)^{n+1}} = \dfrac{[\mathbb{C}_I - \mathbb{C}^*(c_I^n)] : \mathrm{L}_{(\mathrm{DS})}}{1 - c_I^n}$

This leads to a system of nonlinear ordinary differential equations and solved by using suitable numerical methods for initial value problem. Initial condition can be set as

$$\mathbb{C}^*(c_I^{n=0} = 0) = \mathbb{C}_M$$

4. Problem description and Results:



Here, we consider a micro heterogeneous material with ellipsoidal inclusion in infinite material matrix to obtain effective properties of material using various homogenization methods as explained above. Agenda of the task is to compare changes in Elasticity modulus with change in volume fraction of inclusion and compare between Homogenization methods. The parameters the elastic materials are given by.

Matrix (M): E_M = 100MPa; ϑ_M = 0.3

Inclusion (I): E_I = 100,000MPa; ϑ_I = 0.2

The ellipsoidal cross-section is described by the semi-axes a_1 and a_2, whereby the relation between these semi-axes is described by the parameter b, i.e. $a_2 = b * a_1$. In the following

b = 0.4

Elasticity modulus of Ellipsoidal inclusion

$$\mathbb{C}_I = 10^5 \begin{bmatrix} 1.1111 & 0.2778 & 0.2778 & 0 & 0 & 0 \\ 0.2778 & 1.1111 & 0.2778 & 0 & 0 & 0 \\ 0.2778 & 0.2778 & 1.1111 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.4167 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.4167 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.4167 \end{bmatrix}$$

Elasticity modulus of Matrix modulus

$$\mathbb{C}_M = \begin{bmatrix} 134.6154 & 57.6923 & 57.6923 & 0 & 0 & 0 \\ 57.6923 & 134.6154 & 57.6923 & 0 & 0 & 0 \\ 57.6923 & 57.6923 & 134.6154 & 0 & 0 & 0 \\ 0 & 0 & 0 & 38.4615 & 0 & 0 \\ 0 & 0 & 0 & 0 & 36.4615 & 0 \\ 0 & 0 & 0 & 0 & 0 & 36.4615 \end{bmatrix}$$

4.1 Coefficients of Eshelby Tensor

Eshelby tensor of inclusion:

$$\mathbb{S}_I = \begin{bmatrix} 0.4133 & -0.0561 & 0.0714 & 0 & 0 & 0 \\ 0.0510 & 0.5740 & 0.1786 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.3724 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.3571 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.1429 \end{bmatrix}$$

Eshelby tensor of Matrix Material:

$$\mathbb{S}_M = \begin{bmatrix} 0.4315 & -0.0233 & 0.1224 & 0 & 0 & 0 \\ 0.1603 & 0.5539 & 0.3061 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.3524 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.3571 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.1429 \end{bmatrix}$$

4.2 Voigt and Reuss Bounds

Voulme Fraction of Inclusion	Bounds			
	Voigt		Reuss	
	Kappa_Voigt (MPa)	μ_Voigt (MPa)	Kappa_Reuss (MPa)	μ_Reuss (MPa)
0.2	11177.7778	8364.1026	104.1276	48.0658
0.4	22272.2222	16689.7436	138.7501	64.0631
0.6	33366.6667	25015.3846	207.8656	96.0209

4.3 Effective Elasticity Modulus

4.3.1 Effective elasticity tensor with volume fraction of inclusion 0.2

$\mathbb{C}$ Voigt:

$$10^4\begin{bmatrix} 2.2330 & 0.5602 & 0.5602 & 0 & 0 & 0 \\ 0.5602 & 2.2330 & 0.5602 & 0 & 0 & 0 \\ 0.5602 & 0.5602 & 2.2330 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.8364 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.8364 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.8364 \end{bmatrix}$$

$\mathbb{C}$ Reuss:

$$\begin{bmatrix} 168.2154 & 72.0837 & 72.0837 & 0 & 0 & 0 \\ 72.0837 & 168.2154 & 72.0837 & 0 & 0 & 0 \\ 72.0837 & 72.0837 & 168.2154 & 0 & 0 & 0 \\ 0 & 0 & 0 & 168.2154 & 0 & 0 \\ 0 & 0 & 0 & 0 & 168.2154 & 0 \\ 0 & 0 & 0 & 0 & 0 & 168.2154 \end{bmatrix}$$

$\mathbb{C}$ Dilute Distribution:

$$10^4\begin{bmatrix} 0.0180 & 0.0057 & 0.0158 & 0 & 0 & 0 \\ 0.0057 & 0.0169 & 0.0270 & 0 & 0 & 0 \\ 0.0047 & 0.0049 & 2.1337 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0060 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.0060 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.0092 \end{bmatrix}$$

$\mathbb{C}$ Differential Scheme:

$$10^3\begin{bmatrix} 0.0185 & 0.0057 & 0.0045 & 0 & 0 & 0 \\ 0.0057 & 0.0173 & 0.0048 & 0 & 0 & 0 \\ 0.0169 & 0.0295 & 2.1439 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0063 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.0062 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.0098 \end{bmatrix}$$

4.3.2 Effective elasticity tensor with volume fraction of inclusion 0.4

$\mathbb{C}$ Voigt: $$10^4\begin{bmatrix} 4.4525 & 1.1146 & 1.1146 & 0 & 0 & 0 \\ 1.1146 & 4.4525 & 1.1146 & 0 & 0 & 0 \\ 1.1146 & 1.1146 & 4.4525 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.6690 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.6690 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.6690 \end{bmatrix}$$

$\mathbb{C}$ Reuss: $$\begin{bmatrix} 224.1677 & 96.0414 & 96.0414 & 0 & 0 & 0 \\ 96.0414 & 224.1677 & 96.0414 & 0 & 0 & 0 \\ 96.0414 & 96.0414 & 224.1677 & 0 & 0 & 0 \\ 0 & 0 & 0 & 64.0631 & 0 & 0 \\ 0 & 0 & 0 & 0 & 64.0631 & 0 \\ 0 & 0 & 0 & 0 & 0 & 64.0631 \end{bmatrix}$$

$\mathbb{C}$ Dilute Distribution: $$10^4\begin{bmatrix} 0.0225 & 0.0057 & 0.0258 & 0 & 0 & 0 \\ 0.0057 & 0.0204 & 0.0483 & 0 & 0 & 0 \\ 0.0035 & 0.0041 & 2.7906 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0082 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.0082 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.0146 \end{bmatrix}$$

$\mathbb{C}$ Differential Scheme: $$10^3\begin{bmatrix} 0.0251 & 0.0057 & 0.0029 & 0 & 0 & 0 \\ 0.0057 & 0.0223 & 0.0036 & 0 & 0 & 0 \\ 0.0314 & 0.0601 & 4.2986 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0094 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.0093 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.0176 \end{bmatrix}$$

4.3.3 Effective elasticity tensor with volume fraction of inclusion 0.6

$\mathbb{C}$ Voigt:

$$10^4\begin{bmatrix} 6.6721 & 1.6690 & 1.6690 & 0 & 0 & 0 \\ 1.6690 & 6.6721 & 1.6690 & 0 & 0 & 0 \\ 1.6690 & 1.6690 & 6.6721 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.5015 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2.5015 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2.5015 \end{bmatrix}$$

$\mathbb{C}$ Reuss:

$$\begin{bmatrix} 335.8935 & 143.8517 & 143.8517 & 0 & 0 & 0 \\ 143.8517 & 335.8935 & 143.8517 & 0 & 0 & 0 \\ 143.8517 & 143.8517 & 335.8935 & 0 & 0 & 0 \\ 0 & 0 & 0 & 96.0209 & 0 & 0 \\ 0 & 0 & 0 & 0 & 96.0209 & 0 \\ 0 & 0 & 0 & 0 & 0 & 96.0209 \end{bmatrix}$$

$\mathbb{C}$ Dilute Distribution:

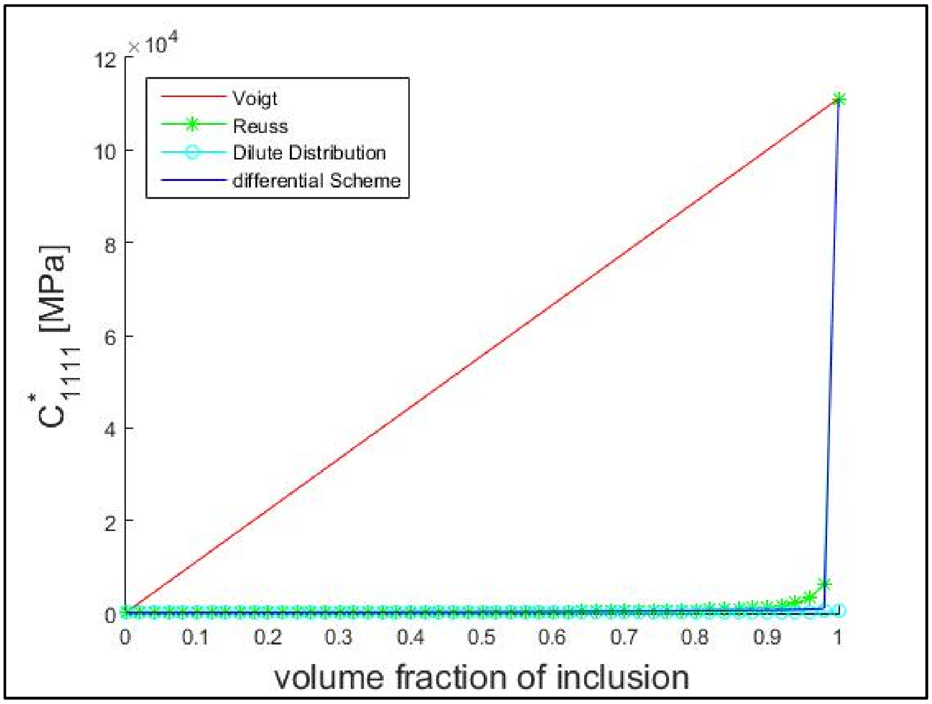
$$10^4\begin{bmatrix} 0.0271 & 0.0057 & 0.0358 & 0 & 0 & 0 \\ 0.0057 & 0.0239 & 0.0695 & 0 & 0 & 0 \\ 0.0024 & 0.0032 & 6.3742 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0104 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.0103 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.0200 \end{bmatrix}$$

$\mathbb{C}$ Differential Scheme:

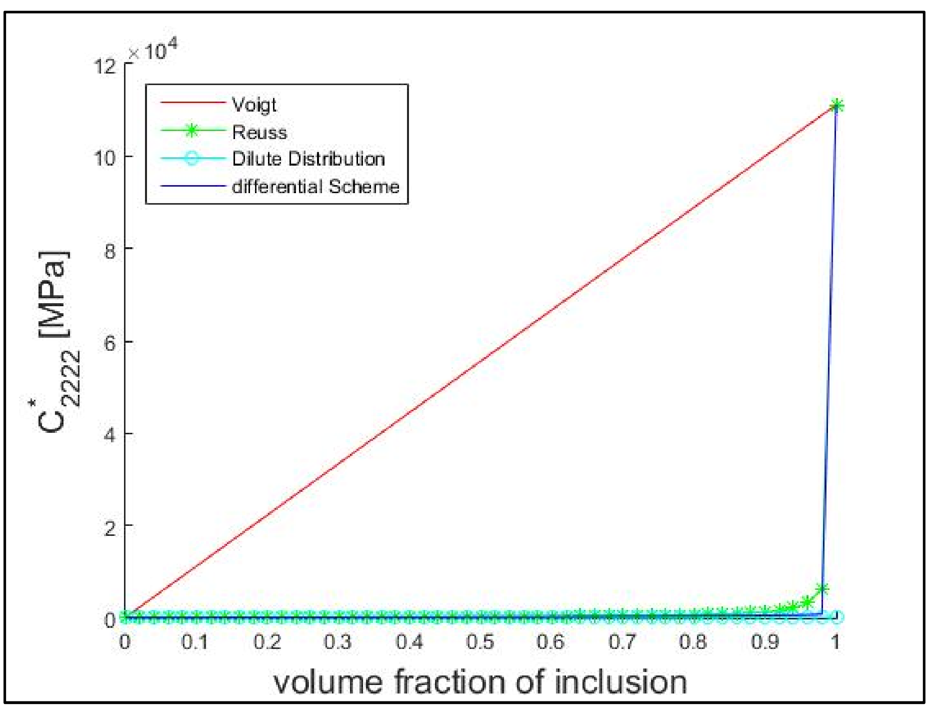
$$10^3\begin{bmatrix} 0.0251 & 0.0057 & 0.0007 & 0 & 0 & 0 \\ 0.0057 & 0.0223 & 0.0020 & 0 & 0 & 0 \\ 0.0517 & 0.1032 & 6.4863 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0138 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.0137 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.0284 \end{bmatrix}$$

4.4 Relation between Effective elasticity Modulus and Volume Fraction of inclusion

Graphs are plotted between $\mathbb{C}^*_{1111}$ and Volume fraction of inclusion



Graphs are plotted between $\mathbb{C}^*_{2222}$ and Volume fraction of inclusion



5. Appendix:

Execution Source Code to obtain results of Bounds for Voigt and Reuss and Effective elasticity tensor for Voigt, Reuss, Dilute Distribution and Differential Scheme or Plot the relations between the coefficients $\mathbb{C}^*_{1111}$ and $\mathbb{C}^*_{2222}$ of the effective elasticity tensor and the inclusion's volume fraction c_I.

In Line 22 if Code = 1 output obtained: Results of Bounds for Voigt and Reuss and Effective elasticity tensor for Voigt, Reuss, Dilute Distribution and Differential Scheme

In Line 22 if Code = N {N = Natural Numbers(2,3,4……)} output obtained: Plot the relations between the coefficients $\mathbb{C}^*_{1111}$ and $\mathbb{C}^*_{2222}$ of the effective elasticity tensor and the inclusion's volume fraction c_I.

Project_5(Matlab *.m File):

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%
%Project-5
%Effective properties of a matrix with ellipsoid inclusion
%c_i -- volume fraction of inclusion
%c_m -- volume fraction of matrix
%E_m -- youngs moudulus of matrix
%E_i -- youngs moudulus of inlusion
%ny_m -- poissons ration of matrix
%ny_i -- poissons ration of inclusion
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%
clc
clear
tol = 1e-5;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%
% Here you can select to display numerical resultd for homogenization
% techniques or plot graphs between them.
% code = 1 ..... display results of Voigt, Reuss, Dilute Distribution and
%              Differetial Scheme
% code = N ..... N = any Natural number N > 1 to plot graphs of Homogenization
%              results
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%
code= 1;

% material properties
c_i = 0.2;  % volume fraction  of inclusion cant take '0' and '1' in differential Scheme
c_m = 1-c_i;
E_m = 100;
ny_m = 0.3;
E_i = 100000;
ny_i = 0.2;

% kapa for the matrix
Kappa_m = E_m / ( 3 * ( 1 - 2 * ny_m));

% kapa for the inclusino
```

```
Kappa_i = E_i / ( 3 * ( 1 - 2 * ny_i));

% miu for the matrix
mu_m = E_m /  ( 2 * ( 1 + ny_m));

% miu for the inclusion
mu_i = E_i /  ( 2 * ( 1 + ny_i));

%Eshelby tensor matrix and inclusions
S_i = Eshelby(ny_i);
S_m = Eshelby(ny_m);

disp('Eshelby tensor of Matrix');
disp(S_m);
disp('Eshelby tensor of inclusion');
disp(S_i);

C_i = mat_prop( Kappa_i, mu_i);
disp('Elasticity Modulous of Ellipsoidal Inclusion');
disp(C_i);
C_m = mat_prop( Kappa_m, mu_m);
disp('Elasticity Modulous of Matrix Material');
disp(C_m);

% Effective Elasticity Tensor Voigt

if 1
    c_v_1 = zeros();
    c_v_2 = zeros();
    switch code
        case 1
            for l = 1:length(c_i)
                %voigt Bounds
                kappa_Voigt  = c_i * Kappa_i  + ( 1 - c_i) * Kappa_m;
                mu_Voigt = c_i * mu_i + ( 1 - c_i) * mu_m;
                disp(sprintf('Voigt Bounds: Volume fraction inclusion = %8.4f',c_i));
                disp(sprintf('Kappa Voigt.................................. %8.4f',kappa_Voigt));
                disp(sprintf('Mu Voigt..................................... %8.4f',mu_Voigt));
                %effextive Elasticity modulous
                C_Eff_V = c_i(l)*C_i + (1-c_i(l))*C_m;
                disp(sprintf('Effective modulous Voigt: Volume fraction inclusion = %8.4f',c_i));
                disp(C_Eff_V);
            end
        otherwise
            for l = 1:length(c_i)
                C_Eff_V = c_i(l)*C_i + (1-c_i(l))*C_m;
                c_v_1(l) = C_Eff_V(1,1);
                c_v_2(l) = C_Eff_V(2,2);
            end
            disp('plot');
            h_C_Eff_1 = figure;
            xlabel('volume fraction of inclusion', 'Fontsize', 16)
            ylabel('C^*_1_1_1_1 [MPa]', 'Fontsize', 16)
            h_C_Eff_2 = figure;
```

```
            xlabel('volume fraction of inclusion', 'Fontsize', 16)
            ylabel('C^*_2_2_2_2 [MPa]', 'Fontsize', 16)
            set( 0, 'currentfigure', h_C_Eff_1);
            hold on;
            h_C_Eff_V_1 = plot( c_i, c_v_1, '-r');
            legend( 'Voigt', 'Location', 'NorthWest');
            hold off;
            set( 0, 'currentfigure', h_C_Eff_2);
            hold on;
            h_C_Eff_V_2 = plot( c_i, c_v_2, '-r');
            legend( 'Voigt','Location', 'NorthWest');
            hold off;
    end
end
        % Effective Elasticity Tensor Reuss
if 1
    c_r_1 = zeros();
    c_r_2 = zeros();
    switch code
        case 1
            for l = 1:length(c_i)
                %Reuss Bounds
                kappa_Reuss  = c_i/Kappa_i  + ( 1 - c_i)/Kappa_m;
                kappa_Reuss  = 1./kappa_Reuss;

                mu_Reuss = c_i/mu_i + ( 1 - c_i)/mu_m;
                mu_Reuss = 1./mu_Reuss;

                disp(sprintf('Reuss Bounds: Volume fraction inclusion = %8.4f',c_i));
                disp(sprintf('Kappa Reuss................................. %8.4f',kappa_Reuss));
                disp(sprintf('Mu Reuss.................................... %8.4f',mu_Reuss));
                %Effective Elasticity Modulous Reuss
                C_Eff_R = inv(inv(C_i)*c_i(l) + inv(C_m)*(1-c_i(l)));
                disp(sprintf('Effective modulous Reuss: Volume fraction inclusion = %8.4f',c_i));
                disp(C_Eff_R);
            end

        otherwise
            for l = 1:length(c_i)
                C_Eff_R = inv(inv(C_i)*c_i(l) + inv(C_m)*(1-c_i(l)));
                c_r_1(l) = C_Eff_R(1,1);
                c_r_2(l) = C_Eff_R(2,2);
            end
            set( 0, 'currentfigure', h_C_Eff_1);
            hold on;
            h_C_Eff_R_1 = plot( c_i, c_r_1, '*-g');
            legend( 'Voigt', 'Reuss','Location', 'NorthWest');
            hold off;
            set( 0, 'currentfigure', h_C_Eff_2);
            hold on;
            h_C_Eff_R_2 = plot( c_i, c_r_2, '*-g');
            legend( 'Voigt', 'Reuss','Location', 'NorthWest');
            hold off;
    end
```

```
end

    % Effective Elasticity Tensor Dilute Distribution
if 1
  c_dd_1 = zeros();
  c_dd_2 = zeros();
  S_m = Eshelby(ny_m);
  switch code
    case 1
      for i = 1:length(c_i)
        L = inv(eye() + S_m.*inv(C_m).*(C_i-C_m));
        C_Eff_DD = C_m + c_i(i) * ( C_i - C_m) .* L;
        disp(sprintf('Effective modulous Dilute Distribution: Volume fraction inclusion = %8.4f',c_i));
        disp(C_Eff_DD);
      end

    otherwise
      for i = 1:length(c_i)
        L = inv(eye() + S_m.*inv(C_m).*(C_i-C_m));
        C_Eff_DD = C_m + c_i(i) * ( C_i - C_m) .* L;
        c_dd_1(i) = C_Eff_DD(1,1);
        c_dd_2(i) = C_Eff_DD(2,2);
      end
      set( 0, 'currentfigure', h_C_Eff_1);
      hold on;
      h1_C_Eff_DD_1 = plot( c_i, c_dd_1, 'o-c');
      legend( 'Voigt', 'Reuss', 'Dilute Distribution', 'Location', 'NorthWest');
      hold off;

      set( 0, 'currentfigure', h_C_Eff_2);
      hold on;
      h1_C_Eff_DD_2 = plot( c_i, c_dd_2, 'o-c');
      legend( 'Voigt', 'Reuss', 'Dilute Distribution', 'Location', 'NorthWest');
      hold off;
  end
end

% Effective Elasticity Tensor Differential scheme
if 1
  c_ds_1 = zeros(length( c_i), 1);
  c_ds_2 = zeros(length( c_i), 1);
  S_m = Eshelby(ny_m);
  switch code
    case 1
      for i = 1:length(c_i)
        [C_Eff_Diff] = diff_analy_ellip(c_i(i), ny_m, Kappa_m, mu_m, Kappa_i, mu_i);
        disp(sprintf('Effective modulous Differential Scheme: Volume fraction inclusion = %8.4f',c_i));
        disp(C_Eff_Diff);
      end

    otherwise
      c_ds_1(1) = C_m(1,1); c_ds_1(end) = C_i(1,1);
      c_ds_2(1) = C_m(2,2); c_ds_2(end) = C_i(2,2);
      for i = 2:length(c_i)-1
```

```
            [C_Eff_Diff] = diff_analy_ellip(c_i(i), ny_m, Kappa_m, mu_m, Kappa_i, mu_i);
            c_ds_1(i) = C_Eff_Diff(1,1);
            c_ds_2(i) = C_Eff_Diff(2,2);
        end
        set( 0, 'currentfigure', h_C_Eff_1);
        hold on;
        h1_C_Eff_DS_1 = plot( c_i, c_ds_1, '-b');
        legend( 'Voigt', 'Reuss', 'Dilute Distribution','differential Scheme', 'Location', 'NorthWest');
        hold off;
        set( 0, 'currentfigure', h_C_Eff_2);
        hold on;
        h1_C_Eff_DS_2 = plot( c_i, c_ds_2, '-b');
        legend( 'Voigt', 'Reuss', 'Dilute Distribution','differential Scheme', 'Location', 'NorthWest');
        hold off;
  end
end
```

Calculation of Eshelby Tensor can be obtained from the Eshelby.m File
Eshelby Tensor(Eshelby.m File)

```
function S = Eshelby(ny)
%obtaining Eshelby tensor
%a1: majour axis
%a2 : minor axis
%b : scale factor
%mu_i:     mu of the inclusion

S = zeros(6,6);
a1 = 10;    %major axis
b = 0.4;    %scale factor
a2 = b*a1;  %minior axis
S(1,1) = 1/(2*(1-ny))*(((a2^2+2*a1*a2)/(a1+a2)^2)+((1-2*ny)*a2/(a1+a2)));
S(2,2) = 1/(2*(1-ny))*(((a2^2+2*a1*a2)/(a1+a2)^2)+((1-2*ny)*a1/(a1+a2)));
S(1,2) = 1/(2*(1-ny))*((a2^2/(a1+a2)^2)-((1-2*ny)*a2/(a1+a2)));
S(2,1) = 1/(2*(1-ny))*((a1^2/(a1+a2)^2)-((1-2*ny)*a1/(a1+a2)));
S(4,4) = 1/(2*(1-ny))*((a1^2+a2^2)/(2*(a1+a2)^2)+(1-2*ny)/2);
S(1,3) = ny/(2*(1-ny))*(2*a2/(a1+a2));
S(2,3) = ny/(2*(1-ny))*(2*a1/(a1+a2));
S(6,6) = a2/(2*(a1+a2));
S(5,5) = a1/(2*(a1+a2));
End
```

Calculation of Coefficient of Elasticity tensor can be obtained from the mat_prop.m File
Coefficient of Elasticity tensor (mat_prop.m file)

```
function C = mat_prop(kappa, mu)
%obtaining the isotropic elasticity modulous
%ci:   volume fraction of the inclusion
%kappa:  kappa of the material
%mu:     mu of the material
  C = zeros(6,6);
  C(1,1) = kappa+(4/3)*mu;
  C(1,2) = kappa-(2/3)*mu;
  C(4,4) = mu;
```

```
    C(5,5) = C(4,4);
    C(2,2) = C(1,1);
    C(3,3) = C(2,2);
    C(6,6) = C(5,5);
    C(1,3) = C(1,2);
    C(2,1) = C(1,2);
    C(3,1) = C(1,3);
    C(2,3) = C(1,2);
    C(3,2) = C(2,3);
end
```

Differential Scheme formulation is coded in function with matlab file name diff_analy_ellip.m and called in execution file to obtain respective result

Differential Scheme(diff_analy_ellip.m File):

```
function [C_Eff_Diff] = diff_analy_ellip(c_i, ny_m, Kappa_m, mu_m, Kappa_i, mu_i)
%obtaining the analytical effective material properties by differential
%scheme method
%ci:   volume fraction of the inclusion
%kappa_m:  kappa of the matrix material
%mu_m:     mu of the matrix material
%kappa_i:  kappa of the inclusion
%mu_i:     mu of the inclusion

        C_m = mat_prop( Kappa_m, mu_m);
        cm = reshape(C_m,[length(C_m)^2,1]);
        z0  = cm;
        % Initial conditions for the ODE
            v2_range = [0,c_i];
            [v2_i,zi] = ode45(@CEffDS,v2_range,z0,[], ny_m, Kappa_m, mu_m, Kappa_i, mu_i);

            z(1:36) = zi(length(zi),1:36);

            C_Eff_Diff = [z(1) z(2) z(3) z(4) z(5) z(6);...
                    z(7) z(8) z(9) z(10) z(11) z(12);...
                    z(13) z(14) z(15) z(16) z(17) z(18);...
                    z(19) z(20) z(21) z(22) z(23) z(24);...
                    z(25) z(26) z(27) z(28) z(29) z(30);...
                    z(31) z(32) z(33) z(34) z(35) z(36)];

 end
% DF MODEL Ordinary Differential Equation
function dfdv = CEffDS(c_i,C_Eff,ny_m, Kappa_m, mu_m, Kappa_i, mu_i)

        mat = vec2mat (C_Eff, 6);
        C_i = mat_prop( Kappa_i, mu_i);
        C_m = mat_prop( Kappa_m, mu_m);
        S_m = Eshelby(ny_m);
        %localization Tensor L
        L = inv(eye() + S_m.*inv(C_m).*(C_i-C_m));
        C_DS = ((1/(1-c_i))*(C_i-mat).*L);
        %right hand side
        dfdv = reshape(C_DS,[length(C_DS)^2,1]);
end
```

6. References:

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