# FEM MULTIPHASE MATERIALS

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M.SC. COMPUTATIONAL MECHANICS

UNDER THE SUPERVISION OF:

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# Contents

# 1 Introduction

As a general rule, the mechanical properties such as ductility, strength, resistance to creep and fatigue of engineering materials are determined by their (micro) structure at different geometric scales. For a vast majority of materials, the microstructure can be characterized as a composite of different phases, sometimes with vastly different properties. As a consequence, the behavior of such multiphase material is determined by the properties of the individual phases and the fashion in which these phases interact. The ultimate goal is to use this type of knowledge and predictive capability as a key ingredient in the design of high-performing, durable and reliable products for a sustainable society.

The classical continuum mechanics is the basis to describe the macroscopic behavior of so-called one component materials, i.e., the body consists of one single constituent, e.g., a solid, a fluid or a gas. The mixture theory has been developed with respect to the description of multi-phase continua (heterogeneously composed continua with internal interactions between the constituents and independent degrees of freedom)

Mixture theory is used to model multiphase systems using the principles of continuum mechanics generalized to several inter penetrable continua. The basic assumption is that, at any instant of time, all phases are present at every material point, and momentum and mass balance equations are postulated. Like other models, mixture theory requires constitutive relations to close the system of equations. Multi-phase continua can be described as a porous structure, i.e. a porous solid skeleton filled with an arbitrary number of fluids. This pore structure has a complex geometry which makes it almost impossible to separate out the constituents and use classical Continuum Mechanics their properties while considering the boundary and initial conditions i.e. the microscopic description of the porous medium. This hindrance can be overcome with the development of a macroscopic theory, which is based on the mixture theory. The constituent bodies of a mixture, in contrast to a porous body, are always separately identifiable at any time during a thermodynamic process. This is achieved by the concept of volume fraction.

# 2 Volume fraction concept

In the volume fraction concept, it is assumed that the porous solid always models a control space and that only the liquids and/or gases contained in the pores can leave the control space. Furthermore it is assumed that the pores are statistically distributed and an arbitrary volume element in the reference and the actual placement is composed of the volume elements of the real constituents. The basis for the description of porous media ,using elements of the theory of mixtures restricted by the volume fraction concept ,is the model of a Macroscopic body neither a geometrical interpretation of the pore structure nor the exact solution of the individual components of the body(constituents) are considered.

Today, the most used theory to describe the thermodynamic behavior of empty or Saturated porous solids is the mixture theory in consideration of the volume fraction concept. This concept is understood as the determination of the fraction of a body occupied by a constituent with the local ratio of the partial volume (volume of the real material of the corresponding constituent) in relation to the total volume. Within the framework of the volume fraction concept it will be assumed that the porous solid always models a control space in which the pores are filled with liquids and/or gases. Furthermore, it is assumed that the pores are statistically distributed.

Thus, one proceeds on the assumption that the constituents, which are bound together, are "smeared" over the control space, which is spanned by the porous solid, i.e., each substitute constituent occupies the total volume of space simultaneously with the other constituents. This approach is based on the concept that the constituents are statistically distributed over the control space. Thus, all geometrical and physical quantities of the constituents $\phi^\alpha$ such as motion, deformation and stress are defined in the total control space and they can be interpreted as the statistically average values of the true quantities of $\phi^\alpha$

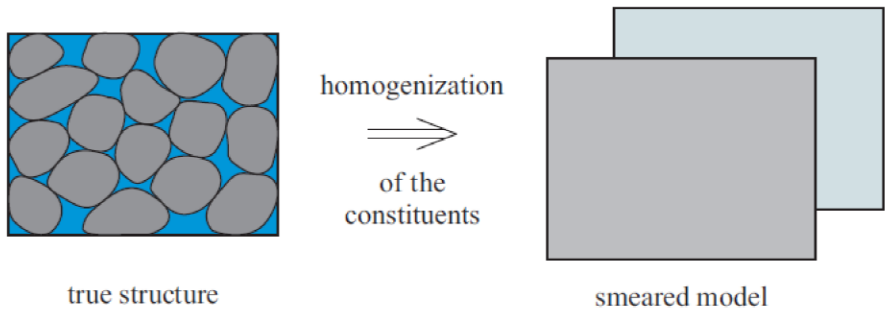


Figure 1: Statistical distribution of a binary porous medium consisting of a granular solid phase and a gas phase

## 2.1 Saturation condition

The Saturation or Volume Fraction Condition states that the sum of all volume fractions in a multi-constituent body is equal to one.

In mixture theory coupling exists between the volume fraction and the field quantities to make the sum of all volume fractions equal to one.

$$n_{0\alpha}^{\alpha} = n_{0\alpha}^{\alpha}(\mathbf{X}_\alpha t = t_0)$$

$$V_0^\alpha = \int_{B_{0S}} n_{0\alpha}^\alpha dV_{0\alpha} \ , \qquad dV_{0\alpha} = dV_{0\alpha}(t = t_0)$$

$$dV_{0\alpha}^\alpha = n_{0\alpha}^\alpha \, dV_{0\alpha}$$

$\therefore$

$$V_0 = \int_{B_{0S}} dV_{0\alpha} = \sum_{\alpha=1}^{\kappa} V_0^\alpha = \int_{B_{0S}} \sum_{\alpha=1}^{\kappa} dV_{0\alpha}^\alpha = \int_{B_{0S}} \sum_{\alpha=1}^{\kappa} n_{0\alpha}^\alpha dV_{0\alpha}^\alpha$$

$\therefore$ at position $\mathbf{X}$

$$\sum_{\alpha=1}^{\kappa} n_{0\alpha}^\alpha = 1$$

similarly at position $\mathbf{x}$

$$\sum_{\alpha=1}^{\kappa} n^{\alpha} = 1$$

If the porous solid is unsaturated, i.e., the pore space is partially material free, at position $\mathbf{X}$

$$V_0 > \sum_{\alpha=1}^{\kappa} V_0^{\alpha} \quad , \quad \sum_{\alpha=1}^{\kappa} n_{0\alpha}^{\alpha} < 1$$

similarly at position $\mathbf{x}$

$$V > \sum_{\alpha=1}^{\kappa} V^{\alpha} \quad , \quad \sum_{\alpha=1}^{\kappa} n^{\alpha} < 1$$

## 2.2 Real and partial densities

In mixture theory, volume fraction is connected with real densities of the constituents $\phi^{\alpha}$. Thus the total mass of the mixture in reference and actual configuration is

$$M_0 = \sum M_0^{\alpha} = \int_{B_{0\alpha}} \sum_{\alpha=1}^{\kappa} \rho_{0\alpha}^{\alpha} dV_{0\alpha}$$

$$M = \sum M^{\alpha} = \int_{B_{\alpha}} \sum_{\alpha=1}^{\kappa} \rho_{\alpha}^{\alpha} \mathrm{dv}$$

where $\rho_{0\alpha}^{\alpha} = \rho_{0\alpha}^{\alpha}(\mathbf{X}_{\alpha}, t = t_0)$and$\rho^{\alpha} = \rho^{\alpha}(\mathbf{x}, t)$ are partial densities at positions $\mathbf{X}_{\alpha}$ and $\mathbf{x}$. $M_0^{\alpha}$ and $M^{\alpha}$ are the partial masses of the constituents $\phi^{\alpha}$ in the reference and actual configurations. Since the densities $\rho_{0\alpha}^{\alpha}$ and $\rho^{\alpha}$ refer to the volume elements $dV_{0\alpha}^{\alpha}$ and $d\mathrm{v}$ at $t = t_0$ and t respectively,

$$\rho_{0\alpha} = \sum_{\alpha=1}^{\kappa} \rho_{0\alpha}^{\alpha}$$

and

$$\rho = \sum_{\alpha=1}^{\kappa} \rho^{\alpha}$$

$\therefore$

$$\rho_{0\alpha}^{\alpha} = \rho_{0\alpha}^{\alpha}(\mathbf{X}_{\alpha}, t = t_0)$$

and

$$\rho^{\alpha} = \rho^{\alpha}(\mathbf{x}, t)$$

the real densities refer to the corresponding partial volume elements.
$\therefore$ The partial densities are given by

$$\rho_{0\alpha}^{\alpha} = n_{0\alpha}^{\alpha} \rho_{0\alpha}^{\alpha R}$$

and

$$\rho = n^{\alpha} \rho^{\alpha R}$$

$\therefore$ The masses is given by

$$M_0^{\alpha} = \int_{B_{0S}} \rho_{0\alpha}^{\alpha} dV_{0\alpha} = \int_{B_{0S}} \rho_{0\alpha}^{\alpha R} dV_{0\alpha}^{\alpha}$$

and

$$M^\alpha = \int_{B_S} \rho^\alpha \mathrm{dv} = \int_{B_S} \rho^{\alpha R} \mathrm{dv}^\alpha$$

Also for special surface elements of the body with reference to the reference and actual configuration as well as to the corresponding partial surface element of the constituents $\phi^\alpha$:

$$dA^\alpha_{0_{SC}} = n^\alpha_{0\alpha} dA_{0_{SC}}$$

$$da^\alpha_{SC} = n^\alpha da_{SC}$$

where SC: statistical cut

# 3 Derivation of Balance equation and Entropy Inequality

## 3.1 Balance Equation of Mass

The balance equation of mass states, that for every constituent $\phi^\alpha$ the rate of mass is equal to a mass production term.

$$(\int_{B_\alpha} \rho^\alpha \, \mathrm{dv})'_\alpha \; = \; \int_{B_\alpha} \hat{\rho}^\alpha \, \mathrm{dv}$$

which can be further reduced as per the following steps

$$\begin{aligned}
(\int_{B_\alpha} \rho^\alpha \, \mathrm{J}_\alpha \, \mathrm{dv})'_\alpha \; &= \; \int_{B_\alpha} \hat{\rho}^\alpha \, \mathrm{dv} \\
\int_{B_\alpha} \Big( (\rho^\alpha)'_\alpha \mathrm{J}_\alpha + \rho^\alpha \mathrm{J}'_\alpha \Big) \, \mathrm{dV} \; &= \; \int_{B_\alpha} \hat{\rho}^\alpha \, \mathrm{dv} \\
\int_{B_\alpha} \Big( (\rho^\alpha)'_\alpha \mathrm{J}_\alpha \; + \; \rho^\alpha (\mathrm{div}\, \mathbf{x}'_\alpha) \mathrm{J}_\alpha \Big) \, \mathrm{dV} \; &= \; \int_{B_\alpha} \hat{\rho}^\alpha \, \mathrm{dv} \\
\int_{B_\alpha} \Big[ \Big( (\rho^\alpha)'_\alpha \; + \; \rho^\alpha (\mathrm{div}\, \mathbf{x}'_\alpha) \Big) \, \mathrm{J}_\alpha \Big] \, \mathrm{dV} \; &= \; \int_{B_\alpha} \hat{\rho}^\alpha \, \mathrm{dv} \\
\int_{B_\alpha} \Big( (\rho^\alpha)'_\alpha \; + \; \rho^\alpha \mathrm{div}\, \mathbf{x}'_\alpha \Big) \, \mathrm{dV} \; &= \; \int_{B_\alpha} \hat{\rho}^\alpha \, \mathrm{dv}
\end{aligned}$$

we know that

$$\rho^\alpha \; = \; \mathrm{n}^\alpha \, \rho^{\alpha R}$$

we get

$$\begin{aligned}
(\mathrm{n}^\alpha \, \rho^{\alpha R})'_\alpha \; + \; \mathrm{n}^\alpha \, \rho^{\alpha R} \, \mathrm{div}\, \mathbf{x}'_\alpha \; &= \; \hat{\rho}^\alpha \\
(\mathrm{n}^\alpha)'_\alpha \, \rho^{\alpha R} \; + \; \mathrm{n}^\alpha (\rho^{\alpha R})'_\alpha \; + \; \mathrm{n}^\alpha \, \rho^{\alpha \mathrm{R}} \, \mathrm{div}\, \mathbf{x}'_\alpha \; &= \; \hat{\rho}^\alpha \\
(\mathrm{n}^\alpha)'_\alpha \; + \; \mathrm{n}^\alpha \frac{(\rho^{\alpha R})'_\alpha}{\rho^{\alpha R}} \; + \; \mathrm{n}^\alpha \, \mathrm{div}\, \mathbf{x}'_\alpha \; &= \; \frac{\hat{\rho}^\alpha}{\rho^{\alpha R}}
\end{aligned}$$

For an Incompressible constituent , $(\rho^{\alpha R})'_\alpha \; = \; 0$ equation reduces to

$$(\mathrm{n}^\alpha)'_\alpha \; + \; \mathrm{n}^\alpha \, \mathrm{div}\, \mathbf{x}'_\alpha \; = \; \frac{\hat{\rho}^\alpha}{\rho^{\alpha R}}$$

Furthermore, if we consider no mass production ($\hat{\rho}^\alpha \;=\; 0$), then the balance of mass can only be expressed in terms of the volume fractions

$$(\mathrm{n}^\alpha)'_\alpha \;+\; \mathrm{n}^\alpha \,\mathrm{div}\,\mathbf{x}'_\alpha \;=\; 0$$

The integral form of the balance of mass

$$\mathrm{J}_\alpha = \frac{\rho^\alpha_{0\alpha}}{\rho^\alpha} = \frac{n^\alpha_{0\alpha}\,\rho^{\alpha R}_{0\alpha}}{n^\alpha\,\rho^{\alpha R}}$$

for incompressible material $\rho^{\alpha R}_{0\alpha} = \rho^{\alpha R}$
$\therefore$

$$\mathrm{J}_\alpha = \frac{n^\alpha_{0\alpha}}{n^\alpha}$$

For binary model $\hat{\rho}^S + \hat{\rho}^F = 0$
for a closed system $\hat{\rho}^S = -\hat{\rho}^F$

## 3.2 Balance of Momentum

The derivation of the final form of the balance equation of momentum begins with the below equation:

$$(\mathbf{I}^\alpha)'_\alpha \;=\; \mathbf{k}^\alpha \;+\; \int_{B\alpha} \hat{\mathrm{s}}\,\mathrm{dv}$$

with the momentum

$$\mathbf{I}_\alpha \;=\; \int_{B\alpha} \rho^\alpha\,\mathbf{x}'_\alpha\,\mathrm{dv}$$

and the vector of external forces

$$\mathbf{k}^\alpha \;=\; \int_{B\alpha} \rho^\alpha\,\mathbf{b}^\alpha\,\mathrm{dv} \;+\; \int_{\partial B\alpha} \mathbf{t}^\alpha\,\mathrm{da}$$

With the help of the material time derivative of $\mathbf{I}^\alpha$

$$\begin{aligned}
(\mathbf{I}^\alpha)'_\alpha \;&=\; \left(\int_{B\alpha} \rho^\alpha\,\mathbf{x}'_\alpha\,\mathrm{dv}\right)'_\alpha \\
&= \int_{B_\alpha} (\rho^\alpha\,\mathrm{J}_\alpha\,\mathbf{x}'_\alpha)'_\alpha\,\mathrm{dV} \\
&= \int_{B_{0\alpha}} \left((\rho^\alpha)'_\alpha\,\mathrm{J}_\alpha\,\mathbf{x}'_\alpha \;+\; \rho^\alpha\,\mathrm{J}'_\alpha\,\mathbf{x}'_\alpha \;+\; \rho^\alpha\,\mathrm{J}_\alpha\,\mathbf{x}''_\alpha\right)\mathrm{dV} \\
&= \int_{B_{0\alpha}} \left((\rho^\alpha)'_\alpha\,\mathrm{J}_\alpha\,\mathbf{x}'_\alpha \;+\; \rho^\alpha\,(\mathrm{div}\,\mathbf{x}'_\alpha)\mathrm{J}_\alpha\,\mathbf{x}'_\alpha \;+\; \rho^\alpha\,\mathrm{J}_\alpha\,\mathbf{x}''_\alpha\right)\mathrm{dV} \\
&= \int_{B_\alpha} \left((\rho^\alpha)'_\alpha\,\mathbf{x}'_\alpha \;+\; \rho^\alpha\,(\mathrm{div}\,\mathbf{x}'_\alpha)\,\mathbf{x}'_\alpha \;+\; \rho^\alpha\,\mathbf{x}''_\alpha\right)\mathrm{dv} \\
&= \int_{B_\alpha} \left[\left((\rho^\alpha)'_\alpha \;+\; \rho^\alpha\,\mathrm{div}\,\mathbf{x}'_\alpha\right)\;\mathbf{x}'_\alpha \;+\; \rho^\alpha\,\mathbf{x}''_\alpha\right]\mathrm{dv} \\
&= \int_{B_\alpha} (\hat{\rho}^\alpha\,\mathbf{x}'_\alpha \;+\; \rho^\alpha\,\mathbf{x}''_\alpha)\,\mathrm{dv}
\end{aligned}$$

By using the *Cauchy's* theorem in combination with the divergence theorem we get

$$\int_{\partial B_\alpha} \mathbf{t}^\alpha \,\mathrm{da} \;=\; \int_{\partial B_\alpha} \mathbf{T}^\alpha \,\mathbf{n}\,\mathrm{da} \;=\; \int_{B_\alpha} \operatorname{div} \mathbf{T}^\alpha \,\mathrm{dv}$$

the whole expression reads

$$\int_{B_\alpha} \hat{\rho}^\alpha \,\mathbf{x}_\alpha^{'} \,\mathrm{dv} \;+\; \int_{B_\alpha} \rho^\alpha \,\mathbf{x}_\alpha^{''} \,\mathrm{dv} \;=\; \int_{B_\alpha} \rho^\alpha \,\mathbf{b}^\alpha \,\mathrm{dv} \;+\; \int_{B_\alpha} \operatorname{div} \mathbf{T}^\alpha \,\mathrm{dv} \;+\; \int_{B_\alpha} \hat{\mathrm{s}} \,\mathrm{dv}$$

The above equation can be reformulated as

$$\int_{B_\alpha} \Big( \operatorname{div} \mathbf{T}^\alpha \;+\; \rho_\alpha \,(\mathbf{b}^\alpha \;-\; \mathbf{x}_\alpha^{''}) \Big) \,\mathrm{dv} \;=\; \int_{B_\alpha} (\hat{\rho}^\alpha \,\mathbf{x}_\alpha^{'} \;-\; \hat{\mathrm{s}}^\alpha) \,\mathrm{dv}$$

where the local form can be expressed as:

$$\operatorname{div} \mathbf{T}^\alpha \;+\; \rho_\alpha \,(\mathbf{b}^\alpha \;-\; \mathbf{x}_\alpha^{''}) \;=\; \hat{\rho}^\alpha \,\mathbf{x}_\alpha^{'} \;-\; \hat{\mathrm{s}}^\alpha$$

where the term $\hat{\rho}^\alpha \,\mathbf{x}_\alpha^{'} \;-\; \hat{\mathrm{s}}^\alpha \;=\; -\hat{\mathrm{p}}^\alpha$ for the constituent $\phi^\alpha$.

## 3.3 Balance of Angular (Moment of) Momentum

In the following derivation of the balance equation of the moment of momentum for non-polar materials is given, beginning with the equation

$$(\mathbf{h}_{(0)}^\alpha)_\alpha^{'} \;=\; \mathbf{m}_{(0)}^\alpha \;+\; \hat{\mathbf{h}}_{(0)}^\alpha$$

with the moment of momentum

$$\mathbf{h}_{(0)}^\alpha \;=\; \int_{B_\alpha} \mathbf{x} \;\times\; \rho^\alpha \,\mathbf{x}_\alpha^{'} \,\mathrm{dv}$$

the moment of external forces

$$\mathbf{m}_{(0)}^\alpha \;=\; \int_{B_\alpha} \mathbf{x} \;\times\; \rho^\alpha \,\mathbf{b}^\alpha \,\mathrm{dv} \;+\; \int_{\partial B_\alpha} \mathbf{x} \;\times\; \mathbf{t}^\alpha \,\mathrm{da}$$

Using *Cauchy's* theorem

$$\begin{aligned}\mathbf{m}_{(0)}^\alpha \;&=\; \int_{B_\alpha} \mathbf{x} \;\times\; \rho^\alpha \,\mathbf{b}^\alpha \,\mathrm{dv} \;+\; \int_{\partial B_\alpha} \mathbf{x} \;\times\; \mathbf{T}^\alpha \,\mathbf{n}\,\mathrm{da} \\ &=\; \int_{B_\alpha} \mathbf{x} \;\times\; \rho^\alpha \,\mathbf{b}^\alpha \,\mathrm{dv} \;+\; \int_{B_\alpha} (\mathbf{x} \;\times\; \operatorname{div} \mathbf{T}^\alpha \;+\; \mathbf{I} \;\times\; \mathbf{T}^\alpha) \,\mathrm{dv}\end{aligned}$$

and the moment resulting from the internal interaction forces

$$\hat{\mathbf{h}}_{(0)}^\alpha \;=\; \int_{B_\alpha} \mathbf{x} \;\times\; \hat{\mathbf{p}}^\alpha \,\mathrm{dv}$$

The material time derivative of the moment of momentum is given by

$$
\begin{aligned}
(\mathbf{h}^{\alpha}_{(0)})'_{\alpha} &= (\int_{B_\alpha} \mathbf{x} \times \rho^{\alpha}\, \mathbf{x}'_{\alpha}\, \mathrm{dv})'_{\alpha} \\
&= \int_{B_{0\alpha}} (\mathbf{x} \times \rho^{\alpha}\, \mathrm{J}_{\alpha}\, \mathbf{x}'_{\alpha})'_{\alpha}\, \mathrm{dV} \\
&= \int_{B_{0\alpha}} \Big[\mathbf{x}'_{\alpha} \times \rho^{\alpha}\, \mathrm{J}_{\alpha}\, \mathbf{x}'_{\alpha} + \mathbf{x} \times \Big((\rho^{\alpha})'_{\alpha}\, \mathrm{J}_{\alpha}\, \mathbf{x}'_{\alpha} + \rho^{\alpha}\, \mathrm{J}'_{\alpha}\, \mathbf{x}'_{\alpha} + \rho^{\alpha}\, \mathrm{J}_{\alpha}\, \mathbf{x}''_{\alpha}\Big)\Big]\, \mathrm{dV}
\end{aligned}
$$

(applying the relation $\mathrm{J}'_{\alpha} = \mathrm{J}_{\alpha}\, \mathrm{div}\, \mathrm{x}'_{\alpha}$)

$$
\begin{aligned}
&= \int_{B_{0\alpha}} \Big[\mathbf{x}'_{\alpha} \times \rho^{\alpha}\, \mathrm{J}_{\alpha}\, \mathbf{x}'_{\alpha} + \mathbf{x} \times \Big((\rho^{\alpha})'_{\alpha}\, \mathrm{J}_{\alpha}\, \mathbf{x}'_{\alpha} + \rho^{\alpha}\, \mathrm{J}_{\alpha}\, (\mathrm{div}\, \mathbf{x}'_{\alpha})\, \mathbf{x}'_{\alpha} + \rho^{\alpha}\, \mathrm{J}_{\alpha}\, \mathbf{x}''_{\alpha}\Big)\Big]\, \mathrm{dV} \\
&= \int_{B_{\alpha}} \Big[\mathbf{x}'_{\alpha} \times \rho^{\alpha}\, \mathbf{x}'_{\alpha} + \mathbf{x} \times \Big((\rho^{\alpha})'_{\alpha}\, \mathbf{x}'_{\alpha} + \rho^{\alpha}\, (\mathrm{div}\, \mathbf{x}'_{\alpha})\, \mathbf{x}'_{\alpha} + \rho^{\alpha}\, \mathbf{x}''_{\alpha}\Big)\Big]\, \mathrm{dv} \\
&= \int_{B_{\alpha}} \Big[\mathbf{x}'_{\alpha} \times \rho^{\alpha}\, \mathbf{x}'_{\alpha} + \mathbf{x} \times \Big([(\rho^{\alpha})'_{\alpha} + \rho^{\alpha}\, \mathrm{div}\, \mathbf{x}'_{\alpha}]\, \mathbf{x}'_{\alpha} + \rho^{\alpha}\, \mathbf{x}''_{\alpha}\Big)\Big]\, \mathrm{dv}
\end{aligned}
$$

(we know that $\mathbf{x}'_{\alpha} \times \rho^{\alpha}\, \mathbf{x}'_{\alpha} = 0;\ (\rho^{\alpha})'_{\alpha} + \rho^{\alpha}\, \mathrm{div}\, \mathbf{x}'_{\alpha} = \hat{\rho}^{\alpha}$)

$$
= \int_{B_{\alpha}} \mathbf{x} \times (\hat{\rho}^{\alpha}\, \mathbf{x}'_{\alpha} + \rho^{\alpha}\, \mathbf{x}''_{\alpha})\, \mathrm{dv}
$$

The whole balance equation of moment of momentum reads now

$$
\int_{B_\alpha} \mathbf{x} \times (\hat{\rho}^{\alpha}\mathbf{x}'_{\alpha} + \rho^{\alpha}\mathbf{x}''_{\alpha})\mathrm{dv} = \int_{B_\alpha} (\mathbf{x} \times \mathrm{div}\,\mathbf{T}^{\alpha} + \mathbf{I} \times \mathbf{T}^{\alpha})\mathrm{dv} + \int_{B_\alpha} \mathbf{x} \times \rho^{\alpha}\mathbf{b}^{\alpha}\mathrm{dv} + \int_{B_\alpha} \mathbf{x} \times \hat{\mathrm{s}}^{\alpha}\mathrm{dv}
$$

which can be reformulated and expressed in the local form

$$
\mathbf{x} \times \Big(\mathrm{div}\, \mathbf{T}^{\alpha} + \rho^{\alpha}\, (\mathbf{b}^{\alpha} - \mathbf{x}''_{\alpha} + \hat{\mathrm{s}}^{\alpha} - \hat{\rho}\, \mathbf{x}'_{\alpha})\Big) + \mathbf{I} \times \mathbf{T}^{\alpha} = 0
$$

We know that ($\mathrm{div}\, \mathbf{T}^{\alpha} + \rho^{\alpha}\, (\mathbf{b}^{\alpha} - \mathbf{x}''_{\alpha} + \hat{\mathrm{s}}^{\alpha} - \hat{\rho}\, \mathbf{x}'_{\alpha} = 0$)
which reduces to

$$
\mathbf{I} \times \mathbf{T}^{\alpha} = 0
$$

and finally leads to the statement

$$
\mathbf{T}^{\alpha} = (\mathbf{T}^{\alpha})^{\mathrm{T}}
$$

which means that the partial *Cauchy* stress tensor $\mathbf{T}^{\alpha}$ is symmetric

### 3.4 Balance of Energy

The balance equation of energy for the constituent $\phi^\alpha$ is given by

$$(\mathrm{E}^\alpha)_\alpha' \,+\, (\mathrm{K}^\alpha)_\alpha' \,=\, \mathrm{W}^\alpha \,+\, \mathrm{Q}^\alpha \,+\, \int_{B_\alpha} \hat{e}^\alpha \,\mathrm{dv}$$

The internal energy is given by

$$\mathrm{E}^\alpha \,=\, \int_{\mathrm{B}\alpha} \rho^\alpha \, \varepsilon^\alpha \mathrm{dv}$$

The kinetic energy is defined as

$$\mathrm{K}^\alpha \,=\, \int_{B_\alpha} \frac{1}{2} \, \rho^\alpha \, \mathbf{x}_\alpha' \,\cdot\, \mathbf{x}_\alpha' \,\mathrm{dv}$$

The increment of the mechanical work is given by

$$\mathrm{W}^\alpha \,=\, \int_{B_\alpha} \mathbf{x}_\alpha' \,\cdot\, \rho^\alpha \, \mathbf{b}^\alpha \,\mathrm{dv} \,+\, \int_{\partial B_\alpha} \mathbf{x}_\alpha' \,\cdot\, \mathbf{T}^\alpha \,\mathbf{da}$$

and the increment of non-mechanical work

$$\mathrm{Q}^\alpha \,=\, \int_{B_\alpha} \rho^\alpha \, \mathrm{r}^\alpha \,\mathrm{dv} \,+\, \int_{\partial B_\alpha} \mathbf{q}^\alpha \,\cdot\, \mathbf{da}$$

The material time derivative of the internal energy $(\mathrm{E}^\alpha)_\alpha'$ can be expressed as

$$\begin{aligned}
(\mathrm{E}^\alpha)_\alpha' \,&=\, (\int_{B_{0\alpha}} \rho^\alpha \, \varepsilon^\alpha \,\mathrm{dv})_\alpha' \\
&=\, \int_{\alpha} (\rho^\alpha \, \mathrm{J}_\alpha \, \varepsilon^\alpha)_\alpha' \,\mathrm{dV} \\
&=\, \int_{B_{0\alpha}} \Big( (\rho^\alpha)_\alpha' \, \mathrm{J}_\alpha \, \varepsilon^\alpha \,+\, \rho^\alpha \, \mathrm{J}_\alpha' \, \varepsilon^\alpha \,+\, \rho^\alpha \, \mathrm{J}_\alpha \, (\varepsilon^\alpha)_\alpha' \Big) \,\mathrm{dV} \\
&=\, \int_{B_{0\alpha}} \Big( (\rho^\alpha)_\alpha' \, \mathrm{J}_\alpha \, \varepsilon^\alpha \,+\, \rho^\alpha \, \mathrm{J}_\alpha \, (\mathrm{div}\, \mathbf{x}_\alpha') \, \varepsilon^\alpha \,+\, \rho^\alpha \, \mathrm{J}_\alpha \, (\varepsilon^\alpha)_\alpha' \Big) \,\mathrm{dV} \\
&=\, \int_{B_\alpha} \Big( (\rho^\alpha)_\alpha' \, \varepsilon^\alpha \,+\, \rho^\alpha \, (\mathrm{div}\, \mathbf{x}_\alpha') \, \varepsilon^\alpha \,+\, \rho^\alpha \, (\varepsilon^\alpha)_\alpha' \Big) \,\mathrm{dv} \\
&=\, \int_{B_\alpha} \Big[ \Big( (\rho^\alpha)_\alpha' \,+\, \rho^\alpha \, (\mathrm{div}\, \mathbf{x}_\alpha') \Big) \, \varepsilon^\alpha \,+\, \rho^\alpha \, (\varepsilon^\alpha)_\alpha' \Big] \,\mathrm{dv} \\
&=\, \int_{B_\alpha} \Big( \hat{\rho}^\alpha \, \varepsilon^\alpha \,+\, \rho^\alpha \, (\varepsilon^\alpha)_\alpha' \Big) \,\mathrm{dv}
\end{aligned}$$

and for the kinetic energy $\mathrm{K}^\alpha$ we gain the material time derivative

$$
\begin{aligned}
(\mathrm{K}^\alpha)'_\alpha &= \Big(\int_{B_\alpha} \frac{1}{2}\,\rho^\alpha\,\mathbf{x}'_\alpha \cdot \mathbf{x}'_\alpha \mathrm{dv}\Big)'_\alpha \\
&= \Big(\int_{B_\alpha} \frac{1}{2}\,\rho^\alpha\,\mathbf{x}'_\alpha \cdot \mathbf{x}'_\alpha\Big)'_\alpha\,\mathrm{dv} \\
&= \int_{B_\alpha} \frac{1}{2}\Big( (\rho^\alpha)'_\alpha\,\mathrm{J}_\alpha\,\mathbf{x}'_\alpha \cdot \mathbf{x}'_\alpha + \rho^\alpha \mathrm{J}'_\alpha\;\mathbf{x}'_\alpha \cdot \mathbf{x}'_\alpha + 2\,\rho^\alpha\,\mathbf{x}''_\alpha \cdot \mathbf{x}'_\alpha\Big)\,\mathrm{dV} \\
&\quad\text{(applying the relation } \mathrm{J}'_\alpha = \mathrm{J}_\alpha\,\mathrm{div}\,\mathbf{x}'_\alpha) \\
&= \int_{B_{0\alpha}} \frac{1}{2}\Big( (\rho^\alpha)'_\alpha\,\mathrm{J}_\alpha\,\mathbf{x}'_\alpha \cdot \mathbf{x}'_\alpha + \rho^\alpha \mathrm{J}_\alpha\,(\mathrm{div}\,\mathbf{x}'_\alpha)\,\mathbf{x}'_\alpha \cdot \mathbf{x}'_\alpha + 2\,\rho^\alpha\,\mathbf{x}''_\alpha \cdot \mathbf{x}'_\alpha\Big)\,\mathrm{dV} \\
&= \int_{B_{0\alpha}} \frac{1}{2}\Big[ \Big((\rho^\alpha)'_\alpha + \rho^\alpha\,\mathrm{div}\,\mathbf{x}'_\alpha\Big)\;\mathbf{x}'_\alpha \cdot \mathbf{x}'_\alpha + 2\,\rho^\alpha\,\mathbf{x}''_\alpha \cdot \mathbf{x}'_\alpha\Big]\,\mathrm{dv} \\
&= \int_{B_\alpha} \Big(\frac{1}{2}\,\hat{\rho}^\alpha\,\mathbf{x}'_\alpha + \rho^\alpha\,\mathbf{x}''_\alpha\Big) \cdot \mathbf{x}'_\alpha\,\mathrm{dv}
\end{aligned}
$$

on further derivation using equations

$$
\begin{aligned}
\int_{\partial B_\alpha} \mathbf{x}'_\alpha \cdot \mathbf{T}^\alpha\,\mathbf{n}\,\mathrm{da} &= \int_{B_\alpha} \mathrm{div}\,(\mathbf{x}'_\alpha \cdot \mathbf{T}^\alpha)\,\mathrm{dv} \\
&= \int_{B_\alpha} (\mathrm{div}\,\mathbf{T}^\alpha \cdot \mathbf{x}'_\alpha + \mathbf{T}^\alpha \cdot \mathrm{grad}\,\mathbf{x}'_\alpha)\,\mathrm{dv} \\
&\text{with } \mathrm{grad}\,\mathbf{x}'_\alpha = \mathrm{L}_\alpha \\
&= \int_{B_\alpha} (\mathrm{div}\,\mathbf{T}^\alpha \cdot \mathbf{x}'_\alpha + \mathbf{T}^\alpha \cdot \mathbf{L}_\alpha)\,\mathrm{dv}
\end{aligned}
$$

and

$$
\int_{\partial B_\alpha} \mathbf{q}^\alpha \cdot \mathbf{n}\,\mathrm{da} = \int_{B_\alpha} \mathrm{div}\,\mathbf{q}^\alpha\,\mathrm{dv}
$$

the whole equation of the balance of energy can be expressed as:

$$
\begin{aligned}
&\int_{B_\alpha} \Big(\hat{\rho}^\alpha\,\varepsilon^\alpha + \rho^\alpha\,(\varepsilon)'_\alpha\Big)\,\mathrm{dv} + \int_{B_\alpha} \Big(\frac{1}{2}\,\hat{\rho}^\alpha\,\mathbf{x}'_\alpha + \rho^\alpha\,\mathbf{x}''_\alpha\Big) \cdot \mathbf{x}'_\alpha\,\mathrm{dv} = \\
&= \int_{B_\alpha} \mathbf{x}'_\alpha \cdot \rho^\alpha\,\mathbf{b}^\alpha\,\mathrm{dv} + \int_{B_\alpha} (\mathrm{div}\,\mathbf{T}^\alpha \cdot \mathbf{x}'_\alpha + \mathbf{T}^\alpha \cdot \mathbf{L}_\alpha)\,\mathrm{dv} + \\
&\qquad \int_{B_\alpha} \rho^\alpha\,\mathrm{r}^\alpha\,\mathrm{dv} + \int_{B_\alpha} \mathrm{div}\,\mathbf{q}^\alpha\,\mathrm{dv} + \int_{B_\alpha} \hat{\varepsilon}^\alpha\,\mathrm{dv}
\end{aligned}
$$

Under the consideration of the symmetric part *Cauchy* stress tensors $\mathbf{T}^\alpha$, which leads to the substitution of the spatial velocity gradient by its symmetric part (i.e, $\mathbf{L}^\alpha = \mathbf{D}_\alpha$), the local form can be written as

$$
\begin{aligned}
\rho^\alpha\,(\varepsilon^\alpha)'_\alpha + \hat{\rho}^\alpha\,(\varepsilon^\alpha + \frac{1}{2}\,\mathbf{x}'_\alpha \cdot \mathbf{x}'_\alpha) &= \Big(\mathrm{div}\,\mathbf{T}^\alpha + \rho^\alpha\,(\mathbf{b}^\alpha - \mathbf{x}''_\alpha) \cdot \mathbf{x}'_\alpha\Big) + \\
&\qquad \mathbf{T}^\alpha \cdot \mathbf{D}_\alpha + \rho^\alpha\,\mathrm{r}^\alpha + \mathrm{div}\,\mathbf{q}^\alpha\,\hat{\varepsilon}^\alpha
\end{aligned}
$$

where $\mathrm{div}\,\mathbf{T}^\alpha + \rho^\alpha\,(\mathbf{b}^\alpha - \mathbf{x}''_\alpha) = -\hat{\mathbf{p}}^\alpha$
above equation becomes

$$
\rho^\alpha\,(\varepsilon^\alpha)'_\alpha - \mathbf{T}^\alpha \cdot \mathbf{D}_\alpha - \rho^\alpha\,\mathrm{r}^\alpha + \mathrm{div}\,\mathbf{q}^\alpha = \hat{\mathrm{e}}^\alpha - \hat{\mathbf{p}}^\alpha \cdot \mathbf{x}'_\alpha - \hat{\rho}^\alpha\,(\varepsilon^\alpha + \frac{1}{2}\,\mathbf{x}'_\alpha \cdot \mathbf{x}'_\alpha)
$$

where $\hat{\mathrm{e}}^{\alpha} - \hat{\mathbf{p}}^{\alpha} \cdot \mathbf{x}_{\alpha}^{'} - \hat{\rho}^{\alpha} (\varepsilon^{\alpha} + \frac{1}{2} \mathbf{x}_{\alpha}^{'} \cdot \mathbf{x}_{\alpha}^{'}) = \hat{\varepsilon}^{\alpha}$
Considering the *Helmholtz free energy function* $\psi^{\alpha} = \varepsilon^{\alpha} - \Theta^{\alpha} \eta^{\alpha}$ and it's material time derivative $(\psi^{\alpha})_{\alpha}^{'} = (\varepsilon^{\alpha})_{\alpha}^{'} - (\Theta^{\alpha})_{\alpha}^{'} \eta^{\alpha} - \Theta^{\alpha} (\eta^{\alpha})_{\alpha}^{'}$, we get the alternative expression for the balance of Energy

$$\rho^{\alpha} \left( (\psi^{\alpha})_{\alpha}^{'} + (\Theta^{\alpha})_{\alpha}^{'} \eta^{\alpha} + \Theta^{\alpha} (\eta^{\alpha})_{\alpha}^{'} \right) - \mathbf{T}^{\alpha} \cdot \mathbf{D}_{\alpha} - \rho^{\alpha} \mathrm{r}^{\alpha} + \operatorname{div} \mathbf{q}^{\alpha} = \\ \hat{\mathrm{e}}^{\alpha} - \hat{\mathbf{p}}^{\alpha} \cdot \mathbf{x}_{\alpha}^{'} - \hat{\rho} (\psi^{\alpha} + \Theta^{\alpha} \eta^{\alpha} + \frac{1}{2} \mathbf{x}_{\alpha}^{'} \cdot \mathbf{x}_{\alpha}^{'})$$

## 3.5 Entropy Inequality

The entropy inequality for mixture is defined as

$$\sum_{\alpha=1}^{m} (\mathrm{H}^{\alpha})_{\alpha}^{'} \geq \sum_{\alpha=1}^{m} \int_{B_{\alpha}} \frac{1}{\Theta^{\alpha}} \rho^{\alpha} \mathrm{r}^{\alpha} \mathrm{dv} - \sum_{\alpha=1}^{m} \int_{B_{\alpha}} \frac{1}{\Theta^{\alpha}} \mathbf{q}^{\alpha} \cdot \mathbf{n} \, \mathrm{da}$$

with the entropy

$$\mathbf{H}^{\alpha} = \int_{B_{\alpha}} \rho^{\alpha} \eta^{\alpha} \mathrm{dv}$$

where $\eta^{\alpha}$ is the partial specific entropy. The material time derivative of the entropy is given by

$$\begin{aligned}
(\mathbf{H}^{\alpha})_{\alpha}^{'} &= (\int_{B_{\alpha}} \rho^{\alpha} \eta^{\alpha} \mathrm{dv})_{\alpha}^{'} \\
&= \int_{B_{\alpha}} (\rho^{\alpha} \eta^{\alpha})_{\alpha}^{'} \mathrm{dv} \\
&= \int_{B_{0\alpha}} (\rho^{\alpha} \mathrm{J}_{\alpha} \eta^{\alpha})_{\alpha}^{'} \mathrm{dV} \\
&= \int_{B_{0\alpha}} \left( (\rho^{\alpha})_{\alpha}^{'} \mathrm{J}_{\alpha} \eta^{\alpha} + \rho^{\alpha} \mathrm{J}_{\alpha}^{'} \eta^{\alpha} + \rho^{\alpha} \mathrm{J}_{\alpha} (\eta_{\alpha})_{\alpha}^{'} \right) \mathrm{dV} \\
&= \int_{B_{0\alpha}} \left( (\rho^{\alpha})_{\alpha}^{'} \mathrm{J}_{\alpha} \eta^{\alpha} + \rho^{\alpha} \mathrm{J}_{\alpha} (\operatorname{div} \mathbf{x}_{\alpha}^{'}) \eta^{\alpha} + \rho^{\alpha} \mathrm{J}_{\alpha} (\eta^{\alpha})_{\alpha}^{'} \right) \mathrm{dV} \\
&= \int_{B_{\alpha}} \left( (\rho^{\alpha})_{\alpha}^{'} \eta^{\alpha} + \rho^{\alpha} (\operatorname{div} \mathbf{x}_{\alpha}^{'}) \eta^{\alpha} + \rho^{\alpha} (\eta^{\alpha})_{\alpha}^{'} \right) \mathrm{dv} \\
&= \int_{B_{\alpha}} \left[ \left( (\rho^{\alpha})_{\alpha}^{'} + \rho^{\alpha} \operatorname{div} \mathbf{x}_{\alpha}^{'} \right) \eta^{\alpha} + \rho^{\alpha} (\eta^{\alpha})_{\alpha}^{'} \right] \mathrm{dv} \\
&= \int_{B_{\alpha}} \left( \hat{\rho}_{\alpha} \eta^{\alpha} + \rho^{\alpha} (\eta^{\alpha})_{\alpha}^{'} \right) \mathrm{dv}
\end{aligned}$$

Together with the relation

$$\begin{aligned}
\int_{\partial B_{\alpha}} \frac{1}{\Theta^{\alpha}} \mathbf{q}^{\alpha} \cdot \mathbf{n} \, \mathrm{da} &= \int_{B_{\alpha}} \operatorname{div} (\frac{1}{\Theta^{\alpha}} \mathbf{q}^{\alpha}) \mathrm{dv} \\
&= \int_{B_{\alpha}} \frac{\operatorname{div} \mathbf{q}^{\alpha} \Theta^{\alpha} - \operatorname{grad} \Theta^{\alpha} \cdot \mathbf{q}^{\alpha}}{(\Theta^{\alpha})^2} \mathrm{dv} \\
&= \int_{B_{\alpha}} \left( \frac{1}{\Theta^{\alpha}} \operatorname{div} \mathbf{q}^{\alpha} - \frac{1}{(\Theta^{\alpha})^2} \operatorname{grad} \Theta^{\alpha} \cdot \mathbf{q}^{\alpha} \right) \mathrm{dv}
\end{aligned}$$

The reformulated local form of the balance of energy can written as:

$$\rho^{\alpha}\,\mathrm{r}^{\alpha} \;=\; -\,\rho^{\alpha}\,(\varepsilon^{\alpha})^{'}_{\alpha} \,+\, \mathbf{T}^{\alpha}\,\cdot\,\mathbf{D}_{\alpha} \,-\, \mathrm{div}\,\mathbf{q}^{\alpha} \,-\, \hat{\mathrm{e}} \,+\, \hat{\mathbf{p}}^{\alpha}\,\cdot\,\mathbf{x}^{'}_{\alpha} \,+ \\ \hat{\rho}^{\alpha}\,(\varepsilon^{\alpha} \,+\, \frac{1}{2}\,\mathbf{x}^{'}_{\alpha}\,\cdot\,\mathbf{x}^{'}_{\alpha})$$

Further if the *Helmholtz free energy function* $\psi^{\alpha} = \varepsilon^{\alpha} - \Theta^{\alpha}\,\eta^{\alpha}$ and it's material time derivative $(\psi^{\alpha})^{'}_{\alpha} = (\varepsilon^{\alpha})^{'}_{\alpha} - (\Theta^{\alpha})^{'}_{\alpha}\,\eta^{\alpha} - \Theta^{\alpha}\,(\eta^{\alpha})^{'}_{\alpha}$, then we get the alternative expression for the Entropy inequality is:

$$\sum_{\alpha=1}^{m}\,\frac{1}{\Theta^{\alpha}}\,[-\,\rho^{\alpha}\,\left((\psi^{\alpha})^{'}_{\alpha} \,+\, (\Theta^{\alpha})^{'}_{\alpha}\,\eta^{\alpha}\right) \,-\, \hat{\rho}^{\alpha}\,(\psi^{\alpha} \,+\, \frac{1}{2}\,\mathbf{x}^{'}_{\alpha}\,\cdot\,\mathbf{x}^{'}_{\alpha}) \,+ \\ \mathbf{T}^{\alpha}\,\cdot\mathbf{D}_{\alpha} \,-\, \hat{\mathbf{p}}^{\alpha}\,\cdot\,\mathbf{x}^{'}_{\alpha} \,-\, \frac{1}{\Theta^{\alpha}}\,\mathbf{q}^{\alpha}\,\cdot\mathrm{grad}\,\Theta^{\alpha} \,+\, \hat{\mathrm{e}}^{\alpha}] \,\geq\, 0$$

# 4 Incompressible binary model

Assumptions used are,

- Since fluid in porous material is less dynamic effects can be neglected, i.e., $\mathbf{x}^{''}_{\alpha} = 0$
- Local temperatures of all the constituents $\phi_{\alpha}$ are equal, i.e., $\theta^{\alpha} = 0$
- Both solid and liquid are incompressible, i.e., $\rho^{SR} = const$ , $\rho^{LR} = const$
- No external heat supply, i.e., $\rho^{\alpha} r^{\alpha} = 0$
- No mass exchange between solid and liquid phases, i.e., $\hat{\rho}^{\alpha} = \hat{\rho}^{\alpha} = 0$

## 4.1 Field equations

Considering the above mentioned assumptions
*Balance equation of mass* for solid,

$$(n^{S})^{'}_{S} \,+\, n^{S} div\,\mathbf{x}^{'}_{S} = 0$$

for liquid,

$$(n^{L})^{'}_{L} \,+\, n^{L} div\,\mathbf{x}^{'}_{L} = 0$$

*Balance equation of momentum* for mixture,

$$div\,\mathbf{T}^{SL} \,+\, \rho^{SL}\,\mathbf{b} = \mathbf{0}$$

for liquid,

$$div\,\mathbf{T}^{L} \,+\, \rho^{L}\,\mathbf{b} = \mathbf{0}$$

*Balance equation of energy* for mixture,

$$\rho^{S}(\epsilon^{S})^{'}_{S} \,+\, \rho^{L}(\epsilon^{L})^{'}_{L} \,-\, \mathbf{T}^{S}\cdot\mathbf{D}_{S} \,-\, \mathbf{T}^{L}\cdot\mathbf{D}_{L} \,+\, div\,\mathbf{q}^{SL} \;=\; -\,\hat{\mathbf{p}}^{L}\cdot\mathbf{w}_{LS}$$

*Time derivative of saturation condition*,

$$(n^S)'_S + (n^L)'_L - grad\, n^L \cdot \mathbf{w}_{LS} = 0$$

where,

$$\mathbf{T}^{SL} = \mathbf{T}^S + \mathbf{T}^L \; ; \; \rho^{SL} = \rho^S + \rho^L \; ; \; \mathbf{q}^{SL} = \mathbf{q}^S + \mathbf{q}^L$$

and the velocity of liquid relative to solid,

$$\mathbf{w}_{LS} = \mathbf{x}'_L - \mathbf{x}'_S$$

using the relation $\epsilon^\alpha = \psi^\alpha + \theta\,\eta^\alpha$ and local statement of balance equation of mass, *balance equation of energy and the time derivative of saturation condition* can be reformulated as

$$\begin{gathered} \rho^S(\psi^S)'_S + \rho^L(\psi^L)'_L + \theta\,[\rho^S(\eta^S)'_S + \rho^L(\eta^L)'_L] + \\ +\theta'_S\,[\rho^S\eta^S + \rho^L\eta^L] + \rho^L\eta^L\, grad\theta \cdot \mathbf{w}_{LS} - \\ -\mathbf{T}^S \cdot \mathbf{D}_S - \mathbf{T}^L \cdot \mathbf{D}_L + div\,\mathbf{q}^{SL} = -\,\hat{\mathbf{p}}^L \cdot \mathbf{w}_{LS} \end{gathered}$$

and

$$div(n^L\mathbf{w}_{LS} + \mathbf{x}'_S) = 0$$

## 4.2 Constitutive theory

In this model, material time derivative of saturation condition is an equation in excess that restricts the motion of incompressible constituents. So to account for this a reaction force must be assigned in unknown field quantities in form of a Lagrange multiplier $\lambda$,

$$\mathfrak{U} = \{\chi_S\,,\,\chi_L\,,\,\theta\,,\,n_S\,,\,n_L\,,\,\lambda\}$$

the known field,

$$\mathfrak{K} = \{\mathbf{b}\}$$

the constitutive relations are formed for the field quantities,

$$\mathfrak{C} = \{\mathbf{T}^S\,,\,\mathbf{T}^L\,,\,\mathbf{q}^{SL}\,,\,\psi^S\,,\,\psi^L\,,\,\eta^S\,,\,\eta^L\,,\,\hat{\mathbf{P}}^L\}$$

For liquid saturated porous solid,

$$\mathfrak{P} = \{\theta\,,\,grad\,\theta\,,\,\mathbf{C}_S\,,\,n^L\,,\,\mathbf{w}_{LS}\,,\,grad\,n^L\}$$

Here viscous effects are not considered, $\mathbf{C}_S = \mathbf{F}_S^T\mathbf{F}_S$ represents the right Cauchy Green stress tensor. By saturation condition and balance of mass

$$\mathrm{J}_S = \sqrt{det\mathbf{C}_S} = \frac{n^S_{0S}}{n^S}$$

where $n^S_{0S}$ is the volume fraction of solid in reference configuration. This means volume fraction of solid is expressed by $\mathbf{C}_S$.

# 5 Entropy inequality

For the above mentioned assumptions and considered binary model along with local temperature of phases and the production terms of balance equations, the entropy inequality results in,

$$\begin{aligned}\rho^S\,(\psi^S)'_S\,-\,(\psi^L)'_L\,-\,\theta'_S\left[\rho^S\,\eta^S\,+\,\rho^L\,\eta^L\right]\,-\,\rho^L\,\eta^L\,grad\,\theta\,\cdot\,\mathbf{w}_{LS}+\\ +\,\mathbf{T}^S\,\cdot\,\mathbf{D}_S\,+\,\mathbf{T}^L\,\cdot\,\mathbf{D}_L\,-\,\frac{1}{\theta}\,\mathbf{q}^{SL}\,\cdot\,grad\,\theta\,-\,\hat{\mathbf{P}}^L\,\cdot\,\mathbf{w}_{LS}\,-\\ -\,\lambda\big[(n^S)'_S\,+\,(n^S)'_S\,-\,grad\,n^L\,\cdot\,\mathbf{w}_{LS}\big]\,\geq\,0\end{aligned}$$

Considering the dependence of tr free Helmoltz energy

$$\psi^S\,=\,\psi^S\left(\theta\,,\,\mathbf{C}_S\right)\quad,\quad\psi^L\,=\,\psi^L\left(\theta\,,\,n^L\right)$$

time derivative of these following their respective motions, their partial derivatives,

$$\rho^S\,(\psi^S)'_S\,=\,\rho^S\,\frac{\partial\psi^S}{\partial\theta}\,+\,2\,\rho^S\,\mathbf{F}_S\,\frac{\partial\psi^S}{\partial\mathbf{C}_S}\,\mathbf{F}_S^T\,\cdot\,\mathbf{D}_S$$

$$\rho^L\,(\psi^L)'_L\,=\,\rho^L\,\frac{\partial\psi^L}{\partial\theta}\left(\theta'_S\,+\,grad\,\theta\,\cdot\,\mathbf{w}_{LS}\right)+\,\rho^L\,\frac{\partial\psi^L}{\partial n_L}\,(n_L)'_L$$

To eliminate the dependencies of process variables $\mathbf{C}_S$ and $n^S$ as well as $n^L$ and their rates in space and time, local statement of balance of mass along with Lagrange multipliers.

$$\lambda^S_{MB}\big[(n^S)'_S\,+\,n^S\left(\mathbf{D}_S\,\cdot\,\mathbf{I}\right)\big]\,=\,0$$

$$\lambda^L_{MB}\big[(n^L)'_L\,+\,n^L\left(\mathbf{D}_L\,\cdot\,\mathbf{I}\right)\big]\,=\,0$$

$\therefore$ considering the above equations along with entropy inequality for binary model constitutive relations and dissipation mechanism can be formulated as,

$$\begin{aligned}-\theta'_S\big\{\rho^S\,\eta^S\,+\,\rho^L\,\eta^L\,+\,\rho^S\,\frac{\partial\psi^S}{\partial\theta}\,+\,\rho^L\,\frac{\partial\psi^L}{\partial\theta}\big\}+\\ +\,\mathbf{D}_S\,\cdot\,\big\{\mathbf{T}^S\,-\,2\,\rho^S\,\mathbf{F}_S\,\frac{\partial\psi^S}{\partial\mathbf{C}_S}\,\mathbf{F}_S^T\,+\,n^S\,\lambda^S_{MB}\,\mathbf{I}\big\}+\\ \mathbf{D}_L\,\cdot\,\big\{\mathbf{T}^L\,+\,n^L\,\lambda^L_{MB}\,\mathbf{I}\big\}-\\ -\,(n^S)'_S\,\big\{\lambda\,-\,\lambda^L_{MB}\big\}-\\ -\,(n^L)'_L\,\big\{\lambda\,-\,\lambda^L_{MB}\,+\,\rho^L\,\frac{\partial\psi^L}{\partial n_L}\big\}-\\ -\,\mathbf{w}_{LS}\,\cdot\,\big\{\hat{\mathbf{P}}^L\,-\,\lambda\,grad\,n^L\,+\,\rho^L\,n^L\,grad\,\theta\,+\,\rho^L\,\frac{\partial\psi^L}{\partial\theta}\,grad\,\theta\big\}-\\ -\,grad\,\theta\,\cdot\,\big\{\frac{1}{\theta}\,\mathbf{q}^{SL}\big\}\,\geq\,0\end{aligned}$$

To satisfy the entropy inequality

$$\rho^S\,\eta^S\,+\,\rho^L\,\eta^L\,+\,\rho^S\,\frac{\partial\psi^S}{\partial\theta}\,+\,\rho^L\,\frac{\partial\psi^L}{\partial\theta}\,=\,0$$

$$\mathbf{T}^S \,-\, 2\,\rho^S\,\mathbf{F}_S\,\frac{\partial\psi^S}{\partial\mathbf{C}_S}\,\mathbf{F}_S^T \,+\, n^S\,\lambda_{MB}^S\,\mathbf{I} \,=\, 0$$
$$\mathbf{T}^L \,+\, n^L\,\lambda_{MB}^L\,\mathbf{I} \,=\, 0$$
$$\lambda \,-\, \lambda_{MB}^L \,=\, 0$$
$$\lambda \,-\, \lambda_{MB}^L \,+\, \rho^L\,\frac{\partial\psi^L}{\partial n_L} \,=\, 0$$

the dissipation mechanism

$$\mathrm{D} \,=\, -\,\mathbf{w}_{LS}\cdot\big\{\hat{\mathbf{P}}^L - \lambda\,grad\,n^L + \rho^L\,n^L\,grad\,\theta + \rho^L\,\frac{\partial\psi^L}{\partial\theta}\,grad\,\theta\big\} - grad\,\theta\cdot\big\{\frac{1}{\theta}\,\mathbf{q}^{SL}\big\} \,\geq\, 0$$

Constitutive relations for the effective stresses tensor of solid and liquid phases,

$$\mathbf{T}_E^S \,=\, 2\,\rho^S\,\mathbf{F}_S\,\frac{\partial\psi^S}{\partial\mathbf{C}_S}\,\mathbf{F}_S^T \,=\, \frac{1}{\mathrm{J}_S}\big[2\,\mu^S\,\mathbf{K}_S \,+\, \lambda^S(log\,\mathrm{J_S})\,\mathbf{I} \,-\, 3\,\alpha_\theta^S\,k^S(\theta \,-\, \theta_0)\,\mathbf{I}\big]$$
$$\mathbf{T}_E^L \,=\, \mathbf{0}$$

## 5.1 Reformulation of balance of energy

The Helmoltz free energy for the solid and liquid phases according to the assumptions of this model.

$$\rho^S\,(\psi^S)_S' \,=\, \rho^S\,\frac{\partial\psi^S}{\partial\theta}\,\theta_S' \,+\, 2\,\rho^S\,\mathbf{F}_S\,\frac{\partial\psi^S}{\partial\mathbf{C}_S}\,\mathbf{F}_S^T\,\cdot\,\mathbf{D}_S$$
$$\rho^L\,(\psi^L)_L' \,=\, \rho^L\,\frac{\partial\psi^L}{\partial\theta}\,\big(\theta_S' \,+\, grad\theta\,\cdot\,\mathbf{w}_{LS}\big)$$

inserting these equations in balance of energy yields

$$n^S\,\lambda\,(\mathbf{D}_S\,\cdot\,\mathbf{I}) \,+\, n^L\,\lambda\,(\mathbf{D}_L\,\cdot\,\mathbf{I}) \,+\, \theta\,\big[\rho^S\,(\eta^S)_S' \,+\, \rho^L\,(\eta^L)_L'\big] \,+\, div\,\mathbf{q}^{SL} \,=\, \hat{\mathbf{p}}^L\,\cdot\,\mathbf{w}_{LS}$$

material time derivative of saturation condition along the trajectory of solid, weighted with Lagrange multiplier gives,

$$n^S\,\lambda\,(\mathbf{D}_S\,\cdot\,\mathbf{I}) \,+\, n^L\,\lambda\,(\mathbf{D}_L\,\cdot\,\mathbf{I}) \,=\, -\,\lambda\,grad\,n^L\,\cdot\,\mathbf{w}_{LS}$$

Thus the balance of energy results as,

$$\theta\,\big[\rho^S\,(\psi^S)_S' \,+\, \rho^L\,(\psi^L)_L'\big] \,+\, div\,\mathbf{q}^{SL} \,=\, -\,\big[\mathbf{p}^L \,+\, grad\,n^L\big]\,\cdot\,\mathbf{w}_{LS}$$

# 6 Solution scheme

For efficient solution algorithm, the set of field equations must be reduced,

- time integration of mass conservation of solid.

$$n^S = \mathrm{J}_S^{-1}\, n_{OS}$$

$\therefore$ from saturation condition,

$$n^L = 1 - n^S$$

- equation of seepage velocity, using the balance equation of momentum for the liquid along with production term of balance equation of momentum for liquid,

$$n^L\,\mathbf{w}_{LS} = -\frac{(n^L)^2}{\gamma^L_{\mathbf{w}_{LS}}}\left[grad\lambda - \rho^{LR}\,\mathbf{b} + \frac{1}{n^L}\left(grad\,P^L_E + \gamma^L_{\nabla\theta}\,grad\theta\right)\right]$$

here $P^L_E = -\frac{1}{3}\left(\mathbf{T}^L_E \cdot \mathbf{I}\right)$ is the effective hydrolic pressure of liquid phase.

If $\mathbf{T}^L_E = 0$

$$n^L\,\mathbf{w}_{LS} = -\frac{(n^L)^2}{\gamma^L_{\mathbf{w}_{LS}}}\left[grad\lambda - \rho^{LR}\,\mathbf{b} + \frac{1}{n^L}\,\gamma^L_{\nabla\theta}\,grad\theta)\right]$$

$\therefore$ the field equations for solving the model are, *Balance equation of momentum fro mixture*

$$div\,\mathbf{T}^{SL} + \rho^{SL}\,\mathbf{b} = \mathbf{0}$$

*Balance equation of energy for the mixture*

$$\begin{gathered}\rho^S(\psi^S)'_S + \rho^L(\psi^L)'_L + \theta\,[\rho^S(\eta^S)'_S + \rho^L(\eta^L)'_L] + \\ +\theta'_S\,[\rho^S\eta^S + \rho^L\eta^L] + \rho^L\eta^L\,grad\theta \cdot \mathbf{w}_{LS} - \\ -\mathbf{T}^S \cdot \mathbf{D}_S - \mathbf{T}^L \cdot \mathbf{D}_L + div\,\mathbf{q}^{SL} = -\,\hat{\mathbf{p}}^L \cdot \mathbf{w}_{LS}\end{gathered}$$

*Material time derivative of the saturation condition*

$$div(n^L\mathbf{w}_{LS} + \mathbf{x}'_S) = 0$$

## 6.1 Weak from of balance of momentum

We know from transport theorem,

$$\mathrm{dv} = \mathrm{J}_S\,\mathrm{dV} \quad , \quad \mathrm{da} = \mathrm{J}_S^{-1}\,\mathbf{F}_S^{T-1}\,\mathbf{n}_{0S}\,\mathrm{dA}$$

Weak form of balance of momentum for the mixture in reference configuration is given by,

$$\int_{B_{0S}}\left[\mathbf{P}^{SL} \cdot Grad\delta\,\mathbf{u}_S - \mathrm{J}_S\,\rho^{SL}\,\mathbf{b} \cdot \delta\,\mathbf{u}_S\right]\mathrm{dV}_{0S} = \int_{\partial B_{0S}} P^{SL} \cdot \delta\,\mathbf{u}_S\,\mathrm{dA}_{0S}$$

where $P^{SL} = \mathbf{P}^{SL}\,\mathbf{n}_{0S}$ and the $\delta\,\mathbf{u}_S$ is the vector weighting quantity.
$\mathbf{P}^{SL} = \mathrm{J}_S\,\mathbf{T}^{SL}\,\mathbf{F}_S^{T-1} = \lambda\,\mathbf{F}_S^{T-1} + \mathbf{P}^{SL}_E$ here $P^{SL}$ is the first Piola-Krichhoff stress tensor and $\mathbf{P}^{SL}_E$ is the effective first Piola-Krichhoff stress tensor.

## 6.2 Weak from of balance of energy

Global form of weak form of energy can be obtained from constitutive relations of specific entropy, production term of balance equation of momentum and constitutive relations of effective stresses,

$$\int_{B_{0S}} \Big\{ - \left( \rho^S_{0S}\, \mathrm{c}^S \,+\, \mathrm{J}_S\, \rho^L\, \mathrm{c}^L \right) \theta^{'}_S \,-\, \mathrm{J}_S\, \rho^L\, \mathrm{c}^L\, \mathbf{F}^{T-1}_S\, Grad\theta \,\cdot\, \mathbf{w}_{LS} - \\ - 3\, \alpha^S_\theta\, \theta\, \mathrm{k}^S\, \mathbf{C}^{-1}_S \,\cdot\, (\mathbf{E}_S)^{'}_S \,-\, \mathrm{J}_S\, \hat{P}^L_E \,\cdot\, \mathbf{w}_{LS} \Big\}\, \delta\theta\, \mathrm{dV}_{0S} + \\ + \int_{B_{0S}} \mathbf{q}^{SL}_{0S} \,\cdot\, Grad\delta\theta\, \mathrm{dV}_{0S} \,=\, \int_{\partial B_{0S}} \delta\theta\, \mathbf{q}^{SL}_{0S} \,\cdot\, \mathbf{n}_{0S}\, \mathrm{dA}_{0S}$$

here $\rho^S_{0S} \;=\; \mathrm{J}^{-1}_S\, \rho^S$ is the partial density of reference configuration solid, $\delta\theta$ is the scalar weighting quantity called virtual temperature and $\mathbf{q}^{SL}_{0S} \;=\; \mathrm{J}_S\, \mathbf{F}^{-1}_S\, \mathbf{q}^{SL}$ is the heat flux of all the constituents in reference to the non-deformed solid.

## 6.3 Weak form of saturation condition

This can be derived by multiplying the local statement of saturation condition with scalar function $\delta\lambda$ over the control space in reference configuration.

$$\int_{B_{0S}} \Big\{ n^L \,\cdot\, \mathrm{w}_{LS_{0S}} \,\cdot\, Grad(\delta\lambda) \,-\, \mathrm{J}\big[ (\mathbf{E}_S)^{'}_S \,\cdot\, \mathbf{C}^{-1}_S \big]\, \delta\lambda \Big\} \mathrm{dV}_{0S} \\ = \int_{\partial B_{0S}} \delta\lambda\, n^L\, \mathbf{w}_{LS_{0S}} \,\cdot\, \mathbf{n}_{0S}\, \mathrm{dA}_{0S}$$

here $\mathbf{w}_{LS_{0S}} \;=\; \mathrm{J}_S\, \mathbf{F}^{-1}_S\, \mathbf{w}_{LS}$

## 6.4 Solution

To solve the boundary value problem, the weak forms are used in standard Gelerkin's procedure,

- Dirichlet or Neumann constraints are taken as boundary conditions.
- Newmark's method is used to time discretization.

General form of linearised weak form,

$$(\delta\mathbf{\Phi})^T\, (D\, \mathbf{K} \,+\, D\, \dot{\mathbf{K}}) \;=\; \delta\mathbf{\Phi})^T\, (\bar{\mathbf{R}} \,+\, \mathbf{R}^{\partial B_{0S}})$$

here $\delta\mathbf{\Phi}^T$ is the vector of unknown variations, $\mathbf{K}$ is the stiffness matrix, $\dot{\mathbf{K}}$ is the damping matrix, $\bar{\mathbf{R}}$ is the matrix of known quantities and $\mathbf{R}^{\partial B_{0S}}$ is the matrix of boundary conditions on the surface of porous media.

Thus the reduced set of unknowns are

$$\mathfrak{K} \;=\; \mathfrak{K}\{\chi\,,\, t\} \;=\; \{\mathbf{u}_S\,,\, \theta\,,\, \lambda\}$$

To solve this set of unknowns balance of momentum, balance of energy and saturation condition are used as,

$$D\mathbf{K} = \begin{bmatrix} M_oM_{\mathbf{u}_s} & M_oM_\theta & M_oM_\lambda \\ E_nM_{\mathbf{u}_s} & E_nM_\theta & E_nM_\lambda \\ S_aM_{\mathbf{u}_s} & S_aM_\theta & S_aM_\lambda \end{bmatrix} \begin{bmatrix} \Delta\mathbf{u}_S \\ \Delta\theta \\ \Delta\lambda \end{bmatrix}$$

$$D\dot{\mathbf{K}} = \begin{bmatrix} 0 & 0 & 0 \\ E_nM_{(\mathbf{u}_s)'_S} & E_nM_{(\theta)'_S} & 0 \\ S_aM_{(\mathbf{u}_s)'_S} & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\mathbf{u}_S \\ \Delta\theta \\ \Delta\lambda \end{bmatrix}$$

$$\mathbf{R}^{\partial B_{0S}} = \begin{bmatrix} R^{M_oM} \\ R^{E_nM} \\ R^{S_aM} \end{bmatrix}$$

The boundary condition acting on the surface $\partial B_{0S}$

$$R^{S_aM} = n^L\,\mathbf{w}_{LS_{0S}} \cdot \mathbf{n}_{0S}$$

$$R^{E_nM} = \mathbf{q}_{0S}^{SL} \cdot \mathbf{n}_{0S}$$

$$R^{M_oM} = (-\,\mathrm{J}_S\,\lambda\,\mathbf{F}_S^{T-1} + \mathbf{P}_E^{SL})\,\mathbf{n}_{0S}$$

# 7 Examples

## 7.1 1-D example

This is an example of a 1-D element with two nodes and fixed at one end. To describe the behavior of different force and displacements applied on the element using local statement of balance of mass, the weak form of balance of momentum and the saturation condition.

Figure 2: 1-D model

### 7.1.1 Case 1.1

In this case a displacement is applied on the second node. We can see that the displacement is 0.1 m and the velocity and the acceleration as shown below,

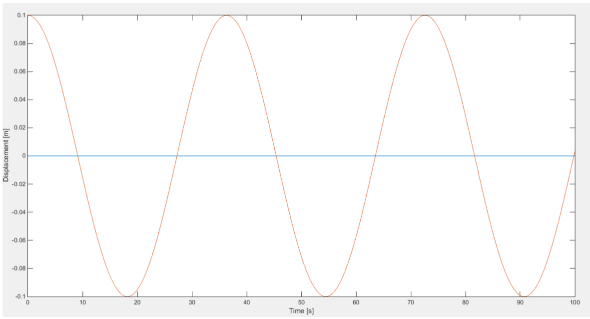

Figure 3: Displacement vs time

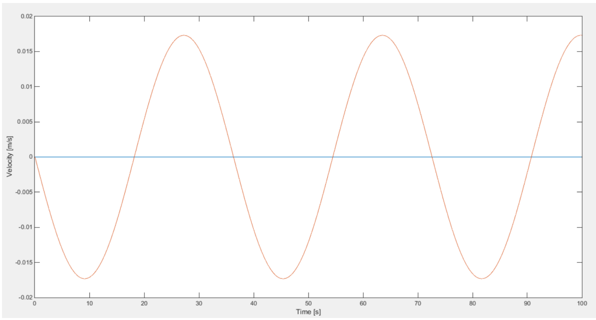

Figure 4: Velocity vs time

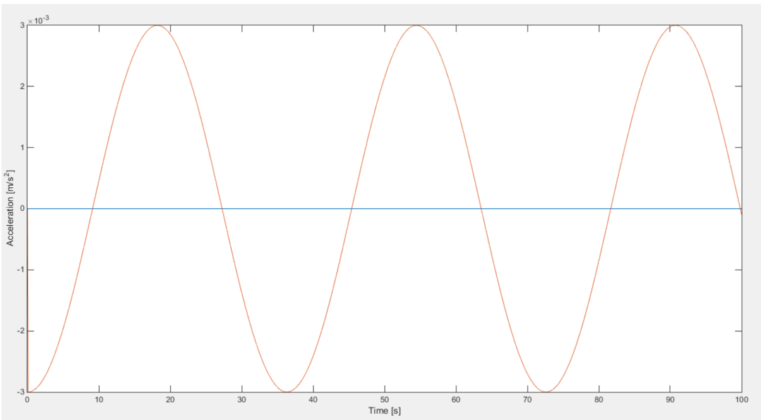

Figure 5: Acceleration vs time

It is seen that the initial displacement of 0.1m is applied outward on node 2 and the difference velocity is negative as the water is flowing into the model and initial acceleration is also negative.

### Case 1.2

Here along with the displacement at node 2, weak form of balance of momentum is considered with $\kappa_{Darcy} = 0.001$

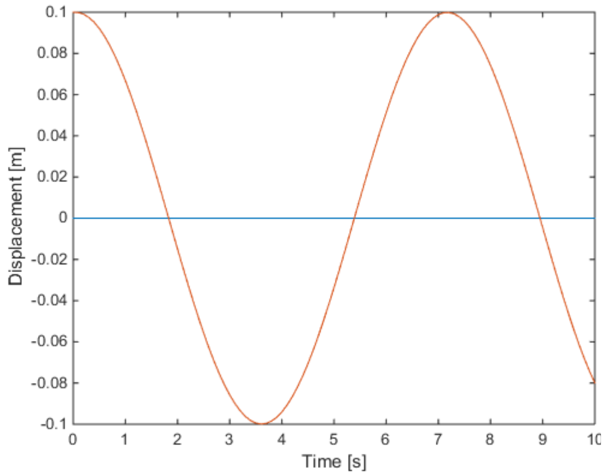


Figure 6: Displacement vs time

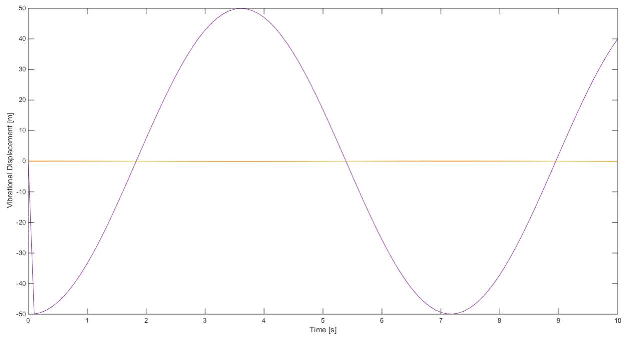

Figure 7: Vibrational Displacement vs time

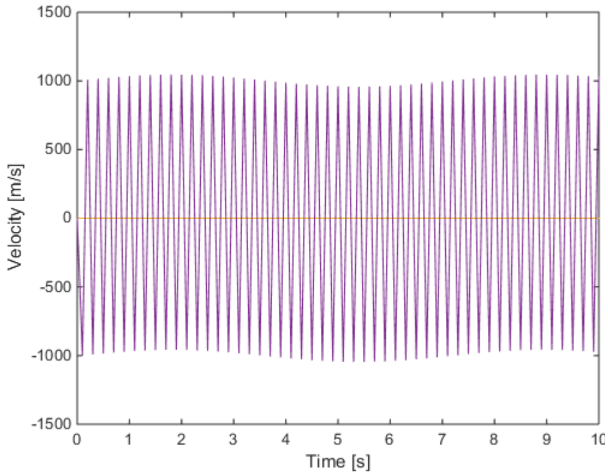


Figure 8: Velocity vs time

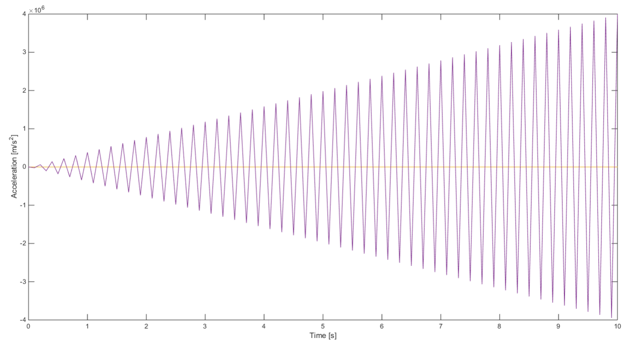

Figure 9: Acceleration vs time

As it can be seen that the initial difference velocity and acceleration still negative. Also the behavior of velocity and acceleration are changing as predicted.

### 7.1.2 Case 2.1

Here a force of 10N is applied on the second node with no displacement.

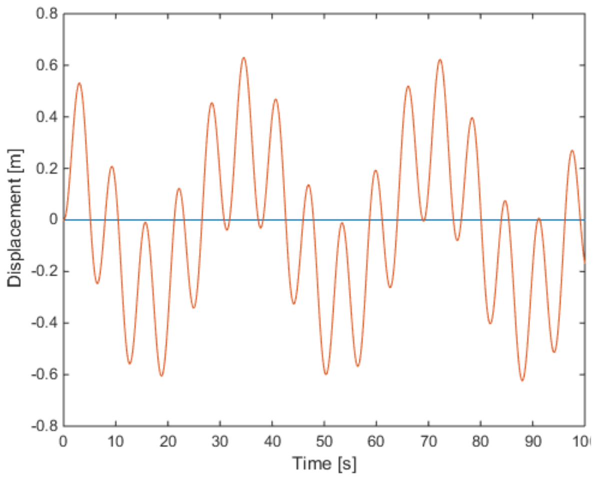


Figure 10: Displacement vs time

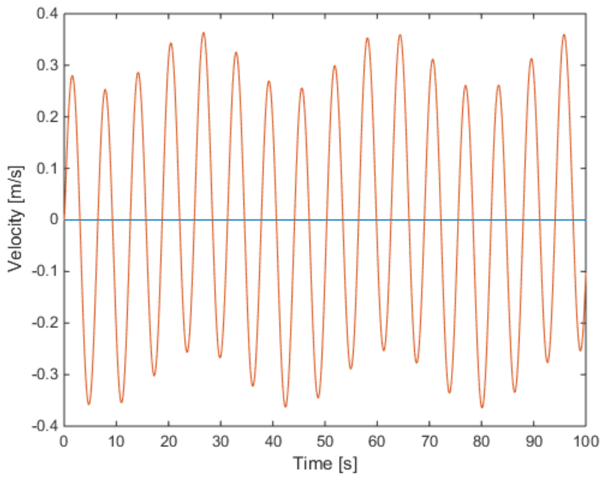


Figure 11: Velocity vs time

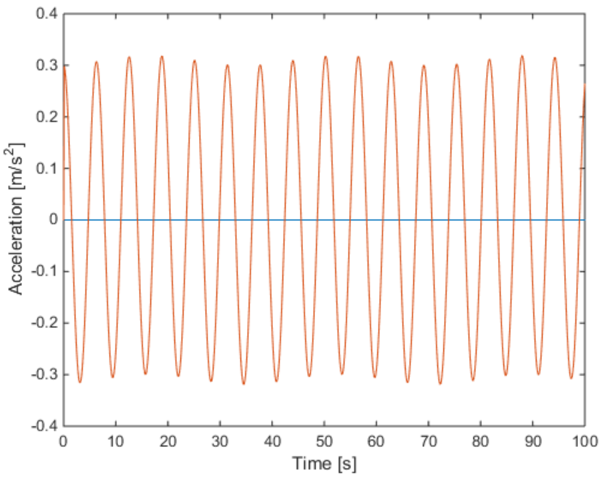


Figure 12: Acceleration vs time

It is seen that the effect of the force is according to the applied Cos function. It can be seen that the water is flowing out of the model, also the velocity and acceleration is according to the prediction.

### Case 2.2

Here along with force of 10N on the second node along with the balance of momentum with $\kappa_{Darcy} = 0.001$

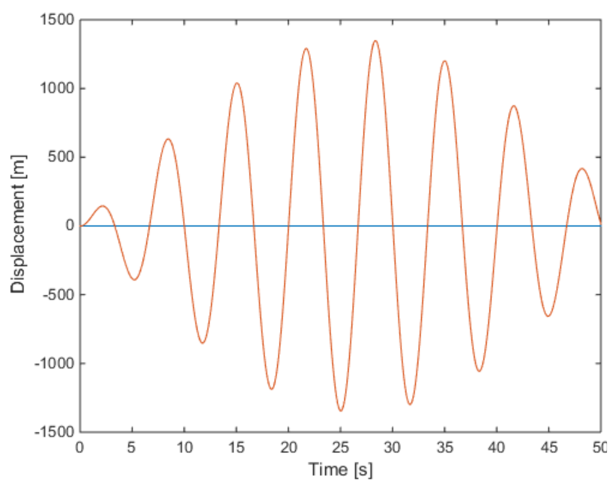


Figure 13: Displacement vs time

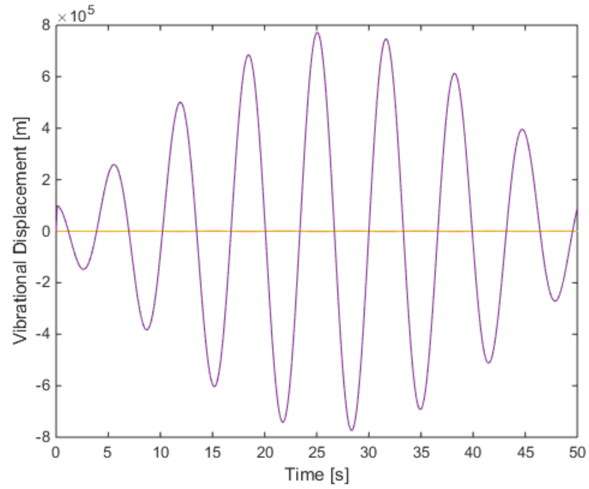


Figure 14: Vibrational Displacement vs time

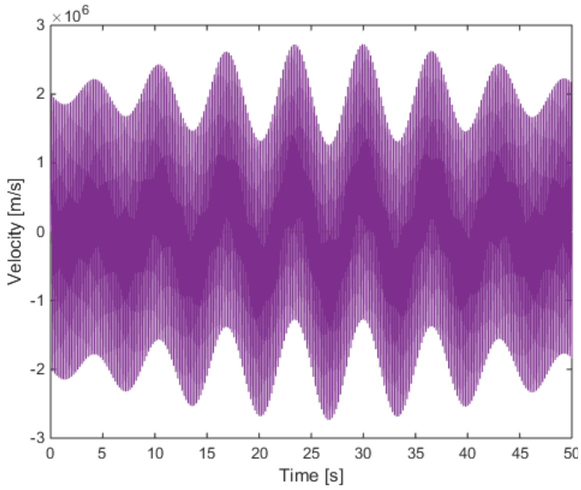


Figure 15: Velocity vs time

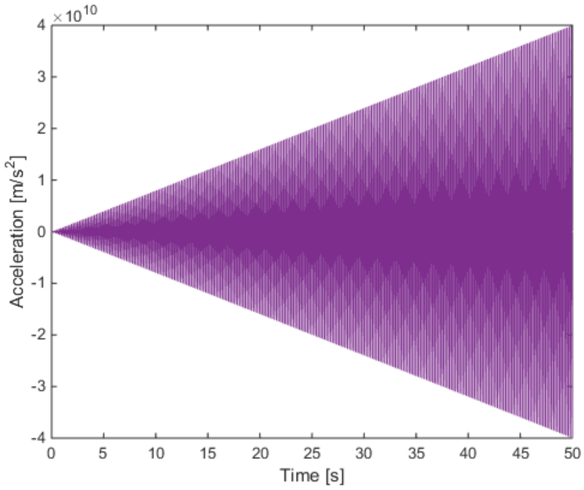


Figure 16: Acceleration vs time

We can see the difference in behavior of displacement, velocity and acceleration between the cases 1 and 2. The force is applied from 0 to a maximum in a gradually increasing and then it is damped gradually. So is the velocity. The acceleration amplitude increases gradually.

### 7.1.3 Case 3.1

Here along with the displacement of 0.1m at the second node a damping factor of 10 is considered.

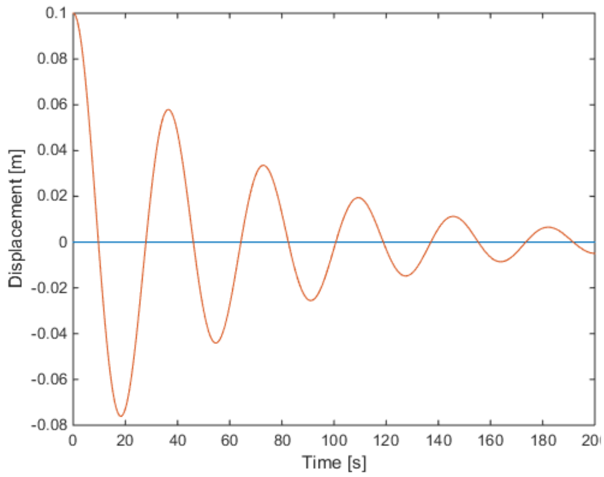


Figure 17: Displacement vs time

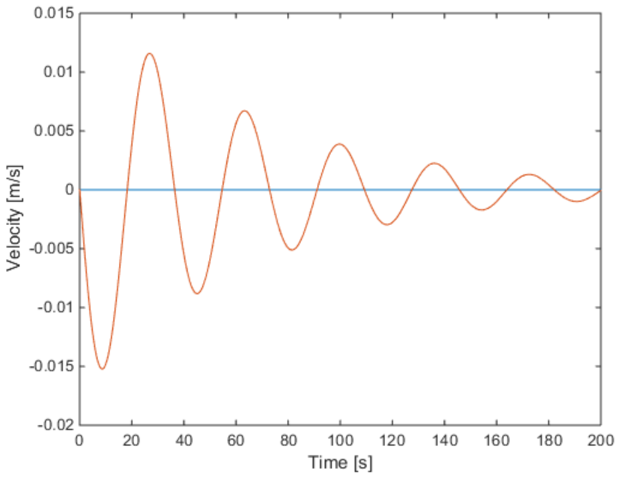


Figure 18: Velocity vs time

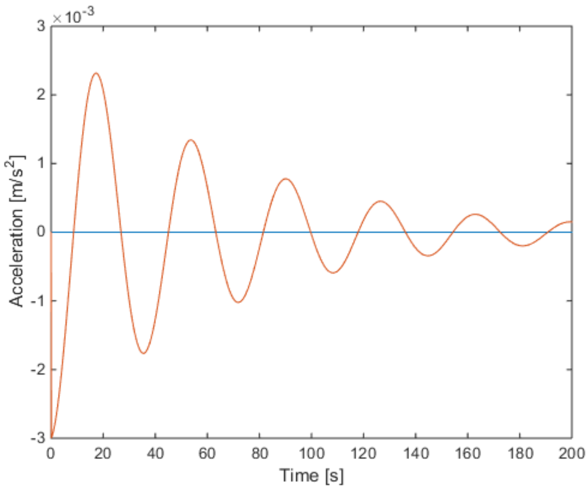


Figure 19: Acceleration vs time

Displacement, velocity and acceleration are damped gradually to a minimum value as seen in the graphs.

### Case 3.2

Here along with the initial displacement, a damping factor of 10 is considered and balance of momentum with $\kappa_{Darcy} = 0.001$

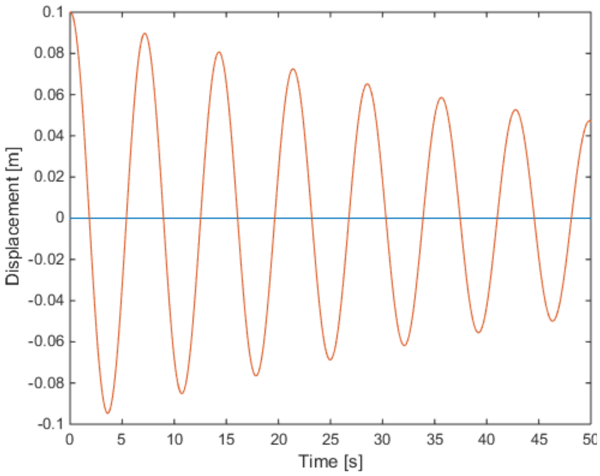


Figure 20: Displacement vs time

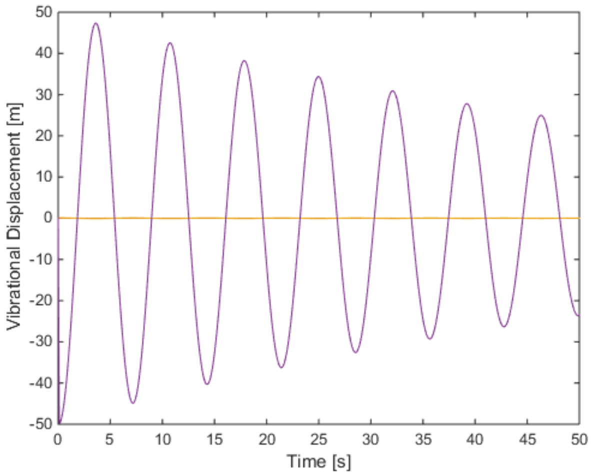


Figure 21: Vibrational Displacement vs time

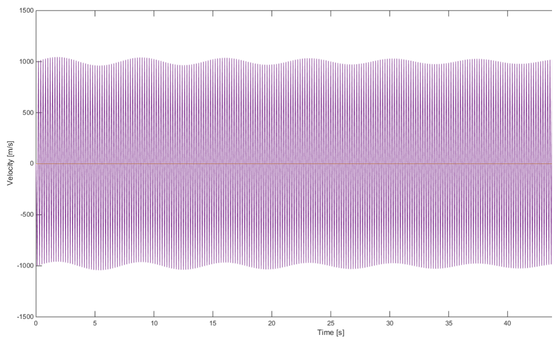

Figure 22: Velocity vs time

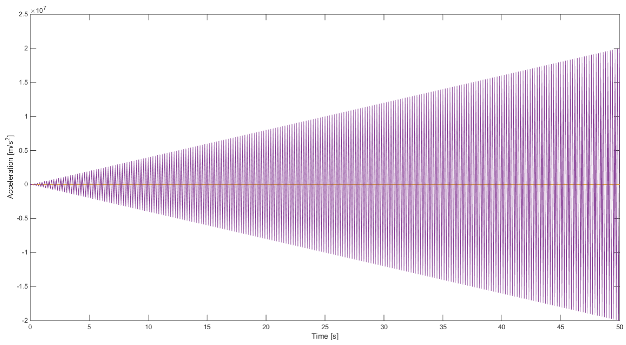

Figure 23: Acceleration vs time

It is seen that the initial displacement of 0.1m is applied on node 2 and the velocity is negative and initial acceleration is negative. Also the displacement, velocity and acceleration are damped gradually as seen in the graphs.

## 7.2 2-D Example

### 7.2.1 Container

This is a 2-D model to describe the behavior of multiphase model with bottom nodes fixed and one side is fixed along x-axis, i.e., free to move along y-axis as shown in figure below,

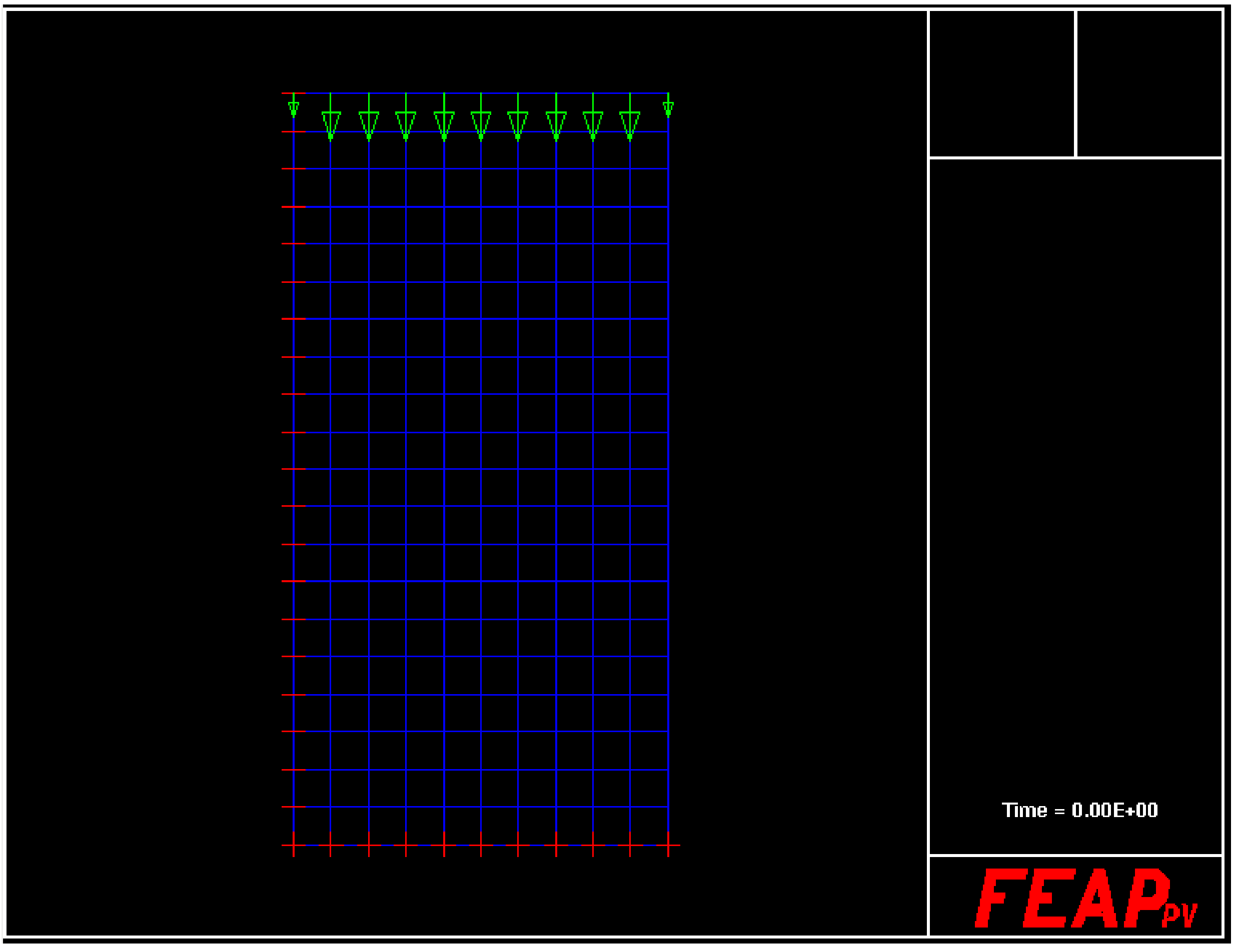


Figure 24: 2-D model

After the load is applied for 24 seconds the results of the simulation are shown below,

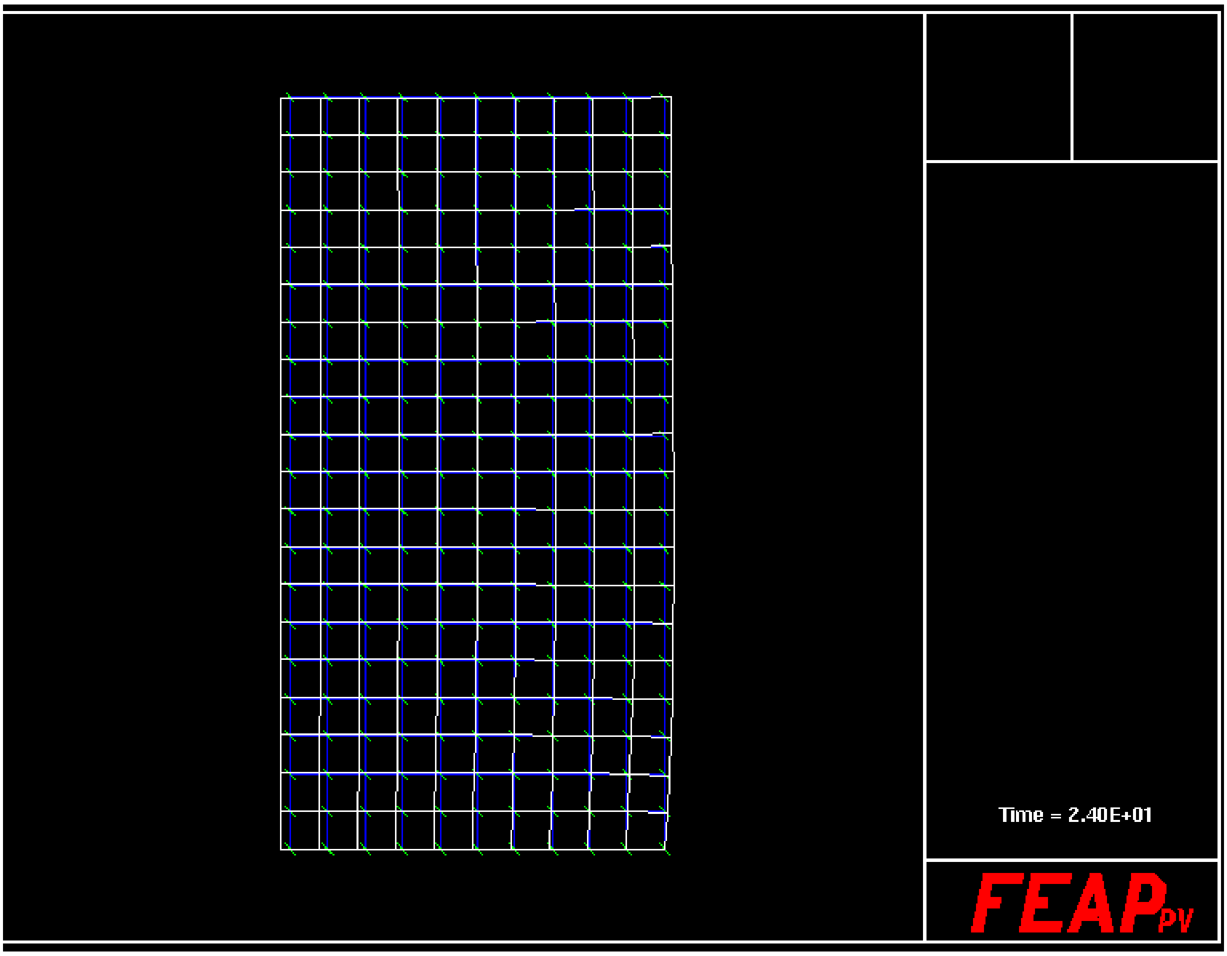


Figure 25: Deformed mesh

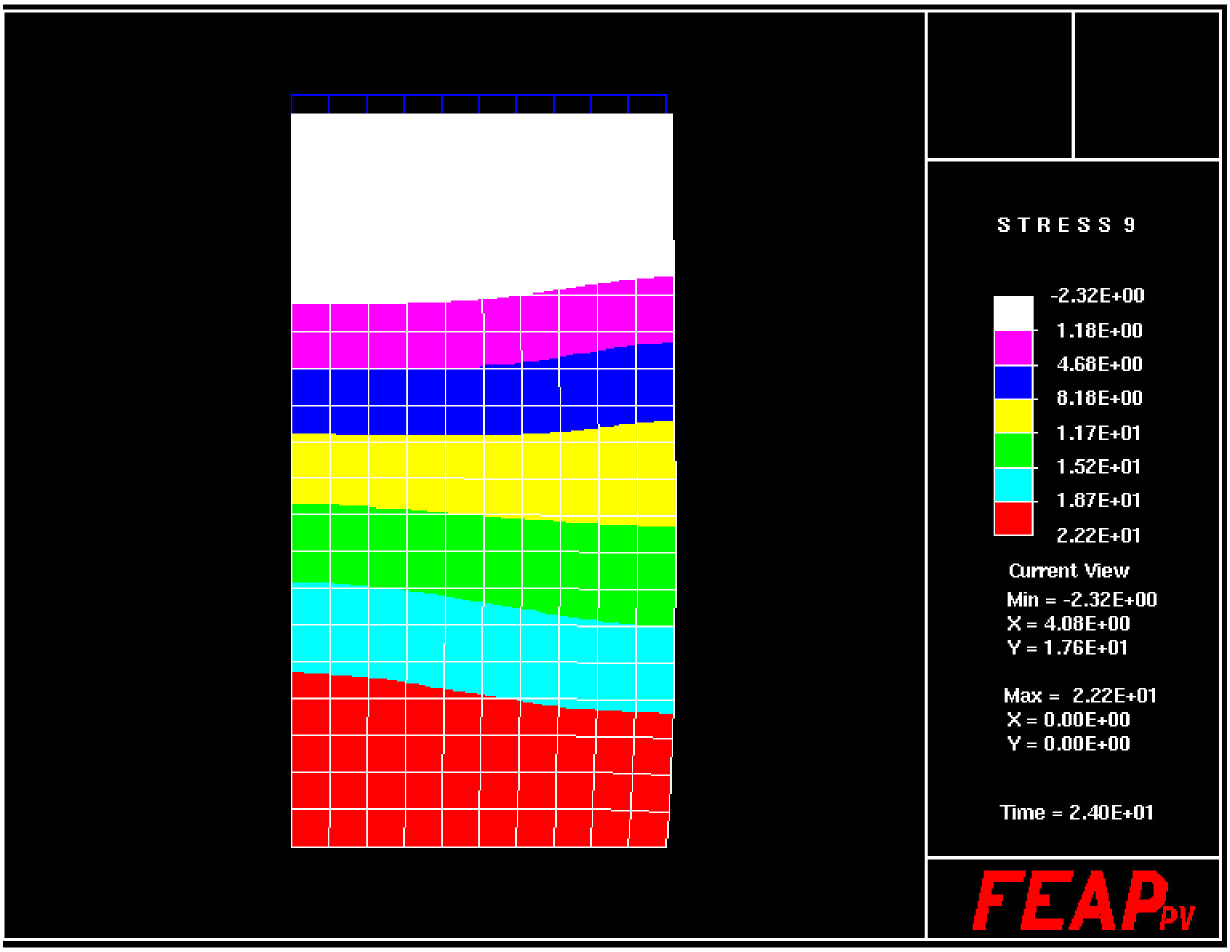


Figure 26: Pressure distribution in the container

It can be seen from the results that, the water flowing out of the container from the side of the container as there is no boundary conditions an the right side of the container.

### 7.2.2 U-Structure

This is an example of U-structure to describe the behavior of multiphase model with boundary conditions as shown in the figure below,

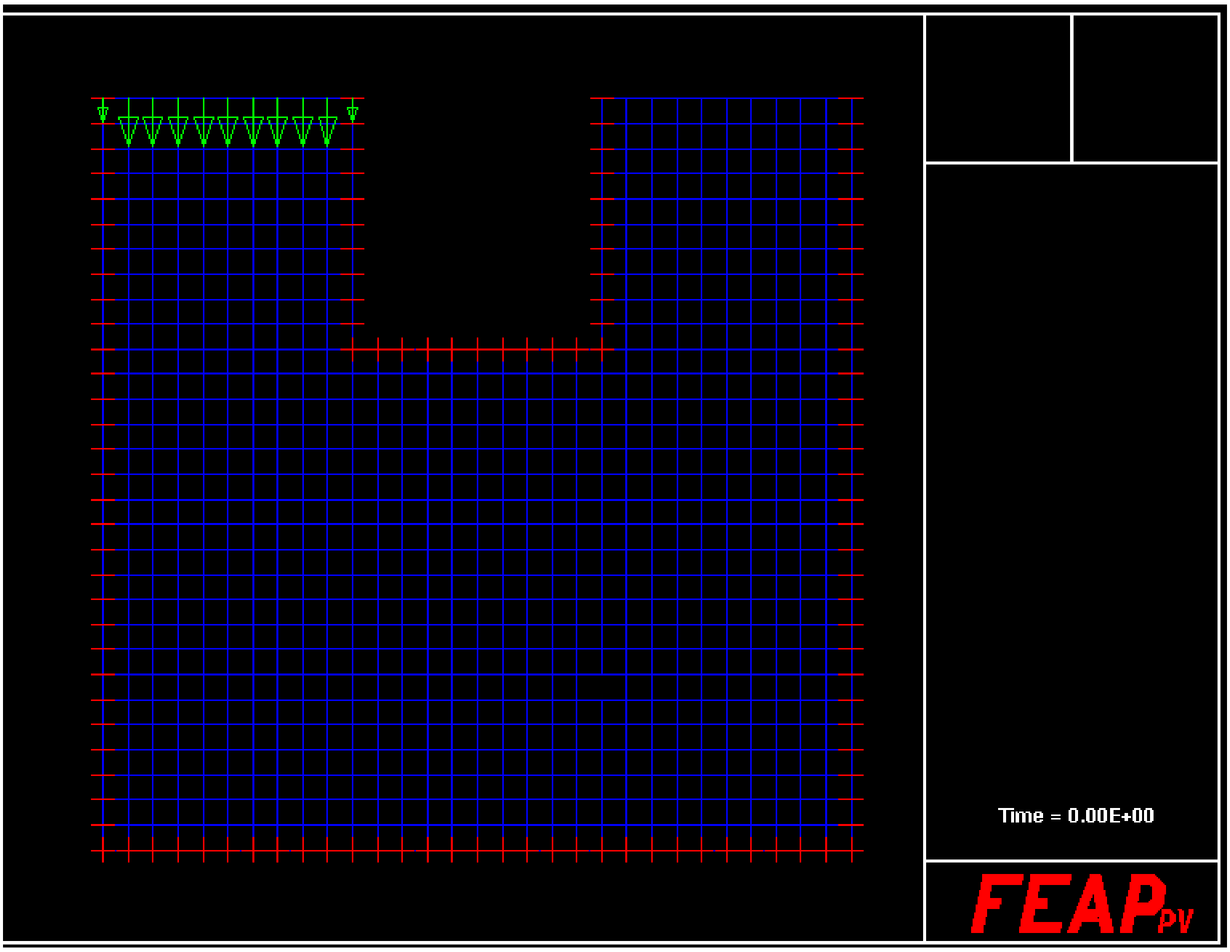


Figure 27: U-Structure

After the load is applied for 3 seconds the results of the simulation are shown below,

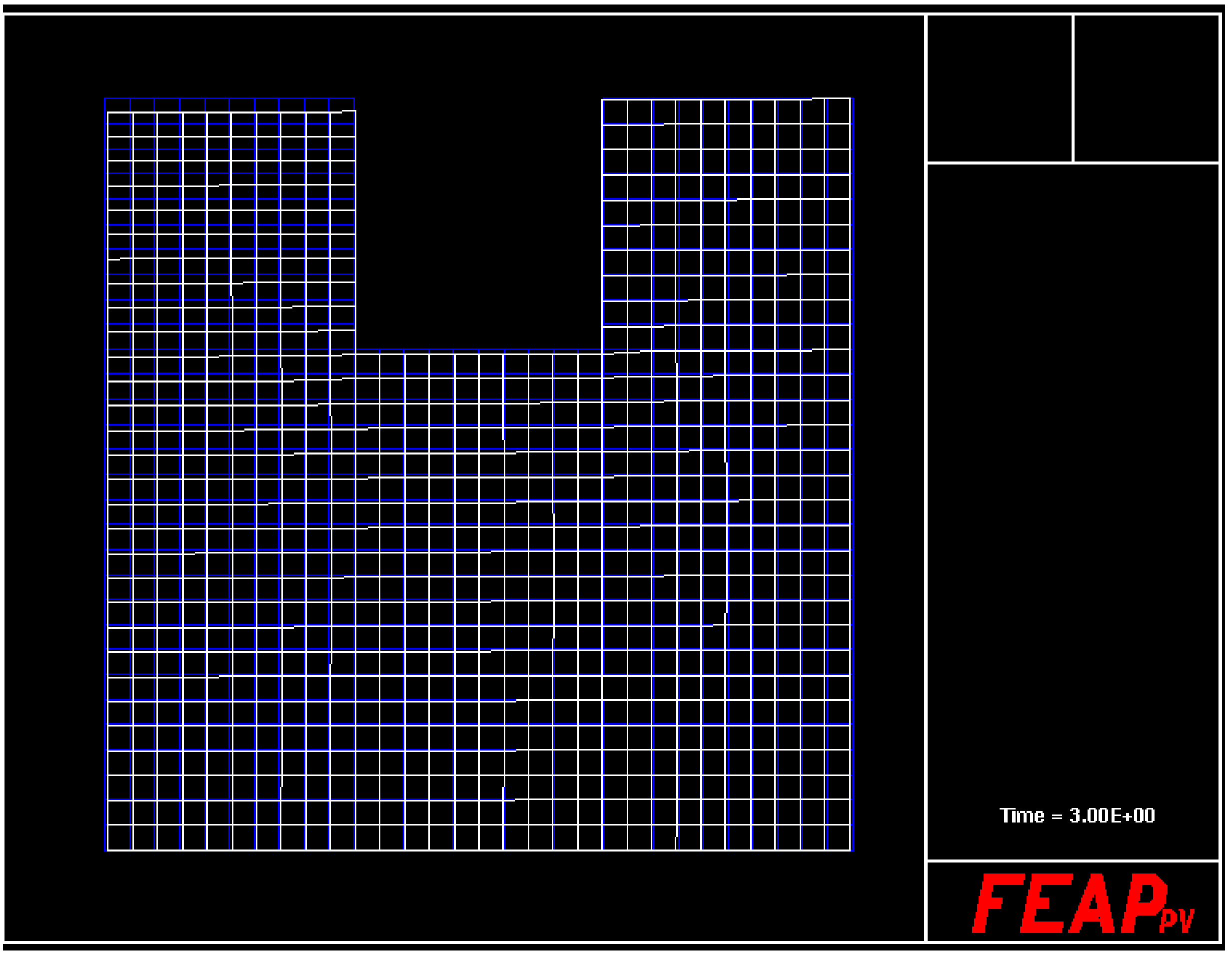


Figure 28: Deformed mesh

It can be seen from the results that, the water flows out of the structure as the load is applied.

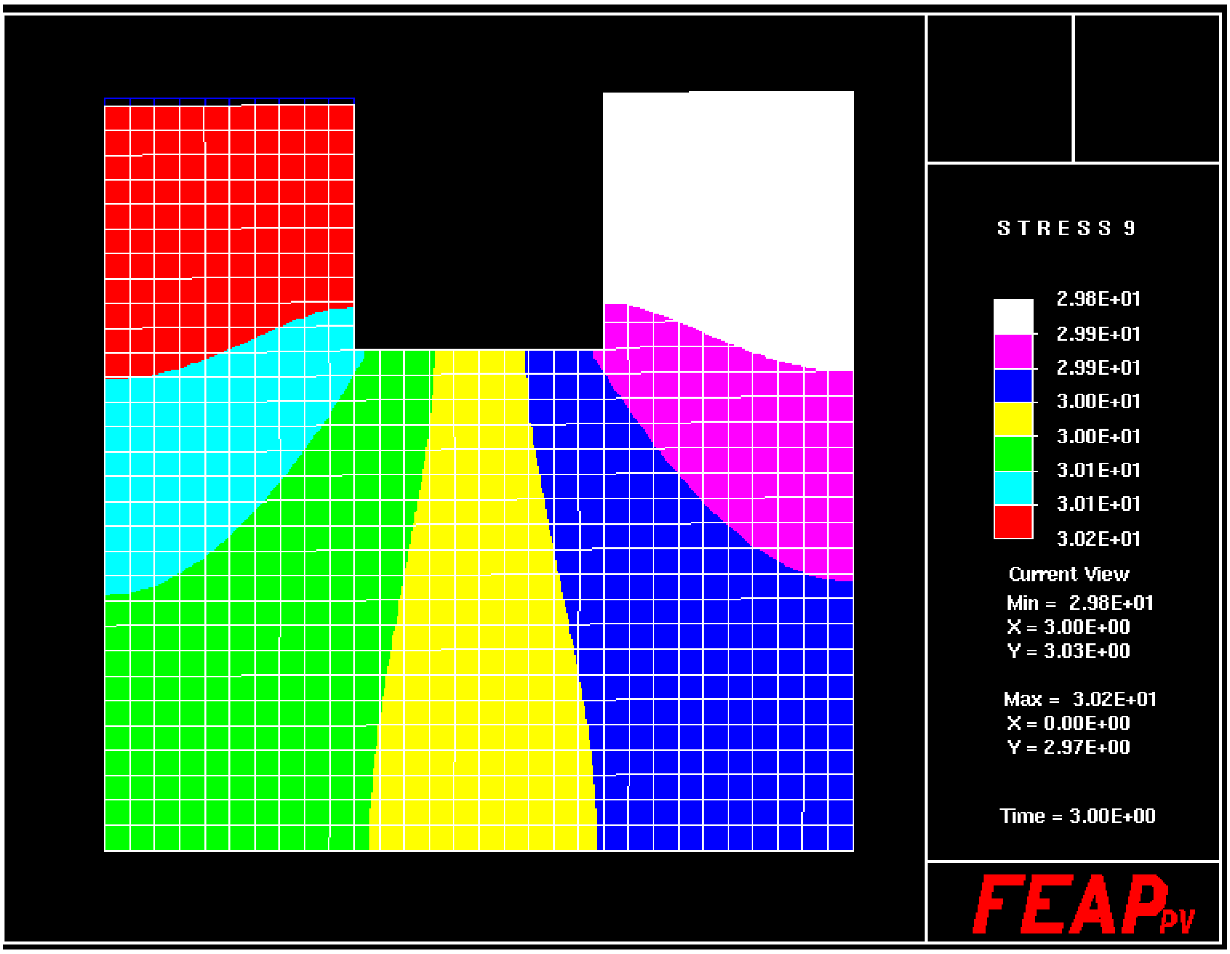


Figure 29: Pressure distribution in the container

# List of Figures

# References


- Lecture script by Prof. Ing. Bluhm
- Introduction to Porous Media theory- R.de Boer