CS-590
Homework 2 – Problem 1

Recurrences:

1. $T(n) = T(n-3) + 3\lg n$. Our guess: $T(n) = O(n \lg n)$.
   Show $T(n) \le cn \lg n$ for some constant $c > 0$.
   Solution:
   $$\begin{aligned} T(n) &= T(n-3) + 3\lg n \\ &\le c(n-3)\lg(n-3) + 3\lg n \\ &\le c(n-3)\lg n + 3\lg n \\ &= (cn - 3c + 3)\lg n \\ &= cn\lg n + (3-3c)\lg n \\ &\le cn \lg n \qquad \text{if } c > 1. \end{aligned}$$

2. $T(n) = 4T\left(\frac{n}{3}\right) + n$. Our guess: $T(n) = O(n^{\log_3 4})$.

   Solution: We get stuck, if we would try a straight substitution proof, assuming that $T(n) \le cn^{\log_3 4}$.
   $$\begin{aligned} T(n) &\le 4\left(c\left(\frac{n}{3}\right)^{\log_3 4}\right) + n \\ &= 4c\left(\frac{n^{\log_3 4}}{4}\right) + n \\ &= cn^{\log_3 4} + n \end{aligned}$$
   which is greater than $cn^{\log_3 4}$. We subtract off a lower-order term and assume that $T(n) \le cn^{\log_3 4} - dn$. We now have
   $$\begin{aligned} T(n) &\le 4\left(c\left(\frac{n}{3}\right)^{\log_3 4} - d\frac{n}{3}\right) + n \\ &= 4\left(\frac{cn^{\log_3 4}}{4} - d\frac{n}{3}\right) + n \\ &= cn^{\log_3 4} - dn\frac{4}{3} + n \\ &= cn^{\log_3 4} - dn - \frac{1}{3}dn + n \end{aligned}$$
   which is less than or equal to $cn^{\log_3 4} - dn$ if $-\frac{1}{3}dn + n \le 0 \Rightarrow d \ge 3$.

3. $T(n) = T\left(\frac{n}{2}\right) + T\left(\frac{n}{4}\right) + T\left(\frac{n}{8}\right) + n$. Our guess: $T(n) = O(n)$.
Show $T(n) \leq cn$ for some constant $c > 0$.
Solution:

$$\begin{aligned}
T(n) &= T\left(\frac{n}{2}\right) + T\left(\frac{n}{4}\right) + T\left(\frac{n}{8}\right) + n \\
&\leq c\frac{n}{2} + c\frac{n}{4} + c\frac{n}{8} + n \\
&= \frac{7cn}{8} + n \\
&= \left(1 + \frac{7c}{8}\right) n \\
&\leq cn \qquad \text{if } c \geq 8
\end{aligned}$$

4. $T(n) = 4T(\frac{n}{2}) + n^2$. Our guess: $T(n) = O(n^2)$.
Show $T(n) \leq cn^2$ for some constant $c > 0$.
Solution:

$$\begin{aligned}
T(n) &= 4T\left(\frac{n}{2}\right) + n^2 \\
&\leq 4c\left(\frac{n}{2}\right)^2 + n^2 \\
&= 4c\frac{n^2}{4} + n^2 \\
&\not\leq cn^2 + n^2
\end{aligned}$$

Guess incorrect. Cannot subtract off lower order term. Adjust the guess.
Our new guess: $T(n) = O(n^2 \lg n)$.
Show $T(n) \leq cn^2 \lg n$ for some constant $c > 0$.

$$\begin{aligned}
T(n) &= 4T\left(\frac{n}{2}\right) + n^2 \\
&\leq 4c\left(\frac{n}{2}\right)^2 \lg\left(\frac{n}{2}\right) + n^2 \\
&= 4c\frac{n^2}{4} \cdot (\lg n - \lg 2) + n^2 \\
&= cn^2 \cdot (\lg n - 1) + n^2 \\
&= cn^2 \lg n - cn^2 + n^2 \\
&\leq cn^2 \lg n \qquad \text{if } c \geq 1
\end{aligned}$$