Numerical pricing of financial derivatives using Jain's high-order compact scheme

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Goal

- High-order compact(HOC) scheme of 4^{th} order for quasilinear parabolic partial differential equations.
- Pricing of European options and then cover some of its more novel applications for pricing interest rate derivatives.
- Crank-Nicolson used as baseline for comparison.
- Generalised Chan-Karolyi-Longstaff-Sanders (CKLS) family of term structure models for pricing interest rate derivatives.

PDE Frameworks: Black-Scholes

- **SDE:** $\frac{dS_t}{S_t} = (r - \delta)dt + \sigma dW_t$
- Payoff for European call:
  $g(S_T) = max(S_T - K, 0) = (S_T - K)^+$
- If $V(S, t)$ denote the price of such a financial instrument at time t, with $V(S, T) = g(S_T)$, then it satisfies:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial_2 V}{\partial S^2} + (r - \delta) S \frac{\delta V}{\delta S} - rV = 0, \quad S > 0, 0 \leqslant t \leqslant T$$

PDE Frameworks: Black-Scholes(Contd.)

- The above PDE can be explicitly solved for $V(S,t)$ giving the analytical solution:

$$V(S,t) = Se^{-\delta(T-t)}\phi(d_1) - Ke^{-r(T-t)}\phi(d_2)$$

- ϕ: Distribution function of the standard normal distribution $N(0,1)$.
- $d_2 = \frac{log(S/K)+((r-\delta)-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$
- $d_1 = d_2 + \sigma\sqrt{T-t}$
- Similarly for put option: $V(S,t) = Ke^{-r(T-t)}\phi(-d_2) - Se^{-\delta(T-t)}\phi(-d_1)$

PDE Frameworks: CKLS Interest rate model

- Spot rate r_t is stochastic at time t, and is governed by the SDE: $dr(t) = \kappa(\theta - r(t))dt + \sigma r(t)^{\gamma} dW_t$ no arbitrage, the price $V(r,t)$ of some financial instrument with payoff $g(r)$ satisfies the PDE:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 r^{2\gamma} \frac{\partial^2 V}{\partial r^2} + \kappa(\theta - r)\frac{\partial V}{\partial r} - rV = 0 \tag{1}$$

- For zero-coupon bond with maturity T and face value $= 1$, $g(r) = 1$.
- For a European call option on the zero-coupon bond with maturity $T_0 < T$ and strike price K, $g(r) = (P(r, T_0, T) - K)^+$.

Numerical methods: Black-Scholes

- The following transformations are made to the Black-Scholes pde(1):

$$
\begin{aligned}
S &= Ke^{x} \\
\tau &= \sigma^2 (T - t)/2 \\
p_\delta &= 2(r - \delta)/\sigma^2 \\
T' &= \sigma^2 T/2 \\
p &= 2r/\sigma^2
\end{aligned}
$$

- This gives us the constant coefficient problem:

$$\frac{\partial V}{\partial \tau} = \frac{\partial^2 V}{\delta x^2} + (p_\delta - 1)\frac{\partial V}{\partial x} - pV, \quad -\infty < x < +\infty, 0 < \tau \leqslant T'$$

Numerical Schemes: Black-Scholes(Contd.)

Boundary conditions:

- A final substitution $u(x,\tau) = e^{p\tau}V(x,\tau)$ is used to reduce above PDE to:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + (p_\delta - 1)\frac{\partial u}{\partial x}, \quad -\infty < x < +\infty, 0 < \tau \leqslant T'$$

- Boundary Conditions:

$$\begin{aligned} u(x,0) &= K(e^x - 1)^+, \quad -\infty < x < +\infty, \\ u(x,\tau) &= 0, x \to -\infty \\ u(x,\tau) &= K(e^{x+(p-2\delta/\sigma^2)\tau} - 1), x \to +\infty \end{aligned}$$

Numerical Schemes: Black-Scholes(Contd.)

- The unbounded along $x - axis$, is truncated to some finite domain $\Omega = (x_{min}, x_{max}) \times [0, T']$.
- Thus we get a uniform mesh of grid points

$$\Omega_\Delta = \{(x_m, \tau_n) \in \Omega, x_m = x_{min} + mh, 0 \leqslant m \leqslant M,$$

$$\tau_n = nk, 0 \leqslant k \leqslant N\}$$

- Taking $b = (1 - p_\delta)$ we have:

$$\frac{\partial^2 u}{\partial x^2} = b\frac{\partial u}{\partial x} + \frac{\partial u}{\partial \tau} = f(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial \tau})$$

Numerical Schemes: Black-Scholes(Contd.)

- The Jain's scheme derived from Numerov discretisation of the form:

$$-\frac{1}{h^2}\delta_x^2 u_m^{n+1/2} + \frac{1}{12}(f_{m+1}^{n+1/2} + 10f_m^{n+1/2} + f_{m-1}^{n+1/2}) = 0$$

- We get the scheme:

$$(\beta_{-1}-\gamma_{-1})u_{m-1}^{n+1}+(1-2\beta_{-1})u_m^{n+1}+(\beta_{-1}+\gamma_{-1})u_{m+1}^{n+1} = (\beta_1+\gamma_1)u_{m-1}^{n}$$

where,

$$\beta_{-1} = \frac{1}{12} - \frac{k}{2h^2} - \frac{b^2k}{24}, \qquad \beta_1 = \frac{1}{12} + \frac{k}{2h^2} + \frac{b^2k}{24}$$

$$\gamma_{-1} = b(\frac{k}{4h} - \frac{h}{24}), \qquad \gamma_1 = b(\frac{k}{4h} + \frac{h}{24})$$

Numerical Schemes: Black-Scholes(Contd.)

- We let $U^n = [u_0^n, u_1^n \cdots u_M^n]$ denote the vector of our numerical solutions at time level n.
- Then we can write the above system in matrix form as:

$$AU^{n+1} = BU^n, \quad n \geqslant 0$$

where,

$$A = tridiag[\beta_{-1} - \gamma_{-1}, 1 - 2\beta_{-1}, \beta_{-1} + \gamma_{-1}]$$
$$B = tridiag[\beta_{-1} + \gamma_{-1}, 1 - 2\beta_{-1}, \beta_{-1} - \gamma_{-1}]$$

Numerical Schemes: CKLS

- We use the substitution $\tilde{\tau} = T - t$ in (2) to get a forward problem where the discount bond price at time $\tilde{\tau}$, denoted by $P(r, \tilde{\tau}, T)$ is the solution of:
- $$\frac{\partial P}{\partial \tilde{\tau}} = \frac{1}{2}\sigma^2 r^{2\gamma} \frac{\partial^2 P}{\partial r^2} + \kappa(\theta - r)\frac{\partial P}{\partial r} - rP$$
  with initial condition $P(r, 0, T) \equiv 1$.
- We again need to truncate the r domain in a fashion similar to the Black-Scholes case to be able to apply the numerical scheme. So, $\Omega_r = (r_{min}, r_{max})$

Numerical Schemes: CKLS(Contd.)

- Thus we get a uniform mesh of grid points

$$\Omega_{\Delta} = \{(r_m, \tau_n) \in \Omega, r_m = r_{min} + mh, 0 \leqslant m \leqslant M,$$

$\tau_n = nk, 0 \leqslant k \leqslant N\}$

- Then we have :

$$\frac{\partial^2 P}{\partial r^2} = \frac{2}{\sigma^2 r^{2\gamma}}(\frac{\partial P}{\partial \tilde{\tau}} - \kappa(\theta - r)\frac{\partial P}{\partial r} + rP)$$

$=$f(r,p, $\frac{\partial P}{\partial \tilde{\tau}}, \frac{\partial P}{\partial r})$

Numerical Schemes: CKLS(Contd.)

- Let P_m^n denote the bond price at the grid point (r_m, t_n), then we obtain the Jain's scheme after some standard approximations of the derivative terms as:

$$b_{m-1}P_{m-1}^{n+1}+b_mP_m^{n+1}+b_{m+1}P_{m+1}^{n+1} = c_{m-1}P_{m-1}^{n}+c_mP_m^{n}+c_{m+1}P_{m+1}^{n}$$

- The next slide contains the equations for $b.$, $c.$

Numerical Schemes: CKLS(Contd.)

$$b_{m\pm1} = r_{m-1}^{-2\gamma} r_m^{-2\gamma} r_{m+1}^{-2\gamma} \left((\sigma)^2 r_m^{2\gamma} \left[\pm hk\,(\xi)_{m\pm1}\, r_{m\pm1}^{2\gamma} + r_{m\pm1}^{2\gamma} \left(4h^2 \pm 3hk\,(\xi)_{m\pm1} + 2h^2 k r_{m\pm1} - 12k\,(\sigma)^2 r_{m\pm1}^{2\gamma} \right) \right] \right)$$
$$\left(\pm h\,(\xi)_m \left(\pm hk\,(\xi)_{m\pm1}\, r_{m\pm1}^{2\gamma} + r_{m\pm1}^{2\gamma} \left[4h^2 \pm 3hk\,(\xi)_{m\pm1} + 2h^2 k r_{m\pm1} - 10k\,(\sigma)^2 r_{m\pm1}^{2\gamma} \right] \right) \right),$$

$$c_{m\pm1} = r_{m-1}^{-2\gamma} r_m^{-2\gamma} r_{m+1}^{-2\gamma} \left((\sigma)^2 r_m^{2\gamma} \left[\pm hk\,(\xi)_{m\pm1}\, r_{m\pm1}^{2\gamma} + r_{m\pm1}^{2\gamma} \left(4h^2 \pm 3hk\,(\xi)_{m\pm1} - 2h^2 k r_{m\pm1} + 12k\,(\sigma)^2 r_{m\pm1}^{2\gamma} \right) \right] \right)$$
$$\left(+h\,(\xi)_m \left(hk\,(\xi)_{m\pm1}\, r_{m\pm1}^{2\gamma} + r_{m\pm1}^{2\gamma} \left[\pm 4h^2 + 3hk\,(\xi)_{m\pm1} \pm 2h^2 k r_{m\pm1} \pm 10k\,(\sigma)^2 r_{m\pm1}^{2\gamma} \right] \right) \right),$$

$$\begin{aligned} b_m &= 4r_m^{-2\gamma} \left(hk\,(\xi)_{m-1}\, r_{m-1}^{-2\gamma} \left(h\,(\xi)_m + (\sigma)^2 r_m^{2\gamma} \right) \right) \\ &\quad + r_{m+1}^{-2\gamma} \left(\left[h^2 k\,(\xi)_m\,(\xi)_{m+1} + (\sigma)^2 \left(-hk\,(\xi)_{m+1}\, r_m^{2\gamma} + r_{m+1}^{2\gamma} \left[10h^2 + 5h^2 k r_m + 6k\,(\sigma)^2 r_m^{2\gamma} \right] \right) \right] \right) \\ c_m &= 4r_m^{-2\gamma} \left(hk\,(\xi)_{m+1}\, r_{m+1}^{-2\gamma} \left(-h\,(\xi)_m + (\sigma)^2 r_m^{2\gamma} \right) \right) + r_{m-1}^{-2\gamma} \left(\left[-h^2 k\,(\xi)_m\,(\xi)_{m-1} - hk\,(\sigma)^2\,(\xi)_{m-1}\, r_m^{2\gamma} \right] \right. \\ &\quad + (\sigma)^2 \left[10h^2 - 5h^2 k r_m - 6k\,(\sigma)^2 r_m^{2\gamma} \right]. \end{aligned}$$

Numerical Schemes: CKLS(Contd.)

- We let $P^n = [P_0^n, P_1^n \cdots P_M^n]$ denote the vector of our numerical solutions at time level n.
- Then we can write (6) in matrix form as:

$$AP^{n+1} = BP^n, \quad n \geqslant 0$$

- where,

$$\begin{aligned} A &= tridiag[b_{m-1}, b_m, b_{m+1}] \\ B &= tridiag[c_{m-1}, c_m, c_{m+1}] \\ P^0 &= \mathbf{1}, \text{ is the initial condition} \end{aligned}$$

Bond Pricing - Crank Nicolson

Crank Nicolson				
M	Price	Error	Order	cpu(s)
10	0.494119	$4.299682*10^{-4}$	-	0.126701
20	0.493887	$9.006585*10^{-4}$	-1.066750	0.001917
40	0.494358	$5.215585*10^{-5}$	4.110079	0.001609
80	0.494334	$3.025054*10^{-6}$	4.107796	0.009457
160	0.494332	$9.626092*10^{-7}$	1.651939	0.193439
320	0.494332	$2.520807*10^{-7}$	1.933065	1.891023
Exact Price $= 0.494332$				

Table: Bond prices under the CIR model for $T = 5$

Bond Pricing - Jains Scheme

Jains Scheme				
M	Price	Error	Order	cpu(s)
10	0.077237	8.437550e-01	-21.674507	0.134101
20	0.496278	3.936669e-03	7.743705	0.002160
40	0.494340	1.558863e-05	7.980338	0.001976
80	0.494332	9.016281e-07	4.111818	0.010164
160	0.494332	6.187253e-08	3.865162	0.147673
320	0.494332	3.841779e-09	4.009452	1.758576
Exact Price = 0.494332				

Table: Bond prices under the CIR model for $T = 5$

Bond Pricing - CIR Model

CIR Model , $\gamma = 0.5$				
M	Price	Error	Order	cpu(s)
10	-0.070230	2.036439e+00	-	0.111856
20	0.068492	1.078395e-02	7.561019	0.004561
40	0.067783	3.310128e-04	5.025855	0.007088
80	0.067761	2.553277e-06	7.018393	0.055595
160	0.067761	5.493039e-08	5.538602	0.865904
320	0.067761	1.704477e-09	5.010203	11.254604
Exact Price = 0.067761				

Table: CIR bond prices for T = 30

Bond Pricing - Vasicek Model

Vasicek Model, $\gamma = 0$				
M	Price	Error	Order	cpu(s)
10	0.225421	1.746344e-02	-23.288505	0.001316
20	0.220957	2.686124e-03	2.700740	0.000553
40	0.220960	2.674648e-03	0.006177	0.001816
80	0.221128	1.912379e-03	0.483980	0.016452
160	0.221373	8.102519e-04	1.238926	0.263427
320	0.221544	3.594945e-05	4.494329	2.778937
CN Price = 0.221552				

Table: Vasicek bond prices for T = 30

Bond Pricing for different values of γ

M	$\gamma = 0.4$	$\gamma = 0.6$	$\gamma = 0.8$	$\gamma = 1$
10	0.263470	-0.045416	0.877085	-7.068155
20	0.495876	0.486904	0.500708	0.566150
40	0.495851	0.493174	0.491772	0.492524
80	0.495872	0.493243	0.491933	0.491273
160	0.495876	0.493244	0.491934	0.491281
320	0.495877	0.493244	0.491934	0.491281
CN	0.495877	0.493244	0.491935	0.491281

Table: Bond prices under CKLS model for different values of the parameter γ

Interest Rate derivatives: Bond options

- European Call option with maturity T_0 and strike price K on a discount bond with maturity T.
- $V(r, \tau^*, T_0)$ is the option price at time τ^*, where $\tau^* = (T_0 - t)$.
- Solve the numerical scheme above for $V(r, \tau^*, T_0)$.
- Initial condition: $V(r, 0, T) = (P(r, T_0, T) - K)^+$.

Bond Options

Bond options - Jain's Scheme $T_0 = 5$				
M	Price	Error	Order	cpu(s)
10	-0.630244	6.959978e+00	-	0.114196
20	0.093379	1.169493e-01	5.895127	0.004168
40	0.107834	1.974350e-02	2.566434	0.003878
80	0.105735	1.048494e-04	7.556915	0.027074
160	0.105745	1.184094e-05	3.146464	0.484453
320	0.105746	5.039296e-07	4.554417	5.361848
CN Price $= 0.105746$				

Table: European bond option prices under the CIR model for option maturities $T_0 = 5$

Bond Options

Bond options - Jain's Scheme $T_0 = 2$				
M	Price	Error	Order	cpu(s)
10	-0.903869	6.502737e+00	-	0.001319
20	0.146344	1.090612e-01	5.897837	0.000740
40	0.170098	3.555198e-02	1.617136	0.002763
80	0.164403	8.793394e-04	5.337366	0.025284
160	0.164417	9.673465e-04	-0.137613	0.412210
320	0.164258	4.934918e-07	10.936791	5.251805
CN Price = 0.164258				

Table: European bond option prices under the CIR model for option maturities $T_0 = 2$

Bond option pricing for different values of γ

M	$\gamma = 0.4$	$\gamma = 0.6$	$\gamma = 0.8$	$\gamma = 1$
10	-0.248573	-0.882009	3.999268	202.308964
20	0.079077	0.096393	0.350941	-0.117450
40	0.096527	0.102405	0.067157	-0.028959
80	0.104636	0.104537	0.103076	0.077795
160	0.106719	0.104502	0.103184	0.102636
CN	0.107358	0.104502	0.103184	0.102636

Table: European call option prices under CKLS for different values of the parameter γ

Interest Rate derivatives: Coupon bonds

- Bond with face value $\check{f}$, which makes annual coupon payments of amount $\check{a}$ at regular intervals.
- Solve the numerical scheme above with initial condition: $P(r, 0, T) = \check{f} + \check{a}$.
- If the time level coincides with a coupon payment date, $P(r, \tau^*, T) = P(r, \tau^*, T) + \check{a}$.

Coupon Bonds - Crank Nicolson

Crank Nicolson				
M	Price	Error	Order	cpu(s)
10	0.669083	2.291907e-01	-	0.002466
20	0.668785	2.286436e-01	0.003448	0.000886
40	0.543442	1.628485e-03	7.133426	0.001454
80	0.544090	4.372947e-04	1.896853	0.014356
160	0.544271	1.045224e-04	2.064793	0.141440
320	0.544317	2.085409e-05	2.325410	1.816633
Exact Price = 0.544328				

Table: Coupon bond prices under the CIR model with an annual coupon of 5% and face value 100

Coupon Bonds - Jains Scheme

Jains Scheme				
M	Price	Error	Order	cpu(s)
10	0.225386	5.859382e-01	-14.778130	0.149883
20	0.672187	2.348919e-01	1.318752	0.002340
40	0.550025	1.046621e-02	4.488186	0.001879
80	0.547292	5.445476e-03	0.942609	0.009910
160	0.545112	1.440180e-03	1.918809	0.156448
320	0.544455	2.323919e-04	2.631618	1.815153
Exact Price = 0.544328				

Table: Coupon bond prices under the CIR model with an annual coupon of 5% and face value 100

Interest Rate derivatives: European option on coupon bond

- Underlying security is a coupon paying bond with maturity T, and the option has a maturity T_0.
- We solve the numerical scheme above with initial condition $V(r, 0, T_0) = (P(r, T_0, T) - K)^+$.
- The coupon amount $\check{a}$ is added if the time level corresponds to a coupon payment.

European option with coupon bonds - Crank Nicolson

Crank Nicolson				
M	Price	Error	Order	cpu(s)
10	0.780053	3.746525e+00	-	0.003316
20	0.777185	3.729079e+00	0.006734	0.001655
40	0.161900	1.486120e-02	7.971125	0.002517
80	0.163737	3.682374e-03	2.012843	0.023734
160	0.164220	7.394183e-04	2.316173	0.270184
320	0.164342	6.084912e-12	26.856579	3.570016
Exact Price = 0.164342				

Table: European option for CIR model on bond with face value 100 with an annual coupon of 10% compounded semiannually

European option with coupon bonds - Jains scheme

Jains Scheme				
M	Price	Error	Order	cpu(s)
10	0.438384	1.667512e+00	-37.995597	0.198804
20	0.745727	3.537657e+00	-1.085097	0.003461
40	0.171579	4.403502e-02	6.327999	0.004141
80	0.170064	3.482024e-02	0.338725	0.026837
160	0.166132	1.089452e-02	1.676324	0.460876
320	0.164657	1.915871e-03	2.507530	5.796097
Exact Price = 0.164342				

Table: European option for CIR model on bond with face value 100 with an annual coupon of 10% compounded semiannually

Interest Rate derivatives: Bermudan option on coupon bond

- A Bermudan call option allows us to exercise the option on only some specified dates.
- We solve the numerical scheme using the initial condition $V(r, 0, T_0) = (P(r, T_0, T) - K)^+$.
- Add the coupon amount $\check{a}$ if the time level corresponds to a coupon payment date.
- If the time level corresponds to one of the exercise dates then, the option price becomes $V(r, \tau^*_{n+1}, T_0) = max(V(r, \tau^*_{n+1}, T_0), (P(r, \tau^*_{n+1}, T) - K)^+)$.

Bermudan option with coupon bonds - Crank Nicolson

Crank Nicolson				
M	Price	Error	Order	cpu(s)
16	0.476050	2.215275e-01	-	0.002723
32	0.475983	2.213535e-01	0.001134	0.000992
64	0.389114	1.548907e-03	7.158958	0.002705
128	0.389575	3.647856e-04	2.086129	0.062997
256	0.483218	2.399198e-01	-9.361288	0.689437
Exact Price = 0.389717				

Table: Bermudan put option under CIR model on a bond with face value 1000 with coupon of 4% compounded annually

Bermudan option with coupon bonds - Jains scheme

Jains Scheme				
M	Price	Error	Order	cpu(s)
16	0.475490	2.200898e-01	0.124460	0.151726
32	0.475993	2.213802e-01	-0.008434	0.003406
64	0.389160	1.429594e-03	7.274776	0.004711
128	0.389665	1.348059e-04	3.406650	0.109706
256	0.483437	2.404822e-01	-10.800831	0.610421
Exact Price = 0.389717				

Table: Bermudan put option under CIR model on a bond with face value 1000 with coupon of 4% compounded annually