MSM HW1

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1. Using index notation, show that
   (a) $(\mathbf{u}\otimes\mathbf{v})\cdot(\mathbf{w}\otimes\mathbf{x}) = (\mathbf{w}\cdot\mathbf{v})\,\mathbf{u}\otimes\mathbf{x}$
   (b) $\mathbf{A}\,(\mathbf{u}\otimes\mathbf{v}) = (\mathbf{A}\mathbf{u})\otimes\mathbf{v}$
   (c) $(\mathbf{u}\otimes\mathbf{v})\,\mathbf{A} = \mathbf{u}\otimes(\mathbf{A}^T\mathbf{v})$
   (d) $(\mathbf{A}\mathbf{B})^T = \mathbf{B}^T\mathbf{A}^T$

   To do this problem, write both the left and right-hand side of the equation in index notation and show that they are identical.

$$\begin{matrix} 11 & 12 & 13 \\ 21 & 22 & 23 \\ 31 & 32 & 33 \end{matrix}$$

$+e_3(u_1v_2 - u_2v_1)$

$w_m v_j \delta_{mj} = w_m v_m$

1) a) $(\vec{u}\otimes\vec{v})\cdot(\vec{w}\otimes\vec{x})$

$(u_i v_j\, \vec{e}_i\otimes\vec{e}_j)\cdot(w_m x_n\, \vec{e}_m\otimes\vec{e}_n) = (w_m v_j\, \vec{e}_m\cdot\vec{e}_j)(u_i x_n\, \vec{e}_i\otimes\vec{e}_n)$

$(u_i v_j\, \vec{e}_i\otimes\vec{e}_j)\cdot(w_m x_n\, \vec{e}_m\otimes\vec{e}_n) = (w_m v_j\, \delta_{mj})(u_i x_n\, \vec{e}_i\otimes\vec{e}_n)$

$= (w_m v_m)\vec{u}\otimes\vec{x}$

$= (\vec{w}\cdot\vec{v})(\vec{u}\otimes\vec{x})$

$= (\vec{u}\otimes\vec{v})\cdot(\vec{w}\otimes\vec{x})$

$A\otimes = b$

b) $A(\vec{u}\otimes\vec{v})$

$= A(u_m\vec{e}_m\otimes v_n\vec{e}_n)$

$= A_{ij}\,\vec{e}_i\otimes\vec{e}_j\,(u_m v_n\,\vec{e}_m\otimes\vec{e}_n)$

$= A_{ij}\,u_m\,\delta_{jm}\,v_n\,\vec{e}_i\otimes\vec{e}_m$

$= (A_{ij}\,u_j\,\vec{e}_i)\otimes(v_n\,\vec{e}_n)$

$= (A\vec{u})\otimes\vec{v}$

c) $(\vec{u}\otimes\vec{v})A$

$= u_m\vec{e}_m\otimes v_n\vec{e}_n\cdot(A_{ij}\,\vec{e}_i\otimes\vec{e}_j)$

$= u_m\quad A_{ij}\,v_n\,\delta_{in}\,\vec{e}_m\otimes\vec{e}_j$

$= u_m\quad A_{ij}\,v_i\,\vec{e}_m\otimes\vec{e}_j$

$= u_m\vec{e}_m\otimes A_{ij}\,v_i\,\vec{e}_j$

$= \vec{u}\otimes(A^T\vec{v})$

d) $(AB)^T$

$= (A_{ij}\,\vec{e}_i\otimes\vec{e}_j\cdot B_{jk}\,\vec{e}_j\otimes\vec{e}_k)^T$

$= (A_{ij}B_{jk}\,(\vec{e}_i\otimes\vec{e}_j)(\vec{e}_j\otimes\vec{e}_k))^T$

$= (A_{ij}B_{jk}\,(\vec{e}_i\otimes\vec{e}_k))^T$

$= A_{ji}B_{kj}\,(\vec{e}_k\otimes\vec{e}_i)$

B^TA^T

$(B_{kj}\,\vec{e}_k\otimes\vec{e}_j)(A_{ji}\,e_j\otimes e_i)$

$= B_{kj}A_{ji}\,(\vec{e}_k\otimes\vec{e}_j)(e_j\otimes e_i)$

$= B_{kj}A_{ji}\,(e_k\otimes\vec{e}_i)$

2. Given the tensor $\mathbf{S} = 2\mathbf{e}_1\otimes\mathbf{e}_1 + 3\mathbf{e}_2\otimes\mathbf{e}_2 + 4\mathbf{e}_2\otimes\mathbf{e}_3 + 4\mathbf{e}_3\otimes\mathbf{e}_2 - 3\mathbf{e}_3\otimes\mathbf{e}_3$, calculate the following by hand. Show all your work and you can check using Matlab.
   (a) The three invariants of $\mathbf{S}$
   (b) The transpose of $\mathbf{S}$ and show that $\mathbf{S}^T = S_{ij}\mathbf{e}_j\otimes\mathbf{e}_i$.
   (c) The inverse of $\mathbf{S}$

2) $S = 2e_1\otimes e_1 + 3e_2\otimes e_2 + 4e_2\otimes e_3 + 4e_3\otimes e_2 - 3e_3\otimes e_3$

$$[S] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & -3 \end{bmatrix}$$

a) $I_1(S) = \text{trace}(S) = S_{ii}$

$= S_{ij}\,\delta_{ij} =$

$= S_{ii} = 2+3-3 = 2$

$tr(S^2) = 4+25+25 = 54$

$I_2(S) = \frac{1}{2}\left(tr(S)^2 - tr(S^2)\right)$

$= \frac{1}{2}\left((2)^2 - 54\right)$

$$
\begin{aligned}
I_2(S) &= \tfrac{1}{2}\left(tr(S)^2 - tr(S^2)\right)\\
&= \tfrac{1}{2}\left((2)^2 - 54\right)\\
&= -25
\end{aligned}
$$

$$
\begin{aligned}
I_3(S) = \det(S) &= \det[S]\\
&= 2\left((3)(-3)-(4)(4)\right) - 0 + 0\\
&= 2(-9-16)\\
&= 2(-25)\\
&= -50
\end{aligned}
$$

b) S^T and show $S^T = S_{ij}\,\vec{e}_j \otimes \vec{e}_i$

$$
[S^T_{ij}] = \begin{bmatrix} S_{11} & S_{21} & S_{31} \\ S_{12} & S_{22} & S_{32} \\ S_{13} & S_{23} & S_{33} \end{bmatrix}
$$

$$
= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & -3 \end{bmatrix}
$$

$$
\begin{aligned}
S^T = {} & S_{11}\,\vec{e}_1\otimes\vec{e}_1 + S_{21}\,\vec{e}_1\otimes\vec{e}_2 + S_{31}\,\vec{e}_1\otimes\vec{e}_3 + \\
& S_{12}\,\vec{e}_2\otimes\vec{e}_1 + S_{22}\,\vec{e}_2\otimes\vec{e}_2 + S_{32}\,\vec{e}_2\otimes\vec{e}_2 + \\
& S_{13}\,\vec{e}_3\otimes\vec{e}_1 + S_{23}\,\vec{e}_3\otimes\vec{e}_2 + S_{33}\,\vec{e}_3\otimes\vec{e}_3
\end{aligned}
$$

$$
= S_{ij}\,\vec{e}_i\otimes\vec{e}_j
$$

c) Inverse of S

$$
S S^{-1} = 1
$$

$$
S_{ij} S^{-1}_{jk} = \delta_{ik}
$$

$$
\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & -3 \end{bmatrix}
\begin{bmatrix} S^{-1}_{11} & S^{-1}_{21} & S^{-1}_{31} \\ S^{-1}_{21} & S^{-1}_{22} & S^{-1}_{32} \\ S^{-1}_{31} & S^{-1}_{23} & S^{-1}_{33} \end{bmatrix}
= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

$$
adj(S) = \begin{bmatrix} -25 & 0 & 0 \\ 0 & -6 & -8 \\ 0 & -8 & 6 \end{bmatrix}
$$

$$
\det S = -50
$$

$$
S^{-1} = \frac{adj(S)}{\det S}
$$

$$
= \begin{bmatrix} -25/-50 & 0/-50 & 0/-50 \\ 0/-50 & -6/-50 & -8/-50 \\ 0/-50 & -8/-50 & 6/-50 \end{bmatrix}
$$

$$
= \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 3/25 & 4/25 \\ 0 & 4/25 & -3/25 \end{bmatrix}
\text{ or }
\begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.12 & 0.16 \\ 0 & 0.16 & -0.12 \end{bmatrix}
$$

3) a) $\det(A^{-1}) = \dfrac{1}{\det(A)}$

$$
\begin{aligned}
\det(A^{-1})\det(A) &= 1\\
\det(A^{-1}A) &= 1\\
\det(I) &= 1 \quad \checkmark
\end{aligned}
$$

b) $\det(Q) = 1$

$\det(Q)\det(Q^T) = \det(Q^T)$

$\det(QQ^T) = \det(Q^T)$

$\det(I) = \det(Q^T)$

$1 = \det(Q^T)$

Since $\det(Q^T) = \det(Q)$

$\det(Q) = 1$

c) $(A^T)^{-1} : A$

$= A^{-T} : A$

$= tr((A^{-T})^T A)$

$= tr(A^T (A^{-1})^T)$

$= tr((AA^{-1})^T)$

$= tr(I)$

$= 3$

4a) $I_1 = tr(A)$ prove $\hat{A}_{ii} = A_{ii}$

$tr\hat{A} = \hat{A}_{ii}$

$\hat{A} = Q^T A Q$

$\hat{A}_{mj} = Q_{im} A_{in} Q_{nj}$

$\hat{A}_{jj} = Q_{ij} A_{in} Q_{nj}$

$= Q_{ij} Q_{nj} A_{in}$

$= \delta_{in} A_{in}$

$\hat{A}_{jj} = A_{ii}$

b) $I_2 = \frac{1}{2}(tr(A)^2 - tr(A^2))$ — $tr(A)$ is invariant so $(tr(A))^2$ is invariant

$= \frac{1}{2}(A_{ii}A_{jj} - A_{ji}A_{ij})$ — prove $tr(A^2)$ is invariant

$= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3$

eigenvalues are invariant

* c) $I_3 = \det(A)$

$\det(A) = \varepsilon_{ijk} A_{1i} A_{2j} A_{3k}$

$= \lambda_1\lambda_2\lambda_3 \leftarrow$ eigenvalues are invariant

5) $A = 2(\underset{\sim}{e}_1 \otimes \underset{\sim}{e}_1 + \underset{\sim}{e}_3 \otimes \underset{\sim}{e}_3) + 3\underset{\sim}{e}_2 \otimes \underset{\sim}{e}_2 + (\underset{\sim}{e}_1 \otimes \underset{\sim}{e}_2 + \underset{\sim}{e}_2 \otimes \underset{\sim}{e}_1)$

$+ (\underset{\sim}{e}_2 \otimes \underset{\sim}{e}_3 + \underset{\sim}{e}_3 \otimes \underset{\sim}{e}_2)$

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

a) $$[A - \lambda I] = \begin{bmatrix} 2-\lambda & 1 & 0 \\ 1 & 3-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{bmatrix}$$

$$\det(A-\lambda I) = (2-\lambda)\big((3-\lambda)(2-\lambda)-1\big) -1\big[((1)(2-\lambda))-0\big]+0$$
$$= (2-\lambda)(6-3\lambda-2\lambda+\lambda^2-1)-2+\lambda$$
$$= (2-\lambda)(5-5\lambda+\lambda^2)-2+\lambda$$
$$= 10-10\lambda+2\lambda^2-5\lambda+5\lambda^2-\lambda^3-2+\lambda$$
$$= 8-14\lambda+7\lambda^2-\lambda^3$$
$$= (-1)(\lambda^3-7\lambda^2+14\lambda-8)$$
$$= (-1)\big[(\lambda-4)(\lambda^2-3\lambda+2)\big]$$
$$= (-1)\big[(\lambda-4)(\lambda-2)(\lambda-1)\big]$$
$$= (4-\lambda)(2-\lambda)(1-\lambda)$$

$$\boxed{\lambda = 4, 2, 1}$$

When $\lambda = 4$

$$\begin{bmatrix} 2-4 & 1 & 0 \\ 1 & 3-4 & 1 \\ 0 & 1 & 2-4 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ -2 & 1 & 0 \\ 0 & 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 2 \\ 0 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} x_1 = t \\ x_2 = 2t \\ x_3 = t \end{matrix}$$

$$1^2+2^2+1^2 = \sqrt{6}$$

$$\begin{bmatrix} t \\ 2t \\ t \end{bmatrix} = t\boxed{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}} \qquad n_1 = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

$\lambda = 2$

$$\begin{bmatrix} 2-2 & 1 & 0 \\ 1 & 3-2 & 1 \\ 0 & 1 & 2-2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} =$$

$$\sqrt{-1^2+0^2+1^2} = \sqrt{2}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} x_1 = -t \\ x_2 = 0 \\ x_3 = t \end{matrix} \qquad \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = t\boxed{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}} \qquad n_2 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$\lambda = 1$

$$\begin{bmatrix} 2-1 & 1 & 0 \\ 1 & 3-1 & 1 \\ 0 & 1 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sqrt{(1)^2+(-1)^2+(1)^2} = \sqrt{3}$$

$$= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} x_1 = t \\ x_2 = -t \\ x_3 = t \end{matrix} \qquad \begin{bmatrix} t \\ -t \\ t \end{bmatrix} = t\boxed{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}} \qquad n_3 = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

b) eigenvectors = $\text{span}\left\{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right\}$

$B = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$ L.I is $B\vec{x} = 0$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 0 & 2 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] = \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 3 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -3 \\ 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \therefore \text{L.I.}$$

orthogonal if

$a_1 \langle v_1, v_1 \rangle + a_2 \langle v_1, v_2 \rangle + a_3 \langle v_1, v_3 \rangle = 0$

Since a_1, a_2, a_3 are $= 0$ and $\{v_1, v_2, v_3\} \neq 0$
the above statement is true

$$\text{spectral } A = \sum_{i=1}^{3} \lambda_i \vec{n}_i \otimes \vec{n}_i \qquad n_1 =$$

$$= 4 \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \otimes \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} + 2 \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \otimes \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} + 1 \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \otimes \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$= 4 \begin{bmatrix} 1/6 & 2/6 & 1/6 \\ 2/6 & 4/6 & 2/6 \\ 1/6 & 2/6 & 1/6 \end{bmatrix} + 2 \begin{bmatrix} 1/2 & 0 & -1/2 \\ 0 & 0 & 0 \\ -1/2 & 0 & 1/2 \end{bmatrix} + \begin{bmatrix} 1/3 & -1/3 & 1/3 \\ -1/3 & 1/3 & -1/3 \\ 1/3 & -1/3 & 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 4/6 & 8/6 & 4/6 \\ 8/6 & 16/6 & 8/6 \\ 4/6 & 8/6 & 4/6 \end{bmatrix} + \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1/3 & -1/3 & 1/3 \\ -1/3 & 1/3 & -1/3 \\ 1/3 & -1/3 & 1/3 \end{bmatrix}$$

$$\text{spectral } A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

c) $A^3 = A \cdot A \cdot A$

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 15 & 21 & 7 \\ 21 & 43 & 21 \\ 7 & 21 & 15 \end{bmatrix}$$

$$\text{spectral } A^3 = \sum_{i=1}^{3} (\lambda_i)^3 \vec{n}_i \otimes \vec{n}_i$$

$$= 4^3 \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \otimes \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} + 2^3 \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \otimes \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} + 1^3 \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \otimes \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$= 64 \begin{bmatrix} 1/6 & 2/6 & 1/6 \\ 2/6 & 4/6 & 2/6 \\ 1/6 & 2/6 & 1/6 \end{bmatrix} + 8 \begin{bmatrix} 1/2 & 0 & -1/2 \\ 0 & 0 & 0 \\ -1/2 & 0 & 1/2 \end{bmatrix} + \begin{bmatrix} 1/3 & -1/3 & 1/3 \\ -1/3 & 1/3 & -1/3 \\ 1/3 & -1/3 & 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 64/6 & 128/6 & 64/6 \\ 128/6 & 256/6 & 128/6 \\ 64/6 & 128/6 & 64/6 \end{bmatrix} + \begin{bmatrix} 4 & 0 & -4 \\ 0 & 0 & 0 \\ -4 & 0 & 4 \end{bmatrix} + \begin{bmatrix} 1/3 & -1/3 & 1/3 \\ -1/3 & 1/3 & -1/3 \\ 1/3 & -1/3 & 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & 21 & 7 \\ 21 & 43 & 21 \\ 7 & 21 & 15 \end{bmatrix}$$

d)

$Q_{ij} = e_i \cdot n_i$

$Q_{11} = \underset{\sim}{e_1} \cdot \underset{\sim}{n_1} = 1/\sqrt{6}$ $\quad Q_{21} = e_2\ n_1 = 2/\sqrt{6}$ $\quad Q_{31} = \mathbf{e}_3\ n_1\ \ 1/\sqrt{6}$

$Q_{12} = \underset{\sim}{e_1}\ \underset{\sim}{n_2} = -1/\sqrt{2}$ $\quad Q_{22} = \underset{\sim}{e_2}\ n_2 = 0$ $\quad Q_{32} = \underset{\sim}{e_3}\ \underset{\sim}{n_2} = 1/\sqrt{2}$

$Q_{13} = \underset{\sim}{e_1}\ \underset{\sim}{n_3} = 1/\sqrt{3}$ $\quad Q_{23} = \underset{\sim}{e_2}\ \underset{\sim}{n_3} = -1/\sqrt{3}$ $\quad Q_{33} = \underset{\sim}{e_3}\ \underset{\sim}{n_3} = 1/\sqrt{3}$

$$n = \begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix} \qquad e = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad Q = \begin{bmatrix} 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

e) $[A] = [Q][\hat{A}][Q]^T$

6 a) $\nabla(\rho \underset{\sim}{u}) = \underset{\sim}{u} \otimes \nabla\rho + \rho \nabla \underset{\sim}{u}$

$$\nabla(\rho \underset{\sim}{u}) = \frac{\partial}{\partial x_j}\left(\rho (u_i \underset{\sim}{e_i})\right) \underset{\sim}{e_j}$$

$$= \frac{\partial \rho}{\partial x_j} \underset{\sim}{e_j}\ u_i \underset{\sim}{e_i} + \rho \frac{\partial u_i}{\partial x_j}\ \underset{\sim}{e_i} \otimes \underset{\sim}{e_j}$$

$$= \nabla \rho \underset{\sim}{u} + \rho \nabla \underset{\sim}{u}$$

$$= \underset{\sim}{u} \otimes \nabla \rho + \rho \nabla \underset{\sim}{u}$$

b) $\nabla \cdot (\rho \underset{\sim}{u}) = \rho \nabla \cdot \underset{\sim}{u} + \underset{\sim}{u}\ \nabla \rho$ $\qquad \underset{\sim}{u} = u_i \underset{\sim}{e_i}$

$$\nabla \cdot (\rho \underset{\sim}{u}) = \frac{\partial}{\partial x_j}\left(\rho (u_i \underset{\sim}{e_i})\right) \cdot \underset{\sim}{e_j}$$

$$= \frac{\partial \rho}{\partial x_j} e_j \cdot u_i e_i + \rho \frac{\partial u_i e_i}{\partial x_j} \underset{\sim}{e_j}$$

$$= \frac{\partial \rho}{\partial x_j} e_j \cdot u_i e_i + \rho \frac{\partial u_i}{\partial x_i} \delta_{ij}$$

$$= \nabla \rho \cdot \underset{\sim}{u} + \rho \nabla \cdot u$$

$$= \rho \nabla \cdot \underset{\sim}{u} + \underset{\sim}{u} \nabla \rho$$

c) $\nabla \cdot (A^T \underset{\sim}{u}) = (\nabla \cdot A) \cdot \underset{\sim}{u} + A : \nabla \underset{\sim}{u}$

$$\nabla \cdot (A^T \underset{\sim}{u}) = \frac{\partial}{\partial x_j}(A^T \underset{\sim}{u}) \cdot \underset{\sim}{e_j} \qquad A^T = A_{ki}\ \underset{\sim}{e_i} \otimes \underset{\sim}{e_k} \quad \underset{\sim}{u} = u_k \underset{\sim}{e_k}$$

$$= \frac{\partial A_{ki}}{\partial x_j} (\underset{\sim}{e_i} \otimes \underset{\sim}{e_k}) \cdot \underset{\sim}{e_j}\ u_k \underset{\sim}{e_k} + A_{ki} \underset{\sim}{e_i} \otimes \underset{\sim}{e_k} \frac{\partial u_k}{\partial x_j} \underset{\sim}{e_k} \cdot \underset{\sim}{e_j}$$

$$= \frac{\partial A_{ki}}{\partial x_j} (\underset{\sim}{e_k} \cdot \underset{\sim}{e_j}) e_i \; u_k \underset{\sim}{e_k} + A_{ki} \, \underset{\sim}{e_i} \otimes \underset{\sim}{e_k} \; \frac{\partial u_k}{\partial x_j} \underset{\sim}{e_k} \cdot \underset{\sim}{e_j}$$

$$= \frac{\partial A_{ki}}{\partial x_j} \delta_{jk} \cdot \underset{\sim}{e_i} \cdot u_k \underset{\sim}{e_k} + A_{ki} (\underset{\sim}{e_i} \otimes \underset{\sim}{e_k}) \underset{\sim}{e_k} \; \frac{\partial u_k}{\partial x_j} \cdot \underset{\sim}{e_j}$$

$$= \frac{\partial A_{ij}}{\partial x_j} \underset{\sim}{e_i} \cdot u_k \underset{\sim}{e_k} + A_{ki} (\underset{\sim}{e_k} \cdot \underset{\sim}{e_k}) \cdot \underset{\sim}{e_i} \; \frac{\partial u_k}{\partial x_j} \underset{\sim}{e_j}$$

$$= (\nabla \cdot A) \cdot \underset{\sim}{u} + A : \nabla \underset{\sim}{u}$$

d) $\nabla \times (\underset{\sim}{u} \times \underset{\sim}{v}) = \underset{\sim}{u} \nabla \cdot \underset{\sim}{v} - \underset{\sim}{v} \nabla \cdot \underset{\sim}{u} + (\nabla \underset{\sim}{u}) \underset{\sim}{v} - (\nabla \underset{\sim}{v}) \underset{\sim}{u}$ $\qquad \underset{\sim}{u} \times \underset{\sim}{v} = u_i v_j \, \varepsilon_{ijk} \, \underset{\sim}{e_k}$

$$\nabla \times (\underset{\sim}{u} \times \underset{\sim}{v}) = \mathbf{e}_i \times \frac{\partial}{\partial x_i} (u_j \underset{\sim}{e_j} \times v_k \underset{\sim}{e_k})$$

$$= \frac{\partial}{\partial x_i} (u_j v_k) \underset{\sim}{e_i} \times (\underset{\sim}{e_j} \times \underset{\sim}{e_k})$$

$$= \left(\frac{\partial}{\partial x_i} u_j v_k + u_j \frac{\partial v_k}{\partial x_i} \right) \left((\underset{\sim}{e_i} \cdot \underset{\sim}{e_k}) \underset{\sim}{e_j} - (\underset{\sim}{e_i} \cdot \underset{\sim}{e_j}) \underset{\sim}{e_k} \right)$$

$$= \left(\frac{\partial}{\partial x_i} u_j v_k + u_j \frac{\partial v_k}{\partial x_i} \right) (\delta_{ik} \underset{\sim}{e_j} - \delta_{ij} \underset{\sim}{e_k})$$

$$= \frac{\partial}{\partial x_i} u_j v_i \, \underset{\sim}{e_j} - u_j \frac{\partial}{\partial x_i} v_k \underset{\sim}{e_k} + u_j \frac{\partial}{\partial x_i} v_i \underset{\sim}{e_j} - u_i \frac{\partial}{\partial x_i} v_k \underset{\sim}{e_k}$$

$$= \underset{\sim}{v} \cdot \nabla \underset{\sim}{u} - (\nabla \cdot \underset{\sim}{u}) \underset{\sim}{v} + (\nabla \cdot \underset{\sim}{v}) \underset{\sim}{u} - \underset{\sim}{u} \cdot \nabla \underset{\sim}{v}$$

$$= \underset{\sim}{u} \nabla \cdot \underset{\sim}{v} - \underset{\sim}{v} \nabla \cdot \underset{\sim}{u} + (\nabla \underset{\sim}{u}) \underset{\sim}{v} - (\nabla \underset{\sim}{v}) \underset{\sim}{u}$$

7a) $\dfrac{\partial \, trA}{\partial A} = I$ $\qquad$ trace$(A) = A_{ii}$

$$\frac{\partial \, trA}{\partial A} = \frac{\partial}{\partial A_{ij}} \left(A_{kl} \; \underset{\sim}{e_k} \otimes \underset{\sim}{e_l} \otimes \underset{\sim}{e_i} \otimes \underset{\sim}{e_j} \right)$$

$$= \delta_{ik} \delta_{jl} \; \underset{\sim}{e_k} \otimes \underset{\sim}{e_l} \otimes \underset{\sim}{e_i} \otimes \underset{\sim}{e_j}$$

$$= I$$

b) $\left(\frac{\partial A^{-1}}{\partial A}\right)_{ijkl} = -\frac{1}{2}\left(A^{-1}_{ik} A^{-1}_{lj} + A^{-1}_{il} A^{-1}_{kj}\right)$

$\left(\frac{\partial A^{-1}}{\partial A} A\right)_{ijkl} = \frac{\partial A^{-1}_{im}}{\partial A_{kl}} A_{mj} + A^{-1}_{im} \frac{\partial A_{mj}}{\partial A_{kl}} = 0$

$\frac{\partial A^{-1}_{im}}{\partial A_{kl}} A_{mj} A^{-1}_{jn} = -A^{-1}_{im} \frac{\partial A_{mj}}{\partial A_{kl}} A^{-1}_{jn}$

$\frac{\partial A^{-1}_{im}}{\partial A_{kl}} \delta_{mn} = -\frac{1}{2}\left(A_{im}^{-1} \delta_{mk} \delta_{jl} A^{-1}_{jn} + A^{-1}_{im} \delta_{ml} \delta_{jk} A^{-1}_{jn}\right)$

$\frac{\partial A_{in}^{-1}}{\partial A_{kl}} = -\frac{1}{2}\left(A_{ik}^{-1} A_{ln}^{-1} + A_{il}^{-1} A_{kn}^{-1}\right)$

$\frac{\partial A_{ij}}{\partial A_{kl}} = -\frac{1}{2}\left(A_{ik}^{-1} A_{lj}^{-1} + A_{il}^{-1} A_{kj}^{-1}\right)$

c) $\frac{\partial \det A}{\partial A} = \det A A^{-1}$

$\frac{\partial \det A}{\partial A} = \frac{\partial}{\partial A}\left(\varepsilon_{ijk} A_{1i} A_{2j} A_{3k}\right)$

$= \varepsilon_{ijk}\left(\frac{\partial A_{1i}}{\partial A} A_{2j} A_{3k} + A_{1i} \frac{\partial A_{2j}}{\partial A} A_{3k} + A_{1i} A_{2j} \frac{\partial A_{3k}}{\partial A}\right)$

$= \frac{\partial A_{1i}}{\partial A} (A^{-1})_{i1} \det(A) + \frac{\partial A_{2j}}{\partial A} (A^{-1})_{j2} \det(A) + \frac{\partial A_{3k}}{\partial A} (A^{-1})_{k3} \det(A)$

$= \det(A)\left[\frac{\partial A_{1i}}{\partial A} (A^{-1})_{i1} + \frac{\partial A_{2j}}{\partial A} (A^{-1})_{j2} + \frac{\partial A_{3k}}{\partial A} (A^{-1})_{k3}\right]$

$= \det(A) \det(A^{-1}) = \det A A^{-1}$