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SLOT	A2+TA2+TAA2+V3
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MODULE - 1
SYSTEM OF
LINEAR EQUATIONS

Q.1 Which of the following equations are linear?

(A) $-3x_1^2 + 5x_2^3 + 4x_3^2 = 4$ (B) $\sqrt{2}x_1 - \sqrt{2}x_2 + 0.5x_3 = 5$

(C) $\sqrt{3}x_1 - 3\sqrt{x_2} + 3x_3 = -1$ (D) $2^2 x_1 + \sin(x_2) + 5x_3 = 17$

A.1 ~~B~~(B) ✓✓

In (A) degree of x_1 & x_3 is 2 & x_2 is 3.
In (C) degree of x_2 is 0.5
In (D) $\sin(x_2)$ is present, hence degree of x_2 is not 1.
$\therefore$ A, C, D are not linear $\therefore$ by process of elimination (B) is correct
(B) all x_1, x_2, x_3 are of degree 1 $\therefore$ it is linear

Q.2 Which system corresponds to the augmented matrix $\begin{bmatrix} 1 & -1 & 6 & 3 \\ 9 & 4 & 0 & -2 \end{bmatrix}$?

(A) $x_1 - x_2 = -3$
$9x_1 + 4x_2 = -2$

(B) $x_1 - x_2 + 6x_3 = 3$
$9x_1 + 4x_2 = -2$

(C) $x_1 - x_2 + 6x_3 + 3x_4 = 0$
$9x_1 + 4x_2 - 2x_4 = 0$

(D) $-x_1 + 4x_2 = 0$
$6x_1 = 0$
$3x_1 - 2x_2 = 0$

A.2 (B) ✓

Only option (B) matches the system represented in augmented matrix
In option (C) the elements of 4th column are coefficients & in the question we are given augmented matrix, not coefficient matrix

Q.3 Which of the following statements best describes the following augmented matrix $\begin{bmatrix} 1 & 0 & 0 & 2 & 3 \\ 0 & 1 & 0 & 1 & 4 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$?

(A) A is consistent with a unique solution (B) A is inconsistent
(C) A is consistent with infinitely many solutions (D) None of the above

A.3 (C) ✓

The system has 4 variables $x, y, z, w = t$ (free variable
$z = 0$; ~~x=3-~~ $X = (3-2t,\ 4-t,\ 0,\ t)$; so for any value of t, we have infinitely many solutions

Q.4 What is the first row of the augmented matrix for the system $\begin{matrix} x + z = 3 \\ y = 4 \\ 2z = 6 \end{matrix}$?

(A) 0, 0, 2, 6 (B) 0, 1, 0, 4 (C) 1, 1, 1, 3 (D) 1, 0, 1, 3

A.4 (D) ✓

The augmented matrix of given system: $\begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 2 & 6 \end{bmatrix} \rightarrow$ matches (D)

Q.4 Which of the following matrices is in reduced row echelon form?

(A) $\begin{pmatrix} 1 & 2 & -1 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{pmatrix}$ (B) $\begin{pmatrix} 1 & 2 & -1 \\ -1 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$ (C) $\begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{pmatrix}$ (D) $\begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

A.4 (D) ✓

(B), (C) are not in row echelon forms, so they can't be reduced REF.
(A) is in REF but not reduced since leading coefficient in row 2 & 3 is not 1.
So by method of elimination, we have (D).

Q.5 Which of the following matrices is not invertible

(A) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (B) $\begin{pmatrix} 3 & 6 \\ 0 & 6 \end{pmatrix}$ (C) $\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$ (D) $\begin{pmatrix} 3 & 6 \\ 0 & 0 \end{pmatrix}$

A.5 (D) ✓

since bottom row has all zero's, the matrix is not invertible

Q.6 Which of the following is not an elementary matrix?

(A) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (B) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ (C) $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ (D) $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

A.6 (B) ✓

(A) is already identity matrix
In (C) $R_3 \to R_3 - R_1$ ~~or R_2~~ leads to identity matrix
In (D) $R_2 \to R_2/2$ leads to identity matrix.
So A, C, D become identity matrix by single row operation, ∴ they are elementary.
(B) single row operation cannot make it identity matrix & also its determinant is 0.

Q.7 What value of b makes the system $\begin{matrix} x+y=0 \\ x+y=b \end{matrix}$ inconsistent?

(A) $b=0$ (B) $b \neq 0$ (C) $b=0.5$ (D) $b=1$

A.7 (B) ✓

$\left(\begin{array}{cc:c} 1 & 1 & 0 \\ 1 & 1 & b \end{array}\right) \quad R_2 \to R_2 - R_1 \sim \left(\begin{array}{cc:c} 1 & 1 & 0 \\ 0 & 0 & b \end{array}\right)$ So if $b=0$, the system will be consistent with infinitely many solutions.
Hence $b \neq 0$ makes the system inconsistent.

Q.8 The matrix $A = \begin{bmatrix} -3 & 2 & 0 \\ 0 & 2000 & 1 \\ 0 & 0 & 79 \end{bmatrix}$ is

(A) Upper Triangular (B) Lower Triangular (C) Both (A) and (B) (D) Neither (A) & (B)

A.8 (A) ✓

P.T.O

Q.9 If a system of equations has no solution, what does the graph look like?
(A) intersecting lines (B) parallel lines (C) skew lines (D) intersecting lines

A.9 (B) ✓

If a system has no solution the the lines representing the equations will never meet on the graph. Therefore parallel lines.

Q.10 Let A be a $n \times n$ matrix & A is non-singular. Then which of the following is incorrect?

(A) A is a product of elementary matrices
(B) A has a right inverse
(C) $AX = 0$ has non-trivial solutions
(D) A has a left inverse

A.10 (C) ✓

If A is a $n \times n$ non-singular matrix, then the homogeneous system $AX = 0$ has only the trivial solution $X = 0$.

Short-descriptive

Q.1 Find the relationship between a & b such that the following system has infinitely many solutions.

$$-x + 2y = a$$
$$-3x + 6y = b$$

A.1 Augmented matrix $A = \begin{pmatrix} -1 & 2 & a \\ -3 & 6 & b \end{pmatrix} \times -1 \sim \begin{pmatrix} 1 & -2 & -a \\ -3 & 6 & b \end{pmatrix} \times 3 +$

$$A \sim \begin{pmatrix} 1 & -2 & -a \\ 0 & 0 & b-3a \end{pmatrix}$$

For the system to have infinitely many solutions $b - 3a = 0$

$\therefore \boxed{b = 3a}$

Q.2 Determine whether the following system has no solution, exactly one solution, or infinitely many solutions.
$$2x + 2y = 2$$
$$x + y = 4$$

A.2 Augmented matrix: $A = \begin{pmatrix} 2 & 2 & 2 \\ 1 & 1 & 4 \end{pmatrix} \rightarrow \div 2 \sim \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 4 \end{pmatrix} \times -1 +$

$$A \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

so bottom row $\Rightarrow 0x + 0y = 3$ which is not possible.

$\therefore$ System has $\boxed{\text{no solution}}$.

Q.3 Find the value of k that makes the system $\begin{bmatrix} 15 & -3 & 6 \\ -10 & k & 9 \end{bmatrix}$ inconsistent?

A.3 $A = \begin{pmatrix} 15 & -3 & 6 \\ -10 & k & 9 \end{pmatrix} \xrightarrow{\div 15} \sim \begin{pmatrix} 1 & -1/5 & 2/5 \\ -10 & k & 9 \end{pmatrix} \xrightarrow{\times 10,+} \sim \begin{pmatrix} 1 & -0.2 & 0.4 \\ 0 & k-2 & 13 \end{pmatrix}$

For the system to be inconsistent $k-2=0$ $\therefore \boxed{k=2}$

Q.4 For which values of α the system $\begin{matrix} x+y-z=1 \\ 2x+y-3z=1 \\ -x+y-z=\alpha \end{matrix}$ is ~~in~~consistent ?

A.4 Augmented matrix : $A = \begin{pmatrix} 1 & 1 & -1 & 1 \\ 2 & 1 & -3 & 1 \\ -1 & 1 & -1 & \alpha \end{pmatrix}$ ~~$\xrightarrow{\div 2} \sim \begin{pmatrix} 1 & 0.5 & -1.5 & 0.5 \\ -1 & 1 & -1 & \alpha \end{pmatrix}$~~

~~$R_2 \to R_1 + R_2$~~

~~$A \sim \begin{pmatrix} 1 & 0.5 & -1.5 & 0.5 \\ 0 & 1.5 & -2.5 & \alpha+0.5 \end{pmatrix} \xrightarrow{\div 1.5} \sim \begin{pmatrix} 1 & 0.5 & -1.5 & 0.5 \\ 0 & 1 & -5/3 & \frac{2\alpha+1}{3} \end{pmatrix}$~~

$A = \begin{pmatrix} 1 & 1 & -1 & 1 \\ 2 & 1 & -3 & 1 \\ -1 & 1 & -1 & \alpha \end{pmatrix} \xrightarrow{\times -2,+} \sim \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & -1 & -1 & -1 \\ -1 & 1 & -1 & \alpha \end{pmatrix} + \sim \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & -2 & \alpha+1 \end{pmatrix} \xrightarrow{\times -2,+}$

$A \sim \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -4 & \alpha-1 \end{pmatrix} \xrightarrow{-,\ \div(-4)} \sim \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & \frac{1-\alpha}{4} \end{pmatrix}$

For the system to be consistent $\frac{1-\alpha}{4} \neq 0 \Rightarrow \alpha \neq 1$

~~So for all values of~~

For $\boxed{\alpha \neq 1}$, the system is consistent.

Q.5 Find the inverse by Gauss-Jordan method $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$.

A.5 $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$; $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$[A|I] : \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right] \xrightarrow{\times -2,+} \sim \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right]_{\times -1} \sim \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & -1 \end{array}\right]_{\times -2} +$

$[A|I] \sim \left[\begin{array}{cc|cc} 1 & 0 & -3 & 2 \\ 0 & 1 & 2 & -1 \end{array}\right]$

$[A|I] = [I|A^{-1}]$

$\therefore \boxed{A^{-1} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}}$

Q.6 Express $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ as a product of elementary matrices.

A.6 Reduction to Row Echlon Form | Elementary Matrix | Inverse of Elementary Matrix

$E = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \times \frac{1}{2}$ $\quad E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$ $\quad E_1^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}_{\times -1}$ $\quad E_2 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ $\quad E_2^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ — —

$E = E_1^{-1} E_2^{-1}$ $\quad \because E = E_1^{-1} \cdot E_2^{-1} \cdots E_k^{-1}$

$\therefore \boxed{\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}$

Q.7 For what condition is the system
$x + 2z = \alpha$
$2x + y + 5z = \beta$
$x - y + z = \gamma$
consistent?

A.7 Augmented matrix: $A = \begin{pmatrix} 1 & 0 & 2 & \alpha \\ 2 & 1 & 5 & \beta \\ 1 & -1 & 1 & \gamma \end{pmatrix} \begin{matrix} \times -2 & \times -1 \end{matrix} \sim \begin{pmatrix} 1 & 0 & 1 & \alpha \\ 0 & 1 & 1 & \beta - 2\alpha \\ 0 & -1 & -1 & \gamma - \alpha \end{pmatrix}$

$A = \begin{pmatrix} 1 & 0 & 1 & \alpha \\ 0 & 1 & 1 & \beta - 2\alpha \\ 0 & -1 & -1 & \gamma - \alpha \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & \alpha \\ 0 & 1 & 3 & \beta - 2\alpha \\ 0 & 0 & 0 & \beta + \gamma - 3\alpha \end{pmatrix}$

~~$A = \begin{pmatrix} 1 & 0 & 1 & \alpha \\ 0 & 1 & 3 & \beta - 2\alpha \\ 0 & 0 & 1 & \frac{\beta + \alpha - 3\alpha}{3} \end{pmatrix}$~~

For system to be consistent $\beta + \gamma - 3\alpha = 0$

i.e $\boxed{3\alpha = \beta + \gamma}$

Q.8 Verify whether $AB = BA$ where $A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$.

A.8 $AB = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$; $BA = \begin{bmatrix} -1 & 2 \\ 0 & 2 \end{bmatrix}$

$AB \neq BA$

Q.9 Find LU decomposition for
$-2x + y = -1$
$4x - y = 5$

A.9 $Ax = b$: $\begin{bmatrix} -2 & 1 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$ $\quad A = \begin{bmatrix} -2 & 1 \\ 4 & -1 \end{bmatrix}$; $B = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$

$A = LU$

$LUx = b$

$Ux = y$

$Ly = b$

Reduction to REF	Elementary Matrix	Inverse of Elementary Matrix
$\begin{bmatrix} -2 & 1 \\ 4 & -1 \end{bmatrix} \times -\frac{1}{2}$	$E_1 = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$	$E_1^{-1} = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & -\frac{1}{2} \\ 4 & -1 \end{bmatrix} \times -4 +$	$E_2 = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}$	$E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix} \times \frac{1}{2} +$	$E_3 = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$	$E_3^{-1} = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$		

$$U = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix} ; \quad L = E_1^{-1} E_2^{-1} E_3^{-1} = \begin{bmatrix} -2 & 0 \\ 4 & 1 \end{bmatrix}$$

LU-Decomposition

$$\boxed{A = LU \qquad \begin{bmatrix} -2 & 1 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix}}$$

$$LUx = b$$

$$\begin{bmatrix} -2 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$$

~~[illegible]~~ $Ux = y$

$$\begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

$$Ly = b$$

$$\begin{bmatrix} -2 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$$

$$x_1 = \frac{1}{2}$$
$$y_1 = 3$$

$$\begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 3 \end{bmatrix}$$

Ans $\boxed{\begin{matrix} y = 3 \\ x = 2 \end{matrix}}$

Q. 10 For $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ & $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Find all vectors x such that $AX = 0$.

A. 10 Here A is coefficient Matrix. By Gauss-Jordan's Method.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix} \underset{2+}{\overset{\times -1}{\sim}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \overset{2+}{\sim} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \overset{\times -2 \; +}{\sim} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ $x + 3z = 0$ (Since 3rd row has no pivot; z is free variable)

$\therefore$ ~~z = t~~ $y - z = 0$

Let $z = t$; $y = z = t$; $x = -3z = -3t$

$$X = \begin{bmatrix} -3t \\ t \\ t \end{bmatrix}$$ Taking $t = 1$ $x = (-3 \;\; 1 \;\; 1)$

$\boxed{X = (-3t \;\; t \;\; t)}$ for any values of t

Module - 2 Multiple Choice Questions

Q.1 Which set is not a vector space

(i) The span of the set of rows in A (ii) The span of the set of columns in A
(iii) The set $\{x : Ax = 0\}$ (iv) The set $\{x : Ax = e_1\}$ where $e_1 = (1, 0, \ldots, 0)$

A.1 (iv)

(i) & (ii) are row & column vector spaces respectively
(iii) is subspace of R^n & ∴ a vector space
So by method of elimination, answer is (iv)

Q.2 Which one is not a vector space?

(i) The kernel of a matrix A (ii) The set of all sequences $(x_1, x_2, x_3, \ldots)$
(iii) The span of a set of vectors in a vector space
(iv) The set of all discontinuous functions defined on the interval $(0, 1)$.

A.2 (iv)

set of all "continuous" functions defined on the interval $(0, 1)$ is a vector space, since sum & scalar multiples of continuous functions are also continuous (& addition is commutative)

Q.3 For what value of λ is the vector $(\lambda, 3, -5)$ in the linear span of $(1, 3, -1)$ & $(-5, -8, 2)$?

(i) 3 (ii) −6 (iii) −13 (iv) none of these

A.3
$$(\lambda, 3, -5) = x(1, 3, -1) + y(-5, -8, 2) \qquad x, y \in R$$

$$\left.\begin{aligned} x - 5y &= \lambda \\ 3x - 8y &= 3 \\ -x + 2y &= -5 \end{aligned}\right\} \Rightarrow \begin{aligned} x &= 17 \\ y &= 6 \end{aligned}$$

$$\therefore\ 17 - 5(6) = \lambda$$
$$\boxed{\lambda = -13}$$

(iii)

Q.4 Which of these assertions is not logically equivalent to all the others? (The matrix A is $n \times n$).

(i) The row vectors in A span R^n (ii) The column vectors in A span R^n
(iii) For each $b \in R^m$, the equation $Ax = b$ has a solution
(iv) A has a pivot position in each row.

A.4 (i)

Row vectors do not span R^n in A

Q.5 Which is not a vector space ?

(i) The set of all polynomials that satisfy $p(0) = 0$

(ii) The set of all polynomials of the form $p(t) = a + bt + ct^2 + t^3$

(iii) The set of all vectors of the form $[a+b,\ b+3a,\ -a-4b]$

(iv) The set of all vectors $[a+b^3,\ a-b^3]$

A.5 (ii)

Polynomial of degree 3 is not vector space. The set doesn't contain 0-element.

Q.6 Which set is not a subspace of R^3?

(i) $\{(x,y,z)^T : 3(x+1)+2(y+1)+z=5\}$ (ii) $\{(x,y,z) : x(y+1)-y(x-2)=z\}$

(iii) $\{(x,y,z)^T : 3(x+1)+2(y+1)+z=6$ (iv) $\{(x,y,z)^T : x(y+z+3) = z(x+y)+y(x-z+1)\}$

A-6 (iv)

$x(y+z+3) = z(x+y)+y(x-z+1) = xz+yz+xy-yz+y$

$xy+xz+3x = xy+xz+y$

$\Rightarrow 3x = y$ $\therefore$ they are linearly dependent

$\therefore$ (iv) is not subspace.

Q.7 Let $u=(1,2,3)$, $v=(0,-1,2)$ & $w=(2,7,0)$. Which assertion is true?

(i) $u \in span(v,w)$ & $w \notin span(u,v)$ (iii) $v \in span(u,w)$ & $u \notin span(v,w)$

(ii) $u \in span(v,w)$ & $w \in span(u,v)$ (iv) $w \in span(u,v)$ & $v \notin span(u,w)$

A.7 (ii)

$u = 1.5v + 0.5w$ $\therefore u \in span(v,w)$

$w = 2u - 3v$ $\therefore w \in span(u,v)$

Q.8 Let S be a linearly independent set of 4 polynomials in P_3. What conclusion can be drawn?

(i) S spans P_3 (ii) one member of S can be discarded to obtain basis of P_3

(iv) S does not span P_3 (iii) The set S can be extended to obtain the basis of P_3.

A. (i), ~~(ii)~~

$S \to$ linearly independent; has 4 polynomials $\therefore$ spans P_3

~~P_3 requires min^m 3 polynomials $\therefore$ 1 can be discarded~~

Q.9 What is the dimension of the space of vectors in R^5 having the form $(a-3b,\ b-a,\ a+2b,\ a,\ b)$? (i) 2 (ii) 3 (iii) 4 (iv) 5

A.9 (i)

$(a-3b,\ b-a,\ a+2b,\ a,\ b) = a\underbrace{(1,-1,1,1,0)}_{v_1} + b\underbrace{(-3,1,2,0,1)}_{v_2}$ $\therefore$ dim = 2

Q.10 Which set is a basis of R^3?

(i) $\{(1,2,3)^T, (4,5,6)^T, (5,7,5)^T\}$ (ii) $\{(1,2,3)^T, (4,5,6)^T, (5,7,9)^T\}$

(iii) $\{(1,2,3)^T, (4,5,6)^T, (7,8,9)^T\}$ (iv) $\{(1,2,3)^T, (4,5,9)^T, (7,8,15)^T\}$

A.10 (i) ✓

$$\left.\begin{aligned}
&\text{(ii)}\quad (5,7,9) = (1,2,3) + (4,5,6) \\
&\text{(iii)}\quad 2\times(4,5,6) = (7,8,9) + (1,2,3) \\
&\text{(iv)}\quad 2\times(4,5,9) = (7,8,15) + (1,2,3)
\end{aligned}\right\} \therefore$$

linearly dependent & do not form basis

Q.11 What is the dimension of the space of polynomials spanned by 4 functions

$u_1(t) = t+t^2$, $u_2(t) = t^3+t^4$, $u_3(t) = t^5 - t^2 - t^4$, $u_4(t) = t+t^3+t^5$?

(i) 1 (ii) 2 (iii) 3 (iv) 4

A.11 (iii) ✓

$u_3(t) = u_4(t) - u_1(t) - u_2(t)$ or $u_4(t) = u_1(t) + u_2(t) + u_3(t)$

$\therefore$ dim = 3 ~~{(n-1) = 5-1 = 4}~~

Q.12 What is the dimension of space of matrices spanned by $\left\{\begin{bmatrix}1&0\\1&1\end{bmatrix}, \begin{bmatrix}1&1\\1&0\end{bmatrix}, \begin{bmatrix}1&1\\1&1\end{bmatrix}, \begin{bmatrix}1&1\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\1&1\end{bmatrix}\right\}$? (i) 1 (ii) 2 (iii) 3 (iv) 4

A.12 $2\times\begin{bmatrix}1&1\\1&1\end{bmatrix} = \begin{bmatrix}1&0\\1&1\end{bmatrix} + \begin{bmatrix}1&1\\0&0\end{bmatrix} + \begin{bmatrix}0&1\\1&1\end{bmatrix}$ $\therefore$ only $\begin{bmatrix}1&0\\1&1\end{bmatrix}, \begin{bmatrix}1&1\\1&0\end{bmatrix}, \begin{bmatrix}1&1\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\1&1\end{bmatrix}$ are linearly independent.

dim = 4 $\boxed{\text{(iv) ✓}}$

Q.13 Which of these statements are false?

(i) $\mathrm{Dim}(P_n) = n+1$ (ii) $\mathrm{Dim}(R^{m\times n}) = m\times n$ (iii) $\mathrm{Dim}(P_n) = n$ (iv) $\mathrm{Dim}(R^m) = m+1$

A.13 (iii), (iv) ✓

Properties of basis & dimensions for general cases

Q.14 What is the dimension of the space of all matrices $\begin{bmatrix}a&b\\c&d\end{bmatrix}$ for which $2a = 3d$?

(i) 1 (ii) 2 (iii) 3 (iv) 4

A.14 (iii) ✓

since $2a = 3d$, only a, b, c or b, c, d are linearly independent

$\therefore$ dim = 3.

Q.15 A linearly independent set can contain a linearly dependent subset – FALSE

Q.16 If a set of vectors is linearly dependent, then we can add vectors to the set & make it linearly independent – FALSE

Q.17 The set of all polynomials p of degree at most 7 s.t $p(7) = 0$ is a vector space – TRUE

Q.18 The set of all vectors $x = (x_1, x_2)$ such that $x_1 \geq 0$ & $x_2 \geq 0$ is a subspace of R^2 - ~~FALSE~~ ~~TRUE~~
- FALSE

Q.19 Every subspace in a vector space V is the span of some vectors in V - TRUE

Q.20 If H is a linearly independent set of vectors in some vector space, then H is a basis for the span of H - ~~TRUE~~ ~~FALSE~~ TRUE

Short Answer Questions:

Q.1 Consider $\{p_1, p_2, p_3, p_4\}$ where $p_1(t) = 1$, $p_2(t) = t$, $p_3(t) = 4 - t$, $p_4(t) = t^3$. Determine whether this set of polynomials is linearly independent or linearly dependent.

A.1 $a p_1(t) + b p_2(t) + c p_3(t) + d p_4(t) = 0$

$a + bt + 4c - ct + dt^3 = 0 \Rightarrow$

$b - c = 0 \rightarrow b = c$

$a + 4c = 0 \rightarrow a = -4c$

$d = 0$

$$\left[\begin{array}{cccc|c} 1 & 0 & 4 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$\boxed{\therefore \text{ Linearly dependent}}$

Q.2 Is the following an example of vector space? $X = R$, $x \oplus y = x + y + 1$; $\alpha \otimes x = \alpha x + \alpha$

A.2 (i) $x \oplus y = y \oplus x$

$x + y + 1 = y + x + 1$ ✓

(ii) $\alpha \otimes (\beta \otimes x) = (\alpha\beta) \otimes x$

$\alpha \otimes (\beta x + \beta) = \alpha\beta x + \alpha\beta$

$\alpha(\beta x + \beta) + \alpha = \alpha\beta x + \alpha\beta$

$\alpha\beta x + \alpha\beta + \alpha \neq \alpha\beta x + \alpha\beta$

(iii) $x \in X$

$\therefore -x \in X$ such that $x \oplus (-x) = 0$

Let $y = -x \in X$

$x \oplus (-x) = x + (-x) + 1$

$0 = 0 + 1$

$0 \neq 1$

$\therefore$ inverse does not exist.

$\therefore$ X is not a vector space.

Q.3 Does the set of all vectors in R^4 that have exactly 2 zero entries span R^4?

A.3 The basic span of R^4 has vectors: $(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)$

These vectors ~~also~~ are in R^4 & span R^4 despite not having exactly 2 ~~zero entries~~

set of vectors having exactly 2 zero entries in R^4:

$\{(1,1,0,0), (0,0,1,1), (1,0,1,0), (1,0,0,1), (0,1,0,1), (0,1,1,0)\}$

$$S = \begin{Bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{Bmatrix}$$

$c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 + c_5v_5 + c_6v_6 = d_1w_1 + d_2w_2 + d_3w_3 + d_4w_4$

$w_1 = (1,0,0,0)^T \quad w_3 = (0,0,1,0)^T$

$w_2 = (0,1,0,0)^T \quad w_4 = (0,0,0,1)^T$

$$\left.\begin{aligned} 1c_1 + 0c_2 + 1c_3 + 1c_4 + 0c_5 + 0c_6 &= 1d_1 + 0d_2 + 0d_3 + 0d_4 \\ 1c_1 + 0c_2 + 0c_3 + 0c_4 + 1c_5 + 1c_6 &= 0d_1 + 1d_2 + 0d_3 + 0d_4 \\ 0c_1 + 1c_2 + 1c_3 + 0c_4 + 0c_5 + 1c_6 &= 0d_1 + 0d_2 + 1d_3 + 0d_4 \\ 0c_1 + 1c_2 + 0c_3 + 1c_4 + 1c_5 + 0c_6 &= 0d_1 + 0d_2 + 0d_3 + 1d_4 \end{aligned}\right\} D$$

Augmented matrix to reduced row echelon form

$$\left[\begin{array}{cccccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & -1/2 & 1/2 & 1/2 & 1/2 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1/2 & -1/2 & 1/2 & -1/2 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1/2 & -1/2 & -1/2 & 1/2 \end{array}\right]$$

$$\left.\begin{aligned} c_1 + c_5 + c_6 &= d_2 \\ c_2 + c_5 + c_6 &= \tfrac{1}{2}(-d_1 + d_2 + d_3 + d_4) \\ c_3 - c_5 &= \tfrac{1}{2}(d_1 - d_2 + d_3 - d_4) \\ c_4 - c_6 &= \tfrac{1}{2}(d_1 - d_2 - d_3 + d_4) \end{aligned}\right\}$$

This system is consisted with system D for all right hand side values. ∴ S spans V

Ans Yes, all vectors in R^4 that have exactly 2 zero entries span R^4.

Q.4 Consider $S = \{(1,3,2,0)^T, (-2,0,6,7)^T, (0,6,10,7)^T, (2,10,-3,1)^T\}$. Find basis for span(S).

A.4
$$S = \begin{bmatrix} 1 & -2 & 0 & 2 \\ 3 & 0 & 6 & 10 \\ 2 & 6 & 10 & -3 \\ 0 & 7 & 7 & 1 \end{bmatrix} \xrightarrow{\text{reduced}} \text{REF} = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

~~Note~~ $= I_4$ all rows have pivots

~~∴ Basis of span = Original pivot columns~~

~~∴ Basis = $\{(1,3,2,0)^T, (-2,0,6,7)^T, (0,6,10,7)^T, (2,10,-3,1)^T\}$~~

Basis : $\{(1,-2,0,1)^T, (0,1,1,0)^T, (0,0,0,1)^T\}$

Q.5 Is there a 2x2 matrix whose powers span $R^{2\times 2}$?

A.5 2x2 matrix should be polynomial in matrix A & would commute with A

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$; $a, b, c, d \in R$ $\therefore$ $ax + by = 0$ & $cx + dy = 0$.

Multiples of I_2 would only commute with all 2x2 matrices. $\therefore$ $\boxed{KI_2}$ is the 2x2 matrix whose powers span $R^{2\times 2}$.

Q.6 If A is 3x4 matrix, prove that the columns of A are linearly ~~in~~dependent.

A.6 $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}_{3\times 4}$ ~~x = 0 is lines~~ → The columns of this matrix are linearly dependent because there exists non-zero x such that

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x = 0 \quad \text{i.e} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Q.7 If A is 4x3 matrix, prove that the rows of A are linearly dependent.

A.7 $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ → The rows of this matrix are linearly dependent because ~~there exists non-zero x such that~~ the last row has all zeroes. So the matrix system will ~~$av_1 + bv_2 + cv_3 + dv_4 = 0$~~ have infinite non-zero solutions.

Q.8 In each case determine whether W is a subspace of R^3.

(i) $W = \{x : x_1^2 = x_1 + x_2\}$ (ii) $W = \{x : x_1 = 2x_3\}$

A.8 (i) $W = \{(x_1, x_2, x_3) ; x_1^2 = x_1 + x_2\}$

Let $\alpha = (a, b, c)$; $\beta = (u, v, w)$; $k \in R$ where $a^2 = a + c$; $u^2 = u + w$

for W to be subspace of R $\alpha + k\beta = (a, b, c) + k(u, v, w)$

$\alpha + k\beta = (a + ku, b + kv, c + kw)$

$\therefore (a + ku)^2 = a + ku + c + kw$

$a^2 + k^2u^2 + 2auk \neq a + ku + c + kw$ $\therefore$ $\boxed{\text{W is not subspace of } R^3}$

(ii) $W = \{(x_1, x_2, x_3) \mid x_1 = 2x_3\}$

For any vectors $\alpha = (a, b, c)$; $\beta = (u, v, w)$ in W & any skalar $k_1, k_2 \in R$

Then $k_1\alpha + k_2\beta = k_1(a, b, c) + k_2(u, v, w) = (k_1a + k_2u, k_1b + k_2v, k_1c + k_2w)$

$\therefore k_1a + k_2u = 2k_1c + 2k_2w$ ✓ $\because$ $a = 2c \Rightarrow k_1a = 2k_1c$, $u = 2w \Rightarrow k_2u = 2k_2w$

$\therefore$ $\boxed{\text{W is a subspace of } R^3}$

Q-9 If $V = R^n$ & $W = \{(x_1, x_2 \dots x_n) ; x_1 = x_n\}$, find a basis & dimension of W.

A-9

$$\alpha v_1 + \alpha v_2 + \alpha v_3 + \dots \alpha_n v_n = 0$$

$$\alpha_1 (1, 0, 0, 0 \dots, 1) + \alpha_2 (0, 1, 0, 0, \dots 0) + \alpha_3 (0, 0, 1, 0, 0, \dots 0)$$

$$+ \dots + \alpha_{n-1} (0, 0, 0, \dots 1, 0) = 0$$

$\because$ α_n does not exist $\because$ it is given $x_1 = x_n \Rightarrow v_1 = v_n$ $\therefore$ v_n is not linearly dependent.

$\therefore$ Basis of W will be $\{v_1, v_2 \dots v_{n-1}\}$

$\therefore$ $\boxed{\dim W = n-1}$

Q.10 Can you exhibit a basis of P_n consisting of all elements of degree n? All of the degree $\leq n-1$?

A-10

$$P_n = (x_1^n, x_2^n, x_3^n \dots x_n^n)$$

$$c_1 x_1^n + c_2 x_2^n + c_3 x_3^n + \dots c_n x_n^n = 0$$

$$\therefore \quad c_1 + c_2 + c_3 + c_4 \dots + c_n = 0$$

So $c_1, c_2, c_3 \dots c_n$ can have any values as long as they satisfy the above eq^n. Iff $c_1 = c_2 = c_3 \dots = c_n = 0$ then we have a basis for P_n with all elements of degree n i.e the ~~vector~~ ie the elements themselves.

As for all elements of degree $\leq n-1$, the basis will be $B = \{P_1(x), P_2(x) \dots P_{n-1}(x)\}$ with dimension $n-1$