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C1+TC1+TCC1+V2	EQUATIONS - DA	Date:

Q.1 Prove that the function $u = x(x^2-3y^2) + (x^2-y^2) + 2xy$ is harmonic. Also find the conjugate harmonic function v & the corresponding analytic function $(u+iv)$.

A.1 $u = x(x^2-3y^2)+(x^2-y^2)+2xy = x^3 - 3xy^2 + x^2 - y^2 + 2xy$

$u_x = 3x^2 - 3y^2 + 2x + 2y$

$u_{xx} = 6x + 2$

$u_y = -6xy - 2y + 2x$

$u_{yy} = -6x - 2$

$u_{xx} + u_{yy} = 6x+2-6x-2 = 0$

$\therefore$ function $u = x(x^2-3y^2)+(x^2-y^2)+2xy$ is harmonic since $u_{xx}+u_{yy}=0$.

Now by Cauchy-Riemann equations;

$v_y = u_x = 3x^2-3y^2+2x+2y$ —(1)

$v_x = -u_y = 6xy+2y-2x$ —(2)

To find analytic function:

$f'(z) = u_x + iv_x = u_x - iu_y$

$= 3x^2-3y^2+2x+2y + i(6xy+2y-2x)$

By milne thomson ; put $x=z$ & $y=0$ in $f'(z)$

$f'(z) = 3z^2 + 2z + \cancel{0} - 2zi$

$\therefore f(z) = \int 3z^2+2z-2zi \,.\, dz + c$

$\boxed{f(z) = z^3 + z^2 - iz^2 + c}$ $\longrightarrow$ analytic function

Now $f(z) = u+iv$ & $z = x+iy$

$\therefore u+iv = x^3-iy^3-3xy^2+i3x^2y + x^2-y^2+i2xy-ix^2+iy^2+2xy+c$

$= \underbrace{(x^3-3xy^2+x^2-y^2+2xy)}_{u \Rightarrow \text{given}} + i(-y^3+3x^2y+2xy-x^2+y^2)+c$

$\therefore$

$\therefore \boxed{v = -y^3+3x^2y+2xy-x^2+y^2+c}$

v is harmonic conjugate of u

Q.2 Prove that the function $v = 3x^2y + x^2 - y^3 - y^2$ is harmonic. Also find the conjugate harmonic function u & the corresponding analytic function $(u+iv)$

A.2 $v = 3x^2y + x^2 - y^3 - y^2$

$v_x = 6xy + 2x$

$v_{xx} = 6y + 2$

$v_y = 3x^2 - 3y^2 - 2y$

$v_{yy} = -6y - 2$

$v_{xx} + v_{yy} = 6y + 2 - 6y - 2 = 0$

$\therefore$ function $v = 3x^2y + x^2 - y^3 - y^2$ is harmonic since $v_{xx} + v_{yy} = 0$.

To find analytic function:

$f'(z) = u_x + iv_x = v_y + iv_x$ $[u_x = v_y$ by CR-equation$]$

$= 3x^2 - 3y^2 - 2y + i(6xy + 2x)$

By Milne Thomson method, put $x = z$ & $y = 0$ in $f'(z)$

$f'(z) = 3z^2 + 2zi$

$\therefore f(z) = \int 3z^2 + 2zi \cdot dz = z^3 + iz^2 + c$

$\boxed{f(z) = z^3 + iz^2 + c}$ $\rightarrow$ analytic function

Now $f(z) = u + iv$; $z = x + iy$

$\therefore u + iv = x^3 - iy^3 - 3xy^2 + i3x^2y + ix^2 - iy^2 - 2xy + c$

$= (x^3 - 3xy^2 - 2xy) + i(\underbrace{3x^2y + x^2 - y^3 - y^2}_{v \rightarrow \text{given}}) + c$

$\therefore \boxed{u = x^3 - 3xy^2 - 2xy + c}$

u is harmonic conjugate of v

Q.3 Find the analytic function $w = u+iv$, if $u = e^x(x\sin y + y\cos y)$. Hence find v.

A.3 $u = e^x x\sin y + e^x y\cos y$

$u_x = e^x x\sin y + e^x \sin y + e^x y\cos y$

$u_{xx} = e^x x\sin y + e^x\sin y + e^x \sin y + e^x y\cos y = e^x x\sin y + 2e^x\sin y + e^x y\cos y$

$u_y = e^x x\cos y + e^x\cos y - e^x y\sin y$

$u_{yy} = -e^x x\sin y - e^x\sin y - e^x\sin y - e^x y\cos y = -e^x x\sin y - 2e^x \sin y - e^x y\cos y$

$\Rightarrow u_{xx} + u_{yy} = 0$ i.e u is harmonic function

$w = u+iv = f(z) \rightarrow$ analytic function

Now $f'(z) = u_x + iv_x = u_x - iu_y$ [$v_x = -u_y$ by CR equation]

$f'(z) = e^x x\sin y + e^x \sin y + e^x y\cos y - i(e^x x\cos y + e^x\cos y - e^x y\sin y)$

By Milne-Thomson method, put $x = z$ & $y = 0$ in $f'(z)$

$f'(z) = -i(e^z z + e^z)$

$\therefore f(z) = -i\int e^z z + e^z \cdot dz = -i\left[\int e^z z\cdot dz + \int e^z\cdot dz\right]$

$= -i\left[z(e^z) - \int e^z\cdot dz + \int e^z \cdot dz\right] + c$

$\boxed{f(z) = -ize^z + c} \rightarrow$ analytic function $w = u+iv$

$f(z) = u+iv = w$; $z = x+iy$

$u+iv = -i(x+iy)e^{(x+iy)} + c = (y-ix)e^x\cdot e^{iy} + c$

$= e^x(y-ix)(\cos y + i\sin y) + c$ [$e^{i\theta} = \cos\theta + i\sin\theta$]

$= e^x(y\cos y + iy\sin y - ix\cos y + x\sin y) + c$

$= \underbrace{e^x(x\sin y + y\cos y)}_{u \Rightarrow \text{given}} + i(e^x y\sin y - e^x x\cos y) + c$

$\therefore \boxed{v = e^x y\sin y - e^x x\cos y + c}$

v is harmonic conjugate of u.

Q.4 Find the analytic function $w = u + iv$, if $v = e^{-x}(x\cos y + y\sin y)$. Hence find u.

A.4

$v = e^{-x}x\cos y + e^{-x}y\sin y$

$v_x = -e^{-x}x\cos y + e^{-x}\cos y - e^{-x}y\sin y$

$v_{xx} = e^{-x}x\cos y - e^{-x}\cos y - e^{-x}\cos y + e^{-x}y\sin y$

$v_y = -e^{-x}x\sin y + e^{-x}\sin y + e^{-x}y\cos y$

$v_{yy} = -e^{-x}x\cos y + e^{-x}\cos y + e^{-x}\cos y - e^{-x}y\sin y$

$\therefore\ v_{xx} + v_{yy} = 0$ i.e v is harmonic function

$w = u + iv = f(z) \rightarrow$ analytic function

Now $f'(z) = u_x + iv_x = v_y + iv_x$ [$u_x = v_y$ by CR-eq^n]

$f'(z) = -e^{-x}x\sin y + e^{-x}\sin y + e^{-x}y\cos y + i(-e^{-x}x\cos y + e^{-x}\cos y - e^{-x}y\sin y)$

By Milne-Thomson, put $x = z$ & $y = 0$ in $f'(z)$

$f'(z) = i(-e^{-z}z + e^{-z})$

$f(z) = i\left[\int e^{-z}dz - \int e^{-z}z\,dz\right] + c$

$\quad = i\left[\int e^{-z}dz + ze^{-z} - \int e^{-z}dz\right] + c$

$\boxed{f(z) = ize^{-z} + c} \longrightarrow$ analytic function $w = u + iv$

$\cancel{u+iv}\ f(z) = u + iv = w$; $z = x + iy$

$u + iv = i(x+iy)e^{-x-iy} = (ix - y)e^{-x}\cdot e^{-iy} = (ix - y)e^{-x}(\cos y - i\sin y)$

$\because \boxed{e^{-i\theta} = \cos\theta - i\sin\theta}$

$\therefore\ u + iv = e^{-x}(ix\cos y + x\sin y - y\cos y + iy\sin y)$

$\quad = e^{-x}(x\sin y - y\cos y) + i\,(\underbrace{e^{-x}(x\cos y + y\sin y)}_{v \rightarrow \text{given}})$

$\therefore\ \boxed{u = e^{-x}(x\sin y - y\cos y)}$

u is harmonic conjugate of v

Q.5 Find the analytic function $w = u+iv$, if $v = e^{-2y}(y\cos 2x + x\sin 2x)$. Hence find u.

A.5 $v = e^{-2y} y\cos 2x + e^{-2y} x\sin 2x$

$v_x = -2e^{-2y} y\sin 2x + e^{-2y}\sin 2x + 2e^{-2y} x\cos 2x$

$v_{xx} = -4e^{-2y} y\cos 2x + 2e^{-2y}\cos 2x + 2e^{-2y}\cos 2x - 4e^{-2y} x\sin 2x$

$= -4e^{-2y} y\cos 2x + 4e^{-2y}\cos 2x - 4e^{-2y} x\sin 2x$

$v_y = e^{-2y}\cos 2x - 2e^{-2y} y\cos 2x - 2e^{-2y} x\sin 2x$

$v_{yy} = -2e^{-2y}\cos 2x - 2e^{-2y}\cos 2x + 4e^{-2y} y\cos 2x + 4e^{-2y} x\sin 2x$

$= 4e^{-2y} y\cos 2x - 4e^{-2y}\cos 2x + 4e^{-2y} x\sin 2x$

$\therefore$ $v_{xx} + v_{yy} = 0$ i.e v is harmonic function

$w = u+iv = f(z)$ → analytic function

By CR-eq^n $u_x = v_y$; $u_y = -v_x$

$f'(z) = u_x + iv_x = v_y + iv_x$

$= e^{-2y}\cos 2x - 2e^{-2y} y\cos 2x - 2e^{-2y} x\sin 2x + i(-2e^{-2y} y\sin 2x + e^{-2y}\sin 2x + 2e^{-2y} x\cos 2x)$

By Milne-Thomson method, put $x = z$ & $y = 0$ in $f'(z)$

$f'(z) = \cos 2z - 2z\sin 2z + i(\sin 2z + 2z\cos 2z)$

$\cancel{f(z) = \frac{\sin 2z}{2} - \frac{i\cos 2z}{2} - 2\int z\sin 2z}$

$f(z) = \int \cos 2z \cdot dz - 2\int z\sin 2z \cdot dz + i\int \sin 2z\, dz + 2i\int z\cos 2z \cdot dz + c$

$= \int \cos 2z \cdot dz + z\cos 2z - \int \cos 2z \cdot dz + i\int \sin 2z\, dz + iz\sin 2z - i\int \sin 2z\, dz + c$

$\boxed{f(z) = z\cos 2z + iz\sin 2z + c}$ → analytic function

$f(z) = z(\cos 2z + i\sin 2z) + c = ze^{i2z} + c$

$\boxed{f(z) = ze^{i2z} + c}$ → analytic equation.

Now $f(z) = u+iv$ & $z = x+iy$

$u+iv = (x+iy)e^{2ix-2y} + c = (x+iy)e^{-2y} \cdot e^{i2x} + c$

$\therefore u+iv = (x+iy)\cdot e^{-2y}\cdot(\cos 2x + i\sin 2x) + c$

$\because \left[e^{i\theta} = \cos\theta + i\sin\theta\right]$

$u+iv = e^{-2y}(x\cos 2x + ix\sin 2x + iy\cos 2x - y\sin 2x) + c$

$= e^{-2y}(x\cos 2x - y\sin 2x) + i\underbrace{e^{-2y}(y\cos 2x + x\sin 2x)}_{v \to \text{given}} + c$

$\therefore \boxed{u = e^{-2y}(x\cos 2x - y\sin 2x) + c}$

u is harmonic conjugate of v

Q.6 Find the analytic function $f(z) = P + iQ$, if $P - Q = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$

A.6 $f(z) = P + iQ$ —— (1)

$if(z) = i(P+iQ) = iP - Q$ —— (2)

$(1) + (2) \rightarrow f(z) + if(z) = P + iQ + iP - Q = (P-Q) + i(P+Q)$

$(1+i)f(z) = (P-Q) + i(P+Q)$

Let $F(z) = (1+i)f(z)$ & $U = u - v$; $V = u + v$ $[P = u, Q = v]$

$\therefore F(z) = U + iV$

Given $P - Q = u - v = U = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$

$U_x = \dfrac{2\cos 2x(\cosh 2y - \cos 2x) - \sin 2x(2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$

$U_x = \dfrac{2[\cosh 2y\cos 2x - 1]}{(\cosh 2y - \cos 2x)^2}$

$U_y = \dfrac{(0)(\cosh 2y - \cos 2x) - \sin 2x(2\sinh 2y)}{(\cosh 2y - \cos 2x)^2} = \dfrac{-2\sinh 2y\sin 2x}{(\cosh 2y - \cos 2x)^2}$

Now $F'(z) = U_x + iV_x = U_x - iU_y$ $[\because V_x = -U_y$ by CR-eqns$]$

$F'(z) = \dfrac{2(\cosh 2y\cos 2x - 1) + i(2\sinh 2y\sin 2x)}{(\cosh 2y - \cos 2x)^2}$

By Milne-Thomson method, put $x = z$ & $y = 0$ in $F'(z)$

$F'(z) = \dfrac{-2(1-\cos 2z)}{(1-\cos 2z)^2} + \dfrac{i(0)}{(1-\cos 2z)^2} = -\mathrm{cosec}^2 z + 0$

$F'(z) = -\operatorname{cosec}^2 z$

$F(z) = \int -\operatorname{cosec}^2 z \, dz + c \;=\; \cot z + ic$

$$\boxed{F(z) = \cot z + ic}$$

$(1+i) f(z) = \cot z + ic$

$f(z) = \dfrac{\cot z}{1+i} + \dfrac{ic}{i+1}$ Let $c_1 = \dfrac{ic}{i+1}$

$f(z) = \dfrac{\cot z (1-i)}{(1+i)(1-i)} + c_1$

$$\boxed{f(z) = \frac{(1-i)}{2} \cot z + c_1} \rightarrow \text{analytic function.}$$

Q.7 Evaluate $\displaystyle\int_0^{2\pi} \frac{a\,d\theta}{a^2+\sin^2\theta}$ $a > 0$

A.7 $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$ (Type-1 problem)

$$\int_0^{2\pi} \frac{a\,d\theta}{a^2+\sin^2\theta} = a \int_0^{2\pi} \frac{2\,d\theta}{2a^2+1-\cos 2\theta}$$

Let $2\theta = \phi \rightarrow 2d\theta = d\phi$

if $\theta = 0 \rightarrow \phi = 0$

if $\theta = 2\pi \rightarrow \phi = 4\pi$

$$I = a \int_0^{4\pi} \frac{d\phi}{2a^2+1-\cos\phi} = 2a \int_0^{2\pi} \frac{d\phi}{2a^2+1-\cos\phi}$$ since cos is even f^n

Let $z = e^{i\phi} \rightarrow \dfrac{dz}{d\phi} = ie^{i\phi} = iz \rightarrow d\phi = \dfrac{dz}{iz}$

$\cos\phi = \dfrac{1}{2}\left(\dfrac{z^2+1}{z}\right)$

$$I = \int_c \frac{2a\frac{dz}{iz}}{2a^2+1-\frac{1}{2}\left(\frac{z^2+1}{z}\right)} = \frac{4a}{i} \int_c \frac{dz}{-z^2+z(4a^2+2)-1}$$

$$I = -\frac{4a}{i} \int_c \frac{dz}{z^2 - z(4a^2+2)+1} \quad \text{—(1)}$$

Let $z^2 - z(4a^2+2) + 1 = 0$

then $z = \dfrac{(4a^2+2) \pm \sqrt{16a^4+16a^2}}{2} = 2a^2+1 \pm 2a\sqrt{a^2+1}$

i.e.

$z = 2a^2+1+2a\sqrt{a^2+1}\ ,\quad 2a^2+1-2a\sqrt{a^2+1}$

Let $\alpha = 2a^2+1+2a\sqrt{a^2+1}$ & $\beta = 2a^2+1-2a\sqrt{a^2+1}$ $\quad a>0$

$\Rightarrow$ β lies inside 'c'

$\left[\underset{z=\beta}{\text{Res}} f(z)\right] = \dfrac{P(\beta)}{Q(\beta)}$ where $\dfrac{P(z)}{Q(z)} = \dfrac{1}{(z-\alpha)(z-\beta)}$

$\therefore \left[\underset{z=\beta}{\text{Res}} f(z)\right] = \dfrac{1}{z-\alpha+z-\beta} = \dfrac{1}{\beta-\alpha}$

Now $\beta-\alpha = -4a\sqrt{a^2+1}$

$\therefore \left[\underset{z=\beta}{\text{Res}} f(z)\right] = \dfrac{1}{-4a\sqrt{a^2+1}}$

(1) $\Rightarrow$ $\dfrac{-4a}{i} \times 2\pi i \times \dfrac{1}{-4a\sqrt{a^2+1}}$

$$\boxed{I = \frac{2\pi}{\sqrt{a^2+1}}\ ,\quad a>0}$$

Q.8 $\displaystyle\int_0^\infty \frac{x^4\,dx}{x^6+1}$

$\rightarrow$ $\displaystyle\int_c \frac{z^4\,dz}{z^6+1} = \int_{-R}^{R} \frac{x^4}{x^6+1}\,dx + \underbrace{\int_{c_1} \frac{z^4\,dz}{z^6+1}}_{\to 0 \text{ when } R\to\infty}$

For $\displaystyle\int_c \frac{z^4}{z^6+1}\,dz$ Let $z^6+1=0$

$z^6 = -1 = e^{i(2n+1)\pi} \Rightarrow z = e^{i(2n+1)\pi/6}$

$n = 0,1,2,3,4,5$

$n=0$ $\quad z = e^{i\pi/6} = \dfrac{\sqrt{3}+i}{2}$ $\qquad n=3;\ z = e^{i7\pi/6} = -\dfrac{(\sqrt{3}+i)}{2}$

$n=1$ $\quad z = e^{i\pi/2} = i$ $\qquad n=4;\ z = \cancel{-\dfrac{(\sqrt{3}+i)}{2}}\ e^{i3\pi/2} = -i$

$n=2$ $\quad z = e^{i5\pi/6} = \dfrac{-\sqrt{3}+i}{2}$ $\qquad n=5;\ z = e^{i11\pi/6} = \dfrac{\sqrt{3}-i}{2}$

$z = e^{i\pi/6}, e^{i\pi/2}, e^{i5\pi/6}$ lie inside 'c'

$\therefore$

$$R_1 = \text{Res}[f(z), z = e^{i\pi/6}] = \frac{P(z)}{Q'(z)} \quad \text{where } P(z) = z^4 \ \& \ Q(z) = z^6+1$$

$$\therefore Q'(z) = 6z^5$$

$$= \frac{e^{i2\pi/3}}{6e^{i5\pi/6}} = \frac{e^{-i\pi/6}}{6}$$

$$R_2 = \text{Res}[f(z), z = e^{i\pi/2}] = \frac{e^{i4\pi/2}}{6e^{i5\pi/2}} = \frac{e^{-i\pi/2}}{6}$$

$$R_3 = \text{Res}[f(z), z = e^{i5\pi/6}] = \frac{e^{-i5\pi/6}}{6}$$

$$\int_C \frac{z^4}{z^6+1} dz = 2\pi i [R_1 + R_2 + R_3] = \frac{\pi i}{3}[e^{-i\pi/6} + e^{-i\pi/2} + e^{-i5\pi/6}]$$

$$= \frac{\pi i}{6}[\sqrt{3} - i + (-i) + (-\sqrt{3} - i)] = \frac{\pi}{2}$$

$$\therefore \int_{-R}^{R} \frac{x^4\,dx}{x^6+1} + \underbrace{\int_{C_1} \frac{z^4}{z^6+1} dz}_{\to 0} = \frac{\pi}{2}$$

$R \to \infty \quad \therefore$

$$\int_{-\infty}^{\infty} \frac{x^4}{x^6+1} dx = \frac{\pi}{2} \rightarrow 2\int_0^{\infty} \frac{x^4\,dx}{x^6+1} = \frac{\pi}{2}$$

$$\therefore \boxed{\int_0^{\infty} \frac{x^4}{x^6+1} dx = \frac{\pi}{4}}$$

Q.9 $$\int_{-\infty}^{\infty} \frac{\sin x}{x^2+4x+5} dx$$

$\sin x \rightarrow e^{iz}$

$$\int_C \frac{e^{iz}}{z^2+4z+5} \cdot dz = \int_{-R}^{R} \frac{e^{ix}}{x^2+4x+5} \cdot dx + \int_{C_1} \frac{e^{iz}}{z^2+4z+5} \cdot dz$$

$\rightarrow \int_C \frac{e^{iz}}{z^2+4z+5} \cdot dz$ $\quad P(z) = e^{iz}$

$Q(z) = z^2+4z+5$; $Q'(z) = 2z+4$

Let $z^2+4z+5 = 0$

$z = \frac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i$

So $z = -2+i$, $z = -2-i$

$z = -2+i$ lies inside 'c' is a simple pole

$\text{Res}[f(z), z=-2+i] = \frac{P(z)}{Q'(z)} = \frac{e^{i(-2+i)}}{2(-2+i)+4} = \frac{e^{-2i}}{2ie}$

$\int_C \frac{e^{iz}}{z^2+4z+5} \cdot dz = 2\pi i \times \frac{e^{-2i}}{2ie} = \frac{\pi e^{-2i}}{e}$

$\int_{-R}^{R} \frac{e^{ix}}{x^2+4x+5} \cdot dx + \int_{C_1} \frac{e^{iz}}{z^2+4z+5} \cdot dz = \frac{\pi}{e} e^{-2i}$

$R \to \infty$ then $\int_{C_1} \to 0$

$\therefore \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+4x+5} dx = \frac{\pi}{e} e^{-2i}$

or

$\int_{-\infty}^{\infty} \frac{(\cos x + i \sin x)\, dx}{x^2+4x+5} = \frac{\pi}{e} [\cos 2 - i \sin 2]$

comparing imaginary parts on both side.

$\int_{-\infty}^{\infty} \frac{\sin x \, dx}{x^2+4x+5} = -\frac{\pi}{e} \sin 2$

Q.10 $\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+b^2} \cdot dx$ $\quad a>0, b>0$

$\cos ax \rightarrow e^{iax}$

$\int_C \frac{e^{iaz}}{z^2+b^2} dz = \int_{-R}^{R} \frac{e^{iax}}{x^2+b^2} dx + \int_{C_1} \frac{e^{iaz}}{z^2+b^2} \cdot dz$

$\rightarrow \int_C \frac{e^{iaz}}{z^2+b^2}\,dz$; Let $z^2+b^2=0$ $\therefore z = \pm ib$

Only $z = ib$ lies inside 'c' & it is simple pole

$P(z) = e^{iaz}$; $Q(z) = z^2+b^2$; $Q'(z) = 2z$

$$\text{Res}[f(z), z = ib] = \frac{P(z)}{Q'(z)} = \frac{e^{-ab}}{2ib}$$

$$\int_C \frac{e^{iaz}}{z^2+b^2}\cdot dz = 2\pi i \times \frac{e^{-ab}}{2ib} = \frac{\pi}{b} e^{-ab}$$

$$\therefore \int_{-R}^{R} \frac{e^{iax}}{x^2+b^2}\,dx + \underbrace{\int_{C_1} \frac{e^{iaz}}{z^2+b^2}\,dz}_{\downarrow\ 0} = \frac{\pi}{b} e^{-ab}$$

$R \rightarrow \infty$ then

So $$\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+b^2}\cdot dx = \frac{\pi}{b} e^{-ab}$$

(or)

$$\int_{-\infty}^{\infty} \frac{(\cos ax + i\sin ax)\,dx}{x^2+b^2} = \frac{\pi}{b} e^{-ab}$$

comparing real parts on both side

$$\boxed{\int_{-\infty}^{\infty} \frac{\cos ax\,dx}{x^2+b^2} = \frac{\pi}{b} e^{-ab}}$$

Q.11 Evaluate $\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta}$ $(a>b>0)$

$$\sin\theta = \frac{1}{2i}\frac{[z^2-1]}{z} \qquad d\theta = \frac{dz}{iz}$$

$$\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \int_c \frac{dz/iz}{\frac{2aiz+bz^2-b}{2iz}} = \frac{2}{b}\int_c \frac{dz}{z^2+\frac{2ai}{b}z-1}$$

For $\int \frac{dz}{z^2+\frac{2aiz}{b}-1}$ Let $z^2+\frac{2aiz}{b}-1=0$

$$z = \frac{-2ai/b \pm \sqrt{-4a^2/b^2+4}}{2} = \frac{-2ai/b \pm 2/b\sqrt{b^2-a^2}}{2}$$

$$z = \frac{-ai \pm \sqrt{b^2-a^2}}{b} = \frac{i}{b}\left(-a \pm \sqrt{a^2-b^2}\right)$$

$(\because a>b>0)$

Singular points $z = \frac{-ai+i\sqrt{a^2-b^2}}{b}$ & $z = \frac{-ai-i\sqrt{a^2-b^2}}{b}$ are simple poles

& only $z = \frac{i}{b}(-a+\sqrt{a^2-b^2})$ lies inside 'c' : $|z|=1$

Let $\alpha = \frac{i}{b}(-a+\sqrt{a^2-b^2})$ & $\beta = \frac{i}{b}(-a-\sqrt{a^2-b^2})$

$$\text{Res}[f(z), z=\alpha] = \lim_{z\to\alpha}(z-\alpha)\frac{1}{(z-\alpha)(z-\beta)} = \lim_{z\to\alpha}\frac{1}{z-\beta} = \frac{1}{\alpha-\beta}$$

$$\therefore \text{Res}[f(z), z=\alpha] = \frac{1}{\frac{2i\sqrt{a^2-b^2}}{b}} = \frac{b}{2i\sqrt{a^2-b^2}} = \frac{-ib}{2\sqrt{a^2-b^2}}$$

$$\frac{2}{b}\int_c \frac{dz}{z^2+\frac{2ai}{b}z-1} = \frac{2}{b}\times 2\pi i \times \frac{(-ib)}{2\sqrt{a^2-b^2}} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

$$\boxed{\int_0^{2\pi}\frac{d\theta}{a+b\sin\theta} = \frac{2\pi}{\sqrt{a^2-b^2}} \quad (a>b>0)}$$

Q.12 $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta}\,d\theta$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2} \quad ; \quad \cos\theta = \frac{z^2+1}{2z} \quad ; \quad \cos 2\theta = \frac{z^4+1}{2z^2}$$

$$\int_0^{2\pi} \frac{\sin^2\theta\,d\theta}{a+b\cos\theta} = \int_0^{2\pi} \frac{1-\cos 2\theta}{2(a+b\cos\theta)} \cdot d\theta = \oint_C \frac{i}{2b} \int_C \frac{(z^2-1)^2}{z^2\left(z^2+\frac{2a}{b}z+1\right)} \cdot dz$$

$$\int_C \frac{(z^2-1)^2}{z^2\left(z^2+\frac{2a}{b}z+1\right)} \cdot dz \qquad \text{Let } z^2\left(z^2+\frac{2a}{b}z+1\right) = 0$$

$z=0$ is pole of order 2 & lies inside 'C' : $|z|=1$

$z = \dfrac{-a \pm \sqrt{a^2-b^2}}{b}$ only $z = \dfrac{-a+\sqrt{a^2-b^2}}{b}$ lies inside 'C' & is a simple pole.

$$\alpha = \frac{-a+\sqrt{a^2-b^2}}{b} \quad ; \quad \beta = \frac{-a-\sqrt{a^2-b^2}}{b}$$

$$\text{Res}[f(z), z\to\alpha] = \cancel{d}\lim_{z\to\alpha} \frac{(z-\alpha)(z^2-1)^2}{z^2(z-\alpha)(z-\beta)} = \lim_{z\to\alpha} \frac{(z^2-1)^2}{z^2(z-\beta)} = \frac{(\alpha^2-1)^2}{\alpha^2(\alpha-\beta)}$$

$$= \frac{\alpha^2 - 2 + 1/\alpha^2}{\alpha-\beta} = \frac{2\sqrt{a^2-b^2}}{b}$$

derivation / calculation shown at last

$$\text{Res}[f(z), z\to 0] = \frac{1}{1!}\lim_{z\to 0} \frac{d}{dz} \frac{z^2\cdot(z^2-1)^2}{z^2\left(z^2+\frac{2a}{b}z+1\right)} = \lim_{z\to 0} \frac{d}{dz} \frac{(z^2-1)^2}{\left(z^2+\frac{2a}{b}z+1\right)}$$

$$= \lim_{z\to 0} \frac{2b(z^2-1)(3az^2+a+b(z^2+3)z)}{(2az+bz^2+b)^2} = \frac{2b\times -1\times a}{b^2} = \frac{-2a}{b}$$

$$\therefore \frac{i}{2b}\int_C \frac{(z^2-1)^2}{z^2\left(z^2+\frac{2a}{b}z+1\right)} \cdot dz = \frac{i}{2b} \times 2\pi i \left[\frac{2\sqrt{a^2-b^2} - 2a}{b}\right] = \frac{2\pi(a-\sqrt{a^2-b^2})}{b^2}$$

$$\boxed{\int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta}\,d\theta = \frac{2\pi(a-\sqrt{a^2-b^2})}{b^2}}$$

$$\frac{\alpha^2+\frac{1}{\alpha^2}-2}{\alpha-\beta} \rightarrow \text{Derivation/calculation for } \operatorname{Res}[f(z), z=\alpha]$$

$$\alpha^2 = \frac{a^2+a^2-b^2-2a\sqrt{a^2-b^2}}{b^2} = \frac{2a^2-b^2-2a\sqrt{a^2-b^2}}{b^2}$$

$$\frac{1}{\alpha^2} = \frac{b^2}{2a^2-b^2-2a\sqrt{a^2-b^2}}$$

Numerator: $\alpha^2+\frac{1}{\alpha^2}-2$

$$\therefore \frac{\left(2a^2-b^2-2a\sqrt{a^2-b^2}\right)\left(2a^2-b^2-2a\sqrt{a^2-b^2}\right)+b^4-2b^2\left(2a^2-b^2-2a\sqrt{a^2-b^2}\right)}{b^2\left(2a^2-b^2-2a\sqrt{a^2-b^2}\right)}$$

$4a^4-4a^2b^2$

$$\frac{\left(4a^4-2a^2b^2-4a^3\sqrt{a^2-b^2}-2a^2b^2+b^4+2ab^2\sqrt{a^2-b^2}-4a^3\sqrt{a^2-b^2}+2ab^2\sqrt{a^2-b^2}+4a^2(a^2-b^2)+b^4-4a^2b^2+2b^4+4ab^2\sqrt{a^2-b^2}\right)}{b^2\left(2a^2-b^2-2a\sqrt{a^2-b^2}\right)}$$

$$\frac{8a^4+4b^4-12a^2b^2-8a^3\sqrt{a^2-b^2}+8ab^2\sqrt{a^2-b^2}}{b^2\left(2a^2-b^2-2a\sqrt{a^2-b^2}\right)}$$

$$\frac{4\left(2a^4-2a^2b^2+b^4-a^2b^2-2a\sqrt{a^2-b^2}\,(a^2-b^2)\right)}{b^2\left(2a^2-b^2-2a\sqrt{a^2-b^2}\right)}$$

$$\frac{4\left(2a^2(a^2-b^2)-b^2(a^2-b^2)-2a\sqrt{a^2-b^2}\,(a^2-b^2)\right)}{b^2\left(2a^2-b^2-2a\sqrt{a^2-b^2}\right)}$$

$$\frac{4(a^2-b^2)\left(2a^2-b^2-2a\sqrt{a^2-b^2}\right)}{b^2\left(2a^2-b^2-2a\sqrt{a^2-b^2}\right)} = \frac{4(a^2-b^2)}{b^2}$$

Denominator: $\alpha-\beta = \frac{2\sqrt{a^2-b^2}}{b}$

$$\frac{\text{Num}}{\text{Den}} : \frac{4(a^2-b^2)}{b^2} \times \frac{b}{2\sqrt{a^2-b^2}}$$

$$= \frac{2\sqrt{a^2-b^2}}{b}$$