# DIGITAL ASSIGNMENT

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| REG. NO. | 18BIT0272 |
| COURSE CODE | MAT2002 |
| COURSE NAME | APPLICATIONS OF DIFFERENTIAL AND DIFFERENCE EQUATIONS |
| SLOT | D2 |
| FACULTY | UMA K |

Note:

Q.4 of both the papers are related to Sturm Louiville problems which has not been taught in class and hence not attempted in this DA.

Q.1 The radial displacement $u$ in a rotating disc at a distance $r$ from the axis is given by $r^2\frac{d^2u}{dr^2} + r\frac{du}{dr} - u + kr^3 = 0$ where $k$ is a constant. Solve the equation under the conditions $\frac{u}{r} = 0$ when $r = 0$, $u = 0$ when $r = a$.

A. Using Cauch-Eular method

Let $r = e^z \rightarrow z = \log r$

$r\frac{du}{dr} = Du$ where $D = \frac{d}{dz}$ & $D^2 = \frac{d^2}{dz^2}$

$r^2\frac{d^2y}{dr^2} = (D^2 - D)u$

Substituting in given equation

$(D^2 - 1)u = -ke^{3z}$

$\rightarrow$ A.E : $m^2 - 1 = 0 \Rightarrow m = \pm 1$

$\therefore$ $C.F = c_1e^z + c_2e^{-z} = c_1r + c_2r^{-1}$

$\rightarrow$ $P.I = \frac{1}{D^2-1}(-ke^{3z}) = -\frac{ke^{3z}}{8} = -\frac{kr^3}{8}$

$\rightarrow$ $u = C.F + P.I$

$u = c_1r + \frac{c_2}{r} - \frac{kr^3}{8}$

$\rightarrow$ when $r = 0 \rightarrow u = 0$ & $r = 0 \rightarrow u = 0$

Using these conditions we get $c_1 = \frac{a^2k}{8}$ & $c_2 = 0$

$\therefore$ $u = \frac{kr}{8}(a^2 - r^2)$

Q.2 Two particles each of mass m gm are suspended from 2 springs of stiffness $k_1$ & $k_2$. After the system comes to rest, the lower mass is pulled downwards and released. Discuss their motion by using the matrix method. If $m_1 = m_2 = 1$ and $k_1 = 6$, $k_2 = 4$.

A. Let $y_1$ & $y_2$ denote the displacement of the upper and lower masses at time t from their respective position of equilibrium.

∴ Stretch of upper mass → $y_1$

Stretch of lower mass → $y_2 - y_1$

∴ Restoring force on upper mass $= -k_1 y_1 + k_2(y_2 - y_1)$

$[m_1 = 1;\ k_1 = 6;\ k_2 = 4]$

$$m_1 y_1'' = -(k_1 + k_2) y_1 + k_2 y_2$$

$$y_1'' = -10y_1 + 4y_2$$

Restoring force on lower mass $= -k_2(y_2 - y_1)$

$[m_2 = 1]$

$$m_2 y_2'' = -4(y_2 - y_1)$$

$$y_2'' = 4y_1 - 4y_2$$

∴ $Y' = AY$

Let $Y = MX$ ; $Y' = MX'$ ; $Y'' = MX''$

∴ $AY = MX'' \Rightarrow MX'' = AMX$

∴ $X'' = (M^{-1}AM)X$

$X'' = DX$

$$A = \begin{bmatrix} -10 & 4 \\ 4 & -4 \end{bmatrix}$$

$\therefore$ C.E $\rightarrow |A - \lambda I| = 0$

$\lambda = -12, -2 \rightarrow$ eigen values

eigen vectors $\rightarrow \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\therefore M = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \; ; \; M^{-1} = \frac{1}{5}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

$$D = M^{-1}AM = \begin{bmatrix} -12 & 0 \\ 0 & -2 \end{bmatrix}$$

$$X'' = DX$$

$$\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} -12 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \begin{array}{l} x_1'' = -12x_1 \\ x_2'' = -2x_2 \end{array}$$

For $x_1$

$(D^2+12)x_1 = 0 \quad ; \quad D = \pm 2\sqrt{3}i$

$\therefore x_1 = c_1 \cos(2\sqrt{3}t) + c_2 \sin(2\sqrt{3}t)$

For $x_2$

$(D^2+2)x_2 = 0$

$\therefore D = \pm\sqrt{2}i$

$x_2 = c_3 \cos(\sqrt{2}t) + c_4 \sin(\sqrt{2}t)$

Now $Y = MX$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} c_1\cos(2\sqrt{3}t) + c_2\sin(2\sqrt{3}t) \\ c_3\cos(\sqrt{2}t) + c_4\sin(\sqrt{2}t) \end{bmatrix}$$

$$y_1 = 2\left[c_1\cos(2\sqrt{3}t) + c_2\sin(2\sqrt{3}t)\right] + c_3\cos(\sqrt{2}t) + c_4\sin(\sqrt{2}t)$$

$$y_2 = 2\left[c_3 \cos(\sqrt{2}t) + c_4 \sin(\sqrt{2}t)\right] - c_1 \cos(2\sqrt{3}t) + c_2 \sin(2\sqrt{3}t)$$

**Q.3** Using the Laplace transform, find the current $i(t)$ in the LC circuit. Assuming $L = 1$ henry, $C = 1$ farad, zero initial current and charge on the capacitor and $E = \begin{cases} t \; ; & 0 < t < 1 \\ 0 \; ; & \text{otherwise} \end{cases}$

**A:** The differential equation for L and C is given by

$$L\frac{d^2q}{dt^2} + \frac{q}{C} = E$$

Putting $L = 1$, $C = 1$, $E = v(t)$ in above equation.

$\cancel{D^2+1}$

$$\frac{d^2q}{dt^2} + q = v(t)$$

Using Laplace transform on both sides

$$s^2 L(q) - s\,q(0) - q'(0) + L(q) = \int_0^\infty v(t)\cdot e^{-st}\,dt = \int_0^1 t\cdot e^{-st}\,dt$$

$$(s^2+1)\,L(q) = -\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}$$

$$L(q) = \frac{-e^{-s}}{s(s^2+1)} - \frac{e^{-s}}{s^2(s^2+1)} + \frac{1}{s^2(s^2+1)}$$

Taking inverse Laplace transform

$$q = L^{-1}\left[\frac{-e^{-s}}{s(s^2+1)}\right] - L^{-1}\left[\frac{e^{-s}}{s^2(s^2+1)}\right] + L^{-1}\left[\frac{1}{s^2(s^2+1)}\right]$$

$$q = \left[\cos(t-1) - 1\right]u(t-1) - \left[(t-1) - \sin(t-1)\right]u(t-1) + t - \sin t$$

$$q = \left[\sin(t-1) + \cos(t-1) - t\right]u(t-1) + t - \sin t$$

$$i(t) = \frac{dq}{dt}$$

$$i(t) = [\cos(t-1) - \sin(t-1) - 1]\,u(t-1) + [\sin(t-1) + \cos(t-1) - t]\,u'(t-1)$$
$$+1 - \cos t$$

Formulae used :-

$$L^{-1}\left[\frac{-e^{-s}}{s(s^2+1)}\right] = -[1-\cos(t-1)]\,u(t-1)$$

$$L^{-1}\left[\frac{e^{-s}}{s^2(s^2+1)}\right] = [(t-1) - \sin(t-1)]\,u(t-1)$$

$$L^{-1}\left[\frac{1}{s^2(s^2+1)}\right] = \int_0^t (1-\cos t)\cdot dt = t - \sin t$$

Q.1 (i) Solve the differential equation by method of variation of parameters : $y'' - 2y' + y = \frac{e^t}{1+t^2}$

A. $(D^2 - 2D + 1)y = \frac{e^t}{(1+t^2)}$

$\rightarrow$ A.E : $m^2 - 2m + 1 = 0 \Rightarrow m = 1, 1$

$\rightarrow$ C.F $= (A+Bt)e^t$

$\rightarrow$ P.I $\rightarrow$ $y_1 = e^t$ ; $y_2 = te^t$

Wronskian function $(W) = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} = e^{2t}$

$\rightarrow$ $u(t) = -\int \frac{e^t}{(1+t^2)} \cdot \frac{te^t}{e^{2t}} \cdot dt = -\frac{1}{2}\log(1+t^2)$

$\rightarrow$ $v(t) = +\int \frac{e^t}{(1+t^2)} \cdot \frac{e^t}{e^{2t}} \cdot dt = \tan^{-1}(t)$

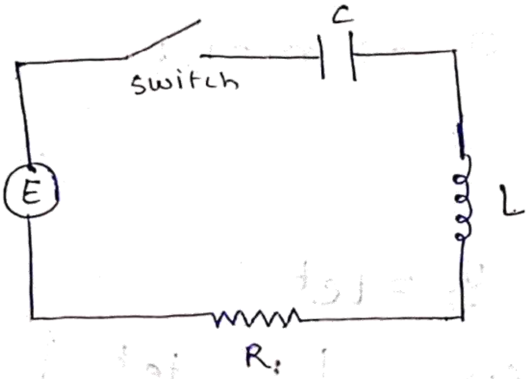
$\therefore$ P.I $= u(t)y_1 + v(t)y_2$

$= -\frac{e^t}{2}\log(1+t^2) + te^t \tan^{-1}(t)$

$\rightarrow$ $y =$ C.F + P.I

$y = (A+Bt)e^t + e^t\left[t\tan^{-1}(t) - \frac{\log(1+t^2)}{2}\right]$

Q.1 (ii) Using Laplace transform find the instantaneous current I for the R-L-C circuit whose governing equation is given by $Lq'' + Rq' + \frac{q}{C} = E$ where the resistance $R = 3\Omega$, capacitance $C = 0.25\ F$, inductance $L = 0.5H$ & electromotive force $E = 2t\ V$, assuming that the current & charge are initially zero.




A. On substituting values of R, L & C in given equation!

$$\frac{d^2q}{dt^2} + 6\frac{dq}{dt} + 8q = 4t$$

Using Laplace transform on both sides

$$s^2L(q) - sq(0) - q'(0) + 6sL(q) - 6q(0) + 8L(q) = \int_0^\infty 4te^{-st}\,dt$$

$$q(0) = q'(0) = 0$$

$$\therefore \quad L(q) = \frac{1}{s^2(s+2)(s+4)} \quad \text{[on simplifying]}$$

Taking inverse Laplace on both sides

$$q = L^{-1}\left[\frac{1}{s^2(s+2)(s+4)}\right] = \frac{1}{8}L^{-1}\left[\frac{1}{s+2}\right] - \frac{1}{32}L^{-1}\left[\frac{1}{s+4}\right] - \frac{3}{32}L^{-1}\left[\frac{1}{s}\right] + \frac{1}{8}L^{-1}\left[\frac{1}{s^2}\right]$$

$$q = \frac{e^{-2t}}{8} - \frac{e^{-4t}}{32} - \frac{3}{32} + \frac{t}{8}$$

Instantaneous current $\rightarrow I(t) = \frac{dq}{dt}$

$$I(t) = \frac{dq}{dt} = -\frac{e^{-2t}}{4} + \frac{e^{-4t}}{8} + \frac{1}{8} \longrightarrow \text{Ans}$$

Q.2 Using Laplace transform solve the differential equations

$$\frac{d^2y}{dt^2} + 4y = f(t) \; ; \; y(0) = 0 \; ; \; y'(0) = 1 \; ; \text{ where}$$

$$f(t) = \begin{cases} 3 \; ; & 0 < t < 4 \\ 2t-5 \; ; & t > 4 \end{cases}$$

A. $(D^2+4)y = f(t)$

Using Laplace transform

$$s^2 L(y) - sy(0) - y'(0) + 4L(y) = \int_0^\infty f(t) \cdot e^{-st} \cdot dt$$

$$(s^2+4)L(y) - 1 = 3\int_0^4 e^{-st} \cdot dt + 2\int_4^\infty te^{-st} \cdot dt - 5\int_4^\infty e^{-st} \cdot dt$$

$$(s^2+4)L(y) - 1 = \frac{3}{s} - \frac{3e^{-4s}}{s} + \frac{8e^{-4s}}{s} + \frac{2e^{-4s}}{s^2} - \frac{5e^{-4s}}{s}$$

$$(s^2+4)L(y) = \frac{3}{s} + \frac{2e^{-4s}}{s^2} + 1$$

$$L(y) = \frac{3}{s(s^2+4)} + \frac{2e^{-4s}}{s^2(s^2+4)} + \frac{1}{s^2+4}$$

Taking inverse Laplace transform on both sides

$$y = 3L^{-1}\left[\frac{1}{s(s^2+4)}\right] + 2L^{-1}\left[e^{-4s} \cdot \frac{1}{s^2(s^2+4)}\right] + L^{-1}\left[\frac{1}{s^2+4}\right]$$

Ans) $y = \frac{3}{4}(1-\cos 2t) + \frac{1}{2}\left[(t-4) - \sin\{2(t-4)\}\right]u(t-4) + \frac{1}{2}\sin 2t$

Formulae used:

$$L^{-1}\left[\frac{1}{s(s^2+4)}\right] = \frac{1}{4}L^{-1}\left[\frac{1}{s} - \frac{s}{s^2+4}\right] = \frac{1}{4}\left[L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{s}{s^2+4}\right)\right]$$

$$L^{-1}\left[\frac{e^{-4s}}{s^2(s^2+4)}\right] = \frac{1}{4}L^{-1}\left[e^{-4s}\left\{\frac{1}{s^2} - \frac{1}{s^2+4}\right\}\right]$$

$$L^{-1}\left[\frac{1}{s^2+4}\right] = \frac{1}{2}L^{-1}\left[\frac{2}{s^2+4}\right] = \frac{1}{2}\sin 2t$$

**Q.3** Solve the system of first order differential equations to find the general solution.

$$x_1' = 2x_1 - x_2 + e^t \quad \text{and} \quad x_2' = 3x_1 - 2x_2 + e^{-t}$$

**A.** Using Diagonalisation of Matrix Method

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} e^t \\ e^{-t} \end{bmatrix}$$

$$x' = Ax + g(t)$$

$$A \rightarrow \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \quad \therefore \quad C.E = |A - \lambda I| = 0$$

We get eigenvalues :- $\lambda = -1, 1$

eigenvectors $\rightarrow$ $\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$M = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} \rightarrow M^{-1} = -\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -3 & 1 \end{bmatrix}$$

$$D = M^{-1}AM = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$X = MY \quad ; \quad X' = MY'$$

$$\therefore \quad MY' = AMY + g(t)$$

$$\rightarrow \quad Y' = (M^{-1}AM)Y + M^{-1}g(t)$$

$$Y' = DY + M^{-1}g(t)$$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} e^t - e^t \\ e^{-t} - 3e^t \end{bmatrix}$$

$$y_1' = -y_1 + \frac{e^{-t} - e^t}{2} \quad ; \quad y_2' = y_2 + \frac{3e^t - e^{-t}}{2}$$

$$y_1 e^t = \int \frac{1 - e^{2t}}{2} \cdot dt = \frac{t}{2} - \frac{e^{2t}}{4} \qquad \left[ I \cdot F = e^{\int 1 \cdot dt} = e^t \right]$$

$$y_1 = \frac{te^{-t}}{2} + \frac{e^t}{4}$$

$$y_2 e^{-t} = \int \frac{3 - e^{-2t}}{2} \cdot dt = \frac{3t}{2} + \frac{e^{-2t}}{4}$$

$$y_2 = \frac{3te^t}{2} + \frac{e^{-t}}{4}$$

$$X = MY$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} \frac{te^{-t}}{2} + \frac{e^t}{4} \\ \frac{3te^t}{2} + \frac{e^{-t}}{4} \end{bmatrix}$$

$$\left.\begin{aligned} x_1 &= \frac{t}{2}(3e^t + e^{-t}) + \frac{e^t + e^{-t}}{4} \\ x_2 &= \frac{3t}{2}(e^t + e^{-t}) + \frac{3e^t + e^{-t}}{4} \end{aligned}\right\} \text{Ans}$$

Q.5 Find the series solution of $y'' + xy' + 4y = 0$ in the neighbourhood of $x = 0$

A. $x = 0$ is Ordinary point

Let $y = \sum_{k=0}^{\infty} a_k x^k$ ; $Dy = \sum k a_k x^{k-1}$

$D^2 y = \sum k(k-1) a_k x^{k-2}$

Substituting in given equation

$$\sum k(k-1) a_k x^{k-2} + \sum k a_k x^k + 4 \sum a_k x^k = 0$$

On comparing coefficients of $x^k$ on both sides

$$(k+2)(k+1) a_{k+2} + k a_k + 4 a_k = 0$$

$$a_{k+2} = \frac{-(k+4)}{(k+1)(k+2)} \cdot a_k$$

For ;

$k = 0$ ; $a_2 = -2a_0$

$k = 1$ ; $a_3 = -\frac{5a_1}{6}$

$k = 2$ ; $a_4 = -\frac{a_2}{2} = +a_0$

$k = 3$ ; $a_5 = -\frac{7a_3}{20} = \frac{7a_1}{24}$

$k = 4$ ; $a_6 = -\frac{8a_4}{30} = -\frac{4}{15} a_0$

$k = 5$ ; $a_7 = -\frac{9a_5}{42} = \frac{-a_1}{16}$

$k = 6$ ; $a_8 = -\frac{10}{56} a_6 = \frac{a_0}{21}$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \ldots\ldots$$

$$y = a_0 + a_1 x - 2a_0 x^2 - \frac{5a_1}{6} x^3 + \cancel{7} a_0 x^4 + \frac{7a_1}{24} x^5 - \frac{4}{15} a_0 x^6 - \frac{a_1}{16} x^7 + \frac{a_0}{10} x^8 \ldots\ldots$$

$\boxed{\text{Ans}}$ $$y = a_0 \left[ 1 - 2x^2 + x^4 - \frac{4}{15} x^6 + \frac{x^8}{10} \ldots \right] + a_1 \left[ x - \frac{5x^3}{6} + \frac{7}{24} x^5 - \frac{x^7}{10} \ldots \right]$$