Q.1(a) What is the condition for a code to correct 'k' or fewer errors? Generate a single error correcting code with $m=4$ & $n=7$

$\Rightarrow$ A code (m,n) can ~~detect~~ correct atmost k errors (k or fewer errors) if & only if the minimum distance between any 2 code words is atleast $2k+1$.

Hamming code (7, 4) :-

modulo 2 addition :-

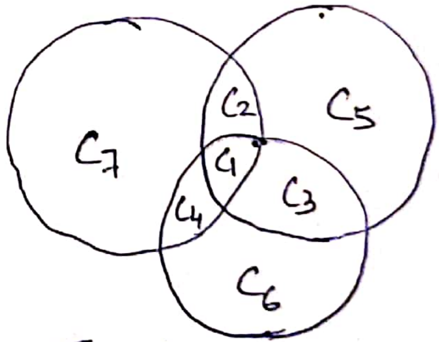
$0 \oplus 0 = 0$

$1 \oplus 0 = 1$

$0 \oplus 1 = 1$

$1 \oplus 1 = 0$

- message digits : $C_1\ C_2\ C_3\ C_4$
- Code words : $C_1 C_2 C_3 C_4 C_5 C_6 C_7$



The circles represent the equations.

Parity check equations :

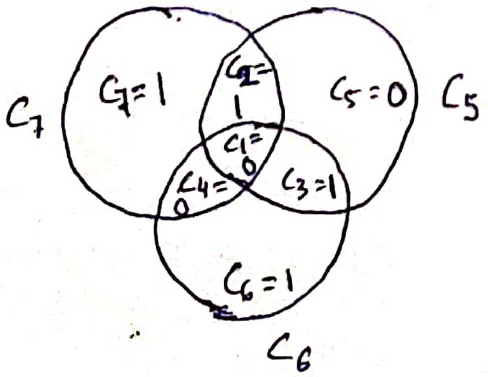
$C_1 \oplus C_2 \oplus C_3 \oplus C_5 = 0$

$C_1 \oplus C_3 \oplus C_4 \oplus C_6 = 0$

$C_1 \oplus C_2 \oplus C_4 \oplus C_7 = 0$

There is an even number of 1's in each circle

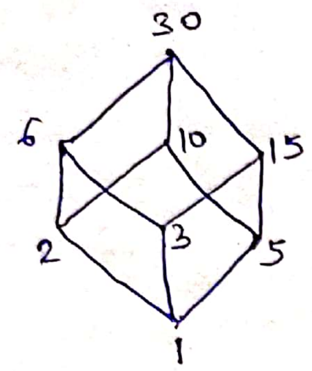
Message : $(C_1 C_2 C_3 C_4) = (0\,1\,1\,0)$



Resultant code word : 0110011

Q.1 (b) Obtain the Hasse diagrams of the lattices (S_n, D) when $n = 30, 45$. Which of these are complemented? Are the lattices distributive? Explain.

$\Rightarrow$ $D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$

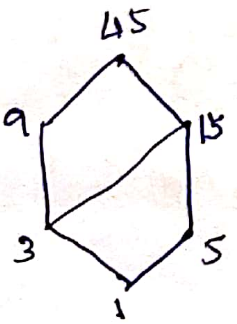


$1' = 30$ & $30' = 1$
$2' = 15$ & $15' = 2$
$3' = 10$ & $10' = 3$
$5' = 6$ & $6' = 5$

Since every element has one element complemented, D_{30} is a complemented lattice.

Here every element has unique complement. Hence D_{30} is a distributive lattice.

$D_{45} = \{1, 3, 5, 9, 15, 45\}$



1 cannot be complement of any element except 0. In this case, the element 3 does not have a complement because ~~3~~ lub $\{3, a\} \neq 1$ for $a = 5, 9, 15$.

Hence D_{45} is neither complemented nor distributive.

Q.2 (a) Prove that every chain is a distributive lattice.

$\Rightarrow$ Let $(L, \leq)$ be a chain & $a, b, c \in L$. Consider the following cases:

(i) $a \leq b$ & $a \leq c$ (ii) $a \geq b$ & $a \geq c$

For (i) $a * (b \oplus c) = a$ .... (1)

$(a * b) \oplus (a * c) = a \oplus a = a$ .... (2)

For (ii) $a * (b \oplus c) = b \oplus c \quad \ldots (3)$

$(a*b) \oplus (a*c) = b \oplus c \quad \ldots (4)$

From 1, 2, 3 & 4

$$a * (b \oplus c) = (a*b) \oplus (a*c)$$

Hence every chain is a distributive lattice.

Q.2(b) Show that in a complemented distributive lattice

$a \leq b \Leftrightarrow a * b' = 0 \Leftrightarrow a' \oplus b = 1 \Leftrightarrow b' \leq a'$

$\Rightarrow$ Let $(L, *, \oplus)$ be any distributive complemented lattice.

Case (i) $a \leq b \Rightarrow a * b' = 0$

Let $a \leq b$ for $a, b \in L$, then $a * b = a$ & $a \oplus b = b$

Now $a \oplus b = b \Rightarrow (a \oplus b) * \bar{b} = b * \bar{b} = 0$

$\Rightarrow (a * \bar{b}) \oplus (b * \bar{b}) = 0$

$\Rightarrow (a * \bar{b}) \oplus 0 = 0$

$\Rightarrow a * \bar{b} = 0$

Conversely ; Let $a * \bar{b} = 0$

$\Rightarrow (a*\bar{b}) \oplus (b * \bar{b}) = 0$ $\quad [\because b*\bar{b} = 0]$

$\Rightarrow (a \oplus b) * \bar{b} = 0 = b * \bar{b}$

$\Rightarrow a \oplus b = b$ i.e $a \leq b$

Hence proved.

Case (ii) $\quad a*\bar{b} = 0 \Leftrightarrow \bar{a} \oplus b = 1$

Let $a*\bar{b} = 0$ for $a, b \in L$

Now $\overline{a*\bar{b}} = \bar{0} = 1$

$\bar{a} \oplus \bar{\bar{b}} = 1$

$\bar{a} \oplus b = 1$

Conversely ; Let $\bar{a} \oplus b = 1$

$\overline{\bar{a} \oplus b} = \bar{1} = 0$

$a * \bar{b} = 0$

Hence proved.

Case(iii) $\quad \bar{a} \oplus b = 1 \Leftrightarrow \bar{b} \leq \bar{a}$

Let $\bar{a} \oplus b = 1$ for $a, b \in L$

Now $(\bar{a} \oplus b) * \bar{b} = \bar{b}$

$(\bar{a} * \bar{b}) \oplus (b * \bar{b}) = \bar{b}$

$(\bar{a} * \bar{b}) \oplus 0 = \bar{b}$

$\bar{a} * \bar{b} = \bar{b} \Rightarrow \bar{b} \leq \bar{a}$

Conversely let $\bar{b} \leq \bar{a}$ $\quad$ i. $\bar{b} * \bar{a} = \bar{b}$ or $\bar{a} * \bar{b} = \bar{b}$

Hence proved.

Q.2 (c) In any Boolean Algebra, show that $a \le b \Rightarrow a+bc = b(a+c)$

$\Rightarrow$ A lattice which is complemented & distributive is called Boolean Algebra. So the properties of lattices such as when $a \le b \Rightarrow a * b = a$ & $a \oplus b = b$ will also apply here.

i.e if $a \le b$, then $a \cdot b = a$ & $a+b = b$

LHS $= a + bc$

$= a \cdot b + bc$

$= b(a+c)$

$=$ RHS

(OR)

RHS $= b(a+c)$

$= ab + bc$

$= a + bc$

$=$ LHS

$\therefore$ Hence Proved.

Q.3(a) Let $(L, \leq)$ be a lattice in which $*, \oplus$ denote the operations of meet & join respectively. For any $a, b \in L$, prove that $a \leq b \Leftrightarrow a * b = a \Leftrightarrow a \oplus b = b$

$\Rightarrow$ (i) $a \oplus b = b$, if & only if $a \leq b$

(ii) $a * b = a$, if only if $a \leq b$

(iii) $a * b = a$, if & only if $a \oplus b = b$

Proof :-

i) Let $a \leq b \Rightarrow b \leq b$ (reflexivity)

$\therefore a \oplus b \leq b$ — (1)

Since $a \oplus b$ is the LUB (a, b), $\therefore b \leq a \oplus b$ — (2)

From 1 & 2 :— $a \oplus b = b$ — (3)

Now Let $a \oplus b = b$

Since $a \oplus b$ is the LUB (a, b), $a \leq a \oplus b$ i.e $a \leq b$ — (4)

From (3) & (4), (i) is proved

(ii) Can be proved in similar manner

Let $a \leq b \Rightarrow a \leq a$ (reflexivity)

$\therefore a \leq a * b$ — (1)

Since $a * b$ is the GLB $(a, b) \Rightarrow a * b \leq a$ — (2)

From (1) & (2) $\Rightarrow a * b = a$ — (3)

Now let $a * b = a$

Since $a * b$ is GLB of (a, b), $a * b \leq b$ i.e $a \leq b$ — (4)

From (3) & (4), (ii) is proved.

(iii) Let $a * b = a$

$b \oplus (a * b) = b \oplus a = a \oplus b$ [commutativity] — (1)

But by Absorption law $b \oplus (a * b) = b$ — (2)

Substituting (2) in (1) we get $b = a \oplus b$

i.e $a \oplus b = b$

Conversely, Let $a \oplus b = b$

$a * (a \oplus b) = a * b$

$a = a * b$ [Absorption law]

Hence (iii) is proved as well.

Q.3(b) Find the code generated by the given parity matrix 'H' when the encoding function is $e: B^3 \to B^6$

$$H = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ $H = [A^T \mid I_{n-m}]$ $\therefore$ $A^T = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ i.e $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Now $G = [I_m \mid A]$

$\therefore$ $G = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$

$B^3 = \{000, 001, 010, 011, 100, 101, 110, 111\}$

$$e(000) = [0\ 0\ 0]\begin{bmatrix}1&0&0&1&1&0\\0&1&0&1&0&1\\0&0&1&0&1&1\end{bmatrix} = [0\ 0\ 0\ 0\ 0\ 0]$$

$$e(001) = [0\ 0\ 1]\begin{bmatrix}1&0&0&1&1&0\\0&1&0&1&0&1\\0&0&1&0&1&1\end{bmatrix} = [0\ 0\ 1\ 0\ 1\ 1]$$

$$e(010) = [0\ 1\ 0]\begin{bmatrix}1&0&0&1&1&0\\0&1&1&1&0&1\\0&0&1&0&1&1\end{bmatrix} = [0\ 1\ 0\ 1\ 0\ 1]$$

$$e(011) = [0\ 1\ 1]\begin{bmatrix}1&0&0&1&1&0\\0&1&1&1&0&1\\0&0&1&0&1&1\end{bmatrix} = [\cancel{0}\ 0\ 1\ 1\ 1\ 1\ 0]$$

$$e(100) = [1\ 0\ 0]\begin{bmatrix}1&0&0&1&1&0\\0&1&0&1&0&1\\0&0&1&0&1&1\end{bmatrix} = [1\ 0\ 0\ 1\ 1\ 0]$$

$$e(101) = [1\ 0\ 1]\begin{bmatrix}1&0&0&1&1&0\\0&1&0&1&0&1\\0&0&1&0&1&1\end{bmatrix} = [1\ 0\ 1\ 1\ 0\ 1]$$

$$e(110) = [1\ 1\ 0]\begin{bmatrix}1&0&0&1&1&0\\0&1&0&1&0&1\\0&0&1&0&1&1\end{bmatrix} = [1\ 1\ 0\ 0\ 1\ 1]$$

$$e(111) = [1\ 1\ 1]\begin{bmatrix}1&0&0&1&1&0\\0&1&0&1&0&1\\0&0&1&0&1&1\end{bmatrix} = [1\ 1\ 1\ 0\ 0\ 0]$$

Therefore the codes are :-

$e(000) = 000000$

$e(001) = 001011$

$e(010) = 010101$

$e(011) = 011110$

$e(100) = 100110$

$e(101) = 101101$

$e(110) = 110011$

$e(111) = 111000$

Q.4(a) Obtain the products of sums canonical form in three variables of Boolean expression $x_1 * x_2$.

$\Rightarrow x_1 * x_2 \Rightarrow (x_1 + (x_2 * x_2')) * ((x_1 * x_1') + x_2)$

$\Rightarrow (x_1 + x_2) * (x_1 + x_2') * (x_1 + x_2) * (x_1' + x_2)$

Remove similar terms since $x_1 * x_1 = x_1$

$\Rightarrow (x_1 + x_2) * (x_1 + x_2') * (x_1' + x_2)$

$\Rightarrow (x_1 + x_2 + (x_3 * x_3')) * (x_1 + x_2' + (x_3 * x_3')) *$
$(x_1' + x_2 + (x_3 * x_3'))$

Final answer $\Rightarrow (x_1 + x_2 + x_3) * (x_1 + x_2 + x_3') * (x_1 + x_2' + x_3) *$
$(x_1 + x_2' + x_3') * (x_1' + x_2 + x_3) * (x_1' + x_2 + x_3')$

Q.4(b) In any Boolean algebra prove that ~~(a+b')(b+c')~~
$(a+b')(b+c')(c+a') = (a'+b)(b'+c)(c'+a)$

$\Rightarrow$ LHS $= (a+b'+0)(b+c'+0)(c+a'+0)$

$= (a+b'+c.c')(b+c'+a.a')(c+a'+b.b')$

$= (a+b'+c).(a+b'+c').(a+b+c').(a'+b+c').$
$(a'+b+c).(a'+b'+c)$

$= \{(a'+b+c)(a'+b+c')\}.\{(a+b'+c)(a'+b'+c)\}.$
$\{(a+b+c').(a+b'+c')\}$

$= (a'+b+c.c').(b'+c+a.a').(a+c'+b.b')$

$= (a'+b+0).(b'+c+0).(c'+a+0)$

$= (a'+b)(b'+c)(c'+a)$

$=$ RHS Hence proved

Q.4 (c) Simplify the following Boolean function using Quine-McCluskey's Method.

$f(w,x,y,z) = wxyz + wxy\bar{z} + w\bar{x}\bar{y}z + wx\bar{y}z + w\bar{x}\bar{y}\bar{z} + \bar{w}x\bar{y}z + \bar{w}\bar{x}y\bar{z} + \bar{w}\bar{x}\bar{y}z$

$\Rightarrow f(w,x,y,z) = \Sigma(1, 2, 5, 8, 9, 13, 14, 15)$

	w	x	y	z	
1	0	0	0	1	✓
2	0	0	1	0	
8	1	0	0	0	✓
5	0	1	0	1	✓
9	1	0	0	1	✓
13	1	1	0	1	✓
14	1	1	1	0	✓
15	1	1	1	1	✓

	w	x	y	z	
(1, 5)	0	–	0	1	✓
(1, 9)	–	0	0	1	✓
(8, 9)	1	0	0	–	
(5, 13)	–	1	0	1	✓
(9, 13)	1	–	0	1	✓
(13, 14)	1	1	–	1	
(14, 15)	1	1	1	–	

	w	x	y	z
(1,5,9,13)	–	–	0	1
(1,9,5,13)	–	–	0	1

∴ Prime implicants

	w	x	y	z
2	0	0	1	0
(8, 9)	1	0	0	–
(13,15)	1	1	–	1
(14,15)	1	1	1	–
(1,5,9,13)	–	–	0	1

∴ essential prime implicants

	1	2	5	8	9	13	14	15	
2		⊗							✓
(8, 9)				⊗	x				✓
(13,15)						x		x	
(14, 15)							⊗	x	✓
(1, 5, 9, 13)	⊗		⊗		x	x			✓

Hence $f(w, x, y, z) = (0010) + (100-) + (111-) + (--01)$

$f(w, x, y, z) = w'x'yz' + wx'y' + wxy + y'z$