Supplementary Material for "PEBBLE: A second order correct Bootstrap method in logistic regression"

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Summary: Proofs of Theorem 3.2 and Theorem 4.1 from the main paper are presented. All auxiliary lemmas are provided along with their proofs. Additional simulation results are also reported.

1. Notations

Let us define the notations considered in the main manuscript. These are useful for proving the lemmas here. Suppose that Φ_V and Φ_V respectively denote the normal distribution and its density with mean $\mathbf{0}$ and covariance matrix $\boldsymbol{V}$. We will write $\Phi_V = \Phi$ and $\phi_{\mathbf{V}} = \phi$ when the dispersion matrix $\mathbf{V}$ is the identity matrix. $C, C_1, C_2, \cdots$ denote generic constants that do not depend on the variables like n, x, and so on. $\boldsymbol{\nu}_1$, $\boldsymbol{\nu}_2$ denote vectors in $\mathscr{R}^p$, sometimes with some specific structures (as mentioned in the proofs). $(\mathbf{e_1}, \ldots, \mathbf{e_p})'$ denote the standard basis of $\mathcal{R}^p$. For a non-negative integral vector $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_l)'$ and a function $f = (f_1, f_2, \ldots, f_l) : \mathcal{R}^l \rightarrow \mathcal{R}^l$, $l \geq 1$, let $|\boldsymbol{\alpha}| = \alpha_1 + \ldots + \alpha_l$, $\boldsymbol{\alpha}! = \alpha_1! \ldots \alpha_l!$, $f^{\boldsymbol{\alpha}} = (f_1^{\alpha_1}) \ldots (f_l^{\alpha_l})$, $D^{\boldsymbol{\alpha}} f_1 = D_1^{\alpha_1} \cdots D_l^{\alpha_l} f_1$, where $D_j f_1$ denotes the partial derivative of f_1 with respect to the jth component of $\boldsymbol{\alpha}$, $1 \leq j \leq l$. We will write $D^{\boldsymbol{\alpha}} = D$ if $\boldsymbol{\alpha}$ has all the component equal to 1. For $\mathbf{t} = (t_1, t_2, \cdots t_l)' \in \mathcal{R}^l$ and α as above, define $\mathbf{t}^{\boldsymbol{\alpha}} = t_1^{\alpha_1} \cdots t_l^{\alpha_l}$. For any two vectors $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathcal{R}^k$, $\boldsymbol{\alpha} \leq \boldsymbol{\beta}$ means that each of the component of $\boldsymbol{\alpha}$ is smaller than that of $\boldsymbol{\beta}$. For a set $\boldsymbol{A}$ and real constants a_1, a_2, $a_1 \boldsymbol{A} + a_2 = \{a_1 y + a_2 : y \in \boldsymbol{A}\}$, ∂A is the boundary of A and A^{ϵ} denotes the $\epsilon-$neighbourhood of A for any $\epsilon > 0$. $\mathcal{N}$ is the set of natural numbers. $C(\cdot), C_1(\cdot), \ldots$ denote generic constants which depend on only their arguments. Given two probability measures P_1 and P_2 defined on the same space $(\Omega, \mathcal{F})$, $P_1 * P_2$ defines the measure on $(\Omega, \mathcal{F})$ by convolution of P_1 & P_2 and $\|P_1 - P_2\| = |P_1 - P_2|(\Omega)$, $|P_1 - P_2|$ being the total variation of $(P_1 - P_2)$. For a function $g : \mathcal{R}^k \rightarrow \mathcal{R}^m$ with $g = (g_1, \ldots, g_m)'$,

$$Grad[g(\boldsymbol{x})] = \left(\left(\frac{\partial g_i(\boldsymbol{x})}{\partial x_j} \right) \right)_{m \times k}.$$

For any natural number m, the class of sets $\mathcal{A}_m$ is the collection of Borel subsets of $\mathcal{R}^m$ satisfying

$$\sup_{B \in \mathcal{A}_m} \Phi((\partial B)^{\epsilon}) = O(\epsilon) \quad \text{as } \epsilon \downarrow 0. \tag{1.1}$$

For Lemma 2.3 below, define $\xi_{1,n,s}(\boldsymbol{t}) = \left(1 + \sum_{i=1}^{s-2} n^{-r/2} \tilde{P}_r\left(i\boldsymbol{t} : \{\bar{\chi}_{\nu,n}\}\right)\right) \exp\left\{ - \boldsymbol{t}' \boldsymbol{E}_n \boldsymbol{t}/2 \right\}$ where $\boldsymbol{E}_n = n^{-1} \sum_{i=1}^n Var(Y_i)$ and $\bar{\chi}_{\nu,n}$ is the average νth cumulant of $Y_1, \ldots, Y_n$. Define $\bar{\rho}_l = n^{-1} \sum_{i=1}^n \mathbf{E}\|Y_i\|^l$, the average lth absolute moment of $\{Y_1, \ldots, Y_n\}$. The polynomials $\tilde{P}_r\left(z : \{\bar{\chi}_{\nu,n}\}\right)$ are defined on the pages of 51 – 53 of Bhattacharya and Rao (1986). Define the identity

$$\xi_{1,n,s}(\boldsymbol{t})\Big(\sum_{j=0}^{\infty}(-\|\boldsymbol{t}\|^2 b_n^2)^j/j!\Big) = \xi_{n,s}(\boldsymbol{t}) + o\big(n^{-(s-2)/2}\big),$$

uniformly in $\|\boldsymbol{t}\| < 1$, where c_n is defined in Lemma 2.3. $\psi_{n,s}(\cdot)$ is the Fourier inverse of $\xi_{n,s}(\cdot)$.

2. Auxiliary lemmas

Lemma 2.1. *Suppose that $Y_1,\dots,Y_n$ are zero mean independent r.v.s with $\mathbf{E}(|Y_i|^t) < \infty$ for $i = 1,\dots,n$ and $S_n = \sum_{i=1}^n Y_i$. Let $\sum_{i=1}^n \mathbf{E}(|Y_i|^t) = \sigma_t$, $c_t^{(1)} = \big(1+\frac{2}{t}\big)^t$ and $c_t^{(2)} = 2(2+t)^{-1}e^{-t}$. Then, for any $t \geq 2$ and $x > 0$,*

$$P[|S_n| > x] \leq c_t^{(1)}\sigma_t x^{-t} + exp(-c_t^{(2)}x^2/\sigma_2)$$

Proof of Lemma 2.1. This inequality was proved in Fuk and Nagaev (1971).

Lemma 2.2. *For any $t > 0$, $\dfrac{1-N(t)}{n(t)} \leq \dfrac{1}{t}$ wher $N(\cdot)$ and $n(\cdot)$ respectively denote the cdf and pdf of real valued standard normal random variable.*

Proof of Lemma 2.2. This inequality is proved in Birnbaum (1942).

Lemma 2.3. *Suppose that $Y_1,\dots,Y_n$ are mean zero independent random vectors in $\mathcal{R}^k$ with $\boldsymbol{E}_n = n^{-1}\sum_{i=1}^n Var(Y_i)$ converging to some positive definite matrix V. Let $s \geq 3$ be an integer and $\bar{\rho}_{s+\delta} = O(1)$ for some $\delta > 0$. Additionally assume Z to be a $N(\mathbf{0}, \boldsymbol{I}_k)$ random vector which is independent of $Y_1,\dots,Y_n$ and the sequence $\{c_n\}_{n\geq 1}$ to be such that $c_n = O(n^{-d})$ & $n^{-(s-2)/\tilde{k}}\log n = o(c_n^2)$ where $\tilde{k} = \max\{k+1, s+1\}$ & $d > 0$ is a constant. Then uniformly for any Borel subset B of $\mathcal{R}^k$,*

$$\Big|\mathbf{P}\big(\sqrt{n}\bar{Y} + c_n Z \in B\big) - \int_B \psi_{n,s}(x)dx\Big| = o\Big(n^{-(s-2)/2}\Big), \tag{2.1}$$

where $\psi_{n,s}(\cdot)$ is defined above.

Proof of Lemma 2.3. For a Borel set $B \subset \mathcal{R}^k$, define $B_n = B - n^{-1/2}\sum_{i=1}^n EV_i$. Again define $V_i = Y_i I\Big(\|Y_i\| \leq \sqrt{n}\Big)$ and $W_i = V_i - EV_i$. Suppose that $\bar{\tilde{\chi}}_{\nu,n}$ is the average cumulant of $W_1,\dots,W_n$ and $D_n = n^{-1}\sum_{i=1}^n Var(W_i)$. Let $\tilde{\xi}_{1,n,s}$, $\tilde{\xi}_{n,s}$ and $\tilde{\psi}_{n,s}$ are respectively obtained from $\xi_{1,n,s}$, $\xi_{n,s}$ and $\psi_{n,s}$ with $\bar{\chi}_{\nu,n}$ replaced by $\bar{\tilde{\chi}}_{\nu,n}$ and E_n replaced by D_n. Then we have

$$\begin{aligned}
&\Big|\mathbf{P}\big(\sqrt{n}\bar{Y}_n + c_n Z \in B\big) - \int_B \psi_{n,s}(x)dx\Big| \\
\leq&\Big|\mathbf{P}\big(\sqrt{n}\bar{Y}_n + c_n Z \in B\big) - \mathbf{P}\Big(\sqrt{n}\bar{V}_n + c_n Z \in B\Big)\Big| \\
&+\Big|\mathbf{P}\big(\sqrt{n}\bar{W}_n + c_n Z \in B_n\big) - \int_{B_n}\tilde{\psi}_{n,s}(x)dx\Big| + \Big|\int_{B_n}\tilde{\psi}_{n,s}(x)dx - \int_B \psi_{n,s}(x)dx\Big| \\
=&I_1 + I_2 + I_3 \quad \text{(say)}.
\end{aligned} \tag{2.2}$$

First we are going to show that $I_1 = o\Big(n^{-(s-2)/2}\Big)$, uniformly for any Borel set B. Now writing G_j and G'_j to be distributions of $n^{-1/2}Y_j$ and $n^{-1/2}V_j$, $j \in \{1, \dots, n\}$, we have uniformly for any Borel set B,

$$I_1 \le \sum_{j=1}^{n} \|G_j - G'_j\| = 2\sum_{j=1}^{n} P\Big(\|Y_j\| > n^{1/2}\Big) = o\Big(n^{-(s-2)/2}\Big), \tag{2.3}$$

due to the fact that $n^{-1}\sum_{j=1}^{n} E\|Y_j\|^{s+\delta} = O(1)$. Next we are going to show $I_3 = o\Big(n^{-(s-2)/2}\Big)$, uniformly for any Borel set B. To that end, define $m_1 = \inf\{j : c_n^{2j} = o\big(n^{-(s-2)/2}\big)\}$. Again note that the eigen values of D_n are bounded away from 0, due to (14.18) in corollary 14.2 of Bhattacharya and Rao (1986) and the fact that E_n converges to some positive definite matrix. Therefore we have

$$I_3 = \Big|\int_{B_n} \tilde{\psi}^{m_1}_{n,s}(x)dx - \int_{B} \psi^{m_1}_{n,s}(x)dx\Big| + o\Big(n^{-(s-2)/2}\Big) = I_{31} + o\Big(n^{-(s-2)/2}\Big) \quad (say), \tag{2.4}$$

uniformly for any Borel set B of $\mathcal{R}^k$, where

$$\psi^{m_1}_{n,s}(x) = \Bigg\{\Big[\sum_{r=0}^{s-2} n^{-r/2}\tilde{P}_r\big(-D : \{\bar{\chi}_{\nu,n}\}\big)\Big]\Big[\sum_{j=0}^{m_1-1} 2^{-j}(j!)^{-1}c_n^{2j}(D'D)^j\Big]\Bigg\}\phi_{E_n}(x) \quad \text{and}$$

$$\tilde{\psi}^{m_1}_{n,s}(x) = \Bigg\{\Big[\sum_{r=0}^{s-2} n^{-r/2}\tilde{P}_r\big(-D : \{\bar{\bar{\chi}}_{\nu,n}\}\big)\Big]\Big[\sum_{j=0}^{m_1-1} 2^{-j}(j!)^{-1}c_n^{2j}(D'D)^j\Big]\Bigg\}\phi_{D_n}(x).$$

Now writing $l(\boldsymbol{u}) = \|\boldsymbol{u}\|/2$, $\boldsymbol{u} \in \mathcal{R}^k$, and $a_n = n^{-1/2}\sum_{i=1}^{n} EV_i$, from (2.3) we have

$$\begin{aligned}
& I_{31} \le \qquad (2.5)\\
& \sum_{r=0}^{s-2}\sum_{j=0}^{m_1-1} \frac{b_n^{2j}}{n^{r/2}}\Bigg[\int_{B_n}\Bigg|\Big\{\tilde{P}_r\big(-D : \{\bar{\chi}_{\nu,n}\}\big)\frac{l(-D)}{j!}\Big\}\phi_{E_n}(x) - \Big\{\tilde{P}_r\big(-D : \{\bar{\bar{\chi}}_{\nu,n}\}\big)\frac{l(-D)}{j!}\Big\}\phi_{D_n}(x)\Bigg|dx\\
& + \int_{B}\Bigg|\Big\{\tilde{P}_r\big(-D : \{\bar{\chi}_{\nu,n}\}\big)\frac{l(-D)}{j!}\Big\}\phi_{E_n}(x) - \Big\{\tilde{P}_r\big(-D : \{\bar{\chi}_{\nu,n}\}\big)\frac{l(-D)}{j!}\Big\}\phi_{E_n}(x - a_n)\Bigg|dx\Bigg]\\
& + o\Big(n^{-(s-2)/2}\Big)\\
= {} & I_{311} + I_{312} + o\Big(n^{-(s-2)/2}\Big) \quad \text{(say)}. \qquad (2.6)
\end{aligned}$$

Now assume $E_n = I_k$, the $k \times k$ identity matrix. Then following the proof of Lemma 14.6 of Bhattacharya and Rao (1986), we have $I_{311} + I_{312} = o\Big(n^{-(s-2)/2}\Big)$, uniformly for any Borel set B of $\mathcal{R}^k$ due to the exponentially decaying term present in the upper bound in that Lemma. Note that Lemma 14.6 of Bhattacharya and Rao (1986) essentially tells that the truncation and the then making the truncated random vectors centered do not essentially change the underlying Edgeworth expansions. Main ingredients of the proof are (14.74), (14.78), (14.79) and bounds similar to (14.80) and (14.86) in Bhattacharya and Rao (1986). The general case when E_n converges to a positive definite matrix, will follow essentially through the same line. Hence from (2.4) and (2.5), we have $I_3 = o\big(n^{-(s-2)/2}\big)$, uniformly for any Borel set B of $\mathcal{R}^k$. Therefore it remains to show that $I_2 = o\left(n^{-(s-2)/2}\right)$, uniformly for any Borel set B. Now let us write $\Gamma_n = \sqrt{n}\bar{W}_n + c_n Z$. Then recall that

$$I_2 = \left|\mathbf{P}(\Gamma_n \in B_n) - \int_{B_n} \tilde{\psi}_{n,s}(x)dx\right|.$$

By Theorem 4 of chapter 5 of Feller (1971), we can say that Γ_n has density with respect to the Lebesgue measure. Let us call that density by $q_n(\cdot)$. Then we have

$$I_2 \leq \int \left|q_n(x) - \tilde{\psi}_{n,s}(x)\right|dx \leq \int \left|q_n(x) - \tilde{\psi}_{n,(\tilde{k}-1)}(x)\right|dx + \int \left|\tilde{\psi}_{n,s}(x) - \tilde{\psi}_{n,(\tilde{k}-1)}(x)\right|dx, \quad (2.7)$$

where $\tilde{k} = \max\{k+1, s+1\}$. Note that $\int \|x\|^j \left|q_n(x) - \tilde{\psi}_{n,(\tilde{k}-1)}(x)\right|dx < \infty$ for any $j \in \mathcal{N}$, since $\tilde{\psi}_{n,(\tilde{k}-1)}(x)$ has negative exponential term and $\bar{W}_n$ is bounded. Therefore by Lemma 11.6 of Bhattacharya and Rao (1986) we have

$$\begin{aligned} I_2 &\leq C(k)\left[\max_{|\boldsymbol{\beta}| \in \{0,\ldots,(k+1)\}} \int \left|D^{\boldsymbol{\beta}}\left(\hat{q}(t) - \tilde{\xi}_{n,(\tilde{k}-1)}(t)\right)\right|dt\right] + \int \left|\tilde{\psi}_{n,s}(x) - \tilde{\psi}_{n,(\tilde{k}-1)}(x)\right|dx \\ &= I_{21} + I_{22}, \quad \text{(say)}. \end{aligned} \quad (2.8)$$

which clearly does not depend on the choice of the Borel set B. Hence enough to show that $I_{21} + I_{22} = o\left(n^{-(s-2)/2}\right)$. Here $\hat{q}_n(\cdot)$ is the Fourier transform of the density $q_n(\cdot)$. Clearly $I_{22} = o\left(n^{-(s-2)/2}\right)$ by looking into the definition of $\tilde{\psi}_{n,s}(\cdot)$. Now define

$$\breve{\xi}_{n,(\tilde{k}-1)}(t) = \left[\sum_{r=0}^{\tilde{k}-3} n^{-r/2}\tilde{P}_r\left(it : \left\{\bar{\tilde{\chi}}_{\nu,n}\right\}\right)\right]\exp\left(\frac{-\boldsymbol{t}'\boldsymbol{D}_n\boldsymbol{t} - c_n^2\|\boldsymbol{t}\|^2}{2}\right).$$

Then we have

$$\begin{aligned} I_{21} &\leq C(k)\max_{|\boldsymbol{\beta}| \in \{0,\ldots,(k+1)\}}\left[\int \left|D^{\boldsymbol{\beta}}\left(\hat{q}_n(t) - \breve{\xi}_{n,(\tilde{k}-1)}(t)\right)\right|dt + \int \left|D^{\boldsymbol{\beta}}\left(\breve{\xi}_{n,(\tilde{k}-1)}(t) - \tilde{\xi}_{n,(\tilde{k}-1)}(t)\right)\right|dt\right] \\ &= I_{211} + I_{212} \quad \text{(say)} \end{aligned} \quad (2.9)$$

First, we are going to show that $I_{212} = o\left(n^{-(s-2)/2}\right)$. Note that

$$\breve{\xi}_{n,(\tilde{k}-1)}(t) - \tilde{\xi}_{n,(\tilde{k}-1)}(t) = \left[\sum_{r=0}^{\tilde{k}-3} n^{-r/2}\tilde{P}_r\left(\boldsymbol{it} : \left\{\bar{\tilde{\chi}}_{\nu,n}\right\}\right)\right]\exp\left(\frac{-\boldsymbol{t}'\boldsymbol{D}_n\boldsymbol{t}}{2}\right)\sum_{j=m_2}^{\infty}\frac{c_n^{2j}\|\boldsymbol{t}\|^{2j}(-1)^j}{2^j j!},$$

where $m_2 = m_2(r) = (s-2)^{-1}m_1(\tilde{k}-3-r)$. Therefore for any $\boldsymbol{\beta} \in \mathcal{N}^k$ with $|\boldsymbol{\beta}| \in \{0,\ldots,k+1\}$ we have

$$D^{\boldsymbol{\beta}}\left(\breve{\xi}_{n,(\tilde{k}-1)}(t) - \tilde{\xi}_{n,(\tilde{k}-1)}(t)\right) =$$

$$\sum^{*}\sum_{r=0}^{\tilde{k}-3}\sum_{j=m_2}^{\infty} C_1(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})\frac{n^{-r/2}(-1)^j c_n^{2j}}{2^j j!}\left[D^{\boldsymbol{\alpha}}\left(\tilde{P}_r\left(\boldsymbol{it} : \left\{\bar{\tilde{\chi}}_{\nu,n}\right\}\right)\right)\right]\left[D^{\boldsymbol{\gamma}}\left(\exp\left(\frac{-\boldsymbol{t}'\boldsymbol{D}_n\boldsymbol{t}}{2}\right)\right)\right]D^{\boldsymbol{\beta}-\boldsymbol{\alpha}-\boldsymbol{\gamma}}\left(\|\boldsymbol{t}\|^{2j}\right) \quad (2.10)$$

where $\sum^*$ is over $\boldsymbol{\alpha},\boldsymbol{\gamma}\in\mathcal{N}^k$ such that $0\le\boldsymbol{\alpha},\boldsymbol{\gamma}\le\boldsymbol{\beta}$. Since the degree of the polynomial $\tilde{P}_r\left(i\boldsymbol{t}:\{\bar{\tilde{\chi}}_{\nu,n}\}\right)$ is $3r$, $D^{\alpha}\left(\tilde{P}_r\left(i\boldsymbol{t}:\left\{\bar{\tilde{\chi}}_{\nu,n}\right\}\right)\right)=0$ if $|\boldsymbol{\alpha}|>3r$. When $|\boldsymbol{\alpha}|\le 3r$, then recalling that $n^{-1}\sum_{i=1}^n E\|Y_i\|^s=O(1)$ and by Lemma 9.5 & Lemma 14.1(v) of Bhattacharya and Rao (1986) we have

$$\left|D^{\alpha}\left(\tilde{P}_r\left(i\boldsymbol{t}:\{\bar{\tilde{\chi}}_{\nu,n}\}\right)\right)\right|\le\begin{cases}C_2(\boldsymbol{\alpha},r)(\bar{\rho}_s)^{r/(s-2)}(1+(\bar{\rho}_2)^{r(s-3)/(s-2)})(1+\|\boldsymbol{t}\|^{3r-|\boldsymbol{\alpha}|}),\text{ if } 0\le r\le(s-2)\\ C_3(\boldsymbol{\alpha},r)n^{(r+2-s)/2}\bar{\rho}_s\left(1+(\bar{\rho}_2)^{r-1}\right)\left(1+\|\boldsymbol{t}\|^{3r-|\boldsymbol{\alpha}|}\right),\ \text{ if }\ r>(s-2).\end{cases}\tag{2.11}$$

Again note that

$$\left|D^{\gamma}\left(\exp\left(\frac{-\boldsymbol{t}'\boldsymbol{D}_n\boldsymbol{t}}{2}\right)\right)\right|\le C_4(\boldsymbol{\gamma})\left(1+\|\boldsymbol{t}\|\right)^{|\boldsymbol{\gamma}|}\|\boldsymbol{D}_n\|^{|\boldsymbol{\gamma}|}\left(\exp\left(\frac{-\boldsymbol{t}'\boldsymbol{D}_n\boldsymbol{t}}{2}\right)\right)\tag{2.12}$$

$$\text{and}\quad\sum_{j=m_2}^{\infty}\left|\frac{c_n^{2j}D^{\boldsymbol{\beta}-\boldsymbol{\alpha}-\boldsymbol{\gamma}}\left(\|\boldsymbol{t}\|^{2j}\right)}{2^j j!}\right|\le C_5(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})c_n^{2m_3}\left[e^{c_n^2/2}+\|\boldsymbol{t}\|^{m_3}\exp(c_n^2\|\boldsymbol{t}\|^2/2)\right],\tag{2.13}$$

where $m_3=m_3(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},r)=\max\{m_2,|\boldsymbol{\beta}-\boldsymbol{\alpha}-\boldsymbol{\gamma}|/2\}$. Now combining (2.11)-(2.13), from (2.10) we have $I_{212}=o\left(n^{-(s-2)/2}\right)$. Last step is to show $I_{211}=o\left(n^{-(s-2)/2}\right)$. Recall that

$$\begin{aligned}I_{211}&=C(k)\max_{|\boldsymbol{\beta}|\in\{0,\dots,(k+1)\}}\left[\int\left|D^{\boldsymbol{\beta}}\left(\hat{q}_n(t)-\breve{\xi}_{n,(\tilde{k}-1)}(t)\right)\right|dt\right]\\&\le C(k)\max_{|\boldsymbol{\beta}|\in\{0,\dots,(k+1)\}}\left[\int_{A_n}\left|D^{\boldsymbol{\beta}}\left(\hat{q}_n(t)-\breve{\xi}_{n,(\tilde{k}-1)}(t)\right)\right|dt+\int_{A_n^c}\left|D^{\boldsymbol{\beta}}\left(\hat{q}(t)-\breve{\xi}_{n,(\tilde{k}-1)}(t)\right)\right|dt\right]\\&=I_{2111}+I_{2112}\quad\text{(say)},\end{aligned}\tag{2.14}$$

where

$$A_n=\left\{\boldsymbol{t}\in\mathcal{R}^k:\|\boldsymbol{t}\|\le C_6(k)\lambda_n^{-1/2}\left(\frac{n^{1/2}}{\eta_{\tilde{k}}^{1/(\tilde{k}-2)}}\right)^{(\tilde{k}-2)/\tilde{k}}\right\},$$

with $C_6(k)$ being some fixed positive constant, λ_n being the largest eigen value of $\boldsymbol{D}_n$, $\eta_{\tilde{k}}=n^{-1}\sum_{i=1}^n E\|\breve{\boldsymbol{B}}_nW_i\|^{\tilde{k}}$ and $\breve{\boldsymbol{B}}_n^2=\boldsymbol{D}_n^{-1}$. Note that

$$D^{\boldsymbol{\beta}}\left(\hat{q}_n(t)-\breve{\xi}_{n,(\tilde{k}-1)}(t)\right)=$$

$$\sum_{\boldsymbol{0}\le\boldsymbol{\alpha}\le\boldsymbol{\beta}}C_7(\boldsymbol{\alpha},\boldsymbol{\beta})D^{\boldsymbol{\alpha}}\left(E\left(e^{i\sqrt{n}\boldsymbol{t}'}\bar{W}_n\right)-\exp\left(\frac{-\boldsymbol{t}'\boldsymbol{D}_n\boldsymbol{t}}{2}\right)\sum_{r=0}^{\tilde{k}-3}n^{-r/2}\tilde{P}_r\left(i\boldsymbol{t}:\left\{\bar{\tilde{\chi}}_{\nu,n}\right\}\right)\right)D^{\boldsymbol{\beta}-\boldsymbol{\alpha}}\left(\exp\left(\frac{-c_n^2\|\boldsymbol{t}\|^2}{2}\right)\right),\tag{2.15}$$

where

$$\left|D^{\boldsymbol{\beta}-\boldsymbol{\alpha}}\left(\exp\left(\frac{-c_n^2\|\boldsymbol{t}\|^2}{2}\right)\right)\right|\le C_8(\boldsymbol{\alpha},\boldsymbol{\beta})c_n^{2|\boldsymbol{\beta}-\boldsymbol{\alpha}|}\|\boldsymbol{t}\|^{|\boldsymbol{\beta}-\boldsymbol{\alpha}|}\exp\left(\frac{-c_n^2\|\boldsymbol{t}\|^2}{2}\right)\quad\text{and}$$

by Theorem 9.11 and the following remark of Bhattacharya and Rao (1986) we have

$$
\begin{aligned}
&\left|D^{\alpha}\left(E\left(e^{i\sqrt{n}\boldsymbol{t}'\bar{W}_n}\right)-\exp\left(\frac{-\boldsymbol{t}'\boldsymbol{D}_n\boldsymbol{t}}{2}\right)\sum_{r=0}^{\tilde{k}-3}n^{-r/2}\tilde{P}_r\left(i\boldsymbol{t}:\left\{\bar{\bar{\chi}}_{\nu,n}\right\}\right)\right)\right| \\
&\leq C_9(k)\lambda_n^{|\alpha|/2}\eta_{\tilde{k}}n^{-(\tilde{k}-2)/2}\left[(\boldsymbol{t}'D_n\boldsymbol{t})^{(\tilde{k}-|\alpha|/2)}+(\boldsymbol{t}'D_n\boldsymbol{t})^{(3(\tilde{k}-2)+|\alpha|)/2}\right]\exp\left(\frac{-\boldsymbol{t}'\boldsymbol{D}_n\boldsymbol{t}}{4}\right). \qquad (2.16)
\end{aligned}
$$

Now note that $\bar{\rho}_{s+\delta}=O(1)$ and E_n converges to a positive definite matrix $\boldsymbol{E}$. Hence applying Lemma 14.1(v) (with $s'=\tilde{k}$) and corollary 14.2 of Bhattacharya and Rao (1986), from (2.15) we have $I_{2111}=o\left(n^{-(s-2)/2}\right)$. Again applying Lemma 14.1(v) and corollary 14.2 of Bhattacharya and Rao (1986) we have $\eta_{\tilde{k}}\leq C_{10}(\tilde{k},s)n^{(\tilde{k}-s)/2}\bar{\rho}_s$ for large enough n and λ_n being converged to some positive number. Therefore we have for large enough n,

$$
A_n^c\subseteq B_n^{\dagger}\ \text{ where }\ B_n^{\dagger}=\left\{\boldsymbol{t}\in\mathcal{R}^k:\|\boldsymbol{t}\|>C_{11}(k,\boldsymbol{E})n^{(s-2)/2\tilde{k}}\right\},
$$

implying

$$
\begin{aligned}
I_{2112}&\leq C(k)\max_{|\boldsymbol{\beta}|\in\{0,\ldots,(k+1)\}}\int_{B_n^{\dagger}}\left|D^{\boldsymbol{\beta}}\left(\hat{q}_n(t)-\breve{\xi}_{n,(\tilde{k}-1)}(t)\right)\right|dt \\
&\leq C(k)\max_{|\boldsymbol{\beta}|\in\{0,\ldots,(k+1)\}}\left[\int_{B_n^{\dagger}}\left|D^{\boldsymbol{\beta}}\left(\hat{q}_n(t)\right)\right|d\boldsymbol{t}+\int_{B_n^{\dagger}}\left|D^{\boldsymbol{\beta}}\left(\breve{\xi}_{n,(\tilde{k}-1)}(t)\right)\right|dt\right] \\
&=I_{21121}+I_{21122}\quad\text{(say)}, \qquad (2.17)
\end{aligned}
$$

for large enough n. To establish $I_{2112}=o\left(n^{-(s-2)/2}\right)$, first we are going to show $I_{21122}=o\left(n^{-(s-2)/2}\right)$. Note that

$$
D^{\boldsymbol{\beta}}\left(\breve{\xi}_{n,(\tilde{k}-1)}(t)\right)=\sum_{\boldsymbol{0}\leq\boldsymbol{\alpha}\leq\boldsymbol{\beta}}C_{12}(\boldsymbol{\alpha},\boldsymbol{\beta})D^{\boldsymbol{\alpha}}\left(\sum_{r=0}^{\tilde{k}-3}n^{-r/2}\tilde{P}_r\left(i\boldsymbol{t}:\left\{\bar{\bar{\chi}}_{\nu,n}\right\}\right)\right)D^{\boldsymbol{\beta}-\boldsymbol{\alpha}}\left(\exp\left(\frac{-\boldsymbol{t}'\tilde{\boldsymbol{D}}_n\boldsymbol{t}}{2}\right)\right),
$$

where $\tilde{\boldsymbol{D}}_n=\boldsymbol{D}_n+c_n^2\boldsymbol{I}_k$. We are going to use bounds (2.11) and (2.12) with $\boldsymbol{D}_n$ being replaced by $\tilde{\boldsymbol{D}}_n$. Note that by Corollary 14.2 of Bhattacharya and Rao (1986) and the fact that $c_n=O(n^{-d})$, $\tilde{\boldsymbol{D}}_n$ converges to the positive definite matrix $\boldsymbol{E}$, which is the limit of $\boldsymbol{E}_n$. Hence those bounds will imply that for large enough n,

$$
\begin{aligned}
I_{21122}&=C(k)\max_{|\boldsymbol{\beta}|\in\{0,\ldots,(k+1)\}}\int_{B_n^{\dagger}}\left|D^{\boldsymbol{\beta}}\left(\breve{\xi}_{n,(\tilde{k}-1)}(t)\right)\right|dt \\
&\leq C_{13}(k,\boldsymbol{E})n^{(\tilde{k}+1-s)/2}\int_{B_n^{\dagger}}\left(1+\|\boldsymbol{t}\|^{3(\tilde{k}-1)}\right)\exp\left(-C_{14}(\boldsymbol{E})\|\boldsymbol{t}\|^2/2\right)d\boldsymbol{t} \\
&\leq C_{15}(k,\boldsymbol{E})n^{(\tilde{k}+1-s)/2}\int_{B_n^{\dagger}}\exp\left(-C_{14}(\boldsymbol{E})\|\boldsymbol{t}\|^2/4\right)d\boldsymbol{t}. \qquad (2.18)
\end{aligned}
$$

Now apply Lemma 2.2 to conclude that $I_{21122} = o\left(n^{-(s-2)/2}\right)$. Only remaining thing to show is $I_{21121} = o\left(n^{-(s-2)/2}\right)$. Note that

$$D^{\beta}\left(\hat{q}_n(\boldsymbol{t})\right) = \sum_{\mathbf{0}\leq\boldsymbol{\alpha}\leq\boldsymbol{\beta}} C_{16}(\boldsymbol{\alpha},\boldsymbol{\beta}) D^{\alpha}\left(E\left(e^{i\sqrt{n}\boldsymbol{t}'\bar{W}_n}\right)\right) D^{\beta-\alpha}\left(\exp\left(\frac{-c_n^2\|\boldsymbol{t}\|^2}{2}\right)\right), \tag{2.19}$$

where

$$\left|D^{\alpha}\left(E\left(e^{i\sqrt{n}\boldsymbol{t}'\bar{W}_n}\right)\right)\right| = \left|D^{\alpha}\left(\prod_{i=1}^{n} E\left(e^{i\boldsymbol{t}'W_i/\sqrt{n}}\right)\right)\right|$$

$$\text{and} \quad \left|D^{\beta-\alpha}\left(\exp\left(\frac{-c_n^2\|\boldsymbol{t}\|^2}{2}\right)\right)\right| \leq C_{17}(\boldsymbol{\alpha},\boldsymbol{\beta})\left(1+\|\boldsymbol{t}\|^{|\beta-\alpha|}\right)\exp\left(\frac{-c_n^2\|\boldsymbol{t}\|^2}{2}\right).$$

Now by Leibniz's rule of differentiation, $D^{\alpha}\left(E\left(e^{i\sqrt{n}\boldsymbol{t}'\bar{W}_n}\right)\right)$ is the sum of $n^{|\alpha|}$ terms. A typical term is of the form

$$\prod_{i\notin C_r} E\left(e^{i\boldsymbol{t}'W_i/\sqrt{n}}\right)\prod_{l=1}^{r} D^{\beta_l}\left(E\left(e^{i\boldsymbol{t}'W_{i_l}/\sqrt{n}}\right)\right),$$

where $C_r = \{i_1,\ldots,i_r\} \subset \{1,\ldots,n\}$, $1 \leq r \leq |\boldsymbol{\alpha}|$. $\boldsymbol{\beta}_1,\ldots,\boldsymbol{\beta}_r$ are non-negative integral vectors satisfying $|\boldsymbol{\beta}_j| \geq 1$ for $j \in \{1,\ldots,r\}$ and $\sum_{j=1}^{r}\boldsymbol{\beta}_i = \boldsymbol{\alpha}$. Note that $\left|D^{\beta_l}\left(E\left(e^{i\boldsymbol{t}'W_{i_l}/\sqrt{n}}\right)\right)\right| \leq n^{-|\beta_l|/2}E\|W_{i_l}\|^{|\beta_l|}$ and $W_{j_l} \leq 2\sqrt{n}$, which implies that

$$\left|\prod_{i\notin C_r} E\left(e^{i\boldsymbol{t}'W_i/\sqrt{n}}\right)\prod_{l=1}^{r} D^{\beta_l}\left(E\left(e^{i\boldsymbol{t}'W_{i_l}/\sqrt{n}}\right)\right)\right| \leq 2^{\sum_{l=1}^{r}|\beta_l|} = 2^{|\alpha|}$$

$$\Rightarrow \left|D^{\alpha}\left(E\left(e^{i\sqrt{n}\boldsymbol{t}'\bar{W}_n}\right)\right)\right| \leq (2n)^{|\alpha|}.$$

Let $K_n = C_{11}(k,\boldsymbol{E})n^{(s-2)/2\tilde{k}}$. Therefore from (2.19), for large enough n we have

$$\begin{aligned}
I_{21121} &\leq \left[\max_{|\beta|\in\{0,\ldots,(k+1)\}}\sum_{\mathbf{0}\leq\boldsymbol{\alpha}\leq\boldsymbol{\beta}} C_{16}(\boldsymbol{\alpha},\boldsymbol{\beta})\right](2n)^{k+1}\left[\int_{B_n^{\dagger}}\left(1+\|\boldsymbol{t}\|^{k+1}\right)\exp\left(\frac{-c_n^2\|\boldsymbol{t}\|^2}{2}\right)d\boldsymbol{t}\right] \\
&\leq C_{18}(k)(2n)^{k+1}\int_{r\geq K_n} r^{k-1}\left(1+r^{k+1}\right)e^{-c_n^2r^2/2}dr \\
&\leq C_{19}(k)(2n)^{k+1}c_n^{-1}\int_{r\geq K_n}\frac{1}{2\sqrt{\pi}c_n^{-1}}e^{-c_n^2r^2/4}dr \\
&\leq C_{20}(k)n^{k+(s-2)/\tilde{k}}\int_{c_nK_n/\sqrt{2}}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz \\
&= o\left(n^{-(s-2)/2}\right).
\end{aligned} \tag{2.20}$$

The second inequality follows by considering polar transformation. Third inequality follows due to the assumptions that $n^{-(s-2)/\tilde{k}}(\log n) = o(c_n^2)$. The last equality is the implication of Lemma 2.2 and $\sqrt{\log n} = o(c_nK_n)$. Therefore the proof of Lemma 2.3 is now complete.

Lemma 2.4. *Suppose that all the assumptions of Lemma 2.2 are true. Define $d_n = n^{-1/2}c_n$ and $A_\delta = \{x \in \mathcal{R}^k : \|\boldsymbol{x}\| < \delta\}$ for some $\delta > 0$. Let $H : \mathcal{R}^k \to \mathcal{R}^m$ $(k \geq m)$ has continuous partial derivatives of all orders on A_δ and $Grad[H(\mathbf{0})]$ is of full row rank. Then uniformly for any Borel subset B of $\mathcal{R}^m$ we have*

$$\left|\mathbf{P}\Big(\sqrt{n}\big(H(\bar{Y}_n + d_n Z) - H(\mathbf{0})\big) \in B\Big) - \int_B \check{\psi}_{n,s}(x)dx\right| = o\Big(n^{-(s-2)/2}\Big), \tag{2.21}$$

where $\check{\psi}_{n,s}(\boldsymbol{x}) = \Big[1 + \sum_{r=1}^{s-2} n^{-r/2} a_{1,r}(Q_n, \boldsymbol{x})\phi_{\check{\boldsymbol{M}}_n^\dagger}(\boldsymbol{x})\Big]\Big[\sum_{j=1}^{m_1-1} c_n^{2j} a_{2,j}(\boldsymbol{x})\Big]$ with $m_1 = \inf\big\{j : c_n^{2j} = o\big(n^{-(s-2)/2}\big)\big\}$ and Q_n being the distribution of $\sqrt{n}\bar{Y}_n$. $a_{1,r}(Q_n, \cdot)$, $r \in \{1, \ldots, (s-2)\}$, are polynomials whose coefficients are continuous functions of first s average cumulants of $\{Y_1, \ldots, Y_n\}$. $a_{2,j}(\cdot)$, $j \in \{1, \ldots, (m-1)\}$, are polynomials whose coefficients are continuous functions of partial derivatives of H of order $(s-1)$ or less. $\check{\boldsymbol{M}}_n^\dagger = \bar{\boldsymbol{B}}\boldsymbol{E}_n\bar{\boldsymbol{B}}'$ with $\bar{\boldsymbol{B}} = Grad[H(\mathbf{0})]$ and $\boldsymbol{E}_n = n^{-1}\sum_{i=1}^n Var(Y_i)$.

Proof of Lemma 2.4. This follows exactly through the same line of the proof of Lemma 3.2 in Lahiri (1989).

Lemma 2.5. *Let $Y_1, \ldots, Y_n$ be mean zero independent random vectors in $\mathcal{R}^k$ with $n^{-1}\sum_{i=1}^n E\|Y_i\|^3 = O(1)$. Suppose that $T_n^2 = E_n^{-1}$ where $E_n = n^{-1}\sum_{i=1}^n Var(Y_i)$ is the average positive definite covariance matrix and E_n converges to some positive definite matrix as $n \to \infty$. Then for any Borel subset B of $\mathcal{R}^k$ we have*

$$\left|\mathbf{P}\Big(n^{-1/2}T_n\sum_{i=1}^n Y_i \in B\Big) - \Phi(B)\right| \leq C_{22}(k)n^{-1/2}\rho_3 + 2\,\Phi\Big((\partial B)^{C_{22}(k)\rho_3 n^{-1/2}}\Big),$$

where $\rho_3 = n^{-1}\sum_{i=1}^n E\|T_n Y_i\|^3$.

Proof of Lemma 2.5. This is a direct consequence of part (a) of corollary 24.3 in Bhattacharya and Rao (1986).

Lemma 2.6. *Suppose that A, B are matrices such that $(A - aI)$ and $(B - aI)$ are positive semi-definite matrices of same order, for some $a > 0$. For some $r > 0$, A^r, B^r are defined in the usual way. Then for any $0 < r < 1$, we have*

$$\|A^r - B^r\| \leq r a^{r-1}\|A - B\|.$$

Proof of Lemma 2.6. More general version of this lemma is stated as corollary (X.46) in Bhatia (1997).

Lemma 2.7. *Suppose that all the assumptions of Lemma 2.4 are true and there $\check{\boldsymbol{M}}_n^\dagger = I_m$, the $m \times m$ identity matrix. Define $\hat{H}_n = \Big[\sqrt{n}\big(H(\bar{Y}_n + d_n Z) - H(\mathbf{0})\big)\Big] + R_n$ where $\mathbf{P}\Big(\|R_n\| = o\big(n^{-(s-2)/2}\big)\Big) = 1 - o\big(n^{-(s-2)/2}\big)$ and s is as defined in Lemma 2.3. Then we have*

$$\sup_{B \in \mathcal{A}_m}\left|\mathbf{P}\Big(\hat{H}_n \in B\Big) - \int_B \check{\psi}_{n,s}(x)dx\right| = o\Big(n^{-(s-2)/2}\Big), \tag{2.22}$$

where the class of sets $\mathcal{A}_m$ is the collection of Borel subsets of $\mathcal{R}^m$ satisfying

$$\sup_{B \in \mathcal{A}_m} \Phi((\partial B)^\epsilon) = O(\epsilon) \quad as \;\; \epsilon \downarrow 0.$$

Proof of Lemma 2.7. Recall the definition of $(\partial B)^{\epsilon}$ which is nothing but the $\epsilon-$neighbourhood of the boundary of the set B. For some $B \subseteq \mathcal{R}^m$ and $\delta > 0$, define $B^{n,s,\delta} = (\partial B)^{\delta n^{-(s-2)/2}}$. Hence using Lemma 2.4, uniformly for any $B \in \mathcal{A}_m$ we have

$$\begin{aligned}
&\left|\mathbf{P}\left(\hat{H}_n \in B\right) - \int_B \check{\psi}_{n,s}(x)dx\right| \\
&\leq \left|\mathbf{P}\left(\hat{H}_n \in B\right) - \mathbf{P}\left(\sqrt{n}\left(H(\bar{Y}_n + d_n Z) - H(\mathbf{0})\right) \in B\right)\right| + o\left(n^{-(s-2)/2}\right) \\
&\leq \mathbf{P}\left(\|R_n\| \neq o\left(n^{-(s-2)/2}\right)\right) + 2\mathbf{P}\left(\sqrt{n}\left(H(\bar{Y}_n + d_n Z) - H(\mathbf{0})\right) \in B^{n,s,\delta}\right) + o\left(n^{-(s-2)/2}\right) \\
&= 2\mathbf{P}\left(\sqrt{n}\left(H(\bar{Y}_n + d_n Z) - H(\mathbf{0})\right) \in B^{n,s,\delta}\right) + o\left(n^{-(s-2)/2}\right) \\
&= 2\int_{B^{n,s,\delta}} \check{\psi}_{n,s}(x)dx + o\left(n^{-(s-2)/2}\right)
\end{aligned} \tag{2.23}$$

for any $\delta > 0$. Now consider the calculations at page 213 of Bhattacharya and Rao (1986). The bound $\left[M_{s'}(f)o(n^{(s-2)/2}) + c_{13}\rho_s\bar{\omega}_f(2e^{-dn} : \mathbf{\Phi})\right]$ is obtained for $\bar{\omega}_{f_{a_n}}\left(2e^{-dn} : \left|\sum_{r=0}^{s+k-2} n^{-r/2} P_r\left(-\Phi_{0,\boldsymbol{D}_n} : \{\chi_{\nu,n}\}\right)\right|\right)$ there. Now assuming $f(\boldsymbol{x}) = I(\boldsymbol{x} \in \boldsymbol{B})$ ($I(\cdot)$ is the indicator function), $a_n = 0$ and replacing $\sum_{r=0}^{s+k-2} n^{-r/2} P_r\left(-\Phi_{0,\boldsymbol{D}_n} : \{\chi_{\nu,n}\}\right)$ by $\check{\psi}_{n,s}(\cdot)$, we have uniformly for any $B \in \mathcal{A}_m$,

$$\int_{B^{n,s,\delta}} \check{\psi}_{n,s}(x)dx \leq C_{21}(s) \sup_{B \in \mathcal{A}_m} \Phi\left(B^{n,s,\delta}\right) + o\left(n^{-(s-2)/2}\right) = o\left(n^{-(s-2)/2}\right),$$

since $\delta > 0$ is arbitrary. Therefore (2.22) follows from (2.23).

Lemma 2.8. *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be positive definite matrices of same order. Then for some given matrix $\boldsymbol{C}$, the solution of the equation $\boldsymbol{AX} + \boldsymbol{XB} = \boldsymbol{C}$ can be expressed as*

$$\boldsymbol{X} = \int_0^\infty e^{-t\boldsymbol{A}} \boldsymbol{C} e^{-t\boldsymbol{B}} dt,$$

where $e^{-t\boldsymbol{A}}$ and $e^{-t\boldsymbol{B}}$ are defined in the usual way.

Proof of Lemma 2.8. This lemma is an easy consequence of Theorem VII.2.3 in Bhatia (1997).

Lemma 2.9. *Let $W_1, \ldots, W_n$ be n independent mean 0 random variables with average variance $s_n^2 = n^{-1}\sum_{i=1}^n \mathbf{E}W_i^2$ and $\mathbf{P}\left(\max\{|W_j| : i \in \{1, \ldots, n\}\} \leq C_{30}\right) = 1$ for some positive constant C_{30} and integer $s \geq 3$. $\bar{\chi}_{\nu,n}$ is the average νth cumulant. Recall the polynomial $\tilde{P}_r$ for any non-negative integer r, as defined in the beginning of this section. Then there exists two constants $0 < C_{31}(s) < 1$ and $C_{32}(s) > 0$ such that whenever $|t| \leq C_{31}(s)\sqrt{n}\min\{C_{30}^{-2}s_n, C_{30}^{-s/(s-2)}s_n^{s/(s-2)}\}$, we have*

$$\begin{aligned}
&\left|\prod_{j=1}^n \mathbf{E}\left(e^{in^{-1/2}tW_j}\right) - \sum_{r=0}^{s-2} n^{-r/2}\tilde{P}_r\left(it : \{\bar{\chi}_{\nu,n}\}\right)e^{-(s_n^2t^2)/2}\right| \\
&\leq C_{32}(s)C_{30}^s s_n^{-s} n^{-(s-2)/2}\left[(s_nt)^s + (s_nt)^{3(s-2)}\right]e^{-(s_n^2t^2)/4}
\end{aligned}$$

Proof of Lemma 2.9. In view of Theorem 9.9 of Bhattacharya and Rao (1986), it is enough to show that for any $j \in \{1,\dots,n\}$, whenever $|t| \le C_{31}(s)\sqrt{n}\min\{C_{30}^{-2}s_n, C_{30}^{-s/(s-2)}s_n^{s/(s-2)}\}$, we have $\Big|\mathbf{E}\big(e^{its_n^{-1}n^{-1/2}W_j}\big)-1\Big| \le 1/2$. This is indeed the case due to the fact that

$$\Big|\mathbf{E}\big(e^{itn^{-1/2}W_j}\big)-1\Big| \le \frac{t^2\mathbf{E}W_j^2}{2ns_n^2}.$$

Lemma 2.10. *Assume the setup of Theorem 2 and let $X_i = y_i x_i$, $i \in \{1,\dots,n\}$. Define $\sigma_n^2 = n^{-1}\sum_{i=1}^n Var(X_i)$ and $\bar{\chi}_{\nu,n}$ as the νth average cumulant of $\{(X_1 - E(X_1)),\dots,(X_n - E(X_n))\}$. $P_r\big(-\Phi_{\sigma_n^2} : \{\bar{\chi}_{\nu,n}\}\big)$ is the finite signed measure on $\mathcal{R}$ whose density is $\tilde{P}_r\big(-D : \{\bar{\chi}_{\nu,n}\}\big)\phi_{\sigma_n^2}(x)$. Let $S_0(x) = 1$ and $S_1(x) = x - [x] - 1/2$, where $[x]$ is the greatest integer $\le x$. Suppose that σ_n^2 is bounded away from both 0 & ∞ and assumptions (C.1)-(C.3) of Theorem 2 hold. Then we have*

$$\begin{aligned}\sup_{x\in\mathcal{R}}\Big|\mathbf{P}\Big(n^{-1/2}\sum_{i=1}^n\big(X_i - E(X_i)\big)\le x\Big) - \sum_{r=0}^{1} n^{-r/2}(-1)^r S_r(n\mu_n + n^{1/2}x)\frac{d^r}{dx^r}\Phi_{\sigma_n^2}(x)\\ - n^{-1/2}P_1\big(-\Phi_{\sigma_n^2} : \{\bar{\chi}_{\nu,n}\}\big)(x)\Big| = o\big(n^{-1/2}\big),\end{aligned} \tag{2.24}$$

where $P_r\big(-\Phi_{\sigma_n^2} : \{\bar{\chi}_{\nu,n}\}\big)(x)$ is the $P_r\big(-\Phi_{\sigma_n^2} : \{\bar{\chi}_{\nu,n}\}\big)$–measure of the set $(-\infty, x]$.

Proof of Lemma 2.10. For any integer α, define $p_n(x) = \mathbf{P}\big(\sum_{i=1}^n X_i = \alpha\big)$ and $x_{\alpha,n} = n^{-1/2}(\alpha - n\mu_n)$. Also define $\tilde{X}_n = n^{-1/2}\sum_{i=1}^n\big(X_i - E(X_i)\big)$ and $q_{n,s}(x) = n^{-1/2}\sum_{r=0}^{s-2}n^{-r/2}\tilde{P}_r\big(-D : \{\bar{\chi}_{\nu,n}\}\big)\phi_{\sigma_n^2}(x)$ for integers $s \ge 3$. Note that

$$\begin{aligned}&\sup_{x\in\mathcal{R}}\Big|\mathbf{P}\Big(n^{-1/2}\sum_{i=1}^n\big(X_i - E(X_i)\big)\le x\Big) - \sum_{r=0}^{1} n^{-r/2}(-1)^r S_r(n\mu_n + n^{1/2}x)\frac{d^r}{dx^r}\Phi_{\sigma_n^2}(x)\\ &\qquad - n^{-1/2}P_1\big(-\Phi_{\sigma_n^2} : \{\bar{\chi}_{\nu,n}\}\big)(x)\Big|\\ &\le \sup_{x\in\mathcal{R}}\Big|\mathbf{P}\Big(\tilde{X}_n \le x\Big) - Q_{n,5}(x)\Big| + \sup_{x\in\mathcal{R}}\Big|Q_{n,3}(x) - \sum_{r=0}^{1} n^{-r/2}(-1)^r S_r(n\mu_n + n^{1/2}x)\frac{d^r}{dx^r}\Phi_{\sigma_n^2}(x)\\ &\qquad - n^{-1/2}P_1\big(-\Phi_{\sigma_n^2} : \{\bar{\chi}_{\nu,n}\}\big)(x)\Big| + \sup_{x\in\mathcal{R}}\Big|Q_{n,5}(x) - Q_{n,3}(x)\Big| \qquad (2.25)\\ &= J_1 + J_2 + J_e \quad \text{(say)},\end{aligned}$$

where $Q_{n,s}(x) = \sum_{\{\alpha : x_{\alpha,n}\le x\}} q_{n,s}(x_{\alpha,n})$ for any integer $s \ge 3$. Now $J_2 = o\big(n^{-1/2}\big)$ follows from Theorem A.4.3 of Bhattacharya and Rao (1986) and dropping terms of order n^{-1}. Now we are going to show $J_1 = O\big(n^{-1}\big)$. Note that

$$J_1 \le \sum_{\alpha\in\Theta}\big|p_n(x_{\alpha,n}) - q_{n,5}(x_{\alpha,n})\big| = J_3 \quad \text{(say)},$$

where Θ has cardinality $\le C_{33}n$, since $\mathbf{P}\big(\big|n^{-1}\sum_{i=1}^n X_i\big| \le C_{33}\big) = 1$ for some constant $C_{33} > 0$, due to the assumption that $\max\{|x_j|^5 : j \in \{1,\dots,n\}\} = O(1)$. Hence $n^{-1}J_3 \le C_{33}\sup_{\alpha\in\Theta}\big|p_n(x_{\alpha,n}) -$

$q_{n,5}(x_{\alpha,n})\big| = C_{33}\sup_{\alpha\in\Theta} J_4(\alpha)$ (say). Hence enough to show $\sup_{\alpha\in\Theta} J_4(\alpha) = O\big(n^{-2}\big)$. Now define $g_j(t) = \mathbf{E}\big(e^{itX_j}\big)$ and $f_n(t) = \mathbf{E}(it\tilde{X}_n)$. Then we have

$$f_n\big(\sqrt{n}t\big) = \sum_{\alpha\in\Theta} p_n\big(x_{\alpha,n}\big)e^{i\sqrt{n}tx_{\alpha,n}}.$$

Hence by Fourier inversion formula for lattice random variables (cf. page 230 of Bhattacharya and Rao (1986)), we have

$$\begin{aligned} p_n\big(x_{\alpha,n}\big) &= (2\pi)^{-1}\int_{\mathcal{F}^*} e^{-i\sqrt{n}tx_{\alpha,n}} f_n\big(\sqrt{n}t\big)dt \\ &= (2\pi)^{-1}n^{-1/2}\int_{\sqrt{n}\mathcal{F}^*} e^{-itx_{\alpha,n}} f_n\big(t\big)dt, \end{aligned} \tag{2.26}$$

where $\mathcal{F}^* = (-\pi,\pi)$, the fundamental domain corresponding to the lattice distribution of $\sum_{i=1}^n X_i$.

Again note that

$$q_{n,s}(x_{\alpha,n}) = (2\pi)^{-1}n^{-1/2}\int_{\mathcal{R}} e^{-itx_{\alpha,n}}\sum_{r=0}^{s-2} n^{-r/2}\tilde{P}_r\big(it:\{\bar{\chi}_{\nu,n}\}\big)e^{-\sigma_n^2t^2/2}dt. \tag{2.27}$$

Now defining the set $E = \Big\{t\in\mathcal{R}: |t|\le C_{31}(s)\sqrt{n}\min\big\{C_{33}^{-2}\sigma_n, C_{33}^{-5/3}\sigma_n^{5/3}\big\}\Big\}$, from (2.26) & (2.27) we have

$$\begin{aligned} sup_{\alpha\in\Theta}J_4(\alpha) \le&(2\pi)^{-1}n^{-1/2}\Bigg[\int_E\Bigg|f_n(t) - \sum_{r=0}^{3} n^{-r/2}\tilde{P}_r\big(it:\{\bar{\chi}_{\nu,n}\}\big)e^{-\sigma_n^2t^2/2}\Bigg|dt \\ &+\int_{\sqrt{n}\mathcal{F}^*\cap E^c}|f_n(t)|dt + \int_{\mathcal{R}\cap(\sqrt{n}\mathcal{F}^*)^c}\Bigg|\sum_{r=0}^{3} n^{-r/2}\tilde{P}_r\big(it:\{\bar{\chi}_{\nu,n}\}\big)e^{-\sigma_n^2t^2/2}\Bigg|dt\Bigg] \\ =&(2\pi)^{-1}n^{-1/2}\big(J_{41}+J_{42}+J_{43}\big) \quad \text{(say)}. \end{aligned} \tag{2.28}$$

Note that $J_{41} = O\big(n^{-3/2}\big)$ by applying Lemma 2.9 with $s = 5$. $J_{43} = O\big(n^{-3/2}\big)$ due to the presence of the exponential term in the integrand and the form of the set $\sqrt{n}\mathcal{F}^*$. Moreover noting the form of the set $\mathcal{F}^*$, we can say that there exists constants $C_{34} > 0$, $0 < C_{35}, C_{36} < \pi$ such that

$$J_{42} \le C_{34}\sup_{t\in\sqrt{n}\mathcal{F}^*\cap E^c}\prod_{i=1}^{n}\big|g_j(n^{-1/2}t)\big| \le C_{34}\sup_{C_{35}\le|t|\le C_{36}}\big|\mathbf{E}(e^{ity_{i_1}})\big|^m \le C_{34}\delta^m, \tag{2.29}$$

for some $0 < \delta < 1$. Recall that $x_{i_j} = 1$ for all $j \in \{1,\ldots,m\}$. The last inequality is due to the fact that there is no period of $\mathbf{E}(e^{ity_{i_1}})$ in the interval $[C_{35}, C_{36}] \cup [-C_{36}, -C_{35}]$. Now $J_{42} = O(n^{-3/2})$ follows from (2.29) since $m \ge (\log n)^2$.

Hence it is left to show that $J_e = o(n^{-1/2})$. Note that with $\mathcal{Z}$ being the set of integers,

$$n^{3/2}J_e \le \sum_{\alpha\in\mathcal{Z}}|\tilde{P}_2\big(-D:\{\bar{\chi}_{\nu,n}\}\big)\phi_{\sigma_n^2}(x_{\alpha,n})| + n^{-1/2}\sum_{\alpha\in\mathcal{Z}}|\tilde{P}_3\big(-D:\{\bar{\chi}_{\nu,n}\}\big)\phi_{\sigma_n^2}(x_{\alpha,n})|,$$

where individual sums in the right hand side is $O(n^{1/2})$ using Lemma A.4.5 of Bhattacharya and Rao (1986). Therefore the proof is complete.

Lemma 2.11. *Let $\breve{W}_1, \ldots, \breve{W}_n$ be iid mean $\mathbf{0}$ non-degenerate random vectors in $\mathcal{R}^l$ for some natural number l, with finite fourth absolute moment and $\limsup_{\|\boldsymbol{t}\|\to\infty} \left|\mathbf{E}e^{i\boldsymbol{t}'\breve{W}_1}\right| < 1$ (i.e. Cramer's condition holds). Suppose that $\breve{W}_i = (\breve{W}_{i1}', \ldots, \breve{W}_{im}')'$ where $\breve{W}_{ij}$ is a random vector in $\mathcal{R}^{l_j}$ and $\sum_{j=1}^m l_j = l$, m being a fixed natural number. Consider the sequence of random variables $\tilde{W}_1, \ldots, \tilde{W}_n$ where $\tilde{W}_i = (c_{i1}\breve{W}_{i1}', \ldots, c_{im}\breve{W}_{im}')'$. $\{c_{ij} : i \in \{1, \ldots, n\}, j \in \{1, \ldots, m\}\}$ is a collection of real numbers such that for any $j \in \{1, \ldots, m\}$, $\left\{n^{-1}\sum_{i=1}^n |c_{ij}|^4\right\} = O(1)$ and $\liminf_{n\to\infty} n^{-1}\sum_{i=1}^n c_{ij}^2 > 0$. Also assume that $\tilde{\boldsymbol{V}}_n = n^{-1}\sum_{i=1}^n Var(\tilde{W}_i)$ converges to some positive definite matrix and $\bar{\chi}_{\nu,n}$ denotes the average νth cumulant of $\tilde{W}_1, \ldots, \tilde{W}_n$. Then we have*

$$\sup_{\boldsymbol{B}\in\mathcal{A}_l}\left|\mathbf{P}\Big(n^{-1/2}\sum_{i=1}^n \tilde{W}_i \in \boldsymbol{B}\Big) - \int_{\boldsymbol{B}}\Big[1 + n^{-1/2}\tilde{P}_r\big(-D : \{\bar{\chi}_{\nu,n}\}\big)\Big]\phi_{\tilde{\boldsymbol{V}}_n}(\boldsymbol{t})d\boldsymbol{t}\right| = o\big(n^{-1/2}\big), \tag{2.30}$$

where the collection of sets $\mathcal{A}_l$ is as defined in section 3.

Proof of Lemma 2.11. First note that $\tilde{W}_1, \ldots, \tilde{W}_n$ is a sequence of independent random variables. Hence (2.30) follows by Theorem 20.6 of Bhattacharya and Rao (1986), provided there exists $\delta_4 \in (0,1)$, independent of n, such that for all $\upsilon \le \delta_4$,

$$n^{-1}\sum_{i=1}^n \mathbf{E}\big\|\tilde{W}_i\big\|^3 \mathbf{1}\Big(\big\|\tilde{W}_i\big\| > \upsilon\sqrt{n}\Big) = o(1) \tag{2.31}$$

and

$$\max_{|\alpha|\le l+2}\int_{\|\boldsymbol{t}\|\ge\upsilon\sqrt{n}}\left|D^{\alpha}\mathbf{E}\exp(i\boldsymbol{t}'\boldsymbol{R}_{1n}^{\dagger})\right|d\boldsymbol{t} = o\Big(n^{-1/2}\Big) \tag{2.32}$$

where $\boldsymbol{R}_{1n}^{\dagger} = n^{-1/2}\sum_{i=1}^n \left(\mathbf{Z}_i - \mathbf{E}\mathbf{Z}_i\right)$ with

$$\mathbf{Z}_i = \tilde{W}_i \mathbf{1}\Big(\big\|\tilde{W}_i\big\| \le \upsilon\sqrt{n}\Big).$$

First consider (2.31). Note that $\max\Big\{|c_{ij}| : i \in \{1, \ldots, n\}, j \in \{1, \ldots, m\}\Big\} = O\big(n^{1/4}\big)$. Therefore, we have for any $\upsilon > 0$,

$$\begin{aligned}
&n^{-1}\sum_{i=1}^n \mathbf{E}\big\|\tilde{W}_i\big\|^3\mathbf{1}\Big(\big\|\tilde{W}_i\big\| > \upsilon\sqrt{n}\Big)\\
\le& n^{-1}\sum_{i=1}^n \mathbf{E}\Big(\sum_{j=1}^m c_{ij}^2\big\|\breve{W}_{ij}\big\|^2\Big)^{3/2}\mathbf{1}\Big(\sum_{j=1}^m c_{ij}^2\big\|\breve{W}_{ij}\big\|^2 > \upsilon^2 n\Big)\\
\le& n^{-1}\sum_{i=1}^n \Big(1 + \sum_{j=1}^m c_{ij}^2\Big)^2\mathbf{E}\Big[\big\|\breve{W}_1\big\|^3\mathbf{1}\Big(\big\|\breve{W}_1\big\|^2 > C_{37}\upsilon^2 n^{1/2}\Big)\Big]\\
=& o(1).
\end{aligned}$$

Now consider (2.32). Note that for any $|\boldsymbol{\alpha}| \le l+2$, $|D^{\boldsymbol{\alpha}}\mathbf{E}\exp(i\boldsymbol{t}'\boldsymbol{R}^{\dagger}_{1n})|$ is bounded above by a sum of $n^{|\alpha|}$-terms, each of which is bounded above by

$$C_{38}(\alpha)\cdot n^{-|\alpha|/2}\max\{\mathbf{E}\|\boldsymbol{Z}_i-\mathbf{E}\boldsymbol{Z}_i\|^{|\alpha|}:k\in\boldsymbol{I}_n\}\cdot\prod_{i\in\boldsymbol{I}_n^{\mathsf{c}}}|\mathbf{E}\exp(i\boldsymbol{t}'\boldsymbol{Z}_i/\sqrt{n})| \tag{2.33}$$

where $\boldsymbol{I}_n\subset\{1,\dots,n\}$ is of size $|\boldsymbol{\alpha}|$ and $\boldsymbol{I}_{\boldsymbol{n}}^{\mathbf{c}}=\{1,\dots,n\}\backslash\boldsymbol{I}_n$. Now for any $\omega>0$ and $\boldsymbol{t}\in\mathcal{R}^{l_j}$, define the set

$$\boldsymbol{B}_n^{(j)}(\boldsymbol{t},\omega)=\Big\{i:1\le i\le n\text{ and }|c_{ij}|\|\boldsymbol{t}\|>\omega\Big\}.$$

Hence for any $\boldsymbol{t}\in\mathcal{R}^l$ writing $\boldsymbol{t}=(\boldsymbol{t}_1',\dots,\boldsymbol{t}_m')'$, $\boldsymbol{t}_j$ is of length l_j, we have

$$\begin{aligned}
&\sup\Big\{\prod_{i\in\boldsymbol{I}_n^{\mathsf{c}}}|\mathbf{E}\exp(i\boldsymbol{t}'\boldsymbol{Z}_k/\sqrt{n})|:\|\boldsymbol{t}\|\ge\upsilon\sqrt{n}\Big\}\\
&=\sup\Big\{\prod_{i\in\boldsymbol{I}_n^{\mathsf{c}}}|\mathbf{E}\exp(i\boldsymbol{t}'\boldsymbol{Z}_k)|:\|\boldsymbol{t}\|^2\ge\upsilon^2\Big\}\\
&\le\max\Bigg\{\sup\Bigg\{\prod_{i\in\boldsymbol{I}_n^{\mathsf{c}}\cap\boldsymbol{B}_n^{(j)}\left(\frac{\boldsymbol{t}_j}{\|\boldsymbol{t}_j\|},\upsilon/\sqrt{2}\right)}\Big[|\mathbf{E}\exp\Big(ic_{ij}\boldsymbol{t}_j'\check{W}_{1j}\Big)|+\mathbf{P}\Big(\|\check{W}_1\|>C_{37}\upsilon^2n^{1/2}\Big)\Big]\\
&:\|\boldsymbol{t}_j\|\ge\upsilon/\sqrt{2}\Bigg\}:j\in\{1,\dots,m\}\Bigg\}
\end{aligned}$$

Now since $\left|\boldsymbol{I}_n^{\mathsf{c}}\right|\ge\left|\boldsymbol{I}_n^{\mathsf{c}}\cap\boldsymbol{B}_n^{(j)}\left(\frac{\boldsymbol{t}_j}{\|\boldsymbol{t}_j\|},\upsilon/\sqrt{2}\right)\right|\ge\left|\boldsymbol{B}_n^{(j)}\left(\frac{\boldsymbol{t}_j}{\|\boldsymbol{t}_j\|},\upsilon/\sqrt{2}\right)\right|-|\alpha|$, due to Cramer's condition we have

$$\begin{aligned}
&\sup\Bigg\{\prod_{i\in\boldsymbol{I}_n^{\mathsf{c}}\cap\boldsymbol{B}_n^{(j)}\left(\frac{\boldsymbol{t}_j}{\|\boldsymbol{t}_j\|},\upsilon/\sqrt{2}\right)}\Big[|\mathbf{E}\exp\Big(ic_{ij}\boldsymbol{t}_j'\check{W}_{1j}\Big)|+\mathbf{P}\Big(\|\check{W}_1\|>C_{37}\upsilon^2n^{1/2}\Big)\Big]:\|\boldsymbol{t}_q\|\ge\upsilon/\sqrt{2}\Bigg\}\\
&\le\theta^{\left|\boldsymbol{B}_n^{(j)}\left(\frac{\boldsymbol{t}_j}{\|\boldsymbol{t}_j\|},\upsilon/\sqrt{2}\right)\right|-|\alpha|},
\end{aligned} \tag{2.34}$$

where $\theta=\left|\mathbf{E}e^{i\boldsymbol{t}'\check{\boldsymbol{W}}_1}\right|\in(0,1)$. Next note that $\liminf_{n\to\infty}n^{-1}\sum_{i=1}^n c_{ij}^2>0$ for all $j\in\{1,\dots,m\}$. Therefore for any $j\in\{1,\dots,m\}$, $\boldsymbol{u}\in\mathcal{R}^{l_j}$ with $|u|=1$, there exists $0<\delta_5<1$ such that for sufficiently large n we have

$$\begin{aligned}
\frac{n\delta_5}{2}&\le\sum_{i=1}^n\left|uc_{ij}\right|^2\\
&\le\max\Big\{\left|c_{ij}\right|^2:1\le i\le n\Big\}\cdot|\boldsymbol{B}_n^{(j)}(u,\omega)|+\Big(n-|\boldsymbol{B}_n^{(j)}(u,\omega)|\Big)\cdot\omega^2\\
&\le C_{38}\cdot n^{1/2}\cdot|\boldsymbol{B}_n^{(j)}(\boldsymbol{u},\omega)|+n\omega^2
\end{aligned}$$

which implies $|\boldsymbol{B}_n^{(j)}(\boldsymbol{u},\omega)| \geq C_{39}\cdot n^{1/2}$ whenever $\omega < \sqrt{\delta_5/2}$. Therefore taking $\delta_4 = \sqrt{\delta_5/3}$, (2.32) follows from (2.33) and (2.34).

Lemma 2.12. *Define $\boldsymbol{M}_n = n^{-1}\sum_{i=1}^n (y_i - p(\boldsymbol{\beta}|\boldsymbol{x}_i))\boldsymbol{x}_i\boldsymbol{x}_i'$. Assume that $n^{-1}\sum_{i=1}^n \|\boldsymbol{x}_i\|^6 = O(1)$ and $\boldsymbol{L}_n$ converges to a positive definite matrix as $n \to \infty$. Then we have*

(a) $P\Big(\|\hat{\boldsymbol{L}}_n - \boldsymbol{L}_n\| \leq C_{100}(p)n^{-1/2}(\log n)^{1/2}\Big) = 1 - o\big(n^{-1/2}\big)$,

(b) $P\Big(\|\hat{\boldsymbol{M}}_n - \boldsymbol{M}_n\| = o\big(n^{-1/2}\big)\Big) = 1 - o\big(n^{-1/2}\big)$ *and*

(c) $P\Big(\|\boldsymbol{M}_n - \boldsymbol{L}_n\| \leq C_{101}(p)n^{-1/2}(\log n)^{1/2}\Big) = 1 - o\big(n^{-1/2}\big)$.

Proof of Lemma 2.12. Let $\hat{L}_{jkn}$, L_{jkn}, $\hat{M}_{jkn}$ and M_{jkn} are (j,k)th element of $\hat{\boldsymbol{L}}_n$, $\boldsymbol{L}_n$, $\hat{\boldsymbol{M}}_n$ and $\boldsymbol{M}_n$ respectively. Let us first prove part (a). Note that it is enough to show that $P\Big(|\hat{L}_{jkn} - L_{jkn}| \leq C_{110}(p)n^{-1/2}(\log n)^{1/2}\Big) = 1 - o\big(n^{-1/2}\big)$. Now by Taylor's theorem we have

$$\hat{L}_{jkn} = n^{-1}\sum_{i=1}^n x_{ij}x_{ik}e^{\boldsymbol{x}_i'\hat{\boldsymbol{\beta}}_n}\big(1+e^{\boldsymbol{x}_i'\hat{\boldsymbol{\beta}}_n}\big)^{-2} = L_{jkn} + n^{-1}\sum_{i=1}^n x_{ij}x_{ik}\big[\boldsymbol{x}_i'(\hat{\boldsymbol{\beta}}_n - \boldsymbol{\beta})\big]\frac{e^{z_i}(1-e^{z_i})}{(1+e^{z_i})},$$

where $|z_i - \boldsymbol{x}_i'\boldsymbol{\beta}| \leq |\boldsymbol{x}_i'(\hat{\boldsymbol{\beta}}_n - \boldsymbol{\beta})|$ for all $i \in \{1,\ldots,n\}$. Now part (a) follows by applying part (a) of Theorem 1. Now let us consider part (c). It is enough to show that $P\Big(|M_{jkn} - L_{jkn}| \leq C_{111}(p)n^{-1/2}(\log n)^{1/2}\Big) = 1 - o\big(n^{-1/2}\big)$. Now since $EM_{jkn} = L_{kjn}$ for any $j,k \in \{1,\ldots,p\}$, Lemma 2.1 with $t=3$ implies $P\Big(|M_{jkn} - L_{jkn}| \leq C_{111}(p)n^{-1/2}(\log n)^{1/2}\Big) = 1 - o\big(n^{-1/2}\big)$ and hence we are done. Now let us consider part (b). Here also enough to show that $P\Big(|\hat{M}_{jkn} - M_{jkn}| = o\big(n^{-1/2}\big)\Big) = 1 - o\big(n^{-1/2}\big)$. Now note that

$$\hat{M}_{jkn} - M_{jkn} = n^{-1}\sum_{i=1}^n (\hat{p}(\boldsymbol{x}_i) - p(\boldsymbol{\beta}|\boldsymbol{x}_i))^2 x_{ij}x_{ik} + 2n^{-1}\sum_{i=1}^n (\hat{p}(\boldsymbol{x}_i) - p(\boldsymbol{\beta}|\boldsymbol{x}_i))(y_i - p(\boldsymbol{\beta}|\boldsymbol{x}_i))x_{ij}x_{ik},$$

where by Taylor's theorem we have

$$\hat{p}(\boldsymbol{x}_i) - p(\boldsymbol{\beta}|\boldsymbol{x}_i) = x_{ij}x_{ik}e^{\boldsymbol{x}_i'\boldsymbol{\beta}}\big(1+e^{\boldsymbol{x}_i'\boldsymbol{\beta}}\big)^{-2}\big[\boldsymbol{x}_i'(\hat{\boldsymbol{\beta}}_n - \boldsymbol{\beta})\big] + o\Big(\|\hat{\boldsymbol{\beta}}_n - \boldsymbol{\beta}\|^2\Big).$$

Therefore, using part (a) of Theorem 1 we have $P\Big(\Big|n^{-1}\sum_{i=1}^n (\hat{p}(\boldsymbol{x}_i) - p(\boldsymbol{\beta}|\boldsymbol{x}_i))^2 x_{ij}x_{ik}\Big| = o\big(n^{-1/2}\big)\Big) = 1 - o\big(n^{-1/2}\big)$. Again Lemma 2.1 with $t=2$ implies that $P\Big(\Big\|n^{-1}\sum_{i=1}^n x_{ij}x_{ik}e^{\boldsymbol{x}_i'\boldsymbol{\beta}}\big(1+e^{\boldsymbol{x}_i'\boldsymbol{\beta}}\big)^{-2}(y_i - p(\boldsymbol{\beta}|\boldsymbol{x}_i))\boldsymbol{x}_i\Big\| = o\big((\log n)^{-1}\big)\Big) = 1 - o\big(n^{-1/2}\big)$, which together with part (a) of Theorem 1 implies that $P\Big(\Big|n^{-1}\sum_{i=1}^n (\hat{p}(\boldsymbol{x}_i) - p(\boldsymbol{\beta}|\boldsymbol{x}_i))(y_i - p(\boldsymbol{\beta}|\boldsymbol{x}_i))x_{ij}x_{ik}\Big| = o\big(n^{-1/2}\big)\Big) = 1 - o\big(n^{-1/2}\big)$. Therefor we are done.

Lemma 2.13. *Assume that $n^{-1}\sum_{i=1}^n \|\boldsymbol{x}_i\|^6 = O(1)$ and $\boldsymbol{L}_n$ converges to a positive definite matrix as $n \to \infty$. Then we have Then we have*

(a) $P_*\Big(\|\boldsymbol{L}_n^* - \hat{\boldsymbol{L}}_n\| \leq C_{102}(p)n^{-1/2}(\log n)^{1/2}\Big) = 1 - o_p\big(n^{-1/2}\big)$ *and*

(b) $P\Big(\|\hat{\boldsymbol{M}}_n^* - \hat{\boldsymbol{M}}_n\| \le C_{103}(p) n^{-1/2}(\log n)^{1/2}\Big) = 1 - o_p\big(n^{-1/2}\big)$.

Proof of Lemma 2.13. Proofs are similar to the proofs of part (a) and (c) of Lemma 2.12.

3. Proof of Theorem 3.2

Recall that here $p = 1$ and hence $q = 1$. Define, $\boldsymbol{B}_n = \sqrt{n}H(\boldsymbol{E}_n \times \mathcal{R})$ with $\boldsymbol{E}_n = (-\infty, z_n]$ and $z_n = \Big(\frac{3}{4n} - \mu_n\Big)$. Here $\mu_n = n^{-1}\sum_{i=1}^n x_i p(\beta|x_i)$. Note that B_n is an interval, as is pointed out in Section 3 just after the description of Theorem 3.2 in the main manuscript. The function $H(\cdot)$ is defined in (7.13) in the proof of Theorem 3.3 which is presented in the main manuscript. We are going to show that there exists a positive constant M_2 such that

$$\lim_{n\to\infty} \mathbf{P}\Big(\sqrt{n}\Big|\mathbf{P}_*(\mathbf{H}_n^* \in \boldsymbol{B}_n) - \mathbf{P}(\mathbf{H}_n \in \boldsymbol{B}_n)\Big| \ge M_2\Big) = 1.$$

Define the set $\boldsymbol{Q} = \Big\{\big|\hat{\boldsymbol{\beta}}_n - \boldsymbol{\beta}\big| = o\big(n^{-1/2}(\log n)\big)\Big\} \cap \Big\{\big|n^{-1}\sum_{i=1}^n \big[(y_i - p(\beta|x_i))^2 - \mathbf{E}(y_i - p(\beta|x_i))^2\big]x_i^2\big| = o\big(n^{-1/2}(\log n)\big)\Big\} \cap \Big\{\big|n^{-1}\sum_{i=1}^n \big[(y_i - p(\beta|x_i))^3 - \mathbf{E}(y_i - p(\beta|x_i))^3\big]x_i^3\big| = o(1)\Big\}$. Now it is easy to see that $\mathbf{P}\Big(\big|\hat{\boldsymbol{\beta}}_n - \boldsymbol{\beta}\big| = o\big(n^{-1/2}(\log n)\big)\Big) = 1$ for all but finitely many n, upon application of Borel-Cantelli lemma and noting that $\max\{|x_i| : i \in \{1,\dots,n\}\} = O(1)$. Again by applying Lemma 2.1, it is easy to show that $\mathbf{P}\Big(\Big\{\big|n^{-1}\sum_{i=1}^n \big[(y_i - p(\beta|x_i))^2 - \mathbf{E}(y_i - p(\beta|x_i))^2\big]\big| = o\big(n^{-1/2}(\log n)\big)\Big\} \cap \Big\{\big|n^{-1}\sum_{i=1}^n \big[(y_i - p(\beta|x_i))^3 - \mathbf{E}(y_i - p(\beta|x_i))^3\big]\big| = o(1)\Big\}\Big) = 1$ which essentially implies $\mathbf{P}(\boldsymbol{Q}) = 1$. Similarly define the Bootstrap version of the set $\boldsymbol{Q}$ as $\boldsymbol{Q}^* = \Big\{\big|\hat{\boldsymbol{\beta}}_n^* - \hat{\boldsymbol{\beta}}_n\big| = o\big(n^{-1/2}(\log n)\big)\Big\} \cap \Big\{\big|n^{-1}\sum_{i=1}^n \big[(y_i - \hat{p}(x_i))^2(\mu_{G^*}^{-2}(G_i^* - \mu_{G^*})^2 - 1)\big]x_i^2\big| = o\big(n^{-1/2}(\log n)\big)\Big\} \cap \Big\{\big|n^{-1}\sum_{i=1}^n \big[(y_i - \hat{p}(x_i))^3(\mu_{G^*}^{-3}(G_i^* - \mu_{G^*})^3 - 1)\big]x_i^3\big| = o(1)\Big\}$. Through the same line, it is easy to establish that $\mathbf{P}\Big(\mathbf{P}^*(\boldsymbol{Q}^*) = 1\Big) = 1$. Hence enough to show

$$\lim_{n\to\infty} \mathbf{P}\Big(\Big\{\sqrt{n}\Big|\mathbf{P}_*(\{\mathbf{H}_n^* \in \boldsymbol{B}_n\} \cap \boldsymbol{Q}^*) - \mathbf{P}(\{\mathbf{H}_n \in \boldsymbol{B}_n\} \cap \boldsymbol{Q})\Big| \ge M_2\Big\} \cap \boldsymbol{Q}\Big) = 1. \tag{3.1}$$

Recall the definitions of $\bar{W}_n$ and $\bar{W}_n^*$ from the proof of Theorem 3.3. Similar to equations (7.13) and (7.14) in the proof of Theorem 3.3, presented in the main manuscript, it is easy to observe that

$$\mathbf{H}_n = \sqrt{n}H(\bar{W}_n) + R_n \text{ and } \mathbf{H}_n^* = \sqrt{n}\hat{H}(\bar{W}_n^*) + R_n^*, \tag{3.2}$$

where $\{|R_n| = O(n^{-1/2}(\log n)^{-1})\} \subseteq \boldsymbol{Q}$ and $\{|R_n^*| = O(n^{-1/2}(\log n)^{-1})\} \subseteq \boldsymbol{Q}^*$. To prove (3.1), first we are going to show for large enough n,

$$\begin{aligned}&\Big\{\Big\{\sqrt{n}\Big|\mathbf{P}_*(\{\mathbf{H}_n^* \in \boldsymbol{B}_n\} \cap \boldsymbol{Q}^*) - \mathbf{P}(\{\mathbf{H}_n \in \boldsymbol{B}_n\} \cap \boldsymbol{Q})\Big| \ge M_2\Big\} \cap \boldsymbol{Q}\Big\}\\ &\supseteq \Big\{\Big\{\sqrt{n}\Big|\mathbf{P}_*(\{\sqrt{n}\hat{H}(\bar{W}_n^*) \in \boldsymbol{B}_n\} \cap \boldsymbol{Q}^*) - \mathbf{P}(\{\sqrt{n}H(\bar{W}_n) \in \boldsymbol{B}_n\} \cap \boldsymbol{Q})\Big| \ge 2M_2\Big\} \cap \boldsymbol{Q}\Big\}.\end{aligned} \tag{3.3}$$

Now due to (3.2), we have

$$\Big|\mathbf{P}\Big(\mathbf{H}_n \in \boldsymbol{B}_n\Big) - \mathbf{P}\Big(\sqrt{n}H(\bar{W}_n) \in \boldsymbol{B}_n\Big)\Big| \leq \mathbf{P}\Big(\sqrt{n}H(\bar{W}_n) \in \big(\partial \boldsymbol{B}_n\big)^{(n\log n)^{-1/2}}\Big) + \mathbf{P}\Big(|R_n| \neq o(n^{-1/2}(\log n)^{-1})\Big)$$

$$\Big|\mathbf{P}_*\Big(\mathbf{H}_n^* \in \boldsymbol{B}_n\Big) - \mathbf{P}_*\Big(\sqrt{n}\hat{H}(\bar{W}_n^*) \in \boldsymbol{B}_n\Big)\Big| \leq \mathbf{P}_*\Big(\sqrt{n}\hat{H}(\bar{W}_n^*) \in \big(\partial \boldsymbol{B}_n\big)^{(n\log n)^{-1/2}}\Big) + \mathbf{P}_*\Big(|R_n^*| \neq o(n^{-1/2}(\log n)^{-1})\Big)$$

To establish (3.3), enough to show $\mathbf{P}\Big(\sqrt{n}\hat{H}(\bar{W}_n) \in \big(\partial \boldsymbol{B}_n\big)^{(n\log n)^{-1/2}}\Big) = o\big(n^{-1/2}\big)$ and $\mathbf{P}\Big(\Big\{\mathbf{P}_* \Big(\sqrt{n}\hat{H}(\bar{W}_n^*) \in \big(\partial \boldsymbol{B}_n\big)^{(n\log n)^{-1/2}}\Big) = o\big(n^{-1/2}\big)\Big\} \cap \boldsymbol{Q}\Big) = 1$ for large enough n. An Edgeworth expansion of $\sqrt{n}\bar{W}_n^*$ with an error $o(n^{-1/2})$ (in almost sure sense) can be established using Lemma 2.11. Then we can use transformation technique of Bhattacharya and Ghosh (1978) to find an Edgeworth expansion $\hat{\eta}_n(\cdot)$ of the density of $\sqrt{n}\hat{H}(\bar{W}_n^*)$ with an error $o(n^{-1/2})$ (in almost sure sense). Now the calculations similar to page 213 of Bhattacharya and Rao (1986) will imply that $\mathbf{P}\Big(\Big\{\mathbf{P}_*\Big(\sqrt{n}\hat{H}(\bar{W}_n^*) \in \big(\partial \boldsymbol{B}_n\big)^{(n\log n)^{-1/2}}\Big) = o\big(n^{-1/2}\big)\Big\} \cap \boldsymbol{Q}\Big) = 1$, since $\boldsymbol{B}_n$ is an interval. Next we are going to show that $\mathbf{P}\Big(\sqrt{n}\hat{H}(\bar{W}_n) \in \big(\partial \boldsymbol{B}_n\big)^{(n\log n)^{-1/2}}\Big) = 0$ for large enough n and to show that we need to utilize the form of $\boldsymbol{B}_n$, as Edgeworth expansion of $\sqrt{n}H(\bar{W}_n)$ similar to $\sqrt{n}\hat{H}(\bar{W}_n^*)$ does not exist due to the lattice nature of $W_1,\ldots,W_n$. To this end define $k_n(\boldsymbol{x}) = \big(\sqrt{n}H(\boldsymbol{x}/\sqrt{n}), x_2\big)'$ where $\boldsymbol{x} = (x_1, x_2)'$. Note that $k_n(\cdot)$ is a diffeomorphism (cf. proof of Lemma 3.2 in Lahiri (1989)). Hence $k_n(\cdot)$ is a bijection and $k_n(\cdot)$ & $k_n^{-1}(\cdot)$ have derivatives of all orders. Therefore, arguments given between (2.15) and (2.18) at page 444 of Bhattacharya and Ghosh (1978) with g_n there replaced by $k_n^{-1}(\cdot)$ will imply that

$$\begin{aligned}\Big|\mathbf{P}\Big(\mathbf{H}_n \in \boldsymbol{B}_n\Big) - \mathbf{P}\Big(\sqrt{n}H(\bar{W}_n) \in \boldsymbol{B}_n\Big)\Big| &\leq \mathbf{P}\Big(\big(\sqrt{n}\bar{W}_n \in \big(\partial k_n^{-1}(\boldsymbol{B}_n \times \mathcal{R})\big)^{d_n(n\log n)^{-1/2}}\Big) + o\big(n^{-1/2}\big)\\ &= \mathbf{P}\Big(\sqrt{n}\bar{W}_{n1} \in \big(\partial \boldsymbol{E}_n'\big)^{d_n(n\log n)^{-1/2}}\Big) + o\big(n^{-1/2}\big),\end{aligned}$$

where $\boldsymbol{E}_n' = \sqrt{n}\boldsymbol{E}_n$ and $d_n \leq \max\Big\{|det\big(Grad\big[k_n(x)\big]\big)|^{-1} : |x| = O\big(\sqrt{\log n}\big)\Big\}$. Now by looking into the form of $H(\cdot)$ in (7.13) in the proof of Theorem 3.3, presented in the main manuscript, it is easy to see that $d_n = O(1)$, say $d_n \leq C_{44}$ for some positive constant C_{44}. Now note that

$$\begin{aligned}&\mathbf{P}\Big(\sqrt{n}\bar{W}_{n1} \in \big(\partial \boldsymbol{E}_n'\big)^{C_{44}(n\log n)^{-1/2}}\Big)\\ &= \mathbf{P}\Big(\Big[n^{-1/2}\sum_{i=1}^{n} y_i x_i - \sqrt{n}\mu_n\Big] \in \big(\sqrt{n}z_n - C_{44}(n\log n)^{-1/2}, \sqrt{n}z_n + C_{44}(n\log n)^{-1/2}\big)\Big)\\ &= \mathbf{P}\Big(\sum_{i=1}^{n} y_i x_i \in \big(3/4 - C_{44}(\log n)^{-1/2}, 3/4 + C_{44}(\log n)^{-1/2}\big)\Big) = 0,\end{aligned}$$

for large enough n, since $\sum_{i=1}^{n} y_i x_i$ can take only integer values. Therefore (3.3) is established. Now recalling that $\hat{\eta}_n(\cdot)$ is the Edgeworth expansion of the density of $\sqrt{n}\hat{H}(\bar{W}_n^*)$ with an almost sure error

$o(n^{-1/2})$, we have for large enough n,

$$\mathbf{P}\Big(\sqrt{n}\Big|\mathbf{P}_*\Big(\sqrt{n}\hat{H}(\bar{W}_n^*) \in \boldsymbol{B}_n\Big) - \int_{\boldsymbol{B}_n} \hat{\eta}_n(x)dx\Big| = o(1)\Big) = 1. \tag{3.4}$$

Now define $U_i = \Big(\big(y_i - p(\beta|x_i)\big)x_iV_i, \big(y_i - p(\beta|x_i)\big)^2 x_i^2\big[V_i^2 - 1\big]\Big)'$, $i \in \{1,\dots,n\}$, where $V_1,\dots,V_n$ are iid continuous random variables which are independent of $\{y_1,\dots,y_n\}$. Also $\mathbf{E}(V_1) = 0$, $\mathbf{E}(V_1^2) = \mathbf{E}(V_1)^3 = 1$ and $\mathbf{E}V_1^8 < \infty$. An immediate choice of the distribution of V_1 is that of $(G_1^* - \mu_{G^*})\mu_{G^*}^{-1}$. Other choices of $\{V_1,\dots,V_n\}$ can be found in Liu (1988), Mammen (1993) and Das et al. (2019). Now since $\max\{|x_i| : i \in \{1,\dots,n\}\} = O(1)$, there exists a natural number n_0 and constants $0 < \delta_2 \le \delta_1 < 1$ such that $\sup_{n\ge n_0} p(\beta|x_n) \le \delta_1$ and $\inf_{n\ge n_0} p(\beta|x_n) \ge \delta_2$. Again $V_1,\dots,V_n$ are iid continuous random variables. Hence writing $p_n = p(\beta|x_n)$, for any $b > 0$ we have

$$\sup_{n\ge n_0}\sup_{\|\boldsymbol{t}\|>b}\Big|\mathbf{E}e^{i\boldsymbol{t}'U_n}\Big| \le \sup_{n\ge n_0}\Big[p_n \sup_{\|\boldsymbol{t}\|>b(1-\delta_1)^2}\Big|\mathbf{E}e^{it_1(1-p_n)V_1 + it_2(-p_n)^2[V_1^2-1]}\Big|$$
$$+ (1-p_n)\sup_{\|\boldsymbol{t}\|>b\delta_2^2}\Big|\mathbf{E}e^{it_1(-p_n)V_1+it_2(-p_n)^2[V_1^2-1]}\Big|\Big] < 1,$$

i.e. uniform Cramer's condition holds. Also the minimum eigen value condition of Theorem 20.6 of Bhattacharya and Rao (1986) holds due to $\max\{|x_i| : i \in \{1,\dots,n\}\} = O(1)$ and $\liminf_{n\to\infty} n^{-1}\sum_{i=1}^n x_i^6 > 0$. Hence applying Theorem 20.6 of Bhattacharya and Rao (1986) and then applying transformation technique of Bhattacharya and Ghosh (1978) we have

$$\Big|\mathbf{P}\Big(\sqrt{n}H(\bar{U}_n) \in \boldsymbol{B}_n\Big) - \int_{\boldsymbol{B}_n} \eta_n(x)dx\Big| = o\big(n^{-1/2}\big), \tag{3.5}$$

where $\bar{U}_n = n^{-1}\sum_{i=1}^n U_i$. Note that in both the expansions $\eta_n(\cdot)$ and $\hat{\eta}_n(\cdot)$ the variances corresponding to normal terms are 1. Also $\hat{H}(\cdot)$ can be obtained from $H(\cdot)$ first replacing $\boldsymbol{L}_n$ by $\hat{M}_n$ and then β by $\hat{\beta}_n$ (cf. equations (7.13) and (7.14) in the proof of Theorem 3.3 which is presented in the main manuscript). Hence we can conclude that for any Borel set $\boldsymbol{C}$,

$$\mathbf{P}\Big(\Big\{\sqrt{n}\Big|\int_{\boldsymbol{C}} \eta_n(x)dx - \int_{\boldsymbol{C}} \hat{\eta}_n(x)dx\Big| = o(1)\Big\} \cap \boldsymbol{Q}\Big) = 1$$

Hence from (3.4) and (3.5), we have

$$\mathbf{P}\Big(\Big\{\sqrt{n}\Big|\mathbf{P}_*\Big(\sqrt{n}\hat{H}(\bar{W}_n^*) \in \boldsymbol{B}_n\Big) - \mathbf{P}\Big(\sqrt{n}H(\bar{U}_n) \in \boldsymbol{B}_n\Big)\Big| = o(1)\Big\} \cap \boldsymbol{Q}\Big) = 1, \tag{3.6}$$

for large enough n. To establish (3.1), in view of (3.3) and (3.6) it is enough to find a positive constant M_3 such that

$$\sqrt{n}\Big|\mathbf{P}\Big(\sqrt{n}H(\bar{W}_n) \in \boldsymbol{B}_n\Big) - \mathbf{P}\Big(\sqrt{n}H(\bar{U}_n) \in \boldsymbol{B}_n\Big)\Big| = \sqrt{n}\Big|\mathbf{P}\Big(\sqrt{n}\bar{W}_{n1} \in \boldsymbol{E}_n\Big) - \mathbf{P}\Big(\sqrt{n}\bar{U}_{n1} \in \boldsymbol{E}_n\Big)\Big| \ge 4M_3.$$

Note that since $\mathbf{E}V_i^2 = \mathbf{E}V_i^3 = 1$ for all $i \in \{1,\dots,n\}$, the first three average moments of $\{W_{11},\dots,W_{n1}\}$ are same as that of $\{U_{11},\dots,U_{n1}\}$. However $\{W_{11},\dots,W_{n1}\}$ are independent lattice random variables whereas $\{U_{11},\dots,U_{n1}\}$ are independent random variables for which uniform Cramer's condition holds. Therefore by Lemma 2.10 and Theorem 20.6 of Bhattacharya and Rao (1986) we have

$$\sup_{x\in\mathcal{R}}\Big|\mathbf{P}\Big(\sqrt{n}\bar{W}_{n1}\le x\Big)-\Phi_{\sigma_n^2}(x)-n^{-1/2}P_1\big(-\Phi_{\sigma_n^2}:\{\bar{\chi}_{\nu,n}\}\big)(x)$$
$$+n^{-1/2}\big(n\mu_n+\sqrt{n}x-[n\mu_n+\sqrt{n}x]-1/2\big)\frac{d}{dx}\Phi_{\sigma_n^2}(x)\Big|=o\big(n^{-1/2}\big)$$
$$\text{and}\quad\sup_{x\in\mathcal{R}}\Big|\mathbf{P}\Big(\sqrt{n}\bar{U}_{n1}\le x\Big)-\Phi_{\sigma_n^2}(x)-n^{-1/2}P_1\big(-\Phi_{\sigma_n^2}:\{\bar{\chi}_{\nu,n}\}\big)(x)\Big|=o\big(n^{-1/2}\big),\tag{3.7}$$

where $P_1\big(-\Phi_{\sigma_n^2}:\{\bar{\chi}_{\nu,n}\}\big)(x)$ is as defined in Lemma 2.10 and $[x]$ is the greatest integer $\le x$. Recall that $\boldsymbol{E}_n=(-\infty,z_n]$ where $z_n=\Big(\dfrac{3}{4n}-\mu_n\Big)$. Therefore for some positive constants C_{46},C_{47},C_{48} we have

$$\sqrt{n}\Big|\mathbf{P}\Big(\sqrt{n}\bar{W}_{n1}\in\boldsymbol{E}_n\Big)-\mathbf{P}\Big(\sqrt{n}\bar{U}_{n1}\in\boldsymbol{E}_n\Big)\Big|=\sqrt{n}\Big|\mathbf{P}\Big(\sqrt{n}\bar{W}_{n1}\le\sqrt{n}z_n\Big)-\mathbf{P}\Big(\sqrt{n}\bar{U}_{n1}\le\sqrt{n}z_n\Big)\Big|$$
$$\ge\big(n\mu_n+nz_n-1/2\big)\big(\sqrt{2\pi}\sigma_n\big)^{-1}e^{-(nz_n^2)/(2\sigma_n^2)}-o(1)=\big(4\sqrt{2\pi}\sigma_n\big)^{-1}e^{-(nz_n^2)/(2\sigma_n^2)}-o(1)$$
$$\ge C_{46}\exp\Big\{-C_{47}n^{-1}\Big(\frac{9}{16}+n^2\mu_n^2-\frac{3n\mu_n}{2}\Big)\Big\}-o(1)\ge C_{48}\exp\Big\{-C_{47}M_1^2\Big\}.$$

The first inequality follows due to (3.7). Second one is due to $\max\{|x_i|:i\in\{1,\dots,n\}\}=O(1)$ and the last one is due to the assumption $\sqrt{n}|\mu_n|<M_1$. Taking $4M_2=C_{48}\exp\Big\{-C_{47}M_1^2\Big\}$, the proof of Theorem 3.2 is now complete.

4. Proof of Theorem 4.1

Note that the matrix $\boldsymbol{L}_n$ converges to some positive definite matrix as $x\to\infty$. Hence by Taylor's theorem and using part (a) of Theorem 3.1 & equation (7.7) in the proof of Theorem 3.3 which, presented in the main manuscript, we have

$$\sqrt{n}\big(f(\hat{\boldsymbol{\beta}}_n)-f(\boldsymbol{\beta})\big)=\big(f'(\beta)\big)'\boldsymbol{L}_n^{-1}\Big[\Lambda_n-\frac{\xi_n}{2}\Big]+n^{-1/2}\Lambda_n'\boldsymbol{L}_n^{-1}f''(\boldsymbol{\beta})\boldsymbol{L}_n^{-1}\Lambda_n+R_{7n},\tag{4.1}$$

where $\mathbf{P}\Big(|R_{7n}|\le C_{49}(p)n^{-1}(\log n)^2\big)\Big)=1-o\big(n^{-1/2}\big)$. Here $\Lambda_n=n^{-1/2}\sum_{i=1}^n(y_i-p(\boldsymbol{\beta}|\boldsymbol{x}_i))\boldsymbol{x}_i$ and $\xi_n=n^{-3/2}\sum_{i=1}^n\boldsymbol{x}_ie^{\boldsymbol{x}_i'\beta}\big(1-e^{\boldsymbol{x}_i'\beta}\big)\big(1+e^{\boldsymbol{x}_i'\beta}\big)^{-3}\Big[\boldsymbol{x}_i'\big(\boldsymbol{L}_n^{-1}\Lambda_n\big)\Big]^2$ which are as defined before. Now due to part (a) of Lemma 2.12 we have

$$\hat{\boldsymbol{L}}_n^{-1}-\boldsymbol{L}_n^{-1}=\boldsymbol{L}_n^{-1}\big(\boldsymbol{L}_n-\hat{\boldsymbol{L}}_n\big)\hat{\boldsymbol{L}}_n^{-1}=\boldsymbol{L}_n^{-1}\big(\boldsymbol{L}_n-\hat{\boldsymbol{L}}_n\big)\boldsymbol{L}_n^{-1}+R_{8n}$$

and due to part (a) & (b) of Lemma 2.12 we have

$$\hat{\boldsymbol{L}}_n^{-1}\hat{\boldsymbol{M}}_n\hat{\boldsymbol{L}}_n^{-1}=\boldsymbol{L}_n^{-1}+\boldsymbol{L}_n^{-1}\big(\boldsymbol{M}_n-\boldsymbol{L}_n\big)\boldsymbol{L}_n^{-1}+\boldsymbol{L}_n^{-1}\boldsymbol{M}_n\big(\hat{\boldsymbol{L}}_n^{-1}-\boldsymbol{L}_n^{-1}\big)+\big(\hat{\boldsymbol{L}}_n^{-1}-\boldsymbol{L}_n^{-1}\big)\boldsymbol{M}_n\boldsymbol{L}_n^{-1}+R_{9n},$$

where $\mathbf{P}\Big(\|R_{8n}\|+\|R_{9n}\|=o\big(n^{-1/2}\big)\Big)=1-o\big(n^{-1/2}\big)$. The above two equations together with part (c) of Lemma 2.12 imply that

$$\hat{\boldsymbol{L}}_n^{-1}\hat{\boldsymbol{M}}_n\hat{\boldsymbol{L}}_n^{-1}=\boldsymbol{L}_n^{-1}+\boldsymbol{L}_n^{-1}\big(\boldsymbol{M}_n-\boldsymbol{L}_n\big)\boldsymbol{L}_n^{-1}-2\boldsymbol{L}_n^{-1}\big(\hat{\boldsymbol{L}}_n-\boldsymbol{L}_n\big)\boldsymbol{L}_n^{-1}+R_{10n},$$

with $\mathbf{P}\Big(\|R_{10n}\| = o\big(n^{-1/2}\Big) = 1 - o\big(n^{-1/2}\big)$. Therefore we have

$$\begin{aligned} s_n^2 &= (f'(\hat{\boldsymbol{\beta}}_n))'\hat{\boldsymbol{L}}_n^{-1}\hat{\boldsymbol{M}}_n\hat{\boldsymbol{L}}_n^{-1}(f'(\hat{\boldsymbol{\beta}}_n)) = (f'(\boldsymbol{\beta}))'\boldsymbol{L}_n^{-1}(f'(\boldsymbol{\beta})) + 2n^{-1/2}\Lambda_n'\boldsymbol{L}_n^{-1}f''(\boldsymbol{\beta})\boldsymbol{L}_n^{-1}(f'(\boldsymbol{\beta})) \\ &\quad + (f'(\boldsymbol{\beta}))'\Big(\boldsymbol{L}_n^{-1}(\boldsymbol{M}_n - \boldsymbol{L}_n)\boldsymbol{L}_n^{-1} - 2\boldsymbol{L}_n^{-1}(\hat{\boldsymbol{L}}_n - \boldsymbol{L}_n)\boldsymbol{L}_n^{-1}\Big)(f'(\boldsymbol{\beta})) + R_{11n}, \end{aligned} \tag{4.2}$$

where $\mathbf{P}\Big(|R_{11n}| \leq C_{52}(p)n^{-1}(\log n)^2)\Big) = 1 - o\big(n^{-1/2}\big)$. Hence from equation (4.2) Lemma 2.12 implies

$$\begin{aligned} s_n^{-1} &= \Big[(f'(\boldsymbol{\beta}))'\boldsymbol{L}_n^{-1}(f'(\boldsymbol{\beta}))\Big]^{-1/2} - \frac{1}{2}\Big[(f'(\boldsymbol{\beta}))'\boldsymbol{L}_n^{-1}(f'(\boldsymbol{\beta}))\Big]^{-3/2}\Big[2n^{-1/2}\Lambda_n'\boldsymbol{L}_n^{-1}f''(\boldsymbol{\beta})\boldsymbol{L}_n^{-1}(f'(\boldsymbol{\beta})) \\ &\quad + (f'(\boldsymbol{\beta}))'\Big(\boldsymbol{L}_n^{-1}(\boldsymbol{M}_n - \boldsymbol{L}_n)\boldsymbol{L}_n^{-1} - 2\boldsymbol{L}_n^{-1}(\hat{\boldsymbol{L}}_n - \boldsymbol{L}_n)\boldsymbol{L}_n^{-1}\Big)(f'(\boldsymbol{\beta}))\Big] + R_{12n}, \end{aligned} \tag{4.3}$$

where $\mathbf{P}\Big(|R_{12n}| = o\big(n^{-1/2}\big)\Big) = 1 - o\big(n^{-1/2}\big)$. Now noting that $b_n = O\big(n^{-d}\big)$ for some $d > 0$, from equations (4.1) and (4.3) we have

$$\begin{aligned} \boldsymbol{T}_n &= s_n^{-1}\Big[\sqrt{n}\Big(f(\hat{\boldsymbol{\beta}}_n) - f(\boldsymbol{\beta})\Big) + b_n(f'(\hat{\boldsymbol{\beta}}_n))'\hat{\boldsymbol{L}}_n^{-1}Z\Big] \\ &= \Big[(f'(\boldsymbol{\beta}))'\boldsymbol{L}_n^{-1}(f'(\boldsymbol{\beta}))\Big]^{-1/2}(f'(\beta))'\boldsymbol{L}_n^{-1}\Big[\Lambda_n + b_nZ\Big] - \frac{1}{2}\Big[(f'(\boldsymbol{\beta}))'\boldsymbol{L}_n^{-1}(f'(\boldsymbol{\beta}))\Big]^{-1/2} \\ &\quad (f'(\beta))'\boldsymbol{L}_n^{-1}\xi_n - \frac{1}{2}\Big[(f'(\boldsymbol{\beta}))'\boldsymbol{L}_n^{-1}(f'(\boldsymbol{\beta}))\Big]^{-3/2}\Big[2n^{-1/2}\Lambda_n'\boldsymbol{L}_n^{-1}f''(\boldsymbol{\beta})\boldsymbol{L}_n^{-1}(f'(\boldsymbol{\beta})) \\ &\quad + (f'(\boldsymbol{\beta}))'\Big(\boldsymbol{L}_n^{-1}(\boldsymbol{M}_n - \boldsymbol{L}_n)\boldsymbol{L}_n^{-1} - 2\boldsymbol{L}_n^{-1}(\hat{\boldsymbol{L}}_n - \boldsymbol{L}_n)\boldsymbol{L}_n^{-1}\Big)(f'(\boldsymbol{\beta}))\Big] \\ &\quad (f'(\beta))'\boldsymbol{L}_n^{-1}\Lambda_n + R_{13n}, \end{aligned} \tag{4.4}$$

where $\mathbf{P}\Big(|R_{13n}| = o\big(n^{-1/2}\big)\Big) = 1 - o\big(n^{-1/2}\big)$. Recall that $W_i = \Big(Y_i\boldsymbol{x}_i', \big[Y_i^2 - \mathbf{E}Y_i^2\big]z_i'\Big)'$ where $Y_i = (y_i - p(\boldsymbol{\beta}|\boldsymbol{x}_i))$ and $z_i = (x_{i1}^2, x_{i1}x_{i2}, \ldots, x_{i1}x_{ip}, x_{i2}^2, x_{i2}x_{i3}, \ldots, x_{i2}x_{ip}, \ldots, x_{ip}^2)'$ with $\boldsymbol{x}_i = (x_{i1}, \ldots, x_{ip})'$, $i \in \{1, \ldots, n\}$. Using this notations, we have seen before in the proof of Theorem 3.3 that

$$\Lambda_n + b_nZ = \sqrt{n}(\bar{W}_{n1} + n^{-1/2}b_nZ) \;\text{ and }\; \xi_n = n^{-1/2}\Big[\sum_{i=1}^{n}\boldsymbol{x}_ie^{\boldsymbol{x}_i'\beta}\big(1 - e^{\boldsymbol{x}_i'\beta}\big)\big(1 + e^{\boldsymbol{x}_i'\beta}\big)^{-3}$$

$$\Big[\bar{W}_{n1}'\boldsymbol{L}_n^{-1}\boldsymbol{x}_i\boldsymbol{x}_i'\boldsymbol{L}_n^{-1}\bar{W}_{n1}\Big]^2\Big] = \sqrt{n}\Big(\bar{W}_{n1}'\tilde{\boldsymbol{M}}_1\bar{W}_{n1}, \ldots, \bar{W}_{n1}'\tilde{\boldsymbol{M}}_p\bar{W}_{n1}\Big)',$$

where $\tilde{\boldsymbol{M}}_k = n^{-1}\sum_{i=1}^{n}x_{ik}e^{\boldsymbol{x}_i'\beta}\big(1 - e^{\boldsymbol{x}_i'\beta}\big)\big(1 + e^{\boldsymbol{x}_i'\beta}\big)^{-3}\Big(\boldsymbol{L}_n^{-1}\boldsymbol{x}_i\boldsymbol{x}_i'\boldsymbol{L}_n^{-1}\Big)$ for $k \in \{1, \ldots, p\}$ and the jth row of $\big(\boldsymbol{M}_n - \boldsymbol{L}_n\big)$ is $\bar{W}_{n2}'\boldsymbol{E}_{jn}$ where $\boldsymbol{E}_{jn}$ is a matrix of order $q \times p$ with $\|\boldsymbol{E}_{jn}\| \leq q$, $j \in \{1, \ldots, p\}$. $\bar{W}_n = (\bar{W}_{n1}, W_{n2})'$ is the mean of W_i's. Therefore writing $\tilde{W}_{n1} = \bar{W}_{n1} + n^{-1/2}b_nZ$ and $\tilde{W}_{n2} = \bar{W}_{n2} + n^{-1/2}b_nZ_1$ with $Z_1 \sim N_q\big(\mathbf{0}, \boldsymbol{I}_q\big)$ being independent of Z & $\{y_1, \ldots, y_n\}$, from (4.4) we have

$$\boldsymbol{T}_n = \sqrt{n}\Big[\sigma_n\Big((f'(\boldsymbol{\beta}))'\boldsymbol{L}_n^{-1},\;\; \mathbf{0}'\Big)\tilde{W}_n + \tilde{W}_n'\boldsymbol{N}\tilde{W}_n\Big] + R_{13n}, \tag{4.5}$$

where $\sigma_n = \left[(f'(\boldsymbol{\beta}))' \boldsymbol{L}_n^{-1}(f'(\boldsymbol{\beta}))\right]^{-1/2}$, $\tilde{W}_n = (\tilde{W}_{n1}', \tilde{W}_{n2}')'$ and $\boldsymbol{N} = \begin{bmatrix} \sum_{i=1}^3 \boldsymbol{N}_i & \mathbf{0} \\ \boldsymbol{N}_4 & \mathbf{0} \end{bmatrix}$ with

$$\boldsymbol{N}_1 = -2^{-1}\sigma_n \sum_{k=1}^{p} \left((f'(\boldsymbol{\beta}))' \boldsymbol{L}_{\cdot kn}^{-1}\right)\tilde{\boldsymbol{M}}_k,\ \boldsymbol{N}_2 = -\sigma_n^3 \boldsymbol{L}_n^{-1}(f''(\boldsymbol{\beta}))\boldsymbol{L}_n^{-1}(f'(\boldsymbol{\beta}))(f'(\boldsymbol{\beta}))'\boldsymbol{L}_n^{-1},$$

$$\boldsymbol{N}_3 = \sigma_n^3\left(n^{-1}\sum_{i=1}^{n}\left((f'(\boldsymbol{\beta}))'\boldsymbol{L}_n^{-1}\boldsymbol{x}_i\right)\left(\boldsymbol{L}_n^{-1}\boldsymbol{x}_i\right)(1-e^{\boldsymbol{x}_i'\boldsymbol{\beta}})(1+e^{\boldsymbol{x}_i'\boldsymbol{\beta}})^{-3}\boldsymbol{x}_i'\right)\boldsymbol{L}_n^{-1}(f'(\boldsymbol{\beta}))(f'(\boldsymbol{\beta}))'\boldsymbol{L}_n^{-1},$$

$$\boldsymbol{N}_4 = -2^{-1}\sigma_n^3\left(\sum_{k=1}^{p}\left((f'(\boldsymbol{\beta}))'\boldsymbol{L}_{\cdot kn}^{-1}\right)E_{kn}\right)\boldsymbol{L}_n^{-1}(f'(\boldsymbol{\beta}))(f'(\boldsymbol{\beta}))'\boldsymbol{L}_n^{-1},$$

and $\boldsymbol{L}_{\cdot kn}^{-1}$ being the kth column of $\boldsymbol{L}_n^{-1}$.

Now let us look into the Bootstrap pivot $\boldsymbol{T}_n^*$. Similar to the original case it can be shown that

$$\sqrt{n}(f(\hat{\boldsymbol{\beta}}_n^*) - f(\hat{\boldsymbol{\beta}}_n)) = (f'(\hat{\boldsymbol{\beta}}_n))'\hat{\boldsymbol{L}}_n^{-1}\left[\hat{\Lambda}_n^* - \frac{\hat{\xi}_n^*}{2}\right] + n^{-1/2}\hat{\Lambda}_n^{*\prime}\hat{\boldsymbol{L}}_n^{-1}f''(\hat{\boldsymbol{\beta}}_n)\hat{\boldsymbol{L}}_n^{-1}\hat{\Lambda}_n^* + R_{7n}^*, \tag{4.6}$$

where $\hat{\Lambda}_n^*$ and $\hat{\xi}_n^*$ are same as Λ_n and ξ_n but after replacing $\boldsymbol{\beta}$ by $\hat{\boldsymbol{\beta}}_n$ and $\mathbf{P}_*\left(|R_{7n}^*| \le C_{53}(p)n^{-1}(\log n)^2\right) = 1 - o_p(n^{-1/2})$. Again it is easy to show that

$$\begin{aligned}\boldsymbol{L}_n^{*-1}\hat{\boldsymbol{M}}_n^*\boldsymbol{L}_n^{*-1} = \hat{\boldsymbol{L}}_n^{-1}\hat{\boldsymbol{M}}_n\hat{\boldsymbol{L}}_n^{-1} + \hat{\boldsymbol{L}}_n^{-1}(\hat{\boldsymbol{M}}_n^* - \hat{\boldsymbol{M}}_n)\hat{\boldsymbol{L}}_n^{-1} - \hat{\boldsymbol{L}}_n^{-1}\hat{\boldsymbol{M}}_n\hat{\boldsymbol{L}}_n^{-1}(\boldsymbol{L}_n^* - \hat{\boldsymbol{L}}_n)\hat{\boldsymbol{L}}_n^{-1} \\ - \hat{\boldsymbol{L}}_n^{-1}(\boldsymbol{L}_n^* - \hat{\boldsymbol{L}}_n)\hat{\boldsymbol{L}}_n^{-1}\hat{\boldsymbol{M}}_n\hat{\boldsymbol{L}}_n^{-1} + R_{10n}^*,\end{aligned} \tag{4.7}$$

where $\mathbf{P}_*\left(\|R_{10n}^*\| = o(n^{-1/2})\right) = 1 - o_p(n^{-1/2})$. Combining (4.6) and (4.7) we have

$$\begin{aligned}\boldsymbol{T}_n^* &= s_n^{*-1}\left[\sqrt{n}\left(f(\hat{\boldsymbol{\beta}}_n^*) - f(\hat{\boldsymbol{\beta}}_n)\right) + b_n(f'(\hat{\boldsymbol{\beta}}_n^*))'\boldsymbol{L}_n^{*-1}Z^*\right] \\ &= \left[(f'(\hat{\boldsymbol{\beta}}_n))'\hat{\boldsymbol{L}}_n^{-1}\hat{\boldsymbol{M}}_n\hat{\boldsymbol{L}}_n^{-1}(f'(\hat{\boldsymbol{\beta}}_n))\right]^{-1/2}(f'(\hat{\boldsymbol{\beta}}_n))'\hat{\boldsymbol{L}}_n^{-1}\left[\hat{\Lambda}_n^* + b_nZ^*\right] \\ &\quad - \frac{1}{2}\left[(f'(\hat{\boldsymbol{\beta}}_n))'\hat{\boldsymbol{L}}_n^{-1}\hat{\boldsymbol{M}}_n\hat{\boldsymbol{L}}_n^{-1}(f'(\hat{\boldsymbol{\beta}}_n))\right]^{-1/2}(f'(\boldsymbol{\beta}))'\hat{\boldsymbol{L}}_n^{-1}\hat{\xi}_n^* \\ &\quad - \frac{1}{2}\left[(f'(\hat{\boldsymbol{\beta}}_n))'\hat{\boldsymbol{L}}_n^{-1}\hat{\boldsymbol{M}}_n\hat{\boldsymbol{L}}_n^{-1}(f'(\hat{\boldsymbol{\beta}}_n))\right]^{-3/2}\Big[2n^{-1/2}\hat{\Lambda}_n^{*\prime}\hat{\boldsymbol{L}}_n^{-1}f''(\hat{\boldsymbol{\beta}}_n)\hat{\boldsymbol{L}}_n^{-1}\hat{\boldsymbol{M}}_n\hat{\boldsymbol{L}}_n^{-1}(f'(\hat{\boldsymbol{\beta}}_n)) \\ &\quad + (f'(\hat{\boldsymbol{\beta}}_n))'\Big(\hat{\boldsymbol{L}}_n^{-1}(\hat{\boldsymbol{M}}_n^* - \hat{\boldsymbol{M}}_n)\hat{\boldsymbol{L}}_n^{-1} - \hat{\boldsymbol{L}}_n^{-1}\hat{\boldsymbol{M}}_n\hat{\boldsymbol{L}}_n^{-1}(\boldsymbol{L}_n^* - \hat{\boldsymbol{L}}_n)\hat{\boldsymbol{L}}_n^{-1} \\ &\quad - \hat{\boldsymbol{L}}_n^{-1}(\boldsymbol{L}_n^* - \hat{\boldsymbol{L}}_n)\hat{\boldsymbol{L}}_n^{-1}\hat{\boldsymbol{M}}_n\hat{\boldsymbol{L}}_n^{-1}\Big)(f'(\hat{\boldsymbol{\beta}}_n))\Big](f'(\hat{\boldsymbol{\beta}}_n))'\hat{\boldsymbol{L}}_n^{-1}\hat{\Lambda}_n^* + R_{13n}^*,\end{aligned} \tag{4.8}$$

where $\mathbf{P}_*\left(|R_{13n}^*| = o(n^{-1/2})\right) = 1 - o_p(n^{-1/2})$. Now recall that $W_i^* = \left(\hat{Y}_i\left[(G_i^* - \mu_{G^*})\mu_{G^*}^{-1}\right]\boldsymbol{x}_i', \hat{Y}_i^2\left[\mu_{G^*}^{-2}(G_i^* - \mu_{G^*})^2 - 1\right]z_i'\right)'$ where $\hat{Y}_i = (y_i - \hat{p}(\boldsymbol{x}_i))$ and $z_i = (x_{i1}^2, x_{i1}x_{i2}, \ldots, x_{i1}x_{ip}, x_{i2}^2, x_{i2}x_{i3}, \ldots, x_{i2}x_{ip}, \ldots, x_{ip}^2)'$ with $\boldsymbol{x}_i = (x_{i1}, \ldots, x_{ip})'$, $i \in \{1, \ldots, n\}$. Therefore writing $\tilde{W}_{n1}^* = \bar{W}_{n1}^* + n^{-1/2}b_nZ^*$ and $\tilde{W}_{n2} = \bar{W}_{n2}^* + n^{-1/2}b_nZ_1^*$ with $(\bar{W}_{n1}^*, \bar{W}_{n2}^*)$ being the mean of W_i^* and $Z_1 \sim N_q(\mathbf{0}, \boldsymbol{I}_q)$ being independent of Z & $\{y_1,\dots,y_n\}$, Z^* & G_i^*'s, from (4.8) we have

$$\boldsymbol{T}_n^* = \sqrt{n}\Big[\hat{\sigma}_n\Big((f'(\hat{\boldsymbol{\beta}}_n))'\hat{\boldsymbol{L}}_n^{-1},\ \ \boldsymbol{0}'\Big)\tilde{W}_n^* + \tilde{W}_n^{*\prime}\boldsymbol{N}^*\tilde{W}_n^*\Big] + R_{13n}^*, \tag{4.9}$$

where $\hat{\sigma}_n^* = s_n^{-1} = \Big[(f'(\hat{\boldsymbol{\beta}}_n))'\hat{\boldsymbol{L}}_n^{-1}\hat{\boldsymbol{M}}_n\hat{\boldsymbol{L}}_n^{-1}(f'(\hat{\boldsymbol{\beta}}_n))\Big]^{-1/2}$, $\tilde{W}_n^* = (\tilde{W}_{n1}^{*\prime}, \tilde{W}_{n2}^{*\prime})'$ and $\boldsymbol{N}^* = \begin{bmatrix}\sum_{i=1}^{3}\boldsymbol{N}_i^* & \boldsymbol{0}\\ \boldsymbol{N}_4^* & \boldsymbol{0}\end{bmatrix}$ with

$$\begin{aligned}
\boldsymbol{N}_1^* &= -2^{-1}\hat{\sigma}_n\sum_{k=1}^{p}\Big((f'(\hat{\boldsymbol{\beta}}_n))'\hat{\boldsymbol{L}}_{\cdot kn}^{-1}\Big)\tilde{\boldsymbol{M}}_k^*\\
\boldsymbol{N}_2^* &= -\hat{\sigma}_n^3\hat{\boldsymbol{L}}_n^{-1}(f''(\hat{\boldsymbol{\beta}}_n))\hat{\boldsymbol{L}}_n^{-1}\hat{\boldsymbol{M}}_n\hat{\boldsymbol{L}}_n^{-1}(f'(\hat{\boldsymbol{\beta}}_n))(f'(\hat{\boldsymbol{\beta}}_n))'\hat{\boldsymbol{L}}_n^{-1}\\
\boldsymbol{N}_3^* &= 2^{-1}\hat{\sigma}_n^3\Big(n^{-1}\sum_{i=1}^{n}\Big((f'(\hat{\boldsymbol{\beta}}_n))'\hat{\boldsymbol{L}}_n^{-1}\hat{\boldsymbol{M}}_n\hat{\boldsymbol{L}}_n^{-1}\boldsymbol{x}_i\Big)\Big(\hat{\boldsymbol{L}}_n^{-1}\boldsymbol{x}_i\Big)(1-e^{\boldsymbol{x}_i'\boldsymbol{\beta}})(1+e^{\boldsymbol{x}_i'\boldsymbol{\beta}})^{-3}\boldsymbol{x}_i'\Big)\hat{\boldsymbol{L}}_n^{-1}(f'(\hat{\boldsymbol{\beta}}_n))\\
&\qquad (f'(\hat{\boldsymbol{\beta}}_n))'\hat{\boldsymbol{L}}_n^{-1} + 2^{-1}\hat{\sigma}_n^3\Big(n^{-1}\sum_{i=1}^{n}\Big((f'(\hat{\boldsymbol{\beta}}_n))'\hat{\boldsymbol{L}}_n^{-1}\boldsymbol{x}_i\Big)\Big(\hat{\boldsymbol{L}}_n^{-1}\boldsymbol{x}_i\Big)(1-e^{\boldsymbol{x}_i'\boldsymbol{\beta}})(1+e^{\boldsymbol{x}_i'\boldsymbol{\beta}})^{-3}\boldsymbol{x}_i'\Big)\\
&\qquad \hat{\boldsymbol{L}}_n^{-1}\hat{\boldsymbol{M}}_n\hat{\boldsymbol{L}}_n^{-1}(f'(\hat{\boldsymbol{\beta}}_n))(f'(\hat{\boldsymbol{\beta}}_n))'\hat{\boldsymbol{L}}_n^{-1}\\
\boldsymbol{N}_4^* &= -2^{-1}\hat{\sigma}_n^3\Big(\sum_{k=1}^{p}\Big((f'(\hat{\boldsymbol{\beta}}_n))'\hat{\boldsymbol{L}}_{\cdot kn}^{-1}\Big)E_{kn}\Big)\hat{\boldsymbol{L}}_n^{-1}(f'(\hat{\boldsymbol{\beta}}_n))(f'(\hat{\boldsymbol{\beta}}_n))'\hat{\boldsymbol{L}}_n^{-1},
\end{aligned}$$

where $\tilde{\boldsymbol{M}}_k^* = n^{-1}\sum_{i=1}^{n}x_{ik}e^{\boldsymbol{x}_i'\boldsymbol{\beta}}(1-e^{\boldsymbol{x}_i'\hat{\boldsymbol{\beta}}_n})(1+e^{\boldsymbol{x}_i'\hat{\boldsymbol{\beta}}_n})^{-3}\Big(\hat{\boldsymbol{L}}_n^{-1}\boldsymbol{x}_i\boldsymbol{x}_i'\hat{\boldsymbol{L}}_n^{-1}\Big)$ for $k\in\{1,\dots,p\}$ and $\hat{\boldsymbol{L}}_{\cdot kn}^{-1}$ is the kth column of $\hat{\boldsymbol{L}}_n^{-1}$. Now using Lemma 2.4 and Lemma 2.7 we can obtain two term Edgeworth expansions for the cdf of $\boldsymbol{T}_n$ and the conditional cdf of $\boldsymbol{T}_n^*$ which can be shown to be close with an error $o_p(n^{-1/2})$ by following the same line of arguments as in case of the Theorem 3.3. Then the conclusion of Theorem 4.1 follows by noting that $\mathcal{A}_1$ can be taken as $\{(-\infty,t]: t\in\mathcal{R}\}$ where the class of subsets $\mathcal{A}_m$ of $\mathcal{R}^m$ is defined in (1.1).

5. Additional simulation results

In this section we provide the extended simulation study results. Recall, in total we considered six simulation scenarios given by:

- Scenario 1: $\boldsymbol{b} = (1, .5, -2, -0.75, 1.5, -1, 1.85, -1.6)$, $b_n^2 = n^{-\frac{1}{p_1+1}}$.
- Scenario 2: $\boldsymbol{b} = (1, .5, 2, 0.75, 1.5, 1, 1.85, 1.6)$, $b_n^2 = n^{-\frac{1}{p_1+1}}$.
- Scenario 3: $\boldsymbol{b} = (1, .5, -2, -0.75, 1.5, -1, 1.85, -1.6)$, $b_n^2 = n^{-\frac{1}{p_1}} * (\log n)^2$.
- Scenario 4: $\boldsymbol{b} = (1, .5, 2, 0.75, 1.5, 1, 1.85, 1.6)$, $b_n^2 = n^{-\frac{1}{p_1}} * (\log n)^2$.
- Scenario 5: $\boldsymbol{b} = (1, .5, -2, -0.75, 1.5, -1, 1.85, -1.6)$, $b_n^2 = n^{-\frac{1}{2p_1}}$.
- Scenario 6: $\boldsymbol{b} = (1, .5, 2, 0.75, 1.5, 1, 1.85, 1.6)$, $b_n^2 = n^{-\frac{1}{2p_1}}$.

Please refer to the main paper for other details regarding the simulation structure. In Table 1, 2, 3, 4, 5, 6, we note down the empirical coverage of 90% confidence region of $\boldsymbol{\beta}$, upper, middle and lower 90%

Confidence intervals (CIs) corresponding to the minimum and maximum components of $\boldsymbol{\beta}$, the average empirical coverages of upper, middle and lower 90% CI over all components of $\boldsymbol{\beta}$ for all considered methods, in scenarios 1, 2, 3, 4, 5 and 6 respectively. Average widths of 90% CI corresponding to all applicable cases are also noted in parenthesis.

In Table 7, 8, 9, 10, 11, 12 we compare the methods in terms of empirical coverages of 90% CIs of odds ratio $f(\boldsymbol{\beta}) = e^{\boldsymbol{x}_0'\boldsymbol{\beta}}$ with all the components of $\boldsymbol{x}_0$ being 1. Based on the results, we recommend that for using PEBBLE, one may use $b_n^2 = n^{-\frac{1}{p_1+1}}$. In that case (scenario 1 and 2), PEBBLE is observed to perform better compared to other methods, in general.

Table 1. Scenario 1: Comparative performance study of PEBBLE with Normal approximation (Normal), Pearson Residual Resampling Bootstrap (PRRB), One-Step Bootstrap (OSB) and Quadratic Bootstrap (QB) for components of $\boldsymbol{\beta}$. Empirical coverages of 90% confidence region of $||\boldsymbol{\beta}||$ (column 1), upper, lower and middle confidence intervals (CIs) of the minimum absolute value of $\boldsymbol{\beta}$ (column 2,3,4), upper, lower and middle CIs of the maximum absolute value of the $\boldsymbol{\beta}$ (column 5,6,7), upper, lower and middle CIs of the all components of $\boldsymbol{\beta}$, on average (column 8,9,10) are presented, computed over 500 experiments. Average widths of the middle CIs are provided in parenthesis.

(n, p)	Methods	$\|\|\beta\|\|$	β_{min} middle (width)	β_{min} upper	β_{min} lower	β_{max} middle (width)	β_{max} upper	β_{max} lower	β avg. middle (width)	β avg. upper	β avg. lower
(30, 3)	PEBBLE	0.924	0.902 (2.98)	0.872	0.928	0.922 (4.12)	0.922	0.904	0.905 (3.25)	0.893	0.920
	Normal	0.948	0.940 (2.31)	0.956	0.896	0.966 (2.85)	0.916	0.998	0.957 (2.43)	0.937	0.940
	PRSB	0.938	0.918 (2.16)	0.926	0.862	0.944 (2.66)	0.918	0.944	0.929 (2.27)	0.913	0.910
	OSB	0.946	0.942 (2.35)	0.934	0.902	0.940 (2.66)	0.920	0.954	0.932 (2.38)	0.917	0.931
	QB	0.968	0.956 (2.50)	0.942	0.922	0.964 (3.06)	0.934	0.978	0.937 (2.53)	0.911	0.947
(50, 3)	PEBBLE	0.898	0.900 (2.20)	0.880	0.924	0.914 (3.13)	0.918	0.906	0.907 (2.36)	0.900	0.919
	Normal	0.944	0.934 (1.76)	0.920	0.916	0.958 (2.21)	0.920	0.968	0.941 (1.82)	0.923	0.932
	PRSB	0.914	0.896 (1.67)	0.892	0.898	0.920 (2.07)	0.918	0.924	0.907 (1.71)	0.903	0.910
	OSB	0.932	0.904 (1.80)	0.892	0.902	0.910 (2.06)	0.916	0.928	0.912 (1.78)	0.907	0.915
	QB	0.938	0.918 (1.86)	0.892	0.924	0.924 (2.13)	0.918	0.934	0.925 (1.84)	0.915	0.927
(50, 4)	PEBBLE	0.894	0.886 (3.05)	0.872	0.924	0.898 (4.12)	0.922	0.886	0.896 (2.85)	0.893	0.906
	Normal	0.926	0.922 (2.14)	0.954	0.892	0.946 (2.65)	0.896	0.982	0.935 (2.04)	0.919	0.923
	PRSB	0.916	0.904 (1.98)	0.928	0.872	0.924 (2.42)	0.890	0.934	0.899 (1.88)	0.901	0.892
	OSB	0.946	0.912 (2.19)	0.938	0.916	0.922 (2.44)	0.908	0.936	0.915 (2.03)	0.920	0.913
	QB	0.940	0.902 (2.09)	0.930	0.916	0.910 (2.40)	0.892	0.936	0.907 (1.98)	0.918	0.909
(100, 3)	PEBBLE	0.888	0.874 (1.24)	0.882	0.89	0.876 (1.77)	0.896	0.886	0.886 (1.41)	0.889	0.891
	Normal	0.934	0.902 (1.08)	0.896	0.898	0.918 (1.41)	0.914	0.892	0.913 (1.19)	0.900	0.902
	PRSB	0.890	0.890 (1.07)	0.888	0.898	0.902 (1.39)	0.916	0.876	0.897 (1.17)	0.895	0.893
	OSB	0.898	0.886 (1.08)	0.886	0.892	0.888 (1.39)	0.916	0.876	0.889 (1.18)	0.897	0.894
	QB	0.892	0.882 (1.06)	0.882	0.894	0.868 (1.33)	0.910	0.874	0.888 (1.16)	0.896	0.895
(100, 4)	PEBBLE	0.900	0.918 (1.86)	0.902	0.930	0.886 (2.33)	0.914	0.868	0.903 (1.78)	0.904	0.898
	Normal	0.934	0.932 (1.39)	0.930	0.912	0.934 (1.66)	0.904	0.928	0.920 (1.36)	0.908	0.908
	PRSB	0.890	0.910 (1.33)	0.906	0.906	0.876 (1.59)	0.892	0.868	0.888 (1.30)	0.888	0.889
	OSB	0.910	0.918 (1.41)	0.922	0.918	0.894 (1.63)	0.902	0.894	0.904 (1.36)	0.906	0.905
	QB	0.934	0.940 (1.49)	0.942	0.924	0.940 (1.85)	0.928	0.918	0.919 (1.42)	0.914	0.910
(100, 6)	PEBBLE	0.930	0.888 (1.82)	0.868	0.906	0.910 (2.88)	0.936	0.894	0.905 (2.13)	0.908	0.911
	Normal	0.870	0.870 (1.24)	0.868	0.870	0.910 (1.69)	0.884	0.940	0.873 (1.35)	0.876	0.890
	PRSB	0.858	0.842 (1.21)	0.858	0.864	0.884 (1.65)	0.872	0.906	0.846 (1.32)	0.865	0.874
	OSB	0.932	0.784 (1.29)	0.828	0.824	0.836 (1.66)	0.850	0.868	0.792 (1.37)	0.837	0.845
	QB	0.954	0.796 (1.37)	0.850	0.828	0.868 (1.84)	0.856	0.894	0.805 (1.45)	0.848	0.854
(200, 3)	PEBBLE	0.886	0.904 (0.89)	0.912	0.902	0.902 (1.27)	0.914	0.912	0.897 (1.06)	0.903	0.912
	Normal	0.900	0.898 (0.78)	0.894	0.920	0.904 (1.03)	0.932	0.868	0.896 (0.89)	0.913	0.893
	PRSB	0.900	0.900 (0.76)	0.898	0.910	0.898 (1.01)	0.924	0.872	0.891 (0.88)	0.909	0.890
	OSB	0.898	0.892 (0.78)	0.890	0.924	0.892 (1.01)	0.932	0.850	0.888 (0.88)	0.908	0.888
	QB	0.866	0.884 (0.75)	0.878	0.924	0.860 (0.93)	0.918	0.836	0.867 (0.82)	0.893	0.879
(200, 4)	PEBBLE	0.878	0.900 (1.11)	0.886	0.910	0.902 (1.60)	0.908	0.874	0.894 (1.15)	0.897	0.897
	Normal	0.912	0.916 (0.89)	0.892	0.904	0.892 (1.18)	0.914	0.882	0.897 (0.93)	0.898	0.897
	PRSB	0.880	0.894 (0.87)	0.882	0.904	0.870 (1.15)	0.912	0.868	0.878 (0.90)	0.892	0.888
	OSB	0.884	0.914 (0.89)	0.882	0.898	0.866 (1.16)	0.924	0.864	0.886 (0.92)	0.895	0.889
	QB	0.916	0.926 (0.92)	0.900	0.910	0.898 (1.23)	0.926	0.874	0.904 (0.96)	0.903	0.899
(200, 6)	PEBBLE	0.936	0.914 (1.34)	0.882	0.934	0.908 (1.84)	0.938	0.864	0.911 (1.62)	0.908	0.903
	Normal	0.812	0.832 (0.89)	0.850	0.864	0.892 (1.17)	0.922	0.858	0.854 (1.01)	0.866	0.871
	PRSB	0.794	0.826 (0.90)	0.854	0.868	0.858 (1.18)	0.914	0.850	0.836 (1.01)	0.859	0.867
	OSB	0.906	0.756 (0.91)	0.808	0.852	0.794 (1.18)	0.894	0.790	0.746 (1.02)	0.814	0.825
	QB	0.902	0.746 (0.88)	0.800	0.844	0.786 (1.15)	0.890	0.776	0.741 (1.01)	0.813	0.822
(200, 8)	PEBBLE	0.842	0.864 (1.76)	0.848	0.944	0.856 (2.30)	0.962	0.774	0.846 (1.94)	0.866	0.876
	Normal	0.406	0.664 (0.94)	0.878	0.668	0.740 (1.19)	0.708	0.958	0.685 (1.00)	0.782	0.794
	PRSB	0.492	0.650 (0.97)	0.876	0.670	0.734 (1.22)	0.702	0.944	0.683 (1.02)	0.780	0.799
	OSB	0.854	0.472 (0.97)	0.798	0.570	0.564 (1.16)	0.632	0.838	0.490 (1.00)	0.680	0.715
	QB	0.848	0.478 (0.98)	0.800	0.574	0.544 (1.14)	0.638	0.842	0.484 (0.98)	0.683	0.714

Table 2. Scenario 2: Comparative performance study of PEBBLE with Normal approximation (Normal), Pearson Residual Resampling Bootstrap (PRRB), One-Step Bootstrap (OSB) and Quadratic Bootstrap (QB) for components of $\boldsymbol{\beta}$. Empirical coverages of 90% confidence region of $||\boldsymbol{\beta}||$ (column 1), upper, lower and middle confidence intervals (CIs) of the minimum absolute value of $\boldsymbol{\beta}$ (column 2,3,4), upper, lower and middle CIs of the maximum absolute value of the $\boldsymbol{\beta}$ (column 5,6,7), upper, lower and middle CIs of the all components of $\boldsymbol{\beta}$, on average (column 8,9,10) are presented, computed over 500 experiments. Average widths of the middle CIs are provided in parenthesis.

(n, p)	Methods	$\|\|\beta\|\|$	β_{min} middle (width)	β_{min} upper	β_{min} lower	β_{max} middle (width)	β_{max} upper	β_{max} lower	β avg. middle (width)	β avg. upper	β avg. lower
(30, 3)	PEBBLE	0.910	0.906 (3.50)	0.886	0.904	0.890 (4.29)	0.836	0.928	0.899 (4.03)	0.856	0.923
	Normal	0.904	0.958 (2.67)	0.958	0.910	0.932 (2.92)	0.998	0.870	0.944 (2.89)	0.981	0.890
	PRSB	0.954	0.860 (2.28)	0.868	0.862	0.880 (2.46)	0.922	0.856	0.881 (2.45)	0.908	0.854
	OSB	0.978	0.904 (2.66)	0.914	0.916	0.926 (2.70)	0.956	0.878	0.910 (2.75)	0.945	0.887
	QB	0.940	0.878 (2.55)	0.912	0.906	0.890 (2.35)	0.954	0.848	0.883 (2.55)	0.940	0.867
(50, 3)	PEBBLE	0.894	0.892 (2.61)	0.858	0.898	0.898 (3.11)	0.870	0.914	0.897 (2.66)	0.867	0.912
	Normal	0.920	0.926 (2.08)	0.914	0.874	0.944 (2.22)	0.974	0.898	0.939 (2.02)	0.940	0.893
	PRSB	0.880	0.846 (1.85)	0.850	0.852	0.894 (1.96)	0.892	0.874	0.874 (1.79)	0.869	0.873
	OSB	0.950	0.882 (2.09)	0.902	0.884	0.916 (2.11)	0.908	0.904	0.895 (1.99)	0.909	0.903
	QB	0.946	0.866 (1.95)	0.886	0.864	0.912 (2.11)	0.904	0.910	0.896 (1.96)	0.905	0.899
(50, 4)	PEBBLE	0.904	0.904 (3.06)	0.882	0.910	0.908 (3.82)	0.866	0.924	0.904 (2.97)	0.873	0.916
	Normal	0.932	0.932 (2.24)	0.928	0.902	0.948 (2.54)	0.982	0.902	0.936 (2.13)	0.946	0.897
	PRSB	0.900	0.870 (1.98)	0.890	0.870	0.916 (2.24)	0.922	0.892	0.880 (1.87)	0.890	0.875
	OSB	0.968	0.912 (2.30)	0.924	0.88	0.928 (2.46)	0.946	0.922	0.916 (2.14)	0.932	0.903
	QB	0.982	0.918 (2.36)	0.922	0.894	0.952 (2.61)	0.958	0.926	0.929 (2.24)	0.941	0.912
(100, 3)	PEBBLE	0.888	0.878 (1.52)	0.908	0.866	0.900 (2.33)	0.878	0.936	0.891 (1.82)	0.883	0.913
	Normal	0.908	0.906 (1.29)	0.908	0.900	0.926 (1.72)	0.962	0.884	0.910 (1.44)	0.931	0.887
	PRSB	0.874	0.872 (1.21)	0.882	0.880	0.898 (1.59)	0.912	0.878	0.877 (1.34)	0.897	0.875
	OSB	0.912	0.884 (1.30)	0.896	0.912	0.908 (1.67)	0.938	0.886	0.899 (1.43)	0.914	0.889
	QB	0.912	0.878 (1.25)	0.888	0.904	0.912 (1.73)	0.934	0.900	0.898 (1.43)	0.911	0.891
(100, 4)	PEBBLE	0.924	0.916 (2.00)	0.904	0.892	0.924 (2.75)	0.884	0.942	0.918 (2.12)	0.884	0.922
	Normal	0.938	0.920 (1.52)	0.888	0.910	0.960 (1.90)	0.974	0.910	0.931 (1.57)	0.933	0.899
	PRSB	0.880	0.840 (1.36)	0.826	0.880	0.912 (1.70)	0.940	0.880	0.870 (1.41)	0.886	0.866
	OSB	0.950	0.898 (1.56)	0.890	0.912	0.954 (1.80)	0.952	0.910	0.910 (1.57)	0.920	0.903
	QB	0.970	0.910 (1.68)	0.912	0.916	0.944 (1.73)	0.948	0.896	0.922 (1.63)	0.929	0.907
(100, 6)	PEBBLE	0.926	0.926 (2.52)	0.916	0.934	0.868 (3.26)	0.812	0.956	0.899 (2.69)	0.862	0.935
	Normal	0.864	0.934 (1.67)	0.932	0.912	0.908 (2.00)	0.998	0.830	0.920 (1.76)	0.965	0.876
	PRSB	0.840	0.846 (1.35)	0.854	0.864	0.846 (1.63)	0.970	0.760	0.842 (1.43)	0.909	0.818
	OSB	0.956	0.938 (1.72)	0.930	0.924	0.920 (1.94)	0.990	0.832	0.922 (1.77)	0.954	0.881
	QB	0.980	0.950 (1.82)	0.934	0.930	0.918 (1.93)	0.990	0.826	0.933 (1.87)	0.962	0.888
(200, 3)	PEBBLE	0.898	0.928 (1.09)	0.918	0.912	0.900 (1.50)	0.908	0.902	0.915 (1.24)	0.905	0.913
	Normal	0.926	0.916 (0.93)	0.910	0.904	0.918 (1.18)	0.906	0.904	0.918 (1.01)	0.911	0.909
	PRSB	0.888	0.900 (0.89)	0.896	0.890	0.896 (1.13)	0.888	0.894	0.899 (0.98)	0.894	0.901
	OSB	0.912	0.914 (0.93)	0.908	0.906	0.882 (1.16)	0.890	0.902	0.905 (1.00)	0.903	0.906
	QB	0.920	0.912 (0.93)	0.912	0.904	0.888 (1.18)	0.898	0.904	0.906 (1.01)	0.907	0.908
(200, 4)	PEBBLE	0.912	0.906 (1.39)	0.936	0.896	0.916 (1.89)	0.900	0.934	0.907 (1.45)	0.902	0.916
	Normal	0.914	0.908 (1.08)	0.892	0.918	0.936 (1.34)	0.928	0.904	0.915 (1.11)	0.909	0.907
	PRSB	0.874	0.874 (1.02)	0.870	0.884	0.906 (1.26)	0.886	0.896	0.880 (1.04)	0.877	0.889
	OSB	0.934	0.900 (1.09)	0.886	0.924	0.924 (1.32)	0.908	0.904	0.902 (1.10)	0.899	0.905
	QB	0.936	0.918 (1.14)	0.904	0.926	0.920 (1.30)	0.908	0.898	0.909 (1.11)	0.904	0.906
(200, 6)	PEBBLE	0.914	0.908 (1.47)	0.924	0.884	0.892 (2.41)	0.822	0.974	0.908 (1.94)	0.876	0.934
	Normal	0.882	0.924 (1.09)	0.902	0.922	0.912 (1.45)	0.994	0.852	0.924 (1.28)	0.952	0.891
	PRSB	0.870	0.862 (0.95)	0.864	0.882	0.884 (1.27)	0.978	0.782	0.873 (1.17)	0.914	0.842
	OSB	0.954	0.906 (1.12)	0.890	0.934	0.922 (1.41)	0.990	0.840	0.916 (1.27)	0.944	0.888
	QB	0.966	0.920 (1.19)	0.912	0.942	0.924 (1.41)	0.982	0.846	0.922 (1.30)	0.948	0.893
(200, 8)	PEBBLE	0.932	0.890 (2.26)	0.884	0.924	0.892 (2.95)	0.856	0.946	0.895 (2.44)	0.855	0.934
	Normal	0.828	0.930 (1.39)	0.952	0.872	0.904 (1.57)	0.990	0.826	0.906 (1.41)	0.968	0.839
	PRSB	0.622	0.772 (0.97)	0.874	0.768	0.764 (1.10)	0.946	0.682	0.748 (0.99)	0.905	0.719
	OSB	0.960	0.882 (1.32)	0.950	0.824	0.760 (1.39)	0.998	0.650	0.811 (1.31)	0.975	0.732
	QB	0.968	0.884 (1.35)	0.960	0.820	0.778 (1.45)	0.998	0.660	0.822 (1.34)	0.978	0.735

Table 3. Scenario 3: Comparative performance study of PEBBLE with Normal approximation (Normal), Pearson Residual Resampling Bootstrap (PRRB), One-Step Bootstrap (OSB) and Quadratic Bootstrap (QB) for components of $\boldsymbol{\beta}$. Empirical coverages of 90% confidence region of $||\boldsymbol{\beta}||$ (column 1), upper, lower and middle confidence intervals (CIs) of the minimum absolute value of $\boldsymbol{\beta}$ (column 2,3,4), upper, lower and middle CIs of the maximum absolute value of the $\boldsymbol{\beta}$ (column 5,6,7), upper, lower and middle CIs of the all components of $\boldsymbol{\beta}$, on average (column 8,9,10) are presented, computed over 500 experiments. Average widths of the middle CIs are provided in parenthesis.

(n, p)	Methods	$\|\|\beta\|\|$	β_{min} middle (width)	β_{min} upper	β_{min} lower	β_{max} middle (width)	β_{max} upper	β_{max} lower	β avg. middle (width)	β avg. upper	β avg. lower
(30, 3)	PEBBLE	0.940	0.904 (6.94)	0.894	0.926	0.900 (9.25)	0.902	0.896	0.909 (7.36)	0.893	0.919
	Normal	0.948	0.940 (2.31)	0.956	0.896	0.966 (2.85)	0.916	0.998	0.957 (2.43)	0.937	0.940
	PRSB	0.938	0.918 (2.16)	0.926	0.862	0.944 (2.66)	0.918	0.944	0.929 (2.27)	0.913	0.910
	OSB	0.946	0.942 (2.35)	0.934	0.902	0.940 (2.66)	0.920	0.954	0.932 (2.38)	0.917	0.931
	QB	0.968	0.956 (2.50)	0.942	0.922	0.964 (3.06)	0.934	0.978	0.937 (2.53)	0.911	0.947
(50, 3)	PEBBLE	0.924	0.896 (5.46)	0.890	0.906	0.894 (7.63)	0.900	0.906	0.901 (5.72)	0.897	0.911
	Normal	0.944	0.934 (1.76)	0.920	0.916	0.958 (2.21)	0.920	0.968	0.941 (1.82)	0.923	0.932
	PRSB	0.914	0.896 (1.67)	0.892	0.898	0.920 (2.07)	0.918	0.924	0.907 (1.71)	0.903	0.910
	OSB	0.932	0.904 (1.80)	0.892	0.902	0.910 (2.06)	0.916	0.928	0.912 (1.78)	0.907	0.915
	QB	0.938	0.918 (1.86)	0.892	0.924	0.924 (2.13)	0.918	0.934	0.925 (1.84)	0.915	0.927
(50, 4)	PEBBLE	0.894	0.854 (7.65)	0.868	0.888	0.816 (9.78)	0.858	0.860	0.871 (6.99)	0.881	0.887
	Normal	0.926	0.922 (2.14)	0.954	0.892	0.946 (2.65)	0.896	0.982	0.935 (2.04)	0.919	0.923
	PRSB	0.916	0.904 (1.98)	0.928	0.872	0.924 (2.42)	0.890	0.934	0.899 (1.88)	0.901	0.892
	OSB	0.946	0.912 (2.19)	0.938	0.916	0.922 (2.44)	0.908	0.936	0.915 (2.03)	0.920	0.913
	QB	0.940	0.902 (2.09)	0.930	0.916	0.910 (2.40)	0.892	0.936	0.907 (1.98)	0.918	0.909
(100, 3)	PEBBLE	0.918	0.902 (3.21)	0.908	0.894	0.884 (4.80)	0.892	0.878	0.899 (3.64)	0.900	0.893
	Normal	0.934	0.902 (1.08)	0.896	0.898	0.918 (1.41)	0.914	0.892	0.913 (1.19)	0.900	0.902
	PRSB	0.890	0.890 (1.07)	0.888	0.898	0.902 (1.39)	0.916	0.876	0.897 (1.17)	0.895	0.893
	OSB	0.898	0.886 (1.08)	0.886	0.892	0.888 (1.39)	0.916	0.876	0.889 (1.18)	0.897	0.894
	QB	0.892	0.882 (1.06)	0.882	0.894	0.868 (1.33)	0.910	0.874	0.888 (1.16)	0.896	0.895
(100, 4)	PEBBLE	0.902	0.890 (5.23)	0.890	0.898	0.852 (6.55)	0.888	0.870	0.889 (4.90)	0.890	0.894
	Normal	0.934	0.932 (1.39)	0.930	0.912	0.934 (1.66)	0.904	0.928	0.920 (1.36)	0.908	0.908
	PRSB	0.890	0.910 (1.33)	0.906	0.906	0.876 (1.59)	0.892	0.868	0.888 (1.30)	0.888	0.889
	OSB	0.910	0.918 (1.41)	0.922	0.918	0.894 (1.63)	0.902	0.894	0.904 (1.36)	0.906	0.905
	QB	0.934	0.940 (1.49)	0.942	0.924	0.940 (1.85)	0.928	0.918	0.919 (1.42)	0.914	0.910
(100, 6)	PEBBLE	0.874	0.858 (4.70)	0.880	0.864	0.778 (7.11)	0.872	0.836	0.838 (5.28)	0.872	0.867
	Normal	0.870	0.870 (1.24)	0.868	0.870	0.910 (1.69)	0.884	0.940	0.873 (1.35)	0.876	0.89
	PRSB	0.858	0.842 (1.21)	0.858	0.864	0.884 (1.65)	0.872	0.906	0.846 (1.32)	0.865	0.874
	OSB	0.932	0.784 (1.29)	0.828	0.824	0.836 (1.66)	0.850	0.868	0.792 (1.37)	0.837	0.845
	QB	0.954	0.796 (1.37)	0.850	0.828	0.868 (1.84)	0.856	0.894	0.805 (1.45)	0.848	0.854
(200, 3)	PEBBLE	0.906	0.898 (2.34)	0.904	0.902	0.904 (3.65)	0.906	0.884	0.899 (2.88)	0.905	0.901
	Normal	0.900	0.898 (0.78)	0.894	0.920	0.904 (1.03)	0.932	0.868	0.896 (0.89)	0.913	0.893
	PRSB	0.900	0.900 (0.76)	0.898	0.910	0.898 (1.01)	0.924	0.872	0.891 (0.88)	0.909	0.890
	OSB	0.898	0.892 (0.78)	0.890	0.924	0.892 (1.01)	0.932	0.850	0.888 (0.88)	0.908	0.888
	QB	0.866	0.884 (0.75)	0.878	0.924	0.860 (0.93)	0.918	0.836	0.867 (0.82)	0.893	0.879
(200, 4)	PEBBLE	0.882	0.888 (3.22)	0.884	0.890	0.862 (4.89)	0.888	0.870	0.887 (3.35)	0.895	0.890
	Normal	0.912	0.916 (0.89)	0.892	0.904	0.892 (1.18)	0.914	0.882	0.897 (0.93)	0.898	0.897
	PRSB	0.880	0.894 (0.87)	0.882	0.904	0.870 (1.15)	0.912	0.868	0.878 (0.90)	0.892	0.888
	OSB	0.884	0.914 (0.89)	0.882	0.898	0.866 (1.16)	0.924	0.864	0.886 (0.92)	0.895	0.889
	QB	0.916	0.926 (0.92)	0.900	0.910	0.898 (1.23)	0.926	0.874	0.904 (0.96)	0.903	0.899
(200, 6)	PEBBLE	0.800	0.834 (3.57)	0.850	0.892	0.806 (5.14)	0.878	0.842	0.808 (4.24)	0.854	0.856
	Normal	0.812	0.832 (0.89)	0.850	0.864	0.892 (1.17)	0.922	0.858	0.854 (1.01)	0.866	0.871
	PRSB	0.794	0.826 (0.90)	0.854	0.868	0.858 (1.18)	0.914	0.850	0.836 (1.01)	0.859	0.867
	OSB	0.906	0.756 (0.91)	0.808	0.852	0.794 (1.18)	0.894	0.790	0.746 (1.02)	0.814	0.825
	QB	0.902	0.746 (0.88)	0.800	0.844	0.786 (1.15)	0.890	0.776	0.741 91.01)	0.813	0.822
(200, 8)	PEBBLE	0.788	0.812 (4.09)	0.836	0.892	0.748 (5.24)	0.882	0.778	0.768 (4.27)	0.842	0.835
	Normal	0.406	0.664 (0.94)	0.878	0.668	0.740 (1.19)	0.708	0.958	0.685 (1.00)	0.782	0.794
	PRSB	0.492	0.650 (0.97)	0.876	0.670	0.734 (1.22)	0.702	0.944	0.683 (1.02)	0.780	0.799
	OSB	0.854	0.472 (0.97)	0.798	0.570	0.564 (1.16)	0.632	0.838	0.490 (1.00)	0.680	0.715
	QB	0.848	0.478 (0.98)	0.800	0.574	0.544 (1.14)	0.638	0.842	0.484 (0.98)	0.683	0.714

Table 4. Scenario 4: Comparative performance study of PEBBLE with Normal approximation (Normal), Pearson Residual Resampling Bootstrap (PRRB), One-Step Bootstrap (OSB) and Quadratic Bootstrap (QB) for components of $\boldsymbol{\beta}$. Empirical coverages of 90% confidence region of $||\boldsymbol{\beta}||$ (column 1), upper, lower and middle confidence intervals (CIs) of the minimum absolute value of $\boldsymbol{\beta}$ (column 2,3,4), upper, lower and middle CIs of the maximum absolute value of the $\boldsymbol{\beta}$ (column 5,6,7), upper, lower and middle CIs of the all components of $\boldsymbol{\beta}$, on average (column 8,9,10) are presented, computed over 500 experiments. Average widths of the middle CIs are provided in parenthesis.

(n, p)	Methods	$\|\|\beta\|\|$	β_{min} middle (width)	β_{min} upper	β_{min} lower	β_{max} middle (width)	β_{max} upper	β_{max} lower	β avg. middle (width)	β avg. upper	β avg. lower
(30, 3)	PEBBLE	0.930	0.880 (7.86)	0.884	0.910	0.862 (9.16)	0.850	0.906	0.877 (8.93)	0.868	0.907
	Normal	0.904	0.958 (2.67)	0.958	0.910	0.932 (2.92)	0.998	0.870	0.944 (2.89)	0.981	0.890
	PRSB	0.954	0.860 (2.28)	0.868	0.862	0.880 (2.46)	0.922	0.856	0.881 (2.45)	0.908	0.854
	OSB	0.978	0.904 (2.66)	0.914	0.916	0.926 (2.70)	0.956	0.878	0.910 (2.75)	0.945	0.887
	QB	0.940	0.878 (2.55)	0.912	0.906	0.890 (2.35)	0.954	0.848	0.883 (2.55)	0.940	0.867
(50, 3)	PEBBLE	0.944	0.908 (6.58)	0.894	0.904	0.896 (7.58)	0.878	0.908	0.904 (6.53)	0.885	0.907
	Normal	0.920	0.926 (2.08)	0.914	0.874	0.944 (2.22)	0.974	0.898	0.939 (2.02)	0.940	0.893
	PRSB	0.880	0.846 (1.85)	0.850	0.852	0.894 (1.96)	0.892	0.874	0.874 (1.79)	0.869	0.873
	OSB	0.950	0.882 (2.09)	0.902	0.884	0.916 (2.11)	0.908	0.904	0.895 (1.99)	0.909	0.903
	QB	0.946	0.866 (1.95)	0.886	0.864	0.912 (2.11)	0.904	0.910	0.896 (1.96)	0.905	0.899
(50, 4)	PEBBLE	0.936	0.890 (7.82)	0.878	0.882	0.846 (9.29)	0.872	0.890	0.879 (7.37)	0.877	0.894
	Normal	0.932	0.932 (2.24)	0.928	0.902	0.948 (2.54)	0.982	0.902	0.936 (2.13)	0.946	0.897
	PRSB	0.900	0.870 (1.98)	0.890	0.870	0.916 (2.24)	0.922	0.892	0.880 (1.87)	0.890	0.875
	OSB	0.968	0.912 (2.30)	0.924	0.880	0.928 (2.46)	0.946	0.922	0.916 (2.14)	0.932	0.903
	QB	0.982	0.918 (2.36)	0.922	0.894	0.952 (2.61)	0.958	0.926	0.929 (2.24)	0.941	0.912
(100, 3)	PEBBLE	0.914	0.910 (4.02)	0.918	0.902	0.890 (6.40)	0.878	0.914	0.901 (4.87)	0.895	0.913
	Normal	0.908	0.906 (1.29)	0.908	0.900	0.926 (1.72)	0.962	0.884	0.910 (1.44)	0.931	0.887
	PRSB	0.874	0.872 (1.21)	0.882	0.880	0.898 (1.59)	0.912	0.878	0.877 (1.34)	0.897	0.875
	OSB	0.912	0.884 (1.30)	0.896	0.912	0.908 (1.67)	0.938	0.886	0.899 (1.43)	0.914	0.889
	QB	0.912	0.878 (1.25)	0.888	0.904	0.912 91.73)	0.934	0.900	0.898 (1.43)	0.911	0.891
(100, 4)	PEBBLE	0.900	0.874 (5.71)	0.900	0.892	0.856 (7.60)	0.860	0.900	0.879 (5.92)	0.884	0.906
	Normal	0.938	0.920 (1.52)	0.888	0.910	0.960 (1.90)	0.974	0.910	0.931 (1.57)	0.933	0.899
	PRSB	0.880	0.840 (1.36)	0.826	0.880	0.912 (1.70)	0.940	0.880	0.870 (1.41)	0.886	0.866
	OSB	0.950	0.898 (1.56)	0.890	0.912	0.954 (1.80)	0.952	0.910	0.910 (1.57)	0.920	0.903
	QB	0.970	0.910 (1.68)	0.912	0.916	0.944 (1.73)	0.948	0.896	0.922 (1.63)	0.929	0.907
(100, 6)	PEBBLE	0.852	0.806 (6.67)	0.826	0.884	0.758 (8.13)	0.820	0.852	0.789 (7.02)	0.834	0.856
	Normal	0.864	0.934 (1.67)	0.932	0.912	0.908 (2.00)	0.998	0.830	0.920 (1.76)	0.965	0.876
	PRSB	0.840	0.846 (1.35)	0.854	0.864	0.846 (1.63)	0.970	0.760	0.842 (1.43)	0.909	0.818
	OSB	0.956	0.938 (1.72)	0.930	0.924	0.920 (1.94)	0.990	0.832	0.922 (1.77)	0.954	0.881
	QB	0.980	0.950 (1.82)	0.934	0.930	0.918 (1.93)	0.990	0.826	0.933 (1.87)	0.962	0.888
(200, 3)	PEBBLE	0.898	0.898 (2.99)	0.916	0.908	0.892 (4.39)	0.896	0.898	0.893 (3.50)	0.904	0.902
	Normal	0.926	0.916 (0.93)	0.910	0.904	0.918 (1.18)	0.906	0.904	0.918 (1.01)	0.911	0.909
	PRSB	0.888	0.900 (0.89)	0.896	0.890	0.896 (1.13)	0.888	0.894	0.899 (0.98)	0.894	0.901
	OSB	0.912	0.914 (0.93)	0.908	0.906	0.882 (1.16)	0.890	0.902	0.905 (1.00)	0.903	0.906
	QB	0.920	0.912 (0.93)	0.912	0.904	0.888 (1.18)	0.898	0.904	0.906 (1.01)	0.907	0.908
(200, 4)	PEBBLE	0.878	0.888 (4.20)	0.900	0.876	0.858 (5.74)	0.888	0.878	0.875 (4.33)	0.887	0.882
	Normal	0.914	0.908 (1.08)	0.892	0.918	0.936 (1.34)	0.928	0.904	0.915 (1.11)	0.909	0.907
	PRSB	0.874	0.874 (1.02)	0.870	0.884	0.906 (1.26)	0.886	0.896	0.880 (1.04)	0.877	0.889
	OSB	0.934	0.900 (1.09)	0.886	0.924	0.924 (1.32)	0.908	0.904	0.902 (1.10)	0.899	0.905
	QB	0.936	0.918 (1.14)	0.904	0.926	0.920 (1.30)	0.908	0.898	0.909 (1.11)	0.904	0.906
(200, 6)	PEBBLE	0.782	0.822 (4.44)	0.862	0.848	0.736 (6.38)	0.810	0.828	0.777 (5.48)	0.835	0.842
	Normal	0.882	0.924 (1.09)	0.902	0.922	0.912 (1.45)	0.994	0.852	0.924 (1.28)	0.952	0.891
	PRSB	0.870	0.862 (0.95)	0.864	0.882	0.884 (1.27)	0.978	0.782	0.873 (1.12)	0.914	0.842
	OSB	0.954	0.906 (1.12)	0.890	0.934	0.922 (1.41)	0.990	0.840	0.916 (1.27)	0.944	0.888
	QB	0.966	0.920 (1.19)	0.912	0.942	0.924 (1.41)	0.982	0.846	0.922 (1.30)	0.948	0.893
(200, 8)	PEBBLE	0.664	0.766 (5.15)	0.830	0.842	0.598 (6.24)	0.756	0.758	0.678 (5.62)	0.786	0.802
	Normal	0.800	0.934 (1.23)	0.946	0.890	0.914 (1.57)	0.994	0.828	0.908 (1.37)	0.970	0.847
	PRSB	0.664	0.780 (0.90)	0.888	0.770	0.804 (1.14)	0.948	0.746	0.776 (1.00)	0.910	0.743
	OSB	0.954	0.906 (1.21)	0.946	0.856	0.802 (1.37)	0.996	0.682	0.821 (1.27)	0.974	0.739
	QB	0.958	0.916 (1.25)	0.950	0.858	0.786 (1.33)	0.996	0.670	0.826 (1.29)	0.976	0.741

Table 5. Scenario 5: Comparative performance study of PEBBLE with Normal approximation (Normal), Pearson Residual Resampling Bootstrap (PRRB), One-Step Bootstrap (OSB) and Quadratic Bootstrap (QB) for components of $\boldsymbol{\beta}$. Empirical coverages of 90% confidence region of $||\boldsymbol{\beta}||$ (column 1), upper, lower and middle confidence intervals (CIs) of the minimum absolute value of $\boldsymbol{\beta}$ (column 2,3,4), upper, lower and middle CIs of the maximum absolute value of the $\boldsymbol{\beta}$ (column 5,6,7), upper, lower and middle CIs of the all components of $\boldsymbol{\beta}$, on average (column 8,9,10) are presented, computed over 500 experiments. Average widths of the middle CIs are provided in parenthesis.

(n, p)	Methods	$\|\|\beta\|\|$	β_{min} middle (width)	β_{min} upper	β_{min} lower	β_{max} middle (width)	β_{max} upper	β_{max} lower	β avg. middle (width)	β avg. upper	β avg. lower
(30, 3)	PEBBLE	0.932	0.912 (3.25)	0.874	0.932	0.928 (4.51)	0.918	0.900	0.914 (3.55)	0.893	0.920
	Normal	0.948	0.940 (2.31)	0.956	0.896	0.966 (2.85)	0.916	0.998	0.957 (2.43)	0.937	0.940
	PRSB	0.938	0.918 (2.16)	0.926	0.862	0.944 (2.66)	0.918	0.944	0.929 (2.27)	0.913	0.910
	OSB	0.946	0.942 (2.35)	0.934	0.902	0.940 (2.66)	0.920	0.954	0.932 (2.38)	0.917	0.931
	QB	0.968	0.956 (2.50)	0.942	0.922	0.964 (3.06)	0.934	0.978	0.937 (2.53)	0.911	0.947
(50, 3)	PEBBLE	0.908	0.904 (2.40)	0.884	0.930	0.908 (3.45)	0.924	0.904	0.910 (2.58)	0.903	0.921
	Normal	0.944	0.934 (1.76)	0.920	0.916	0.958 (2.21)	0.920	0.968	0.941 (1.82)	0.923	0.932
	PRSB	0.914	0.896 (1.67)	0.892	0.898	0.920 (2.07)	0.918	0.924	0.907 (1.71)	0.903	0.910
	OSB	0.932	0.904 (1.80)	0.892	0.902	0.910 (2.06)	0.916	0.928	0.912 (1.78)	0.907	0.915
	QB	0.938	0.918 (1.86)	0.892	0.924	0.924 (2.13)	0.918	0.934	0.925 (1.84)	0.915	0.927
(50, 4)	PEBBLE	0.910	0.892 (3.35)	0.878	0.922	0.902 (4.55)	0.920	0.876	0.902 (3.13)	0.897	0.906
	Normal	0.926	0.922 (2.14)	0.954	0.892	0.946 (2.65)	0.896	0.982	0.935 (2.04)	0.919	0.923
	PRSB	0.916	0.904 (1.98)	0.928	0.872	0.924 (2.42)	0.890	0.934	0.899 (1.88)	0.901	0.892
	OSB	0.946	0.912 (2.19)	0.938	0.916	0.922 (2.44)	0.908	0.936	0.915 (2.03)	0.920	0.913
	QB	0.940	0.902 (2.09)	0.930	0.916	0.910 (2.40)	0.892	0.936	0.907 (1.98)	0.918	0.909
(100, 3)	PEBBLE	0.894	0.884 (1.34)	0.878	0.890	0.880 (1.93)	0.900	0.884	0.889 (1.52)	0.888	0.891
	Normal	0.934	0.902 (1.08)	0.896	0.898	0.918 (1.41)	0.914	0.892	0.913 (1.19)	0.900	0.902
	PRSB	0.890	0.890 (1.07)	0.888	0.898	0.902 (1.39)	0.916	0.876	0.897 (1.17)	0.895	0.893
	OSB	0.898	0.886 (1.08)	0.886	0.892	0.888 (1.39)	0.916	0.876	0.889 (1.18)	0.897	0.894
	QB	0.892	0.882 (1.06)	0.882	0.894	0.868 (1.33)	0.910	0.874	0.888 (1.16)	0.896	0.895
(100, 4)	PEBBLE	0.904	0.918 (2.04)	0.904	0.926	0.884 (2.58)	0.912	0.870	0.905 (1.94)	0.905	0.899
	Normal	0.934	0.932 (1.39)	0.930	0.912	0.934 (1.66)	0.904	0.928	0.920 (1.36)	0.908	0.908
	PRSB	0.890	0.910 (1.33)	0.906	0.906	0.876 (1.59)	0.892	0.868	0.888 (1.30)	0.888	0.889
	OSB	0.910	0.918 (1.41)	0.922	0.918	0.894 (1.63)	0.902	0.894	0.904 (1.36)	0.906	0.905
	QB	0.934	0.940 (1.49)	0.942	0.924	0.940 (1.85)	0.928	0.918	0.919 (1.42)	0.914	0.910
(100, 6)	PEBBLE	0.940	0.898 (1.95)	0.880	0.908	0.906 (3.12)	0.942	0.886	0.907 (2.29)	0.909	0.910
	Normal	0.870	0.870 (1.24)	0.868	0.870	0.910 (1.69)	0.884	0.940	0.873 (1.35)	0.876	0.890
	PRSB	0.858	0.842 (1.21)	0.858	0.864	0.884 (1.65)	0.872	0.906	0.846 (1.32)	0.865	0.874
	OSB	0.932	0.784 (1.29)	0.828	0.824	0.836 (1.66)	0.850	0.868	0.792 (1.37)	0.837	0.845
	QB	0.954	0.796 (1.37)	0.850	0.828	0.868 (1.84)	0.856	0.894	0.805 (1.45)	0.848	0.854
(200, 3)	PEBBLE	0.894	0.908 (0.96)	0.906	0.904	0.912 (1.39)	0.918	0.904	0.905 (1.14)	0.901	0.910
	Normal	0.900	0.898 (0.78)	0.894	0.920	0.904 (1.03)	0.932	0.868	0.896 (0.89)	0.913	0.893
	PRSB	0.900	0.900 (0.76)	0.898	0.910	0.898 (1.01)	0.924	0.872	0.891 (0.88)	0.909	0.890
	OSB	0.898	0.892 (0.78)	0.890	0.924	0.892 (1.01)	0.932	0.850	0.888 (0.88)	0.908	0.888
	QB	0.866	0.884 (0.75)	0.878	0.924	0.860 (0.93)	0.918	0.836	0.867 (0.82)	0.893	0.879
(200, 4)	PEBBLE	0.882	0.888 (3.22)	0.884	0.890	0.862 (4.89)	0.888	0.870	0.887 (3.35)	0.895	0.890
	Normal	0.912	0.916 (0.89)	0.892	0.904	0.892 (1.18)	0.914	0.882	0.897 (0.93)	0.898	0.897
	PRSB	0.880	0.894 (0.87)	0.882	0.904	0.870 (1.15)	0.912	0.868	0.878 (0.90)	0.892	0.888
	OSB	0.884	0.914 (0.89)	0.882	0.898	0.866 (1.16)	0.924	0.864	0.886 (0.92)	0.895	0.889
	QB	0.916	0.926 (0.92)	0.900	0.910	0.898 (1.23)	0.926	0.874	0.904 (0.96)	0.903	0.899
(200, 6)	PEBBLE	0.800	0.834 (3.57)	0.850	0.892	0.806 (5.14)	0.878	0.842	0.808 (4.24)	0.854	0.856
	Normal	0.812	0.832 (0.89)	0.850	0.864	0.892 (1.17)	0.922	0.858	0.854 (1.01)	0.866	0.871
	PRSB	0.794	0.826 (0.90)	0.854	0.868	0.858 (1.18)	0.914	0.850	0.836 (1.01)	0.859	0.867
	OSB	0.906	0.756 (0.91)	0.808	0.852	0.794 (1.18)	0.894	0.790	0.746 (1.02)	0.814	0.825
	QB	0.902	0.746 (0.88)	0.800	0.844	0.786 (1.15)	0.890	0.776	0.741 91.01)	0.813	0.822
(200, 8)	PEBBLE	0.788	0.812 (4.09)	0.836	0.892	0.748 (5.24)	0.882	0.778	0.768 (4.27)	0.842	0.835
	Normal	0.406	0.664 (0.94)	0.878	0.668	0.740 (1.19)	0.708	0.958	0.685 (1.00)	0.782	0.794
	PRSB	0.492	0.650 (0.97)	0.876	0.670	0.734 (1.22)	0.702	0.944	0.683 (1.02)	0.780	0.799
	OSB	0.854	0.472 (0.97)	0.798	0.570	0.564 (1.16)	0.632	0.838	0.490 (1.00)	0.680	0.715
	QB	0.848	0.478 (0.98)	0.800	0.574	0.544 (1.14)	0.638	0.842	0.484 (0.98)	0.683	0.714

Table 6. Scenario 6: Comparative performance study of PEBBLE with Normal approximation (Normal), Pearson Residual Resampling Bootstrap (PRRB), One-Step Bootstrap (OSB) and Quadratic Bootstrap (QB) for components of $\boldsymbol{\beta}$. Empirical coverages of 90% confidence region of $\|\boldsymbol{\beta}\|$ (column 1), upper, lower and middle confidence intervals (CIs) of the minimum absolute value of $\boldsymbol{\beta}$ (column 2,3,4), upper, lower and middle CIs of the maximum absolute value of the $\boldsymbol{\beta}$ (column 5,6,7), upper, lower and middle CIs of the all components of $\boldsymbol{\beta}$, on average (column 8,9,10) are presented, computed over 500 experiments. Average widths of the middle CIs are provided in parenthesis.

(n, p)	Methods	$\|\|\beta\|\|$	β_{min} middle (width)	β_{min} upper	β_{min} lower	β_{max} middle (width)	β_{max} upper	β_{max} lower	β avg. middle (width)	β avg. upper	β avg. lower
(30, 3)	PEBBLE	0.916	0.906 (3.85)	0.894	0.910	0.894 (4.70)	0.830	0.928	0.905 (4.42)	0.860	0.925
	Normal	0.904	0.958 (2.67)	0.958	0.910	0.932 (2.92)	0.998	0.870	0.944 (2.89)	0.981	0.890
	PRSB	0.954	0.860 (2.28)	0.868	0.862	0.880 (2.46)	0.922	0.856	0.881 (2.45)	0.908	0.854
	OSB	0.978	0.904 (2.66)	0.914	0.916	0.926 (2.70)	0.956	0.878	0.910 (2.75)	0.945	0.887
	QB	0.940	0.878 (2.55)	0.912	0.906	0.890 (2.35)	0.954	0.848	0.883 (2.55)	0.940	0.867
(50, 3)	PEBBLE	0.914	0.902 (2.88)	0.866	0.904	0.906 (3.44)	0.870	0.914	0.909 (2.93)	0.871	0.914
	Normal	0.920	0.926 (2.08)	0.914	0.874	0.944 (2.22)	0.974	0.898	0.939 (2.02)	0.940	0.893
	PRSB	0.880	0.846 (1.85)	0.850	0.852	0.894 (1.96)	0.892	0.874	0.874 (1.79)	0.869	0.873
	OSB	0.950	0.882 (2.09)	0.902	0.884	0.916 (2.11)	0.908	0.904	0.895 (1.99)	0.909	0.903
	QB	0.946	0.866 (1.95)	0.886	0.864	0.912 (2.11)	0.904	0.910	0.896 (1.96)	0.905	0.899
(50, 4)	PEBBLE	0.920	0.914 (3.37)	0.884	0.910	0.912 (4.20)	0.868	0.930	0.912 (3.27)	0.876	0.920
	Normal	0.932	0.932 (2.24)	0.928	0.902	0.948 (2.54)	0.982	0.902	0.936 (2.13)	0.946	0.897
	PRSB	0.900	0.870 (1.98)	0.890	0.870	0.916 (2.24)	0.922	0.892	0.880 (1.87)	0.890	0.875
	OSB	0.968	0.912 (2.30)	0.924	0.880	0.928 (2.46)	0.946	0.922	0.916 (2.14)	0.932	0.903
	QB	0.982	0.918 (2.36)	0.922	0.894	0.952 (2.61)	0.958	0.926	0.929 (2.24)	0.941	0.912
(100, 3)	PEBBLE	0.898	0.894 (1.65)	0.910	0.866	0.902 (2.58)	0.870	0.940	0.901 (2.00)	0.881	0.915
	Normal	0.908	0.906 (1.29)	0.908	0.900	0.926 (1.72)	0.962	0.884	0.910 (1.44)	0.931	0.887
	PRSB	0.874	0.872 (1.21)	0.882	0.880	0.898 (1.59)	0.912	0.878	0.877 (1.34)	0.897	0.875
	OSB	0.912	0.884 (1.30)	0.896	0.912	0.908 (1.67)	0.938	0.886	0.899 (1.43)	0.914	0.889
	QB	0.912	0.878 (1.25)	0.888	0.904	0.912 (1.73)	0.934	0.900	0.898 (1.43)	0.911	0.891
(100, 4)	PEBBLE	0.930	0.916 (2.20)	0.912	0.908	0.924 (3.06)	0.888	0.944	0.920 (2.34)	0.890	0.928
	Normal	0.938	0.920 (1.52)	0.888	0.910	0.960 (1.90)	0.974	0.910	0.931 (1.57)	0.933	0.899
	PRSB	0.880	0.840 (1.36)	0.826	0.880	0.912 (1.70)	0.940	0.880	0.870 (1.41)	0.886	0.866
	OSB	0.950	0.898 (1.56)	0.890	0.912	0.954 (1.80)	0.952	0.910	0.910 (1.57)	0.920	0.903
	QB	0.970	0.910 (1.68)	0.912	0.916	0.944 (1.73)	0.948	0.896	0.922 (1.63)	0.929	0.907
(100, 6)	PEBBLE	0.950	0.928 (2.77)	0.914	0.938	0.880 (3.59)	0.828	0.950	0.904 (2.95)	0.866	0.937
	Normal	0.864	0.934 (1.67)	0.932	0.912	0.908 (2.00)	0.998	0.830	0.920 (1.76)	0.965	0.876
	PRSB	0.840	0.846 (1.35)	0.854	0.864	0.846 (1.63)	0.970	0.760	0.842 (1.43)	0.909	0.818
	OSB	0.956	0.938 (1.72)	0.930	0.924	0.920 (1.94)	0.990	0.832	0.922 (1.77)	0.954	0.881
	QB	0.980	0.950 (1.82)	0.934	0.930	0.918 (1.93)	0.990	0.826	0.933 (1.87)	0.962	0.888
(200, 3)	PEBBLE	0.900	0.920 (1.18)	0.920	0.912	0.906 (1.65)	0.906	0.904	0.915 (1.35)	0.905	0.915
	Normal	0.926	0.916 (0.93)	0.910	0.904	0.918 (1.18)	0.906	0.904	0.918 (1.01)	0.911	0.909
	PRSB	0.888	0.900 (0.89)	0.896	0.890	0.896 (1.13)	0.888	0.894	0.899 (0.98)	0.894	0.901
	OSB	0.912	0.914 (0.93)	0.908	0.906	0.882 (1.16)	0.890	0.902	0.905 (1.00)	0.903	0.906
	QB	0.920	0.912 (0.93)	0.912	0.904	0.888 (1.18)	0.898	0.904	0.906 (1.01)	0.907	0.908
(200, 4)	PEBBLE	0.924	0.912 (1.53)	0.928	0.900	0.920 (2.11)	0.898	0.930	0.911 (1.59)	0.899	0.919
	Normal	0.914	0.908 (1.08)	0.892	0.918	0.936 (1.34)	0.928	0.904	0.915 (1.11)	0.909	0.907
	PRSB	0.874	0.874 (1.02)	0.870	0.884	0.906 (1.26)	0.886	0.896	0.880 (1.04)	0.877	0.889
	OSB	0.934	0.900 (1.09)	0.886	0.924	0.924 (1.32)	0.908	0.904	0.902 (1.10)	0.899	0.905
	QB	0.936	0.918 (1.14)	0.904	0.926	0.920 (1.30)	0.908	0.898	0.909 (1.11)	0.904	0.906
(200, 6)	PEBBLE	0.926	0.912 (1.61)	0.926	0.880	0.902 (2.68)	0.828	0.972	0.909 (2.14)	0.879	0.931
	Normal	0.882	0.924 (1.09)	0.902	0.922	0.912 (1.45)	0.994	0.852	0.924 (1.28)	0.952	0.891
	PRSB	0.870	0.862 (0.95)	0.864	0.882	0.884 (1.27)	0.978	0.782	0.873 (1.12)	0.914	0.842
	OSB	0.954	0.906 (1.12)	0.890	0.934	0.922 (1.41)	0.990	0.840	0.916 (1.27)	0.944	0.888
	QB	0.966	0.920 (1.19)	0.912	0.942	0.924 (1.41)	0.982	0.846	0.922 (1.30)	0.948	0.893
(200, 8)	PEBBLE	0.944	0.894 (2.47)	0.882	0.922	0.900 (3.23)	0.864	0.938	0.896 (2.67)	0.863	0.931
	Normal	0.828	0.930 (1.39)	0.952	0.872	0.904 (1.57)	0.990	0.826	0.906 (1.41)	0.968	0.839
	PRSB	0.622	0.772 (0.97)	0.874	0.768	0.764 (1.10)	0.946	0.682	0.748 (0.99)	0.905	0.719
	OSB	0.960	0.882 (1.32)	0.950	0.824	0.760 (1.39)	0.998	0.650	0.811 (1.31)	0.975	0.732
	QB	0.968	0.884 (1.35)	0.960	0.820	0.778 (1.45)	0.998	0.660	0.822 (1.34)	0.978	0.735

Table 7. Scenario 1: Comparative performance study of PEBBLE with Normal approximation (Normal), Pearson Residual Resampling Bootstrap (PRRB), One-Step Bootstrap (OSB) and Quadratic Bootstrap (QB) for the odds ratio $e^{\boldsymbol{x}_0'\boldsymbol{\beta}}$ with components of $\boldsymbol{x}_0$ being 1. Empirical coverages of 90% middle, upper and lower CIs of the four methods are presented, computed over 500 experiments. Average width of the middle CIs are provided in parenthesis.

(n, p)	Methods	middle (width)	upper	lower
(30, 3)	PEBBLE	0.906 (1.13)	0.926	0.890
	Normal	0.902 (0.78)	0.958	0.854
	PRSB	0.892 (0.81)	0.978	0.844
	OSB	0.902 (0.84)	0.986	0.856
	QB	0.922 (0.94)	0.994	0.870
(50, 3)	PEBBLE	0.892 (0.45)	0.922	0.890
	Normal	0.914 (0.36)	0.930	0.870
	PRSB	0.896 (0.36)	0.922	0.862
	OSB	0.912 (0.38)	0.940	0.880
	QB	0.932 (0.40)	0.966	0.888
(50, 4)	PEBBLE	0.896 (0.53)	0.856	0.930
	Normal	0.894 (0.33)	0.912	0.904
	PRSB	0.880 (0.32)	0.912	0.898
	OSB	0.918 (0.36)	0.932	0.886
	QB	0.916 (0.34)	0.928	0.868
(100, 3)	PEBBLE	0.868 (0.15)	0.894	0.892
	Normal	0.890 (0.12)	0.904	0.876
	PRSB	0.886 (0.12)	0.894	0.868
	OSB	0.882 (0.13)	0.908	0.868
	QB	0.866 (0.12)	0.896	0.846
(100, 4)	PEBBLE	0.894 (0.74)	0.918	0.874
	Normal	0.906 (0.51)	0.932	0.882
	PRSB	0.876 (0.51)	0.916	0.858
	OSB	0.910 (0.53)	0.946	0.878
	QB	0.934 (0.59)	0.972	0.896
(100, 6)	PEBBLE	0.910 (1.20)	0.924	0.886
	Normal	0.868 (0.71)	0.932	0.838
	PRSB	0.834 (0.70)	0.910	0.804
	OSB	0.804 (0.72)	0.906	0.806
	QB	0.820 (0.81)	0.930	0.820
(200, 3)	PEBBLE	0.920 (0.17)	0.914	0.914
	Normal	0.882 (0.14)	0.918	0.850
	PRSB	0.856 (0.14)	0.910	0.844
	OSB	0.850 (0.14)	0.920	0.838
	QB	0.802 (0.12)	0.904	0.808
(200, 4)	PEBBLE	0.904 (0.58)	0.916	0.896
	Normal	0.896 (0.43)	0.882	0.910
	PRSB	0.880 (0.43)	0.882	0.898
	OSB	0.898 (0.43)	0.890	0.904
	QB	0.934 (0.47)	0.926	0.916
(200, 6)	PEBBLE	0.922 (0.21)	0.920	0.886
	Normal	0.860 (0.17)	0.892	0.846
	PRSB	0.840 (0.17)	0.888	0.840
	OSB	0.770 (0.17)	0.854	0.798
	QB	0.774 (0.18)	0.860	0.796
(200, 8)	PEBBLE	0.892 (0.25)	0.844	0.936
	Normal	0.766 (0.16)	0.800	0.812
	PRSB	0.726 (0.17)	0.804	0.794
	OSB	0.544 (0.16)	0.742	0.718
	QB	0.528 (0.15)	0.740	0.698

Table 8. Scenario 2: Comparative performance study of PEBBLE with Normal approximation (Normal), Pearson Residual Resampling Bootstrap (PRRB), One-Step Bootstrap (OSB) and Quadratic Bootstrap (QB) for the odds ratio $e^{\boldsymbol{x}_0'\boldsymbol{\beta}}$ with components of $\boldsymbol{x}_0$ being 1. Empirical coverages of 90% middle, upper and lower CIs of the four methods are presented, computed over 500 experiments. Average width of the middle CIs are provided in parenthesis.

(n, p)	Methods	middle (width)	upper	lower
(30, 3)	PEBBLE	0.912 (0.53)	0.852	0.986
	Normal	0.898 (0.40)	0.900	0.928
	PRSB	0.880 (0.43)	0.918	0.854
	OSB	0.896 (0.47)	0.934	0.892
	QB	0.852 (0.38)	0.906	0.854
(50, 3)	PEBBLE	0.920 (0.38)	0.862	0.928
	Normal	0.882 (0.28)	0.884	0.900
	PRSB	0.850 (0.26)	0.872	0.852
	OSB	0.870 (0.30)	0.908	0.884
	QB	0.854 (0.28)	0.894	0.866
(50, 4)	PEBBLE	0.928 (0.89)	0.882	0.934
	Normal	0.902 (0.36)	0.910	0.904
	PRSB	0.882 (0.34)	0.930	0.868
	OSB	0.912 (0.40)	0.926	0.900
	QB	0.914 (0.41)	0.936	0.898
(100, 3)	PEBBLE	0.898 (0.33)	0.954	0.866
	Normal	0.904 (0.23)	0.946	0.866
	PRSB	0.884 (0.22)	0.946	0.854
	OSB	0.894 (0.23)	0.954	0.862
	QB	0.900 (0.23)	0.950	0.868
(100, 4)	PEBBLE	0.886 (0.51)	0.870	0.912
	Normal	0.910 (0.34)	0.890	0.908
	PRSB	0.866 (0.33)	0.876	0.870
	OSB	0.914 (0.36)	0.910	0.908
	QB	0.922 (0.38)	0.930	0.910
(100, 6)	PEBBLE	0.906 (0.73)	0.862	0.934
	Normal	0.906 (0.48)	0.902	0.902
	PRSB	0.848 (0.41)	0.872	0.840
	OSB	0.920 (0.54)	0.958	0.900
	QB	0.934 (0.55)	0.962	0.902
(200, 3)	PEBBLE	0.898 (0.26)	0.908	0.900
	Normal	0.896 (0.22)	0.906	0.900
	PRSB	0.890 (0.22)	0.908	0.886
	OSB	0.890 (0.22)	0.914	0.894
	QB	0.906 (0.23)	0.922	0.898
(200, 4)	PEBBLE	0.908 (0.22)	0.918	0.904
	Normal	0.892 (0.18)	0.916	0.886
	PRSB	0.888 (0.17)	0.914	0.856
	OSB	0.888 (0.18)	0.922	0.880
	QB	0.892 (0.19)	0.922	0.886
(200, 6)	PEBBLE	0.908 (0.25)	0.882	0.944
	Normal	0.902 (0.20)	0.894	0.910
	PRSB	0.858 (0.18)	0.856	0.878
	OSB	0.906 (0.21)	0.916	0.912
	QB	0.920 (0.22)	0.920	0.916
(200, 8)	PEBBLE	0.896 (0.31)	0.866	0.914
	Normal	0.878 (0.21)	0.874	0.904
	PRSB	0.772 (0.16)	0.768	0.844
	OSB	0.908 (0.22)	0.862	0.930
	QB	0.910 (0.23)	0.882	0.930

Table 9. Scenario 3: Comparative performance study of PEBBLE with Normal approximation (Normal), Pearson Residual Resampling Bootstrap (PRRB), One-Step Bootstrap (OSB) and Quadratic Bootstrap (QB) for the odds ratio $e^{\boldsymbol{x}_0'\boldsymbol{\beta}}$ with components of $\boldsymbol{x}_0$ being 1. Empirical coverages of 90% middle, upper and lower CIs of the four methods are presented, computed over 500 experiments. Average width of the middle CIs are provided in parenthesis.

(n, p)	Methods	middle (width)	upper	lower
(30, 3)	PEBBLE	0.904 (2.53)	0.910	0.898
	Normal	0.902 (0.78)	0.958	0.854
	PRSB	0.892 (0.81)	0.978	0.844
	OSB	0.902 (0.84)	0.986	0.856
	QB	0.922 (0.94)	0.994	0.870
(50, 3)	PEBBLE	0.914 (1.06)	0.926	0.886
	Normal	0.914 (0.36)	0.930	0.870
	PRSB	0.896 (0.36)	0.922	0.862
	OSB	0.912 (0.38)	0.940	0.880
	QB	0.932 (0.40)	0.966	0.888
(50, 4)	PEBBLE	0.864 (0.97)	0.858	0.898
	Normal	0.894 (0.33)	0.912	0.904
	PRSB	0.880 (0.32)	0.912	0.898
	OSB	0.918 (0.36)	0.932	0.886
	QB	0.916 (0.34)	0.928	0.868
(100, 3)	PEBBLE	0.884 (0.41)	0.888	0.898
	Normal	0.890 (0.12)	0.904	0.876
	PRSB	0.886 (0.12)	0.894	0.868
	OSB	0.882 (0.13)	0.908	0.868
	QB	0.866 (0.12)	0.896	0.846
(100, 4)	PEBBLE	0.862 (2.07)	0.88	0.862
	Normal	0.906 (0.51)	0.932	0.882
	PRSB	0.876 (0.51)	0.916	0.858
	OSB	0.910 (0.53)	0.946	0.878
	QB	0.934 (0.59)	0.972	0.896
(100, 6)	PEBBLE	0.816 (2.86)	0.872	0.834
	Normal	0.868 (0.71)	0.932	0.838
	PRSB	0.834 (0.70)	0.910	0.804
	OSB	0.804 (0.72)	0.906	0.806
	QB	0.820 (0.81)	0.930	0.820
(200, 3)	PEBBLE	0.898 (0.50)	0.886	0.910
	Normal	0.882 (0.14)	0.918	0.850
	PRSB	0.856 (0.14)	0.910	0.844
	OSB	0.850 (0.14)	0.920	0.838
	QB	0.802 (0.12)	0.904	0.808
(200, 4)	PEBBLE	0.876 (1.73)	0.892	0.872
	Normal	0.896 (0.43)	0.882	0.910
	PRSB	0.880 (0.43)	0.882	0.898
	OSB	0.898 (0.43)	0.890	0.904
	QB	0.934 (0.47)	0.926	0.916
(200, 6)	PEBBLE	0.902 (0.47)	0.922	0.910
	Normal	0.860 (0.17)	0.892	0.846
	PRSB	0.840 (0.17)	0.888	0.840
	OSB	0.770 (0.17)	0.854	0.798
	QB	0.774 (0.18)	0.860	0.796
(200, 8)	PEBBLE	0.862 (0.54)	0.852	0.906
	Normal	0.766 (0.16)	0.80	0.812
	PRSB	0.726 (0.17)	0.804	0.794
	OSB	0.544 (0.16)	0.742	0.718
	QB	0.528 (0.15)	0.740	0.698

Table 10. Scenario 4: Comparative performance study of PEBBLE with Normal approximation (Normal), Pearson Residual Resampling Bootstrap (PRRB), One-Step Bootstrap (OSB) and Quadratic Bootstrap (QB) for the odds ratio $e^{\boldsymbol{x}_0'\boldsymbol{\beta}}$ with components of $\boldsymbol{x}_0$ being 1. Empirical coverages of 90% middle, upper and lower CIs of the four methods are presented, computed over 500 experiments. Average width of the middle CIs are provided in parenthesis.

(n, p)	Methods	middle (width)	upper	lower
(30, 3)	PEBBLE	0.910 (1.01)	0.858	0.976
	Normal	0.898 (0.40)	0.900	0.928
	PRSB	0.880 (0.43)	0.918	0.854
	OSB	0.896 (0.47)	0.934	0.892
	QB	0.852 (0.38)	0.906	0.854
(50, 3)	PEBBLE	0.934 (0.72)	0.900	0.924
	Normal	0.882 (0.28)	0.884	0.900
	PRSB	0.850 (0.26)	0.872	0.852
	OSB	0.870 (0.30)	0.908	0.884
	QB	0.854 (0.28)	0.894	0.866
(50, 4)	PEBBLE	0.912 (0.67)	0.886	0.916
	Normal	0.902 (0.36)	0.910	0.904
	PRSB	0.882 (0.34)	0.930	0.868
	OSB	0.912 (0.40)	0.926	0.900
	QB	0.914 (0.41)	0.936	0.898
(100, 3)	PEBBLE	0.906 (0.92)	0.916	0.882
	Normal	0.904 (0.23)	0.946	0.866
	PRSB	0.884 (0.22)	0.946	0.854
	OSB	0.894 (0.23)	0.954	0.862
	QB	0.900 (0.23)	0.950	0.868
(100, 4)	PEBBLE	0.848 (1.45)	0.860	0.892
	Normal	0.910 (0.34)	0.890	0.908
	PRSB	0.866 (0.33)	0.876	0.870
	OSB	0.914 (0.36)	0.910	0.908
	QB	0.922 (0.38)	0.930	0.910
(100, 6)	PEBBLE	0.822 (2.04)	0.832	0.888
	Normal	0.906 (0.48)	0.902	0.902
	PRSB	0.848 (0.41)	0.872	0.840
	OSB	0.920 (0.54)	0.958	0.900
	QB	0.934 (0.55)	0.962	0.902
(200, 3)	PEBBLE	0.900 (0.71)	0.894	0.908
	Normal	0.896 (0.22)	0.906	0.900
	PRSB	0.890 (0.22)	0.908	0.886
	OSB	0.890 (0.22)	0.914	0.894
	QB	0.906 (0.23)	0.922	0.898
(200, 4)	PEBBLE	0.902 (0.61)	0.908	0.882
	Normal	0.892 (0.18)	0.916	0.886
	PRSB	0.888 (0.17)	0.914	0.856
	OSB	0.888 (0.18)	0.922	0.880
	QB	0.892 (0.19)	0.922	0.886
(200, 6)	PEBBLE	0.886 (0.73)	0.892	0.902
	Normal	0.902 (0.20)	0.894	0.910
	PRSB	0.858 (0.18)	0.856	0.878
	OSB	0.906 (0.21)	0.916	0.912
	QB	0.920 (0.22)	0.920	0.916
(200, 8)	PEBBLE	0.742 (0.69)	0.804	0.836
	Normal	0.848 (0.15)	0.822	0.944
	PRSB	0.754 (0.12)	0.742	0.876
	OSB	0.832 (0.15)	0.762	0.972
	QB	0.832 (0.15)	0.772	0.966

Table 11. Scenario 5: Comparative performance study of PEBBLE with Normal approximation (Normal), Pearson Residual Resampling Bootstrap (PRRB), One-Step Bootstrap (OSB) and Quadratic Bootstrap (QB) for the odds ratio $e^{\boldsymbol{x}_0'\boldsymbol{\beta}}$ with components of $\boldsymbol{x}_0$ being 1. Empirical coverages of 90% middle, upper and lower CIs of the four methods are presented, computed over 500 experiments. Average width of the middle CIs are provided in parenthesis.

(n, p)	Methods	middle (width)	upper	lower
(30, 3)	PEBBLE	0.906 (1.23)	0.926	0.892
	Normal	0.902 (0.78)	0.958	0.854
	PRSB	0.892 (0.81)	0.978	0.844
	OSB	0.902 (0.84)	0.986	0.856
	QB	0.922 (0.94)	0.994	0.870
(50, 3)	PEBBLE	0.904 (0.49)	0.922	0.892
	Normal	0.914 (0.36)	0.93	0.87
	PRSB	0.896 (0.36)	0.922	0.862
	OSB	0.912 (0.38)	0.94	0.88
	QB	0.932 (0.40)	0.966	0.888
(50, 4)	PEBBLE	0.908 (0.60)	0.856	0.936
	Normal	0.894 (0.33)	0.912	0.904
	PRSB	0.880 (0.32)	0.912	0.898
	OSB	0.918 (0.36)	0.932	0.886
	QB	0.916 (0.34)	0.928	0.868
(100, 3)	PEBBLE	0.868 (0.17)	0.886	0.902
	Normal	0.890 (0.12)	0.904	0.876
	PRSB	0.886 (0.12)	0.894	0.868
	OSB	0.882 (0.13)	0.908	0.868
	QB	0.866 (0.12)	0.896	0.846
(100, 4)	PEBBLE	0.888 (0.82)	0.918	0.870
	Normal	0.906 (0.51)	0.932	0.882
	PRSB	0.876 (0.51)	0.916	0.858
	OSB	0.910 (0.53)	0.946	0.878
	QB	0.934 (0.59)	0.972	0.896
(100, 6)	PEBBLE	0.908 (1.29)	0.920	0.884
	Normal	0.868 (0.71)	0.932	0.838
	PRSB	0.834 (0.70)	0.910	0.804
	OSB	0.804 (0.72)	0.906	0.806
	QB	0.820 (0.81)	0.930	0.820
(200, 3)	PEBBLE	0.922 (0.19)	0.908	0.922
	Normal	0.882 (0.14)	0.918	0.850
	PRSB	0.856 (0.14)	0.910	0.844
	OSB	0.850 (0.14)	0.920	0.838
	QB	0.802 (0.12)	0.904	0.808
(200, 4)	PEBBLE	0.904 (0.64)	0.92	0.902
	Normal	0.896 (0.43)	0.882	0.910
	PRSB	0.880 (0.43)	0.882	0.898
	OSB	0.898 (0.43)	0.890	0.904
	QB	0.934 (0.47)	0.926	0.916
(200, 6)	PEBBLE	0.924 (0.22)	0.922	0.888
	Normal	0.860 (0.17)	0.892	0.846
	PRSB	0.840 (0.17)	0.888	0.840
	OSB	0.770 (0.17)	0.854	0.798
	QB	0.774 (0.18)	0.860	0.796
(200, 8)	PEBBLE	0.898 (0.26)	0.846	0.948
	Normal	0.766 (0.16)	0.800	0.812
	PRSB	0.726 (0.17)	0.804	0.794
	OSB	0.544 (0.16)	0.742	0.718
	QB	0.528 (0.15)	0.740	0.698

Table 12. Scenario 6: Comparative performance study of PEBBLE with Normal approximation (Normal), Pearson Residual Resampling Bootstrap (PRRB), One-Step Bootstrap (OSB) and Quadratic Bootstrap (QB) for the odds ratio $e^{\boldsymbol{x}_0'\boldsymbol{\beta}}$ with components of $\boldsymbol{x}_0$ being 1. Empirical coverages of 90% middle, upper and lower CIs of the four methods are presented, computed over 500 experiments. Average width of the middle CIs are provided in parenthesis.

(n, p)	Methods	middle (width)	upper	lower
(30, 3)	PEBBLE	0.908 (0.98)	0.85	0.986
	Normal	0.898 (0.40)	0.900	0.928
	PRSB	0.880 (0.43)	0.918	0.854
	OSB	0.896 (0.47)	0.934	0.892
	QB	0.852 (0.38)	0.906	0.854
(50, 3)	PEBBLE	0.926 (0.75)	0.866	0.932
	Normal	0.882 (0.28)	0.884	0.900
	PRSB	0.850 (0.26)	0.872	0.852
	OSB	0.870 (0.30)	0.908	0.884
	QB	0.854 (0.28)	0.894	0.866
(50, 4)	PEBBLE	0.928 (1.00)	0.886	0.936
	Normal	0.902 (0.36)	0.910	0.904
	PRSB	0.882 (0.34)	0.93	0.868
	OSB	0.912 (0.40)	0.926	0.900
	QB	0.914 (0.41)	0.936	0.898
(100, 3)	PEBBLE	0.902 (0.37)	0.956	0.864
	Normal	0.904 (0.23)	0.946	0.866
	PRSB	0.884 (0.22)	0.946	0.854
	OSB	0.894 (0.23)	0.954	0.862
	QB	0.900 (0.23)	0.950	0.868
(100, 4)	PEBBLE	0.896 (0.57)	0.876	0.916
	Normal	0.910 (0.34)	0.890	0.908
	PRSB	0.866 (0.33)	0.876	0.870
	OSB	0.914 (0.36)	0.910	0.908
	QB	0.922 (0.38)	0.930	0.910
(100, 6)	PEBBLE	0.906 (0.80)	0.856	0.932
	Normal	0.906 (0.48)	0.902	0.902
	PRSB	0.848 (0.41)	0.872	0.84
	OSB	0.920 (0.54)	0.958	0.90
	QB	0.934 (0.55)	0.962	0.902
(200, 3)	PEBBLE	0.896 (0.29)	0.906	0.896
	Normal	0.896 (0.22)	0.906	0.900
	PRSB	0.890 (0.22)	0.908	0.886
	OSB	0.890 (0.22)	0.914	0.894
	QB	0.906 (0.23)	0.922	0.898
(200, 4)	PEBBLE	0.912 (0.23)	0.922	0.908
	Normal	0.892 (0.18)	0.916	0.886
	PRSB	0.888 (0.17)	0.914	0.856
	OSB	0.888 (0.18)	0.922	0.880
	QB	0.892 (0.19)	0.922	0.886
(200, 6)	PEBBLE	0.916 (0.27)	0.884	0.948
	Normal	0.902 (0.20)	0.894	0.910
	PRSB	0.858 (0.18)	0.856	0.878
	OSB	0.906 (0.21)	0.916	0.912
	QB	0.920 (0.22)	0.920	0.916
(200, 8)	PEBBLE	0.898 (0.33)	0.870	0.924
	Normal	0.878 (0.21)	0.874	0.904
	PRSB	0.772 (0.16)	0.768	0.844
	OSB	0.908 (0.22)	0.862	0.930
	QB	0.910 (0.23)	0.882	0.930

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