Homework 1

January 26, 2022

0.0.1 Programming Questions

[19]:
```python
import numpy as np

# Generate Random Data
X = np.random.normal(size=(100,5))
y = X.dot(np.array([1, 0, 0, 1, -1])) + np.random.normal(size=(100,))
```

0.0.2 Multiple Linear Regression without Intercept

We will be using the formula $B = (X^T X)^{-1} X^T y$ on our mock dataset with 5 predictors. We can then see below that our model would be:

$y = 1.0909X_1 - 0.01868X_2 + 0.0025X_3 + 1.03890X_4 - 1.2044X_5$

[30]:
```python
# Coefficients without an intercept
coefficients = np.linalg.inv(X.transpose().dot(X)).dot(X.transpose()).dot(y)
```

[30]:
```
array([ 1.09091575, -0.01868566,  0.00257115,  1.03890498, -1.20442473])
```

0.0.3 Multiple Linear Regression With Intercept

We will repeat the same steps as above but this time including an intercept in our model

$y = -0.1573 + 1.0675X_1 - 0.0227X_2 + 0.03402X_3 + 1.01877X_4 - 1.19504X_5$

[59]:
```python
X_with_bias = np.hstack((np.ones(shape=(100,1)), X))
coefficients_with_int = np.linalg.inv(X_with_bias.transpose().dot(X_with_bias)).
 ↪dot(X_with_bias.transpose()).dot(y)

coefficients_with_int
```

[59]:
```
array([-0.15735914,  1.06752257, -0.0227478 ,  0.03402595,  1.01877311,
       -1.19504765])
```

Question 1

1.1a: $f(x) = e^x$

$$f'(x) = e^x$$

1.1b: $f(x) = \log(1+x)$ (assume base e)

$$f'(x) = \frac{1}{1+x}$$

1.1c: $f(x) = \log(1+e^x)$

$$f'(x) = \frac{1}{1+e^x} \cdot e^x$$

1.2: First 3 terms of taylor expansion at $x=1$

$$f(x) = f(1) + \frac{f'(1)(x-1)^2}{2!} + \frac{f''(1)(x-1)^3}{3!}$$

1.3: For infinite series $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$, find range of α s.t. it converges.

This is a p series, so it will converge iff

$$\alpha > 1$$

Question 2

2.1.1 : what is the eigendecomp. of symmetric matrix $A_{n \times n}$

$$A = U \Lambda U^T$$

- Λ is a diagonal matrix of our eigenvalues
- Columns of U are our eigenvectors.
- we can do transpose instead of inverse on U^T b/c our matrix is symmetric, i.e. $U^{-1} = U^T$

$$A^{-1/2} = (U \Lambda U^T)^{-1/2} = U \Lambda^{-1/2} U^T$$

where $\Lambda^{-1/2} = \text{diag}(\lambda_1^{-1/2}, \ldots, \lambda_n^{-1/2})$

2.2 What is a symmetric positive definite matrix $A_{n \times n}$

This is when a matrix is symmetric and its eigenvalues are all positive

2.3a F

2.3b T

2.3c T

2.3d F

2.3e F

2.3f F

Question 3

3.1a the $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$

$$E(\bar{x}) = E\left(\frac{1}{n}\sum_i x_i\right) = \frac{1}{n} E\left(\sum_i x_i\right)$$

$$= \frac{1}{n} \cdot n\mu = \mu$$

$\therefore$ unbiased.

3.1b $Var(x) = E(\bar{x}_n^2) - [E(\bar{x}_n)]^2$

$$E(\bar{x}_n^2) = Var(x) - E(x_n)^2$$

$$= \sigma^2 - \mu^2$$

3.1c give an unbiased estimate of σ^2

$$\frac{\sum_i (x_i - \bar{x})^2}{n-1}$$

Sample variance is an unbiased estimator of the population variance.

3.1d A consistent estimator is given when if we have parameter $\hat{\theta}$, if it converges in probability to θ, then it is consistent

$$\hat{\theta} = \bar{X} \xrightarrow{P} \mu$$

by weak law of large numbers.

$\therefore$ it is consistent.

3.2a) $L(\beta) = \|y - X\beta\| = (y - X\beta)^T (y - X\beta)$

3.2b $\beta = (X^T X)^{-1} X^T y$