7.2 Eigenvalues and Eigenvectors (cont'd)

Example. Determine l_2 induced norm of $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$

Solution

$$A^t A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 6 & 4 \\ -1 & 4 & 5 \end{bmatrix}$$

$$\text{Solve } \det(A^t A - \lambda I) = 0$$

$$0 = -\lambda(\lambda^2 - 14\lambda + 42)$$

$$\text{Then } \lambda = 0, \lambda = 7 \pm \sqrt{7}$$

$$||A||_2 = [\rho(A^t A)]^{1/2} = \sqrt{\max(0, 7+\sqrt{7}, 7-\sqrt{7})} = \sqrt{7+\sqrt{7}}$$

Convergent Matrices

Definition. An $n \times n$ matrix A is **convergent** if $\lim_{k\to\infty}(A^k)_{ij} = 0$ for each $i = 1,2,,\ldots n$ and $j = 1,2,,\ldots n$.

Example. Show that $A = \begin{bmatrix} 1/2 & 0 \\ 1/4 & 1/2 \end{bmatrix}$ is a convergent matrix.

Theorem 7.17 The following statements are equivalent.

(i) A is **convergent** matrix.
(ii) $\lim_{n\to\infty} ||A^n|| = 0$ for some natural norm.
(iii) $\lim_{n\to\infty} ||A^n|| = 0$ for all natural norm.
(iv) $\rho(A) < 1$
(v) $\lim_{n\to\infty} A^n \boldsymbol{x} = \boldsymbol{0}$ for every $\boldsymbol{x}$.

7.1 The Jacobi and Gauss-Seidel Iterative Methods

The Jacobi Method

***Two assumptions made on Jacobi Method*:**

1. The system given by

$$\begin{gathered} a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots a_{nn}x_n = b_n \end{gathered}$$

Has a unique solution.

2. The coefficient matrix A has no zeros on its main diagonal, namely, $a_{11}, a_{22}, \dots, a_{nn}$ are nonzeros.

<u>Main idea of Jacobi</u>

To begin, solve the 1st equation for x_1, the 2nd equation for x_2 and so on to obtain the rewritten equations:

$$\begin{gathered} x_1 = \frac{1}{a_{11}}(b_1 - a_{12}x_2 - a_{13}x_3 - \cdots a_{1n}x_n) \\ x_2 = \frac{1}{a_{22}}(b_2 - a_{21}x_1 - a_{23}x_3 - \cdots a_{2n}x_n) \\ \vdots \\ x_n = \frac{1}{a_{nn}}(b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots a_{n,n-1}x_{n-1}) \end{gathered}$$

Then make an initial guess of the solution $\boldsymbol{x}^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, \dots x_n^{(0)})$. Substitute these values into the right hand side the of the rewritten equations to obtain the *first approximation*, $\left(x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, \dots x_n^{(1)}\right)$.

This accomplishes one **iteration**.

In the same way, the *second approximation* $\left(x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, \dots x_n^{(2)}\right)$ is computed by substituting the first approximation's x-vales into the right hand side of the rewritten equations.

By repeated iterations, we form a sequence of approximations $\boldsymbol{x}^{(k)} = \left(x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, \dots x_n^{(k)}\right)^t, \quad k = 1,2,3, \dots$

The Jacobi Method. For each $k \geq 1$, generate the components $x_i^{(k)}$ of $\boldsymbol{x}^{(\boldsymbol{k})}$ from $\boldsymbol{x}^{(\boldsymbol{k-1})}$ by

$$x_i^{(k)} = \frac{1}{a_{ii}}\left[\sum_{\substack{j=1, \\ j\neq i}}^{n} (-a_{ij}x_j^{(k-1)}) + b_i\right], \qquad \text{for } i = 1,2,,\ldots n$$

Example. Apply the Jacobi method to solve

$$\begin{aligned} 5x_1 - 2x_2 + 3x_n &= -1 \\ -3x_1 + 9x_2 + x_n &= 2 \\ 2x_1 - x_2 - 7x_n &= 3 \end{aligned}$$

Continue iterations until two successive approximations are identical when rounded to three significant digits.

Solution To begin, rewrite the system

$$\begin{aligned} x_1 &= \frac{-1}{5} + \frac{2}{5}x_2 - \frac{3}{5}x_3 \\ x_2 &= \frac{2}{9} + \frac{3}{9}x_1 - \frac{1}{9}x_3 \\ x_3 &= -\frac{3}{7} + \frac{2}{7}x_1 - \frac{1}{7}x_2 \end{aligned}$$

Choose the initial guess $x_1 = 0, x_2 = 0, x_3 = 0$

The first approximation is

$$\begin{aligned} x_1^{(1)} &= \frac{-1}{5} + \frac{2}{5}(0) - \frac{3}{5}(0) = -0.200 \\ x_2^{(1)} &= \frac{2}{9} + \frac{3}{9}(0) - \frac{1}{9}(0) = 0.222 \\ x_3^{(1)} &= -\frac{3}{7} + \frac{2}{7}(0) - \frac{1}{7}(0) = -0.429 \end{aligned}$$

Continue iteration, we obtain

n	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$x_1^{(k)}$	0.000	-0.200	0.146	0.192			
$x_2^{(k)}$	0.000	0.222	0.203	0.328			
$x_2^{(k)}$	0.000	-0.429	-0.517	-0.416			

The Jacobi Method in Matrix Form

Consider to solve an $n \times n$ size system of linear equations $A\boldsymbol{x} = \boldsymbol{b}$ with $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ for $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$.

We split A into

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} - \begin{bmatrix} 0 & \cdots & 0 & 0 \\ -a_{21} & \cdots & 0 & 0 \\ \vdots & & \ddots & \vdots \\ -a_{n1} & \cdots & -a_{n,n-1} & 0 \end{bmatrix} - \begin{bmatrix} 0 & -a_{12} & \cdots & -a_{1n} \\ 0 & 0 & & \vdots \\ \vdots & \vdots & \ddots & -a_{n-1,n} \\ 0 & 0 & \cdots & 0 \end{bmatrix} = D - L - U$$

$A\boldsymbol{x} = \boldsymbol{b}$ is transformed into $(D - L - U)\boldsymbol{x} = \boldsymbol{b}$

$$D\boldsymbol{x} = (L + U)\boldsymbol{x} + \boldsymbol{b}$$

Assume D^{-1} exists and $D^{-1} = \begin{bmatrix} \frac{1}{a_{11}} & 0 & \cdots & 0 \\ 0 & \frac{1}{a_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{a_{nn}} \end{bmatrix}$

Then

$$\boldsymbol{x} = D^{-1}(L + U)\boldsymbol{x} + D^{-1}\boldsymbol{b}$$

The matrix form of Jacobi iterative method is

$$\boldsymbol{x}^{(k)} = D^{-1}(L+U)\boldsymbol{x}^{(k-1)} + D^{-1}\boldsymbol{b} \qquad\qquad k = 1,2,3,\ldots$$

Define $T = D^{-1}(L+U)$ and $\boldsymbol{c} = D^{-1}\boldsymbol{b}$, Jacobi iteration method can also be written as

$$\boldsymbol{x}^{(k)} = T\boldsymbol{x}^{(k-1)} + \boldsymbol{c} \qquad\qquad k = 1,2,3,\ldots$$

<u>Numerical Algorithm of Jacobi Method</u>

Input: $A = [a_{ij}]$, $\boldsymbol{b}, \boldsymbol{XO} = \boldsymbol{x}^{(0)}$, tolerance TOL, maximum number of iterations N.

Step 1 Set $k = 1$

Step 2 while ($k \leq N$) do Steps 3-6

Step 3 For for $i = 1,2,,\ldots n$

$$x_i = \frac{1}{a_{ii}}\left[\sum_{\substack{j=1,\\ j\neq i}}^{n}(-a_{ij}\boldsymbol{XO}_j) + b_i\right],$$

Step 4 If $||\boldsymbol{x} - \boldsymbol{XO}|| < TOL$, then OUTPUT $(x_1, x_2, x_3, \ldots x_n)$;

STOP.

Step 5 Set $k = k + 1$.

Step 6 For for $i = 1,2,,\ldots n$

Set $\boldsymbol{XO}_i = x_i$.

Step 7 OUTPUT $(x_1, x_2, x_3, \ldots x_n)$;

STOP.

Another stopping criterion in Step 4: $\frac{||\boldsymbol{x}^{(k)} - \boldsymbol{x}^{(k-1)}||}{||\boldsymbol{x}^{(k)}||}$

The Gauss-Seidel Method

Main idea of Gauss-Seidel

With the Jacobi method, the values of $x_i^{(k)}$ obtained in the kth iteration remain unchanged until the entire $(k+1)$th iteration has been calculated. With the Gauss-Seidel method, we use the new values $x_i^{(k+1)}$ as soon as they are known. For example, once we have computed $x_1^{(k+1)}$ from the first equation, its value is then used in the second equation to obtain the new $x_2^{(k+1)}$, and so on.

Example. Derive iteration equations for the Jacobi method and Gauss-Seidel method to solve

$$5x_1 - 2x_2 + 3x_n = -1$$
$$-3x_1 + 9x_2 + x_n = 2$$
$$2x_1 - x_2 - 7x_n = 3$$