

Power Series Method

$$\sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

↑
Power Series about $x=0$.

Series :-

$$1) e^x : 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{m=0}^{\infty} \frac{x^m}{m!}$$

Convergence of Power Series :

1) Convergence on a point

2) Convergence on whole plane

3) Convergence on a circular plane with Radius R ($|x| < R$)
OR Radius of convergence.

Radius of convergence :

Cauchy's theorem on limits :

$$i) \frac{1}{R} = \lim_{n \rightarrow \infty} |a_n|^{1/n} \quad ii) \frac{1}{R} = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$2) \cos x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$3) \sin x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$1. \quad y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots, \quad |x| < R$$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + \dots, \quad |x| < R$$

$$a_1 + 2a_2 x + 3a_3 x^2 + \dots = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots, \quad |x| < R$$

$$\Rightarrow a_1 = a_0 \quad ; \text{coeff. of } x^0$$

$$2a_2 = a_1 \quad ; \text{coeff. of } x^1$$

$$\Rightarrow a_2 = \frac{a_1}{2} = \frac{a_0}{2} = \frac{a_0}{2!}$$

$$\text{coeff. of } x^2 : \quad 3a_3 = a_2$$

$$\Rightarrow a_3 = \frac{a_2}{3} = \frac{a_0}{3 \cdot 2} = \frac{a_0}{3!}$$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$$= a_0 + a_0 x + \frac{a_0}{2!} x^2 + \frac{a_0}{3!} x^3 + \dots$$

$$= a_0 \left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right]$$

$$= a_0 e^x$$

$$2. \quad \frac{d^2 y}{dx^2} + y = 0$$

$$\cancel{y = A \cos x + B \sin x}$$

$$y = \cancel{A \cos x + B \sin x}$$

$$y = -A \sin x + B \cos x = C \cos(x + \phi)$$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n'=0}^{\infty} (n'+2)(n'+1) a_{n'+2} x^{n'} + \sum_{n=0}^{\infty} a_n x^n = 0 \quad [n-2 = n']$$

$$\Rightarrow \sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} + a_n] x^n = 0 \quad |x| < R$$

$$\Rightarrow (n+2)(n+1) a_{n+2} + a_n = 0 \quad \forall n = 0, 1, 2, \dots$$

\uparrow
Recursion formula

For $n = 0$

$$2 \cdot 1 a_2 + a_0 = 0 \Rightarrow a_2 = -\frac{a_0}{2!}$$

For $n = 1$

$$3 \cdot 2 \cdot a_3 + a_1 = 0 \Rightarrow a_3 = -\frac{a_1}{3!}$$

For $n = 2$

$$4 \cdot 3 \cdot a_4 + a_2 = 0 \Rightarrow a_4 = -\frac{a_2}{4 \cdot 3} = \frac{a_0}{4!}$$

For $n = 3$

$$5 \cdot 4 \cdot a_5 + a_3 = 0 \Rightarrow a_5 = -\frac{a_1}{5!}$$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \dots$$

$$= a_0 + a_1 x - \frac{a_0}{2!} x^2 - \frac{a_1}{3!} x^3 + \frac{a_0}{4!} x^4 + \frac{a_1}{5!} x^5 + \dots$$

$$= a_0 \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \right] + a_1 \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right]$$

$$= a_0 \cos x + a_1 \sin x.$$

3. Find radius of convergence of : (converges for all)

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

4. Find radius of convergence of :

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$R = \lim_{n \rightarrow \infty} \left| \frac{a_{2n}}{a_{2n+2}} \right|$$

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad p \text{ os about } x = x_0$$

$$5. 4y'' - 4y' + y = 0$$

$$y(1) = 0, \quad y'(1) = 1$$

$$y = \sum_{n=0}^{\infty} a_n (x-1)^n$$

$$y' = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}$$

$$4 \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} - 4 \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} + \sum_{n=0}^{\infty} a_n (x-1)^n = 0.$$

$$\cancel{4 \sum_{n=0}^{\infty} a_n (n+1) a_{n+1} (x-1)^n} - 48$$

$$n-2 = m \\ n-1 = p$$

$$4 \sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} (x-1)^m - 4 \sum_{p=0}^{\infty} (p+1) a_{p+1}$$

$$(x-1)^p + \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

~~$$4 \cdot 2 \cdot 1 a_2 (x-1)^0 - 4 \cdot 1 a_1 (x-1)^0 +$$~~

$$a_0 (x-1)^0 = 0$$

$$8a_2 - 4a_1 + a_0 = 0$$

For $n=0$,

~~$$4 \cdot 1 a_0 (x-1)^0 - 4 \cdot 2 \cdot a_2 (x-1)^2 + a_1 (x-1)^1 = 0$$~~

$$4(n+2)(n+1)a_{n+2} - 4(n+1)a_{n+1} + a_n = 0, \quad n=0, 1, 2,$$

$n=0$

$$4 \cdot 2 \cdot 1 \cdot a_2 - 4 \cdot a_1 + a_0 = 0$$

$$\Rightarrow 8a_2 = a_0 + 4a_1$$

$$\therefore a_2 = \frac{1}{2}, \quad a_3 = \frac{1}{8}, \quad a_4 = \frac{1}{48}$$

$$y = \sum_{n=0}^{\infty} a_n (x-1)^n, \quad |x-1| < R$$

$$y = a_0 + a_1 (x-1) + a_2 (x-1)^2 + a_3 (x-1)^3 + \dots$$

$$y = 0 \text{ at } x=1$$

$$\boxed{a_0 = 0, a_1 = 1}$$

$$y = (x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{8}(x-1)^3 + \frac{1}{48}(x-1)^4 + \dots$$

$$6. xy' = y$$

separable

$$x \frac{dy}{dx} = y$$

$$\Rightarrow \frac{dy}{y} = \frac{dx}{x}$$

$$\Rightarrow \ln y = \ln x + c$$

$$\Rightarrow y =$$

$$3. 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{L^n}$$

$$a_n = \frac{1}{L^n} \quad a_{n+1} = \frac{1}{L^{n+1}}$$

$$\frac{1}{R} = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{L^n}{L^{n+1}} \right|$$

$$\frac{1}{R} = \lim_{n \rightarrow \infty} \left| \frac{L^n}{(n+1)L^n} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{1}{n+1} \right|$$

$$\frac{1}{R} = 0$$

$$R = \frac{1}{0} = \infty \quad [R = \infty]$$

$$4. \quad 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$a_{2n} = \frac{(-1)^n}{(2^n)}$$

$$a_{2n+1} = \frac{(-1)^{n+1}}{(2n+2)}$$

$$\frac{1}{R} = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1}}{(2n+2)} \times \frac{(2n)}{(-1)^n} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{(-1)^n \cdot (-1)}{(2n+2)(2n+1)(2n)} \times \frac{(2n)}{(-1)^n} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1}}{(2n+2)(2n+1)} \right| = \lim_{n \rightarrow \infty} \left| \frac{1}{2(n+1)(n+2)} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{(-1)}{\left(\frac{1}{n} + \frac{2}{n}\right)\left(\frac{2}{n} + \frac{1}{n}\right)} \right| = \frac{1}{4}$$

$$R = 4$$

$$1. \quad y'' + y' + x^2 y = 0$$

$$y(0) = 1 \quad y'(0) = 2$$

$$y = 1 + 2x - \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{6} + \dots$$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$2. \quad x^2 y' = y$$

$$y = \left(\sum_{n=0}^{\infty} a_n x^n \right), \quad |x| < R$$

$$\Rightarrow y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$3. y' = 1 + y^2$$

$$y = \sum_{n=0}^{\infty} a_n x^n \quad y' = \sum_{n=0}^{\infty} n a_n x^{n-1} \quad \begin{matrix} n-1=n' \\ n=n'+1 \end{matrix}$$

$$= \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$= 1 + \left(\sum_{n=0}^{\infty} a_n x^n \right)^2 = 1 + (a_0 + a_1 x + a_2 x^2 + \dots)^2$$

$$= 1 + a_0^2 + a_1^2 x^2 + 2a_0 a_1 x^2 + \dots$$

$$a_1 + 2a_2 x + 3a_3 x^2 + \dots = 1 + a_0^2 + a_1^2 x^2 + 2a_0 a_1 x^2$$

$$a_1 = 1 + a_1^2$$

$$2a_2 = a_1^2 + 2a_1 a_2$$

Power Series apply only on

1. linear homogeneous equation
2. we apply power series method about an ordinary point of the given equation.

* General second order linear inhomogeneous diff. eqn

$$y'' + P(x)y' + Q(x)y = 0 \quad \text{--- (1)}$$

A point $x = x_0$ is called an ordinary point of (1), if both $P(x)$ and $Q(x)$ were analytic at $x = x_0$.

Example of analytic function are $f(x) = x^2, \sin x, \cos x, e^x$

New analytic function $f(x) = \frac{1}{x}, e^{1/x}, \frac{x}{x-1}$

Definition.

If any one of $P(x)$ or $Q(x)$ or both $P(x)$ and $Q(x)$ fails to be analytic at some point, $x = x_0$, then $x = x_0$ is called a singular point of the differentiation equation.

$$x^2 y'' + xy' + y = 0$$

$$\Rightarrow y'' + \frac{y'}{x} + \frac{y}{x^2} = 0 ; P(x) = \frac{1}{x}, Q(x) = \frac{1}{x^2}$$

$\Rightarrow x=0$ is a singular pt.

$$\Rightarrow x(x-1)y'' + xy' + y = 0$$

$$\Rightarrow y'' + \frac{1}{x-1} y' + \frac{1}{x(x-1)} y = 0$$

$$\Rightarrow P(x) = \frac{1}{x-1}, Q(x) = \frac{1}{x(x-1)}$$

$\Rightarrow x=0, 1$ are singular points.

Note.

We can always assume a "power series sol" of a given diff eqn about an ordinary point.

$$\text{ex: } x^2 y' = y$$

$$\Rightarrow y' - \frac{1}{x^2} y = 0$$

$$y'' + P(x)y' + Q(x)y = 0 \quad \text{--- (1)}$$

Definition.

Suppose $x = x_0$ is a singular point of (1).
Now $x = x_0$ is called a regular singular point of (1) if both

$$\lim_{x \rightarrow x_0} (x - x_0)P(x)$$

$$\text{ii) } \lim_{x \rightarrow x_0} (x - x_0)^2 Q(x) \text{ exist.}$$

$$1. \cancel{x(x-1)y'' + xy' + y = 0}.$$

Note

If $x = x_0$ is a regular singular point of (1), we can assume Frobenius series sol'n of (1) about $x = x_0$.

$$y = (x - x_0)^m \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad \text{--- (1)(A)}$$

(A) becomes
if $m = 0$, Power series

$\underline{m=0}, n=0, +ve, -ve$ integer fraction exponent

$$\text{If } x = 2, \cancel{y'' + 2y' + 2y = 0}.$$

$$y = (x - x_0)^m.$$

$$y = (x - x_0)^2 [a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots]$$

$$= b_0 + b_1(x - x_0) + b_2(x - x_0)^2$$

$$b_0 = 0, b_1 = 0, b_2 = a_0$$

Power series is a special case of Frobenius Series

$$1. \quad x(x-1)y'' + xy' + y$$

$$P(x) = x(x-1) \quad Q(x) = \frac{1}{x(x-1)}$$

$$x_0 = 0.$$

$$\lim_{x \rightarrow 0} x \cdot \frac{1}{x-1} = 0$$

$$\lim_{x \rightarrow 0} x^2 \cdot \frac{1}{x(x-1)} = 0$$

$x = 0$ is a regular singular point of B.

Frobenius series sol'

$$2x^2y'' + x(2x+1)y' - y = 0 \quad \text{--- (1)}$$

$x = 0$ is singular pt. of.

$$\lim_{x \rightarrow 0} x \cdot \left(\frac{2x+1}{2x} \right) = \frac{1}{2}$$

$$\lim_{x \rightarrow 0} x^2 \left(-\frac{1}{2x^2} \right) = -\frac{1}{2}$$

$x = 0$ is a regular singular pt.

$$y = \sum_{n=0}^{\infty} a_n x^{n+m} \quad a_0 \neq 0$$

$$y' = \sum_{n=0}^{\infty} (n+m) a_n x^{n+m-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+m)(n+m-1) a_n x^{n+m-2}$$

$$y = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + \dots$$

$$y' = m a_0 x^{m+1} + (m+1) a_1 x^{m+2} + \dots$$

$$y = a_0 + a_1 x + a_2 x^2$$

$$y' = a_1 + 2a_2 x$$

$$2x^2 y'' + x(2(x+1)) y' - y = 0 \quad \text{--- (1)}$$

$$\Rightarrow 2x^2 \sum_{n=0}^{\infty} (m+n-1) a_n x^{m+n-2} + 2x^2 \sum_{n=0}^{\infty} (m+n) a_n x^{n+m-1}$$

$$+ x \sum_{n=0}^{\infty} (m+n) a_n x^{n+m-1} - \sum_{n=0}^{\infty} x^{n+m} = 0$$

$$\Rightarrow 2 \sum_{n=0}^{\infty} (n+m)(n+m-1) a_n x^{n+m} + 2 \sum_{n=0}^{\infty} a_n (n+m) x^{n+m+1}$$

$$+ \sum_{n=0}^{\infty} (n+m) a_n (x^{n+m} - \sum_{n=0}^{\infty} a_n x^{n+m}) = 0$$

$$= \sum_{n=0}^{\infty} [2(n+m)(n+m-1) + n+m - 1] a_n x^{n+m}$$

$$+ 2 \sum_{n=1}^{\infty} a_{n-1} (n-1+m) x^{m+n} = 0.$$

$$\Rightarrow 2 \sum_{n=0}^{\infty} \{ 2(n+m)^2 - (n+m) - 1 \{ a_n + 2(n+m-1)a_{n-1} \}$$

$$+ (2m^2 - m - 1) a_0 x^m = 0.$$

Co-eff. of lowered order term in y for the series

$$= (2m^2 - m - 1) a_0 = 0$$

$$= 2m^2 - m - 1 = 0 \quad \because a_0 \neq 0$$

Indicates eqⁿ

$$n = 1, -\frac{1}{2}$$

Co-eff. of $x^{n+m} = 0$. $n = 1, 2, \dots$

$$2(n+m) + 1 \{ (n+m-1) a_n + 2(n+m-1) a_{n-1} \} = 0$$

Recursion formula

$$a_n = \frac{-2}{2n+2m+1} a_{n-1} + a_n \rightarrow$$

$$m=1,$$

$$a_n = \frac{-2}{2n+3} a_{n-1}, n \geq 1$$

$$\cancel{n+m-1} = n+1$$

$$m=1,$$

$$a_{1/2} = \frac{-2}{2n+3} a_{n-1}$$

$$n=1$$

$$a_1 = \frac{-2}{2n+3} a_0$$

$$a_2 = \frac{-2a_{n-1}}{7} = -\frac{2}{7} \left(-\frac{2}{5} a_0 \right)$$

$$= \frac{4}{35} a_0$$

$$y = a_0 x \left(1 - \frac{2}{5} x + \frac{4}{35} x^2 + \dots \right)$$

$$y = 2^m (a_0 + a_1 x + a_2 x^2 + \dots)$$

$$= x' \left(a_0 - \frac{2}{5} a_0 x + \frac{4}{35} a_0 x^2 + \dots \right)$$

that is what we want for m=1? it's not true

General soln:

$$\text{If } N_{(y_1, y_2)} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0, \text{ linearly independent.}$$

$$\frac{R}{n} = -1/2$$

$$a = -\frac{1}{n} a_{n-1}$$

$$a_1 = -a_0 \quad a_2 = a_0/2 \quad a_3 = -a_0/6$$

$$y = \frac{a_0}{\sqrt{x}} \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} \right)$$

$$= a_0 x^{-1/2} e^{-x}$$

$$y_2(x) = x^{-1/2} e^{-x}$$

$$2. \quad 4xy'' + 2y' + y = 0$$

$$P(x) = \frac{1}{2x} \quad Q(x) = \frac{1}{4x}$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+m}, \quad a_0 \neq 0.$$

$$y' = \sum_{n=0}^{\infty} a_n (n+m+1) x^{n+m}$$

$$y'' = \sum_{n=0}^{\infty} (n+m)(n+m-1) x^{n+m-2}$$

$$y = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2}$$

$$y' = m a_0 x^{m-1} + (m+1) a_1 x^m + \dots$$

$$y = a_0 + a_1 x + a_2 x^2$$

$$y = a_0 + 2a_2 x^2$$

$$4ay'' + 2y' + y = 0$$

$$\Rightarrow 4a \left(\sum_{n=0}^{\infty} (n+m)(n+m-1)x^{m+n-2} \right) + 2 \left(\sum_{n=0}^{\infty} (n+m)a_n x^{n+m} \right)$$

$$\Rightarrow \sum_{n=0}^{\infty} 4(n+m)(n+m-1)$$

$$m = 0, 1/2.$$

$$a_n = -\frac{a_{n-1}}{2(m+n)(2m+2n-1)}$$

$$m = 0,$$

$$a_n = \frac{1}{2n(2n-1)} a_{n-1}$$

$$n=1 \quad a_1 = \frac{1}{2 \times 1} = \frac{1}{2} a_0$$

$$n=2 \quad a_2 = \frac{1}{4(2 \times 2 - 1)} a_{2-1} \\ = \frac{1}{12} a_1$$

$$y_1(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} - \dots$$

$$= 1 - \frac{(\sqrt{x})^2}{2!} + \frac{(\sqrt{x})^4}{4!} - \frac{(\sqrt{x})^6}{6!} - \dots$$

$$= \cos \sqrt{x}$$

$$y_2(x) = x^{1/2} \left(1 - \frac{x}{3!} + \frac{x^2}{5!} \dots \right)$$

$$= \sqrt{x} - \frac{(\sqrt{x})^3}{3!} + \frac{(\sqrt{x})^5}{5!}$$

$$= \sin \sqrt{x}$$

$$y'' = -1$$

H/W.

$$1. 2x^2y'' - xy' + (1-x^2)y = 0$$

$$2. 2x^2y'' + xy' - (x+1)y = 0$$

$$3. 9x(1-x)y'' - 12y' + 4y = 0$$

$$4. (2x+x^3)y'' - y' - 6xy = 0.$$

$$1. 2x^2y'' - xy' + (1-x^2)y = 0$$

$$P(x) = \frac{1}{x} \quad Q(x) = \frac{1-x^2}{2x^2}$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+m} \quad [|x| < R]$$

$$\begin{aligned} & \text{at } x=0 \\ & x \neq 0 \quad 2^n \varphi(x) = \frac{1}{2} \end{aligned}$$

Regularity

$$\therefore 2x^2 \sum_{n=0}^{\infty} (n+m)(n+m-1)a_n x^{n+m-2} - x \sum_{n=0}^{\infty} (n+m)a_m x^{n+m}$$

$$+ (1-x^2) \sum_{n=0}^{\infty} a_n x^{n+m} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} \left[2(n+m)(n+m-1) - (n+m)+2 \right] a_n x^{n+m} - \sum_{n=0}^{\infty} a_n x^{n+m+2} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} (2n+2m-1)(n+m-1) a_n x^{n+m} - \sum_{n=2}^{\infty} a_{n-2} x^{n+m} = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} \left[(2n+2m-1)(n+m-1) a_n - a_{n-2} \right] x^{n+m} + (2m-1)(m-1)x^m + (2m+1)^{m+1} a_0 x^{m+1} = 0$$

$$(2m-1)(m-1) = 0 \quad \therefore a_0 \neq 0$$

$$(2m+1)m a_1 = 0 \Rightarrow a_1 = 0$$

$$m_1 = 1/m = 1/2$$

$$a_n = \frac{a_{n-2}}{(2n+2m-1)(n+m-1)}$$

when $m=1$,

$$a_n = \frac{a_{n-2}}{(2n+1)n} \Rightarrow a_1 = 0$$

$$a_2 = a_0/10 ; a_3 = 0 ; a_4 = \frac{a_0}{360} ; a_5 = 0$$

$$\begin{aligned} y_1 &= x \left(a_0 + \frac{a_0}{10} x^2 + \frac{a_0}{360} x^4 + \dots \right) \\ &= a_0 x \left(1 + \frac{x^2}{10} + \frac{x^4}{360} + \dots \right) \end{aligned}$$

ii) when $m = \frac{1}{2}$

$$a_n = \frac{a_{n-2}}{2n(n-\frac{1}{2})} = \frac{a_{n-2}}{n(2n-1)}$$

$$a_1 = 0$$

$$a_2 = \frac{a_0}{6} \quad a_4 = \frac{a_0}{168}$$

$$a_3 = 0 \quad a_5 = 0$$

$$\therefore y = x^{1/2} \left(a_0 + \frac{a_0}{6} x^2 + \frac{a_0}{168} x^4 + \dots \right)$$

$$= a_0 \left(x^{1/2} + \frac{x^{5/2}}{6} + \frac{x^{9/2}}{168} + \dots \right)$$

$$y_2(x) = x^{1/2} + \frac{1}{6} x^{5/2} + \frac{1}{168} x^{9/2} + \dots$$

$$y = A y_1(x) + B y_2(x)$$

$$= A \left(x + \frac{x^3}{10} + \frac{x^7}{360} + \dots \right) + B \left(\frac{1}{x^{1/2}} + \frac{1}{6} x^{3/2} + \frac{1}{168} x^{7/2} + \dots \right)$$

Cauchy Euler eqn.

$$a_1 (ax+b)^n \frac{d^ny}{dx^n} + a_2 (ax+b)^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \dots + a_n (ax+b) \frac{dy}{dx} + a_{n+1} y = 0$$

$$y = \sum_{n=0}^{\infty} a_n (x-x_0)^{n+m}$$

$$1. \quad 4x^2 y'' - 8x^2 y' + (4x^2 + 1)y = 0$$

$$\text{After } y'' - 2y' + \left(\frac{4x^2+1}{4x^2}\right)y = 0.$$

$$P(x) = -2 \quad Q(x) = 1 + \frac{1}{4x^2}$$

$$\lim_{x \rightarrow 0} x P(x) = 0$$

$$\lim_{x \rightarrow 0} x^2 Q(x) = x^2 \left(1 + \frac{1}{4x^2}\right)$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+m}$$

$$y' =$$

$$y'' =$$

$$\Rightarrow 4x^2 \sum_{n=0}^{\infty} (n+m)(m+n-1) a_n x^{n+m-2} - 8x^2 \sum_{n=0}^{\infty} (n+m)a_n x^n$$

$$+ (4x^2 + 1) \sum_{n=0}^{\infty} a_n x^{n+m} = 0$$

$$\Rightarrow 4(n+m)(n+m-1)a_n - 8(n+m)a_{n-1} + 4a_{n-2} = 0$$

$$a_0 = 0; a_1 = 0$$

$$a_{n-2} = \frac{8(n+m)a_{n-1} - 4(n+m)(n+m-1)a_n}{4}$$

$$\Rightarrow 4x^2 \sum_{n=0}^{\infty} (n+m)(m+n-1) a_n x^{n+m-2} - 8x^2 \sum_{n=0}^{\infty} (n+m)a_n x^n$$

$$+ (4x^2 + 1) \sum_{n=0}^{\infty} a_n x^{n+m} + \sum_{n=0}^{\infty} a_n x^{n+m} = 0$$

$$\Rightarrow 4 \sum_{n=0}^{\infty} (n+m)(n+m-1) a_n x^{n+m} - 8 \sum_{n=0}^{\infty} (n+m)a_n x^{n+m-1}$$

$$+ 4 \sum_{n=0}^{\infty} a_n x^{n+m+2} + \sum_{n=0}^{\infty} a_n x^{n+m} = 0$$

$$\Rightarrow 4 \sum_{n=0}^{\infty} (n+m)(n+m+1) a_n x^{n+m} - 8 \sum_{n=1}^{\infty} (n+m-1) a_{n-1} x^{n+m}$$

$$+ 4 \sum_{n=2}^{\infty} a_{n-2} x^{n+m} + \sum_{n=0}^{\infty} a_n x^{n+m} = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} [4(n+m)(n+m-1)a_n - 8(n+m-1)a_{n-1} + 4a_{n-2} + a_n] x^{n+m} + 4m(m-1)a_0 x^m + 4(m+1)m a_1 x^{m+1} - 8ma_0 x^{m+1} + a_0 x^m + a_1 x^{m+1} = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} [4(n+m)^2 - 4(n+m)+1] a_n - 8[(n+m-1)a_{n-1}] + 4a_{n-2}] x^{n+m} + a_0 \{4m(m-1) + 1\} x^m + \cancel{4m(m-1)} \cancel{a_0} x^{n+m} + \{4m(m+1)a_1 + a_1 - 8ma_0\} x^{m+1} = 0$$

Individual eqn:

$$4m(m-1) + 1 = 0$$

$$\therefore a_0 \neq 0$$

$$m = 1/2, -1/2$$

$$c_0 - \text{coeff. of } x^{m+1} = 0$$

$$\Rightarrow \{4m(m+1) + 1\} a_1 - 8m a_0 = 0$$

$$m = 1/2$$

$$2(\frac{1}{2} + 1) + 1 \{ a_1 - 8 \times \frac{1}{2} a_0 \} = 0$$

$$\Rightarrow 4a_1 - 4a_0 = 0$$

$$\Rightarrow a_1 = a_0$$

Recursion formula :

$$\text{coeff. of } x^{n+m} = 0$$

$$\Rightarrow \{2(n+m)-1\}^2 a_n - 8(n+m-1)a_{n-1} + 4a_{n-2} = 0$$

$$\Rightarrow n^2 a_n - (2n-1)a_{n-1} + a_{n-2} = 0 \quad \# n > 2$$

$$a_2 = \frac{a_0[3-1]}{4} = \frac{a_0}{2}$$

$$a_3 = \frac{a_0}{6}, \quad a_4 = \frac{a_0}{24}$$

$$4x^2y'' - 8x^2y' + (4x^2 + 1)y = 0 \quad \textcircled{1}$$

$$y = x^m (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots)$$

$$= x^{1/2} (a_0 + a_0 x + \frac{a_0 x^2}{2!} + \frac{a_0 x^3}{3!} + \frac{a_0 x^4}{4!} + \dots)$$

$$= a_0 x^{1/2} e^x$$

$$\boxed{y_2(x) = x^{1/2} e^x}$$

\textcircled{2}

$$y = y_1(x)v(x)$$

$$y' = y_1'v + y_1v'$$

$$y'' = y_1''v + 2y_1'v' + y_1v''$$

$$4x^2(y_1''v + 2y_1'v' + y_1v'') - 8x^2(y_1'v + y_1v') \\ + (4x^2 + 1)y_1v = 0$$

$$\Rightarrow \underbrace{[4x^2y_1'' - 8x^2y_1' + (4x^2 + 1)y_1]}_{+ 4x^2y_1v'' = 0} v + [8x^2y_1' - 8x^2y_1]v$$

$$\Rightarrow 8x^2(y_1' - y_1)v' + 4x^2y_1v'' = 0$$

$$\Rightarrow 2(y_1' - y_1)v' + y_1v'' = 0$$

$$\Rightarrow \frac{2(y_1' - y_1)}{y_1} + \frac{v''}{v'} = 0, \quad x \neq 0$$

$$\Rightarrow \frac{y_1'}{y_1} - 2 + \frac{v''}{v'} = 0$$

Integrating

$$2\log y_1 - 2x + \ln v'' = \text{constant } C$$

$$\Rightarrow \ln(v'y_1^2) = 2x + C$$

$$\Rightarrow v'y_1^2 = ke^{2x}, \quad C = -\ln k$$

$$\Rightarrow v' = \frac{ke^{2x}}{y_1^2} = \frac{ke^{2x}}{xe^{2x}} = \frac{k}{x}$$

$$\Rightarrow \frac{dv}{dx} = \frac{k}{x}$$

$$\Rightarrow dv = k \frac{dx}{x}$$

$$\Rightarrow \boxed{v = K \ln x}$$

$$y = k \ln x y_1(x)$$

$$y_2(x) = y_1(x) \ln x$$

General Soln:

H/W

$$y = Ay_1(x) + By_2(x)$$

$$xy'' + y' - xy = 0$$

$$20. xy'' - xy' + \frac{3}{4}y = 0$$

$$P(x) = -\frac{1}{x} \quad Q(x) = \frac{3}{4x^2}$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+m} = 0$$

$$\begin{aligned} & x^2 \sum_{n=0}^{\infty} (n+m)(m+n-1) a_n x^{n+m-2} - x \sum_{n=0}^{\infty} (n+m) a_n x^{n+m-1} \\ & + \frac{3}{4} \sum_{n=0}^{\infty} a_n x^{n+m} = 0 \end{aligned}$$

$$\begin{aligned} \Rightarrow & \sum_{n=0}^{\infty} (n+m)(n+m-1) a_n x^{n+m} - \sum_{n=0}^{\infty} (n+m) a_n x^{n+m} \\ & + \frac{3}{4} \sum_{n=0}^{\infty} a_n x^{n+m} = 0 \end{aligned}$$

$$a_n(n+m)(n+m-1) - a_n(n+m) + \frac{3}{4}a_n = 0$$

$$a_n = 0$$

$$\text{Let } n+m = t$$

$$a_{t-m} t(t-1) - a_{t-m} t + \frac{3}{4}a_{t-m} = 0$$

$$\Rightarrow a_{t-m} \left[t(t-1) - t + \frac{3}{4} \right] = 0$$

$$\Rightarrow a_{t-m} \left[t^2 - t - t + \frac{3}{4} \right] = 0$$

$$t^2 - 2t + \frac{3}{4} = 0$$

$$t = \frac{1}{2}, \frac{3}{2}$$

~~Recurrence~~

$$\cdot n=0 \Rightarrow m = \frac{1}{2}/3/2$$

Recursion formula :

$$\text{coeff. of } x^{n+m} = 0$$

$$\Rightarrow [(n+m)^2 - 2(n+m) + \frac{3}{4}] a_n = 0 \quad \forall n \geq 1$$

$$\Rightarrow [(n+m - \frac{3}{2})(n+m - \frac{1}{2}) a_n] = 0 \quad \forall n \geq 1$$

$$m = 3/2$$

$$n(n+1) a_n = 0, \quad n \geq 1$$

$$\Rightarrow a_n = 0 \quad \forall n \geq 1$$

$$m = 1/2$$

$$(n-1)n a_n = 0, \quad n \geq 1$$

$$n=1, \quad 0 \cdot 1 \cdot a_1 = 0$$

a_1 is arbitrary.

$$n \geq 2, \quad n(n-1) \neq 0$$

$$\Rightarrow a_n = 0$$

$$y = x^m (a_0 + a_1 x + a_2 x^2 + \dots)$$

$$= x^{3/2} (a_0) = a_0 x^{3/2}$$

$$y_1(x) = x^{3/2}$$

$$m = 1/2$$

$$y = x^m (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots)$$

$$= x^{1/2} (a_0 + a_1 x)$$

$$= a_0 x^{1/2} + a_1 x^{3/2}$$

$$\boxed{y_2(x) = x^{1/2}}$$

General solⁿ:

$$y = A y_1 + B y_2$$

$$3 \cdot x y'' - y' + 4x^3 y = 0$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+m}$$

$$y' = \sum_{n=0}^{\infty} (n+m) a_n x^{n+m-1}$$

$$y'' = \sum_{n=0}^{\infty} (m+n-1)(n+m) a_n x^{n+m-2}$$

$$= x \sum_{n=0}^{\infty} (m+n-1)(n+m) a_n x^{n+m-2} - \sum_{n=0}^{\infty} (n+m) a_n x^{n+m-1}$$
$$+ 4x^3 \sum_{n=0}^{\infty} a_n x^{n+m}$$

$$\sum_{n=0}^{\infty} (m+n-1)(n+m) a_n x^{n+m-1} - \sum_{n=0}^{\infty} (n+m) a_n x^{n+m-1}$$
$$+ 4 \sum_{n=0}^{\infty} a_n x^{n+m+3}$$

$$\rightarrow [(m+n-1)(n+m) - (n+m)] a_n x^{n+m-1}$$
$$+ 4 a_n x^{n+m+3} = 0$$

~~$n+m-1$~~

$$\Rightarrow [(m+n-1)(n+m) - (n+m)] a_n x^{n+m-1} + 4a_{n-4} x^{n+m-1} =$$

$$\Rightarrow (m+n-1)(n+m)a_n - (m+n)a_n + 4a_{n-4} = 0.$$

Put $m=4$

~~$$(m+3)(m+4) - (m+4)a_4 + 4a_0 = 0$$~~
~~$$(m+4)[(m+3) - a_4] + 4a_0 = 0$$~~

Put $n=0$,

$$(m-1)(m-m) - 1 \cdot m a_0 = 0.$$

$$m^2 - m - m = 0$$

$$m^2 - 2m = 0$$

$$m = 0, 2$$

If $m=0$,

$$(n-1)a_n - n \cdot a_n + 4a_{n-4} = 0.$$

$$n=0,$$

~~a₀~~ a₀ is arbitrary.

$$n=1,$$

$$0 \cdot 1 - 1 \cdot a_1 + 4a_{1-4} = 0$$

$$a_1 = 0$$

$$n=2,$$

~~$$1 \cdot 2a_2 - 2 \cdot a_2 + 8a_{2-4} = 0$$~~
~~$$2a_2 - 2a_2 + 8a_0 = 0$$~~
~~$$8a_0 = 0$$~~

$$a_2 = 1$$

$$n=3,$$

$$(3-1) \cdot 3a_3 - 3a_3 = 0$$

$$-a_3 = 0$$

$$n=4$$

$$3 \cdot 4a_4 - 4 \cdot a_4 + 4a_{4-4} = 0$$

~~$$8a_4 + 4a_0 = 0$$~~

$$a_4 = -\frac{4a_0}{8}$$

~~$m_1 = 2$~~ ,

~~$\sin^2 x$~~

~~y_1~~

case-I :-

If $m_1 - m_2$ is not an integer or zero:
The eqⁿ has two linearly independent

solutions $y_1 = x^{m_1} \sum_{n=0}^{\infty} a_n x^n$, $a_0 \neq 0$

$$y_2 = x^{m_2} \sum_{n=0}^{\infty} b_n x^n$$
, $b_0 \neq 0$

Ex:- $2x^2 y'' + x(2x+1)y' - y = 0$

case-II :-

If $m_1 - m_2 = 0$, there exist exist
second Frobenius series soln

$$y_1 = x^{m_1} \sum_{n=0}^{\infty} a_n x^n$$

$$y_2 = (y_1 \log x + x^{m_1} \sum_{n=0}^{\infty} c_n x^n), c \neq 0$$

Ex:- $4x^2 y'' - 8x^2 y' + (4x^2 + 1)y = 0$

case III :-

If $m_1 - m_2 = \text{integer}$ two linearly independent
solⁿ are $y_1 = x^{m_1} \sum_{n=0}^{\infty} a_n x^n$

$$y_2 = (y_1 \log x + x^{m_1} \sum_{n=0}^{\infty} c_n x^n)$$

- A) In some cases $c \neq 0$, so the soln.
contains a logarithm term. This case

corresponds to when second. Frobenius series

soln can't exist

$$\text{Ex: } x(1+x)y'' + 3xy' + y = 0.$$

B) In some cases $c=0$, so the soln does not contain logarithm term. In this case there does exist a second Frobenius series soln.

$$\text{Ex: } x^2y'' - xy' + \frac{3}{4}y = 0.$$

Legendre differential eqn

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

n is a constant

$$y = \sum_{m=0}^{\infty} a_m x^m$$

$$(1-x^2) \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - 2x \sum_{m=1}^{\infty} m a_m x^{m-1} +$$

$$n(n+1) \sum_{m=2}^{\infty} a_m x^m = 0$$

$$\Rightarrow \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - \sum_{m=2}^{\infty} m(m-1)a_m x^m - 2 \sum_{m=1}^{\infty} m a_m x^m +$$

$$n(n+1) \sum_{m=0}^{\infty} a_m x^m = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2} x^m - \sum_{m=0}^{\infty} m(m-1)a_m x^m - 2 \sum_{m=0}^{\infty} m a_m x^m$$

$$\Rightarrow \sum_{m=0}^{\infty} [(m+2)(m+1)a_{m+2} - \{m(m+1) + 2m - n(n+1)\} a_m] x^m = 0$$

Coeff of $x^m = 0$, if $m \geq 0$

$$\Rightarrow a_{m+2} = \frac{m(m-1) + 2m - n(n+1)}{(m+2)(m+1)} a_m$$

$$= \frac{(m-n)(m+n+1)}{(m+2)(m+1)} a_m$$

$$a_{m+2} = -\frac{(n-m)(n+m+1)}{(m+2)(m+1)} a_m, \quad m \geq 0$$

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

$$m=0, \quad a_0 = -\frac{n(n+1)}{2!} a_0$$

$$m=1,$$

$$m=1, \quad a_1 = -\frac{(n-1)(n+2)}{3!} a_1$$

$$m=2, \quad a_2 = -\frac{(n-2)(n+3)}{4!} a_2$$

$$m=3, \quad a_3 = +\frac{n(n-2)(n+1)}{4!} a_0$$

$$a_4 = +\frac{(n-1)(n-3)(n+2)(n+4)}{5!} a_0$$

$$y = \cancel{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5},$$

$$= a_0 + a_1 x - \frac{n(n+1)}{2!} a_0 x^2 - \frac{(n-1)(n+2)}{3!} a_1 x^3 + \frac{n(n-2)(n+1)(n+3)}{4!} x^4 a_0$$

$$= a_0 \left[1 - \frac{n(n+1)}{2} x^2 + \frac{n(n-2)(n+1)(n+3)}{4!} x^4 + \dots \right] + \frac{n(n-2)(n-4)(n+1)(n+3)(n+5)}{4!} x^6 + \dots$$

$$+ a_1 \left[x - \frac{(n-1)(n+2)}{3!} x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!} x^5 \right]$$

$$- \frac{(n-1)(n-3)(n-5)(n+2)(n+4)(n+6)}{7!} x^7$$

$$= a_0 y_0(x) + a_1 y_1(x)$$

$$y_0(x) = 1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 \dots$$

$$y_1(x) = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5 \dots$$

i) If n is zero or an integer.

Both y_0 and y_1 are infinite series.

$$R = \lim_{m \rightarrow \infty} \left| \frac{a_m}{a_{m+2}} \right|$$

$$\text{Ans.} \\ = 1$$

ii) If n is zero or an integer

If n is zero or an integer, then one of the infinite series terminates and the other remains infinite. The terminated series give a polynomial f^n , these polynomials ($P_n(x)$) satisfying the cond'n $P_n(1) = 1$ are called Legendre Polynomial.

$$n=0, \quad y_0(x) = 1 \quad | \quad (1-x^2)y'' - 2xy' = 0 \\ P_0(x) = 1 \times m_0$$

$$P_0(1) = 1$$

$$\Rightarrow m_0 = 1$$

$$\boxed{P_0(x) = 1}$$

$$m=1, \quad y_1(x) = x \Rightarrow (1+x^2)y'' + 2xy' + 1+2x=0$$

$$P_1(x) = m_1 x^n$$

$$P_1(x) = 1$$

$$\therefore m_1 = 1$$

$$\boxed{P_1(x) = x}$$

$$n=2$$

~~$$y_2(x) = C_2 x^2$$~~

$$y_2(x) = 1 - \frac{2+3}{2} x^2 = 1 - 3x^2$$

$$P_2(x) = m_2 x (1 - 3x^2)$$

$$P_2(x) = 1$$

$$m_2 = -1/2$$

$$\boxed{P_2(x) = \frac{3x^2 - 1}{2}}$$

$$n=3,$$

$$y_1(x) = x - \frac{2+5}{3!} x^3$$

$$= x - \frac{7}{6} x^3 = \frac{3x - 5x^3}{3}$$

$$P_3(x) = m_3 \left(\frac{3x - 5x^3}{3} \right)$$

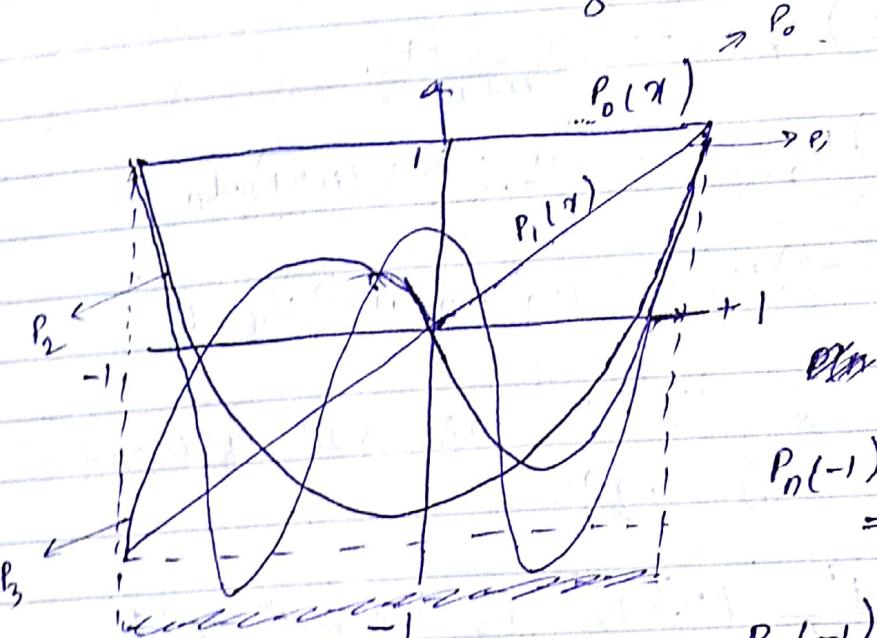
$$P_3(x) = 1 \quad m_3 = -\frac{3}{2}$$

$$P_3(x) = -\frac{3}{2} \left(x - \frac{5}{3}x^3 \right)$$

$n=4,$

$$P_4(x) = 1 - \frac{4 \cdot 5}{2!} x^2 + 4 \cdot 2 \cdot 6 \cdot$$

$$P_4(x) = \frac{35x^4 - 30x^2 + 3}{8}$$



$$P_n(-1) = 1, n \rightarrow \text{even}$$

$$= -1, n \rightarrow \text{odd}$$

$$P_n(-1) = (-1)^n$$

$$P_n(1) = 1$$

Rodrigue's formula

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

$$\text{Proof: } u = (x^2 - 1)^n$$

$$u_1 = \frac{d}{dx} (x^2 - 1)^n$$

$$= n(x^2 - 1)^{n-1} x^{2n-2} = \frac{2nx(x^2 - 1)^n}{(x^2 - 1)}$$

$$\therefore (x^2 - 1) u_1 = 2nxu$$

Leibnitz formula

$$(uv)_n = \sum_{i=0}^n n c_i u_{n-i} v_i$$

Differentiating $(n+1)$ times,

$$(v_1)_{n+1} (x^2 - 1)^{n+1} + {}^{n+1}c_1 (v_1)_n \times 2x + {}^{n+1}c_2 (v_1)_{n-1} x^2$$

$$= 2n \left[(v_1)_{n+1} x + {}^{n+1}c_1 (v_1)_n \right]$$

~~$$(1-x^2) v_{n+2} - 2x v_{n+1} + n(n+1) v_n = 0$$~~

$$(1-x^2) v_n'' - 2x v_n' + n(n+1) v_n = 0$$

$y = v_n$ is the solⁿ of legendre diff eqⁿ.

$y = \frac{d^n}{dx^n} (x^2 - 1)^n$ is the solⁿ of ①.

$y = a \times \frac{d^n}{dx^n} (x^2 - 1)^n$ is also a solⁿ of ①

Now $y = 1$ at $x = 1$

$$1 = a \left[\frac{d^n}{dx^n} (x^2 - 1)^n \right]_{x=1}$$

$$= a \left[\frac{d^n}{dx^n} \{ (x+1)^n (x-1)^n \} \right]$$

$$(UV)_n = \sum_{i=0}^n n_{c_i} \cdot u_{n-i} \cdot v_i \\ = n_{c_0} u_n v_0 + n_{c_1} u_{n-1} v_1$$

$$\boxed{x^n - 1 = \sum_{k=0}^n (x+1)^k - \sum_{k=0}^{n-1} (x+1)^k}$$

$$1 = a \left[n! (x+1)^n + n(n-1)! (x-1) n(x+1)^{n-1} \right. \\ \left. + \dots + (-)(x-1)^2 + \dots + (-)(x-1)^n \right]_{n=1}$$

$$1 = a x n! 2^n$$

$$\Rightarrow a = \frac{1}{2^n n!}$$

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

$$P_0(x) = \frac{1}{2^0 0!} \frac{d^0}{dx^0} (x^2 - 1)^0 \\ = 1$$

$$P_1(x) = \frac{1}{2^1 1!} \frac{d^1}{dx^1} (x^2 - 1)^1 \\ = \frac{1}{2} \times 2x = x$$

$$P_2(x) = \frac{1}{8} (12x^2 - 4)$$

$$(x^2 - 1)^n = \sum_{m=0}^n n_{cm} (x^2)^{n-m} (-1)^m$$

$$\frac{d^n}{dx^n} (x^2 - 1)^n = \frac{d}{dx^n} \sum n_{cm} x^{2n-2m} (-1)^m$$

$$P_n(x) = \frac{1}{2^n n!} \sum_{m=0}^n n_{cm} (-1)^m \frac{(2n-2m)!}{(2n-2m-1)\dots} \\ (2n-2m-n+1) x^{2n-2m-1}$$

$$= \frac{1}{2^n n!} \sum_{r=0}^N n_{CH} (-1)^r (2n-2r) \dots \quad (1)$$

$$P_n(x) = \frac{1}{2^n n!} \sum_{r=0}^N n_{CH} \frac{(-1)^r (2n-2r) \dots (n-2r+1)}{(n-2r) \dots 2!} x^{n-2r}$$

$$= \frac{1}{2^n n!} \sum_{r=0}^N n_{CH} \frac{(-1)^r (2n-2r)!}{(n-2r)!} x^{n-2r}$$

~~n - 2r ≥ 0~~

$$n - 2r \geq 0$$

$$n \leq \frac{n}{2}, \quad N = \left[\frac{n}{2} \right]$$

Orthogonal func

A set of function $\{\phi_i(x)\}$ is said to be orthogonal in $[a, b]$ w.r.t weight function $w(x) > 0$, if

$$\int_a^b w(x) \phi_i(x) \phi_j(x) dx = 0 \text{ for } i \neq j \quad \text{--- (1)}$$

The above set is called orthonormal if (1) is satisfied together with the following condition

$$\int_a^b w(x) \phi_i(x) \phi_i(x) dx = 1$$

$$\bar{a} = (a_1, a_2, a_3)$$

$$\bar{b} = (b_1, b_2, b_3)$$

$$\bar{a} \cdot \bar{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = 0$$

* Legendre polynomials $\{P_n(x)\}_{n=0}^{\infty}$ are orthogonal in $[-1, 1]$ w.r.t $w(x) = 1$

We have to prove $\int_{-1}^1 P_n(x) P_m(x) dx = 0, m \neq n$

Let $P_n(x)$ and $P_m(x)$ are the solⁿ of Legendre diff eqⁿ given below

$$(1-x^2)y'' + 2xy' + n(n+1)y = 0$$

$$(1-x^2)P_n'' - 2xP_n' + n(n+1)P_n = 0 \quad \text{--- (i)} \times P_m$$

$$(1-x^2)P_n'' - 2xP_m' + m(m+1)P_m = 0 \quad \text{--- (ii)} \times P_n$$

$$\textcircled{1} \times P_m - \textcircled{2} \times P_n$$

$$(1-x^2)(P_m' P_n - P_n' P_m) - 2x(P_n' P_m - P_m' P_n) + \\ \{n(n+1) - m(m+1)\} P_m P_n = 0$$

$$\Rightarrow \frac{d}{dx} [(1-x^2)(P_n' P_m - P_m' P_n)] + (n-m)(n+m+1) P_m P_n = 0$$

Integrating w.r.t x for $[-1, 1]$

$$\Rightarrow (1-x^2)(P_n' P_m - P_m' P_n) \Big|_{-1}^1 + (-n-m)(n+m+1) \int_{-1}^1 P_m P_n dx = 0$$

$$\Rightarrow (n-m)(n+m+1) \int_{-1}^1 P_m P_n dx = 0$$

$$\Rightarrow \int_{-1}^1 P_m P_n dx = 0 \quad \text{if } m \neq n.$$

$$ii) \int_{-1}^1 [P_n(x)]^2 dx = ?.$$

$$\int_{-1}^1 P_n(x) P_n(x) dx = \frac{1}{2^n n!} \int_{-1}^1 \frac{d^n (x^2 - 1)^n}{dx^n} P_n(x) dx$$

$$= \frac{1}{2^n n!} \left[P_n(x) \frac{d^{n-1}}{dx^{n-1}} (x^2 - 1)^n \right]_{-1}^1$$

$$\left. \int_{-1}^1 P_n'(x) \frac{d^{n-1}}{dx^{n-1}} (x^2 - 1)^n dx \right]$$

$$= \frac{1}{2^n n!} (-1)^n \left[P_n'(x) \frac{d^{n-2}}{dx^{n-2}} (x^2 - 1)^n \Big|_{-1}^{+1} - \int_{-1}^1 P_n''(x) \frac{d^{n-2}}{dx^{n-2}} (x^2 - 1)^n dx \right]$$

$$= \frac{1 \times (-1)^2}{2^n n!} \int_{-1}^1 P_n''(x) \frac{d^{n-2}}{dx^{n-2}} (x^2 - 1)^n dx$$

$$= \frac{1}{2^n n!} (-1)^n \int_{-1}^1 P_n''(x) (x^2 - 1)^n dx$$

$$P_n = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

$$\Rightarrow P_n^n(x) = \frac{1}{2^n n!} \frac{d^{2n}}{dx^{2n}} (x^2 - 1)^n$$

$$= \frac{1}{2^n n!} (2n)!$$

$$\text{P } \frac{(-1)^n}{2^n n!} \int_{-1}^1 \frac{1}{2^n n!} (2n)! \times (-1)^n (1-x^2)^n dx.$$

$$= \frac{(-1)^{2n} (2n)!}{(2^n n!)^2} \int_{-1}^1 (1-x^2)^n dx.$$

$$I_1 = \frac{(2n)!}{(2^n n!)^2} \times 2 \times I_2.$$

$$I_1 = \int_{-1}^1 (1-x^2)^n dx = 2 \int_0^1 (1-x^2)^n dx = 2I_2$$

$$x = \sin \theta \Rightarrow dx = \cos \theta d\theta.$$

$$I_2 = \int_0^1 (1-x^2)^n dx.$$

$$= \int_0^{\pi/2} \cos^{2n+1} \theta d\theta$$

$$= \frac{1}{2} B\left(\frac{0+1}{2}, \frac{2n+1+1}{2}\right)$$

$$= \frac{1}{2} B\left(\frac{1}{2}, n+1\right).$$

$$= \frac{1}{2} \frac{\Gamma(1/2) \Gamma(n+1)}{\Gamma(n+1 + \frac{1}{2})}.$$

$$\# \int_0^{\pi/2} \sin^p \theta \cos^q \theta d\theta = \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right).$$

$$B(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}.$$

$$\Gamma\left(n+1 + \frac{1}{2}\right) = \left(n + \frac{1}{2}\right) \Gamma\left(n + \frac{1}{2}\right).$$

$$= \left(n + \frac{1}{2}\right) \left(n - \frac{1}{2}\right) \cdots \frac{3}{2} \frac{1}{2} \Gamma\left(\frac{1}{2}\right).$$

$$= \frac{(2n+1)(2n)(2n-1)\dots 3 \cdot 1}{2^{n+1}} \sqrt{\pi}$$

$$= (2n+1)(2n)(2n-1)(2n-2)\dots 3 \cdot 2 \cdot 1 \sqrt{\pi}.$$

$$\frac{2n(2n-2)\dots 4 \cdot 2 \times 2^{n+1}}{2^{n+1}}$$

$$= \frac{(2n+1)!}{2^n n! \times 2^{n+1}} \sqrt{\pi}$$

$$= \frac{(2n+1)!}{n! 2^{2n+1}} \sqrt{\pi}.$$

$$I = \frac{(2n)!}{(2^n n!)^2} \times 2 \times 1_2 = \frac{2}{2n+1}$$

$$\int_{-1}^1 P_m(x) P_n(x) dx = \begin{cases} 0 & , m \neq n \\ \frac{2}{2n+1} & , m = n \end{cases}$$

Q Find Legendre series expansion of.

$$f(x) = x^3 + x \text{ in } [-1, 1]$$

$$f(x) = \sum_0^\infty a_n P_n(x) = a_0 P_0 + a_1 P_1 + a_2 P_2 \dots$$

$$a_n \times \frac{2}{2n+1} = \int_{-1}^1 f(x) P_n(x) dx = a_0 \int_{-1}^1 P_0(x) P_n(x) dx$$

$$+ a_1 \int_{-1}^1 P_1(x) P_n(x) dx \dots$$

$$+ a_n \int_{-1}^1 P_n(x) P_n(x) dx + \dots$$

$$f(x) = x^3 + x.$$

$$a_0 = \frac{2 \times 0 + 1}{2} \int_{-1}^1 (x^3 + x) P_0(x) dx = 0,$$

$$= \frac{1}{2} \int_{-1}^1 (x^3 + x) dx = 0.$$

$$a_2 = 0.$$

$$a_1 = \frac{2 \times 1 + 1}{2} \int_{-1}^1 (x^3 + x) P_1(x) dx.$$

$$= \frac{3}{2} \int_{-1}^1 (x^3 + x) x dx = \frac{8}{5}.$$

$$a_4 = 0$$

$$a_3 = \frac{2 \times 3 + 1}{2} \int_{-1}^1 (x^3 + x) P_3(x) dx = \frac{82}{5}.$$

$$P_n(x) = \frac{1}{2^n n!} \cdot \frac{d^n}{dx^n} (x^2 - 1)^n.$$

→ If $f(x)$ is continuous over $[-1, 1]$
 Then $f(x)$ can be written as linear
 combination of Legendre polynomials

Q Find the Legendre Series expansion

i) $f(x) = e^x$

ii) $f(x) = \begin{cases} 0 & [-1, 0) \\ x & [0, 1] \end{cases}$

$$P_0(x) = 1$$

$$1 = P_0$$

$$P_1(x) = x$$

$$x = P_1$$

$$P_2(x) = \frac{3x^2 - 1}{2} \quad x^2 = \frac{2P_2 + 1}{3} = \frac{2P_2 + P_0}{5}$$

$$P_3(x) = \frac{5x^3 - 3x}{2} \quad x^3 = \frac{2P_3 + 3P_1}{5}$$

$$\text{eg } f(x) = x^3 + x$$

$$= \frac{2P_3 + 3P_1}{5} + P_0 = \frac{8}{5}P_1 + \frac{2}{5}P_3$$

MIII

$$a_n = \frac{2n+1}{2} \int_{-1}^1 f(x) P_n(x) dx.$$

$$= \frac{2n+1}{2} \int_{-1}^1 f(x) \frac{1}{2^n n!} \frac{d^n (x^2 - 1)^n}{dx^n} dx$$

$$= \frac{2n+1}{2} \frac{1}{2^n n!} \int_{-1}^1 f(x) \frac{d^n}{dx^n} (x^2 - 1)^n dx$$

$$a_n = \frac{2n+1}{2} \frac{1}{2^n n!} (-1)^n \int_{-1}^1 f^{(n)}(x) (x^2 - 1)^n dx.$$

$f(x) \rightarrow \text{odd}, n \rightarrow \text{even}$

$f^{(n)}(x) \rightarrow \text{odd}$

Generating f^n

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n.$$

Proof

$$\frac{1}{\sqrt{1-2xt+t^2}} = (1-2xt+t^2)^{-1/2}$$

$$= [1 + \{ -t(2x+t) \}]^{-1/2}.$$

$$= \sum_{r=0}^{\infty} (-1)^r \frac{1 \cdot 3 \cdots (2r+1)}{r! 2^r} (-1)^r (2x+t)^r$$

$$= \sum_{r=0}^{\infty} \frac{1 \cdot 2 \cdot 3 \cdots (2r)}{r! 2^r \cdot 2^r r!} t^r (2x+t)^r$$

$$= \sum_{r=0}^{\infty} \frac{1 \cdot 2 \cdot 3 \cdots (2r)}{r! 2^r \cdot 2^r r!} t^r (2x-t)^r.$$

$$= \sum_{r=0}^{\infty} \frac{(2r)!}{2^{2r} (r!)^2} t^r (2x-t)^r$$

$$(2x-t)^r = \sum_{p=0}^r {}^r C_p (2x)^{r-p} t^p (-1)^p.$$

$$(1-2xt+t^2)^{-1/2} = \sum_{r=0}^{\infty} \frac{(2r)!}{2^{2r} (r!)^2} \sum_{p=0}^r {}^r C_p (-1)^p (2x)^{r-p} t^p$$

$$\text{let } r+p = n.$$

For a fixed r $p = n - r$

$$0 \leq p \leq r \Rightarrow 0 \leq n - r \leq r \Rightarrow r \leq n, r \geq \frac{n}{2}$$

If n is even $\frac{n}{2} \leq r \leq n$.

If n is odd $\frac{n+1}{2} \leq r \leq n$.

Coefficient of t^n in ①.

$$= \sum_{r=m}^n \frac{(2r)!}{2^{2r} (r!)^2} r \binom{n}{n-r} (-1)^{\frac{n-r}{2}} \binom{2r-n}{2r}$$

Now change the variable from.

$$r \text{ to } k = n - r$$

Coeff of t^n .

$$= \sum_{k=m'}^0 \frac{(2(n-k))!}{2^{2(n-k)} ((n-k)!)^2} \binom{n-k}{k} (-1)^{\frac{k}{2}} \binom{m'}{m} \left| \begin{array}{l} m' = \begin{cases} n-n/2 & \text{even} \\ n-(n+1)/2 & \text{odd} \end{cases} \\ (n-k) \wedge n-2k \end{array} \right.$$

$$m' = n - m.$$

$$\boxed{\sum_{k=0}^{m'} \frac{(2n-2k)!}{2^{2n-2k} ((n-k)!)^2} \binom{n-k}{k} (-1)^{\frac{n-2k}{2}} n^{-2k}}$$

$$\begin{aligned}
 &= \sum_{k=0}^{m=[n/2]} \frac{(-1)^k}{2^n k!} \frac{(2n-2k)!}{(n-k)!(n-2k)!} x^{n-2k} \\
 &= P_n(x)
 \end{aligned}$$

— x —

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n, \quad |t| < 1$$

- i) $P_n(1) = ?$
- ii) $P_n(-1) = ?$
- iii) $P_n(0) = ??$

i) $x = 1,$

$$\begin{aligned}
 LHS &= \frac{1}{\sqrt{1-2t+t^2}} = \frac{1}{\sqrt{(1-t)^2}} = (1-t)^{-1} \\
 &= 1 + t + t^2 + t^3 + \dots \\
 &= \sum_{n=0}^{\infty} t^n
 \end{aligned}$$

$$RHS = \sum_{n=0}^{\infty} P_n(1)t^n = P_0(1) + P_1(1)t + P_2(1)t^2 + \dots$$

ii) $x = -1$

$$\begin{aligned}
 LHS &= \frac{1}{\sqrt{1+2t+t^2}} = (1+t)^{-1} \\
 &= \sum_{n=0}^{\infty} (-1)^n t^n
 \end{aligned}$$

iii) $x=0$,

$$\text{LHS} = \frac{1}{\sqrt{1+t^2}} = (1+t^2)^{-1/2}$$

~~(1, 3)~~
~~(2, 4)~~
~~(2, 6)~~
~~(4)~~

$$\text{RHS} = \sum_{n=0}^{\infty} P_n(0) t^n \quad P_{2n}(0) = \frac{(-1)^n 1 \cdot 3 \cdots (2n-1)}{2^n n!}$$
$$P_0(0) = 0$$

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n, \quad |t| < 1$$

Differentiating w.r.t. to

$$\left(-\frac{1}{2}\right) (1-2xt+t^2)^{-3/2} (-2x+2t) = \sum_n n P_n(n) t^{n-1}$$
$$\Rightarrow \frac{x-t}{1-2xt+t^2} \cdot \frac{1}{\sqrt{1-2xt+t^2}} = \sum_n n P_n(n) t^{n-1}$$

$$\Rightarrow (x-t) \sum_n P_n(n) t^n = (1-2xt+t^2) \sum_n n P_n(x) t^{n-1}$$

Equating co-eff. of t^n from both sides

$$x P_n(x) - P_{n-1}(x) = (n+1) P_{n+1}(x) -$$
$$2x_n P_n(x) + (n-1) P_{n-1}(x)$$

$$\Rightarrow [(n+1) P_{n+1}(x) + (2n+1)x P_n(x) + n P_{n-1}(x) = 0]$$

$$P_0(x) = 1$$
$$P_1(x) = x$$

$$2 P_2(x) - 3 \cdot x P_1(x) + P_0(x) = 0$$

$$\rightarrow P_2 = \frac{3x^2 - 1}{2}$$

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n, |t| < 1 \quad (1)$$

Diffr. w.r.t t^n

$$\left(-\frac{1}{2}\right) (1-2xt+t^2)^{-3/2} (-2x+2t) = \sum_n nP_n(x)t^{n-1} \quad (2)$$

Diffr. (1) w.r.t x

$$\left(-\frac{1}{2}\right) (1-2xt+t^2)^{-3/2} (-2t) = \sum_n P_n(x)t^n \quad (3)$$

$$(-2t) \times (2) - (-2x+2t) \times (3) = 0$$

$$\Rightarrow (-2t) \sum_n nP_n(x)t^{n-1} = (-2x+2t) \sum_n P_n'(x)t^n$$

$$\Rightarrow \sum_n nP_n(x)t^n = (x-t) \sum_n P_n'(x)t^n$$

Equating coeff of t^n from both sides:

$$nP_n(x) = xP_n'(x) - P_{n-1}'(x)$$

Tchebychev differential eqⁿ

$$(1-x^2)y'' - xy' + n^2y = 0$$

$$y = \sum_{m=0}^{\infty} a_m x^m$$

$$y' = \sum_{m=1}^{\infty} m a_m x^{m-1}$$

$$y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2}$$

$$m+2=2$$

$$m+1=1$$

$$(1-x^2) \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - x \sum_{m=1}^{\infty} m a_m x^{m-1} + n^2 \sum_{m=0}^{\infty} a_m x^m = 0.$$

$$\Rightarrow \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} - \sum_{m=2}^{\infty} m(m-1) a_m x^m - \sum_{m=1}^{\infty} m a_m x^m \\ + n^2 \sum_{m=0}^{\infty} a_m x^m = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m - \sum_{m=2}^{\infty} (m-1)m a_m x^m - \sum_{m=1}^{\infty} m a_m x^m \\ + n^2 \sum_{m=0}^{\infty} a_m x^m$$

$$\Rightarrow \sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m - \sum_{m=0}^{\infty} (m-1)m a_m x^m - \sum_{m=0}^{\infty} m a_m x^m \\ + n^2 \sum_{m=0}^{\infty} a_m x^m$$

$$\Rightarrow \sum_{m=0}^{\infty} \left[(m+2)(m+1) a_{m+2} - (m-1)m a_m - m a_m + n^2 a_m \right] x^m = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} \left[(m+2)(m+1) a_{m+2} + [m^2 + m - m + n^2] a_m \right] x^m = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} \left[(m+2)(m+1) a_{m+2} + [n^2 - m^2] a_m \right] x^m = 0$$

$$\boxed{a_{m+2} = -\frac{n^2 - m^2}{(m+2)(m+1)} a_m}$$

$$m=0,$$

$$a_2 = -\frac{n^2}{2+1} a_0$$

$$m=1, a_3 = -\frac{n^2 - 1}{3+1} a_1$$

$$Y(x) = a_0 \left[1 - \frac{n^2 x^2}{2!} + \frac{n^2(n^2-2^2)}{4!} x^4 \dots \right]$$

$$+ a_1 \left[x - \frac{(n^2-1^2)}{3!} x^3 + \frac{(n^2-1^2)(n^2-3^2)}{5!} x^5 \dots \right]$$

$$= a_0 y_0(x) + a_1 y_1(x)$$

$$y_0(x) = 1 - \frac{n^2 x^2}{2!} + \frac{n^2(n^2-2^2)}{4!} x^4 \dots$$

$$y_1(x) = x - \frac{(n^2-1^2)}{3!} x^3 + \frac{(n^2-1^2)(n^2-3^2)}{5!} x^5 \dots$$

n=0

$$y_0(x) = 1$$

n=3

1 -

n=1

$$y(x) = x$$

n=2

$$y = \left(1 - \frac{2^2 x^2}{2} \right)$$

$T_n(x) = n^{\text{th}}$ deg. Tschirnhausen poly.

$$T_n(x) = 1 + n.$$

$$n=1,$$

$$T_1(x) = 1$$

$$T_1(x) = x \times m_1 \quad m_1 = 1$$

$$\boxed{T_1(x) = x}$$

$$T_2(x) = m_2 \times \left(1 - \frac{2^2 x^2}{2} \right) = (1 - 2x^2) \times m_2 = (1 - 2x^2)$$

$$T_3(x) = m_3 \times 4 = 4x^3 - 3x$$

~~$$T_4(x) = 8x^4 - 8x^2 + 1$$~~

$n \neq 0$, i.e., find radius of convergence for both $y_0(x)$ and $y_1(x)$

Radius of convergence:

$$\frac{a_{m+2}}{a_m} = -\frac{n^2 - m^2}{(m+2)(m+1)} a_m \times \frac{m(m-1)}{-n^2 - (m+2)},$$

$$= 1$$

$$(1-x^2)y'' - ny' + n^2y = 0$$

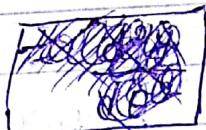
$$x = \cos \theta$$

$$\frac{dx}{d\theta} = -\sin \theta$$

$$y' = \frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$$

$$= \frac{dy}{d\theta} \left(-\frac{1}{\sin \theta}\right)$$

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left(\frac{dy}{d\theta} \left(-\frac{1}{\sin \theta}\right) \right)$$



$$= \frac{d}{d\theta} \left(-\frac{1}{\sin \theta} \frac{dy}{d\theta} \right) \cdot \frac{d\theta}{dx} =$$

$$= \left[\frac{1}{\sin^2 \theta} \cos \theta \frac{dy}{d\theta} - \frac{1}{\sin \theta} \frac{d^2y}{d\theta^2} \right] \left(-\frac{1}{\sin \theta} \right)$$

by substituting it,

$$-\frac{\sin^2 \theta}{\sin \theta} \left[\frac{1}{\sin^2 \theta} \cos \theta \frac{dy}{d\theta} - \frac{1}{\sin \theta} \frac{d^2y}{d\theta^2} \right]$$

$$\frac{+\cos\theta}{\sin^n\theta} \frac{dy}{d\theta} + n^2y = 0$$

$$\Rightarrow \frac{d^2y}{d\theta^2} + n^2y = 0$$

$$y = A\cos n\theta + B\sin n\theta$$

$$y = A\cos(n\cos^{-1}x) + B\sin(n\cos^{-1}x)$$

$$T_n(x) = \cos(n\cos^{-1}x)$$

(Then)
Polynomial

$$y = \sin(n\cos^{-1}x)$$

$$\cos^{-1}x = \theta$$

$$\cos\theta = x, \quad y = \sin(n\cos^{-1}x) \\ = \sin(n\theta)$$

$$y_1 = \cos(n\cos^{-1}x)$$

$$y_2 = \sin(n\cos^{-1}x)$$

$$\cos n\theta = \operatorname{Re}\{e^{in\theta}\}$$

$$= \operatorname{Re}\{(\cos\theta + i\sin\theta)^n\}$$

$$= \operatorname{Re}\{\cos^n\theta + n_1\cos^{n-1}\theta(i\sin\theta) + {}^nC_2 \cos^{n-2}(i\sin\theta)^2 + {}^nC_3 \cos^{n-3}(i\sin\theta)^3 + \dots\}$$

$$= \operatorname{Re}\left[\cos^n\theta - {}^nC_2 \cos^{n-2}\theta \sin^2\theta + {}^nC_4 \cos^{n-4}\theta \sin^4\theta \dots + i\left\{n_1\cos^{n-1}\theta \sin\theta - {}^nC_3 \cos^{n-3}\theta \sin^3\theta + \dots\right\}\right]$$

$$\cos n\theta = \cos^n \theta - {}^n c_2 \cos^{n-2} \theta \sin^2 \theta + {}^n c_4 \cos^{n-4} \theta \sin^4 \theta$$

$$T_n(x) = \cos(n \cos^{-1} x) = x^n - {}^n c_2 x^{n-2} (1-x^2) + {}^n c_4 x^{n-4} (1-x^2)^2 + \dots$$

↳ polynomial in x of degree n

$$\text{coeff. of } x^n = 1 + {}^n c_2 + {}^n c_4 + \dots + {}^n c_n \\ = \frac{1}{2} {}^{n-1} c_2 + \dots + {}^n c_n$$

$$(1+1)^n = 1 + {}^n c_1 + {}^n c_2 + \dots + {}^n c_n$$

$$(1-1)^n = 1 - {}^n c_1 + {}^n c_2 + \dots + \dots$$

$$\Rightarrow 2^n = 2 [1 + {}^n c_2 + {}^n c_4 + \dots]$$

$$\Rightarrow 1 + {}^n c_2 + {}^n c_4 + \dots = 2^{n-1}$$

$$T_0(x) = 1$$

$$T_1(x) = x$$

$$T_2(x) =$$

⋮

$$\sin(n \cos^{-1} x) = \sin(n\theta) = \operatorname{Im}(e^{in\theta}) \\ = {}^n c_1 \cos^{n-1} \theta \sin \theta - {}^n c_3 \cos^{n-3} \theta \sin^3 \theta \\ \dots$$

$$\sin(n \cos^{-1} x) = {}^n c_1 x^{n-1} \sqrt{1-x^2} - {}^n c_3 x^{n-3}$$

$$(1-x^2) \sqrt{1-x^2} + {}^n c_5 x^{n-5} (1-x^2)^2 \sqrt{1-x^2}$$

$$\rightarrow \frac{\sin(\cos^{-1} x)}{\sqrt{1-x^2}} = {}^n c_1 x^{n-1} - {}^n c_3 x^{n-3} (1+x^2) + \dots$$

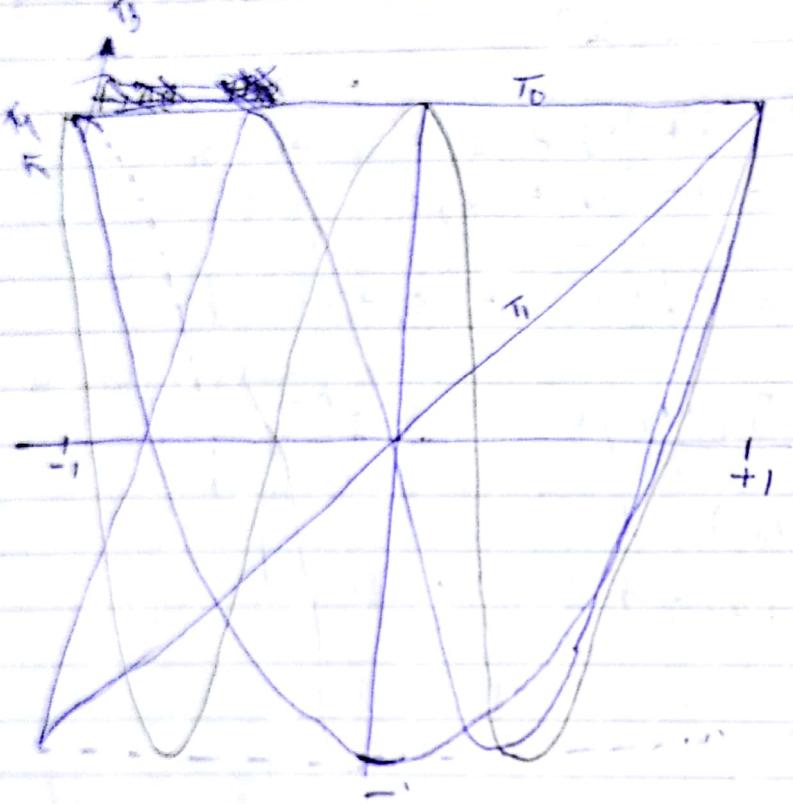
↳ polynomial of deg(n-1)

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$T_n(x)$
L.T.O of first kind.

$$U_n(x) = \frac{\sin[(n+1)\cos^{-1}x]}{\sqrt{1-x^2}} = \text{poly of deg } n$$

- Tchebyshev poly of second kind



$$T_n(x) = \cos(n\cos^{-1}x) = 0$$

$$\Rightarrow \cos((2k+1)\pi/2) \quad k \in \mathbb{Z}/0$$

$$n\pi s^{-1} = (2k+1)\pi/2$$

$$\cos^{-1}x = (2k+1)\pi/2n$$

$$x = \cos[(2k+1)\pi/2n]$$

Ans

$$k = 1$$

$$c_{1 \text{ term}} = 1$$

$$\text{coeff. of } x^{100} = 1$$

Recursion Formula

$$1) T_n(x) = \cos(n\cos^{-1}x)$$

$$\begin{aligned}T_{n+1}(x) + T_{n-1}(x) &= \cos[(n+1)\cos^{-1}x] + \cos[n\cos^{-1}x] \\&= 2\cos(n\cos^{-1}x)\cos(\cos^{-1}x) \\&= 2T_n(x)x\end{aligned}$$

$$\boxed{T_{n+1}(x) + T_{n-1}(x) - 2xT_n(x) = 0}$$

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

$$T_0(x) = 1$$

$$T_1(x) = x$$

$$n = 1,$$

$$T_2(x) = 2x \cdot x - 1 = 2x^2 - 1$$

$$n = 2,$$

$$T_3(x) = 2x(2x^2 - 1) - x = 4x^3 - 3x$$

$$n = 3,$$

$$T_4(x) = 8x^4 - 8x^2 + 1$$

$$2) (1-x^2) T_n'(x) = -nxT_n(x) + nT_{n-1}(x)$$

$$T_n(x) = \cos(n\cos^{-1}x)$$

LHS

$$(1-x^2) T_n'(x)$$

$$\text{Put } x = \cos\theta ; \quad \text{LHS} = \sin^2\theta \frac{d}{d\theta}(\cos n\theta) \frac{d\theta}{dx}$$

$$= -n \frac{\sin^2\theta \sin n\theta}{-\sin^2\theta}$$

$$= n \sin n\theta \sin\theta = \frac{n}{2} [\cos(n-1)\theta - \cos(n+1)\theta]$$

$$\begin{aligned}
 & 2 \sin n\theta \cos \theta \\
 &= \cos(n-1)\theta - \cos(n+1)\theta \\
 &= \cos(n-1)\theta - [\cos n\theta \cos \theta - \sin n\theta \sin \theta] \\
 & 2 \sin \theta \sin \theta = \cos(n-1)\theta - \cos n\theta \cos \theta + \sin n\theta \sin \theta
 \end{aligned}$$

$$\Rightarrow \sin n\theta \sin \theta = \cos(n-1)\theta - \cos n\theta \cos \theta$$

$$-n \cos(n-1)\theta - n \cos n \cos \theta$$

$$\therefore (1-x^2) T_n'(x) = n T_{n-1}(x) - n x T_n(x)$$

$$\text{ii) } \int T_n(x) dx = ??$$

$$x = \cos \theta$$

$$\begin{aligned}
 I &= - \int \cos n\theta \sin n\theta d\theta = -\frac{1}{2} \int [\sin(n+1)\theta - \sin(n-1)\theta] d\theta \\
 &= -\frac{1}{2} \left[-\frac{\cos(n+1)\theta}{n+1} + \frac{\cos(n-1)\theta}{n-1} \right] + C \\
 &= \frac{1}{2} \left[-\frac{T_{n-1}(x)}{n-1} + \frac{T_{n+1}(x)}{n+1} \right] + C
 \end{aligned}$$

Orthogonality prop.

$$(1-x^2)y'' - 2xy' + n^2y = 0$$

$$x = \cos \theta$$

$$\frac{d^2y}{d\theta^2} + n^2 y_n = 0 \quad \text{--- (1)}$$

$$\frac{d^2y_m}{d\theta^2} + m^2 y_m = 0 \quad \text{--- (2)}$$

$$\textcircled{1} \times y_m - \textcircled{2} \times y_n$$

$$y_m y_n'' - y_n y_m'' + (n^2 - m^2) y_m y_n = 0$$

Integrating w.r.t θ over $[0, \pi]$

$$\frac{d}{d\theta} (y_m y_n' - y_n y_m') d\theta + (n^2 - m^2) \int_0^\pi y_m y_n d\theta = 0$$

$$[y_m y_n' - y_n y_m']_0^\pi + (n^2 - m^2) \int_0^\pi y_m y_n d\theta = 0$$

$$(n^2 - m^2) \int_0^\pi y_m y_n d\theta = 0$$

$$\int_0^\pi y_m(\theta) y_n(\theta) d\theta = 0, \quad m \neq n$$

$$\Rightarrow \int_{-1}^1 \frac{T_m(x) T_n(x)}{\sqrt{1-x^2}} dx = 0.$$

$$\int_0^\pi \cos^n \theta d\theta \quad y_m(\theta) = \cos m\theta \\ y_n(\theta) = \cos n\theta$$

$$= \begin{cases} \frac{1}{2}\pi, & n \neq 0 \\ \pi, & n = 0 \end{cases}$$

$$= \begin{cases} \int_{-1}^1 \frac{T_n^2(x)}{\sqrt{1-x^2}} dx = \begin{cases} \frac{\pi}{2}, & n \neq 0 \\ \pi, & n = 0 \end{cases} \end{cases}$$

Tchebyshev Series expansion

Let $f(x)$ be continuous having continuous derivative on $[-1, 1]$

$$e^x = f(x) = \sum_{n=0}^{\infty} a_n T_n(x)$$

$$\int_{-1}^1 \frac{f(x) T_m(x)}{\sqrt{1-x^2}} dx = a_m x \cdot \frac{\pi}{2}$$

$$\Rightarrow a_n = \frac{2}{\pi} \int_{-1}^1 \frac{f(x) T_n(x)}{\sqrt{1-x^2}} dx$$

$$a_0 = \frac{1}{\pi} \int_{-1}^1 \frac{f(x)}{\sqrt{1-x^2}} dx$$

1. Expand $\sin(\cos^{-1}x)$ in Tchebyshov series.

$$f(x) = \sin(\cos^{-1}x)$$

$$\sin(\cos^{-1}x) = \sum_{n=0}^{\infty} a_n T_n(x)$$

$$a_0 = \frac{1}{\pi} \int_{-1}^1 \frac{\sin(\cos^{-1}x)}{\sqrt{1-x^2}} dx$$

$$\begin{aligned} \cos^{-1}x &= \theta \\ dx &= d\theta \\ \sqrt{1-x^2} &= \sin\theta \end{aligned}$$

$$x = \cos\theta$$

$$a_0 = \frac{1}{\pi} \int_{-1}^1 \frac{\sin(\cos^{-1}\cos\theta)}{\sqrt{1-\cos^2\theta}} d\theta$$

$$= \frac{1}{\pi} \int_{-1}^1 \frac{\sin\theta}{\sin\theta} d\theta$$

$$= \frac{1}{\pi} [x]_{-1}^1 = \frac{2}{\pi}$$

$$a_m = 0.$$

$$a_2 = \frac{2}{\pi} \int_{-1}^1 \frac{\sin(\cos^{-1}x)}{\sqrt{1-x^2}} \cos(2\cos^{-1}x) dx$$

$$x = \cos\theta$$

$$= -\frac{4}{9\pi}$$

$$a_3 = 0$$

$$a_4 = -\frac{4}{T_0 n}$$

2. Expand $x^3 + 3x^2 - 5x + 2$ in T_n 's

$$\begin{aligned} & x^3 + 3x^2 - 5x + 2 \\ &= \frac{T_3 + 3T_2}{4} + 3 \frac{T_2 + T_0}{2} - 5T_1 + 2T_0 \\ &= \left(\frac{3}{2}\right)T_0 - \left(\frac{13}{4}\right)T_1 + \frac{3}{2}T_2 + \frac{1}{4}T_3 \end{aligned}$$

Generating f^n

$$\begin{aligned} & \sum_{n=0}^{\infty} T_n(x) t^n = ?? \\ & x = \cos \theta \\ &= \sum_{n=0}^{\infty} (\cos n\theta) t^n = \operatorname{Re} \left\{ \sum_{n=0}^{\infty} e^{in\theta} t^n \right\} \\ &= \operatorname{Re} \left\{ \sum_{n=0}^{\infty} (t e^{i\theta})^n \right\} \\ &= \operatorname{Re} \left\{ \frac{1}{1 - t e^{i\theta}} \right\} \end{aligned}$$

$$\frac{1 - t^4}{1 - 2xt + t^2}$$

Homogeneous

$$1. \quad y'' - 5y' + 6y = 0$$

$$2. \quad y''' + 9y = 0$$

$$3. \quad y''' + 8y' + 9y = 0, \quad y(0) = 2, \quad y'(0) = -3$$

$$4. \quad y''' - 5y'' + 7y' - 3y = 0$$

$$5. \quad y''' - 4y' + 13y = 0$$

dx

$$1. y'' - 5y' + 6y = 0 \quad (\text{linear eqn with const. coeff.})$$

$$\begin{aligned} y &= e^{mx} \\ y' &= m e^{mx} \\ y'' &= m^2 e^{mx} \end{aligned}$$

$$\begin{aligned} m^2 e^{mx} - 5m e^{mx} + 6e^{mx} &= 0 \\ \Rightarrow (m^2 - 5m + 6)e^{mx} &= 0 \\ \Rightarrow m^2 - 5m + 6 &= 0 \\ \Rightarrow m &= 3, 2. \end{aligned}$$

$y_1 = e^{3x}$ and $y_2 = e^{2x}$ are two solns.

If y_1 and y_2 are linearly independent.

General soln $y = Ay_1 + By_2$. A, B are arbitrary constants.

$$\begin{aligned} \text{Wronskian } (y_1, y_2) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ w(y_1, y_2) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \end{aligned}$$

If $w(y_1, y_2) = 0$ then y_1 and y_2 are linearly dependent. If $w(y_1, y_2) \neq 0$, then y_1 and y_2 are linearly independent.

$$w(y_1, y_2) = -e^{5x} \neq 0$$

$$\boxed{y = Ay_1^{3x} + By_2^{5x}}$$

$$(i) \quad y''' - 5y'' + 7y' - 3y = 0$$

$$y = e^{mx}$$

1-5+

$$m^3 - 5m^2 + 7m - 3 = 0$$

$$m = 1,$$

$$m = 1, 1, 0$$

$y_1 = e^{x^n}$

$$y = e^x (Ax + B) + Ce^{-3x}$$

Inhomogeneous Eqn

$$F(D)y = x(n) \quad \text{--- (1)}$$

$$F(D) = a \frac{d^2}{dx^2} + b \frac{d}{dx} + c$$

$$i) \quad X(x) = x^m$$

$$ii) \quad X(x) = e^{ax^n}$$

$$iii) \quad X(x) = \sin(ax+b) / \cos(ax+b)$$

$$iv) \quad X(x) = e^{ax} V(x), \quad V(x) = x^m / \sin(ax+b)$$

complementary (CP)

$$\leftarrow F(D)y = 0 \quad \text{--- (2)}$$

If $y = y_h(x)$ is the general sol'n of (2)

particular $y = y_p(x)$ is a particular sol'n of (1)
integrate Then $y = y_h(x) + y_p(x)$ is the general
(P.T) sol'n of (1).

$$1. (D^2 - 9) y = e^{3x} \cos x$$

$$P.O.I = \frac{1}{D^2 - 9} [e^{3x} \cos x]$$

Formula : $\frac{1}{F(D)} [e^{ax} v(x)] = e^{ax} \frac{1}{F(D+a)} [v(x)]$

$$P.O.I = \frac{1}{D^2 - 9} [e^{3x} \cos x]$$

$$= e^{3x} \frac{1}{(D+3)^2 - 9} [\cos x]$$

$$= e^{3x} \frac{1}{D^2 + 6D} [\cos x]$$

$$= e^{3x} \frac{D^2 - 6D}{D^4 - 36D^2} [\cos x]$$

$$= e^{3x} \frac{[-\cos x + 6 \sin x]}{37}$$

$$\rightarrow D^2 = -1$$

$$2. (D^2 + 2D + 2) y = xe^{-x}$$

$$3. (D^2 - 1) y = e^x \sin x$$

$$4. (D^2 + 4) y = x \cos \frac{x}{2}$$

$$5. (D^2 + 1) y = x^2 \sin 2x$$

$$6. \frac{d^2y}{dx^2} - y = xe^x \sin x$$

$$2. (D^2 + 2D + 2)y = xe^{-x}$$

$$P.O.I = \frac{1}{(D^2 + 2D + 2)} [xe^{-x}]$$

$$P.O.I = e^{-x} \cdot \frac{1}{(D-1)^2 + 2(D-1) + 2} [x]$$

$$= e^{-x} \cdot \frac{1}{D^2 + 1} [x]$$

$$= e^{-x} (1 + D^2)^{-1} [x]$$

$$= e^{-x} (1 - D^2 + \dots) [x]$$

$$= e^{-x} \cdot x$$

$$3. (D^2 - 1)y = e^x \sin \frac{x}{2}$$

$$P.O.I = \frac{1}{(D^2 - 1)} e^x \sin \frac{x}{2}$$

$$= e^x \cdot \frac{1}{(D+1)^2 - 1} [\sin \frac{x}{2}]$$

$$= e^x \cdot \frac{1}{(D+1)^2 - 1} [\sin \frac{x}{2}]$$

$$= e^x \cdot \frac{1(D^2 - 2D)}{(D^2 + 2D)(D^2 - 2D)} [\sin \frac{x}{2}]$$

$$= e^x \cdot \frac{1}{D(2+D)} [\sin \frac{x}{2}]$$

$$= e^x \cdot \frac{(D^2 - 2D)}{D^4 - 4D^2} [\sin \frac{x}{2}]$$

$$= e^x \cdot \frac{(D^2 - 2D)}{\frac{1}{16} + 4x \frac{1}{4}} [\sin \frac{x}{2}]$$

$D^2 = -\frac{1}{4}$

$$= e^x \cdot \frac{-\frac{1}{2}}{\frac{1}{16}} [\sin \frac{x}{2} - \cos \frac{x}{2}]$$

$$40. \quad (D^2 + 4) y = x \cos x$$

$$PI = \frac{1}{D^2 + 4} [x \cos x]$$

$$x = \frac{1}{D^2 + 4} [x \cos x] \quad y = \frac{1}{D^2 + 4} [x \sin x]$$

$$x + iy = \frac{1}{D^2 + 4} [xe^{ix}]$$

$$= e^{ix} \frac{1}{(D+i)^2 + 4} [z]$$

$$= e^{ix} \frac{1}{D^2 + 2iD + 3} [z] = e^{ix} \frac{1}{3 \left(1 + \frac{2iD + D^2}{3} \right)} [z]$$

$$= \frac{e^{ix}}{3} \left(1 + \frac{2iD + D^2}{3} \right)^{-1} [z]$$

$$= \frac{e^{ix}}{3} \left[1 - \frac{2iD + D^2}{3} + \dots \right] [z]$$

$$= \frac{e^{ix}}{3} \left(z - \frac{2i}{3} \right)$$

$$50. \quad (D^2 + 1)Y = x^2 \sin 2x$$

$$PI = \frac{1}{D^2 + 1} [x^2 \sin 2x]$$

$$Y = \frac{1}{D^2 + 1} [x^2 \sin 2x] \quad X = \frac{1}{D^2 + 1} [x^2 \cos 2x]$$

~~$x+i^y \neq 0$~~

$$x+i^y = \frac{1}{D^2 + 1} [x^2 e^{2ix}]$$

$$= e^{2ix} \frac{1}{(D^2 + 2i)^2 + 1} [x^2]$$

$$= e^{2ix} \frac{1}{D^2 - 3 + 4Di} [x^2]$$

$$= e^{2ix} \frac{1}{-3 \left(\frac{D^2 + 4Di}{3} \right)} [x^2]$$

$$= \frac{e^{2ix}}{-3} \left\{ 1 - \left(\frac{D^2 + 4Di}{3} \right) \right\} [x^2]$$

$$= -\frac{e^{2ix}}{3} \left\{ 1 + \frac{D^2 + 4iD}{3} + \left(\frac{D^2 + 4iD}{3} \right)^2 + \dots \right\} [x^2]$$

$$= -\frac{e^{2ix}}{3} \left\{ 1 + \frac{D^2 + 4iD}{3} + \left(\frac{4iD}{3} \right)^2 + \dots \right\} [x^2]$$

$$= -\frac{e^{2ix}}{3} \left[x^2 + \frac{1}{3}x^2 + \frac{4}{3} 2x + \left(\frac{-16}{9} \right) \cdot 2 \right]$$

~~$X = -\frac{x^2 \sin 2x}{3} + \frac{2x \sin 2x}{9} - \frac{32}{27} \sin 2x - \frac{8x \cos 2x}{9}$~~

$$Y = \left(-\frac{x^2}{3} + \frac{26}{27} \dots \right) \sin 2x - \frac{8x}{9} \cos 2x$$

$$6. \frac{d^2y}{dx^2} - y = xe^x \sin x \Rightarrow (D^2 - 1)y = xe^x \sin x$$

$$y = \frac{1}{(D^2 - 1)} [xe^x \sin x] \quad X = \frac{1}{(D^2 - 1)} [xe^x \cos x]$$

$$\textcircled{6} \quad X + iy = \frac{1}{(D^2 - 1)} (xe^x e^{ix}) = \frac{1}{D^2 - 1} (xe^{(i+1)x})$$

$$= \operatorname{Im} \left\{ e^{(i+1)x} \frac{1}{(D+1+i)^2 - 1} [x] \right\}$$

$$= \operatorname{Im} \left\{ e^{(i+1)x} \cdot \frac{1}{D^2 + (1+i)^2 + 2D(1+i) - 1} [x] \right\}$$

$$= e^{(i+1)x} \frac{1}{(D^2 + D(2i+2) + (2i-1))}$$

~~$$= \operatorname{Im} \left\{ e^{x(1+i)} \frac{x(1+i)}{D^2 + 1 - i + 2D + 2Di + 2i^2 - 1} [x] \right\}$$~~

$$= \operatorname{Im} \left\{ e^{x(1+i)} \frac{1}{(2i-1)^2 + 2D(1+i) + D^2} [x] \right\}$$

$$= \operatorname{Im} \left\{ \frac{e^{x(1+i)}}{2i-1} \left(x - \frac{2(1+i)}{2i-1} \right) \right\}$$

$$= \operatorname{Im} \left\{ \frac{e^{x(1+i)}}{(4i^2-1)^2} \left((2i+1)^2 [x(2i-1) - 2(1+i)] \right) \right\}$$

$$= \operatorname{Im} \left\{ \frac{1}{25} e^x (\cos x + i \sin x) (1 - 4 + 4i) [-x - 2 + i(2x - 2)] \right\}$$

~~$$= \operatorname{Im} \left\{ \frac{e^x}{25} \left[\frac{1}{3} \cos x - 4 \sin x \right] + i \left[(-3 \sin x + 4 \cos x) \right] \right\}$$~~

$$\{ (x+2) + i(2-2x) \}$$

$$= -\frac{e^x}{25} [(x+2)(4 \cos x - 3 \sin x) + (2-2x)(3 \cos x + 4 \sin x)]$$

$$= -\frac{e^x}{25} [\cos x (4x+8-6+8x) - \sin x (-3x-6-8+8x)]$$

$$= -\frac{e^x}{25} [(10x+2) \cos x + (5x-14) \sin x]$$

$$y''' - 5y'' + 7y' - 3y = 0 \quad 1, 3$$

$$e^x \quad e^{3x} \\ e^x (Ax + B) + Ce^{3x}$$

Inhomogeneous Eq"

$$F(D)y = x(x) \quad \text{--- } ①$$

$$F(D) = a \frac{d^2}{dx^2} + b \frac{d}{dx} + c, \quad a, b, c \text{ constant}$$

$$\text{i)} \quad x(x) = x^m$$

$$\text{ii)} \quad x(x) = e^{ax}$$

$$\text{iii)} \quad x(x) = \sin(ax+b) / \cos(ax+b)$$

$$\text{iv)} \quad x(x) = e^{ax} v(x), \quad v(x) = x^m / \sin(ax+b) /$$

$$F(D)y = 0 \quad \text{--- } ②$$

→ complementary f = C.F.

Let $y = y_u(x)$ is the general solution of ② and

$y = y_p(x)$ is a particular soln of ①.

Then $y = y_u(x) + y_p(x)$ is the general soln of ①

$$m = \frac{\alpha}{\beta} = e^{\frac{\alpha}{\beta}x} \\ y_1 = e^{(\alpha+i\beta)x} = A + iB \quad A(x) + iB(x)$$

$$\frac{d^2}{dx^2} (A+iB) + P \frac{d}{dx} (A+iB) + Q(A+iB) = 0$$

$$\left(\frac{d^2A}{dx^2} + P \frac{dA}{dx} + QA \right) + i \left(\frac{d^2B}{dx^2} + P \frac{dB}{dx} + QB \right) = 0$$

$$\left. \begin{aligned} \frac{d^2A}{dx^2} + P \frac{dA}{dx} + QA &= 0 \\ \frac{d^2B}{dx^2} + P \frac{dB}{dx} + QB &= 0 \end{aligned} \right\} \Rightarrow A(x), B(x) \text{ satisfy eq" } ①$$

$$m = m_1, m_2, m_3 \\ Ae^{m_1 x} + Be^{m_2 x} + Ce^{m_3 x}$$

$$(Ax^2 + Bx + C)e^{m_1 x} \\ e^{m_2 x} [(Ax + B) \cos \beta x + (Cx + D) \sin \beta x]$$

$$\frac{d^2y}{dx^2} + p \frac{dy}{dx} + qy = x(x) \quad \text{--- (1)}$$

1. Comp Func C.F. \rightarrow General soln of homogeneous eqn
2. Particular integral (P.I)
3. Gen soln to (1) is $y = PF + CF$

$$\begin{aligned} & \left(\frac{d^2y}{dx^2} + y \right) = e^x \\ \downarrow & \\ \text{C.F.} & \rightarrow A \sin x + B \cos x = y_1(x) \\ \text{P.I.} & \rightarrow \frac{1}{2} e^x \end{aligned}$$

$$y = A \sin x + B \cos x + \frac{1}{2} e^x \quad \boxed{\text{General soln of the inhomogeneous eqn}}$$

- i) $x(x) = x^n$
- ii) $x(x) = e^{ax}$
- iii) $x(x) = \sin(ax+b) / \cos(ax+b)$
- iv) $x(x) = e^{ax} v(x) \quad v(x) = \sin(ax+b) / x$

$$\frac{d^2y}{dx^2} + p \frac{dy}{dx} + qy = x(x)$$

$$\Rightarrow (D^2 - 1)y = e^{2x} \quad A e^{2x} + B e^{-2x}$$

$$(4-1)e^{2x} = 3e^{2x} \quad F(D) = D^2 + pD + q$$

$\boxed{3}$

$$F(D)e^{2x} = F(2)e^{2x}$$

$$\begin{aligned} \frac{1}{F(D)} F(D)[e^{ax}] &= \frac{1}{F(a)} F(a)[e^{ax}] \\ e^{ax} &= F(a) \frac{1}{F(D)} [e^{ax}] \\ &= (a^2 + p + q) e^{ax} \\ &= F(a) e^{ax} \end{aligned}$$

$$\therefore \boxed{\frac{1}{F(a)} [e^{ax}] = \frac{1}{F(D)} [e^{ax}] \quad F(a) \neq 0}$$

$$(D^2 - 1)y = e^{2x} \quad P.I. = \frac{1}{D^2 - 1} [e^{2x}] = \frac{1}{2^2 - 1} e^{2x} = \frac{1}{3} e^{2x}$$

8

Solve

$$(D+3)^2 y = 25 e^{-3x}$$

$$P.I. = \frac{1}{(D+3)^2} 25 e^{-3x} = \frac{25}{36} e^{-3x}$$

$$m^2 + 6m + 9 = 0$$

$$(Ax+B) e^{-3x} + \frac{25}{36} e^{-3x}$$

8 Solve

$$(D^2 - 4) y = e^{2x}$$

$$m^2 - 4 = 0$$

$$Ae^{\sqrt{2}x} + Ae^{-\sqrt{2}x} = 0$$

$$P.I. = \frac{1}{D^2 - 4} [e^{2x}]$$

$$(D^2 + 1)y = x^3$$

$$y = Ax^3 + Bx \quad \text{Method of undetermined co-efficient}$$

$$6Ax^2 + Ax^3 + Bx = x^3$$

$$A = 1 \quad B = -6$$

$$y = \underline{x^3 - 6x}$$

$$\begin{aligned} P.I. &= \frac{1}{(D^2 + 1)} [x^3] \\ &= (1 + D^2)^{-1} [x^3] \\ &= (1 - D^2 + D^4 - \dots) [x^3] \\ &\approx x^3 - 6x \end{aligned}$$

$$8 \quad (D^2 + 2D) y = e^{-x^2}$$

$$m^2 + 2m = 0 \quad m = -2, 0$$

$$\begin{aligned} & \frac{1}{(D^2 + 2D)^{-1}} [e^{-x^2}] \\ &= \frac{1}{D(D+2)^{-1}} [x^2] \\ &= \frac{1}{2D} \left[1 - \frac{D}{2} + \frac{D^2}{4} - \dots \right] [x^2] \\ &= \frac{1}{2D} (x^2 - x + \frac{1}{2}) \\ &= \frac{1}{2} \left(\frac{x^3}{3} - \frac{x^2}{2} + \frac{1}{2}x \right) \quad (D = 0) \\ y(x) &= \sin \end{aligned}$$

$$8 \quad (D^2 - 4) y = \sin 2x$$

$$y = A \sin 2x \quad = -\frac{1}{8} \sin 2x$$

$$-4A \sin 2x - 4A \sin 2x = \sin 2x$$

$$A = -\frac{1}{8}$$

$$F(D^2) y = \sin(bx+c)$$

$$\text{P.I.} \quad \frac{1}{F(D^2)} [\sin(bx+c)] = \frac{1}{F(-b^2)} [\sin(bx+c)]$$

$F(-b^2) \neq 0$

$$8 \quad (D^2 - D - 2) y = \sin 2x$$

$$\text{P.I.} : \frac{1}{(D^2 - D - 2)} [\sin 2x]$$

$$= \frac{D^2 - 2 + D}{(D^2 - 2)^2 - D^2} [\sin 2x]$$

$$= \frac{(D^2 + D - 2)}{(-4 - 2)^2 + 4} [\sin 2x] = \frac{(D^2 + D - 2)}{32} [\sin 2x]$$

$$= \frac{1}{32} [-4 \sin 2x + 2 \cos 2x - 2 \sin 2x]$$

$$= \frac{1}{32} [2 \cos 2x - 6 \sin 2x]$$

$$\mathcal{L} (x^2 + 2x + 1) = e^x + e^{-x}$$

$$\mathcal{L} \frac{x^2 y}{x^n} + y = \cos^2 x = \frac{\cos 2x + 1}{2}$$

$$F(D^2+1) \left(\frac{\cos(2x)+1}{2} \right)$$

$$= \frac{1}{2} \cdot \left(-\frac{1}{3} \right) \left(\frac{\cos(2x)+1}{2} \right)$$

$$= \frac{1}{-3} \cdot \frac{\cos 2x}{2} + \frac{1}{F(D^2+1)} \left(\frac{1}{2} \right)$$

$$+ \frac{1}{2}$$

$$x(x) = e^{ax} v(x) \quad v(x) = x^n, \cos(bx+c), \sin(bx+c)$$

$$D[e^{ax} v(x)] = e^{ax} Dv + av(x)e^{ax} \\ = e^{ax} (D+a)v(x)$$

$$F(D) = D^2 + PD + Q$$

$$F(D) [e^{ax} v(x)]$$

$$D [e^{ax} (D+a)v(x)]$$

$$= ae^{ax} (D+a)v(x)$$

$$+ e^{ax} (D+a)(D)v(x)$$

$$e^{ax} (D+a)^2 v(x)$$

$$= e^{ax} (D+a)^2 v(x) + P e^{ax} (D+a)v(x) + Q e^{ax} v(x)$$

$$= \cancel{e^{ax} v(x)} e^{ax} [(D+a)^2 + P(D+a) + Q] v(x)$$

$$e^{ax} F(D+a) v(x)$$

$$F(D) [e^{ax} v(x)] = e^{au} \underbrace{F(D+a)}_{V_1(u)} v(u)$$

$$e^{ax} v(x) = \frac{1}{F(D)} [e^{ax} V_1(u)]$$

$$V_1(u) = F(D+a) v(u)$$

$$\Rightarrow v(x) = \frac{1}{F(D+a)} [V_1(x)]$$

$$\frac{1}{F(D)} [e^{ax} V_1(u)] = e^{ax} \frac{1}{F(D+a)} [V_1(u)]$$

$$(D^2 + 2D + 2)y = xe^{-x}$$

$$P.I. = \frac{1}{D^2 + 2D + 2} [xe^{-x}]$$

$$= e^{-x} \frac{1}{(D-1)^2 + 2(D-1) + 2} [x]$$

$$= e^{-x} \frac{1}{D^2 + 1} [x]$$

$$= e^{-x} (1 - D^2 + D^4 \dots) [x]$$

$$= xe^{-x}$$

$$Q \quad (D^2 - 1) y = e^x$$

$$\frac{1}{D^2 - 1} [e^x]$$

$$D^2 + 2D$$

$$= e^x \frac{1}{(D+1)^2 - 1} \quad [1]$$

$$D(D+2)$$

$$= e^x \frac{1}{2D\left(1 + \frac{D}{2}\right)} \quad [1]$$

$$\frac{1}{2} \left[\frac{1}{D} - \frac{1}{D+2} \right] \quad [1]$$

$$= \frac{e^x}{2} \frac{1}{D} \left(1 - \frac{D^2}{2} + \dots \right) \quad [1]$$

$$\frac{1}{2} x - \frac{1}{4(1+D_2)}$$

$$= \frac{x e^x}{2}$$

$$- \frac{i}{4} (1+D_2)^{-1}$$

$$\frac{x}{2} - \frac{1}{4}$$

$$e^x \cdot \left(\frac{x}{2} - \frac{1}{4} \right)$$

$$C. F. \leftarrow$$

$$A e^{nx} + B e^{-nx}$$

$$(A+Bi)e^{nx}$$

$$8 \quad (D^2 + 4) y = \sin 2x$$

$$L \cdot I = \frac{1}{D^2 + 4} (\sin 2x)$$

$$Y = \frac{1}{D^2 + 4} [\sin 2x]$$

$$X = \frac{1}{D^2 + 4} [\cos 2x]$$

$$x + iy = \frac{1}{D^2 + 4} [e^{i(2x)}]$$

$$= e^{i(2x)} \frac{1}{(D+2i)^2 + 4} [1] \quad D^2 + 4iD \\ D(D+4i)$$

$$= e^{i(2x)} \frac{1}{D(D+4i)} [1]$$

$$= e^{i(2x)} \frac{1}{4iD} \left[1 + \frac{D^2}{4^2} + \dots \right] \left(1 + \frac{D}{4i} \right)^{-1}$$

$$= e^{i(2x)} \frac{1}{(4i)D} [1]$$

$$= \frac{e^{i(2x)}}{4i} x$$

$$= \frac{(\cos 2x + i\sin 2x) x}{4i}$$

$$\text{Im } \rightarrow x \frac{\cos 2x}{4i} = -i \frac{x \cos 2x}{4}$$

$$Y = -\frac{x \cos 2x}{4}$$

① Find a power series solution of the initial value problem,

$$\frac{d^2y}{dx^2} + x^2 \frac{dy}{dx} + xy = 0, \quad y(0) = 2, \quad y'(0) = 4$$

write atleast first four nonzero terms in each series.

② Find a power series soln of the initial value problem,

$$(x^2 - 1) \frac{d^2y}{dx^2} + 3x \frac{dy}{dx} + xy = 0, \quad y(0) = 4, \quad y'(0) = 6$$

③ solve $y'' + xy = 0$, together with $y(1) = 2$ and $y'(1) = 0$ using power series.