29/01/18

Determinant

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1(b_2c_3 - c_2b_3) - b_1(a_2c_3 - c_2a_3) + c_1(a_2b_3 - b_2a_3)$$

Properties :-

1) $$\begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\Delta$$

By Exchanging 2 rows $= -\Delta$

2) 2 Row or more identicle , $\Delta = 0$

3) $$\begin{vmatrix} a_1 + a_1' & b_1 + b_1' & c_1 + c_1' \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1' & b_1' & c_1' \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

4) $$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 + ca_2 & b_1 + cb_2 & c_1 + cc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$R_1' \to R_1 + cR_2$

$= \Delta.$

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5) $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} ca_1 & cb_1 & cc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = c\Delta$

1. Evalute the determinant :

$$\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \end{vmatrix}$$

$C_1' \to C_1 - C_2$

$C_2' \to C_2 - C_3$

$$= \begin{vmatrix} 0 & 0 & 1 \\ \alpha-\beta & \beta-\gamma & \gamma \\ \alpha^2-\beta^2 & \beta^2-\gamma^2 & \gamma^2 \end{vmatrix}$$

$$\cancel{= (\alpha-\beta)\{-(\gamma\beta^2-\gamma^2)\} + \alpha^2-\beta^2\{\beta-\gamma\}}$$

$$\cancel{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)}$$

$$\cancel{= (\alpha-\beta)(\beta-\gamma)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & \gamma \\ (\alpha+\beta) & (\beta+\gamma) & \gamma \end{vmatrix}}$$

$$= (\alpha-\beta) \times (\beta-\gamma) \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & \gamma \\ (\alpha+\beta) & (\beta+\gamma) & \gamma^2 \end{vmatrix}$$

$$= (\alpha-\beta)(\beta-\gamma) \begin{vmatrix} 1 & 1 \\ (\alpha+\beta) & (\beta+\gamma \end{vmatrix}$$

$$= (\alpha-\beta)(\beta-\gamma)(\beta+\gamma)(\alpha+\beta)$$

$$= (\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)$$

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2. $\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix}$

$$= \begin{vmatrix} 2(a+b+c) & a & b \\ 2(a+b+c) & b+c+2a & b \\ 2(a+b+c) & a & c+a+2b \end{vmatrix} \quad [C_1' \to C_1+C_2+C_3]$$

$$= 2(a+b+c) \begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}, \quad \begin{bmatrix} R_2' \to R_2 - R_1 \\ R_3' \to R_3 - R_1 \end{bmatrix}$$

$$= 2(a+b+c) \begin{vmatrix} 0 & -(a+b+c) & 0 \\ 0 & a+b+c & -(a+b+c) \\ 1 & a & c+a+2b \end{vmatrix}$$

$$= 2(a+b+c)\,(a+b+c)^2$$

$$= 2(a+b+c)^3$$

3. Evaluate :

$$\begin{vmatrix} (b+c)^2 & c^2 & b^2 \\ c^2 & (c+a)^2 & a^2 \\ b^2 & a^2 & (a+b)^2 \end{vmatrix}$$

$$= \frac{1}{a^2b^2c^2} \begin{vmatrix} (ab+ac)^2 & b^2c^2 & b^2c^2 \\ (ac)^2 & (bc+ab)^2 & a^2c^2 \\ a^2b^2 & a^2b^2 & (ac+bc)^2 \end{vmatrix} \quad \begin{matrix} C_1' \to a^2C_1 \\ C_2' \to b^2C_2 \\ C_3' \to c^2C_3 \end{matrix}$$

$$= \frac{1}{a^2b^2c^2} \begin{vmatrix} (ab+ac)^2 - b^2c^2 & 0 & b^2c^2 \\ 0 & (bc+ab)^2 - a^2c^2 & a^2c^2 \\ a^2b^2 - (ac+bc)^2 & a^2b^2 - (ac+bc)^2 & (ac+bc)^2 \end{vmatrix}$$

$[C_1' \to C_1 - C_3 \,;\, C_2' \to C_2 -$

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$$= \frac{(ab+bc+ca)^2}{a^2b^2c^2}\begin{vmatrix} ab+ac-bc & 0 & b^2c^2 \\ 0 & bc+ab-ac & a^2c^2 \\ ab-ac-bc & ab-ac-bc & (ac+bc)^2 \end{vmatrix}$$

$$= \frac{1}{a^2b^2}(ab+bc+ca)^2\begin{vmatrix} ab+ac-bc & 0 & b^2 \\ 0 & bc+ab-ac & a^2 \\ ab-ac-bc & ab-ac-bc & (a+b)^2 \end{vmatrix}$$

$$= \frac{1}{a^2b^2}(ab+bc+ca)^2\begin{vmatrix} ab+ac-bc & 0 & b^2 \\ 0 & bc+ab-ac & a^2 \\ -2ac & -2bc & 2ab \end{vmatrix}$$

$[R_3' \to R_3 - R_1 - R_2]$

$$= \frac{2}{a^2b^2}(ab+bc+ca)\begin{vmatrix} ab+ac-bc & 0 & b^2 \\ 0 & bc+ab-ac & a^2 \\ -ac & -bc & ab \end{vmatrix}$$

$$\left[\begin{matrix} R_1' \to R_1 - \frac{b}{a}R_3 \\ R_2' \to R_2 - \frac{a}{b}R_3 \end{matrix}\right]$$

$$= \frac{2}{a^2b^2}(ab+bc+ca)^2\begin{vmatrix} ab+ac & \frac{b^2c}{a} & 0 \\ \frac{a^2c}{b} & bc+ab & 0 \\ -ac & -bc & ab \end{vmatrix}$$

[Expanding in terms of 3^{rd} column]

$$= \frac{2}{a^2b^2}(ab+bc+ca)^2 \cdot ab\left[(ab+ac)(bc+ab) - abc^2\right]$$

$$= \frac{2}{ab}(ab+bc+ca)^2\left[ab(b+c)(c+a) - abc^2\right]$$

$$= 2(ab+bc+ca)^2\left[(b+c)(c+a) - c^2\right]$$

$$= 2(ab+bc+ca)^3.$$

Multiplication

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \begin{vmatrix} a_1' & b_1' \\ a_2' & b_2' \end{vmatrix} = \begin{vmatrix} a_1 a_1' + b_1 b_1' & a_1 a_2' + b_1 b_2' \\ a_2 a_1' + b_2 b_1' & a_2 a_2' + b_2 b_2' \end{vmatrix}$$

Symmetric determinant

$$\begin{vmatrix} a & b \\ b & c \end{vmatrix} \qquad \begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix} = 0$$

Skew-symmetric determinant

$$\begin{vmatrix} 0 & a \\ -a & 0 \end{vmatrix} . \qquad a_{ij} = -a_{ji}$$

$$a_{ii} = -a_{ii} \qquad i = j$$

$$\Rightarrow a_{ii} = 0$$

All odd order skew-symmetric det $= 0$.

$$\Delta = \begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix}$$

$$\Delta^T = \begin{vmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{vmatrix}$$

$$\Delta = \Delta^T$$

Δ^T = Transpose of Δ

$$\Delta = -\Delta$$

$$\Delta = 0$$

$$\begin{vmatrix} 0 & a \\ -a & 0 \end{vmatrix} = a^2$$

$$\begin{vmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{vmatrix} = (af - be + cd)^2$$

Factor :

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \end{vmatrix}$$

if $\alpha = \beta$ $\Delta = 0$

$\therefore (\alpha - \beta)$ is factor

if $\beta = \gamma$ $\Delta = 0$

$\therefore (\beta - \gamma)$ is factor

if $\alpha = \gamma$ $\Delta = 0$

$\therefore (\alpha - \gamma)$ is factor

$\Delta = K(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)$

for K $\alpha = 1$, $\beta = 0$, $\gamma = -1$

$\Delta = K(1)(1)(-2) = -2K$.

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{vmatrix}$$

$= -2$. $\therefore K = 1$

Prove that :

$$\begin{vmatrix} a^2 & (s-a)^2 & (s-a)^2 \\ (s-b)^2 & b^2 & (s-b)^2 \\ (s-c)^2 & (s-c)^2 & c^2 \end{vmatrix} = 2s^3(s-a)(s-b)(s-c)$$

if $s = 0$, 3 row same.

$\therefore (s-0)^3$ is a factor.

$s = \dfrac{a+b+c}{2}$

Let $s = a$

$a = \dfrac{a+b+c}{2}$

$$\left.\begin{aligned} a &= \frac{a+b+c}{2} \\ a-c &= b \\ a-b &= c \end{aligned}\right\} \text{ (1)}$$

$$\begin{vmatrix} a^2 & 0 & 0 \\ (a-b)^2 & b^2 & (a-b)^2 \\ (a-c)^2 & (a-c)^2 & c^2 \end{vmatrix}$$

Putting (1),

$$\Delta = \begin{vmatrix} a^2 & 0 & 0 \\ a^2 & b^2 & c^2 \\ b^2 & b^2 & c^2 \end{vmatrix}$$

similarly $(s-a)$ is a factor
$(s-b)$ "
$(s-c)$, "

$$\Delta = Ks^2 s(s-a)(s-b)(s-c)$$

Adjoint :

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$A_1 = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} \qquad B_1 = -\begin{vmatrix} a_2 & B_2 \\ a_3 & c_3 \end{vmatrix} \qquad C_1 = \begin{vmatrix} a_2 & b_2 \\ a_3 & c_3 \end{vmatrix}$$

$$\Delta' = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \frac{1}{a_1}\begin{vmatrix} a_1 A_1 & B_1 & C_1 \\ a_1 A_2 & B_2 & C_2 \\ a_1 A_3 & B_3 & C_3 \end{vmatrix}$$

$$\Delta\Delta' = \begin{vmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{vmatrix} = \Delta^3$$

1. Without expanding find the ~~[illegible]~~ value :

$$\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ac \\ 1 & c & c^2-ab \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \frac{1}{abc}\begin{vmatrix} a & a^2 & abc \\ b & b^2 & abc \\ c & c^2 & abc \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + \begin{vmatrix} 1 & a^2 & a \\ 1 & b^2 & b \\ 1 & c^2 & c \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$ (Interchanging 3rd colomn and 2nd column)

$= 0.$

Matrix

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

→ Skew - symmetric matrix

$$\begin{bmatrix} \cancel{a} & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$$

$a_{ij} = -a_{ji}$

$a_{ij} = 0$ if $i = j$

1. A can be represented as sum of symmetric and skew-symmetric matrix.

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

$$(A + A^T)^T = A^T + A = A + A^T$$

$\therefore$ $A + A^T$ is a symmetric matrix

$$(A - A^T)^T = A^T - A = -(A - A^T)$$

$\therefore$ $A - A^T$ is a skew symmetric matrix.

2. $A = P - Q$

P is singular matrix

Q is a skew - symmetric matrix

$$A^T = P^T + Q^T = P \cancel{-} Q$$

$$A + A^T = P - Q \cancel{+} + P + Q = 2P$$

$\varphi = \frac{1}{2}(A - A^T)$

$A = [a_{ij}]_{p\times q}$ $B = [b_{ij}]_{q\times r}$ $A \cdot B = [c_{ij}]_{p\times r}$

$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ $\cancel{B \neq \begin{bmatrix} a_1' & b_1' \\ b_1' \end{bmatrix}}$

$B = \begin{bmatrix} a_1' & b_1' \\ a_2' & b_2' \end{bmatrix}$

$A \cdot B = \begin{bmatrix} a_1 a_1' + b_1 a_2' & a_1 b_1' + b_1 b_2' \\ a_2 a_1' + b_2 a_2' & a_2 b_1' + b_2 b_2' \end{bmatrix}$

1. $(AB)^T = \cancel{A^T} \cdot B^T \cdot A^T$

proof: $(i,j)^{th}$ element of $(AB)^T$

$= (j,i)^{th}$ element of AB

$=$ Product of j^{th} row of A and i^{th} element of B.

$=$ Sum of product of j^{th} column A A^T and i^{th} row of B^T.

$=$ Sum of product of i^{th} row of B^T and j^{th} column of A^T

$= (i,j)^{th}$ element of $B^T A^T$

$= (AB)^T = B^T A^T$

2. $A \cdot (B \cdot C) = (AB) \cdot C$

3. $adj(A) =$ Transpose of the cofactor of A

cofactor of $A = ?$

$\begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}$

$= \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$

$A^{-1} = \frac{1}{\Delta} adj(A)$

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If $AB = BA = I$

the B is the inverse of A.

$$A \cdot adj(A) = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_1A_1 + b_1B_1 + cC_1 & a_1A_2 + b_1B_2 + c_1C_2 & a_1A_3 + b_1B_3 + c_1C_3 \\ A_2 a_2A_1 + b_2B_1\ c_2C_1 & a_2A_2 + b_2B_2 + c_2C_2 & a_2A_3 + b_2B_3 + c_2C_3 \\ a_3A_1 + b_3B_1 + c_3C_1 & a_3A_2 + b_3B_2 + c_3C_2 & a_3A_3 + b_3B_3 + c_3C_3 \end{bmatrix}$$

$$= \begin{bmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{bmatrix} = \Delta \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \Delta I$$

$$A \cdot \frac{1}{\Delta} adj(A) = I$$

$$\frac{1}{\Delta} adj\, A \cdot A = I$$

$$A^{-1} = \frac{1}{\Delta} adj\, A$$

1. Find :

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}^{-1}$$

cofactor of A $= \begin{bmatrix} \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} \\ \begin{vmatrix} 3 & 4 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 4 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} \end{bmatrix}$

$$= \begin{bmatrix} 1 & -5 & 3 \\ -2 & 8 & -5 \\ -1 & 6 & -4 \end{bmatrix}$$

$$adj(A) = \begin{bmatrix} 1 & -2 & -1 \\ -5 & 8 & 6 \\ 3 & -5 & -4 \end{bmatrix}$$

$$A^{-1} = -1 \begin{bmatrix} 1 & -2 & -1 \\ -5 & 8 & 6 \\ 3 & -5 & 4 \end{bmatrix}$$

2. $x - y + 2z = 1$
$x + y + z = 2$
$2x - y + z = 5$

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \\ 2 & -1 & 1 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

$$AX = B$$
$$A^{-1}(AX) = A^{-1}B$$
$$\Rightarrow (A^{-1}A)X = A^{-1}B$$
$$\Rightarrow IX = A^{-1}B$$
$$\Rightarrow X = A^{-1}B$$

$$\text{cofactor}(A) = \begin{bmatrix} \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} \\ -\begin{vmatrix} -1 & 2 \\ -1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} \\ \begin{vmatrix} -1 & 2 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & -3 \\ -1 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$$

$$\text{adj } A = \begin{bmatrix} 2 & -1 & -3 \\ 1 & -3 & 1 \\ -3 & -1 & 2 \end{bmatrix}$$

$$\Delta = |A| = -5$$

$$X = A^{-1}B = \frac{1}{-5} \begin{bmatrix} 2 & -1 & -3 \\ 1 & -3 & 1 \\ -3 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

$$= -\frac{1}{5} \begin{bmatrix} -15 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$$

Orthogonal matrix

If $AA^T = A^TA = I$

If A is said to be orthogonal matrix.

$$A = \frac{1}{3}\begin{bmatrix} -1 & -2 & 2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix}$$

$$AA^T = \frac{1}{3}\cdot\frac{1}{3}\begin{bmatrix} -1 & -2 & 2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = I$$

1. If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ show $A^3 = A^{-1}$.

$$\Rightarrow A^2 = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -2 & 2 & -3 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^4 = I$$

$$A^{-1}A^4 = A^{-1}I$$

$$A^3 = A^{-1}.$$

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Rank of a matrix

1. $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$

$\Delta_3 = 0$

$\Delta_2 \neq 0$

Rank = 2.

$= \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad R_3' \to R_3 - R_1 - R_2$

Order of highest order non-zero minor of a matrix is said to the rank of the matrix.

2. $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

$\sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 3 & 1 & -3 & -1 \\ 1 & 1 & -3 & -1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

$R_3' \to R_3 - R_1$

$R_4' \to R_4 - R_1$

$\sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. $\quad R_4' \to R_4 - R_2$

$R_3' \to R_3 - 3R_2$

$\sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\quad C_3' \to C_3 + 3C_2$

$C_4' \to C_4 + C_2$

→ Rank = 2.

3. Find Rank:

$\begin{bmatrix} 2 & 4 & 4 & -6 \\ 0 & 0 & 5 & -2 \\ 3 & 6 & 8 & -1 \\ 1 & 2 & -1 & 0 \end{bmatrix}$

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Eigen value or Eigen vectors of a matrix.

Given matrix = A

$|A - \lambda I| = 0$ $\qquad AX = \lambda X$

$(A - \lambda I) X = 0$

1. Find the eigen value and commonly eigen vector of the matrix.

$$\begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

$c_3' \to c_3 - c_1$

$c_4' \to c_4 - c_1$

$$|A - \lambda I| \begin{vmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix} = 0$$

$= (2-\lambda)\ (-1+\lambda-3) + 2(-1-\lambda-1) + 3(3-1+\lambda)$

$= (2-\lambda)(-1+\lambda-3) + 2(-1-\lambda-1) + 3(3-1+\lambda)$

$= (2-\lambda)\ (\lambda^2 - 4) - 2(\lambda+2) + 3(\lambda+2)$

$0 = (\lambda+2)[(2-\lambda)(\lambda-2) - 2 + 3]$

$0 = (\lambda+2)[-(\lambda^2 - 4\lambda + 4) + 1]$

$0 = (\lambda+2)[-\lambda^2 + 4\lambda - 3]$

$= -(\lambda+2)(\lambda^2 - 4\lambda + 3)$

$= -(\lambda+2)(\lambda-3)(\lambda-1)$

$\therefore$ The eigen values of the given matrix are $-2, 3, 1$.

$$\begin{bmatrix} 2+2 & -2 & 3 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$4x_1 - 2x_2 + 3x_3 = 0$
$x_1 + 3x_2 + x_3 = 0$

$$\frac{x_1}{-2-9} = \frac{x_2}{-1} = \frac{x_3}{12+2}$$

$$\Rightarrow \frac{x_1}{-11} = \frac{x_2}{-1} = \frac{x_3}{14} = k.$$

~~11,1,~~ $\begin{pmatrix} 11 \\ 1 \\ -14 \end{pmatrix}$ is the eigen vector ~~corrsengel to~~ corresponding to the eigen val -2.

— x —

1. Find the eigen value and eigen ~~values~~ vectors of the matrix

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Cayley Hamilton theorem

Statement :- Every characteristics equation of a square matrix satifies the matrix itself.

$$|A - \lambda I| = 0$$

$$(A - \lambda I) X = 0$$

$$(-1)^n \lambda^n + k_1 \lambda^{n-1} + K_2 \lambda^{n-2} + \dots + K_n = 0$$

$$\text{Adj}(A - \lambda I) = P_0 \lambda^{n-1} + P_1 \lambda^{n-2} + \dots + P_n$$

where $P_0, P_1, P_2, \dots P_n$ is a $n \times n$ matrix.

$\Rightarrow (A - \lambda I)\,\text{Adj}(A - \lambda I)$

$= (A - \lambda I)(P_1 \lambda^{n-1} + P_2 \lambda^{n-2} + \dots + P_n) = \cancel{|A - \lambda I| I_n}$

$= |A - \lambda I| I_n$

$$= \left[(-1)^n \lambda^n + k_1 \lambda^{n-1} + k_2 \lambda^{n-2} + \dots + k_n\right] I_n$$

comparing both sides the different parts of λ.

$$A^n \left[-P_0 = (-1)^n I_n \right]$$

$$A^{n-1} \left[AP_0 - P_1 = k_1 I_n \right] \quad \left[\text{Adding all eq}^n\right]$$

$$A^{n-2} \left[AP_1 - P_2 = k_2 I_n \right]$$

$$\vdots$$

$$AP_n = k_n I_n$$

$$(-1)^n A^n + k_1 A^{n-1} + k_2 A^{n-2} + \dots + k_n I_n = 0$$

1.

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix} \quad |A - \lambda I| = 0$$

characteristic eq^n :

$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 3-\lambda & -3 \\ -2 & -4 & -4-\lambda \end{vmatrix} = 0$$

$$= (1-\lambda)\left[(3-\lambda)(-4-\lambda) - 12\right] - 1\left[-4-\lambda-6\right] +$$

$$3\left[-4 + 2(3-\lambda)\right]$$

$$= (1-\lambda)\left[-12 - 3\lambda + 4\lambda + \lambda^2 - 12\right] + (4 + \lambda + 6) +$$

$$\left[-12 + 18 - 6\lambda\right]$$

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$$= (1-\lambda)\left[\lambda^2+\lambda-24\right] + 10+\lambda + 6-6\lambda$$

$$= \cancel{\lambda^2}+\lambda-24-\lambda^3-\cancel{\lambda^2}+24\lambda+16-5\lambda$$

$$= -\lambda^3+20\lambda-8$$

$$\lambda^3-20\lambda+8=0$$

$$A^3-20A+8I_3=0$$

$$A^{-1}\left(A^3-20A+8I_3\right)=0$$

$$A^2-20I_3+8A^{-1}=0$$

$$\Rightarrow 8A^{-1} = 20I_3 - A^2$$

$$\Rightarrow A^{-1} = \frac{1}{8}\left(20I_3-A^2\right)$$

~~Ag~~ Now,

$$A^2 = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1-6 & 1+3-12 & 3-3-12 \\ 1+3+6 & 1+9+12 & 3-9+12 \\ -2-4+8 & -2-12+16 & -6+12+16 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & -8 & -12 \\ 10 & 22 & 6 \\ 2 & 2 & 22 \end{bmatrix}$$

$$A^{-1} = \frac{1}{8}\left[\begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{bmatrix} - \begin{bmatrix} -4 & -8 & -12 \\ 10 & 22 & 6 \\ 2 & 2 & 22 \end{bmatrix}\right]$$

$$= \frac{1}{8}\begin{bmatrix} 24 & 8 & 12 \\ -10 & -2 & -6 \\ -2 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 3/2 \\ -5/4 & -1/4 & -3/4 \\ -1/4 & -1/4 & -1/4 \end{bmatrix}$$

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Properties :-

Two eigen vectors corresponding to two ~~distinct~~ difficult eigen values of a square matrix are independent

$x_1 \quad x_2 \qquad \lambda_1 \neq \lambda_2$

$A x_1 = \lambda_1 x_1$

$A x_2 = \lambda_2 x_2$

$c_1 x_1 + c_2 x_2 = 0 \Rightarrow c_1 = 0 = c_2$

$c_1 x_1 + c_2 x_2 = 0 \quad — (1)$

$c_1 A x_1 + c_2 A x_2 = 0$

$c_1 \lambda_1 x_1 + c_2 \lambda_2 x_2 = 0$

$c_1 \lambda_1 x_1 + c_2 \lambda_1 x_2 = 0 \rightarrow$ [multiplying λ_1 in eq. (1)]

$c_2 (\lambda_2 - \lambda_1) x_2 = 0 \rightarrow c_2 = 0$

$c_1 x_1 + 0 = 0$

$\Rightarrow c_1 = 0 \quad (x_1 \neq 0)$

$A x_1 = \lambda_1 x_1$

$(A x_1)^T = \lambda_1 x_1^T$

$x_1^T A^T = \lambda_1 x_1^T$

$\Rightarrow \bar{x}_1^T \bar{A}^T = \bar{\lambda} \bar{x}_1^T$

$\Rightarrow \bar{x}_1^T . A = \bar{x}_1 \bar{x}^T$

$\Rightarrow \bar{x}_1^T A x_1 = \bar{\lambda}_1 \bar{x}_1^T x_1$

$\bar{x}_1^T \lambda_1 x_1 = \bar{\lambda}_1 \bar{x}_1^T x_1$

$\Rightarrow (\bar{\lambda}_1 - \lambda_1) \bar{x}_1^T x_1 = 0$

$\Rightarrow \bar{\lambda}_1 = \lambda_1 (\text{sin } \bar{x}_1^T x_1 \neq 0)$

$\Rightarrow X_1$ is real.

1. Find the eigen vector and eigen values of the matrix

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

verify that the eigen vectors of a real symmetric matrix are orthogonal.

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Vector space

V is said to be a vector space over a field F if it satisfies the following conditions

i) $\alpha + \beta \in V \ \forall \ \alpha, \beta \in V$

ii) $\alpha + \beta = \beta + \alpha$ (Commutative property) $\forall \ \alpha, \beta \in V$

iii) $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$ (Associative property) $\forall \ \alpha, \beta, \gamma \in V$

iv) $\alpha + \theta = \alpha$ for all $\alpha \in V$, $\theta \in V$.
(θ is called zero element in V)

v) $\alpha + (-\alpha) = \theta \ \ \forall \ \alpha \in V$

vi) $c\alpha \in V$ if $\alpha \in V$, $c \in F$

vii) $c(d\alpha) = (cd)\alpha$, $\forall \ c, d \in F$, $\alpha \in V$.

viii) $(c+d)\alpha = c\alpha + d\alpha$, $c, d \in F$, $\alpha \in V$

ix) $c(\alpha + \beta) = c\alpha + c\beta$, $c \in F$, $\alpha, \beta \in V$

x) $1 \cdot \alpha = \alpha$ for all $\alpha \in V$,
1 is called the identity element.

Ex. $\mathbb{R}^n$ is a vector space over a field R

$(x_1, x_2, \ldots, x_n) + (a_1, a_2, \ldots, a_n)$
$= (a_1 + x_1, a_2 + x_2, \ldots, a_n + x_n)$

$c(x_1, x_2, \ldots, x_n) = (cx_1, cx_2, \ldots, cx_n)$, $c \in R$.

$\theta = (0, 0, \ldots, 0)$

complex number (C^n)

$(a+ib)+(c+id) = [a+c+i(b+d)]$

$p(a+ib) = pa + ipb$

set of all polynomials of degree $\leq n$.

$f = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$

$g = a_0 + a_1 x + a_2 x^2 + \dots + a_m x^m$

$h = b_0 + b_1 x + b_2 x^2 + \dots + b_m x^m +$

$b_{m+1} x^{m+1} + \dots + b_r x^r, \quad (m < r)$

$g+h = a_0 + b_0 + (a_1+b_1)x + (a_2+b_2)x^2 + \dots +$

$(a_m + b_m) x^m_r + b_{m+1} x^{m+1} + \dots + b_r x^r.$

$g+h = (a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2 + \dots$

$+ (a_r + b_r)x + a_{r+1} x^{r+1}$

$g+h = (a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2 + \dots +$

$(a_m+b_m)x^m + \dots + a_m x^n$ if $m = r$.

$c \cdot g = a_0 c + a_1 c x + a_2 c x^2 + \dots + a_m c x^m.$

Subspace

V is a vector space over a field F.
then W is said to be a subspace of V if $W \leq V$
and W is a vector space over a field F.

Suppose W is a subset of a vector space V ~~over a~~ over a field F. then W is said to be a subspace of V over field F.

i) $\alpha + \beta \in W$ if $\alpha, \beta \in W$

ii) $c \cdot \alpha \in W$ if $c \in F, \alpha \in W$.

Let S be a set of solution of the eqn

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = 0$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = 0$$

we want to show that S is a subspace of $\mathbb{R}^3$ over $\mathbb{R}$

(a,b,c) is a solution of these eqn.

(a',b',c') is a solution of this eqn

$$a_{11}a + a_{12}b + a_{13}c = 0$$
$$a_{21}a + a_{22}b + a_{23}c = 0$$
$$a_{11}a' + a_{12}b' + a_{13}c' = 0$$
$$a_{21}a' + a_{22}b' + a_{23}c' = 0$$

$$a_{11}(a+a') + a_{12}(b+b') + a_{13}(c+c') = 0$$
$$a_{21}(a+a') + a_{22}(b+b') + a_{13}(c+c') = 0$$

$(a,b,c) \in S$
$(a',b',c') \in S$

$(a+a', b+b', c+c') \in S$

$a_{11}ad + b_{2}bd + a_{13}cd = 0$

$d(a,b,c) \in S$

$a_{21}ad + a_{22}bd + a_{33}cd = 0.$

$\therefore$ S is a subspace of $\mathbb{R}^3$ over $\mathbb{R}$

If w_1, w_2 are two subspace of a vector space V over a field F then show that $w_1 \cap w_2$ is also a subspace of V over a field F.

$$\alpha \in w_1 \cap w_2 , \ \beta \in w_1 \cap w_2$$

$\Rightarrow \alpha \in w_1 , \ \alpha \in w_2 , \ \beta \in w_1, \ \beta \in w_2$

$\Rightarrow \alpha + \beta \in w_1 , \ \alpha + \beta \in w_2$

$\Rightarrow \alpha + \beta \in w_1 \cap w_2$

$c \cdot \alpha \in w_1 , \ c\alpha \in w_2$

$\Rightarrow c\alpha \in w_1 \cap w_2$

$\Rightarrow w_1 \cap w_2$ is a subspace of V over c field F.

consider a vector space $W : c\alpha , \ \alpha \in V$

$= \{ c\alpha , \ c \in F \} , \alpha \in W$

$c_1 \alpha + c_2 \alpha = (c_1 + c_2) \alpha \in W$

$c(c_1 \alpha) \in W$

$(c, c_1) \alpha$ ie $W,$

$\alpha\beta \in W, \ \alpha \in F, \ \beta \in W$

~~$\in \alpha$~~ $c\alpha + d\beta \in W$

$c\alpha + d\beta \notin W$

$e \cdot \gamma, \ \gamma \in W,$

—x—

~~Eg~~ $(1, 1, 1)$

$(2, 1, 1)$

$c(1,1,1) + d(2,1,1)$

$= (c+2d, c+d, c+d) \in W$

—x—

$S = \{(a, b, c) \mid a^2 + b^2 = c^2\}$

S.K. Mapa
Linear and abstract algebra.

$(-3, 4, 5)$

$(3, -4, 5)$

$(-3, 4, 5) + (3, -4, 5) = (0, 0, 10)$

—x—.

W_1, W_2 V over a field F

$Q = \{\alpha(0,1,0)\}$, $\alpha \in F$ in a subgroup of R^3

$P \cup Q$ $\alpha = a(1, 0, a)$ $a \in F$

$\beta = b(0, 1, 0)$, $b \in F$

$\alpha + \beta = (a, b, 0) \notin P \cup Q$

1. In a vector space V over a field F

i) $0 \cdot \alpha = \theta$ for all $\alpha \in N$

ii) $c \cdot \theta = \theta$ for all $c \in F$

iii) $(-1)\alpha = -\alpha$

iv) $c\alpha = \theta \rightarrow c = 0$ or $\alpha = \theta$

i) $0 + 0 = 0$

$(0+0) \cdot \alpha = 0 \cdot \alpha$

$\Rightarrow 0 \cdot \alpha + 0 \cdot \alpha = 0 \cdot \alpha$

$\Rightarrow -0 \cdot \alpha + (0 \cdot \alpha + 0 \cdot \alpha) = -0 \cdot \alpha + 0 \cdot \alpha$

$\Rightarrow (-0\cdot\alpha + 0\cdot\alpha) + 0\cdot\alpha = \theta$

$\Rightarrow \quad 0 + 0\cdot\alpha = \theta.$

$\Rightarrow \quad 0\cdot\alpha = \theta$

ii) $\quad 0 + \theta = \theta$

$c\cdot(\theta + \theta) = c\cdot\theta$

$\Rightarrow \quad c\cdot\theta + c\cdot\theta = c\cdot\theta$

$\Rightarrow \quad -c\cdot\theta + (c\cdot\theta + c\cdot\theta) = -c\cdot\theta + c\cdot\theta$

$\Rightarrow \quad (-c\cdot\theta + c\cdot\theta) + c\cdot\theta = \theta$

$\Rightarrow \quad \theta + c\cdot\theta = \theta$

$\Rightarrow \quad c\cdot\theta = \theta$

iii) $\quad 1 + (-1) = 0$

$1\cdot\alpha + (-1)\cdot\alpha = 0\cdot\alpha = \theta$

$\Rightarrow \quad \alpha + (-1)\alpha = \theta$

$\Rightarrow \quad -\alpha + [\alpha + (-1)\alpha] = -\alpha + \theta$

$\Rightarrow \quad (-\alpha + \alpha) + (-1)\cdot\alpha = -\alpha$

$\Rightarrow \quad 0 + (-1)\alpha = -\alpha$

$\Rightarrow \quad (-1)\alpha = -\alpha.$

iv) $\quad c\cdot\alpha = \theta \quad$ suppose, $c \neq 0.$

$c^{-1}(c\cdot\alpha) = c^{-1}\cdot\theta = \theta.$

$\Rightarrow \quad (c^{-1}\cdot c)\cdot\alpha = \theta$

$\Rightarrow \quad 1\cdot\alpha = \theta$

$\Rightarrow \quad \alpha = \theta.$

By contrapositivity we can say that

$\alpha \pm \theta$

$\Rightarrow c = 0.$

2. If U, W be two subspace of a vector space V over a field F. The the subset $\{u+w / u \in U, w \in W\}$ forms a subspace of V and is the smallest subspace of V containing the subspace U and W.

$$U+W = \{u+w \mid u \in U, w \in W\}$$

$$\alpha, \beta \in U+W$$

$$\alpha = u_1 + w_1 \mid u_1 \in U, w_1 \in W$$

$$\beta = u_2 + w_2 \mid u_2 \in U, w_2 \in W$$

$$\alpha+\beta = (u_1+u_2) + (w_1+w_2)$$

$$= u_3 + w_3 \qquad u_3 = u_1 + u_2 \in U$$

$$w_3 = w_1 + w_2 \in W$$

we have to check whether

$$c \cdot \alpha \in U+W$$

$$c \cdot \alpha = cu_1 + cw_1 \in U+W$$

P is a subspace containing U and W.

$$\alpha \in U+W$$

$$\alpha = u+w \mid u \in U, w \in W$$

$$u \in U \Rightarrow u \in P$$

$$w \in W \Rightarrow w \in P$$

$$\left.\begin{array}{l} u+w \in P \\ cu \in P \end{array}\right\} \Rightarrow \alpha = u+w \in P$$

$$\Rightarrow \alpha \in U+W$$

$$\Rightarrow \alpha \in P$$

$$\Rightarrow U+W \subseteq P$$

$\therefore$ $U+W$ is smallest subspace of vector space V.

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3. Let S be a finite set of ~~vat~~ vectors in a vector space V over a field F. The set of all linear combination of the vectors in S form a subspace of V and this is the smallest subgroup containing S.

$S = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$

$\alpha = \sum_{i=1}^{n} c_i \alpha_i$ $\quad \therefore W = \{\sum_{i=1}^{n} c_i \alpha_i, c_i \in F \forall i\}$

$\beta = \sum_{i=1}^{n} d_i \alpha_i$

$\alpha \in W, \beta \in W$

$\alpha + \beta = \sum_{i=1}^{n} ((c_i + d_i)) \alpha_2$, and $c_i + d_i \in F$

$\in W$

$c \cdot \alpha = \sum_{i=1}^{n} c \cdot c_i \alpha_i$, $c, c_i \in F$

$\Rightarrow c, \alpha \in W$

P is subspace of V over a field F containing S.

$\alpha \in W$ implies

$\alpha = \sum_{i=1}^{n} a_i \alpha_i \in W$

$\alpha_1 \in P \Rightarrow a_i \alpha_i \in P$

$\Rightarrow \sum_{i=1}^{n} a_i \alpha_i \in P$

$\alpha \in W \Rightarrow \alpha \in P$

$\Rightarrow W \subseteq P$

W is the smallest subspace containg P.

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Defination : The smallest subspace containing a finite set S of vectors of a vector space V is said to be the linear space of S and a denoted by $L(S)$.

Result : If S and T be two finite subsets of vectors of a vector space V and $S \subseteq T$ then $L(S) \subseteq L(T)$.

$$S = \{\alpha_1, \alpha_2, \dots \alpha_n\}$$

we shall show that

$$\alpha \in L(S) \Rightarrow \alpha \in L(T)$$

$$\alpha \in L(S) \Rightarrow \alpha = \sum_{i=1}^{n} c_i \alpha_i$$

$$\alpha_i \in S \rightarrow \alpha_i \in T \Rightarrow \alpha_i \in L(T)$$

$$\alpha = \sum_{i=1}^{n} c_i \alpha_i \in L(T)$$

Defination: If S and T be two finite subgroup of a vector space V over a field F and each elements of T be linear combination of the vector of S then $L(T) \subseteq L(S)$

Result :-

$$S = \{\alpha_1, \alpha_2, \dots \alpha_n\}$$

$$T = \beta_i = c_{i1}\alpha_1 + c_{i2}\alpha_2 + \dots + c_{in}\alpha_n.$$

$$c_i \in F \;\forall i$$

$$T = \{\beta_1, \beta_2, \dots \beta_n\}$$

$$\alpha = \sum_{i=1}^{m} d_i \beta_i \in L(T)$$

$$= \sum_{i=1}^{n} b_i \alpha_i \in L(S)$$

$$\Rightarrow \alpha \in L(T) \Rightarrow \alpha \in L(S)$$

$$\therefore L(T) \subseteq L(S).$$

4. Determine the subspace of $\mathbb{R}^3$ spanned by the vectors $\alpha = (1,2,3)$, $\beta = (3,1,0)$. Examine if $\gamma = (2,1,3)$, $\delta = (-1,3,6)$ are in the subgroup.

$$\{a\alpha + b\beta , a,b \in F\}$$

$$= \{a(1,2,3) + b(3,1,0), a,b \in F\}$$

$$= \{(a+3b, 2a+b, 3a), a,b \in F\}$$

$$a+3b = 2, \quad 2a+b = 1, \quad 3a = 3$$
$$\Rightarrow a = 1$$

$$2+b = 1 \Rightarrow b = -1.$$

This is inconsistant equation.

$$a + 3b = -1 \quad \text{—(i)}$$

$$2a + b = 3 \quad \text{—(ii)}$$

$$3a = 6 \text{—(iii)} \Rightarrow a = 2, \ b = -1 \Big\} \text{ This satisfies eq. (i),}$$

$\therefore$ It is consistant.

Defination: A finite set of vectors $\{\alpha_1, \alpha_2, \dots \alpha_n\}$ of a vector space V over a field F is said to be linearly dependent if there exists scalars $c_1, c_2, \dots c_n$ not all zero, in F ~~sat~~ such that

$$c_1\alpha_1 + c_2\alpha_2 + \dots + c_n\alpha_n = 0 \quad \text{—(1)}.$$

The set said to be linearly independent in V if the equality (1) is satisfied only when $c_1 = c_2 = \dots\dots c_n = 0$.

1. Examine linear dependent of the of the set of vowels $S = \{\alpha_1, \alpha_2, \alpha_3\}$ in $\mathbb{R}^3$ where $\alpha_1 = \{1,0,1\}$ $\alpha_2 = \{0,1,1\}$ and $\alpha_3 = (1,1,1)$.

$c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = (0,0,0)$

$c_1(1,0,1) + c_2(0,1,1) + c_3(1,1,1)$

$$\left.\begin{aligned} c_1 + c_3 &= 0 \\ c_2 + c_3 &= 0 \\ c_1 + c_2 + c_3 &= 0 \end{aligned}\right\} \Rightarrow c_2 = 0,\ c_3 = 0,\ c_1 = 0$$

Definition: Let V be a vector space over a field F, A set S of vectors in V is said to be a basis of V if

i) S is linearly independent and.

ii) S generated V

$$V = L(S)$$

2. Prove that the set $S = \{(1,0,1), (0,1,1), (1,1,0)\}$ is a axis of $\mathbb{R}^3$.

$c_1(1,0,1) + c_2(0,1,1) + c_3(1,1,0) = (0,0,0)$

$c_1 + c_3 = 0$, $c_1 + c_2 = 0$, $c_2 + c_3 = 0$

$2(c_1 + c_2 + c_3) = 0$

$\Rightarrow c_1 + c_2 + c_3 = 0$

$\Rightarrow c_1 = c_2 = c_3 = 0.$

Any point in $\mathbb{R}^3$ is (x, y, z)

$= c_1(1,0,1) + c_2(0,1,1) + c_3(1,1,0)$

$c_1 + c_3 = x$ — (i)

$c_2 + c_3 = y$ — (ii)

$c_1 + c_2 = z$ — (iii)

Adding (1), (2) (3)

$2(c_1 + c_2 + c_3) = x + y + z$

$c_1 + c_2 + c_3 = \frac{1}{2}(x + y + z)$

$c_2 = \frac{1}{2}(x + y + z) - x$

$c_1 = \frac{1}{2}(x+y+z) - y$

$c_3 = \frac{1}{2}(x+y+z) - z.$

$\therefore$ This set forms a basis.

3. Let $S = \{\alpha, \beta\}$ and $T = \{\alpha, \beta, \alpha+\beta\}$ where $\alpha, \beta \in \mathbb{R}^n$.

Show that $L(S) = L(T)$

$S \subseteq T \Rightarrow L(S) \subseteq L(T)$

we have to show that $L(T) \subseteq L(S)$.

$S \subseteq T \Rightarrow L(S) \subseteq L(T)$

$\gamma = c_1\alpha + c_2\beta + c_3(\alpha+\beta)$

$= (c_1+c_3)\alpha + (c_2+c_3)\beta \in L(S)$

Deletion theorem

If a vector space V over a field F be separated by a linearly dependent set $\{\alpha_1, \alpha_2, \alpha_3 \dots \alpha_n\}$, then V can also be generated by a suitable proper subset of $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$.

1. Let $S = \{x, y, z \in \mathbb{R}^3 \mid x+y+z=0\}$

Prove that S is a subspace of $\mathbb{R}^3$. Find a basis for S.

$\alpha_1 = x_1 + y_1 + z_1 \in S$

$\Rightarrow x_1 + y_1 + z = 0$

$\alpha_2 = x_2 + y_2 + z_2 \in S$

$\Rightarrow x_2 + y_2 + z_2 = 0$

$\alpha_1 + \alpha_2 = (x_1+x_2) + (y_1+y_2) + (z_1+z_2) \in S$

$c\alpha_1 = cx_1 + cy_1 + cz_1 \in R^3$

if $cx_1, cy_1, cz_1 \in R$

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$\Rightarrow S$ is a subspace of $\mathbb{R}^3$.

$\alpha = \{-x-y, x, y\} \in S$

$= \{x(-1,1,0) + y(-1,0,1)\} \in S$

Linear combination of the vectors

$(-1, 1, 0), (-1, 0, 1) \in S$.

The vectors $(-1, 1, 0)$, $(-1, 0, 1)$ generates S.

$c_1(-1, 1, 0) + c_2(-1, 0, 1) = (0, 0, 0)$

$$\left.\begin{aligned} c_1 - c_2 &= 0 \\ c_1 &= 0 \\ c_2 &= 0 \end{aligned}\right\} \quad c_1 = c_2 = 0$$

— x —

Def : The no. of vectors in a basis of a vector space V is said to be the dimension of V and is denoted by dim (V). If V be a null space it has no basis and it is said to be of dimension 0.

Problem :- Let $S = \{(x, y, z) \mid x + y + z = 0\}$. Prove that S is a subspace of $\mathbb{R}^3$. Find a basis for S and dim(S).

$(-1, 0, 1)$ $(-1, 1, 0)$ $\dim(S) = 2$

Result :- If U be a subspace of finites dimensional under vector space V and $\dim(V) = n$, then U is finite dimensional and $\dim(n) \leq n$

$U = \{\theta\}$

$\alpha \in V$ $L(\alpha) \neq V$

$L(\alpha, \beta) = V$

$\dim.(V) \leq n$

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continue using pen also

Let U and W be two subspaces of a finites dimension vector space V over a feild F thas

$$\dim(V+W) = \dim(V) + \dim(W) - \dim(V \cap W)$$

$$\dim(U \cap W) \leq \dim(U) \leq \dim(U+W) \leq \dim(V)$$

$$\dim(U \cap W) \leq \dim(W) \leq \dim(U+W) \leq \dim(V)$$

$$U \cap W \cdot \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$$

$\{\alpha_1, \alpha_2, \ldots, \alpha_r, \beta_1, \beta_2, \ldots, \beta_s\}$ basis vectors of U

$\{\alpha_1, \alpha_2, \ldots \alpha_r, \gamma_1, \gamma_2, \ldots, \gamma_t\}$ basis vector of W

we want to prove the basis vectors of $U+W$ are :

$$B = \{\alpha_1, \alpha_2, \ldots, \alpha_r, \beta_1, \beta_2, \ldots \beta_s, \gamma_1, \gamma_2, \ldots, \gamma_t\}$$

$$L(B) = U+W$$

$$\alpha \in L(B) \Rightarrow \alpha \in U \text{ or } \alpha \in W \Rightarrow \alpha \in U+W$$

$$\Rightarrow L(B) \subseteq U+W$$

Suppose, $\alpha \in U+W$, then there exists $u \in U$ and $w \in W$.

such that $\alpha = u + w$

$$u \in L(\alpha_1, \alpha_2, \ldots \alpha_r, \beta_1, \beta_2, \ldots \beta_s)$$

$$w \in L(\alpha_1, \alpha_2, \ldots \alpha_r, \gamma_1, \gamma_2, \ldots \gamma_t)$$

$$L(B) = U+W$$

The vector of B are independent.

$$\sum_{i=1}^{r} a_i \alpha_i + \sum_{i=1}^{s} b_i \beta_i + \sum_{i=1}^{t} c_i \gamma_i = 0$$

$$\underbrace{\sum_{i=1}^{r} a_i \alpha_i + \sum_{i=1}^{s} b_i \beta_i}_{\in U} = -\sum_{i=1}^{t} c_i \gamma_i$$

$$-\sum c_i \gamma_i \in U \cap W \Rightarrow -\sum_{i=1}^{t} c_i \gamma_i = \sum_{i=1}^{r} d_i \alpha_i$$

$$\sum_{i=1}^{t} c_i \gamma_i + \sum_{i=1} d_i \alpha_i = \theta$$

$$c_i = 0,\ d_i = 0\ \forall i$$

$$\sum_{i=1}^{\gamma} a_i \alpha_i + \sum_{i=1}^{M} b_i \beta_i = \theta$$

$$a_i = 0 = b_i\ \forall i$$

and $c_i = 0\ \forall i$

The vector of B are ~~l.i.~~ independent

$$\sum_{i=1}^{s} a_i \alpha_i + \sum_{i=1}^{M} b_i \beta_i + \sum_{i=1}^{t} c_i \gamma_i = \theta$$

$$\underbrace{\sum_{i=1}^{\gamma} a_i \alpha_i + \sum_{i=1}^{M} b_i \beta_i}_{\in U} = \underbrace{-\sum_{i=1}^{t} c_i \gamma_i}_{\in W}$$

$$\sum c_i \gamma_i \in U \cap W$$

$$-\sum_{i=1}^{t} c_i \gamma_i = \sum_{i=1}^{M} d_i \alpha_i$$

$$\sum_{i=1}^{t} c_i \gamma_i + \sum_{i=1}^{M} d_i \alpha_i = \theta$$

$$c_i = 0,\ d_i = 0\ \forall i$$

$$\sum_{i=1}^{M} \alpha_i a_i + \sum_{i=1}^{s} b_i \beta_i = \theta$$

$$a_i = 0 = b_i\ \forall i \text{ and } c_i = 0\ \forall i$$

$U+W$ has the basis values $\{\alpha_1, \alpha_2, \ldots \alpha_M, \beta_1, \beta_2, \ldots \beta_s, \gamma_1, \gamma_2, \ldots, \gamma_t\}$

$$\begin{aligned} \dim(U+W) &= t + s + M \\ &= M + s + M + t - M \\ &= \dim(V) + \dim(W) - \dim(U \cap W) \end{aligned}$$

Def^n: Two ~~subgroup~~ subspaces U and W of a vector space V are said to be complement of each other if $U \cap W = \{\theta\}$ and $U + W = V$. If V be a finish dimensinal vector space and U and W are complements of each other in V the $\dim(V) = \dim(U) + \dim(W)$.

If $U = L\{(1,2,1), (2,1,3)\}$ and
$W = L\{(1,0,0), (0,0,1)\}$

Q) Show that U and W are subspace of $\mathbb{R}^3$.
Determine $\dim(V)$, $\dim(W)$, $\dim(U+W)$

$$c_1(1,2,1) + c_2(2,1,3) = (0,0,0)$$

$$\left.\begin{array}{l} c_1 + 2c_2 = 0 \\ 2c_1 + c_2 = 0 \\ c_1 + 3c_2 = 0 \end{array}\right\} \quad \begin{array}{l} c_1 + c_2 = 0 \Rightarrow c_1 = -c_2 \\ -c_2 + 3c_2 = 0 \\ \Rightarrow c_2 = 0 \\ c_1 = 0 \end{array}$$

$\dim(U) = 2$.

$$c_1(1,0,0) + c_2(0,0,1) = (0,0,0)$$

$$c_1 = 0$$
$$c_2 = 0$$

$\dim(W) = 0$

$$c_1(1,2,1) + c_2(2,1,3) = d_1(1,0,0) + d_2(0,0,1)$$

$$c_1 + 2c_2 = d_1$$

$$2c_1 + c_2 = 0 \Rightarrow c_2 = -2c_1$$

$$c_1 + 3c_2 = d_2 \Rightarrow c_1 - 4c_1 = d_1$$
$$\Rightarrow -3c_1 = d_1$$
$$-5c_1 = d_2$$

$$-3c_1(1,0,0) - 5c_1(0,0,1)$$
$$= c_1(-3,0,0) + c_1(0,0,-5)$$
$$= c_1(-3,0,-5)$$

$U \cap W = L\{(-3,0,-5)\}$ $\dim(U \cap W) = 1$

$\dim(V+W) = \dim(V) + \dim(W) - \dim(U \cap W)$

$= 2 + 2 - 1 = 3$

$\dim(U \cap W) = 1$

Q) $U = L\{(1,2,1),(2,1,3)\}$

Find two different complements of U in $\mathbb{R}^3$

$W = L\{(1,0,0)\}$

$(x,y,z) \in \mathbb{R}^3$

$(x,y,z) = c_1(1,2,1) + c_2(2,1,3) + c_3(1,0,0)$

$x = c_1 + 2c_2 + c_3$

$y = 2c_1 + c_2$

$z = c_1 + 3c_2$

$y = 2c_1 + c_2$

$2z = 2c_1 + 6c_2$

$y - 2z = -5c_2$

$c_2 = -\frac{1}{5}(y-2z)$

$2c_1 = y + \frac{1}{5}(y-2z)$

$c_1 = \frac{y}{2} + \frac{1}{10}(y-2z)$

$x = c_1 + 2c_2 + c_3$

$= \frac{y}{2} + \frac{1}{10}(y-2z) - \frac{2}{5}(y-2z) + c_3$

$= \frac{y}{2} - \frac{3}{10}(y-2z) + c_3$

$c_3 = x - \frac{y}{2} + \frac{3}{10}(y-2z)$

$c_2 = \frac{2z-y}{5} = 0$

Replacement theorem

If $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be a basis of a vector space V over a field F and a non zero vector β of V is expressed as $\beta = c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + \dots + c_n\alpha_n$, $c_i \in F$.

Then if $c_3 \neq 0$, $\{\alpha_1, \alpha_2, \dots \alpha_{j-1}, \beta, \alpha_{j+1}, \dots, \alpha_n\}$ new basis of V [that is β can replace α_j]

$$\beta = c_1\alpha_1 + c_2\alpha_2 + \dots + c_{j-1}\alpha_{j-1} + c_j\alpha_j + c_{j+1}\alpha_{j+1} + \dots + c_n\alpha_n.$$

$$\alpha_j = c_j^{-1}\beta - c_3^{-1}c_1\alpha_1 - c_3^{-1}c_2\alpha_2 - \dots - c_3^{-1}c_{j-1}\alpha_{j-1} - c_3^{-1}c_{j+1} - \dots - c_j^{-1}c_n\alpha_n$$

$$d_1\alpha_1 - d_2\alpha_2 + \dots + d_{j-1}\alpha_{j-1} + d_j\beta + d_{j+1}\alpha_{j+1} + \dots + d_n\alpha_n = \theta$$

$$d_1\alpha_1 + d_2\alpha_2 + \dots + d_{j-1}\alpha_{j-1} + d_j(c_1\alpha_1 + c_2\alpha_2 + \dots + c_n\alpha_n) + d_{j+1}\alpha_{j+1} + \dots d_n\alpha_n = 0$$

$$\Rightarrow (d_1 + d_jc_1)\alpha_1 + (d_2 + d_jc_2)\alpha_2 + (d_{j-1} + d_jc_{j-1})\alpha_{j-1} + d_jc_j\alpha_j$$

$$\Rightarrow (d_1 + d_jc_1)\alpha_1 + (d_2 + d_jc_2)\alpha_2 + (d_{j-1} + d_jc_{j-1})\alpha_{j-1} + d_jc_j\alpha_j + (d_{j+1} + d_jc_{j+1})\alpha_{j+1} + \dots + (d_n + d_jc_n)\alpha_n = \theta$$

$$\Rightarrow d_1 + d_jc_1 = 0,\ d_2 + d_2c_2 = 0 \dots d_{j-1} + d_jc_{j+1} = 0,$$

$$d_jc_j = 0 \Rightarrow d_j = 0$$

$$d_{j+1} + d_jc_{j+1} = 0$$

$$d_n + d_jc_n = 0.$$

$d_1 = 0, \quad d_2 = 0, \ldots, \quad d_{j-1} = 0, \quad d_{j+1} = 0, \ldots$

$d_n = 0.$

$$\alpha = b_1 \alpha_1 + b_2 \alpha_2 + \cdots + b_n \alpha_n$$

$$= b_1 \alpha_1 + b_2 \alpha_2 + \cdots + b_{j-1} \alpha_{j-1}$$

$$+ b_j (c_j^{-1} \beta - c_j^{-1} c_1 \alpha_1 - c_j^{-1} c_2 \alpha_2 - \cdots -$$

$$c_j^{-1} c_{j-1} \alpha_{j-1} - c_j^{-1} c_{j+1} - \cdots - c_j^{-1} c_n \alpha_n) +$$

$$b_n \alpha_n$$

$$= (b_1 - c_j^{-1} c_1) \alpha_1 + (b_2 - c_j^{-1} c_2) \alpha_2 + (b_{j-1} - c_j^{-1} c_{j-1}) \alpha_{j-1}$$

$$+ b_j c_j^{-1} \beta + \cdots + (b_n - c_j^{-1} c_n) \alpha_n$$

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