

- Computer can perform four basic arithmetic operations like addition, subtraction, multiplication, division and a logical operation which is comparison.
 - Computer has a finite word size, it can represent a number with finite precision.

number finite has to be ded off finite of digits after decimal point.

mantissa

exponent

sign bit

sign bit

Round Off Error

(123456.789)

or a factor

acc. to [redacted] is added so as to remove the sign bit and express [redacted]

- Computation has to finished within a finite time, we cannot leave the computer doing the same process for an infinite period of time.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} +$$

Truncation error
since truncating the series
after a finite no. of terms.

formulation of problem and unavailability of accurate data.

24/9/18

$$x = \bar{x} + e_x \quad \text{local error}$$

$$y = \bar{y} + e_y$$

$$p = x \cdot y = \bar{x}\bar{y} + (\bar{x}e_y + \bar{y}e_x + e_x e_y)$$

absolute error = |computed value - exact value|

Relative error $\frac{\text{Absolute error}}{\text{Exact value}}$

12345.678 — computed value
 12345.675 — exact value
 $= \frac{0.003}{12345.675}$
 (Insignificant).
 0.123 — computed
 0.120 — exact
 $[\text{Abs. err.}] = 0.003$

Relative error $= \frac{0.003}{0.120} = \frac{3}{120} = 0.025$
 $= 2.5\%$
 (Significant)

Rel. Error. $\hat{=}$ absolute error divided by
exact value Iterative method

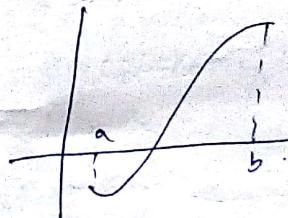
Let $x = f^{-1}(0)$ be a root of $f(x) = 0$ and let I be an interval containing the point $x = f^{-1}(0)$. Let $f(x)$ and $\phi(x) = f(f(x))$ be continuous in I , where $\phi(x)$ is defined by the equation $\phi(x) = f(x)$, which is equivalent to $f(\phi(x)) = 0$. If $|f'(x)| < 1$ for all x in I , the seq. of approx. x_0, x_1, x_2, \dots defined by $x_{n+1} = \phi(x_n)$ converges to the root x , provided that the initial approximation x_0 is chosen from I .

If $y = f(w)$ cont. over $[a, b]$

If $f(a) \cdot f(b)$ are of opp signs, then there is a number in $[a, b]$ such that $f(z) = 0$ (There exists odd no. of roots).

Eqs involving trig, hyperbolic or logarithmic terms are transcendental eqns.

Bisection Method



input a and b are so chosen that $f(a) \cdot f(b)$ are of opp sign.
 $f(a) \cdot f(b) < 0$

$$\textcircled{1} \quad m = \frac{a+b}{2}$$

\textcircled{2} if $f(m) = 0$ $\Rightarrow m$ is a root \rightarrow go to 6.

\textcircled{3} if $f(a) \cdot f(m) < 0$ then $b = m$
 else $a = m$.

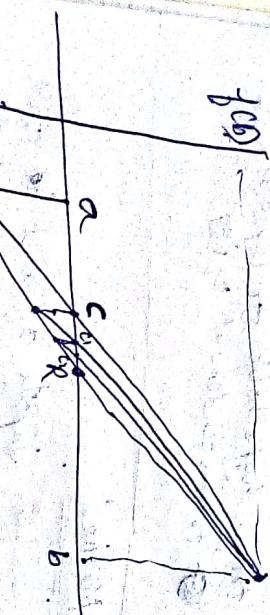
\textcircled{4} Repeat steps 1, 2, 3 until the size of the interval $|a-b|$ is less than the required precision.

$$\textcircled{5} \quad M = (a+b)/2$$

\textcircled{6} Print m \textcircled{7} End

The method fails only when the curve touches the x -axis. It converges very slowly.

Method of false position or Regula falsi method



$$y - f(a) = \frac{f(b) - f(a)}{b - a} (x - a)$$

$$\Rightarrow (x - a) = \frac{(b - a)y - f(a)}{f(b) - f(a)}$$

$$\Rightarrow c - a = \frac{(b - a)f(b) - f(a)}{f(b) - f(a)}$$

$$\Rightarrow c = \frac{(a - b)f(a) + a}{f(b) - f(a)}$$

$$= \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

Over bisection / regular false method takes less number of iterations to converge to the root.

Analysis of any numerical algorithm considers the following facts:

1) Whether the method is guaranteed to provide solution.

2) How fast will it provide solution.

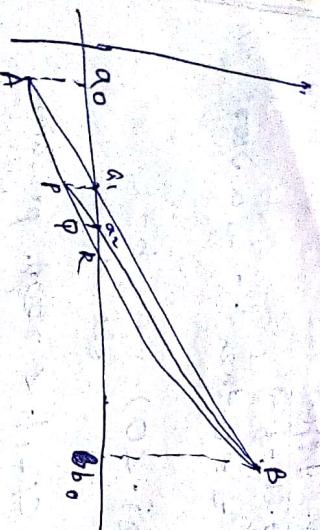
3) Amount of error associated with the solution.

$$1) c = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

2) If $f(c) = 0$ then c is a root else $a = c$

3) If $|c_i - c_{i+1}| < \text{reqd. precision}$
then c_{i+1} is the root.

for results corrected upto 5 decimal places,
precision value will be 0.00005



From $\Delta_s a_{1,0} A$ and $a_{1,0} B$,

$$\frac{a_1 - a_0}{|f(a_0)|} = \frac{b_0 - a_1}{|f(b_0)|} \quad \text{---} \textcircled{1}$$

from $\Delta_s a_{2,0} P$ and $a_{2,0} B$,

$$\frac{a_2 - a_1}{|f(a_1)|} = \frac{b_0 - a_2}{|f(b_0)|} \quad \text{---} \textcircled{2}$$

$$a_2 - a_1 |f(c_0)| = f(a_1) (b_0 - a_2)$$

$$\Rightarrow (a_2 - a_1) |f(b_0)| + a_2 |f(a_1)| = b_0 |f(a_1)|$$

$$\Rightarrow (a_2 - a_1) |f(b_0)| + (a_2 - a_1) |f(a_1)| = (b_0 - a_1) |f(a_1)|$$

$$\Rightarrow (a_2 - a_1) (|f(b_0)| + |f(a_1)|) = (b_0 - a_1) |f(a_1)|$$

$$= (a_1 - a_0) |f(b_0)| |f(a_1)|$$

$$= |f(a_0)| |f(b_0)| |f(a_1)|$$

$$\therefore a_2 - a_1 = \frac{(a_1 - a_0) |f(b_0)| |f(a_1)|}{(|f(a_0)| |f(b_0)| + |f(a_1)|)}$$

$$= a_1 - a_0 \left\{ \frac{(|f(a_1)| / |f(a_0)|) / (1 + |f(a_1)|)}{(|f(a_0)| / |f(b_0)|) / (1 + |f(a_1)|)} \right\} <$$

Solve $e^{-n-n}=0$ using bisection method.

Iteration no(i)	a_i	b_i	m_i	$f(m_i)$	$f(a_i)^*$	$f(m_i)^*$
0	0	1	0.5	0.10653	+ ve.	
1	0.5	1	0.75	-0.272	- ve.	
2	0.5	0.75	0.625	-0.8972	- ve.	
3	0.5	0.625	0.5625	0.00728	+ ve.	
4	0.5	0.5625	0.5135	-0.045	- ve.	
5	0.5625	0.5135	0.53125	-0.0172	- ve.	
6	0.5625	0.53125	0.546875			

includes `iostream.h`
include <std.h>
include <iomanip.h>
include <math.h>
include <double.h>

then ($exp(-n)-n$);

} // just main()

float low, up, root, fl, ul,
point, ("Lower limit: ");
scanf ("%f", &low);
scanf ("Upper limit: ");
scanf ("%f", &ul);

$f(x)$ has a simple root if $f'(x) \neq 0$ and $f''(x) > 0$

else

$\{$

No root exist by Rmthm
return 0;

How fast does the method converge to the root.

Order of convergence is n if $e_{in} = K e^{-n}$

K is const. of proportionality

$$\text{Let } e_i = 0.01 \text{ & } n = 2$$

$$e_{in} \propto (0.01)^n = 0.0001$$

$$\log |e_i| = \log |K| + n \log |e_i|$$

$$n = \frac{\log |e_{in}| - \log |K|}{\log |e_i|}$$

$$n = \left\{ \frac{\log |e_{in}| - \log |K|}{\log |e_i|} \right\}_{\text{approx}}$$

Fixed Point Iteration Method / Newton's Method
using repeated substitution.
Given $f(x) = 0$, where root has to be computed.

Use recursive form $x_n = g(x_{n-1})$

Let us assume the initial approx. or the root of $f(x) = 0$ be x_0 , we generate a sequence of approx to the root as follows:

$$x_1 = g(x_0)$$

$$x_2 = g(x_1)$$

$$x_n = g(x_{n-1})$$

Continue until $|x_{n+1} - x_n| < \text{reqd. precision}$.

$$x = \pm \sqrt{n+6} - 1$$

$$x^2 = x+6$$

$$x = 1 + 6n \quad \text{--- (2)}$$

$$nx = x^2 - 6 \quad \text{--- (3)}$$

$$x = \sqrt{n+6}$$

$$\left\{ \begin{array}{l} g^{(n)} = \sqrt{n+6} \\ \text{at } x = 3 \quad |g'(n)| < 1 \end{array} \right.$$

$$x_1 = 1 + 6 \cdot 1 = 7$$

$$x_2 = 1 + 6 \cdot 2 = 13$$

$$x_3 = 1 + 6 \cdot 3 = 19$$

$$\begin{aligned} x_4 &= 1 + 6 \cdot 4 = 25 \\ x_5 &= 1 + 6 \cdot 5 = 31 \end{aligned}$$

$$g^{(n)} = 1 + \frac{6}{n}, \quad g'(n) = -\frac{6}{n^2}$$

$$\begin{aligned} x_6 &= 1 + 6 \cdot 6 = 37 \\ x_7 &= 1 + 6 \cdot 7 = 43 \end{aligned}$$

$$g(n) = n^{2/6}, \quad g'(n) = \frac{2}{6n^{5/6}}$$

$$x_8 = 1 + 6 \cdot 8 = 49$$

$$x_9 = 35.5 \rightarrow \text{diverges}$$

$$\begin{aligned} x_1 &= \sqrt{1+6} = \sqrt{7} = 2.645 \\ x_2 &= \sqrt{2+6} = 2.99 \\ x_3 &= \sqrt{2.645+6} = 2.989 \\ x_4 &= \sqrt{2.99+6} = 2.989 \end{aligned}$$

$\rightarrow 3$

$$x = -\sqrt{n+6}$$

$$\left\{ \begin{array}{l} g^{(n)} = -\sqrt{n+6} \\ |g'(n)| < 1 \end{array} \right.$$

$$x_0 = 1$$

$$x_1 = -\sqrt{1+6} = -2.645$$

$$x_2 = -\sqrt{2+6} = -2.99$$

$$x_3 = -\sqrt{2.645+6} = -2.989$$

$$x_4 = -\sqrt{2.99+6} = -2.989$$

$$x_5 = -\sqrt{2.989+6} = -2.989$$

$$x_6 = -\sqrt{2.989+6} = -2.989$$

$\rightarrow -3$

Theorem: Let $x = R$ be a root of the eqn. $f(x) = 0$,

which is written as $x = g^{(n)}$,
let both $g^{(n)}$ & $g'(n)$ exist and continuous
over an interval $[a, b]$ containing $x = R$.

If $|g'(n)| < 1$ over the interval $[a, b]$
initial approx. x_0 is also in $[a, b]$,
then the sequence of approx. x_1, x_2, x_3, \dots
will converge to the root $x = R$.

Proof: Let $x = R$ be a root of $x = g^{(n)}$,

$$R = g^{(n)}$$

Let x_0 be the initial approx. to the root,
then the seq. of approx. can be generated as

$$\left. \begin{array}{l} x_1 = g(x_0) \\ x_2 = g(x_1) \\ x_3 = g(x_2) \\ \vdots \\ x_n = g(x_{n-1}) \end{array} \right\}$$

①

Let $e_0, e_1, e_2, \dots, e_{n+1}$ be the errors associated with x_0, x_1, \dots, x_{n+1} .

$$e_0 = R - x_0$$

$$\left. \begin{array}{l} e_1 = R - x_1 = g(R) - g(x_0) \\ e_2 = R - x_2 = g(R) - g(x_1) \end{array} \right\}$$

∴

$$e_{n+1} = R - x_{n+1} = g(R) - g(x_n)$$

Using mean value theorem, we can rewrite the eqn in ③ as

$$e_0 = R - x_0$$

$$e_1 = (R - x_0)g'(c_1) ; x_0 < c_1 < R$$

$$e_2 = (R - x_1)g'(c_2) ; x_1 < c_2 < R$$

$$e_{n+1} = (R - x_n)g'(c_{n+1}) ; x_{n+1} < c_{n+1} < R$$

Now from ④.

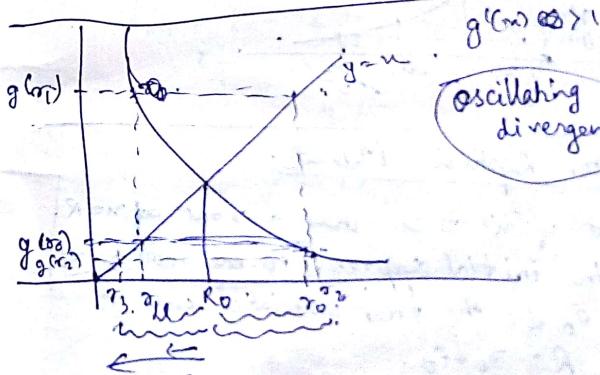
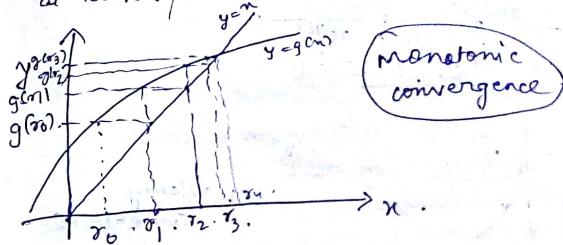
$$e_1 = e_0 g'(c_1)$$

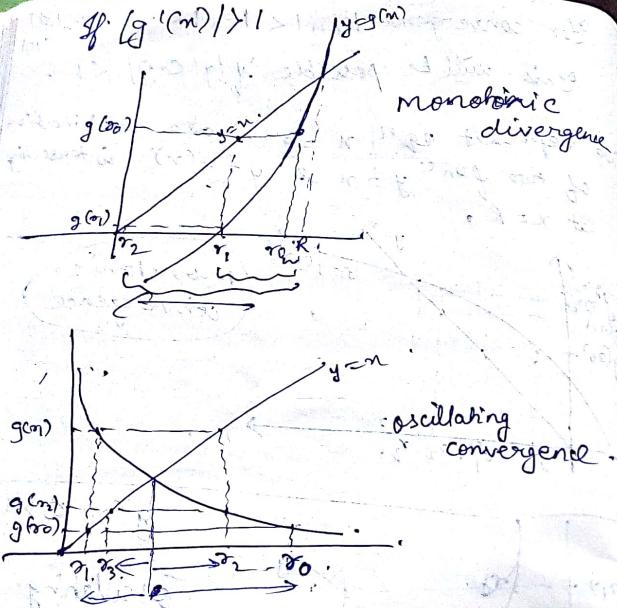
$$e_2 = e_1 g'(c_2)$$

$$e_{n+1} = e_n g'(c_{n+1})$$

For convergence $|e_{n+1}| < |e_n| < \dots < |e_1| < |e_0|$
This will be possible if $|g'(c_i)| < 1$.

We represent eqn $x = g(n)$ as a combination of two func. $y = n$ & $y = g(n)$ intersecting at $x = R$.





Newton Raphson Method.

Let $y = 0$ has a root at $x=R$.
Let the initial approximation to the root be x_0 .
Let e_0 be the error associated with x_0 .

$$R = x_0 + e_0.$$

$$\text{Now } f(R) = 0$$

$$\text{or } f(x_0 + e_0) = 0$$

Using Taylor series expansion,

$$f(x_0) + e_0 f'(x_0) + \frac{e_0^2}{2!} f''(x_0) + \dots = 0$$

we assume $|e_0| \ll 1$
∴ we neglect e_0^2 and higher order terms in the series.

$$f(x_0) + e_0 f'(x_0) = 0$$

$$\text{or } e_0 = -\frac{f(x_0)}{f'(x_0)}$$

We can generate the next approximation to the root as $x_1 = x_0 + e_0 = x_0 - \frac{f(x_0)}{f'(x_0)}$

continuing this way, we generate a sequence of approximations as -

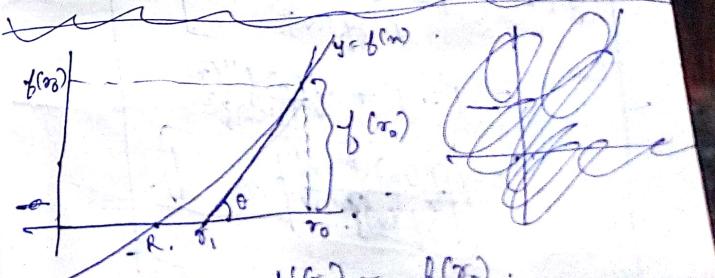
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$\vdots$$

continue until $|x_{n+1} - x_n| < \text{required precision.}$

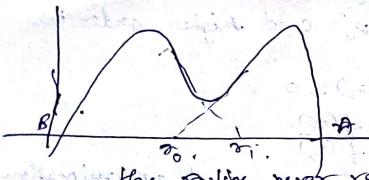
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$



$$\tan \theta = f'(x_0) = \frac{f(x_0)}{x_0 - x_1}$$

$$x_0 - x_1 = \frac{f(x_0)}{f'(x_0)} \Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

If $|f'(x_1)| \ll 1$, the method fails.



meaning, the ordinate never reaches A or B.

$$NR \text{ method: } x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\text{Iterative method: } x_{n+1} = g(x_n)$$

$$\therefore g(x_n) = x_n - \frac{f(x_n)}{f'(x_n)}$$

For convergence,

$$|g'(x_n)| < 1$$

$$g'(x_n) = 1 - \frac{(f'(x_n))^2 - f(x_n)f''(x_n)}{\sum f'(x_n)^2}$$

$$= 1 - 1 + \frac{f(x_n)f''(x_n)}{\sum f'(x_n)^2}$$

$$\therefore \left| \frac{f(x_n)f''(x_n)}{\sum f'(x_n)^2} \right| < 1$$

$$f(x) = x^3 - x - 3$$

$$f'(x) = 3x^2 - 1$$

$$f''(x) = 6x$$

Solve $f(x) = 0$ using Newton Raphson method

$$\text{Take } x_0 = 0$$

i	x_i	σ_{iH}	$ x_{i+1} - x_i $	$f(x_i) \cdot f'(x_i)$
0	0	-3	3	-3
1	-3	-1.96	1.04	
2	-1.96	-1.146	0.82	
3	-1.146	-0.003 -0.1274	-8.56 -3.393	
4	-0.1274	-3.198		

$$\begin{array}{|c|c|c|c|} \hline & f'(x_i) & f''(x_i) & \\ \hline & 3 & 6 & \\ \hline 0 & < 1 & & \\ \hline 0.718 & < 1 & & \\ \hline 0.91 & < 1 & & \\ \hline 2.02 & > 1 & & (\text{Stop}) \\ \hline \end{array}$$

Order of convergence for N-R method

$$\tau_{i+1} = R - \epsilon_i$$

$$\tau_i = R - \epsilon_i \Rightarrow R = \tau_i + \epsilon_i$$

$$f(\tau_i + \epsilon_i) = 0$$

Expanding by Taylor's formula

$$f(\tau_i) + \epsilon_i f'(\tau_i) + \frac{\epsilon_i^2}{2!} f''(\tau_i) + \frac{\epsilon_i^3}{3!} f'''(\tau_i) + \dots = 0$$

We assume $|\epsilon_i| \ll 1$, so we neglect ϵ_i^3 and higher order terms.

$$f(\tau_i) + \epsilon_i f'(\tau_i) + \frac{\epsilon_i^2}{2!} f''(\tau_i) = 0$$

$$\text{or } \frac{f(\tau_i)}{f'(\tau_i)} + \epsilon_i + \frac{\epsilon_i^2 f''(\tau_i)}{2 f'(\tau_i)} = 0$$

$$-\frac{f(\tau_i)}{f'(\tau_i)} = (R - \tau_i) + \frac{\epsilon_i^2 f''(\tau_i)}{2 f'(\tau_i)}$$

$$\Rightarrow \tau_i + \frac{f(\tau_i)}{f'(\tau_i)} = R + \frac{\epsilon_i^2 f''(\tau_i)}{2 f'(\tau_i)}$$

$$\Rightarrow \tau_{i+1} = R + \frac{\epsilon_i^2 f''(\tau_i)}{2 f'(\tau_i)}$$

$$\Rightarrow \tau_{i+1} - R = \frac{\epsilon_i^2 f''(\tau_i)}{2 f'(\tau_i)} \Rightarrow -\epsilon_{i+1} = \frac{\epsilon_i^2 f''(\tau_i)}{2 f'(\tau_i)}$$

$$\text{or } |\epsilon_{i+1}| = |\epsilon_i|^2 \left| \frac{f''(\tau_i)}{2 f'(\tau_i)} \right|$$

\therefore order of convergence = 2

$$\rightarrow \log |\epsilon_{i+1}| = \log |\epsilon_i| + 2 \log |\epsilon_i|$$

$$\text{or } \frac{\log |\epsilon_{i+1}|}{\log |\epsilon_i|} = \frac{\log |\epsilon_i|}{\log |\epsilon_i|} + 2$$

For $e \rightarrow 0$,

$$f(x) = x^2 - x - e$$

$$f'(x) = 2x - 1$$

$$\text{so } = 2 \cdot S.$$

i	x_i	x_{i+1}	$ x_{i+1} - x_i $	$ g'(x) $	$\frac{\text{order of convergence}}{n}$
0	2.5	3.0625	0.5625	<1	-
1	3.0625	3.000762	0.0617	<1	4.84
2	3.000762	2.999625	0.0038	<1	2.00
3	2.999625	2.999999	0.0009	<1	1.00
4	2.999999	3.000001	0.00005	<1	2.107
5	3.000001	2.999996	0.00001	<1	1.02

For fixed point iteration method.

$$(R - \sigma_{i+1}) = (R - \sigma_i) g'(c_i), \quad \forall c_i \in [x_0, R]$$

Let $|g'(c_i)| < 1$ for all c_i 's in the interval $[x_0, R]$
and let $|g'(c_0)| = k$.

We can write,

$$R - \sigma_{i+1} = k(R - \sigma_i) \quad \textcircled{1}$$

$$R - \sigma_{i+2} = k(R - \sigma_{i+1}) \quad \textcircled{2}$$

$$\textcircled{1} \Rightarrow \textcircled{2}$$

$$\frac{(R - \sigma_{i+1})}{(R - \sigma_i)} = \frac{(R - \sigma_i)}{(R - \sigma_{i+1})}$$

$$\Rightarrow (R - \sigma_{i+1})^2 = (R - \sigma_i)(R - \sigma_{i+2})$$

$$\Rightarrow R^2 - 2R\sigma_{i+1} + (\sigma_{i+1})^2 = R^2 + \sigma_i\sigma_{i+2} - R(\sigma_i + \sigma_{i+2})$$

$$\Rightarrow R(\sigma_i - 2\sigma_{i+1} + \sigma_{i+2}) = \sigma_i\sigma_{i+2} - (\sigma_{i+1})^2$$

$$\Rightarrow R = \frac{\sigma_i\sigma_{i+2} - (\sigma_{i+1})^2}{\sigma_i + \sigma_{i+2} - 2\sigma_{i+1}} = \frac{(\sigma_{i+1})^2 - \sigma_i\sigma_{i+2}}{2\sigma_{i+1} - \sigma_i - \sigma_{i+2}}$$

i	σ_i	σ_{i+1}	σ_{i+2}
0	1	2.64575	$\sigma_0, \sigma_1, \sigma_2 (3.604)$
1	2.6	2.94037	

$$\begin{array}{l|l|l|l}
R & 3.0046 & 3.00076 & \\
\sigma_0 & 3.00076 & 3.00013 & \} 3.000006 \\
\sigma_1 & 3.00013 & 3.00001 & \\
\sigma_2 & 3.00001 & 3.000001 &
\end{array}$$

$$\begin{aligned} f(x) &= x^3 - 4x^2 - 3x + 18 \\ &= x^2(x+2) - 6x(x+2) + 9(x+2) \\ &= (x+2)(x^2 - 6x + 9) = (x+2)(x-3)^2 \end{aligned}$$

$$\begin{aligned} f'(x) &= 3x^2 - 8x - 3 \\ &\approx 2.5. \end{aligned}$$

i	σ_i	σ_{i+1}	$e_i = \sigma_{i+1} - \sigma_i $
0	2.5	2.7647	0.264706
1	2.7647	2.86974	0.105035
2	2.86974	2.93515	0.0660

The value converges very fast to the root as at $n=3$

$$f^{(n)} \rightarrow 0, \quad f'^{(n)} \rightarrow 0$$

Let $f(n) = (x-a)^n g(n)$
i.e. there exists multiple roots of order n at $x=a$.

$$\begin{aligned} \therefore f^{(n-1)}(a) &= f^{(n-2)}(a) = \dots = f''(a) = f'(a) \\ &= f(a) = 0. \end{aligned}$$

$$f'(n) = n(x-a)^{n-1} g(n) + (x-a)^n g'(n)$$

$$\Rightarrow (x-a)f'(n) = n(x-a)^n g(n) + (x-a)^{n-1} g'(n)$$

$$= n f(x) + \underbrace{(x-a)^{n+1} g'(m)}_{\text{As } x \rightarrow a, \rightarrow \text{ tends to } 0}$$

$$(x-a) = n f(x)/f'(x)$$

$$\text{or } a = x - n \frac{f(x)}{f'(x)}$$

For multiple roots ~

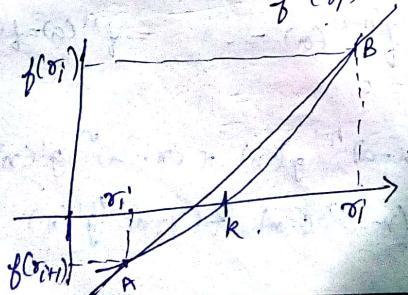
i	σ_i	$\sigma_{i+1} = \sigma_i - \frac{f(\sigma_i)}{f'(\sigma_i)}$	$e_i = \sigma_{i+1} - \sigma_i $
0	2.5	3.0294	0.5294
1	3.0294	3.00009	0.0293
2	3.00009	3.000000	0.00009

SECANT METHOD :

for N-R method we have to provide the functional representation of both $f(x)$ & $f'(x)$.

$f'(x)$ may not be easy to compute.

$$\sigma_{i+1} = \sigma_i - \frac{f(\sigma_i)}{f'(\sigma_i)}$$



'It is on off'
Slope of AB ↑
 $= f(\sigma_i) + f(\sigma_{i+1})$
 $\sigma_i \rightarrow \approx f'(\sigma_i)$
(only if $\sigma_i \approx 1$)

$$\begin{aligned}\sigma_{i+1} &= \sigma_i - \frac{f(\sigma_i)(\sigma_{i+1} - \sigma_i)}{f(\sigma_i) - f(\sigma_{i+1})} \\ &= \frac{f(\sigma_i)\sigma_{i+1} - f(\sigma_{i+1})\sigma_i}{f(\sigma_i) - f(\sigma_{i+1})}\end{aligned}$$

It differs from Regular falsi
in the respect of interval checking
hence it has the possibility of
divergence.

R is the actual root of $f(x)=0$

$$\therefore f(R)=0$$

$$\text{Let } \sigma_{i-1} = R - e_{i-1}$$

$$\sigma_i = R - e_i$$

$$\sigma_{i+1} = R - e_{i+1}$$

$$\sigma_{i+1} = \frac{\sigma_{i-1} f(\sigma_i) - \sigma_i f(\sigma_{i-1})}{f(\sigma_i) - f(\sigma_{i-1})}$$

$$\Rightarrow R - e_{i+1} = \frac{(R - e_{i-1})f(R - e_i) - (R - e_i)f(R - e_{i-1})}{f(R - e_i) - f(R - e_{i-1})}$$

$$\begin{aligned}\Rightarrow R - e_{i+1} &= (R - e_{i-1})\{f(R) - e_i f'(R) + \frac{e_i^2}{2} f''(R)\} \\ &\quad - (R - e_i)\{f(R) - e_{i-1} f'(R) + \frac{e_{i-1}^2}{2} f''(R)\}\end{aligned}$$

$$\begin{aligned}&= (R - e_{i-1})\{ -e_i f'(R) + \frac{e_i^2}{2} f''(R)\} \\ &\quad - (R - e_i)\{ -e_{i-1} f'(R) + \frac{e_{i-1}^2}{2} f''(R)\} \\ &= f(R)(e_{i-1} - e_i)\end{aligned}$$

$$\begin{aligned}
 & + R e_{i-1} f'(R) - R \frac{\epsilon_{i-1}^2}{2} f''(R) + \epsilon_i \frac{\epsilon_{i-1}^2}{2} f''(R) \\
 & \cancel{f'(R)(\epsilon_{i-1} - \epsilon_i)} \\
 & \approx R \cancel{\frac{f''(R)(\epsilon_{i-1}^2 + \epsilon_i^2)}{2}} + R f'(R)(\epsilon_{i-1} - \epsilon_i) \\
 & \cancel{f'(R)(\epsilon_{i-1} - \epsilon_i)} \\
 & = R f'(R) - R \cancel{\frac{f''(R)(\epsilon_{i-1} + \epsilon_i)}{2}} \\
 & \cancel{f'(R)} \\
 & \approx R f'(R) (\epsilon_{i-1} - \epsilon_i) - \epsilon_{i-1} \epsilon_i f''(R) (\epsilon_{i-1} - \epsilon_i) \\
 & \approx R - \frac{\epsilon_{i-1} \epsilon_i f''(R)}{2 f'(R)} \\
 & \therefore R - \epsilon_{i+1} \approx R - \frac{\epsilon_{i-1} \epsilon_i f''(R)}{2 f'(R)} \\
 & - \epsilon_{i+1} \approx - \frac{\epsilon_{i-1} \epsilon_i f''(R)}{2 f'(R)} \\
 & \therefore \epsilon_{i+1} \approx \frac{\epsilon_{i-1} \epsilon_i f''(R)}{2 f'(R)} \quad \text{--- (1)}
 \end{aligned}$$

Let us assume the order of convergence be n :

$$\begin{aligned}
 & \therefore \epsilon_{i+1} = R \epsilon_i^n - \text{--- (2)} \\
 & \& \epsilon_i = k \epsilon_{i-1}^n - \text{--- (3)} \\
 & \therefore \epsilon_{i+1} = \left(\frac{\epsilon_i}{k}\right)^{1/n} - \text{--- (4)}
 \end{aligned}$$

with (1), (2) & (4), we can write:

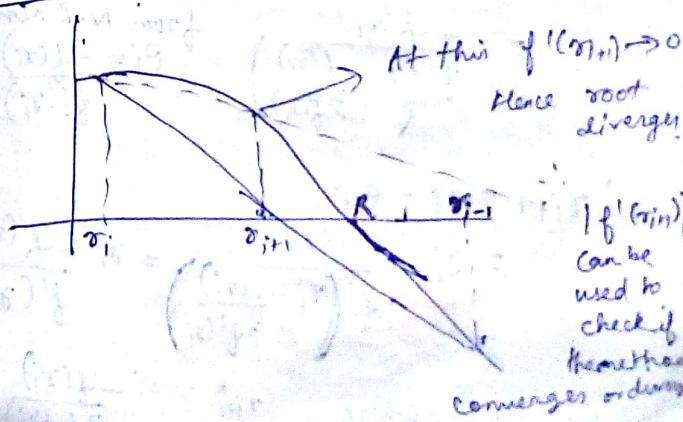
$$\begin{aligned}
 k \epsilon_i^n & \approx \left(\frac{\epsilon_i}{k}\right)^{1/n} \epsilon_i \cdot \frac{f''(R)}{2 f'(R)} \\
 & \approx \epsilon_i^{1+1/n} \cdot \frac{f''(R)}{2 k^{1/n} f'(R)} \\
 \therefore \epsilon_i^n & \approx \epsilon_i^{1+1/n} \cdot \frac{f''(R)}{2 k^{1+1/n} f'(R)}
 \end{aligned}$$

Approx. equating powers of ϵ_i from both sides:

$$\begin{aligned}
 n &= 1 + \frac{1}{n} \\
 n^2 - n - 1 &= 0 \\
 \Rightarrow n &= \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2} = \frac{1 \pm 2.236}{2} \\
 n &= 1.618 \quad \text{--- (0.61)} \\
 \text{Method of } & \text{converges.} \quad \text{better than regular false}
 \end{aligned}$$

\rightarrow method diverges.

Spec. Case:



Multipoint Iteration Method

Let R be the actual root of $f(x) = 0$
 & x_i be an approx to the root.

$$\therefore x_i = R - e_i \\ \text{or } R = x_i + e_i$$

$$\text{Now } f(R) = f(x_i + e_i) = 0$$

Expanding by Taylor's formula

$$f(x_i) + e_i f'(x_i) + \frac{e_i^2}{2} f''(x_i) + \frac{e_i^3}{3!} f'''(x_i) + \dots = 0$$

$$\Rightarrow f(x_i) + e_i \left\{ f'(x_i) + \frac{e_i}{2} f''(x_i) + \frac{e_i^2}{3!} f'''(x_i) + \dots \right\} = 0$$

$$\approx f'(x_i + e_i/2)$$

$$\Rightarrow f(x_i) + e_i f'(x_i + e_i/2) = 0$$

$$\therefore e_i = \frac{-f(x_i)}{f'(x_i + e_i/2)}$$

$$= \frac{-f(x_i)}{f'(x_i - \frac{f(x_i)}{2f'(x_i)})} \quad \text{from N-R method}$$

$$\therefore x_{i+1} = x_i + e_i \\ = x_i - \frac{f(x_i)}{f'(x_i - \frac{f(x_i)}{2f'(x_i)})} = x_i - \frac{f(x_i)}{f'(x_{i+1}^*)}$$

$$\text{where } x_{i+1}^* = x_i - \frac{f(x_i)}{2f'(x_i)}$$

for a given x_i

$$x_{i+1}^* = x_i - \frac{f(x_i)}{2f'(x_i)}$$

$$x_{i+1} = x_i - \frac{f(x_i)}{4f'(x_{i+1}^*)}$$

$$R = x_k + e_k$$

$$R = x_{k+1} + e_{k+1}$$

$$x_{k+1}^* = x_k - \frac{f(x_k)}{2f'(x_k)} = R - e_k - \frac{f(x_k)}{2f'(x_k)} \\ = (R - e_k) - \left\{ \frac{f(R - e_k)}{2f'(R - e_k)} \right\}$$

$$\frac{f(R - e_k)}{f'(R - e_k)} = \frac{f(R) - e_k f'(R) + \frac{e_k^2}{2} f''(R) - \frac{e_k^3}{3!} f'''(R) + \dots}{f'(R) - e_k f''(R) + \frac{e_k^2}{2} f'''(R) - \dots}$$

$$= \frac{-e_k + \frac{e_k^2}{2} \frac{f''(R)}{f'(R)} - \frac{e_k^3}{3!} \frac{f'''(R)}{f'(R)}}{1 - \frac{e_k f''(R)}{f'(R)} + \frac{e_k^2}{2} \frac{f'''(R)}{f'(R)}}$$

$$\text{let } \frac{f''(R)}{f'(R)} = a_1 \quad \text{and } \frac{f'''(R)}{f'(R)} = a_2$$

$$\Rightarrow = \frac{\left(e_k + \frac{e_k^2}{2} a_1 - \frac{e_k^3}{3!} a_2 \right)}{\left(1 - e_k a_1 + \frac{e_k^2}{2} a_2 \right)}$$

$$= \left(-e_k + \frac{e_k^2}{2} a_1 - \frac{e_k^3}{3!} a_2 \right) \left(1 - e_k a_1 + \frac{e_k^2}{2} a_2 \right)$$

$$\approx \left(-e_k + \frac{e_k^2}{2} a_1 - \frac{e_k^3}{3!} a_2 \right) \left(1 + e_k a_1 - \frac{e_k^2}{2} a_2 \right)$$

$$= -e_k + \frac{e_k^2}{2} a_1 - \frac{e_k^3}{3!} a_2 + \frac{e_k^3 a_1}{2} - e_k^2 a_1 - \frac{e_k^4 a_1}{3!} \\ + \frac{e_k^2 a_2}{2} - \frac{e_k^4}{4} a_1 a_2 + \frac{e_k^2}{2 \cdot 3!} a_2^2$$

$$= e_k + \frac{e_k^2 a_1}{2} - \frac{e_k^3}{3!} a_2 + \frac{e_k^3 a_1^2}{2} - e_k^2 a_1 + \frac{e_k^3}{2} a_1$$

$$= -e_k - \frac{e_k^2 a_1}{2} + e_k^3 \left(\frac{a_1^2}{2} + \frac{a_2}{3} \right).$$

$$\therefore \gamma_{k+1} = (R - e_k) - \frac{1}{2} \left\{ -e_k - \frac{a_1 e_k^2}{2} + \left(\frac{a_1^2}{2} + \frac{a_2}{3} \right) \right\} \\ = R - \frac{e_k}{2} + \frac{a_1}{2} e_k^2 - \left(\frac{a_1^2}{2} + \frac{a_2}{6} \right) e_k^3.$$

$$\sigma_{k+1} > \gamma_k - \frac{f(\gamma_k)}{f'(\gamma_{k+1})}$$

$$R - e_{k+1} = R - e_k - \frac{f(R - e_k)}{f'(R - \left[\frac{e_k}{2} - \frac{a_1}{2} e_k^2 + \left(\frac{a_1^2}{2} + \frac{a_2}{6} \right) e_k^3 \right])}$$

$$\frac{f(R - e_k)}{f'(R - \left[\frac{e_k}{2} - \frac{a_1}{2} e_k^2 + \left(\frac{a_1^2}{2} + \frac{a_2}{6} \right) e_k^3 \right])}$$

$$= f(R) - e_k f'(R) + \frac{e_k^2}{2} f''(R) - \frac{e_k^3}{3!} f'''(R) \quad (\text{By Taylor series})$$

$$= -e_k \cancel{\left(\frac{e_k}{2} - \frac{a_1}{2} e_k^2 + \left(\frac{a_1^2}{2} + \frac{a_2}{6} \right) e_k^3 \right)} + \cancel{\left(\frac{e_k^2}{2} f''(R) - \frac{e_k^3}{3!} f'''(R) \right)} \\ = -e_k \cancel{\left(\frac{e_k}{2} - \frac{a_1}{2} e_k^2 + \left(\frac{a_1^2}{2} + \frac{a_2}{6} \right) e_k^3 \right)} - e_k^3 \cancel{\left(\frac{f''(R)}{2} - \frac{f'''(R)}{3!} \right)} a_2$$

$$a_1 = \frac{f''(R)}{8 f'''(R)}$$

$$= \left[-e_k + \frac{e_k^2 a_1}{2} - \frac{e_k^3 a_2}{6!} \right] \left[1 + \frac{e_k a_1}{2} - \frac{a_1^2 e_k^2}{4} - \frac{e_k^3 a_2}{4!} \right]$$

$$= -e_k - \frac{e_k^2 a_1}{2} + \frac{e_k^2 a_1}{2} + \frac{a_1^2 e_k^3}{4} + \frac{e_k^3 a_2}{4} - \frac{e_k^3 a_2}{6}.$$

$$\therefore R - e_{k+1} = R - e_k - (-e_k + \frac{a_1^2}{2} e_k^3 - \frac{e_k^3 a_2}{6})$$

$$\Rightarrow -e_{k+1} = e_k^3 \left(\frac{a_2 - a_1^2}{6} \right)$$

$$\Rightarrow e_{k+1} = e_k^3 \left(\frac{a_1^2 - a_2}{6} \right)$$

find up to e_k^4

\therefore order of convergence = 3

$$(n+2)(n-3)(n+1)$$

$$(n^2 - n - 6)(n+1)$$

$$n^3 - n^2 - 6n + n^2 - n - 6$$

$$n^3 - 7n - 6$$

For multiple roots, order of convergence becomes < 3 .
In any case $f' = 1.1$

Chebyshev Method.

Let τ_k be the initial approx to the root R of $f(x) = 0$.

$$\therefore R = \tau_k + e_k.$$

$$f(R) = f(\tau_k + e_k) = 0$$

$$\Rightarrow f(\tau_k) + e_k f'(\tau_k) + \frac{e_k^2}{2} f''(\tau_k) + \frac{e_k^3}{6} f'''(\tau_k) = 0$$

neglecting e_k^3 and higher order terms

$$-e_k f'(\tau_k) = f(\tau_k) + \frac{e_k^2}{2} f''(\tau_k).$$

$$\Rightarrow -e_k = \frac{f(\tau_k)}{f'(\tau_k)} + \frac{e_k^2}{2} \frac{f''(\tau_k)}{f'(\tau_k)}$$

$$\therefore R = \tau_k + e_k$$

$$= \tau_k - \frac{f(\tau_k)}{f'(\tau_k)} - \frac{e_k^2 \cdot f''(\tau_k)}{2 \cdot f'(\tau_k)}$$

From N-R method.

$$e_k = -\frac{f(\tau_k)}{f'(\tau_k)}$$

Order of convergence
= 3

$$\therefore R = \tau_{k+1} = \tau_k - \frac{f(\tau_k)}{f'(\tau_k)} - \frac{\frac{f(\tau_k)^2}{2} f''(\tau_k)}{3 f'(\tau_k)}$$

$$|f'(\tau_k)| < \epsilon \quad \left[\text{condition for checking the method converges} \right]$$

Newton, Chebyshev method have a very high chance of diverging from the actual root. They are computation intensive.

Find The order of convergence.

(this method)
finding the complex roots of a polynomial eqn.

$$x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1} x + a_n = 0$$

let $x^2 + px + q$ be a factor of ~~poly~~ poly.

$$(x^2 + px + q) (x^{n-2} + b_1 x^{n-3} + b_2 x^{n-4} + \dots + b_{n-3} x + b_{n-2}) = 0$$

let p_0 be the approximated value,

then
$$(x^2 + p_0 x + q_0) (x^{n-2} + b_1 x^{n-3} + b_2 x^{n-4} + \dots + b_{n-3} x + b_{n-2}) + (R_n + S) = 0$$

Linear Remainder term.

This should be equal to zero
if the original polynomial is
exactly divisible by $(x^2 + px + q)$.

$$\begin{aligned} x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1} x + a_n &= (x^2 + px + q) \\ &\quad (x^{n-2} + b_1 x^{n-3} + b_2 x^{n-4} + b_3 x^{n-5} + b_4 x^{n-6} \\ &\quad + \dots + b_{n-2}) + R_n + S \end{aligned}$$

$$\Rightarrow x^n + (b_1 + p)x^{n-1} + (b_2 + pb_1 + q)x^{n-2} + \dots + (b_{n-2} + pb_{n-3} + qb_{n-2})x + R_n + S$$

Equating the coefficients
from both sides we get -

$$a_1 = b_1 + p \Rightarrow b_1 = a_1 - p$$

$$a_2 = b_2 + pb_1 + q \Rightarrow b_2 = a_2 - pb_1 - q$$

$$a_3 = b_3 + pb_2 + qb_1 \Rightarrow b_3 = a_3 - pb_2 - qb_1$$

$$a_j = b_j + pb_{j-1} + qb_{j-2} \Rightarrow b_j = a_j - pb_{j-1} - qb_{j-2}$$

$$\Rightarrow R = a_{n-1} - pb_{n-2} - qb_{n-3} \quad \textcircled{1}$$

$$\Rightarrow S = a_n - qb_{n-2} \quad \textcircled{2}$$

$$a_{n-1} - qb_{n-3} = pb_{n-2}$$

Let the initial approx. value for $p \& q$ be (p_0, q_0)

By equating (7) & (5) to 0
we generate new value for
 $p \& q$. (say) (p_1, q_1)

$$\begin{aligned} \text{max}(|p_0 - p_1|, |q_0 - q_1|) &\leq \epsilon \\ \text{and} \\ \text{max}(|p_{i-1} - p_i|, |q_{i-1} - q_i|) &\leq \epsilon \end{aligned}$$

Bairstow's method

$$[R(p, q)], [S(p, q)]$$

R & S are ff. of $p \& q$.

Let, (p_0, q_0) be the approx. to (p, q) .
and Δp & Δq be the corrections needed
to get the true values of (p, q) .

$$R(p_0 + \Delta p, q_0 + \Delta q) = 0$$

$$\Rightarrow R(p_0, q_0) + \Delta p \frac{\partial R}{\partial p} \Big|_{(p_0, q_0)} + \Delta q \frac{\partial R}{\partial q} \Big|_{(p_0, q_0)} = 0$$

$$S(p_0 + \Delta p, q_0 + \Delta q) = 0$$

$$\Rightarrow S(p_0, q_0) + \Delta p \frac{\partial S}{\partial p} \Big|_{(p_0, q_0)} + \Delta q \frac{\partial S}{\partial q} \Big|_{(p_0, q_0)} = 0$$

neglecting the higher order terms.

$$\Delta p \frac{\partial R}{\partial p} \Big|_{(p_0, q_0)} + \Delta q \frac{\partial R}{\partial q} \Big|_{(p_0, q_0)} = -R(p_0, q_0)$$

$$\Delta p \frac{\partial S}{\partial p} \Big|_{(p_0, q_0)} + \Delta q \frac{\partial S}{\partial q} \Big|_{(p_0, q_0)} = -S(p_0, q_0)$$

$$\Delta p = \left[S(p_0, q_0) \frac{\partial R}{\partial p} \right] - R(p_0, q_0) \frac{\partial S}{\partial p}$$

$$\Delta q = \left[S(p_0, q_0) \frac{\partial R}{\partial q} \right] + R(p_0, q_0) \frac{\partial S}{\partial p}$$

$$\frac{\partial R}{\partial p} = -p \frac{\partial b_{n-2}}{\partial p} - b_{n-2} - q \frac{\partial b_{n-3}}{\partial p}$$

Partial derivatives of equation (2) are:
w.r.t p & q .

$$c_1 = \frac{\partial b_1}{\partial p} = -1 \quad p \cdot d_1 = \frac{\partial b_1}{\partial q} = 0$$

$$c_2 = \frac{\partial b_2}{\partial p} = -p \frac{\partial b_1}{\partial p} - b_1 = p - b_1 \quad d_2 = \frac{\partial b_2}{\partial q} = 1$$

$$c_3 = -pc_2 - b_2 - qc_1 \quad d_3 = -p^2 - b_1 - qd_2$$

$$c_j = -b_{j-1} - pc_{j-1} - qc_{j-2}$$

$$\frac{\partial R}{\partial p} = -b_{n-2} - pc_{n-2} - qc_{n-3}$$

$$\frac{\partial S}{\partial p} = -qc_{n-2}$$

$$\frac{\partial R}{\partial q} = -pd_{n-2} - b_{n-3} - qd_{n-3}$$

$$\frac{\partial S}{\partial q} = -b_{n-2} - qd_{n-2}$$

Express Δp & Δq in terms of a_i, b_i, c_i

$$\begin{aligned}\Delta p = & \frac{(a_{n-2} - q_0 b_{n-2})(-p_0 d_{n-2} - b_{n-3} - q_0 d_{n-3})}{(-a_{n-1} - p_0 b_{n-2} - q_0 b_{n-3}) b_{n-2} q_0} \\ & \frac{(-b_{n-2} - p_0 c_{n-2} - q_0 c_{n-3})(-b_{n-2} - q_0 d_{n-3})}{(-q_0 c_{n-2})(p_0 d_{n-2} - b_{n-3} - q_0 d_{n-3})} \\ = & a_0 p d_{n-2} - a_0 b_{n-3} - a_0 d_{n-3} q_0 + q_0^2 b_{n-2} d_{n-3} \\ & + a_{n-1} b_{n-2} + \dots \\ = & \text{④}\end{aligned}$$

Solution of linear simultaneous eqns:

$$x_1 + 5x_2 + 3x_3 = 10 \quad \text{①}$$

$$x_1 + 3x_2 + 2x_3 = 5 \quad \text{②}$$

$$2x_1 + 4x_2 - 6x_3 = -4 \quad \text{③}$$

$$\text{①} - \text{②} \rightarrow$$

$$2x_2 + x_3 = 5 \quad \text{④}$$

$$\text{②} \times 2 - \text{③} \rightarrow$$

$$2x_2 + 10x_3 = 14 \quad \text{⑤}$$

$$\text{⑤} - \text{④} \rightarrow 9x_3 = 9$$

Back substitution

Gaussian elimination method.

$$x_1 + 5x_2 + 3x_3 = 10$$

$$\boxed{x_1} + 3x_2 + 2x_3 = 5 \quad \text{②} \xrightarrow{\text{②} - \text{①}} -2x_2 - x_3 = -5$$

$$\boxed{2x_1} + 4x_2 - 6x_3 = -4 \quad \text{③} \xrightarrow{\text{③} - 2\text{①}} -6x_2 - 7x_3 = -24$$

$$\Rightarrow x_1 + 5x_2 + 3x_3 = 10 \quad \text{①}$$

$$\begin{aligned}-2x_2 - x_3 &= -5 \quad \text{④} \\ \boxed{-6x_2} - 12x_3 &= -24 \quad \text{⑤}\end{aligned}$$

$$\Rightarrow x_1 + 5x_2 + 3x_3 = 10 \quad \text{①}$$

$$\begin{aligned}-2x_2 - x_3 &= -5 \quad \text{④} \\ -9x_3 &= -9 \quad \text{⑥} \quad (\text{⑤} - 3\text{④})\end{aligned}$$

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = g_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = g_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = g_3$$

Steps

$$(①) \times m_{12} + (②) \rightarrow \text{generating new } (③)$$

$$m_{11} = \frac{a_{11}}{a_{11}} \quad i=2, \dots, n$$

such that x_1 is eliminated

Coefficients change acc. to : $a_{ij} = a_{ij} - m_{11}$

$$j = 1, 2, \dots, n$$

$$m_{12} = \frac{a_{12}}{a_{22}}$$

$$a_{ij} = a_{ij} - m_{12} + a_{2j} \quad i=3, \dots, n$$

$$j=2, \dots, n$$

Ans.

(At k stage)

$$a_{kk}x_k + a_{k+1,k}x_{k+1} + \dots + a_{n,k}x_{n-1} + a_{nk}x_n = g_k$$

$$= a_{nk(n+1)}$$

$$\xrightarrow{(k+1)} a_{(k+1),k}x_k + a_{(k+1),(k+1)}x_{k+1} + \dots + a_{(k+1),n}x_{n-1} + a_{(k+1),n}x_n = a_{(k+1)(n+1)}$$

$$\xrightarrow{(n)} a_{nk}x_k + a_{n,n-1}x_{n-1} + \dots + a_{n,n-i}x_{n-i} + a_{nn}x_n = a_{n(n+1)}$$

$$m_{ik} = \frac{a_{ik}}{a_{kk}} \quad i=(k+1), \dots, n$$

$$a_{ij} = a_{ij} - m_{ik} \cdot a_{ki} \quad i=(k+1), \dots, n$$

$$j=k, \dots, (n+1)$$

Point if $|a_{kk}| \ll |a_{kk}|$
then m_{ik} is very large
(overflow may occur)

To avoid this,

partial pivoting is done.
which means row having largest
first non-zero element is swapped with
the first row of concerned
matrix

$$\begin{bmatrix} a_{11} & a_{12} & \dots & \dots & a_{1n}; & a_{1(n+1)} \\ a_{21} & a_{22} & \dots & \dots & a_{2n}; & a_{2(n+1)} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & \dots & a_{nn}; & a_{n(n+1)} \\ n \times (n+1) & & & & & \end{bmatrix}$$

\Downarrow

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n}; & a_{1(n+1)} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n}; & a_{2(n+1)} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n}; & a_{3(n+1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn}; & a_{n(n+1)} \\ n \times (n+1) & & & & & \end{bmatrix}$$

$$x_n = \frac{a_{n(n+1)}}{a_{nn}}$$

$$x_{n-1} = \frac{a_{(n-1)(n+1)} - a_{(n-1)n} x_n}{a_{(n-1)(n-1)}}$$

$$\text{In general, } x_i = \underline{a_{i(n+1)} - \sum_{j=i+1}^n a_{ij} * x_j}$$

$$i = (n-1), (n-2), \dots, 2, 1$$

Input the coefficient matrix $A[n][n]$

for ($k=1$; $k < n$; $k++$)

big = 0;

for ($i=k$; $i < n$; $i++$)

//finding
the coeff.
of largest
magn. along
the column k .

if ($|\text{abs}(a[i][k])| > \text{big}$)

{ big = $a[i][k]$;

$p = i$;

}

}

//swapping
the row
containing
the largest
magnitude
coeff. factor
in the k th
column
with KK
row.

Temp = $a[p][k]$;

$a[k][k] = a[p][k]$;

$a[p][k] = -\text{temp}$;

}

for ($i=k+1$; $i < n$; $i++$)

$m[i][k] = a[i][k]/a[k][k]$;

for ($j=k$; $j < (k+1)$; $j++$)

$a[i][j] = a[i][j] - m[i][k] * a[k][j]$;

}

}

//elimination process .

$x[n] = a[n] \cdot [n+1] / a[n] \cdot [n]$
 for ($i = (k-1)$; $i >= 1$; $i--$)
 {
 sum = 0;
 for ($j = 1$; $j < i$; $j++$)
 sum = sum + a[i] * [j];
 x[i] = (a[i] * [n+1] - sum) / a[i];

$$\begin{array}{r}
 \overline{4x^3 + 7x^2 + 8x + 1} \\
 \underline{-4x^3 - 56x^2 - 4x} \\
 \hline
 0 \quad -49x^2 + 4x + 1 \\
 \underline{-49x^2 - 49x - 49} \\
 \hline
 x^2 + 3x + 3 \\
 \underline{x^3 - x^2} \\
 \hline
 x^3 - 2x^2
 \end{array}$$

~~2x^3 + 12x^2 - x~~

$$x^3$$

$$\begin{aligned} x_1 + 5x_2 + 3x_3 &= 10 \\ x_1 + 3x_2 + 2x_3 &= 5 \\ -2x_1 + 4x_2 - 6x_3 &= -4. \end{aligned}$$

Gaussian elimination

$$\begin{bmatrix} 1 & 5 & 3 & 1 & 10 \\ 1 & 3 & 2 & 5 & 1 \\ 2 & 4 & -6 & 1 & 4 \end{bmatrix} \xrightarrow{\text{Row operations}} \begin{bmatrix} 2 & 4 & -6 & 1 & 4 \\ 1 & 3 & 2 & 5 & 1 \\ 1 & 3 & 3 & 1 & 10 \end{bmatrix} \quad R_1 \leftrightarrow R_3$$

$$\Rightarrow \left[\begin{array}{ccc|c} 2 & 4 & -6 & -4 \\ 1-1 & 3-2 & 2+3 & 5+2 \\ 1-1 & 5-2 & 3+3 & 10+2 \end{array} \right] \xrightarrow{R_2 \leftrightarrow R_2 - R_1/2} \xrightarrow{R_3 \leftrightarrow R_3 - R_1/2}$$

$$\Rightarrow \left[\begin{array}{cccc} 2 & 4 & -6 & 14 \\ 0 & 1 & 5 & -4 \\ 0 & 3 & 6 & 12 \end{array} \right] \Rightarrow \left[\begin{array}{cccc} 2 & 4 & -6 & 14 \\ 0 & 3 & 6 & 12 \\ 0 & 1 & 5 & 7 \end{array} \right] \quad R_2 \leftrightarrow R_3$$

$$\Rightarrow \left[\begin{array}{cccc|c} 2 & 4 & -6 & 1 & -4 \\ 0 & 3 & 6 & 1 & 12 \\ 0 & 1 & -5 & 2 & 7-4 \end{array} \right] R_3 \rightarrow R_3 - R_2/3,$$

$$\rightarrow \left[\begin{array}{ccccc|c} 2 & 4 & -6 & 1 & -4 \\ 0 & 3 & 6 & 1 & 12 \\ 0 & 0 & 3 & 1 & 3 \end{array} \right]$$

$$(a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1(n-1)}x_{n-1} + a_{1n}x_n) \\ + (a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2(n-1)}x_{n-1} + a_{2n}x_n) \\ + \dots + (a_{(n-1)1}x_1 + a_{(n-1)2}x_2 + a_{(n-1)3}x_3 + \dots + a_{(n-1)(n-1)}x_{n-1} + a_{(n-1)n}x_n)$$

$$M_{ik} = \frac{a_{ik}}{a_{kk}}$$

where the denominator is the product of multipliers required for getting the coefficient of x_k from the i^{th} row.

\Rightarrow Total no. of divisions reqd. for getting multipliers
 $= (n-1) + (n-2) + \dots + \frac{n(n-1)}{2}$

No. of multipliers reqd.: for the elimination process for eliminating the coeff. of x_n from 2^{nd} row to n^{th} row.

$$\text{no. of multiplications} = (n+1)(n-1) = n^2 - 1$$

\rightarrow Suppose at any stage of elimination we are left with k eqns with k unknowns for eliminating the coefficient along first column.
 no. of multiplications required will be $(k^2 - 1)$.

Total no. of multiplications reqd. for entire elimination process to reduce the system of eqns into upper triangular form

$$\sum_{k=n}^2 (k^2 - 1) = (n^2 + (n-1)^2 + \dots + 2^2) - (n-1) \\ = (n^2 + (n-1)^2 + \dots + 2^2 + 1^2) - n \\ = \frac{1}{6}n(n+1)(2n+1) - n \\ = \frac{2n^3 + 3n^2 - 5n}{6} = \underline{\underline{O(n^3)}}.$$

Calculate same for back propagation

$$= 1 + 2 + 3 + \dots + n \\ = \underline{\underline{n(n+1)}}$$

Graves Jordan elimination

similar to abv:

No. of multipliers req. $\leq (n-1)(k+1)$

$$= \sum_{k=n}^2 (n-1)(k+1) = (n-1)((n+1) + n + (n-1) + \dots + 2) \\ = (n-1)\left(\frac{n(n+1)}{2} - 1\right)$$

$$\begin{bmatrix} a_{11} & 0 & 0 & \dots & 0 & a_{1m} \\ 0 & a_{22} & 0 & \dots & 0 & a_{2m} \\ 0 & 0 & a_{33} & \dots & 0 & a_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} & a_{nm} \\ 0 & 0 & 0 & \dots & 0 & a_{nm} \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 & a_{1m}/a_{11} \\ 0 & 1 & 0 & \dots & 0 & a_{2m}/a_{22} \\ 0 & 0 & 1 & \dots & 0 & a_{3m}/a_{33} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & a_{nm}/a_{nn} \\ 0 & 0 & 0 & \dots & 0 & a_{nm}/a_{nm} \end{bmatrix}$$

$$M_{ik} = \frac{a_{ik}}{a_{kk}}$$

$$a_{ij} = a_{ij} - \text{min}(a_{ij})$$

$$x_n = \frac{a_{nn}}{a_{nn}}$$

$$x_{i+1} = \frac{a_{(i+1)n}}{a_{nn}} - \sum_{j=1}^i \frac{a_{ij}x_j}{a_{nn}}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & & & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Let solution of this system be
 $[c_{11} c_{21} \dots c_{n1}]^T$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & & & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} c_{11} \\ c_{21} \\ \vdots \\ c_{n1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

Let solⁿ of it be $[c_{12} c_{22} \dots c_{n2}]^T$

We can write

$$A \begin{bmatrix} c_{12} \\ c_{22} \\ \vdots \\ c_{n2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

$$\rightarrow A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \text{Let solution be } [c_{1n} c_{2n} \dots c_{nn}]^T$$

We can write :

$$A \begin{bmatrix} c_{1n} \\ c_{2n} \\ \vdots \\ c_{nn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Combining all :

$$A \begin{bmatrix} c_{11} & c_{12} & c_{13} & \dots & c_{1n} \\ c_{21} & c_{22} & c_{23} & \dots & c_{2n} \\ \vdots & & & & \vdots \\ c_{n1} & c_{n2} & c_{n3} & \dots & c_{nn} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

Inverse of a matrix :

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

$n \times n$

Identity matrix

By applying Gauss Jordan elimination

$$\begin{bmatrix} 1 & 0 & \dots & 0 & 1 & c_{11} & c_{12} & \dots & c_{1n} \\ 0 & 1 & \dots & 0 & 1 & c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & -1 & 0 & 1 & c_{n1} & c_{n2} & \dots & c_{nn} \end{bmatrix}$$

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L-U Decomposition Method

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$AX = B$$

$$\text{Let } A = LU$$

$$L = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix} \quad U = \begin{bmatrix} U_{11} & U_{12} & \cdots & U_{1n} \\ 0 & U_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & U_{nn} \end{bmatrix}$$

$$AX = B$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & \cdots & U_{1n} \\ 0 & U_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & U_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$UX = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Iterative method for solⁿ of linear simultaneous eqn:



$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

:

:

:

$$a_{nn}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

$$\begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \\ \vdots \\ x_n^{(0)} \end{bmatrix}^T = \mathbf{x}^0$$

$$\mathbf{x}^{(1)} = \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ \vdots \\ x_n^{(1)} \end{bmatrix}^T.$$

$$a_{11}x_1^{(1)} = b_1 - (a_{12}x_2^{(0)} + \dots + a_{1n}x_n^{(0)})$$

$$a_{22}x_2^{(1)} = b_2 - (a_{21}x_1^{(0)} + \dots + a_{2n}x_n^{(0)})$$

:

:

$$a_{nn}x_n^{(1)} = b_n - (a_{n1}x_1^{(0)} + a_{n2}x_2^{(0)} + \dots + a_{n(n-1)}x_{n-1}^{(0)})$$

Check if

$$\max_{1 \leq i \leq n} \{ |x_i^{(1)} - x_i^{(0)}| \} < \epsilon$$

then stop.

else continue the iterations.

Tacolor's method.

$$a_{11}x_1^{(k+1)} = b_1 - (a_{12}x_2^{(k)} + \dots + a_{1n}x_n^{(k)})$$

$$a_{22}x_2^{(k+1)} = b_2 - (a_{21}x_1^{(k)} + \dots + a_{2n}x_n^{(k)})$$

$$L = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ a_{21} & 0 & \dots & 0 & 0 \\ a_{31} & a_{32} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & 0 \end{bmatrix} \quad D = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ 0 & 0 & a_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{bmatrix}$$

$$A = L + D + U$$

$$\boxed{a_{nn}x_n^{(k+1)} = b_n - (a_{n1}x_1^{(k)} + a_{n2}x_2^{(k)} + \dots + a_{n(n-1)}x_{n-1}^{(k)})}$$

$$\boxed{x_1^{(0)} = x_2^{(0)} = \dots = x_n^{(0)} = 0}$$

In matrix notation: $Ax = B$

$$L = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ a_{21} & 0 & \dots & 0 & 0 \\ a_{31} & a_{32} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & 0 \end{bmatrix}$$

$$U = \begin{bmatrix} \dots & & & & 0 & 0 & \dots & 0 \\ 0 & a_{12} & a_{13} & \dots & a_{1n} & 0 & \dots & 0 \\ 0 & 0 & a_{23} & \dots & a_{2n} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{bmatrix}$$

In matrix notation, the set of eqns. in ① can be written as.

vector α col

$$Dx^{(k+1)} = B - (L+U)x^{(k)}$$

$$\text{or } x^{(k+1)} = D^{-1}\{B - (L+U)x^{(k)}\}$$

H is a $n \times n$ matrix depending on A and C is again a column vector defined as $Cx^{(k+1)} - x^{(k)} = D^{-1}\{B - (L+U)x^{(k)}\} - x^{(k)}$ on $A \leftarrow B$.

$$= D^{-1}\{B - (L+U+D)x^{(k)}\}$$

$$= D^{-1}\{B - Ax^{(k)}\}$$

$$= D^{-1}\{B - D^T B\}$$

$$\bullet J = D^T D$$

$$x^{(k+1)} = D^{-1} \{B - (L+U)x^{(k)}\}$$

Let $x^{(k+1)} = x + E^{(k+1)}$
 $\leftarrow x^{(k)} = x + E^{(k)} \cdot E^{(k+1)} =$
 are the error
 vectors
 associated with
 $x^{(k+1)} \leftarrow x^{(k)}$

$$x + E^{(k+1)} = D^{-1} \{B - (L+U)(x + E^{(k)})\}$$

$$\Rightarrow E^{(k+1)} = D^{-1} \{B - (L+U)x - (L+U)E^{(k)}\}$$

$$\begin{aligned} \Rightarrow E^{(k+1)} &= D^{-1} \{B - AX - (L+U)E^{(k)}\} \\ &= -D^{-1}(L+U)E^{(k)} \end{aligned}$$

For convergence

(\cancel{D})

$$\|(-D^{-1}(L+U))\| < 1$$

$$\Rightarrow \|(-D^{-1}(A-D))\| < 1$$

$$\Rightarrow \|(I - D^{-1}A)\| < 1$$

Let $(I - D^{-1}A)y = \lambda y$ where y is eigenvector.

or $(D - A)y = \lambda Dy$ λ is eigenvalue of A .

~~defn of convergence~~

$$\text{or } - \sum_{\substack{j=1 \\ j \neq i}}^n a_{ij} y_j = \lambda a_{ii} y_i \quad i=1, \dots, n$$

since y is a non-zero vector, $|y_i| \neq 0$

$$\therefore |\lambda| = \left\{ \sum_{\substack{j=1 \\ j \neq i}}^n |a_{ij}| \right\} / |a_{ii}| \quad i=1, \dots, n$$

for convergence $|\lambda| < 1$

$$\left\{ \sum_{j=1}^n |a_{ij}| \right\} / |a_{ii}| < 1$$

$$\text{or } \boxed{\sum_{j=1}^n |a_{ij}| < |a_{ii}|} \quad i=1, \dots, n$$

Partial pivoting can be used to
converge
non converge

$$\begin{array}{l} x+6y = 4 \\ x-2y = 14 \\ 9x+4y = -17 \end{array}$$

$$A = \begin{bmatrix} 1 & 6 & 0 \\ 1 & -2 & 6 \\ 9 & 4 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 6 & 0 & 4 \\ 1 & -2 & 6 & 14 \\ 9 & 4 & 1 & -17 \end{array} \right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 9 & 4 & 1 & -17 \\ 1 & -2 & 6 & 14 \\ 1 & 6 & 0 & 4 \end{array} \right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 9 & 4 & 1 & -17 \\ 1 & 6 & 0 & 4 \\ 1 & -2 & 6 & 14 \end{array} \right]$$

$$A = \begin{bmatrix} 9 & 2 & 1 \\ 1 & 6 & 0 \\ 1 & -2 & -6 \end{bmatrix}, B = \begin{bmatrix} -17 \\ 14 \\ 0 \end{bmatrix}, D = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$D^{-1} = \begin{bmatrix} 1/6 & 0 & 0 \\ 0 & 1/6 & 0 \\ 0 & 0 & 1/6 \end{bmatrix}$$

$$x^{(0)} = y^{(0)} = z^{(0)} = 0$$

$$\begin{aligned} x^{(1)} &= -1.9, \quad y^{(1)} = 0.67, \quad z^{(1)} = -2.3 \\ x^{(2)} &= 1.93, \quad y^{(2)} = 0.98, \quad z^{(2)} = -2.8 \\ x^{(3)} &= -2.006, \quad y^{(3)} = 0.99, \quad z^{(3)} = -2.9 \\ x^{(4)} &= -1.996, \quad y^{(4)} = 1.001, \quad z^{(4)} = -3.0 \\ x^{(5)} &= -2.000, \quad y^{(5)} = 0.999, \quad z^{(5)} = -3.0 \\ x^{(6)} &= -2.000, \quad y^{(6)} = 1.0, \quad z^{(6)} = -3.0 \end{aligned}$$

Gauss-Seidel Method

$$\begin{aligned} a_{11}x_1^{(R+1)} &= b_1 - (a_{12}x_2^{(R)} + \dots + a_{1n}x_n^{(R)}) \\ a_{22}x_2^{(R+1)} &= b_2 - (a_{21}x_1^{(R+1)} + \dots + a_{2n}x_n^{(R)}) \\ a_{33}x_3^{(R+1)} &= b_3 - (a_{31}x_1^{(R+1)} + a_{32}x_2^{(R+1)} + \dots + a_{3n}x_n^{(R)}) \end{aligned}$$

$$a_{nn}x_n^{(R+1)} = b_n - (a_{n1}x_1^{(R+1)} + a_{n2}x_2^{(R+1)} + \dots + a_{n(n-1)}x_{(n-1)}^{(R+1)})$$

$$\begin{aligned} a_{11}x_1^{(R+1)} \\ a_{22}x_2^{(R+1)} \\ a_{33}x_3^{(R+1)} \end{aligned} \quad \begin{aligned} &= ① \\ &= ② \\ &= ③ \end{aligned}$$

$$a_{11}x_1^{(R+1)} + a_{12}x_2^{(R+1)} + \dots + a_{1n}x_n^{(R+1)} = b_1$$

$$a_{22}x_2^{(R+1)} + a_{23}x_3^{(R+1)} + \dots + a_{2n}x_n^{(R+1)} = b_2$$

$$a_{33}x_3^{(R+1)} + a_{31}x_1^{(R+1)} + \dots + a_{3n}x_n^{(R+1)} = b_3$$

$$(L+D)x^{(R+1)} = B - UX^{(R)}$$

$$X^{(R+1)} = (L+D)^{-1}(B - UX^{(R)})$$

$$X^{(R+1)} - X^{(R)} = (L+D)^{-1}(B - UX^{(R)}) - X^{(R)}$$

$$X^{(R+1)} - X^{(R)} = (L+D)^{-1}\{B - UX^{(R)} - (L+D)X^{(R)}\}$$

$$= (L+D)^{-1}\{B - AX^{(R)}\}$$

$$E^{(R+1)} = (L+D)^{-1}\{B - UX^{(R)} - A(DX^{(R)})\}$$

$$E^{(R+1)} = (L+D)^{-1}B - (L+D)^{-1}UX^{(R)}$$

$$E^{(R+1)} = -(L+D)^{-1}UX^{(R)}$$

To do
Calc. conv. criteria of this method as earlier.

Iterative method to determine A^{-1}
 Let B be an approximate inverse of A

$$\therefore AB \neq I$$

$$\text{Let } AB = I + E \Rightarrow E = AB - I \quad \text{①}$$

premultiplying both sides with A^{-1}

~~$$A^{-1}AB = A^{-1}(I+E)$$~~

$$\Rightarrow A^{-1} + A^{-1}E \text{ or } B = A^{-1}(I+E)$$

Post multiplying both sides with $(I+E)^{-1}$

$$A^{-1} = B(I+E)^{-1} \approx B(I - E + E^2 - E^3 \dots)$$

if $\|E\| \ll 1$ we can write

$$A^{-1} \approx B(I-E) \\ \approx B(I - AB + I) \approx B(2I - AB)$$

Let $B^{(R+1)}$ & $B^{(R)}$ be the approx. inverse at $(R+1)$ & R th iter.

$$B^{(R+1)} = B^{(R)} \cdot (2I - AB^{(R)})$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad A^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\text{Let } B^{(0)} = \begin{bmatrix} 1.8 & -0.9 \\ -0.9 & 0.9 \end{bmatrix}$$

$$B^{(1)} = \begin{bmatrix} 1.98 & -0.91 \\ -0.91 & 0.99 \end{bmatrix}$$

$$B^{(2)} = \begin{bmatrix} 1.998 & -0.9999 \\ -0.9999 & 0.9999 \end{bmatrix}$$

Premultiplying both sides with A .

$$AB^{(R+1)} = AB^{(R)}(2I - AB^{(R)}) = 2I\{AB^{(R)}\} - \{AB^{(R)}\}^2$$

$$\text{or } AB^{(R+1)} - I = 2I\{AB^{(R)}\} - \{AB^{(R)}\}^2 - I$$

$$= -\{\{AB^{(R)} - I\}^2\} \xrightarrow{\text{cr (By ①)}} \\ \Rightarrow E^{(R+1)} = -\{E^{(R)}\}^2 \quad \text{Order=2}$$

$$x - \sin(x+y) = 0$$

$$y - \cos(x+y) = 0 \quad (0.951701, 0.307028)$$

Solⁿ of non linear simultaneous eqns

$$f_1(x_1, x_2, \dots, x_n) = 0$$

$$f_2(x_1, x_2, \dots, x_n) = 0$$

$$\vdots$$

$$f_n(x_1, x_2, \dots, x_n) = 0$$

$$\rightarrow \left\{ \begin{array}{l} x_1^{(k+1)} = F_1(x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)}) \\ x_2^{(k+1)} = F_2(x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)}) \\ \vdots \\ x_n^{(k+1)} = F_n(x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)}) \end{array} \right.$$

$$\max_{1 \leq i \leq n} \{ |x_i^{(k+1)} - x_i^{(k)}| \} < \epsilon$$

for convergence.

$$\sum_{j=1}^n \left| \frac{\partial F_i}{\partial x_j} \right| < 1 \quad i = 1, 2, \dots, n$$

Newton-Raphson method.

Let $x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)}$ be the initial approx.

& $\Delta x_1, \Delta x_2, \dots, \Delta x_n$ be the corrections needed.

$$\therefore f_1(x_1^{(0)} + \Delta x_1, x_2^{(0)} + \Delta x_2, \dots, x_n^{(0)} + \Delta x_n) = 0$$

$$f_2(x_1^{(0)} + \Delta x_1, x_2^{(0)} + \Delta x_2, \dots, x_n^{(0)} + \Delta x_n) = 0$$

$$f_n(x_1^{(0)} + \Delta x_1, x_2^{(0)} + \Delta x_2, \dots, x_n^{(0)} + \Delta x_n) = 0$$

~~Expanding by Taylor series~~

$$f_1(x^{(0)} + \Delta x_1 \frac{\partial f_1}{\partial x_1}|_{x^{(0)}}, x^{(0)} + \Delta x_2 \frac{\partial f_1}{\partial x_2}|_{x^{(0)}}, \dots, x^{(0)} + \Delta x_n \frac{\partial f_1}{\partial x_n}|_{x^{(0)}}) = 0$$

$$\max_{1 \leq i \leq n} (|\Delta x_i|) < \epsilon$$

$$f_2(x^{(0)} + \Delta x_1 \frac{\partial f_2}{\partial x_1}|_{x^{(0)}}, x^{(0)} + \Delta x_2 \frac{\partial f_2}{\partial x_2}|_{x^{(0)}}, \dots, x^{(0)} + \Delta x_n \frac{\partial f_2}{\partial x_n}|_{x^{(0)}}) = 0$$

$$\Delta x_1 g_x + \Delta x_2 g_y = -f_x$$

$$\Delta x_1 g_x + \Delta x_2 g_y = -g$$

$$\Delta x_1 = \frac{[-f_x \quad g_y]}{|f_x \quad g_y|} = \frac{[f_x \quad g_y]}{|f_x \quad g_y|}$$

$$\Delta x_2 = \frac{[f_x \quad -f_y]}{|f_x \quad g_y|} = \frac{[f_x \quad g_y]}{|f_x \quad g_y|}$$

Eigenvalue and Eigenvector of a square matrix

$$AX = \lambda X$$

λ is called eigen value
 X is called eigenvector

A is a non-zero scalar
 X is a non-zero column matrix or vector.

λ is called eigen value of A & X is its corresponding eigenvector.

λ & X represents ~~at~~ magnitude & dir. resp.

$AX = \lambda I X$ where I is an identity matrix of dimension same as A .

$$AX - \lambda I X = 0 \Rightarrow (A - \lambda I)X = 0$$

$$\det(A - \lambda I) = 0$$

characteristic eqn. for computing eigen value

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

$$\lambda I = \begin{bmatrix} \lambda & 0 & \dots & 0 \\ 0 & \lambda & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{bmatrix}$$

$\det(A - \lambda I)$ is a poly of λ of degree n .

1) For a $n \times n$ matrix, there are n different eigen values & com. n diff. eigenvectors.

2) If the eigen values are distinct, the corresponding eigenvectors are orthogonal.

$$X^T = X^{-1}$$

3) Two matrices A & B are called similar if there exist a non-singular matrix P such that $B = P^{-1}AP$.

If two matrices A & B are similar then they have the same set of eigenvalues.

$$B = P^{-1}AP$$

$$AX = \lambda X$$

Premultiplying with P^{-1}

$$P^{-1}AX = \lambda P^{-1}X$$

$$\text{Let } P^{-1}X = Y \Rightarrow P P^{-1}X = PY \Rightarrow X = P$$

$$P^T A X = \lambda Y$$

$$\text{or } P^T A P Y = \lambda Y$$

$$\Rightarrow B Y = \lambda Y$$

not only A & B have same set of eigenvalues corr. eigenvectors are related as $X = PY$

Find all eigenvalues of the following matrix

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & 1 \end{bmatrix}$$

$$\det \begin{vmatrix} 1-\lambda & 2 & -2 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda) \{ (\lambda-1)(\lambda+1) - 3 \} - 2 \{ -1-\lambda \} - 2 \{ 3-1+\lambda \} \cdot$$

$$\Rightarrow (1-\lambda)(\lambda^2-4) + 2(\lambda+2) - 2(2+\lambda) = 0$$

$$\lambda = -2$$

$$\Rightarrow (\lambda+2)(\lambda-2)(\lambda+1) + 2 - 2 = 0$$

$$\therefore \boxed{\lambda = -2, \lambda = 2, 1}$$

$$AX = \lambda X$$

$$\Rightarrow \begin{bmatrix} x_1 + 2x_2 - 2x_3 \\ x_1 + x_2 + x_3 \\ x_1 + 3x_2 - x_3 \end{bmatrix} = \begin{bmatrix} -y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\Rightarrow 2x_2 = 2x_3$$

$$x_1 = -x_3$$

$$x_1 = -3x_2$$

$$\therefore x_3 = y \quad \therefore x_2 = y \\ x_1 = -y$$

$$X = \boxed{\begin{bmatrix} -y \\ y \\ y \end{bmatrix}}$$

$$\Rightarrow X^T \vec{0} = \boxed{1}$$

$$\Rightarrow \begin{bmatrix} -y & y & y \end{bmatrix} \begin{bmatrix} -y \\ y \\ y \end{bmatrix} = 1$$

$$\Rightarrow y^2 + y^2 + y^2 = 1$$

$$\Rightarrow 3y^2 = 1 \quad \boxed{y = \pm \frac{1}{\sqrt{3}}}$$

Power method for finding the largest eigenvalue of a sq. matrix.

$$AX = \lambda X$$

Let us start the iterations with a trial eigenvector $X^{(0)}$

$$AX^{(0)} = Y^{(1)} = C^{(1)}X^{(1)}$$

$$AX^{(1)} = Y^{(2)} = C^{(2)}X^{(2)}$$

max.
value
in $X^{(1)}$

$$AX^{(k)} = Y^{(k+1)} = C^{(k+1)}X^{(k+1)}$$

Continue until $|C^{(k)} - C^{(k+1)}| < \epsilon$.

Then $C^{(k+1)}$ is the largest magn. Eigenval. of A and $X^{(k+1)}$ is the corr. eigenv.

$$A = \begin{bmatrix} 10 & 5 & 6 \\ 5 & 20 & 4 \\ 6 & 4 & 30 \end{bmatrix} \quad X^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$AX^{(0)} = \begin{bmatrix} 21 \\ 29 \\ 40 \end{bmatrix} = 40 \begin{bmatrix} 2/40 \\ 29/40 \\ 1 \end{bmatrix} = 40 \begin{bmatrix} 0.5 \\ 0.725 \\ 1.0 \end{bmatrix} = C^{(1)}X^{(1)}$$

$$AX^{(1)} = \begin{bmatrix} 14.875 \\ 21.125 \\ 36.050 \end{bmatrix} = 36.050 \begin{bmatrix} 0.413 \\ 0.596 \\ 1.000 \end{bmatrix} = C^{(2)}X^{(0)}$$

$$C^{(1)} = 33.718, \quad X^{(1)} = \begin{bmatrix} 0.341 \\ 0.417 \\ 1.000 \end{bmatrix}$$

$$C^{(2)} = 33.714, \quad X^{(2)} = \begin{bmatrix} 0.341 \\ 0.417 \\ 1.000 \end{bmatrix}$$

corrected upto 2 dec. places.

Proof of Power method:

Let us assume $\lambda_1, \lambda_2, \dots, \lambda_n$ be the n-eigenvalues of a $n \times n$ matrix A and $|\lambda_1| > |\lambda_2| > \dots > |\lambda_n|$

Let v_1, v_2, \dots, v_n be the corr. eigen-vectors.

Let us assume another vector V in the eigenvectors space which is represented as a linear combination of v_1, v_2, \dots, v_n

$$\begin{aligned} V &= c_1 v_1 + c_2 v_2 + \dots + c_n v_n \\ AV &= A \{c_1 v_1 + c_2 v_2 + c_3 v_3 + \dots + c_n v_n\} \\ &= \{c_1 A v_1 + c_2 A v_2 + \dots + c_n A v_n\} \\ &= \{c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + \dots + c_n \lambda_n v_n\} \\ &= \lambda_1 \{c_1 v_1 + c_2 \left(\frac{\lambda_2}{\lambda_1}\right) v_2 + \dots + c_n \left(\frac{\lambda_n}{\lambda_1}\right) v_n\} \end{aligned}$$

$$= \lambda_1 v_1$$

$$Av_1 = A \{ c_1 v_1 + c_2 \left(\frac{\lambda_2}{\lambda_1}\right) v_2 + \dots + c_n \left(\frac{\lambda_n}{\lambda_1}\right) v_n \}$$

$$= c_1 Av_1 + c_2 A \left(\frac{\lambda_2}{\lambda_1}\right) v_2 + \dots + c_n A \left(\frac{\lambda_n}{\lambda_1}\right) v_n$$

~~cancel c1~~

$$= \lambda_1 \{ c_1 v_1 + c_2 \left(\frac{\lambda_2}{\lambda_1}\right)^2 v_2 + \dots + c_n \left(\frac{\lambda_n}{\lambda_1}\right)^2 v_n \}$$

$$Av_k = A \{ c_1 v_1 + c_2 \left(\frac{\lambda_2}{\lambda_1}\right)^k v_2 + \dots + c_n \left(\frac{\lambda_n}{\lambda_1}\right)^k v_n \}$$

$$\text{As } k \rightarrow \infty \quad \Rightarrow \lambda_1 \{ c_1 v_1 + c_2 \left(\frac{\lambda_2}{\lambda_1}\right)^{k_1} v_2 + \dots + c_n \left(\frac{\lambda_n}{\lambda_1}\right)^{k_1} v_n \}$$

$$v_k = c_1 v_1 + c_2 \left(\frac{\lambda_2}{\lambda_1}\right)^{k_1} v_2 + \dots + c_n \left(\frac{\lambda_n}{\lambda_1}\right)^{k_1} v_n$$

$$v_{k+1} = c_1 v_1 + c_2 \left(\frac{\lambda_2}{\lambda_1}\right)^{k_1+1} v_2 + \dots + c_n \left(\frac{\lambda_n}{\lambda_1}\right)^{k_1+1} v_n$$

$$A c_1 v_1 = \lambda_1 c_1 v_1$$

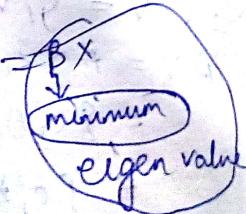
$$\Rightarrow Av_1 = \lambda_1 v_1$$

Premultiplying - with A^{-1}

$$A^{-1} A x = \lambda A^{-1} x$$

$$\Rightarrow x = \lambda A^{-1} x$$

$$\Rightarrow A^{-1} x = \frac{1}{\lambda} x$$



The largest eigenvalue is removed from the matrix using a method called deflation method to generate a new matrix A_1 where $\underbrace{A_1}_{\text{nxn matrix}} = A - \frac{\lambda_1 x_1 x_1^T}{x_1^T x_1}$

$$A_1 = A - \frac{\lambda_1 x_1 x_1^T}{x_1^T x_1}$$

To find the eigenvalue of A , corr. to x_1 , we calculate,

$$A_1 x_1 = Ax_1 - \frac{\lambda_1 x_1 x_1^T}{x_1^T x_1} x_1$$

$$= Ax_1 - \lambda_1 x_1 = 0$$

This proves that eigenvalue corresponding to x_1 is ~~not zero~~ $\Rightarrow 0$.

Transform method for finding all the eigenvalues and corresponding eigenvectors.

$\lambda_1, \lambda_2, \dots, \lambda_n$ be the n eigenvalues of a $n \times n$ matrix A and x_1, x_2, \dots be the corresponding eigenvectors.

$$AX_1 = \lambda_1 X_1 \quad \text{Let } X_1 = \begin{bmatrix} x_{11} \\ x_{12} \\ \vdots \\ x_{1n} \end{bmatrix}$$

$$AX_2 = \lambda_2 X_2$$

\vdots

$$AX_n = \lambda_n X_n$$



$$A \begin{bmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{bmatrix} \cdots \begin{bmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

$V \iff$

$$\begin{aligned} & V^{-1} A V = V D \\ & V^{-1} A V = V^T V D = D. \end{aligned}$$

Transform method for finding all the eigenvalues & corr. eigenvectors for symmetric matrices.

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

$$V = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \quad \text{Why ??}$$

$$V^T = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

$$\begin{aligned} V^{-1} A V &= \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \\ &= \begin{bmatrix} a\cos^2 \theta + b\sin \theta \cos \theta & a\sin \theta \cos \theta + b\cos^2 \theta \\ -a\sin \theta \cos \theta + b\cos \theta \sin \theta & a\sin^2 \theta + b\sin \theta \cos \theta \end{bmatrix} \\ &= \begin{bmatrix} a\cos^2 \theta + 2b\sin \theta \cos \theta + b\sin^2 \theta & b\cos^2 \theta + \cancel{a\sin \theta \cos \theta} \\ -a\sin \theta \cos \theta + b\cos \theta \sin \theta & a\sin^2 \theta - 2b\sin \theta \cos \theta + \cancel{b\sin^2 \theta} \end{bmatrix} \\ &= \begin{bmatrix} a\cos^2 \theta + b\sin^2 \theta + 2b\sin \theta \cos \theta & b\cos^2 \theta + a\sin^2 \theta \\ b\cos^2 \theta + \frac{(a-b)\sin^2 \theta}{2} & a\sin^2 \theta - b\sin \theta \cos \theta \end{bmatrix} \end{aligned}$$

$$bc \cos \theta + (c-a) \frac{\sin \theta}{2} = 0 \quad (\text{Non-diagonal elements})$$

$$b + (c-a) \frac{\tan \theta}{2} = 0 \quad (\text{zero})$$

$$\tan \theta = \frac{2b}{a-c}$$

$$\theta = \tan^{-1} \left(\frac{2b}{a-c} \right)$$

$$\theta = \frac{1}{2} \tan^{-1} \left(\frac{2b}{a-c} \right), \quad a \neq c.$$

Jacobi's method for finding all the eigenvalues and corresponding eigenvectors for symmetric matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$A_1 = P_1^T A P_1$$

$$A_2 = P_2^T A_1 P_2 = P_2^T P_1^T A P_1 P_2$$

$$A_2 = \underbrace{P_2^T P_1^T}_{V-1} \cdots P_2 P_1 A \underbrace{P_1 P_2 \cdots P_{2-1} P_2}_{V}$$

with V
Purely diagonal matrix

Let $a_{rs}^{(k-1)} (s > r)$ be the largest magnitude off-diagonal element of the intermediate similarity matrix A_{k-1}

then the P_k is formed as

$$\begin{aligned} p_{rr}^{(k)} &= p_{ss}^{(k)} = \cos \theta \\ -p_{rs}^{(k)} &= p_{sr}^{(k)} = \sin \theta \end{aligned} \quad \text{where } \theta = \frac{1}{2} \tan^{-1} \left(\frac{2a_{rs}^{(k)}}{a_{rr}^{(k)} - a_{ss}^{(k)}} \right)$$

$$p_{ij} = 1 \text{ for } i \neq s, r$$

$$p_{ij} = 0 \text{ otherwise.}$$

$$A_k = P_k^T A_{k-1} P_k$$

$$\begin{aligned} a_{ir}^{(k)} &= a_{ri}^{(k)} = a_{ir}^{(k-1)} \cos \theta + a_{is}^{(k-1)} \sin \theta \\ a_{is}^{(k)} &= a_{si}^{(k)} = -a_{ir}^{(k-1)} \sin \theta + a_{is}^{(k-1)} \cos \theta \end{aligned}$$

$$a_{rr}^{(k)} = a_{rr}^{(k-1)} \cos^2 \theta + 2a_{rs}^{(k-1)} \cos \theta \sin \theta + a_{ss}^{(k-1)} \sin^2 \theta$$

$$a_{ss}^{(k)} = a_{ss}^{(k-1)} \sin^2 \theta - 2a_{rs}^{(k-1)} \cos \theta \sin \theta + a_{rr}^{(k-1)} \cos^2 \theta$$

$$a_{rs}^{(k)} = a_{sr}^{(k)} = 0$$

$$a_{ij}^{(k)} = a_{ij}^{(k-1)}, \text{ otherwise}$$

$$V_0 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$V_1 = V_0 P_1 \rightarrow V_R = V_{R-1} P_k$$

$$V_2 = V_1 P_2 \quad V_{IR}^{(k)} = V_{IR}^{(k-1)} \cos \theta + V_{IS}^{(k-1)} \sin \theta$$

$$V_{js} = -V_{j1} \cos \theta \sin \theta + V_{j2} \cos^2 \theta$$

$j = 1, 2, \dots, n$

$$A = \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix}$$

Largest magnitude off. ~~diag~~ diag element $= a_{13} = a_{31} = 2$

$$\begin{aligned} P_{11} = P_{33} &= \cos \theta \\ -P_{13} = P_{31} &= \sin \theta \end{aligned} \quad \left. \right\} \text{where } \theta = \frac{1}{2} \tan^{-1} \left(\frac{2a_{13}}{a_{11} - a_{33}} \right) = \frac{1}{2} \tan^{-1} \left(\frac{4}{1-1} \right)$$

$$= \frac{1}{2} \times \frac{\pi}{2} = \frac{\pi}{4}$$

$$P_1 = \begin{bmatrix} \cos \frac{\pi}{4} & 0 & -\sin \frac{\pi}{4} \\ 0 & 1 & 0 \\ \sin \frac{\pi}{4} & 0 & \cos \frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$A_1 = P_1^T A P_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}}, 0 \\ 0, 1 \\ \frac{1}{\sqrt{2}}, 0 \end{bmatrix}$$

3

$$= \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$A_2 = P_2^T A_1 P_2 = \begin{bmatrix} \sqrt{5}_2 & \sqrt{5}_2 & 0 \\ 0 & \sqrt{5}_2 & 0 \\ \sqrt{5}_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \sqrt{5}_2 & 0 & 0 \\ 0 & \sqrt{5}_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} V &= \begin{bmatrix} \sqrt{5}_2 & 0 & -\sqrt{5}_2 \\ 0 & 1 & 0 \\ \sqrt{5}_2 & 0 & \sqrt{5}_2 \end{bmatrix} \begin{bmatrix} \sqrt{5}_2 & -\sqrt{5}_2 & 0 \\ \sqrt{5}_2 & \sqrt{5}_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1_2 & -\sqrt{5}_2 & 0 \\ \sqrt{5}_2 & 0 & -\sqrt{5}_2 \\ 0 & 0 & 1 \end{bmatrix} \\ &\quad \begin{bmatrix} \sqrt{5}_2 & 0 & 0 \\ 0 & \sqrt{5}_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

~~Step 2~~
Ruthmayer method for arbitrary matrix

$$A = LU \quad \text{where } U =$$

$$\begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ 0 & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & v_{nn} \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Let us apply the transformation

$$A_1 = U A U^{-1} = U L$$

$$\begin{aligned} &\text{i.e.} \\ &\text{again } \det A_1 = Q_0 L_1 U_1 \text{ where } (Q_0^{(1)} = 1) \\ &\text{and } A_2 = U_1 A_1 U_1^{-1} = U_1 L_1 \end{aligned}$$

- 1 This is continued until A_k matrix is reduced to upper triangular form.
- 2 an upper triangular elements will represent the eigenvalues of A .
- 3 i.e. eigenvalues of A