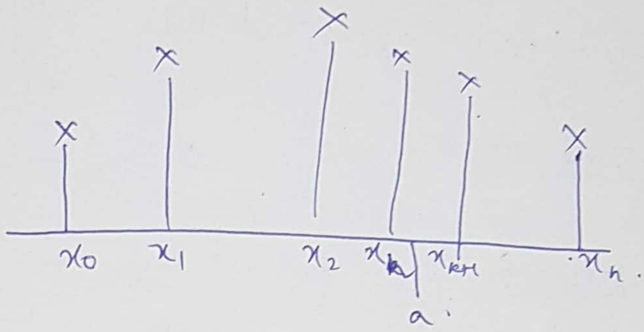
Interpolation

x_i	$y_i\ f(x)$
x_0	y_0
x_1	y_1
$\vdots$	
x_n	y_n



$$\frac{y-y_k}{x-x_k} = \frac{y_{k+1}-y_k}{x_{k+1}-x_k}$$

$$y = y_k + \left(\frac{y_{k+1}-y_k}{x_{k+1}-x_k}\right)\cdot(x-x_k)$$

$$\therefore\ y\Big|_{\text{at } x=a} = y_k + \left(\frac{y_{k+1}-y_k}{x_{k+1}-x_k}\right)(a-x_k)$$

Attempt is made to represent the graph using nth degree polynomial.

2 types of situations may arise.

→ The independent variable (i.e. x_i values) are (i) equispaced $x_i = x_0 + ih$.

(ii) non-equispaced

For equispaced, x_i values we follow diff. finite diff. formulae like Newton Forward / Backward diff. formula / Gauss Central difference formulae.

For not equispaced cases, we follow,

→ divided diff. method.

→ Lagrange interpolation

→ Iterative interpolation.

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Finite difference operators.

Forward diff. (Δ).

First order forward diff. is defined as

$$\Delta y_i = y_{i+1} - y_i$$

2nd order

$$\Delta^2 y_i = \Delta y_{i+1} - \Delta y_i$$

kth order forward diff.

$$\Delta^k y_i = \Delta^{k-1} y_{i+1} - \Delta^{k-1} y_i$$

Forward diff. Table

x_i	y_i	Δy_i	$\Delta^2 y_i$	$\Delta^3 y_i$	$\Delta^4 y_i$
x_0	y_0	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$.
x_1	y_1	Δy_1	$\Delta^2 y_1$	$\Delta^3 y_1$	
x_2	y_2	Δy_2	$\Delta^2 y_2$		
x_3	y_3	Δy_3			
x_4	y_4				

$$\Delta^2 y_i = (y_{i+2} - y_{i+1}) - (y_{i+1} - y_i)$$

$$= 1 y_{i+2} - 2 y_{i+1} + 1 y_i$$

$$\Delta^3 y_i = \Delta^2 y_{i+1} - \Delta^2 y_i = (y_{i+3} - 2y_{i+2} + y_{i+1}) - (y_{i+2} - 2y_{i+1} + y_i)$$

$$= 1 y_{i+3} - 3 y_{i+2} + 3 y_{i+1} - 1 y_i$$

Backward difference (∇)

First order backward diff.

$$\nabla y_{i+1} = y_{i+1} - y_i$$

Second order backward diff.

$$\nabla^2 y_{i+1} = \nabla y_{i+1} - \nabla y_i$$

kth order backward diff.

$$\nabla^k y_{i+1} = \nabla^{k-1} y_{i+1} - \nabla^{k-1} y_i$$

Backward Difference table.

x_i	y_i	∇y_i	$\nabla^2 y_i$	$\nabla^3 y_i$	$\nabla^4 y_i$
x_0	y_0				
x_1	y_1	∇y_1			
x_2	y_2	∇y_2	$\nabla^2 y_2$		
x_3	y_3	∇y_3	$\nabla^2 y_3$	$\nabla^3 y_3$	
x_4	y_4	∇y_4	$\nabla^2 y_4$	$\nabla^3 y_4$	$\nabla^4 y_4$

Shift operator (E)

This operator shifts a functional value y_i to a higher index

$$\therefore E y_i = y_{i+1}$$

Inverse shift operator E^{-1}

$$y_i = E^{-1} y_{i+1}$$

Equivalence amount finite diff. operators.

$$\boxed{E = (1+\Delta)}$$

$$\Delta y_i = y_{i+1} - y_i$$
$$= E y_i - y_i$$
$$\equiv (E-1) y_i$$

or $\Delta \equiv (E-1)$

$E = 1 + \Delta$

$$\nabla y_{i+1} = y_{i+1} - y_i$$
$$= y_{i+1} - E^{-1} y_{i+1}$$
$$\equiv (1 - E^{-1}) y_{i+1}$$
$$\therefore \nabla \equiv (1 - E^{-1})$$
$$\text{or } E \equiv (1-\nabla)^{-1} = \frac{1}{1-\nabla}$$

1) $\Delta \equiv E\nabla$

$$\Delta y_i = y_{i+1} - y_i$$
$$= E y_i - E y_{i-1}$$
$$\equiv E(y_i - y_{i-1})$$
$$\equiv E(\nabla y_i)$$

2) $\cancel{\nabla - \Delta = -\Delta\nabla}$ $\cancel{-(y_{i+1} - y_i)}$

$\cancel{\nabla y_i - \Delta y_i = y_i - y_{i-1} - (y_{i+1} - y_i)}$

$= \cancel{y_i - y_{i-1} - (E y_i -}$

$=$

2) $\nabla - \Delta = -\Delta\nabla$

$$E = 1+\Delta = \frac{1}{1-\nabla}$$
$$(1+\Delta)(1-\nabla) = 1$$
$$1 - \nabla + \Delta - \Delta\nabla = 1$$
$$\nabla - \Delta = -\Delta\nabla$$

3) $\Delta + \nabla = \Delta/\nabla - \nabla/\Delta$

$$E^{-1} = 1 - \nabla$$
$$\frac{\nabla}{\Delta} = 1 - \nabla$$
$$\nabla = 1 - \frac{\nabla}{\Delta}$$

$$E \equiv (1+\Delta)$$
$$\Rightarrow \frac{\Delta}{\nabla} - 1 = \Delta$$

$\therefore \boxed{\Delta + \nabla = \Delta/\nabla - \nabla/\Delta}$

Equivalence amount finite diff operators.

$$\Delta^{r} y_k = \nabla^{r} y_{k+r}$$

for $r = 1$

$$\Delta y_k = y_{k+1} - y_k$$
$$= \nabla y_{k+1}$$

For $r = 2$

$$\Delta^2 y_k = \Delta y_{k+1} - \Delta y_k$$
$$= \nabla y_{k+2} - \nabla y_{k+1}$$
$$= \nabla^2 y_{k+2}$$

By mathematical induction.

$$y_p = E^p y_0 \qquad x_p = x_0 + p\underline{h}$$

$$(x_i - x_{i-1})$$

$$= (1+\Delta)^p y_0 .$$

$$= \left(1 + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \frac{p(p-1)(p-2)}{3!}\Delta^3 + \cdots\right) y_0$$

$$= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + p(p-1)(p-2)\Delta^3 y_0 + \cdots$$

Newton's forward diff. formula.

This formula is useful for interpolation near the top end of the tabular values.

Newton's backward diff. interpolation formula for interpolation near the trailing end or bottom of the tabular values

$$y_p = E^{-p} y_n .$$

$$= (1-\nabla)^{-p} y_n$$

$$= \left(1 - p\nabla + \frac{p(p-1)}{2!}\nabla^2 - \frac{p(p-1)(p-2)}{3!}\Delta^3 + \cdots\right) y_n .$$

$$= \left(y_n - p\nabla y_n + \frac{p(p-1)}{2!}\nabla^2 y_n - \frac{p(p-1)(p-2)}{3!}\nabla^3 y_n + \cdots\right)$$

Gauss. Central diff. formula.

backward version is used for interpolation near the central part of the table but above the central row

Let the interpolating polynomial be.

$$y_p = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_{-1}) + a_3(x-x_0)(x-x_1)\cdot(x-x_{-1})$$

$$+ a_4 (x-x_0)(x-x_{-1})(x-x_1)(x-x_{-2})$$

at $x = x_i$, $y_p = y_i$ $\quad i = -n, -n-1, \ldots -1, 0, 1, 2, \ldots, n$

→ at $x = x_0$, $a_0 = y_0$.

→ at $x = x_{-1}$, $y_p = y_{-1}$

$\therefore y_{-1} = y_0 + a_1(x_{-1} - x_0)$

$y_{-1} - y_0 = a_1(-h)$

or $-\Delta y_{-1} = -a_1 h$ or $a_1 = \frac{\Delta y_{-1}}{h}$.

→ at $x = x_1$, $y = y_1$

$y_1 = a_0 + a_1(x_1 - x_0) + a_2(x_1 - x_0)(x_1 - x_{-1})$

$= y_0 + \frac{\Delta y_{-1} \cancel{h}}{\cancel{h}} + a_2 \cdot 2h^2$.

$a_2 2h^2 = y_1 - y_0 - \Delta y_{-1} = \Delta y_0 - \Delta y_{-1} = \Delta^2 y_{-1}$

or $\boxed{a_2 = \frac{\Delta^2 y_{-1}}{2h^2}}$ $\quad \boxed{a_3 = \frac{\Delta^3 y_{-2}}{3!\, h^3}}$ $\quad \boxed{a_4 = \frac{\Delta^4 y_{-2}}{4!\, h^4}}$

x_i	y_i	Δy_i	$\Delta^2 y_i$	$\Delta^3 y_i$	$\Delta^4 y_i$	$\Delta^5 y_i$	$\Delta^6 y_i$
x_{-3}	y_{-3}	Δy_{-3}	$\Delta^2 y_{-3}$	$\Delta^3 y_{-3}$	$\Delta^4 y_{-3}$	$\Delta^5 y_{-3}$	$\Delta^6 y_{-3}$
x_{-2}	y_{-2}	Δy_{-2}	$\Delta^2 y_{-2}$	$\Delta^3 y_{-2}$	$\Delta^4 y_{-2}$	$\Delta^5 y_{-2}$	
x_{-1}	y_{-1}	Δy_{-1}	$\Delta^2 y_{-1}$	$\Delta^3 y_{-1}$	$\Delta^4 y_{-1}$		
x_0	y_0	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$			
x_1	y_1	Δy_1	$\Delta^2 y_1$				
x_2	y_2	Δy_2					
x_3	y_3						

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Forward difference version

$$y_p = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1)$$
$$+ a_3(x-x_0)(x-x_1)(x-x_{-1})$$
$$+ a_4(x-x_0)(x-x_1)(x-x_{-1})(x-x_2)$$

→ at $x = x_0$, $y = y_0$

$$a_0 = y_0.$$

$$a_1 = \frac{\Delta y_0}{h}$$

$$a_2 = \frac{\Delta^2 y_{-1}}{h^2}.$$

$$a_3 = \frac{\Delta^3 y_{-1}}{h^3}$$

$\vdots$

$y_0 + p\Delta y + \frac{p(p-1)}{2}$ $\frac{(p+1)p(p-1)}{3!}$ $\frac{(p+1)p(p-1)(p-2)}{4!}$

Let the temp. vs sp. heat of ethyl alcohol table is given below. Find sp ht. at $x = 15°c$, $25°c$.

$x°c$	Sp-heat	Δ	Δ^2	Δ^3	Δ^4
0	0.51	0.04	-0.02	0.02	-0.01
10	0.55	0.02	0	0.01	
$x_0 = 20$	$y_0 = 0.57$	0.02	0.01		
30	0.59	0.03			
40	0.62				

$$-\frac{\frac{7}{5} \times \frac{2}{5} \times \frac{3}{5}}{6}$$

When x_i values are not equispaced.

1) Divided difference method.
2) Lagrange interpolation method.
3) Aitken's iterative interpolation method.

Divided difference method:

First order divided difference is defined as:

$$y[x_i, x_{i+1}] = \frac{y_{i+1} - y_i}{x_{i+1} - x_i} = \frac{y_{i+1}}{x_{i+1} - x_i} + \frac{y_i}{x_i - x_{i+1}}$$

Similarly 2nd order divided diff

$$y[x_i, x_{i+1}, x_{i+2}] = \frac{y[x_{i+1}, x_{i+2}] - y[x_i, x_{i+1}]}{x_{i+2} - x_i}$$

kth order divided diff.

$$y[x_i, x_{i+1}, \dots, x_{i+k}] = \frac{y[x_{i+1}, x_{i+2}, \dots, x_{i+k}] - y[x_i, x_{i+1}, \dots x_{i+k-1}]}{x_{i+k} - x_i}$$

x_i	y_i	$y[x_i, x_{i+1}]$	$y[x_i, x_{i+1}, x_{i+2}]$
x_0	y_0	$y[x_0, x_1]$	$y[x_0, x_1, x_2]$
x_1	y_1	$y[x_1, x_2]$	$y[x_1, x_2, x_3]$
x_2	y_2	$y[x_2, x_3]$	$y[x_2, x_3, x_4]$
x_3	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	
x_{n-1}	y_{n-1}	$y[x_{n-1}, x_n]$	
x_n	y_n		

For first column, no. of division $= n$

$n-1$

$\vdots$

$\therefore$ total no. of divisions $= \frac{n(n+1)}{2}$

$$y[x_i, x_{i+1}, x_{i+2}] = \frac{y[x_{i+1}, x_{i+2}] - y[x_i, x_{i+1}]}{x_{i+2} - x_i}$$

$$= \frac{\left\{\frac{y_{i+2}}{x_{i+2}-x_{i+1}} + \frac{y_{i+1}}{x_{i+1}-x_{i+2}}\right\} - \left\{\frac{y_{i+1}}{x_{i+1}-x_i} + \frac{y_i}{x_i - x_{i+1}}\right\}}{x_{i+2}-x_i}$$

$$= \frac{\frac{y_{i+2}}{(x_{i+2}-x_{i+1})} + \frac{y_{i+1}(x_{i+1}-x_i-x_{i+1}+x_{i+2})}{(x_{i+1}-x_{i+2})(x_{i+1}-x_i)} - \frac{y_i}{(x_i-x_{i+1})}}{x_{i+2}-x_i}$$

$$= \frac{y_{i+2}}{(x_{i+2}-x_{i+1})(x_{i+2}-x_i)} + \frac{y_{i+1}}{(x_{i+1}-x_{i+2})(x_{i+1}-x_i)} + \frac{y_i}{(x_i-x_{i+1})(x_i-x_{i+2})}$$

kth order divided difference

$$= \frac{y_{i+k}}{(x_{i+k}-x_{i+k-1})(x_{i+k}-x_{i+k-2})\cdots(x_{i+k}-x_i)} + \frac{y_{i+k-1}}{(x_{i+k-1}-x_{i+k})(x_{i+k-1}-x_{i+k-2})\cdots(x_{i+k-1}-x_i)}$$

$$\cdots + \frac{y_i}{(x_i-x_{i+k})(x_i-x_{i+k-1})\cdots(x_i-x_{i+1})}$$

$$y[x, x_0] = \frac{y-y_0}{x-x_0}.$$

or $y - y_0 = (x-x_0)\, y[x, x_0]$

or $y = y_0 + (x-x_0)\, y[x, x_0]$ — ①

$$y[x, x_0, x_1] = \frac{y[x_0, x_1] - y[x, x_0]}{x_1 - x}$$

$\Rightarrow y[x_0, x_1] - y[x, x_0] = (x_1 - x)\, y[x, x_0, x_1]$

$y[x, x_0] = y[x_0, x_1] + (x - x_1)\, y[x, x_0, x_1]$ — ②.

From ① & ②

$y = y_0 + (x-x_0)\left\{y[x_0, x_1] + (x-x_1)\, y[x, x_0, x_1]\right\}$.

— ③

~~$y[x_0 x_3\ x_1]$~~

$$y[x, x_0, x_1, x_2] = \frac{y[x_0, x_1, x_2] - y[x, x_0, x_1]}{x_2 - x}$$

$$y[x, x_0, x_1] = (x - x_2)\, y[x, x_0, x_1, x_2] + y[x_0, x_1, x_2]$$

(3) $\Rightarrow$ $y = y_0 + (x-x_0)\, y[x_0, x_1] + (x-x_0)(x-x_1)\, y[x, x_0, x_1]$

$+ (x-x_0)(x-x_1)(x-x_2)\, y[x, x_0, x_1, x_2]$

Continuing for n times.

$$y = y_0 + \overset{1}{(x-x_0)}\, y[x_0, x_1] + (x-x_0)\overset{2}{(x-x_1)}\, y[x_0, x_1, x_2]$$
$$+ (x-x_0)(x-x_1)\overset{3}{(x-x_2)}\, y[x_0, x_1, x_2, x_3] + \cdots$$
$$\cdots$$
$$+ (x-x_0)(x-x_1)\overset{n}{\cdots}(x-x_{n-1})\, y[x_0, x_1, x_2, \ldots, x_n]$$
$$+ \underbrace{(x-x_0)(x-x_1)\cdots\cdots(x-x_n)\, y[x, x_0, x_1, x_2, \ldots x_n]}_{\text{Remainder term.}}$$

No. of multiplications for computing y

$= 1 + 2 + \cdots + n$ (marked in blue in the above exp).

$= \frac{n(n+1)}{2}$.

$\therefore$ Total no. of operations $= \frac{n(n+1)}{2} + \frac{n(n+1)}{2} = n(n+1)$

Lagrange's Method

Here we assume the interpolating f^n as a weighted sum of $(n+1)$ nth degree poly.

$\therefore y(x) = y_0 b_0(x) + y_1 b_1(x) + \dots + y_n b_n(x)$.

each $b_i(x)$ is a polynomial of degree n

Since this polynomial satisfies the cond^n $y(x_i) = y_i$

$i = 0, 1, 2 \dots, n$.

$$y_0 = y_0 b_0(x_0) + y_1 b_1(x_0) + \dots + y_n b_n(x_0)$$

$$y_1 = y_0 b_0(x_1) + y_1 b_1(x_1) + \dots + y_n b_n(x_1)$$

$$\vdots$$

$$y_n = y_0 b_0(x_n) + y_1 b_1(x_n) + \dots + y_n b_n(x_n)$$

we choose $b_i(x)$ in such a way that

$$b_i(x_j) = 1 \text{ for } i = j$$
$$= 0 \quad \text{otherwise}.$$

$$b_i(x) = \underbrace{c_i (x - x_0)(x - x_1) \dots (x - x_{i-1})(x - x_{i+1}) \dots (x - x_n)}_{= 0 \text{ if } i \neq j}$$

$b_i(x_i) = 1$

$\Rightarrow c_i (x_i - x_0)(x_i - x_1) \dots (x_i - x_{i-1})(x_i - x_{i+1}) \dots (x_i - x_n) = 1$

$$\therefore c_i = \frac{1}{\prod\limits_{\substack{j=1 \\ i \neq j}} (x_i - x_j)}.$$

$$\therefore b_i(x) = \frac{1}{\prod\limits_{\substack{j=1 \\ j \neq i}}^{n} (x_i - x_j)} (x - x_0)(x - x_1) \dots (x - x_{i-1})(x - x_{i+1}) \dots (x - x_n)$$

$$\therefore y(x) = \sum_{i=0}^{n} y_i \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{(x - x_j)}{(x_i - x_j)}.$$

$(2n+2)(n+1)$ or $(2n+1)(n+1)$.

$= 2(n+1)^2$

x_i	y_i	$y[x_i, x_{i+1}]$	$y[x_i, x_{i+1}, x_{i+2}]$		
0	1	13	-6	1	0
1	14	1	-2	1	0
2	15	-5	2	1	
4	5	1	6		
5	6	13			
6	19				

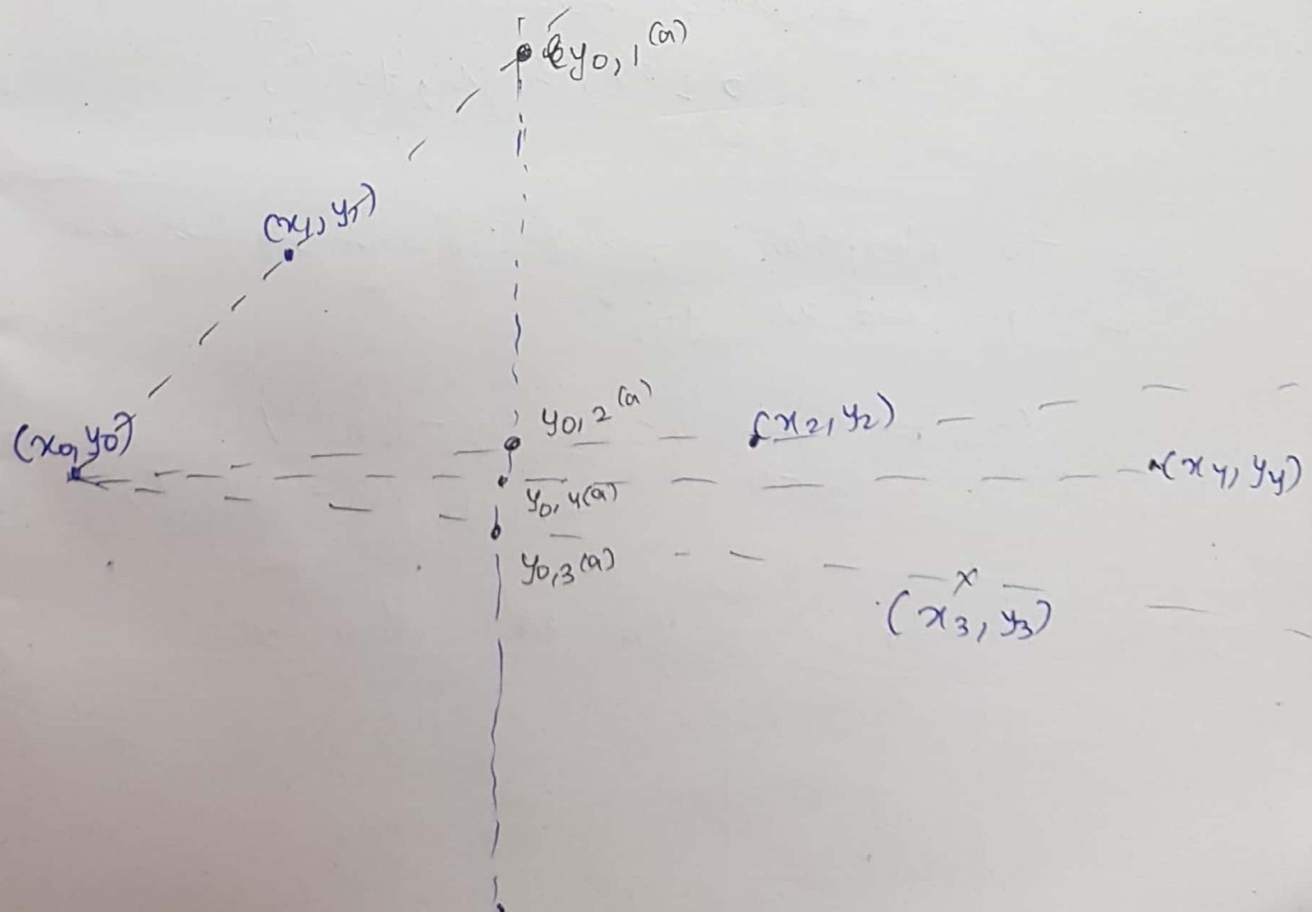
find y at $x = 3$. (By lagrange's formula)

$$y(3) = y_0 + (3-x_0)\, y[x_0, x_1] + (3-x_0)(3-x_1)\, y[x_0, x_1, x_2] + (3-x_0)(3-x_1)(3-x_2)\, y[x_0, x_1, x_2, x_3]$$

$$= 1 + (3-0)\,13 + (3-0)(3-1)(-6) + (3-0)(3-1)(3-2) \times 1$$

$$= 1 + 39 - 36 + 6$$

$$= 10$$



Equation of each st. line

$$\frac{y-y_i}{x-x_i} = \frac{y_i-y_0}{x_i-x_0}$$

$$\frac{y_{0,i}(a)-y_i}{a-x_i} = \frac{y_i-y_0}{x_i-x_0}.$$

$$\therefore\ y_{0,i}(a) = (a-x_i)\frac{(y_i-y_0)}{x_i-x_0} + y_i$$

$$= \frac{1}{(x_i-x_0)}\left\{\cancel{y_i x_i} - y_i x_0 + a y_i - a y_0 - \cancel{x_i y_i} + x_i y_0\right\}$$

$$= \frac{1}{(x_i-x_0)}\left\{a(y_i-y_0) + x_i y_0 - x_0 y_i\right\}.$$

next iteration: we replace the values of y_i with $y_{0,i}(a)$.

Then we fit st. lines between $(x_1, y_{0,1}(a))$ & $\{x_i, y_{0,i}(a)\}$. $i = 2, 3, \ldots, n$.

Let the interpolation result for each of these lines be represented as $y_{0,1,i}$

$$\therefore\ y_{0,1,i}(a) = \frac{1}{(x_i-x_1)}\left\{a(y_{0,i}(a) - y_{0,1}(a)) + x_i y_{0,1}(a) - x_0 y_{0,i}(a)\right\}$$

x_i	y_i	$y_{0,i}(3)$	$y_{0,1,i}(3)$	$y_{0,1,2,i}(3)$

Curve fitting using least square error method.

Let the eq^n of the curve, to be fitted be

$$y = a_1 b_1(x) + a_2 b_2(x) + \dots + a_k b_k(x)$$

where each $b_i(x)$ is a chosen polynomial ($x, x^2, \dots$) or transcendental ($\sin x, \cos x, \log x, e^{-x}$) terms

$$xy^r = c$$

$$\log x + r\log y = \log c$$

Let $\log x = X$, $\log y = Y$

$\log c = C$.

$$\rightarrow Y = \frac{1}{r}(C - X).$$

for the given $(n+1)$ tabular values $y(x_i) = y_i$, $i = 0, 1, \dots, n$. the vertical distance between the fitted curve and tabular y_i value is expressed as

$$e_i = \{g(x_i) - y_i\}. \quad i = 0, 1, \dots, n$$

Sum of the squares of all terms.

$$E = \sum_{i=0}^{n} e_i^2 = \sum_{i=0}^{n} \{g(x_i) - y_i\}^2.$$

$$E = \sum_{i=0}^{n} \{a_1 b_1(x_i) + a_2 b_2(x_i) + \dots + a_k b_k(x_i) - y_i\}^2$$

For minimum E,

$$\frac{\partial E}{\partial a_1} = \frac{\partial E}{\partial a_2} = \dots = \frac{\partial E}{\partial a_k} = 0$$

$$\frac{\partial E}{\partial a_1} = 2\{a_1 \Sigma b_1(x_i) + a_2 \Sigma b_2(x_i) + \dots + a_k \Sigma b_k x_i - \Sigma y_i\} \Sigma b_1(x_i) = 0$$

$$= \{a_1\{\Sigma b_1(x_i)\}^2 + a_2 \Sigma b_2(x_i) b_1(x_i) + \dots + a_k \Sigma b_1(x_i) b_k(x_i)\} - \Sigma y_i b_1(x_i)\} = 0$$

$$\Rightarrow a_1 \{\Sigma b_1(x_i)\}^2 + a_2 \Sigma b_2(x_i) b_1(x_i) + \dots + a_k \Sigma b_1(x_i) b_k(x_i) = \Sigma y_i b_1(x_i).$$

$$\frac{\partial E}{\partial a_2} \to a_1 \Sigma b_1(x_i)\, b_2(x_i) + a_2 \{\Sigma b_2(x_i)\}^2 + \cdots$$

$$+ a_k \Sigma b_2(x_i) b_k(x_i) = \Sigma y_i\, b_2(x_i)$$

$\vdots$

$$\frac{\partial E}{\partial a_k} \to a_1 \Sigma b_1(x_i) b_k(x_i) + a_2 \{\Sigma b_2(x_i)\, b_k(x_i)\} + \cdots$$

$$+ a_k \{\Sigma b_k(x_i)\}^2 = \Sigma y_i\, b_k(x_i)$$

Obtain the values of $a_1, a_2, \ldots, a_k$ from the above k-simultaneous equation

Ex:

Fit a curve of the form $y = ab^x$ to the following tabular values:

$y = ab^x$

$y = a_1 g_1(x) + a_2 g_2(x) + \cdots + a_k g_k(x)$

$\log_{10} y = \log_{10} a + x \log_{10} b$

i	x_i	y_i
0	2	8.3
1	3	15.4
2	4	33.1
3	5	65.2
4	6	127.4

$Y = A + Bx$

$g_1(x) = 1$

$g_2(x) = x$

i	x_i	y_i	$Y_i = \log_{10} y_i$	x_i^2	$x_i Y_i$
0	2	8.3	0.9191	4	1.8382
1	3	15.4	1.1875	9	3.5625
2	4	33.1	1.5198	16	6.0792
3	5	65.2	1.8142	25	9.0710
4	6	127.4	2.1052	36	12.6310
			7.5458		Σ 33.1821

$a_1 \Sigma\{g_1(x)\}^2 + a_2 \Sigma g_1(x) g_2(x) + \cdots + a_k \Sigma g_1(x_i) g_k(x_i) = \Sigma y_i g_1(x_i)$

$a_1 \Sigma g_1(x_i) g_2(x_i) + a_2 \Sigma\{g_2(x_i)\}^2 + \cdots + a_k \Sigma g_2(x_i) g_k(x_i) = \Sigma y_i g_2(x_i)$

$\vdots$

$a_1 \Sigma_1(x_i) g_k(x_i) + a_2 \Sigma g_2(x_i) g_k(x_i) + \cdots + a_k \Sigma\{g_k(x_i)\}^2 = \Sigma y_i g_k(x_i)$

Here,

$A \cdot \Sigma\{g_1(x_i)\}^2 + B \Sigma g_1(x_i) g_2(x_i) = \Sigma y_i g_1(x_i)$

$A \Sigma g_1(x_i) g_2(x_i) + B \Sigma\{g_2(x_i)\}^2 = \Sigma y_i g_2(x_i)$

$\rightarrow A \Sigma(1)^2 + B \Sigma x_i = \Sigma Y_i$

$\rightarrow A \Sigma x_i + B \Sigma x_i^2 = \Sigma x_i Y_i$

$A\,5 + B\,20 = 7.5458$

$A\,20 + B\,90 = 33.1821$

$\frac{n(2n+1)(2n+1)}{6}$

$\frac{6 \times 7 \times 13}{6} - 1$

$B = \dfrac{\frac{33.1821}{4} - 7.5458}{2.5} = \dfrac{8.2955 - 7.5458}{2.5}$

$= \dfrac{0.7493}{2.5000} = \dfrac{2.9972}{10}$

$B = 0.29972.$

$A \times n + B\,sx = sy$

$A\,sx + B\,ssx = sxy$

$A = \begin{vmatrix} sy & sx \\ sxy & ssx \end{vmatrix} \Big/ \begin{vmatrix} n & sx \\ sx & ssx \end{vmatrix}$

$n \times ssx - (sx)^2$

$A = 0.3096.$

$B = \begin{vmatrix} n & sy \\ sx & sxy \end{vmatrix}$

$nsxy - sx\,y$

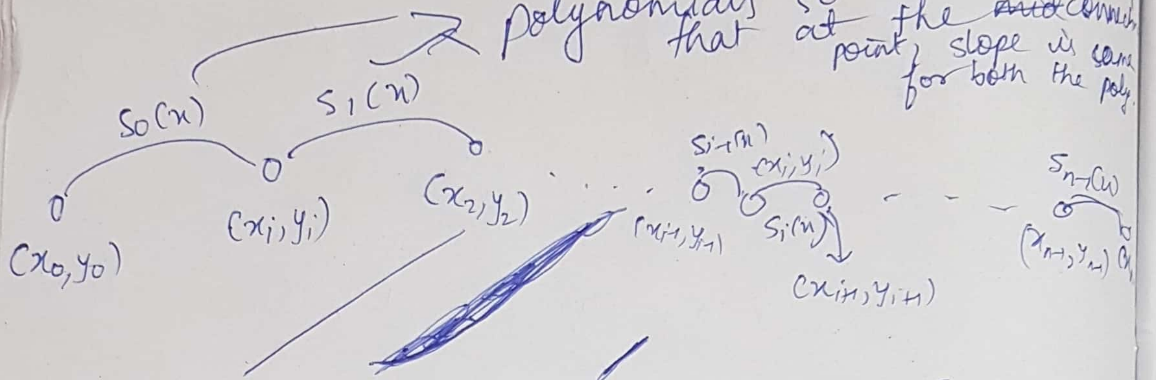
$a = 10^A = (10)^{0.3096} = 2.0396$

$b = 10^B = 1.9948.$

$\therefore\ y = 2.0396\,(1.9948)^x$

Spline approximation.

polynomials so chosen such that at the connecting point, slope is same for both the poly.



Min^m degree polynomial to approx. $S_i(x)$ will be three.

Let $S_i(x) = a_i(x-x_i)^3 + b_i(x-x_i)^2 + c_i(x-x_i) + d_i$

For $(n+1)$ tabular data there will be n different $S_i(x)$, $4n$ unknowns which require $4n$ conditions.

$$\left.\begin{aligned} S_i(x_i) &= y_i \quad \text{—} (1) \\ S_i(x_{i+1}) &= y_{i+1}; \quad (2) \end{aligned}\right\} \text{2n conditions.}$$

$i = 0, \dots, n-1$

Slopes and curvature of the neighbouring segments at the common knot point are equal.

i.e.

$$\left.\begin{aligned} (3) — \quad S_i'(x_{i+1}) &= S_{i+1}'(x_{i+1}) \\ (4) — \quad S_i''(x_{i+1}) &= S_{i+1}''(x_{i+1}) \end{aligned}\right\} \begin{aligned} &i = 0, 1, 2, \dots (n-2) \\ &(2n-2) \text{ cond}^n \end{aligned}$$

Other 2 condition are

$$\left.\begin{aligned} S_0''(x_0) &= 0 \\ S_{n-1}''(x_n) &= 0 \end{aligned}\right\} \text{last 2 cond}^n$$

At $x = x_i$, $y = y_i$

$y_i = d_i$ or $d_i = y_i$ — (7)

Putting (7) in (2).

$$S_i(x_{i+1}) = a_i(x_{i+1} - x_i)^3 + b_i(x_{i+1} - x_i)^2 + c_i(x_{i+1} - x_i) + y_i = y_{i+1}$$

Let $(x_{i+1} - x_i) = \Delta x_i$ & $y_{i+1} - y_i = \Delta y_i$

$$\therefore \Delta y_i = a_i(\Delta x_i)^3 + b_i(\Delta x_i)^2 + c_i \Delta x_i$$

$$\text{or } \frac{\Delta y_i}{\Delta x_i} = a_i(\Delta x_i)^2 + b_i(\Delta x_i) + c_i \quad (8) \quad i = 0, 1, \ldots, n-1$$

$$S_i'(x) = 3a(x - x_i)^2 + 2b_i(x - x_i) + c_i$$

Let $S_i'(x_i) = m_i$, $i = 0, 1, \ldots, n-1$.

$$S_i'(x_i) = m_i = c_i$$

$$S_i'(x_{i+1}) = m_{i+1} = 3a_i(x_{i+1} - x_i)^2 + 2b_i(x_{i+1} - x_i) + m_i$$

$$= 3a_i \Delta x_i^2 + 2b_i \Delta x_i + m_i = m_{i+1} \quad \text{— (9)}$$

Solving (8) & (9).

$$a_i = \frac{1}{\Delta x_i^2}\left\{m_i + m_{i+1} - \frac{2\Delta y_i}{\Delta x_i}\right\}$$

$$b_i = \frac{1}{\Delta x_i}\left\{-2m_i - m_{i+1} + \frac{3\Delta y_i}{\Delta x_i}\right\}.$$

$$S_i''(x) = 6a_i(x - x_i) + 2b_i$$

$$S_i''(x_{i+1}) = S_{i+1}''(x_{i+1})$$

$$6a_i(x_{i+1} - x_i) + 2b_i = 2b_{i+1}$$

$$6a_i(\Delta x_i) + 2b_i = 2b_{i+1} \quad i = 0, 1, \ldots, n-2$$

$$\frac{6}{\Delta x_i^2}\left\{m_i + m_{i+1} - 2\frac{\Delta y_i}{\Delta x_i}\right\}.\Delta x_i + 2\frac{1}{\Delta x_i}\left\{-2m_i - 2m_{i+1} + 3\frac{\Delta}{\Delta}\right.$$

$$= \frac{2}{\Delta x_{i+1}}\left\{-2m_{i+1} - 2m_{i+2} + 3\frac{\Delta y_{i+1}}{\Delta x_{i+1}}\right\}$$

or $$\frac{m_i}{\Delta x_i} + 2\left\{\frac{1}{\Delta x_i} + \frac{1}{\Delta x_{i+1}}\right\}m_{i+1} + \frac{m_{i+2}}{\Delta x_{i+1}} = 3\left(\frac{\Delta y_{i+1}}{\Delta x_{i+1}} + \frac{\Delta y_i}{\Delta x_i}\right)$$

$i = 0, 1, \ldots, n-2$

(5) $\rightarrow$ $2m_0 + m_1 = 3\frac{\Delta y_0}{\Delta x_0}$

(6) $\rightarrow$ $m_{n-1} + 2m_n = \frac{3\Delta y_{n-1}}{\Delta x_{n-1}}$

Tridiagonal matrix to be solved using Gauss - Elimination

Numerical Differentiation.

x_i	y_i
x_0	y_0
x_1	y_1
⋮	
x_n	y_n

fit a n^{th} degree polynomial to these tabular data.

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$

where $x = x_0 + ph$ or $p = \left(\frac{x-x_0}{h}\right) \Rightarrow dp = \frac{1}{h}$.

$$\frac{dy}{dx} = \frac{dy}{dp} \cdot \frac{dp}{dx} = \frac{1}{h}\frac{d}{dp}\left\{y_0 + p\Delta x_0 + \frac{p(p-1)}{2!}\Delta^2 x_0 + p\frac{(p-1)(p-2)}{3!}\Delta^3 x_0 + \cdots\right\}$$

$$= \frac{1}{h}\left\{\Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 x_0 + \cdots\right\}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dp}\left(\frac{dy}{dx}\right)\frac{dp}{dx}.$$

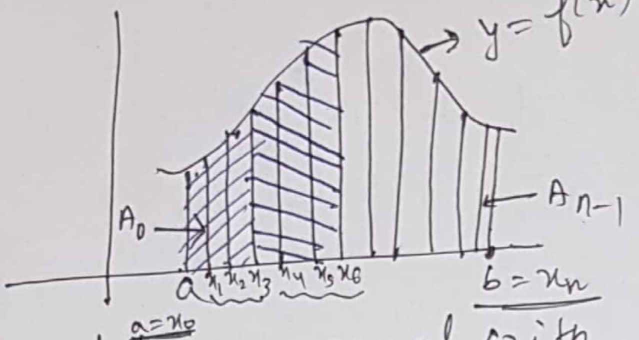
$$= \frac{1}{h}\frac{d}{dp}\left(\frac{dy}{dx}\right) = \frac{1}{h^2}\frac{d}{dp}\left\{\Delta x_0 + \frac{2p-1}{2!}\Delta^2 x_0 + \frac{(3p^2-6p+2)}{3!}\Delta^3 x_0\right.$$

$$= \frac{1}{h^2}\left\{\Delta^2 x_0 + (p-1)\Delta^3 x_0 + \cdots\right\}.$$

$$\left.\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left\{\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\right.$$

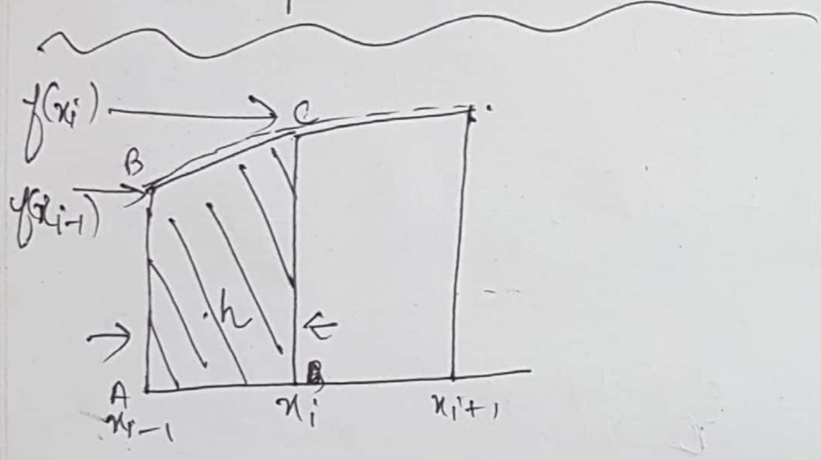
Numerical Integration

$\int_a^b f(x)dx$



When the strips are equal width, the corresponding integration formulae are called
Newton - Cotes formulae
otherwise the formulae are called gaussian formulae.

Four points are taken, at a time.



for the interval $[x_{i-1}, x_i]$ the function $f(x)$ is approximated as a st. line.
$\therefore$ The area under the curve over the interval $[x_{i-1}, x_i]$ will be equal to the area of the trapezium ABCD

$$A_{i-1} = \frac{h}{2}\{f(x_{i-1}) + f(x_i)\}.$$

$$I_{trapezoid} = \sum_{i=1}^{n} A_{i-1}$$

$$= \frac{h}{2}\{(f(x_0) + f(x_1)) + (f(x_1) + f(x_2)) + \cdots + (f(x_{n-1}) + f(x_n))\}$$

$$= \frac{h}{2}\left\{ f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n) \right\}$$

To calculate the error :

for the interval $[x_{i-1}, x_i]$

$$I_{trapezoid} = \frac{h}{2}\left\{ f(x_{i-1}) + f(x_i) \right\}$$

$$I_{actual} = \int_{x_{i-1}}^{x_i} f(x)\,dx .$$

$$= \left[F(x)\right]_{x_{i-1}}^{x_i} = F(x_i) - F(x_{i-1})$$. where $\int f(x)dx = F(x)$

since $x_i = x_{i-1} + h$.

$$I_{actual} = F(x_{i-1}+h) - F(x_{i-1})$$

$$= \left[F(x_{i-1}) + hF'(x_{i-1}) + \frac{h^2}{2}F''(x_{i-1}) + \frac{h^3}{3!}F'''(x_{i-1}) + \cdots \right] - F(x_{i-1})$$

$$= h f(x_{i-1}) + \frac{h^2}{2} f'(x_{i-1}) + \frac{h^3}{3!} f''(x_{i-1}) + \cdots \quad -(1).$$

$$I_{trapezoid} = \frac{h}{2}\left\{ f(x_{i-1}) + f(x_i) \right\}$$

$$= \frac{h}{2}\left\{ f(x_{i-1}) + f(x_{i-1}+h) \right\}$$

$$= \frac{h}{2}\left\{ f(x_{i-1}) + f(x_{i-1}) + h f'(x_{i-1}) + \frac{h^2}{2} f''(x_{i-1}) + \cdots \right\}$$

$$= h f(x_{i-1}) + \frac{h^2}{2} f'(x_{i-1}) + \frac{h^3}{4} f''(x_{i-1}) + \cdots \quad -(2).$$

~~I_{actual}~~ Neglecting higher order terms from (1) & (2).

$$I_{actual} - I_{trapezoid} = \left(\frac{h^3}{6} - \frac{h^3}{4}\right) f''(x_{i-1}) = -\frac{h^3}{12} f''(x_{i-1})$$

Summing up the errors for individual intervals, the total error over $[x_0, x_n]$.

is $E = \sum_{i=1}^{n} \frac{-h^3}{12} f''(x_{i-1}) = -\frac{h^3}{12} \sum_{i=1}^{n} f''(x_{i-1})$

Let $\bar{f}''$ be the average second order derivative of $f(x)$ over $[x_0, x_n]$ $\therefore \bar{f}'' = \frac{\left\{\sum_{i=1}^{n} f''(x_{i-1})\right\}}{n}$.

or $E = -\frac{h^3}{12} n \bar{f}'' = -\frac{h^2}{12} (nh) \bar{f}''$

$$\boxed{E = -\frac{(b-a) h^2}{12} \bar{f}''}$$

Truncation error

(We are computing by assuming a lower degree polynomial & neglecting the higher order terms

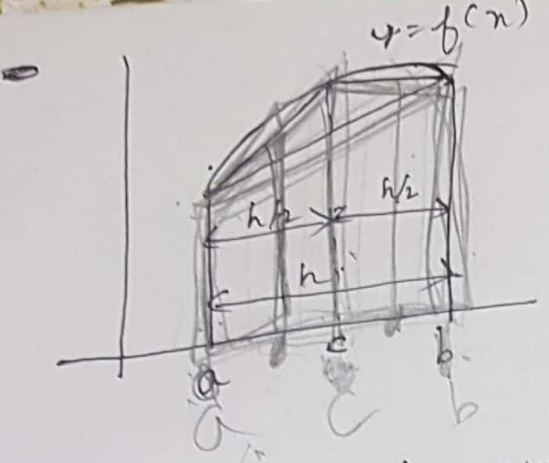
$E \propto h^2$

If we reduce the width by half then Error will be reduced by a factor of 1/4.

$h < 1$

$h_1 = h/2 < 1/2$

$\therefore h_1^2 < 1/4$.



$I_0 = \frac{h}{2}\{f(a)+f(b)\}$

$I_1 = \frac{h}{4}\{f(a)+2f(c)+f(b)\}$

Continue to decrease the (h) to (h/2).

until $\{diff(I_0 - I_1)\} >$ reqd. precision.

$s0 = f(0) + f(1)$.

$\int_0^1 \frac{1}{1+x^2}$

$\boxed{h} = (b-a)$

$n = 1$. $s1 = 0$

$s0 = f(a) + f(b)$.

$I1 = (h * s0)/2$.

Repeat
{
$s1 = 0$; $I0 = I1$.

$h = h/2$

for $\{i=1,\ i \le n,\ i = i+2\}$.

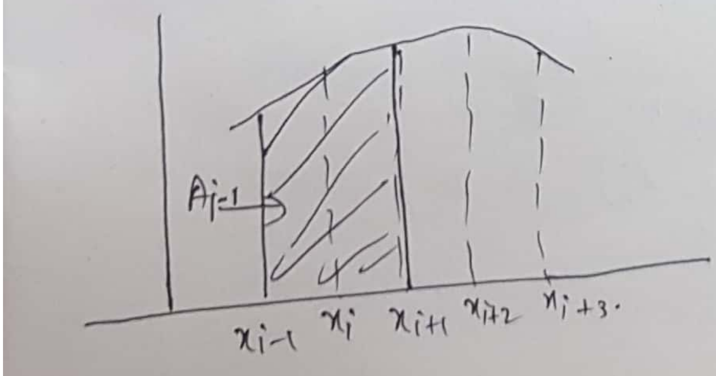
{ $s1 = s1 + f(a + i*h)$ }

$n = 2*n$

$s0 = s0 + 2*s1$ $I1 = (h * s0)/2$

until $((Fabs\,(I1 - I0)) < \epsilon)$.

Simpson's 1/3 rule.



x_{i-1}	$f(x_{i-1})$	$\Delta f(x_{i-1})$	$\Delta^2 f(x_{i-1})$.
x_i	$f(x_i)$	$\Delta f(x_i)$	
x_{i+1}	$f(x_{i+1})$		

Let us fit a Newton's forward diff. polynomial through the above 3 points

$\therefore y = f(x_{i-1}) + p\Delta f(x_{i-1}) + \frac{p(p-1)}{2}\Delta^2 f(x_{i-1})$.

$(x_i = x_{i-1} + ph)$

$$x_i = x_{i-1} + ph$$

$$\int_{x_{i-1}}^{x_{i+1}} y\,dx = h \int_{0}^{2} y\,dp.$$

at $x = x_{i-1}$, $p = 0$

at $x = x_{i+1}$, $p =$

$$= h \int_{0}^{2} \left\{ f(x_{i-1}) + p\,\Delta f(x_{i-1}) + \frac{p(p-1)}{2} \Delta^2 f(x_{i-1}) \right\} dp.$$

$$\boxed{\begin{aligned} \Delta f(x_{i-1}) &= f(x_i) - f(x_{i-1}) \\ \Delta^2 f(x_{i-1}) &= f(x_{i+1}) - 2f(x_i) + f(x_{i-1}) \end{aligned}}$$

$$= h\left\{ \left[p\, f(x_{i-1})\right]_0^2 + \left[\frac{p^2}{2!} \Delta f(x_{i-1})\right]_0^2 + \left[\frac{p^3}{6} - \frac{p^2}{4}\right]_0^2 \Delta^2 f(x_{i-1}) \right\}$$

$$= h\Big\{ 2 f(x_{i-1}) + 2\{f(x_i) - f(x_{i-1})\} + \frac{1}{3}\{f(x_{i-1}) - 2f(x_i) + f(x_{i+1})\} \Big\}$$

$$= \frac{h}{3}\{f(x_{i-1}) + 4f(x_i) + f(x_{i+1})\}$$

$$A_{i-1} = \frac{h}{3}\{f(x_{i-1}) + 4f(x_i) + f(x_{i+1})\}.$$

$$A = \sum_{i=1,3,5,\ldots,n-1} \frac{h}{3}\{f(x_{i-1}) + 4f(x_i) + f(x_{i+1})\}$$

$$= \frac{h}{3}\{f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 4f(x_{n-1}) + f(x_n)\}.$$

Develop algo similar to trapezoidal such that $f(x_i)$ is computed only once for each i.

$$I_{actual} = \int_{x_{i-1}}^{x_{i+1}} f(x)\,dx = F(x_{i+1}) - F(x_{i-1}).$$

where $\int f(x)\,dx = F(x)$

$$= F(x_{i-1} + 2h) - F(x_{i-1})$$

$$= F(x_{i-1}) + 2h F'(x_{i-1}) + \frac{(2h)^2}{2} F''(x_{i-1}) + \frac{(2h)^3}{3!} F'''(x_{i-1})$$

$$+ \frac{(2h)^4}{4!} F^{iv}(x_{i-1}) + \frac{(2h)^5}{5!} F^{v}(x_{i-1}) + \dots] - F(x_{i-1})$$

$$= 2h f(x_{i-1}) + 2h^2 f'(x_{i-1}) + \frac{4}{3} h^3 f''(x_{i-1}) + \frac{2}{3} h^4 f'''(x_{i-1}) + \frac{4}{15} h^5 f^{iv}(x_{i-1}) + \dots$$

$$A_{i-1} = \frac{h}{3}\left\{ f(x_{i-1}) + 4 f(x_{i-1} + h) + f(x_{i-1} + 2h) \right\}$$

$$= \frac{h}{3}\Big[f(x_{i-1}) + 4\left\{ f(x_{i-1}) + h f'(x_{i-1}) + \frac{h^2}{2!} f''(x_{i-1}) + \frac{h^3}{3!} f'''(x_{i-1}) + \frac{h^4}{4!} f^{iv}(x_{i-1}) \right\}$$

$$+ f(x_{i-1}) + 2h f'(x_{i-1}) + \frac{4h^2}{2!} f''(x_{i-1}) + \frac{8h^3}{3!} f'''(x_{i-1}) + \frac{(2h)^4}{4!} f^{iv}(x_{i-1})\Big]$$

$$= \frac{h}{3}\Big[6 f(x_{i-1}) + 6h f'(x_{i-1}) + 4h^2 f''(x_{i-1}) + 2h^3 f'''(x_{i-1}) + \frac{5h^4}{6} f^{iv}(x_{i-1}) \Big]$$

$$= 2h f(x_{i-1}) + 2h^2 f'(x_{i-1}) + \frac{4}{3} h^3 f''(x_{i-1}) + \frac{2h^4}{3} f'''(x_{i-1}) + \frac{5h^5}{18} f^{iv}(x_{i-1})$$

Neglecting higher order terms:

$$\therefore I_{actual} - I_{simpson} = \left(\frac{4}{15} - \frac{5}{18}\right) h^5 f^{iv}(x_{i-1}).$$

$$= \frac{-1}{90} h^5 f^{iv}(x_{i-1}) \quad \text{over } [x_{i-1}, x_{i+1}]$$

Summing up the errors for individual intervals, the total error over $[x_0, x_n]$.

Let $\bar{f}^{(iv)}$ be the mean $f^{(iv)}$ over $[x_0, x_n]$

$$\therefore \bar{f}^{(iv)} = \left\{ \sum_{i=1}^{n} f^{iv} \right\} / (n/2)$$

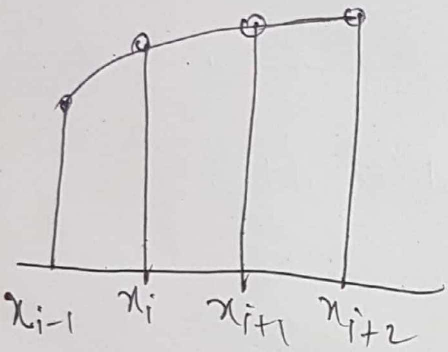
Total error $= \sum_{i=1,3,5} e_{i+1} = \sum_{i=1,3,5} -\frac{h^5}{90} f^{(iv)}_{x_{i+1}}$

$$= -\frac{h^5}{90} \sum_{i=1,3,5} f^{(iv)}(x_{i+1})$$

$$= -\frac{h^5}{90} \cdot \frac{n}{2} \bar{f}^{(iv)}$$

$$= -\frac{h^4}{180}(b-a)\, \bar{f}^{(iv)}$$

Similarly.



$$A_{i+1} = \frac{3}{8} h \{ f(x_{i-1}) + 3 f(x_i) + 3 f(x_{i+1}) + f(x_{i+2}) \}$$

Simpson's 3/8 rule

Truncation error $= -\frac{3}{80}(b-a) h^5 \bar{f}^{(iv)}$

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Boole's Method

This method fits a 4^{th} degree polynomial through consecutive 5 points on the function and incremental area under that curve is.

$$I_{incremental} = \frac{2h}{45}\left(7y_{i-1} + 32y_i + 12y_{i+1} + 32y_{i+2} + 7y_{i+3}\right)$$

$$\text{Truncation error} = -\frac{2(b-a)}{945} h^6 . f^{vi}$$

If $h \to h/2$, error reduces by a factor 2^6.

$$\int_a^b f(x)dx = I_{trapezoidal}(h) + a_1 h^2 + a_2 h^4 + a_3 h^6 + \dots \quad -①$$

(a_1: $f''(x)$, a_2: $f^{iv}(x)$, a_3: $f^{vi}(x)$)

odd order derivatives are removed because their magnitude are negligible (HOW?)

So chosen ~~function are taken.~~

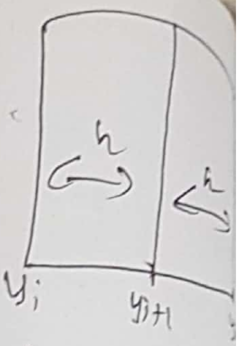
$$\int_a^b f(x)dx = I_{trapezoidal}(2h) + a_1(2h)^2 + a_2(2h)^4 + a_3(2h)^6 + \dots \quad -②$$

$4 \times ① - ②$

$$(4-1)\int_a^b f(x)dx = 4I_{trapezoidal}(h) - I_{trapezoidal}(2h) + b_1' h^4 + b_2' h^6 + \dots$$

$$\int_a^b f(x)dx = \frac{4I_{trapezoidal}(h) - I_{trapezoidal}(2h)}{3} + b_1 h^4 + b_2 h^6 + b_3 h^6 + \dots$$

$$I_{trapezoid}^{(h)} = \frac{h}{2}(y_i + 2y_{i+1} + y_{i+2})$$



$$I_{trapezoid}^{(2h)} = 2h\frac{(y_i + y_{i+2})}{2}$$

$$\frac{4I_{trapezoid}^{(h)} - I_{trapezoid}^{(2h)}}{3} = \frac{h}{3}(y_i + 4y_{i+1} + y_{i+2})$$

$$= I_{simpson's\ 1/3rd}^{(h)}.$$

$$\int_a^b f(x)\,dx = I_{simpson's\ 1/3}^{(h)} + b_1h^4 + b_2h^6 + b_3h^8 + \dots \quad -③$$

$$\int_a^b f(x)\,dx = I_{simpsons\ 1/3rd}^{(2h)} + b_1(2h)^4 + b_2(2h)^6 + b_3(2h)^8 + \dots \quad ④$$

$(③ \times 16 - ④)/15$

$$\int_a^b f(x)\,dx = \frac{16 I_{simpsons\ 1/3rd}^{(h)} - I_{simpson's\ 1/3rd}^{(2h)}}{15} + c_1h^6 + c_2h^8 + \dots$$

$$\begin{tikzcd}
I_{trapezoidal}^{(h)} \arrow[dr] & & \\
I_{trapezoidal}^{(h/2)} \arrow[r] \arrow[dr] & I_{simpsons\ 1/3}^{(h/2)} \arrow[dr] & \\
I_{trapezoidal}^{(h/4)} \arrow[r] & I_{simpson's\ 1/3rd}^{(h/4)} \arrow[r] & I_{Boole}(h/4). \\
\vdots & &
\end{tikzcd}$$

Let us represent the integral values computed by trapezoidal method as a Romberg term $R(h,0)$

We can generate higher order Romberg terms as

$$R(h,1) = \frac{4R(h,0) - R(2h,0)}{3}$$

$$R(h,2) = \frac{16R(h,1) - R(2h,1)}{15}$$

Let at $(k-1)$th iteration, Romberg term be $R(h,k-1)$

$$\therefore \int_a^b f(x)\,dx = R(h,k-1) + d_1 h^{2k} + d_2 h^{2k+2} + d_3 h^{2k+4} + \ldots \quad \text{--- (5)}$$

$$\int_a^b f(x)\,dx = R(2h,k-1) + d_1 (2h)^{2k} + d_2 (2h)^{2k+2} + d_3 (2h)^{2k+4} + \ldots \quad \text{--- (6)}$$

$((5) \times 4^k - (6)) / (4^k - 1)$

$$\int_a^b f(x)\,dx = \frac{4^k R(h,k-1) - R(2h,k-1)}{4^k - 1} + l_1 (h)^{2k+2} + \ldots$$

$$R(h,k) = \frac{4^k R(h,k-1) - R(2h,k-1)}{4^k - 1}$$

Boole's method.

i	$R(h/2^i, 0)$	$R(h/2^i, 1)$	$R(h/2^i, 2)$	$R(h/2^i, 3)$
0	$R(h,0)$			
1	$R(h/2,0)$	$R(h/2,1)$		
2	$R(h/4,0)$	$R(h/4,1)$	$R(h/4,2)$	
3	$R(h/8,0)$	$R(h/8,1)$	$R(h/4,2)$	$R(h/8,3)$

$$\int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4} = 0.785398$$

Trapezoidal Rule:

$h = 0.5$ $I = 0.775$

$h = 0.25$ $I = 0.782794$

$h = 0.125$ $I = 0.784747$

$h = 0.0625$ $I = 0.785235$

Simpson's rule,

$h = 0.5$ $I = 0.783333$

$h = 0.25$ $I = 0.785392$

$h = 0.125$ $I = 0.785398$

$h = 0.0625$ $I = 0.785398$

precise

Romberg's rule.

h	R(h,0)	R(h,1)	R(h,2)
0.5	0.775		
0.25	0.782794	0.785392	
0.125	0.784747	0.785398	0.785396.
~~0.0625~~			

$$I = \sum_{i=0}^{n} w_i f(x_i) + E$$

$(n+1)$ w_i's & $(n+1)$ x_i's.

$(2n+2)$ variables / unknowns

Gauss Quadrature formula.

To choose $(2n+2)$ conditions to find out $(2n+2)$ unknowns gauss assumed that the integration formula will be such that it will integrate the polynomial terms $x^k, k=0, \ldots, 2n+1$. with zero error.

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n} w_i f(x_i) + E.$$

Let $n = 1$

$$\int_{-1}^{1} f(x)\,dx = w_0 f(x_0) + w_1 f(x_1)$$

$f(x) = x^0$

$$\int_{-1}^{1} x^0\,dx = w_0 + w_1 = 2.$$

$$\int_{-1}^{1} x^1\,dx = w_0 x_0 + w_1 x_1 = 0$$

$$\int_{-1}^{1} x^2\,dx = w_0 x_0^2 + w_1 x_1^2 = 2/3$$

$$\int_{-1}^{1} x^3\,dx = w_0 x_0^3 + w_1 x_1^3 = 0.$$

It can be proved that if the limits of integration is $[-1, 1]$ x_i values are the roots of legendre's polynomial $P_{n+1}(x) = 0$.

n	x_i	w_i
1	± 0.57735	1.000
2	± 0.77459, 0.0000	0.55555, 0.88888.

If limits of integration is different from $[-1, 1]$ we have to use the transformation as

$$\int_a^b f(x)\,dx = \frac{(b-a)}{2} \int_{-1}^{1} f\left\{\frac{(b-a)t + (b+a)}{2}\right\} dt.$$

$$\int_a^b f(x)\,dx = \frac{(b-a)}{2}\left[\sum_{i=0}^{n} w_i\, f\left\{\frac{(b-a)t_i + (b+a)}{2}\right\}\right]$$

$$\int_0^1 \frac{dx}{1+x^2} = \int_{-1}^{1} f\left\{\frac{(1-0)t+1}{2}\right\} dt.$$

$$= \frac{(1-0)}{2} \cdot \left[\sum_{i=0}^{n} w_i f\left\{\frac{t_i+1}{2}\right\}\right]$$

for $n=1$,

$t_0 = -0.57735$, $t_1 = 0.57735$, $\int_0^1 \frac{dx}{1+x^2} = 0.73$

$n=2$,

$t_0 = -0.77450$, $t_1 = 0$, $t_2 = 0.77459$.

$w_0 = 0.55555$, $w_1 = 0.88888$, $w_2 = 0.55555$,

$$\int_0^1 \frac{dx}{1+x^2} = 0.785257$$

Solution of differential eqns

$\frac{dy}{dx} = f(x, y)$ with some specified initial condns.

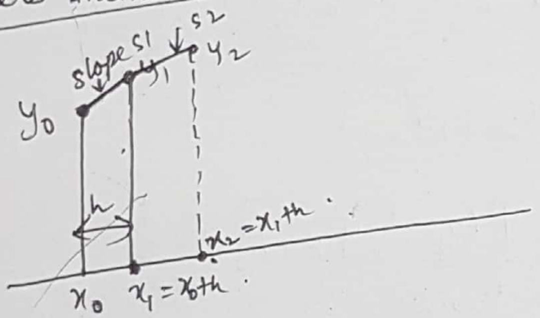
$\frac{\partial y}{\partial x} = f(x, y, z) \qquad \frac{\partial y}{\partial z} = g(x, y, z)$

$y(x_0) = y_0$

At $x = x_0,\ y = y_0$

$y = g(x, y)$.

Euler's method.



$s1 = f(x_0, y_0)$

The line segment (y_0, y_1) intersects (x_1, y_1)

and so on

$$\tan\theta = \frac{y_1 - y_0}{x_1 - x_0} = s1$$

$$\Rightarrow y_1 = y_0 + h.s1$$

$$y_2 = y_1 + h.s2$$

At any point (x_{k-1}, y_{k-1}) of solution curve.

$$x_k = x_{k-1} + h$$

$$y_k = y_{k-1} + h f(x_{k-1}, y_{k-1})$$

Error estimate for Euler method. $\qquad y(x_k) = y_k$.

(x_{k-1}, y_{k-1})

$x_k = x_{k-1} + h \qquad y_k = y(x_{k-1} + h)$

$$= y(x_{k-1}) + h y'(x_{k-1}) + \frac{h^2}{2!} y''(x_{k-1}) + \dots$$

$$\boxed{y_k \approx} \leftarrow = y_{k-1} + h f(x_{k-1}, y_{k-1}) + \frac{h^2}{2!} f'(x_{k-1}, y_k)$$

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Truncation error associated with euler's method $= \frac{h^2}{2} f'(x_{k-1}, y_{k-1}) +$ higher order terms

Let the error associated with y_{k-1} & y_k be e_{k-1} & e_k resp. whereas their true values are y'_{k-1} & y'_k resp.

$$\therefore \; y_{k-1} = y'_{k-1} + e_{k-1}$$
$$y_k = y'_k + e_k;$$

$$y_k = y_{k-1} + h f(x_{k-1}, y_{k-1})$$
$$\Rightarrow y'_k + e_k = y'_{k-1} + e_{k-1} + h f(x_{k-1}, y'_{k-1} + e_{k-1})$$
$$= y'_{k-1} + e_{k-1} + h f(x_{k-1}, y'_{k-1}) + h e_{k-1} f'(x_{k-1}, y'_{k-1})$$
$$= y'_{k-1} + h f(x_{k-1}, y'_{k-1}) + e_{k-1}\{1 + h f'(x_{k-1}, y'_{k-1})\}$$

Now $y'_k = y'_{k-1} + h f(x_{k-1}, y'_{k-1})$

$$\therefore e_k = e_{k-1}\{1 + h f'(x_{k-1}, y'_{k-1})\} + \cdots$$

Neglecting higher order terms.

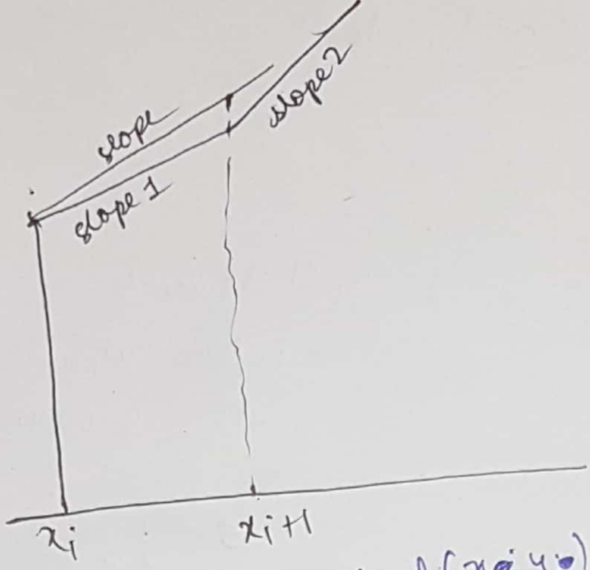
$$|e_k| = |e_{k-1}| \, |1 + h f'(x_{k-1}, y'_{k-1})|$$

for stability of the method

$$|1 + h f'(x_{k-1}, y'_{k-1})| < 1.$$

Single step methods

required for generating initial points.



slope = $\frac{\text{slope 1} + \text{slope 2}}{2}$.

$$y_{i+1}^{0} = y_i + h f(x_i, y_i)$$

$$y_{i+1} = y_i + h\left(\frac{\text{slope 1} + \text{slope 2}}{2}\right).$$

$$y_{i+1}^{(1)} = y_i + h/2 \{ f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{(0)}) \}$$

The iteration formula ~~can be~~.

$$y_{i+1}^{(k+1)} = y_i + h/2 \{ f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{(k)}) \}.$$

until $|y_{i+1}^{(k+1)} - y_{i+1}^{(k)}| < \epsilon$

To compute next point, k no. of iterations are reqd.

$$y_k = y_{k-1} + h f(x_{k-1}, y_{k-1}) + \frac{h^2}{2} f'(x_{k-1}, y_{k-1}) + \cdots$$

$$= y_{k-1} + h f(x_{k-1}, y_{k-1}) + \frac{h^2}{2} \left\{ \frac{f(x_k, y_k) - f(x_{k-1}, y_{k-1})}{h} \right\} + \text{higher order terms}$$

$$= y_{k-1} + \frac{h}{2} \{ f(x_{k-1}, y_{k-1}) + f(x_k, y_k) \} + \frac{h^3}{3!} \{\text{some func}\}.$$

∴ Truncation error is of order "cube".

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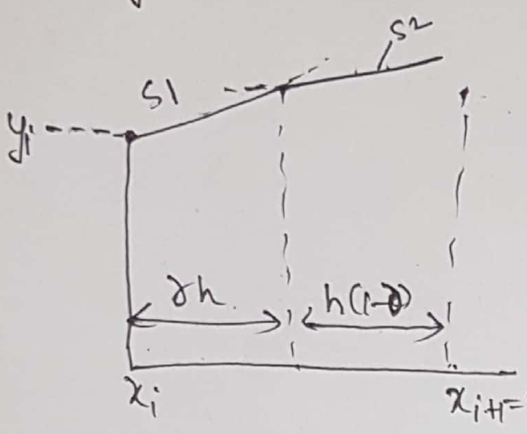
Runge Kutta Method. (2nd order)



(x_0, y_0)

$S = w_1 S1 + w_2 S2$

where $w_1, w_2 < 1$

& $w_1 + w_2 = 1$

$S = \frac{1}{2}\{s1 + s2\}$

$s1 = h f(x_0, y_0)$

$s2 = h f(x_0 + h, y_0 + s1)$

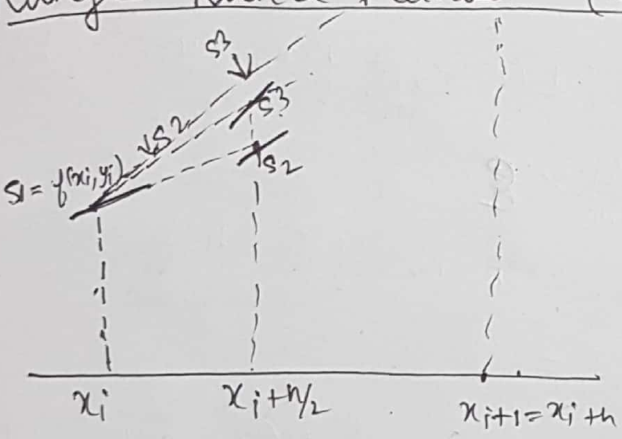
If $\gamma = 1/2$, $w_2 =$, $w_1 =$

$\frac{w_2}{w_1} = \frac{1+\gamma}{1-\gamma}$.

$y_1^{(0)} = y_0 + \frac{(s1+s2)}{2}$

$y_1^{(k+1)} = y_0 + \frac{s1}{2} + h f(x_0+h, y_0+h f(x_1, y_1^{(k)}))$

Runge Kutta Method (4th order)



$S1 = f(x_i, y_i)$

$S2 = f(x_i + h/2, y_i + h/2\, s1)$

$S3 = f(x_i + h/2, y_i + h/2\, s2)$

$S4 = f(x_i + h, y_i + h s3)$

$$S = \frac{1}{6}\{s1 + 2s2 + 2s3 + s4\}.$$

$$y_{i+1}^{(0)} = y_i + h.S$$

For computing $y_{i+1}^{(k)}$

$$S4 = f(x_i + h, y_i + h f(x_{i+1}, y_{i+1}^{(0)}))$$

$$y_{i+1}^{(1)} = y_i + h/6\{s1 + 2s2 + 2s3 + f(x_i+h, y_i + h f(x_{i+1}, y_{i+1}^{(0)}))\}$$

$$y_{i+1}^{(k)} = y_i + h/6\{s1 + 2s2 + 2s3 + f(x_i+h, y_i + h f(x_{i+1}, y_{i+1}^{(k-1)}))\}.$$

continue until $|y_{i+1}^{(k)} - y_{i+1}^{(k-1)}| < \varepsilon$

Sol^n of 2nd

$\frac{d^2y}{dx^2}$

Let $\frac{dy}{dx} =$

$\frac{dz}{dx}$

$\frac{dy}{dx}$

for any k

next poin

x

y

z

where $k_1 =$

$l_1 =$

$k_2 =$

l_2

k_3

l_3

k_4

l_4

Sol^n of 2nd order ordinary differential equ^{n}s.

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right).$$

with initial cond^n

$$y(x_0) = y_0$$

$$\& \quad \left.\frac{dy}{dx}\right|_{x_0} = z_0.$$

Let $\frac{dy}{dx} = z$

$$\frac{dz}{dx} = f(x, y, z)$$

$$\frac{dy}{dx} = z = g(x, y, z) \text{ (say)}$$

with initial cond^{n}s
$y(x_0) = y_0$
$z(x_0) = z_0$.

for any known point (x_i, y_i, z_i). on the sol^n curve, next point can be given as

$$x_{i+1} = x_i + h. \quad \Rightarrow \text{~~(will h be multiplied here)~~}$$

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$z_{i+1} = z_i + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

where
$$k_1 = h\, g(x_i, y_i, z_i)$$
$$l_1 = h\, f(x_i, y_i, z_i)$$
$$k_2 = h\, g(x_i + h/2, y_i + k_1/2, z_i + l_1/2).$$
$$l_2 = h\, f(x_i + h/2, y_i + k_1/2, z_i + l_1/2)$$
$$k_3 = h\, g(x_i + h/2, y_i + k_2/2, z_i + l_2/2)$$
$$l_3 = h\, f(x_i + h/2, y_i + k_2/2, z_i + l_2/2)$$
$$k_4 = h\, g(x_i + h, y_i + k_3, z_i + l_3)$$
$$l_4 = h\, f(x_i + h, y_i + k_3, z_i + l_3)$$

Multistep method / predictor corrector method

1) Adams Bashforth method.
2) Milne's method.
3) Hamming's method.

$h = b - a = 1$

$s1 = 0, \; n = 1.$

$s0 = f(0) + f(1)$

$I1 = s0 * h/2$

$sI = 0, \; I0 = I1.$

$h = h/2$ $\qquad s1 = f(1/2).$

$n = 2$

$s0 = f(0) + 2f(1/2) + f(1).$

$I1 = \dfrac{(h * s0)}{2}$

$s1 = 0.$ $\qquad f(x_0) + 4f(x_1) + f(x_2)$

$h = h/2$

$s1 =$ $\qquad f(x_0) + 4f\left(\frac{x_1 + x_0}{2}\right) + 2f(x_1) + 4f(x_1 \ldots) + f(x$

$f(x_0) + 4f(x_{0.25}) + 2f(\cancel{x}_{.5}) + 4f(x_1) + 2f(1.25)$
$+ 4f$