Combinatorics

Graph Theory & Combinatorics

1) Selection
2) Path ≃ Arrangements

based on elements of set, which is discrete i.e on elements of the set ≃ $G(V,E)$

~~i.e on~~

From a solution set of a very large cardinality, one solution needs to be chosen.

Q) How to solve a problem?

1. Techniques to be used.
2. How to break the problem into smaller managable sizes? ⟹ Divide and Conquer.
3. Unseen challenges. (Size of solution.)
4. Ex: For a Counting Problem.

→ Are the outcomes enumerated in the problem.

i.e (a) Unordered set (Combination)

(b) Ordered set (Sequences)

1) Fundamental Principles:

(I) Addition ~~Problem~~ Principle: If there are r_1 different objects in the first set, r_2 different objects in the second set, ... & r_m different objects in the m^{th} set.

If the sets are disjoint, then the no. of ways to select an object from any of the m-sets is –

$$\boxed{r_1 + r_2 + r_3 \dots r_m}$$

(II) Multiplication ~~Problem~~ Principle:

Suppose a problem or procedure can be broken up into m-succesive (ordered) stages with, r_1 different outcomes in the 1^{st} stage

r_2 " " " " 2^{nd} "

⋮

r_m " " " " m^{th} "

If the number of outcomes at each stage is independent of the choices of in previous stages and if the composite outcomes are all distinct then the total procedure has :.

$r_1 \times r_2 \times r_3 \ldots r_m$ different outcomes.

Note : The addition principle requires disjoint set of objects & the multiplication requires that the procedure break into ordered stages and that the composite outcomes are distinct.

(*) The sum $(a+b)$ is the no. of items resulting when a set $|a|$ items is added to a set $|b|$ items.

(*) The product $(a \times b)$ is the no. of sequences (A, B). When A can be any of a items and B can be any of b items.

Eg: There are 5 different S books
6 " F "
8 " T "

How many ways are there to pick an (unordered) set of two books not both of the same type?

(*) "Unordered" means that there is no 1st book in pair, & so as outcomes cannot be broken into first stage (1st book) & a second stage, i.e (multiplications principle does not apply).

(*) Then the problem is to recast or decompose the problem in such a way that the multiplication / addition principle can be used.

(*) We can break the problem into smaller parts to which the principles apply, which raises the question of what parts are into which we should break the problem.

(**) If one recognises a similarity between the current problem and another previously solved problem. The current problem is easy to solve.

Ordered set $= \{(a,b), (b,a) \ldots\}$

Un " " $= \{\{a,b\} \ldots$

NP complete:
$P_1 \xrightarrow[\text{trans.}]{\text{linearly}} P_2$ (P_2 soln. exists P_1 can be solved similarly)

· Possible Cases

2 Books chosen:
1 from S & 1 from F

1 from S & 1 from F	S & T	F & T
5 × 6	5 × 8	6 × 8
Total no. of ways = 30	40	48

Total = 30 + 40 + 48
= 118

Simple Arrangements & Selections

Def: A permutation of n-distinct objects is an arrangement or ordering of the n-objects.

⊛ An r-permutation of n-distinct object, is an arrangement using r of the n-objects $\longrightarrow P(n,r) = \frac{n!}{(n-r)!}$

⊛ An r-Combination of n-distinct objects is an unordered selection or subset of r-out of the n-objects $\longrightarrow C(n,r) = \binom{n}{r}$ → Also known as the binomial coefficient of $(x+y)$.

$$= \frac{n!}{(n-r)!\,r!}$$

Arrangements with repeated letters:

(A) How many ways are there to arrange the 7 letters in the word SYSTEMS.

(B) In how many ways of these arrangements, do the 3 Ss appear consecutively? → 5!

~~P(7,7)~~

Note: For large values of n-

$$n! \simeq \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} = 1$$

Arrangements and Selections with Repeatations:-

Theorem: If there are n-objects with r_1 of type 1,
r_2 " " 2,
...
& r_m " " m.

where $r_1+r_2+\dots r_m = n$.

Then the no. of arrangements of these n-objects, denoted by:

$P(n; r_1, r_2 \dots r_n)$ is:

$$\binom{n}{r_1}\binom{n-r_1}{r_2}\dots\binom{n-r_1-r_2\dots-r_{m-1}}{r_m} = \frac{n!}{r_1!\,r_2!\dots r_n!}$$

$$\left[\text{Using} - \binom{n-1}{r} + \binom{n-1}{r-1} = \binom{n}{r}\right]$$

Theorem: The number of selections with repeatations of r objects chosen from n type of objects is:

$$C(n+r-1, r) = \binom{n+r-1}{r}$$

Ex: How many ways are there to form a sequence of 10 letters from 4 a, 4 b, 4 c & 4 d, if each letter appears at least twice?

There are 2 categories of letters, frequencies that sum to 10 with each letter two or more times.

① There are 4 appearances of 1 letter & 2 appearances of each letter —

→ 4 case-[$C(4,1)$] for choosing which letter occurs 4 times & $P(10; 4, 2, 2, 2) = 18980$.

② 3 appearances of 2 letters & 2 appearances of the other 2 letters.

→ There are $C(4,2) = 6$ cases for choosing which of the 4 letter occurs 3 times & $P(10; \underbrace{3, 3}_{\text{Arrangement of 3 of 2 letter}}, \underbrace{2, 2}_{\text{Arrange of 2 of 2 letters}})$

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As both cases are distinct then;

$$4 \times P(10; 4, 2, 2, 2) + 6 \times P(10; 3, 3, 2, 2)$$

$$= 226,800$$

Distribution

→ Equivalent to an arrangement or selection problem with repetition.

How to model distribution problems:

Distribution of distinct objects are equivalent to arrangements.

&

Distribution of identical objects are equivalent to selection.

BASIC MODELS

Distinct Objects :- The process of distributing r ~~of~~ distinct objects into n-different boxes is equivalent to putting the r distinct objects in a row and stamping one of the n-different box name on each object.

⊛ The resulting sequence of box names is an arrangement of length r formed from n-items (box-names) with repetitions.

∴ there are :-

$$\underbrace{n \times n \times n \cdots n}_{r \text{ times}} = (n)^r$$

distribution of the r-distinct objects.

⊛ If r_i objects must go in a box i, $1 \leq i \leq n$.

then there are → $P(r; r_1, r_2 \ldots r_n)$ distributions.

Identical boxes

The process of distributing r identical objects into n-different boxes is equivalent to choosing a (unordered) ~~set~~ subset of r box names with repetitions from the n-choices of boxes.

The total no. of distribution of the r-identical objects are:

Check Proof ← $$\boxed{\binom{n+r-1}{r} = \frac{(r+n-1)!}{r!(n-1)!}}$$

Ex 1) Distributing a combination of identical & distinct objects

① How many ways are there to distribute 4 identical oranges & 6 distinct apples into 5 distinct boxes–
(different variety)

② How many ways are there " " " ~~at least~~ " such that at least 2 objects in each boxes.

Ans ①

Distribute 4 identical oranges into 5 distinct boxes $= \binom{4+5-1}{4} = 70$

" " 6 distinct apples " " " " $= 5^6$

$$Ans = 5^6 \times \binom{4+5-1}{4}$$

② Case-1
2 (identical) oranges in each of 2 boxes and 0 oranges in other 3 boxes $\longrightarrow C(5,2)$

6 (distinct) apples can be distributed in 3 other boxes →
$P(6;2,2,2)$

$\therefore Total_1 = C(5,2) \times P(6;2,2,2)$

Case -2

2 (identical) oranges in 1 box → $C(5,1)$
1 orange in each of 2 other boxes → $C(4,2)$ } $C(5,1) \times C(4,2)$
(Equivalent)

orange $\langle 2,1,1,0,0 \rangle$ → arranging the numbers among 5 boxes
box1 box2 b3 b4 b5.

$= P(5;1,2,2) = 30$

2 boxes still empty will get 2 apples
2 " with 1 orange each gets 1 apple. } $P(6;2,2,1,1) = 180$

$\therefore Total_2 = P(5;1,2,2) \times P(6;2,2,1,1)$

Case-3

1 (identical) orange in 4 boxes of 5 boxes → $C(5,4)$
Apples (distinct) can be distributed in:
$P(6;2,1,1,1,1)$

$Total_3 = C(5,4) \times P(6;2,1,1,1,1)$

$Total = total1 + total2 + total3$
$= 8100$

Ex 2 : How many integer solutions are there to the eq —

Integer Programming App

$$x_1 + x_2 + x_3 + x_4 = 12$$

with ① $x_i \geq 0 \longrightarrow \binom{12+4-1}{4-1}$

② $x_i \geq 1 \longrightarrow \binom{12-1}{4-1}$

③ $x_1 \geq 2,\ x_2 \geq 2,\ x_3 \geq 4,\ x_4 \geq 0$ $\quad \binom{(12-2-2-4)+4-1}{4-1} = \binom{7}{3} = 35$

Ans An ~~iter~~ integer solⁿ to an eq. means:
→ order set of integer values for x_i's ~~sum~~ summing to 12

Diophantite Equations

Equivalent forms for selection with repeatitions —

1) The number of ways to select r-objects with repeatition from n-different types of objects.

2) The no. of ways to distribute r-identical objects into n-distinct boxes.

3) The no. of non-negetive integer solutions to
$$x_1 + x_2 + x_3 + \cdots + x_n = r$$

Ex: What fraction of binary sequences of length 10 consists of a positive no. of 1s followed by 0's followed by 1's & followed by 0s.

Ex: 1111 0111 000 | Total no. of binary sequences → 2^{10}

$\underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}$
$x_1 \quad x_2 \quad x_3 \quad x_4$

$$x_1 + x_2 + x_3 + x_4 = 10$$
$$x_i \geq 1$$
$$\binom{10-1}{4-1} : \binom{9}{3}$$

This is similar to putting:
1 ball in each box with a false bottom to conceal the ball in each box & the r.

count the ways to distribute without restriction the remaining $(r-n)$ balls into the n-boxes,

i.e $\binom{(r-n)+n-1}{r-n} = \frac{[(r-n)+n-1]!}{(r-n)!\,(n-1)!} = \binom{r-1}{n-1}$

$\therefore$ For each $4(n)$ box,
distribution of 10 r's is given as —

$C(10-1, 4-1) = 84$ sequences

$= \frac{84}{1024} \simeq 8\%$

Synopsis of the Distribution

'Distribute r-objects from n-items or into n-boxes

	Arrangement (Ordered outcome) OR Distribution of distinct objects.	Combination. (Unordered Outcome). OR Distribution of identical objects.
No Repeatition	$P(n,r)$	$C(n,r)$
Unlimited "	n^r	$C(n+r-1, r)$
Restricted "	$P(n; r_1, r_2, \dots r_m)$	—

PIGEONHOLE PRINCIPLE (applied in Security)

Simple form:

Theorem: If $(n+1)$ objects are put in n-boxes, then at least one box contains two or more of the objects.

Proof: If each of the ~~objects~~ boxes contain at most one of the objects, then the total no. of objects, then the total no. of objects is at most n. ~~Some~~ Since, we start with $(n+1)$ objects, some boxes contains at least two of the objects.

Note: ⊛ The principle / proof does not help in finding a box that contains two or more of the objects

⊛ The principle merely ~~guar~~ guarantees the exis[...]

of a box containing two objects.

(*) The principle cannot guarantee if there are only n (or fewer) objects

Analogical Principles

1. If n objects are put in n boxes and no box is empty then each box contains exactly one object.
2. If n objects are put into n boxes and no box gets more than one object, then each box has an object in it.

Applications

1. There are n married couples. How many of the 2n people must be selected in order to guarantee that at least one couple is selected?

2. Given m integers, $a_1, a_2 \dots a_m$ there exists integers k and l, with $0 \leq k < l \leq m$ such that

$a_{k+1} + a_{k+2} + \dots a_l$, i.e $\sum_{i=k+1}^{l} a_i$ is divisible by m.

i.e there exists consecutive a's in the sequence $a_1, a_2 \dots a_n$, whose sum is divisible by m.

Ex If m = 7, & let integers be:

2, 4, 6, 3, 5, 5, & 6

Computing sums:

2
2+4 = 6
2+4+6 = 12
15
20
25
31

Remainders when divided by 7:

2, 6, 5, 1, 6, 4 & 3

$(a_1+a_2) \bmod 7$ $(a_1+a_2+a_3+a_4+a_5) \bmod 7$

$\therefore k=2,\ l=5$

i.e $(6+3+5) \bmod 7 = 0$

Proof: Let us consider the m-sums —

$a_1, a_1+a_2, a_1+a_2+a_3 \dots \sum_{i=1}^{m} a_i$

If any of these sums be divisible by m, then the conclusion holds.

(*) We may suppose that each of these sums has a non-zero remainder when divided by m, & so has a number of 1, 2, ... (m-1).

(*) Since, there are m sums & (m-1) remainders, two of the sums have the same remainder, when divided by m.

∴ there are integers k and l, both k < l, such that —

$$\sum_{i=1}^{k} a_i = bm + r \rightarrow (1)$$

$$\sum_{i=1}^{l} a_i = cm + r \rightarrow (2)$$

(2) - (1) ⟶ $a_{k+1} + \dots a_l = (c-b)\, m$

$\therefore \left(\sum_{i=k+1}^{l} a_i\right)$ is divisible by m.

Chinese Remainder Theorem

Let m and n be relatively prime +ve numbers (integers)

Let a, b be integers where $0 \le a \le (m-1)$ & $0 \le b \le (n-1)$

Then there is a +ve integer x, such that the remainder when

x is divided by m is a

x " " n is b

i.e x can be written as :

$x = pm + a$ (for some integers p and q)

$x = qn + b$

Soln Let us consider the integers,

$a, m+a, 2m+a \dots (n-1)m+a$

Suppose Each of these integers has a remainder a when divided by m.

Suppose that two of them has the same remainder

r when divided by n.

i.e $im + a$ }
$jm + a$ } $\longrightarrow$ be two numbers. $0 \le i \le j \le (n-1)$

Then there are integers q_i and q_j such that.

$$im + a = q_i n + r \longrightarrow (1)$$

$$jm + a = q_j n + r \longrightarrow (2)$$

$\underline{(2) - (1)}$:

$$(j-i)\,m = (q_j - q_i)\,n \text{ --- } (3)$$

Eq (3) states that n is a factor of the number $(j-i)m$.

Since, n has no common factor. other than 1 with m. it follows that n is a factor of $(j-i)$.

(*) However $0 \le i \le j \le (n-1)$

$$\Rightarrow 0 < (j-i) \le (n-1)$$

(*) This contradiction ~~occurs~~. arises from our supposition ~~that~~ that two of the numbers:

$$a,\ m+a \ \dots\ ,(n-1)m+a$$

has the same remainder when divided by n.

$\rightarrow$ Each of these n-numbers has a different remainder, when divided by n. By pigeonhole principle, each of the n numbers $0, 1 \dots (n-1)$, occurs ~~has~~ as a remainder, in particular b does.

Let p be an integer with $0 \le p \le (n-1)$ such that the number $x = pm + a$, has a remainder b when divided by n.

Then for some integer q,

$$x = qn + b.$$

So, $x = pm + a$ & $x = qn + b$.

& x has the required properties.

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Note The fact that a rational number (a/b) has a decimal expansion that eventually repeats is a consequence of the pigeonhole principle.

Pigeonhole Principle (strong form)

Theorem: Let $q_1\ q_2 \dots q_n$ be +ve integers..

If $A = (q_1 + q_2 + q_3 \dots - n + 1)$

i.e $\sum_{i=1}^{n} q_i - n + 1$

objects are put into n-boxes, then ~~#~~ either the first box contains at least q_i objects ... & nth box contain at least q_n objects.

Proof:

Suppose that we distribute A objects among n boxes.

If for each $i = 1, 2 \dots n$, then ith box contains fewer than q_i objects, then the total no. of objects does not exceed:

$$(q_i - 1) + (q_2 - 1) + \dots (q_n - 1)$$

$$= \sum_{i=1}^{n}(q_i - 1) = \sum_{i=1}^{n} q_i - n.$$

∴ this number is less than the no. of objects distributed, we conclude that for some $i = 1, 2 \dots n$ the ith box will contain at least q_i objects.

(*) It is possible to distribute $\left(\sum_{i=1}^{n} q_i - n\right)$ objects among n boxes, in such a way that for no $i = 1, 2 \dots n$.. is it true that the ith box contains q_i or more objects. This can be done by putting $(q_i - 1)$ objects in the ith box.

(*) The simple form of the Pig. Prin. is obtained by taking $q_i = 2\ \forall\ i = 1, 2 \dots n$

the $\sum_{i=1}^{n} q_i - n + 1 = 2n - n + 1 = (n+1)$ $\square$ / QED

Note
(Averaging Principle) If $n(r-1)+1$ objects are put into n-boxes, then at least one of the boxes contain r or more of the objects.

(*) If the average of the n non-negative integers, $m_1, m_2, \ldots m_n$ is greater than $(r-1)$ i.e

$$\frac{\sum_{i=1}^{n} m_i}{n} > (r-1)$$

Then at least one of the integers is greater than or equal to r.

For $i = 1, 2, \ldots n$, let m_i be the number of objects in the i^{th} box.

$$\text{If } \frac{\sum m_i}{n} = \frac{n(r-1)+1}{n}$$

$$= (r-1) + 1/n$$

$\therefore$ the average is greater than $(r-1)$, one of the integers m_i is at least r i.e of the boxes contains at least r objects.

(*) If the averaging of n non-negative integers $m_1, m_2 \ldots m_n$ is less than $(r+1)$ i.e

$$\frac{\sum_{i=1}^{n} m_i}{n} < (r+1)$$

then at least one of the integers is less than $(r+1)$.

(*) If the avg. of the n-non negative is at least equal to r then at least one of the integers.

$m_1, m_2 \ldots m_n$ satisfies:

$m_i \geq r$.

Ex: A basket of fruits is being arranged out of apples, bananas and oranges. What is the smallest no. of pieces of fruits that should be in basket in order to gurantee at least 8 A / 6 B / 9 O?

Strong form of P.P. : $8+6+9-3+1$
$= 21$

Application (Erdos & Szekeres) : Every sequence $a_1, a_2, \ldots a_{n^2+1}$ of (n^2+1) real numbers contain either an increasing subsequence of length $(n+1)$ or decreasing subsequence of length $(n+1)$

Subsequence: If $b_1, b_2 \ldots b_m$ is a sequence then, $b_{i_1}, b_{i_2}, \ldots b_{i_n}$ is a subsequence, provided that $1 \leq i_1 < i_2 \ldots < i_n \leq m$

Again if $b_{i_1} \leq b_{i_2} \leq \cdots \leq b_{i_k}$ is an increasing subsequence (not decreasing) an sequence ~~subseq~~., then
Eg: if b_2, b_4, b_5, b_6 is not an increasing b_2, b_6, b_5 is not an increasing subsequence. $\boxed{\text{Ramsay Number?}}$

Graph Colouring (Application : ~~Sample~~ Map Colouring)

Def: A k-vertex colouring of a graph $G(V, E)$ is a partition of its vertex set.

V as $V_1 \cup V_2 \cdots \cup V_k$ such that each V_i is an independent set (not neccessarily non-empty).

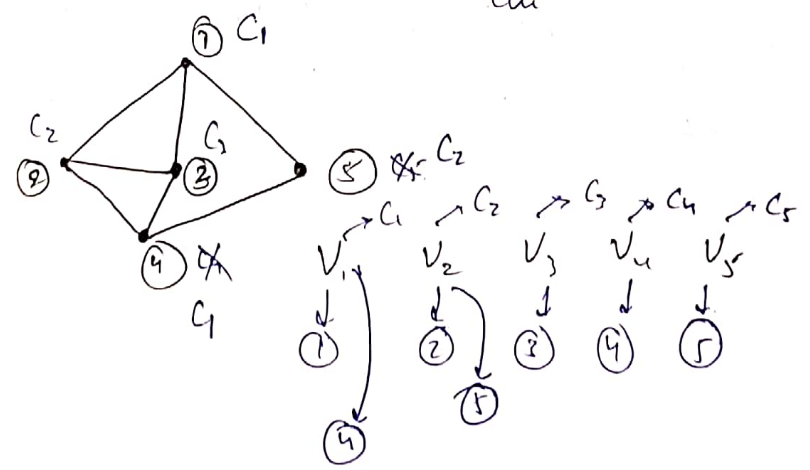
(*) The V_i's are called the <u>colour classes</u>

f a function $f : V \to \{1, 2, \cdots k\}$, such that
$f(v) = i$ for $v \in V_i$, $1 \le i \le k$.

& $f(u) \neq f(v)$ where $\{u, v\} \in E(G)$ is called a colour function.

(*) If G has a k-vertex colouring, it is said to be k-vertex colourable or simple k-colourable. & the minimum integer k for which G is k colourable is called the chromatic no. of G. & denoted by $\chi(G)$.
chi

Ex :



Bipartite & min - 2 colours

(*) If $\chi(G) = k$; we say that G is k chromatic

(*) If G is k-chromatic, but $\chi(G - v) = k - 1$; for every $v \in V$, then G is said to be k-critical graph.

If G is k-chromatic but $\chi(G-e) = k-1$ for each $e \in E$, then G is said to be k-edge critical or simply k-minimal.

Def: A k-edge colouring of a graph $G=(V,E)$ is a partition: $E = E_1 \cup E_2 \cup \dots \cup E_k$ of its edge set such that set E_i (possibly empty) is an independent set of edges.

The E_is are called <u>colour classes</u>; if a function $f: E \to \{1,2 \dots k\}$ such that $f(e) = i$ for each $e \in E_i$, $1 \leq i \leq k$,

$$f(e_i) \neq f(e_j),$$ whenever $\{e_i, e_j\}$ have a common vertex or adjacent to each other.

If G has a k-edge coloring, it is said to be k-edge colorable; the minimum integer k for which G is k-edge colourable is the edge-chromatic number $(\chi'(G))$

If $\chi'(G) = k$, then G is said to be k-edge chromatic, if further $\chi'(G-e) = k-1$, for every $e \in E$, then G is said to be k-edge-minimal.

Ex:
$$\chi(\overline{K_p}) = 1$$
$$\chi(C_{2n+1}) = 3, \quad n \geq 1$$
$$\chi(C_{2n}) = 2, \quad n \geq 1$$
$$\chi(K_p) = p$$

Cycle

Theorems: If G is a graph, then
$$\chi(G) \geq \frac{|V|^2}{|V|^2 - 2|E|}$$

Proof: Let $\chi(G) = k$

Thus, there exists a particular $V = V_1 \cup V_2 \dots \cup V_k$ independent sets such that no two vertices in V_i $(1 \leq i \leq k)$ are adjacent.

Total no. of edges in
$$|E(G)| \leq \binom{|V|}{2} - \sum_{i=1}^{k} \binom{|V_i|}{2}$$
edge between vertices of same colour class, which does not exist in relation. E_i $V_k \to V_j \in V_i$

$$\therefore 2|E(G)| \le |V|^2 - |V| - \sum_{i=1}^{k}|V_i|^2 + \sum_{i=1}^{k}|V_i|$$

$$= |V|^2 - \sum_{i=1}^{k}|V_i|^2. \quad \left[\because \sum_{i=1}^{n}|V_i| = |V|\right]$$

As $\sum_{i=1}^{k} p^2 \geq \frac{\left(\sum_{i=1}^{n} p_i\right)^2}{k} \longrightarrow$ Cauchy Schwartz Inequality

$$\Rightarrow 2|E(G)| \le |V|^2 - \frac{\left(\sum |V_i|\right)}{k}$$

$$= |V|^2 - \frac{|V|^2}{k}$$

$$\Rightarrow \boxed{\chi(G) = k \geq \frac{|V|^2}{|V|^2 - 2|E|}}$$

<u>Theorem</u>: If it is a subgraph of G, then:

$$\chi(G) \geq \chi(H)$$

<u>Corollary</u>: If G contains a complete subgraph k_n, the $\chi(G) \geq n$,

* The above gives a <u>lower band</u> of $\chi(G)$.

<u>Theorem</u>: For any graph G,

$\chi(G) \le \Delta(G) + 1$, where $\Delta(G)$ = max. degree of graph

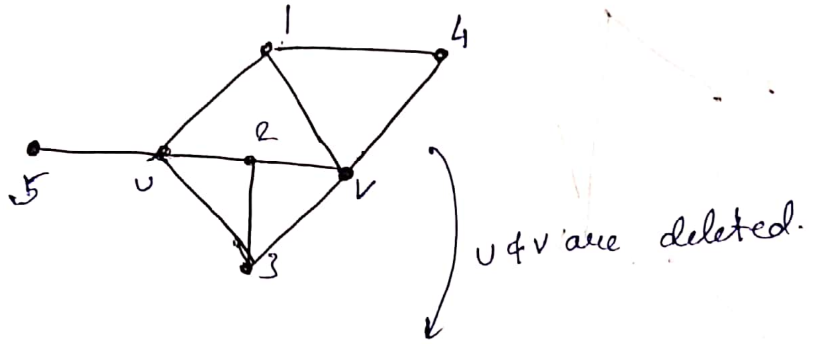
Proof: Done by induction.

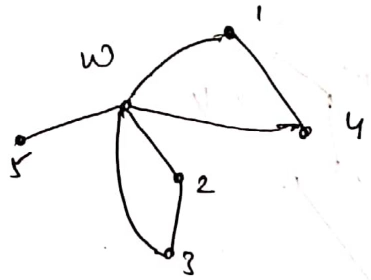
Def: Let G be a graph & let u & v be two non-adjacent vertices in G.

Let H be a graph obtained from $G - \{u, v\}$ by adding a new vertex w & joining it with every vertex which is adjacent to u & v.

(If multiple edges are formed, replace them by single edges)

H is called graph obtained from G & is denoted by $G \circ u \circ v$





Theorem: Let G be a graph & let u & v be non-adjacent edges, then,

$$\chi(G) = \min\{\chi(G + (u,v)), \chi(G \cdot uv)\}$$

Algorithm: For determining chromatic number.

S1) set $G = G_0$, & let π_0 be the sequence: G_0.

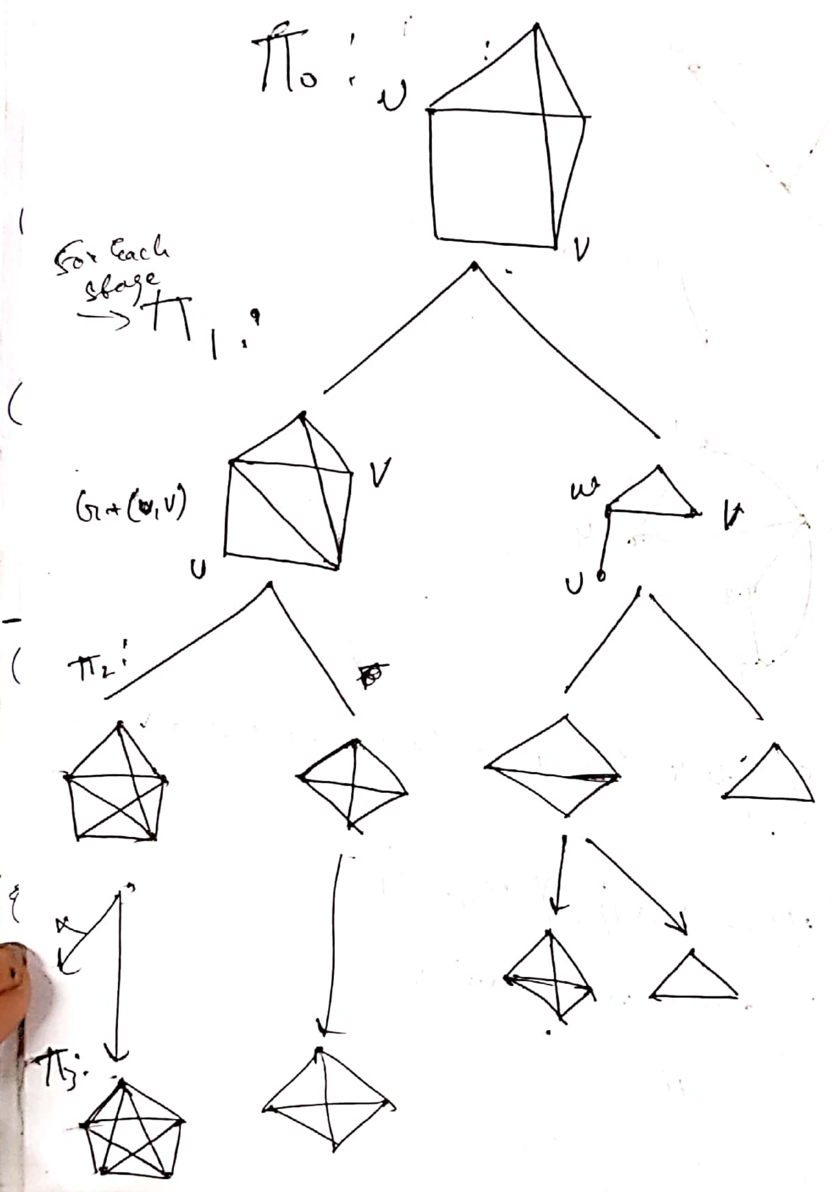
S2) Having constructed a sequence - $\pi_i : G_0, G_1; G_2 \cdots G_n$ of graph,
goto(S3) if $G_l (0 \le l \le m)$ is incomplete
goto(S4) if every G_i is complete

S3) Replace every incomplete $G_l (0 \le l \le m)$ in π_i by $G_i + (u, v)$ & $G_i \cdot uv$, where u & v are arbitrary chosen non-adjacent vertices in G_l.
→ Call this resulting sequence π_{i+1} and goto(S2)

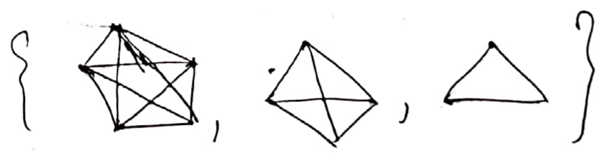
S4) ~~Step~~ STOP

$$\chi(G) = \min\{|V(G_1)|, |V(G_2)|, \cdots |V(G_m)|\}$$ from prev. theor.

⊛ By definition of $G+\{U,V\}$ & $G.U.V$ it follows that some stage one of is sure to obtain a sequence of complete graph.



Π_3 :



$\chi(G) = 3$

(II) Algo for vertex colouring : (Greedy)

S1) Order the vertex of G in some order

Order : $V_1, V_2 \cdots V_p$ (Decreasing)

S2) Color V_1 with color 1

S3) Having coloured the vertices $v_1, v_2 \ldots v_i$ with k-colours $i = 1 \ldots k$,

let $S \subseteq \{1, 2 \ldots k\}$ be the set of colours that have been used to color the vertices adjacent with v_{i+1}, then color v_{i+1},

① $n_i = \min [\{1, 2 \dots k\} - S]$ if $S \neq \{1, 2 \dots k\}$

② with $(k+1)$ if $S = \{1, 2 \dots k\}$

* S3 ensures that no two adjacent vertices have the same color.
* At each stage i, we have at most $1 + \deg_G(v_i)$ colors
* The algo may not color with min. no. of colors.
* The vertices of G can be ordered in such a way that the sequential algo uses only $\chi(G)$ colours. where knowledge of $\chi(G)$ is already known

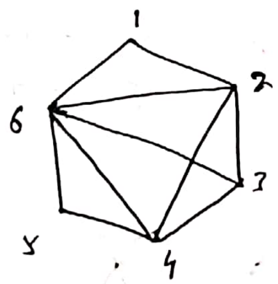
S4) Stop when v_p is colored.

$\boxed{\text{Spectral Graph}}$

Defn: A maximal complete subgraph G is called a clique of G.

* The order of a maximum ~~of~~ clique is the clique number of G & denoted by $\omega(G)$. np complete prob

Eg:



$\{1, 2, 6\} \rightarrow$ clique

$\{2, 3, 4, 6\} \rightarrow$ maximal clique, i.e. $\omega(G) = 4$.

Proposition: For any graph $\chi(G) \geq \omega(G)$, where $\omega(G)$ is the clique no. of G.

Ramsay Number

⊛ Of 6 or more people, there are 3 ~~pairs~~ persons, each pair of whom are ~~equal~~ acquainted or there are 3 persons each of whom are unacquainted.

Dividing

$K_6 \to K_3, K_3$

We distinguish acquainted (red) pairs & unacquainted pairs (blue) by coloured edges.

$\therefore K_6 \to K_3, K_3$ is the assertion that no matter how the edges of K_6 are coloured with the colours red/blue, there is always a red K_3 or a blue K_3, in short a monochromatic triangle.

Proof Suppose the edges of K_6 has been coloured red/blue in any way.

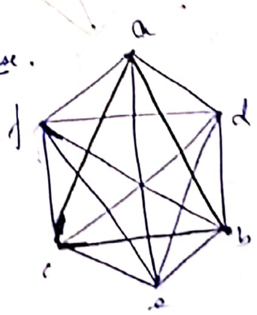
Consider one of the pts. p of K_6.

It meets 5 edges. Since, each of these 5 edges is coloured red/blue (from strong form of Pigeonhole Principle), that either 3 of these are coloured red/3 of them are coloured blue.

Consider the edges which join a, b, c in pairs. If all of these are blue, then a, b, c are coloured blue & a blue K_3 emerges.

⊛ **Note**: The assertion $K_5 \to K_3, K_3$ is false.

This is because there is some way to color the edges of K_5 without creating a red K_3/ blue K_3.



Ramsay Theorem

More generally, it can be stated that

If $m \geq 2$ & $n \geq 2$, are integers, then there is a positive integer p, such that:

$$k_p \to k_m, k_m$$

* If $k_p \to k_m, k_n$ then $k_q \to k_m, k_n$, for some integer $q \geq p$

Ramsay Number:
$r(m, n) \to$ smallest integer p such that:

$$k_p \to k_m, k_n$$

Eg: $r(3,3) \to 6$

Generalization

If $n_1, n_2, n_3 \geq 2$ (integers), then there exists an integer p, such that

$$k_p \to k_{n_1}, k_{n_2}, k_{n_3}$$

i.e if each of the edges of k_p are coloured red, blue or green, then either there exists a red k_{n_1} or a blue k_{n_2} or a green k_{n_3}.

* The smallest p for which this assertion holds is the Ramsay no. $r(n_1, n_2, n_3)$

Eg: $r(3,3,3) = 17$.

Generalization - I

The Ramsay no. $r(n_1, n_2, n \ldots n_k)$ are defined in a ~~smalle~~ similar way of Ramsay's conjecture in its full generality for pairs asserts that these numbers exists, i.e. there is an integer p such that,

$$k_p \to k_{n_1}, k_{n_2}, \ldots k_{n_k}$$

Generalisation - 2

K_n^t, $t \geq 1$

↳ denotes the collection of all subsets of t elements of a set of n elements.

Given integers $t \geq 2$, & integers $q_1, q_2 \cdots q_k \geq t$, there exists an integer p such that.

$$K_p^t \to K_{q_1}^t, K_{q_2}^t, K_{q_3}^t \cdots K_{q_k}^t$$

i.e there exists an integer p, such that if each of the t-element subsets of a p-element set is assigned one of k colours,

$c_1, c_2 \cdots c_k$, then there are:

(a) q_1 elements, all of whose t subsets are assigned color c_1

OR

(b) q_k elements, all of whose t elements subsets are assigned the color c_k.

Smallest integer (p)

$$p \to r_t(q_1, q_2 \cdots q_k)$$

Suppose $t = 1$

Then $r_1(q_1, q_2 \cdots q_k)$ is the smallest number p, such that if the elements of a set of p-elements are coloured with one of the colors $c_1, c_2 \cdots c_k$ then there exists:

(a) there are q_1 elements of color c_1

OR

(b) there are q_k elements of color c_k,

∴ the strong form of the pigeonhole principle

$$r_1(q_1, \cdots q_k) = q_1 + q_2 \cdots - k + 1$$

Note: The determination of the general Ramsay number, i.e $\boxed{p = r_t(q_1, q_2 \cdots q_k)}$ is DIFFICULT.

Recurrence Relations & Generating Function

$$S_n = f_0 + f_1 + f_2 \cdots f_n = f_{n+2} - 1$$

where $f_i \rightarrow$ fibonacci no.

Induction way

for $n = 0$

$$S_0 = f_2 - 1 = 2 - 1 = 1$$

$n = n + 1$

$$S_{n+1} = f_0 + \cdots f_{n+1} = (f_{n+2} - 1) + f_{n+1}$$

$$= f_{n+1} + f_{n+2} - 1$$

$$= f_{n+3} - 1$$

Obtain a formula for the F.N.

$$f_n = f_{n-1} + f_{n-2}$$

$$\Rightarrow f_n - f_{n-1} - f_{n-2} = 0 \qquad \cancel{n \geq 2} \quad n \geq 2$$

Let $f_n = q^n$

$$q^n - q^{n-1} - q^{n-2} = 0$$

$$q^n (1 - q^{-1} - q^{-2}) = 0$$

$$1 - q^{-1} - q^{-2} = 0$$

$$\Rightarrow q^2 - q - 1 = 0$$

$$q = \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}$$

$$f = c_1 q_1 + c_2 q_2$$

$$f_n = \left(\frac{1+\sqrt{5}}{2}\right)^n, \left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$\therefore f_n = c_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$$

$f_0 = 1 \;\} \; c_1 + c_2 = 1.$ $\quad | \quad f_1 = c_1\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right) = 1$

$\cancel{c_1 + c_2} + \sqrt{5}(c_1 - c_2) = \cancel{2}\ 1$

Scanned by CamScanner

$c_1 - c_2 = 1/\sqrt{5}$, $c_1 + c_2 = 1$

$$2c_1 = \frac{1+\sqrt{5}}{\sqrt{5}}$$

$$c_1 = \frac{1+\sqrt{5}}{2\sqrt{5}}, \quad c_2 = \frac{\sqrt{5}-1}{2\sqrt{5}}$$

$$f_n = \left(\frac{\sqrt{5}+1}{2\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{\sqrt{5}-1}{2\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$$

Generating Function

① GF can be treated as an algebraic objects whose formal manipulation allows to count the no. of possibilities for a problem by means of algebra.

② GF are Taylor series (power series expansion) of infinetly differentiable function.

Let $h_0, h_1, h_2 \ldots h_n \ldots$

be an infinite sequence of numbers.

Its generating func. is defined as:

$$g(x) = h_0 + h_1x + h_2x^2 \ldots + h_nx^n + \ldots$$

Eg: Let k be an integer & let the sequence $h_0, h_1, h_2 \ldots h_n$ be defined by letting h_n equal the number of non-negative integer solution of $e_1 + e_2 + \ldots e_k = n$

$$h_n = \binom{n+k-1}{k-1} \quad (n \geq 0)$$

$$\therefore g(x) = \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} x^n$$

Def: The generating fn. for the no. of n-combinations of apples, bananas, oranges and pears, where in n-combination

apples → even, bananas → odd, oranges → (0-4), pears → (≥ 1)

$e_1 + e_2 + e_3 + e_4 = n$

$$g(x) = (1+x^2+x^4\ldots)(x+x^3+x^5\ldots) \times (1+x+x^2+x^3+x^4)(x+x^2+\ldots)$$

$$\frac{1}{1-x^2} \qquad \frac{x}{1-x^2} \qquad \frac{1-x^5}{1-x} \qquad \frac{x}{1-x}$$

$$g(x) = \left(\frac{1}{1-x^2}\right)\left(\frac{x}{1-x^2}\right) \cdot \frac{1-x^5}{1-x} \cdot \frac{x}{1-x}$$

$$= \frac{x^2}{(1-x^2)^2} \cdot \frac{(1-x^5)}{(1-x)^2}$$

→ Taylor Series Exp:

$$g(x) = \sum_{n\geq 0}^{\infty} \binom{n+k-1}{k-1} x^n = \frac{1}{(1-x)^k} \quad \Big| \ \underline{n \geq 2}$$

$x^2 + \cancel{x^3} + 2x^3 + x^4 \ldots$
$+x^3 + x^4$
x^4

$\ldots \ldots k = 4$

$\binom{n+3}{3} x^n$

$n \geq 4 \quad$ 4n

Let $h_0, h_1, \ldots h_k, \ldots h_n$

be a sequence of numbers

The sequence is said to satisfy a linear recurrence relation of order k,

$h_n = a_1 h_{n-1} + a_2 h_{n-2} \cdots a_k h_{n-k} + b_n \quad (n > k)$

where $a_k \neq 0$ & quantity b_n

and b_n may depend on n.

<u>Ex</u>: The Fibonacci series; $f_0, f_1 \ldots f_n$

satisfies the linear recurrence relation:

$f_n = f_{n-1} + f_{n-2}$

$\therefore a_1 = 1, a_2 = 1, b_n = 0$

Homogenous Condition if

$b_n =$ zero constant

$a_1, a_2 \dots a_n \rightarrow$ Constants

Ex: Solve the recurrence relation:

$$h_n = 2h_{n-1} + h_{n-2} - 2h_{n-3}$$

$$h_0 = 1,\ h_1 = 2,\ \& \ h_2 = 0$$

Let $h_n = q^n$

$$q^n = 2q^{n-1} + q^{n-2} - 2q^{n-3}$$

$$\Rightarrow \cancel{2q^3} \Rightarrow q^3 - 2q^2 - q + 2 = 0$$

$$\begin{array}{r|l} & q^2 - q - 2 \\ \hline q-1 & q^3 - 2q^2 - q + 2 \\ & -q^3 + q^2 \\ \hline & -q^2 - q + 2 \\ & +q^2 \pm q \\ \hline & -2q + 2 \\ & \mp 2q + 2 \\ \hline & 0 \end{array}$$

$q = 1$

$q = 2, 1$

$(q-1)(q^2-q-2)$

$= (q-1)(q+1)(q-2)$

$$h_n = C_1 1^n + C_2(-1)^n + C_3 2^n$$

$h_0 = \quad C_1 + C_2 + C_3 = 1 \quad \Rightarrow C_1 + C_2 = 1 - C_3$

$h_1 = \quad C_1 - C_2 + 2C_3 = 2$

$h_2 = \quad C_1 + C_2 + 4C_3 = 0 \qquad 1 - C_3 + 4C_3 = 0$

$\Rightarrow C_3 = -1/3$

$C_1 + C_2 = 1 + 1/3 = 4/3$

$C_1 - C_2 = 2 + 2/3 = 8/3$

$2C_1 = 12/3 = 2$

$C_2 = 4/3 - 2$

$= -2/3$

$$h_n = 2 - \frac{2}{3}(-1)^n - \frac{2^n}{3}$$

Planar Graph

2/4/19

Let $G(V, E)$ be a graph & let S be any surface (eg: plane, sphere). Let $P = \{p_1, p_2, \ldots p_{|V|}\}$ be a set of n-distinct points of S. p_i corresponding to $v_i \in V$.

From Jordan's Theorem

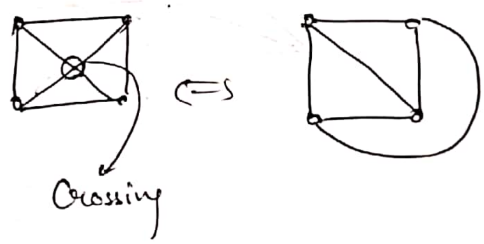
If $e_i = v_j v_k \in E$, draw an arc J_i on S from p_j to p_k. Such that J_i does not pass through any other p_i. Then $P \cup \{J_1, J_2 \ldots J_m\}$ is called a drawing of G on S, or a drawing representing G on S.

Def: A drawing of a graph G is a f^n defined in $V(G) \cup E(G)$ that assigns each vertex $v \in V$ a pt. $f(v)$ in a plane & assigns each edge with end pts. $u, v \in V$ a polygonal $f(u), f(v)$ – curve.

⊛ The images of vertices are distinct. A pt in $f(e) \cap f(e')$ that is not ~~to~~ a common pt. is called a crossing.

Def: A graph is planar if it has a drawing without crossings

Ex:



⊛ A curve is closed if its first & last points are same. It is simple if it has no repeated pts. except first = last.

Def: An open-set in the plane is a set $U \subseteq \mathbb{R}^2$ such that for every $p \in U$, all pts. within some small distance from p belong to U.

A region is an open set U that contains a polygonal u-v curve for every pair $u, v \in U$.

The faces (f) of a plane graph of a plane that contains

· no point used in the embedding. $\boxed{\text{region} \equiv \text{faces}}$

- A plane representation of the graph G ~~denotes~~ divides the plane into regions.

- A region is characterised by the set of edges (or the set of vertices) forming its boundary.

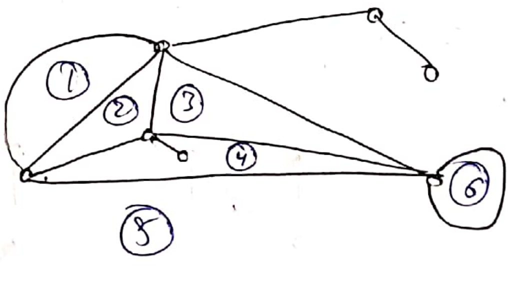
- A region is not defined in a non-planar graph or even in a planar graph not embedded in a plane.

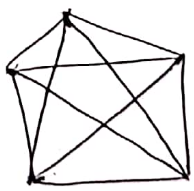
  A region is a property of the specific plane representation of a graph & not of an abstract graph.

- A finite graph G has one unbounded face.

- The faces/regions are pairwise disjoint.

- Points $p, q \in \mathbb{R}^2$ lying in no edges of G are in the same face iff there is a polygonal p, q - curve that crosses no edge.



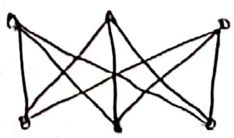


(Non-planar)

Kuratowski's Graph

Theorem: The complete graph K_5 & complete bipartite graph $K_{3,3}$ are non-planar.

K_5



$K_{3,3}$

* Both graphs are regular
* Both are non-planar
* K_5 is a non-planar graph with smallest no. of vertices & $K_{3,3}$ is the non-planar bipartite graph with smallest no. of edges.

∴ Both are simplest non-planar graph.

<u>Euler's</u> <u>Theorem</u> (Properties of Planar Graphs.)

① If G is a planar graph with n-vertices, m-edges & f region/faces & k-component.

Check proofs.

$$\boxed{n-m+f = k+1}$$

② In a connected planar graph with n-vertices, m-edges, f regions/faces, $\boxed{n-m+f=2}$

* A connected graph has a <u>single component</u>.

<u>Theorem</u>: Let G be a simple planar graph with n-vertices & m-edges, where $n \geq 3$, Then $\boxed{m \leq 3n-6}$.

<u>Theorem</u>: If G is a simple planar graph, then G has a vertex v of degree less than 6.

A non-planar connected graph G with minimum number of edges that contain no subdivision of K_5 or $K_{3,3}$ is <u>simple & 3-connected</u>.
→ i.e a minimum of 2 vertices are required to disconnect a the graph.

<u>Subdivision</u>: A subdivision/expansion of a graph G is a graph resulting from the subdivision or expansion of edges.

Eg: $u \overset{e}{—} v \longrightarrow u \underset{e_1}{—} w \underset{e_2}{—} v$ (expansion)

Kuratowski's Theorem

A graph is planar iff it does not have any subdivision of K_5 or $K_{3,3}$.

Detection of planarity → series/parallel Reduction

S1: (a) Every disconnected graph is a planar iff its components are planar.

(B) For every disconnected graph consider only one component at a time.

(b) Similarly a separable graph (graph having vertex connectivity is one) is planar iff each of its blocks is planar.

(B) For every separable graph consider one block at a time.

S2: Since addition/deletion of self loops does not affect planarity, delete all self loops.

S3: Since parallel edges do not affect planarity, keep only one edge among the parallel edges.

S4: Eliminate all the edges in series because elimination of a vertex of degree 2 by merging two edges in series does not affect planarity.

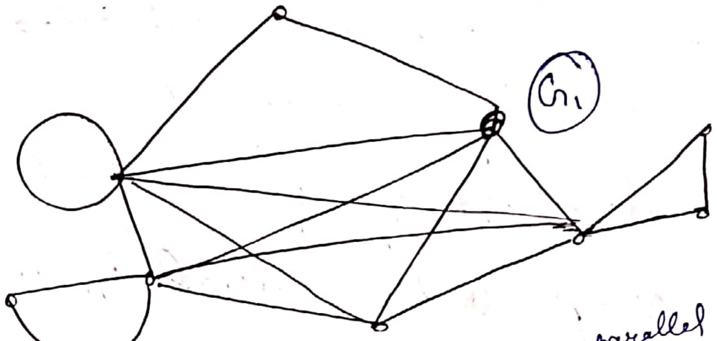
After repeated S3 & S4, the non-seperable connected graph G can be reduced to a new graph H which would be like –

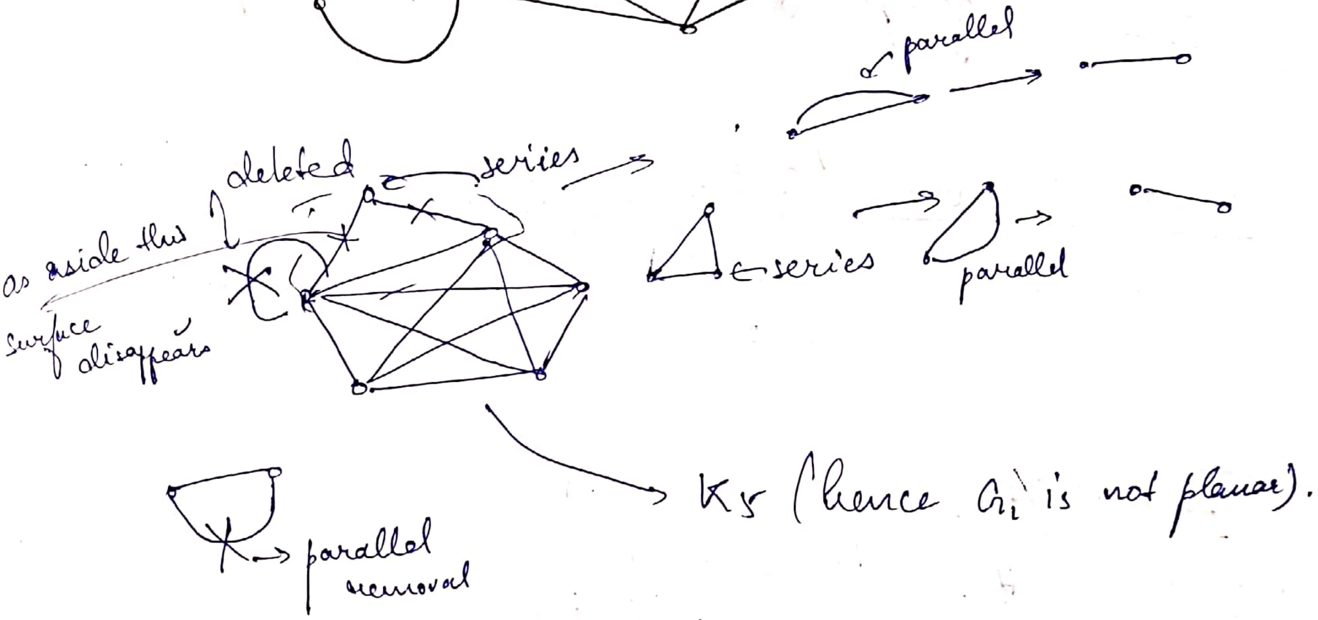
planar graph || (1) Single edge (2) A complete graph of 4 vertices, i.e K_4.

(3) A non-seperable simple graph with $|V| \geq 5$ & $|E| \geq 7$

→ If such a graph then investigate further.

(Astrophysics application)





<u>Dual of a graph</u>

Suppose G be a graph of n no. of faces / regions $(R_1, R_2 \cdots R_n)$ & $v_1, v_2 \dots v_n$ are n-points on n-different regions respectively

① These pts. are joined by line segments according to the following procedure —

① Draw a line segment between v_i & v_j by intersecting the common edge between R_i & R_j.

② If there is more than one edge, common between 2 regions R_i & R_j then draw line segment between v_i & v_j for every common edges.

③ If an edge lies entirely within one region R_i, then draw a self loop intersecting the edge exactly once from the pt. v_i on the region R_i

Def: A graph G^* (Geometric Dual) is obtained from G whose vertices are $v_1, v_2 \dots v_n$ & edges are line segments of the

- pts.

Note. (on G^*)

① A self loop in G yields a pendent ~~edge~~ vertex (of degree 1) in G^*.

② Parallel edges in G produce edges in series in G^*

Theorem: The dual G^* of a planar graph is planar

Theorem: A graph G has a dual iff G is planar.

4-Color Theorem / Conjecture || check.

Hamilton Theorem, Travelling Salesman, (Check Networks Graphs) Careful later

(coming in sems.

Traversibility

Def: A circuit/trail (an edge is traversed once) in a graph G is called an Eulerian ckt. if C contain every edge of G.

Def: A connected graph is Eulerian if it has a closed trail containing all edges.

OR

A connected graph that contains an Eulerian ckt is called an Eulerian graph.

Def: For a connected graph G, an open trail which contains every edge of G is an Eulerian Trail. (Douglas West)

Theorem: If every vertex of graph G has degree at least 2, then G contains a cycle.

Proof: Let P be a maximal path in G & let k be an end pt. of P. Since P cannot be extended, every neighbour of k must already be a vertex of P.

Since k has degree at least 2, it has a neighbour v in V(P) via an edge not in P.

The edge uv completes a cycle with the portion of P from v to u.

Note: If $V(G) = \mathbb{Z}$ & $E(G) = \{ij : |i-j| = 1\}$, then every vertex of G has degree 2 but G has no cycles (i.e no extendible path)

Theorem: A graph G is Eulerian iff it has at most one non-trivial component & its vertices all have even degree.

Proof: Necessity: Suppose that G has an Eulerian circuit C. Each passage of C through a vertex uses two incident edges. & the first edge is paired with the last at the first vertex. Hence every vertex has an even degree

Two edges can be in the same trail only when they lie in the same component, so there is at most one non-~~trivial~~trivial component.

Sufficiency: Assuming that the condition holds we obtain an Eulerian Circuit using induction on the no. of edges, m.

Basic step: $m=0$, A closed trail consisting of one vertex suffices

Induction Step $m>0$
With even degrees, each vertex in the non-trivial component of G has degree at least 2.
This means that it has a cycle.

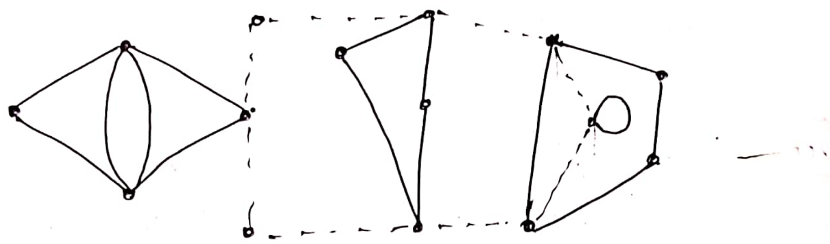
Let G' be the graph obtained from G by deleting E(C) where C is the cycle.
Since C has 0 or 2 edges at each vertex, each component of G' is also an even graph
Since each component also is connected and has fewer than m edges one can apply the induction hypothesis,

to calculate that each component of G' has an Eulerian circuit.

To combine these into an Eulerian ckt. of G, we traverse C, but when a component of G' is entended for the first time, we detour along an Eulerian Ckt. of that component.

When we complete the traversal of C, we have completed an Eulerian ckt. of G.

Eg:



Result Let G and H be non-trivial connected graph.

Then $G \times H$ is Eulerian iff both G & H are Eulerian or every vertex of G and H are odd.

Finding connectivity of Graphs using matrices

Theorem: Let G be a graph. with vertices labelled as: $v_1, v_2 \ldots v_n$ & let A be its corresponding adjacency matrix.

For any +ve integer k, the (i, j) entry of A^k is equal to the number of walk(s) from v_i to v_j that use exactly k-edges.

Proof For $k=1$, the result is true as $[A_{ij}] = 1$ where there is one edge walk from v_i to v_j.

Now suppose, that for every $i \neq j$ the (i, j) entry for A^{k-1} is the no. of walks from v_i to v_j that use exactly $(k-1)$ edges.

For each k-edge walk, from v_i to v_j there exists a vertex v_h, such that the walk can be thought of as a $(k-1)$ edge walk from v_i to v_h combined with an edge from v_h to v_j.

The total no. of these k-edge walks is:

$$\sum_{v \in V} \#(k-1) \text{ edge walks from } v_i \text{ to } v_k$$

(number of)

By induction hypothesis, this is rewritten as:

$$\sum_{v_k \in V} [A^{k-1}]_{i,j} = \sum_{h=1}^{|V|} [A^{k-1}]_{i,k} [A]_{h,j} = [A^k]_{i,j}$$

Note:

① If a Graph G with n-vertices is a connected graph, then the length of the longest possible path in G is (n-1) that can be generated by A^{n-1}

② Any entry (i, j) of $B = A + A^2 + \cdots A^k$ ($k \geq 1$) ~~goes~~ gives the no. of walks less than equal to k.

③ $S = M + M^2 + M^3 + \cdots + M^{n-1}$ where n is the no. of vertices in G.
If there is a zero in Matrix S, then it is impossible to connect the pair of vertices in n^{th} step or more.

Thus the graph is not connected.

④ Given a graph G, if an Euler path/ckt exists, how to find it?
(Fleury's Algorithm)

Bridge: An edge is removed converts a connected graph to a disconnected graph is called a bridge.

① Loops cannot be ~~a~~ bridges.
② If two or more edges, share both end pts., then removing any one of them cannot make the graph disconnected,
∴ None of the above are ~~bad~~ bridges

$\boxed{\text{Do not burn the bridges}}$

Proposition
If G contains no vertices of odd degree then G contains ~~two~~ bridges.

Fleury's Algo

Let G be an Eulerian graph

S1: Choose any vertex v ∈ G (a) set current_vertex = v
(b) current_trail = { }

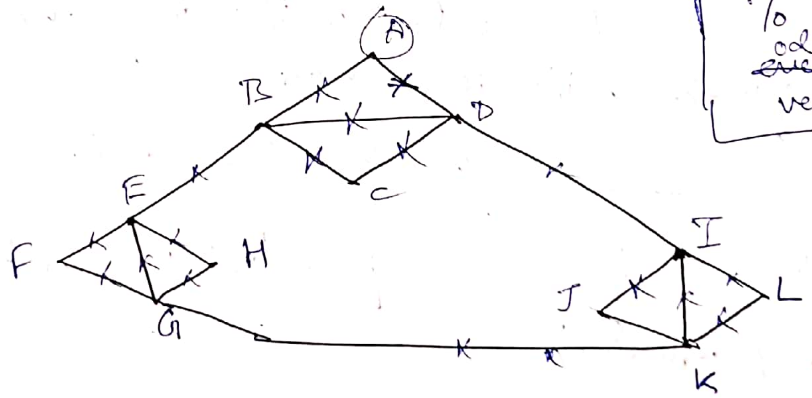
S2: Select an edge e = {v, u}, incident to current_vertex. but choosing a bridge only if there is no alternative.

S3: current_trail = Current trail ∪ {e}
current_vertex = u.

Note: If e is a loop then current vertex remains same.

S4: Delete e from G.
Delete any isolated vertices.

S5: Repeat S2 to S4 untill all edges have been deleted from G
S6: O/P: Current_trail is the Eulerian trail.



To check no ~~even~~ odd degree vertex

Current_vertex

A	J	E
D	K	F
C	I	G
D	L	H
I	K	E
	G	B
		A

Current_trail

A D B C D I J K I L K G
E F G H E B A

Traversibility in Digraphs

Def: ~~A directed/digraph G is a ...~~ If G is a

Eulerian Diagraph

Def: An Eulerian trail in a digraph is a trail containing all edges.

An Eulerian ckt. is a digraph is a closed trail containing all edges.

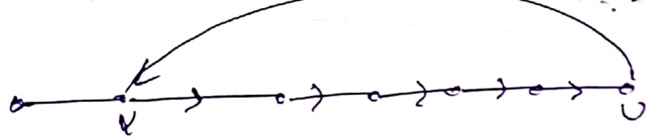
A digraph is Eulerian if it has an Eulerian ckt.

Theorem: If G is a digraph $\delta^+(G) \geq 1$ (ie the out-degree) then G contains a cycle.

The same conclusion holds when $\delta^-(G) \geq 1$

Proof: Let P be a maximal path in G & let u be the last vertex of P. Since P cannot be extended every successor of u must already be in P.

Since $\delta^+(G) \geq 1$, u has a successor v in P.



The edge uv completes a cycle with the portion of P from v to u.

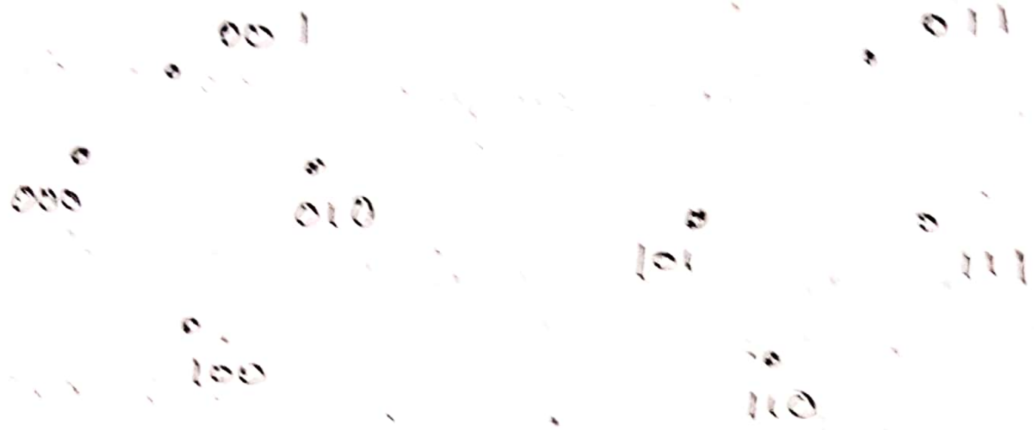
Theorem: A diagraph is Eulerian iff $\delta^+(v) = \delta^-(v)$ for each vertex $v \in V(G)$ & the underlying graph has at most one non-trivial component.

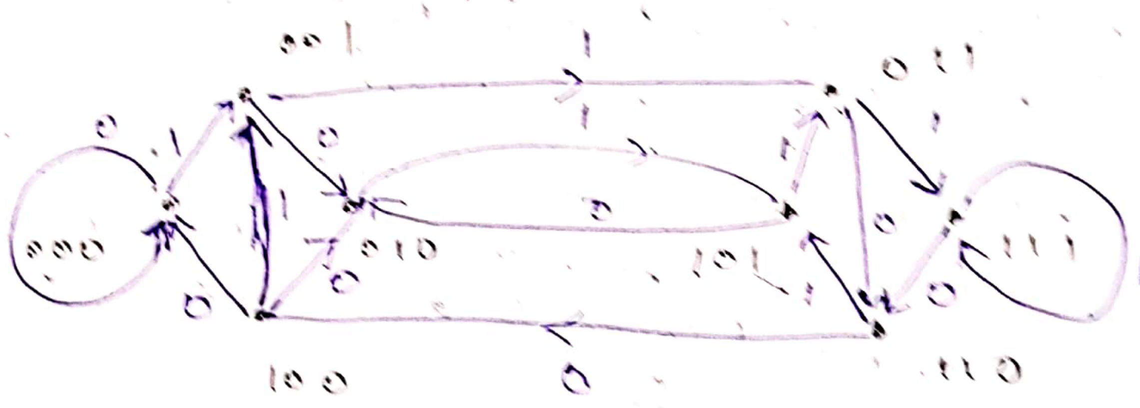
Application (de Bruijn Cycle) [Douglas West]: There are 2^n binary strings of length n of binary digits. Is there a cyclic of arrangement of 2^n binary digits such that 2^n strings of n-consecutive digits are all distinct.

Eg: $n = 4$ 0000 1111 0110 0101

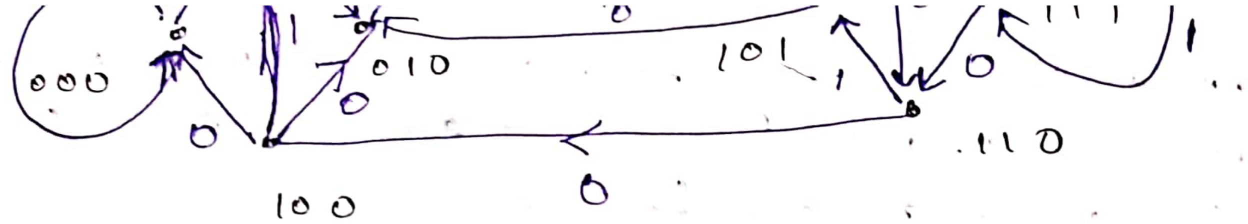
① To obtain such a circular arrangement, define a digraph D_n whose vertices are the $(n-1)$-tuples:

② Put an edge from v_i to v_j if the last $(n-2)$ entries of ② agree with the first $(n-2)$ entries of v_j (label edge with last entry of ②)





MST, Travel Sales Prob., Theorem...



Cormen { MST, Trav. Sales Prob., Trees → Cormen

Chromatic Polynomials:

Given a graph G & a set of {1,2,...k} colours, how many k-colourings of G are there?

For a k-colouring graph (k → non -ve integer) the vertices of a graph G is given as:

if $\boxed{\chi(G) > k}$ → Not possible

$$P_G(k) \text{ or } P_k(G)$$

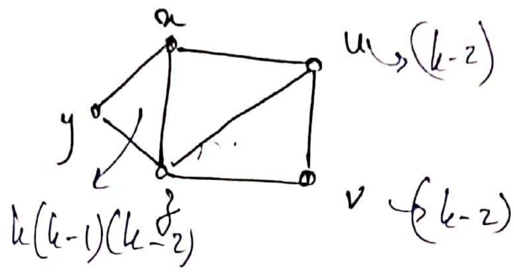
(a) If $\chi(G) > k$, then $P_k(G) = 0$

(b) If K_n graph:

$$P_{K_n}(k) = k(k-1)(k-2)\cdots(k-(n-1)) \text{ ways.}$$

(c) For a null graph →

$$P_{N_n}(k) = k^n$$



$$P_G(k) = k(k-1)(k-2)(k-2)(k-2)$$
$$= k(k-1)(k-2)^3$$

For a tree T →

$$P_T(k) = k(k-1)^{n-1}$$

Theorem

→ Let x, y be two non-adjacent vertices in the graph G. Let $G+(x,y)$ be the graph obtained from G by adding the edge (x,y).

→ Let $G=(x,y)$ [same as $G \circ \{x,y\}$] be the graph obtained from G by coalesing x & y into a single vertex $\overline{xy}$, that is adjacent to any vertex that was adjacent to x or y in G.

Then $P_k(G) = P_k\left(G+(x,y)\right) + P_k\left(G=(x,y)\right) \rightarrow (1)$

Proof: All k-colourings of the vertices of G can be broken into two disjoint cases.

(a) x and y are assigned different colours $(G+(x,y))$ forces a new edge between x & y, forcing x & y to have different colour.

(b) $G=(x,y)$ forces both the vertices to have the same colours.

Corollary: Let x & y be two adjacent vertices in the graph G.

Let $G-(x,y)$ be the graph obtained from G by deleting edge $\{x,y\}$ & ~~$G=(x,y)$~~ $G=(x,y)$ be the graph where x & y are coalesced as $\overline{xy}$.

$$\therefore P_k(G) = P_k(G-(x,y)) - P_k(G=(x,y)) \longrightarrow (2)$$

Proof:

Let H be the graph of $G-(x,y)$:

$\therefore$ x & y are non-adjacent in H.

Again $\boxed{H+(x,y) = G}$ ~~as~~ $H+(x,y)$ is converse of $G-(x,y)$

$$\& \; H=(x,y) \equiv G=(x,y)$$

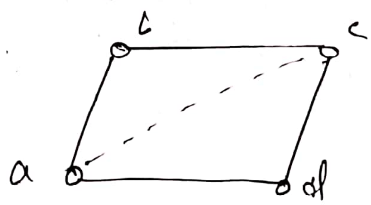
Applying (1) to H

$$P_k(H) = P_k(H+(x,y)) + P_k(H=(x,y))$$

$$P_k(G-(x,y)) = P_k(G) + P_k(G=(x,y))$$

$$\Rightarrow P_k(G) = P_k(G-(x,y)) - P_k(G=(x,y))$$

Chromatic polynomial of 4-circuit $(C_4 = abcda)$



For non-adjacent vertices a, c

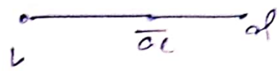
(A) (i) The graph $(C_4)_{+(a,c)} \rightarrow$ triangles (a,b,c) & (a,c,d)

Triangle $(a,b,c) \longrightarrow k(k-1)(k-2) = P_k(G+(x,y))$

(ii) There are $(k-2)$ choices for d:

$$P_k((C_4)_{+(a,c)}) = k(k-1)(k-2)^2$$

(3) $(C_4)_{\div(a,c)} \to$



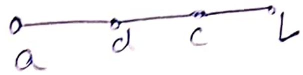
$$P_k((C_4)_{\div(a,c)}) = k(k-1)^2$$

$$P_k((C_4)_{+(a,c)}) + P_k((C_4)_{\div(a,c)})$$
$$= k(k-1)(k-2)^2 + k(k-1)^2$$
$$= k(k-1)[(k-2)^2 + (k-1)]$$

For corollary:

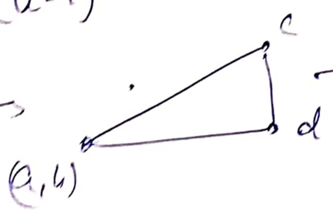
k^2+1-2k
$-k+2$
$(k-2)^2+(k-1)$

$(C_4)_{-(a,b)} \to$



$$= k(k-1)^3$$

$(C_4)_{\div(a,b)} \to$



$\to k(k-1)(k-2)$

$$P_k((C_4)_{-(a,b)}) - P_k((C_4)_{\div(a,b)})$$
$$= k(k-1)^3 - k(k-1)(k-2)$$
$$= k(k-1)[(k-1)^2 - (k-2)]$$