NUMERICAL ANALYSIS

DAY-2

$x = \bar{x} + e_x \quad y = \bar{y} + e_y \quad z = x + y = (\bar{x} + \bar{y}) + (e_x + e_y)$

$$xy = \bar{x}\bar{y} + \underbrace{x e_y + y e_x + e_x e_y}_{\text{error}}$$

Absolute error: | computed value − exact value |

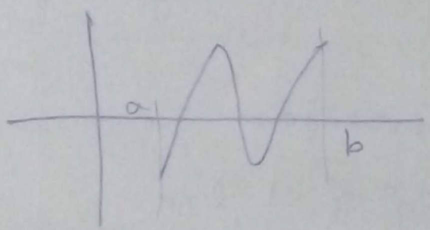
relative error: $\dfrac{\text{Absolute error}}{\text{exact value}}$

12345.678 → computed
12345.675 → exact
0.003 → error

0.123 → computed
0.120 → exact
0.003 "

$\dfrac{0.003}{0.120} \quad < \dfrac{0.003}{12345.675}$

$y = f(x)$ continuous at ~~con~~ $[a, b]$.



If $f(a), f(b)$ are of opposite sign then $\exists\, c \in (a, b)$
$f(c) = 0$.

Mean Value Theorem

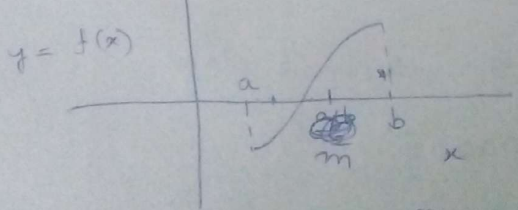
Taylor Series "

Solution of non linear eqn.

$a_0 x^{n+1} + a_1 x^n + \dots + a_{n+1} = 0$

$x + \log_e x + \sin x = 0$ ← transcendental eqn

Bisection method



1) Input a and b are chosen such that they are of opposite sign
$f(a) * f(b) < 0$

m midpt of a, b

2) If $f(m) = 0$ then m is a root, exit.

3) if $f(a) * f(m) < 0$ then $b = m$
else $a = m$

Continue steps 1) 2) 3) width size of the interval $[a, b]$

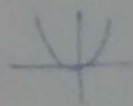
is less than the required precession

4) $m = \frac{(a+b)}{2}$

5) Print m

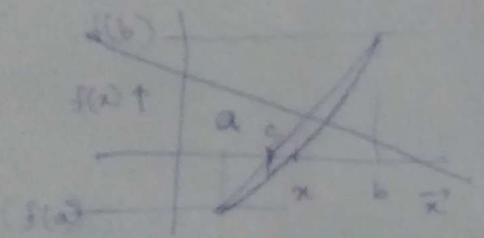
6) End

Method fails when $f(a), f(b)$ always same sign i.e. curve touches x-axis



~~Method of fa...~~

Method of ~~function~~ false position or regular falsi method



~~Line join~~

Eq^n of st. line:

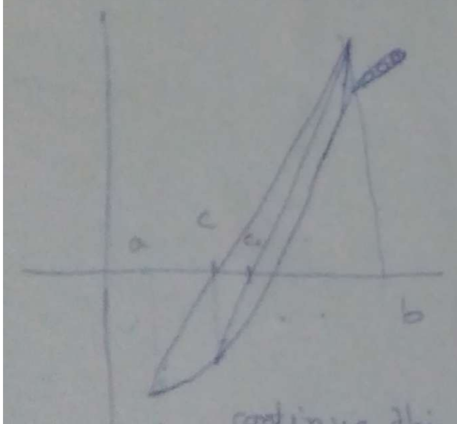
$$\frac{y - f(a)}{x - a} = \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow \quad x - a = \frac{(y - f(a))(b - a)}{(f(b) - f(a))}$$

$$\Rightarrow \quad c - a = \frac{(b - a)(-f(a))}{f(b) - f(a)}$$

$$\Rightarrow \quad c = \frac{af(b) - \cancel{af(a)} - bf(a) + \cancel{af(a)}}{f(b) - f(a)}$$

$$c = \frac{af(b) - bf(a)}{f(b) - f(a)}$$



continue this process.

if $|c_n - c_{n-1}| <$ required ~~position~~ precession.

if accurate for 5 decimal place

precession $= 0.000001$

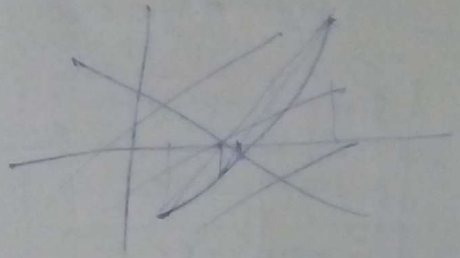
Advantage Convergence faster.

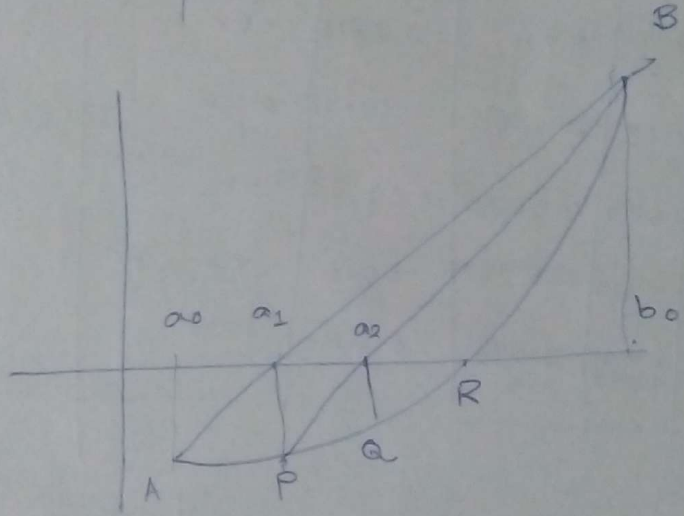
analyses of any numerical algorithm consideres the following facts.

1. whether the method is guaranteed to provi[de] sollution

2. How fast will it provide the solution?

3. Ammount of error associated with the solution.





From a_0, a_1, A and a_1, b_0, B

$$\frac{a_0 - a_1}{|f(a_0)|} = \frac{b_0 - a_1}{|f(b)|} \longrightarrow (1)$$

From a_1, a_2, P and a_2, b_0, B

$$\frac{a_2 - a_1}{|f(a_1)|} = \frac{b_0 - a_2}{|f(b)|} \longrightarrow (2)$$

$\Rightarrow (a_2 - a_1)\,|f(b)| = |f(a_1)|\,(b_0 - a_2)$

$\Rightarrow (a_2 - a_1)\,|f(b_0)| + a_2\,|f(a_1)| = b_0|$

$\Rightarrow (a_2 - a_1)\,|f(b_0)| + a_2|f(a_1)| - a$

$= (b_0 - a_1) \, |f(a_1)|$

$\Rightarrow (a_0 - a_1) \{ |f(b_0)| + |f(a_0)| \}$

$= (a_0 - a_1) \dfrac{|f(b_0)|}{|f(a_0)|} \times |f(a_1)|$

$$\Rightarrow (a_2 - a_1) = (a_0 - a_1) \left\{ \frac{\dfrac{|f(a_1)|}{|f(a_0)|}}{\dfrac{|f(b_0)| + f(a_0)|}{|f(b_0)|}} \right\} < 1$$

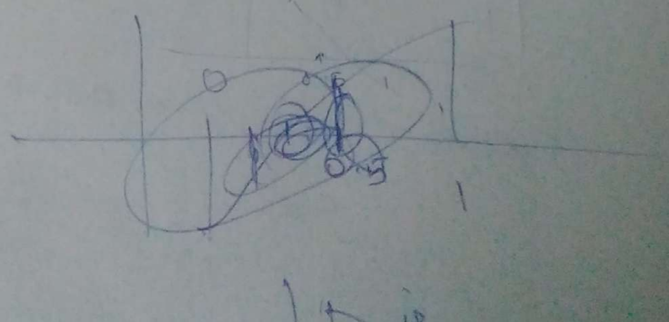
Solve: $e^{-x} - x = 0$ using bisection method.

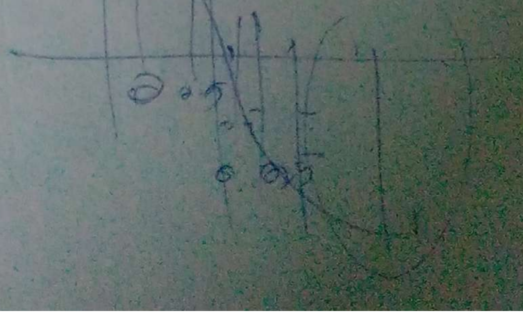
Let $a = 0, b = 1$

$f(0) = 1$

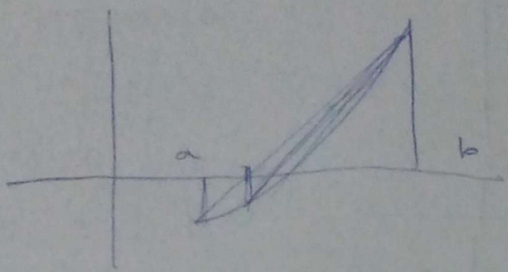
$f(1) = -0.632$

iteration no (i)	a_i	b_i	m_i	$f(m_i)$	$f(a_i) * f(m_i)$
1	0	1	0.5	0.10653	+ve
2	0.5	1	0.75	−0.2776	−ve
3	0.5	0.75	0.625	−0.89738	−ve
4	0.5	0.625	0.5625	0.00738	+ve
5	0.5625	0.625	0.59375	−0.0415	−ve
6	0.5625	0.59375	0.598125	−0.01717	−ve





NUMERICAL ANALYSIS




```
scanf("%f%f", &a,&b) ;
if(f(a)*f(b) < 0)
{
    do
    {
        oldc = c;
        float c = (a*f(b) - b*f(a)) / (b - a) ;

        if(f(c) < = 0.00001)
        {
            root=c;
            break;
        }
        else if(f(c) * f(a)<0)
        {
            b=c;
        }
        else
        {
            a=c;
        }
    } while(abs(oldc - c)>=0.00001)
    if(root != 0)
        printf("% f", root)
    else
        printf("% f", c)
```

}
else
    printf ("Error");

How fast the method converge to the root?

=> order of convergence:

$e_{i+1} = k e_i^n$

$e_i = 0.01$ and $n = 2$.

$e_{i+1} \propto (0.01)^2 \propto 0.001$

$n = 1$

$|k| < 1$

$|e_{i+1}| < |e_i|$.

$e_i = |oldc_i - c_i|$

$e_{i+1} = |oldc_{i+1} - c_{i+1}|$

$\log|e_{i+1}| = \log|k| + n \log|e_i|$

$n = \left| \frac{\log|e_{i+1}| - \log|k|}{\log|e_i|} \right|$

Fixed point iteration method / iterative method using repetative method:

given $f(x) = 0$

substitute as $x = g(x)$.

assume initial approximation of root of $f(x) = 0$ be x_0. we generate a sequence of approximation is the root as follow:

$$x_1 = f(x_0)$$

$$x_2 = f(x_1)$$

$$x_3 = g(x_2)$$

$$\vdots$$

$$x_{n+1} = g(x_n)$$

$|x_{n+1} - x_n| <$ required position.

$x^2 - x - 6 = 0$.

$\Rightarrow x^2 - 3x + 2x - 6 = 0$

$\Rightarrow (x-3)(x+2) = 0$

$x = \pm\sqrt{x+6}$

$x = \sqrt{x+6} \longrightarrow$ (i)

$x = 1 + 6/x \longrightarrow$ (ii)

$x = x^2 - 6 \longrightarrow$ (iii)

(i)

$x = \sqrt{x+6}$

$x_0 = 1$ $\quad x_1 = \sqrt{1+6} = \sqrt{7} = 2.645$

$x_1 = \sqrt{7}$ $\quad x_2 = \sqrt{6+\sqrt{7}} = 2.94$

$x_3 = 2.989$

$x_4 = 2.998$

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$x = -\sqrt{x+6}$

$x_0 = 1$ $\quad x_1 = -\sqrt{7} = -2.645$

$x_2 = -\sqrt{6-2.645} = -1.831$

$x_3 = -2.04$

$x_4 = -1.989$

$x_5 = -2.002$.

$x = 1 + 6/x$

$x_0 = 1$ $\quad x_1 = 7$.

$x_1 = 7$ $\quad x_2 = 1 + 6/7 = 1.857$

$x_2 = 1.857$ $\quad x_3 = 1 + \frac{1}{1.857} = 4.23$.

$x_3 = 4.23$ $\quad x_4 = 1 + \frac{6}{4.23} = 2.413$.

$x_4 = 2.413$ $\quad x_5 = 1 + \frac{6}{2.413} = 3.481$

Theorem: Let $x = R$ be a root of the eq^n $f(x) = 0$ which is rewritten as $x = g(x)$. Let both $g(x)$ and $g'(x)$ exist and continuous at individual (a,b) ~~contain~~ containing $x = R$. If $|g'(x)| < 1$, over the interval (a,b) and the initial approximation of the root x_0 is also in (a,b). Then the sequence of approximations $x_1, x_2, x_3,$ will converge at $x = R$.

③ $x = 1 + \frac{6}{x}$ $\quad g(x) = 1 + \frac{6}{x}$

$$g'(x) = -6/x^2.$$

$x = 3$ $\quad g'(x) = -\frac{6}{9} = -\frac{2}{3}$.

$$|g'(x)| < 1.$$

④ $x = x^2 - 6$ $\quad g(x) = x^2 - 6$

$$g'(x) = 2x.$$

Proof: $x = R$ be a root of $x = g(x)$.

$$R = g(R) \longrightarrow (1).$$

Let x_0 be the initial approximation is the root then approximations can be generated as,

$$\left.\begin{aligned} x_1 &= g(x_0) \\ x_2 &= g(x_1) \\ x_3 &= g(x_2) \\ x_4 &= g(x_3) \\ &\vdots \\ x_{n+1} &= g(x_n) \end{aligned}\right\} \longrightarrow (2)$$

Let $e_0, e_1, e_2, \ldots, e_{n+1}$ be the error associated with $x_0, x_1, x_2, \ldots, x_{n+1}$ then

$e_0 = R - x_0$

$e_1 = R - x_1 = g(R) - g(x_0) \longrightarrow$ ②

$e_2 = R - x_2 = g(R) - g(x_1) \longrightarrow$ ③

$e_{n+1} = R - x_{n+1} = g(R) - g(x_n)$

~~equation~~ ③

Using mean value theorem equation ③ written as :

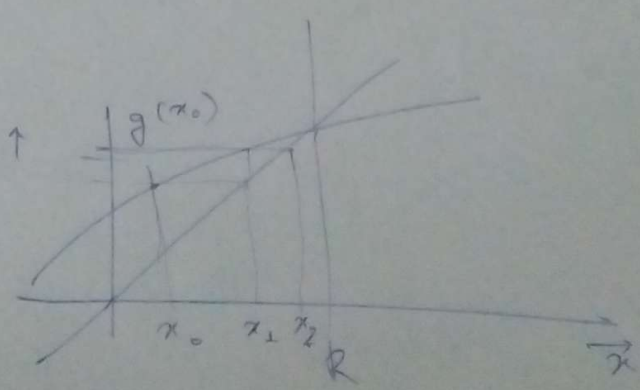
$e_0 = R - x_0$

$e_1 = (R - x_0)\, g'(c_1) \qquad x_0 < c_1 < R.$

$e_2 = (R - x_1)\, g'(c_2) \qquad x_1 < c_2 < R.$

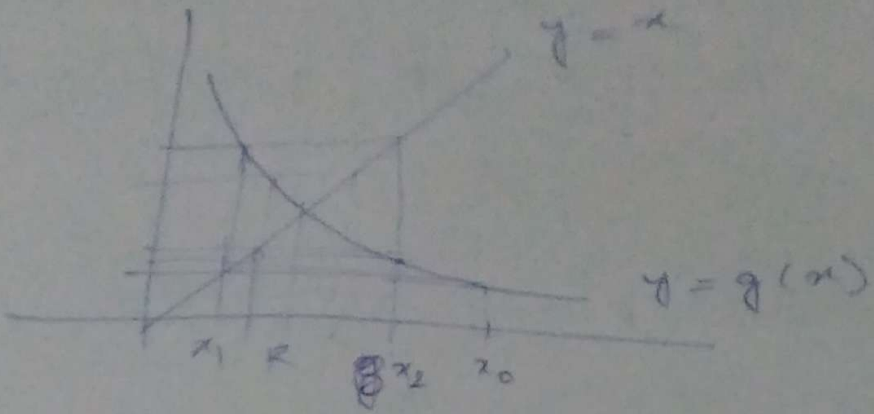
$e_{n+1} = (R - x_n)\, g'(c_n) \qquad x_n < c_n < R$

For convergence $|e_0| < |e_1| < |e_2| < \dots < |e_n|$ will be possible $|g'(c_i)| < 1$



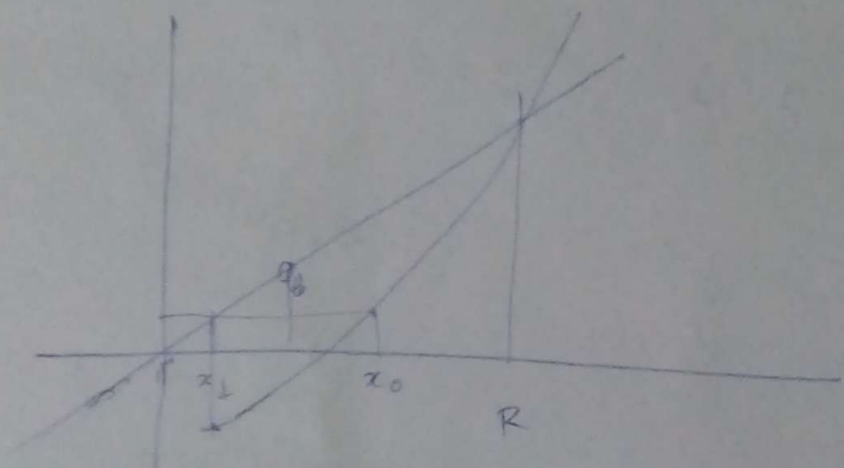
$x = g(x)$ as combination of the function $y = x,\ y = g(x)$

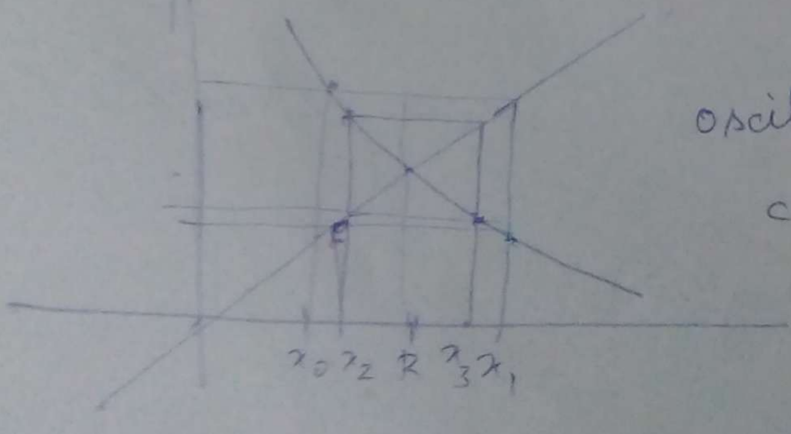
$g(x) < 1$ over the individual continuity



$0 > g'(x) > -1$

oscillating manner.





oscillating convergence

Newton- Raphson method :

$g(x)$
$g'(x)$.
$f(x)=0$.

Let the initial approximation to the root
~~be the~~
Let e_0 be the error associated with x_0.

$R = x_0 + e_0$

now $f(R) = 0$

$f(x_0 + e_0) = 0$.

~~Applying~~ Using Taylor theorem

$$f(x_0) + e_0 f'(x_0) + \frac{e_0^2}{2!} f''(x_0) + \ldots \ldots = 0$$

$e_0 << 1$ $\quad f(x_0) + e_0 f'(x_0) = 0$

$$e_0 = -\frac{f'(x_0)}{f(x_0)} \ldots$$

we can ~~approximat~~ generate the next approximation by the root,

$$x_1 = x_0 + e_0 = x_0 - \frac{f'(x_0)}{f(x_0)} .$$

~~continue~~

$$x_1 = x_0 - \frac{f'(x_0)}{f(x_0)}$$

$$x_2 = x_1 - \frac{f'(x_1)}{f(x_1)}$$

$|x_{n+1} - x_n|$ < required preccession

$$x_{n+1} = x_n - \frac{f'(x_n)}{f(x_n)}$$

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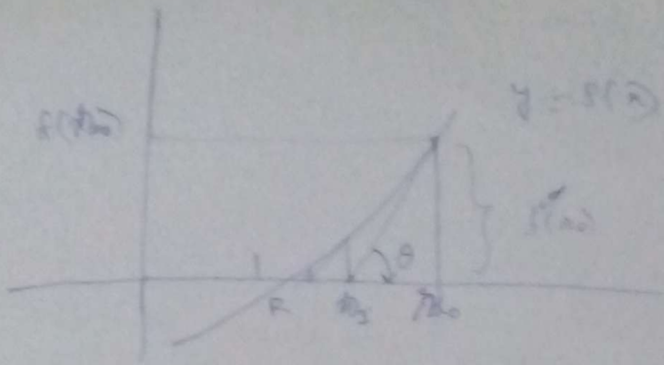
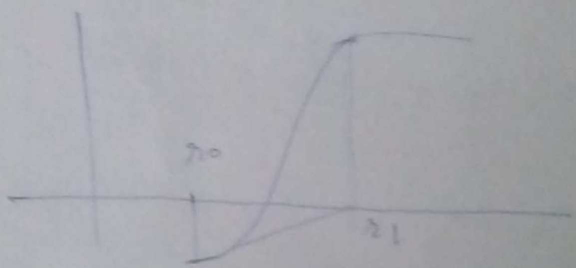


$$\tan\theta = f'(x_0)$$

$$\Rightarrow \frac{f(x_0)}{x_0 - x_1} = f'(x_0)$$

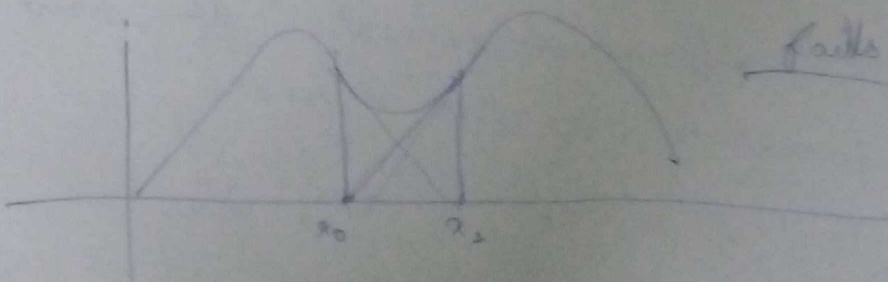
$$\Rightarrow x_0 - x_1 = \frac{f(x_0)}{f'(x_0)}$$

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$



fails

$|f'(x_0)| < 1$



fails

$f(x) = 0$ has a root at $x = R$

then

Newton

Iterative method

$$x_{n+1} = g(x_n)$$

$$g(x) = x - \frac{f(x)}{f'(x)}$$

$$|g'(x)| < 1$$

$$g'(x) = 1 - \frac{f'(x)}{f'(x)} + \frac{f(x) \cdot f''(x)}{\{f'(x)\}^2}$$

$$\left| \cancel{1 - \frac{f'(x)}{f'(x)}} + \frac{f(x) \cdot f''(x)}{\{f'(x)\}^2} \right| < 1$$

$$\left| \frac{f(x) \cdot f''(x)}{\{f'(x)\}^2} \right| < 1$$

8.8.18

Numerical Analysis

$f(x) = x^3 - x - 3$ $\quad f'(x) = 3x^2 - 1$

Solve $f(x) = 0$ using Newton Raphson method Take $x_0 = 0$.

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$0 - \frac{-3}{-1}$

~~-3~~

$-3 - \frac{-27}{26}$

i	x_i	x_{i+1}	$\lvert x_{i+1} - x_i \rvert$	
		(-3)	3	$0 < 1$
0	0			$0.718 < 1$
1	(-3)	-1.96	1.04	$0.91 > 1$
2	-1.96	-1.146	0.814	$2.02 > 1$
3	-1.146	~~0.000~~ -0.274	~~0.3~~	
4	-0.274	-3.198		

Continuously oscillating in $(0, -3)$.

$$\left| \frac{f(x_i)\, f''(x_i)}{\{f'(x_i)\}^2} \right| < 1$$

$\cancel{0 < 1}$

$\cancel{0.718 < 1}$

Order of convergence for N-R method

$$e_{i+1} \propto e_i^n$$

$|e_i| << 1 \quad n > 1$

$x_{i+1} = R - e_{i+1}$

$R = x_i + e_i$

$x_i = R - e_i$

Expanding Taylor's Formula;

$$f(x_i) + e_i f'(x_i) + \frac{e_i^2}{2!} f''(x_i) + \frac{e_i^3}{3!} f'''(x_i) + \dots = 0.$$

Assume $|e_i| \ll 1$. so we neglect e_i^3 and higher order terms;

$$f(x_i) + e_i f'(x_i) + \frac{e_i^2}{2!} f''(x_i) = 0$$

$$\Rightarrow \frac{f(x_i)}{f'(x_i)} + e_i + \frac{e_i^2}{2!} \frac{f''(x_i)}{f'(x_i)} = 0$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$\Rightarrow -\frac{f(x_i)}{f'(x_i)} = R - x_i + e_i^2 \frac{f''(x_i)}{2f'(x_i)}$$

$$\Rightarrow x_i - \frac{f(x_i)}{f'(x_i)} = R + e_i^2 \frac{f''(x_i)}{2f'(x_i)}$$

$$\Rightarrow x_{i+1} - R = e_i^2 \frac{f''(x_i)}{2f'(x_i)}$$

$$\Rightarrow -e_{i+1} = e_i^2 \frac{f''(x_i)}{2f'(x_i)}$$

$$\Rightarrow |e_{i+1}| = |e_i|^2 \left| \frac{f''(x_i)}{2f'(x_i)} \right|$$

~~If we neglect~~ cons

$$\Rightarrow |e_{i+1}| = |k||e_i|^2.$$

$$\Rightarrow \log|e_{i+1}| = \log|k| + 2\log|e|$$

$$\Rightarrow \frac{\log|e_{i+1}|}{\log|e_i|} = \frac{\log|k|}{\log|e_i|} + 2.$$

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$f(x) = x^2 - x - 6 = 0$

$\Rightarrow (x-3)(x+2) = 0$

$f'(x) = 2x - 1$

$r_0 = 2.5$

i	r_i	r_{i+1}	$\left\lvert \frac{f(r_i) f''(r_i)}{f'(r_i)^2} \right\rvert$	$e_i = \lvert r_{i+1} - r_i \rvert$	$n = \left\lvert \frac{\log \lvert e_m \rvert}{\log \lvert e_i \rvert} \right\rvert$
0	2.5	3.0625	<1	0.5625	—
1	3.0625	3.000762	<1	0.617	4.84
2	3.0076	2.99695	<1	0.0038	2.00
3	2.99695	3.000001	<1	0.00305	1.00 ~~1.00~~
4	3.000001	2.999996	<1	0.000005	2.107
5	2.999996	3.00000	<1	0.000004	1.02

For fixed pt iteration method

$$R - r_{i+1} = (R - r_i)\, g'(e_i) \qquad r_i < e_i < R$$

Let $|g'(e_i)| < 1$ for all e_is in the interval $[r_i, R]$

and let $|g'(e_i)| = k$.

we can write

$$R - r_{i+1} = k(R - r_i) \rightarrow (1)$$

$$R - r_{i+2} = k(R - r_{i+1}) \rightarrow (2).$$

$(1) \div (2)$

$$\frac{R - r_{i+1}}{R - r_{i+2}} = \frac{R - r_i}{R - r_{i+1}}$$

$$\Rightarrow (R - r_{i+1})^2 = (R - r_i)(R - r_{i+2})$$

$R^2 - 2Rx_{i+1} + x_{i+1}^2$

$= R^2 - x_iR - x_{i+2}R + x_i x_{i+2}$

$\Rightarrow R(2x_{i+1} - x_i - x_{i+2}) = -x_i x_{i+2} + x_{i+1}^2$

$$\boxed{R = \frac{x_{i+1}^2 - x_i x_{i+2}}{2x_{i+1} - x_i - x_{i+2}}}$$

Aitken's accelerator formula

i	x_i	$x_{i+1} = g(x_i)$		
0	1	2.64575	x_0, x_1, x_2	3.0046
1	2.64575	2.94037		
2	3.0046	3.0076	3.000006	
3	3.0076	3.0013		
4	3.000006	3.000001		

$f(x) = x^3 - 4x^2 - 3x + 18$

$f(x) = x^2(x+2) - 6x(x+2) + 9(x+2)$

$= (x^2 - 6x + 9)(x+2)$

$= (x-3)^2(x+2)$

$f'(x) = 3x^2 - 8x - 3$

$x_i \to 3$

$f(x_i) \to 0$

$f'(x_i) \to 0$

i	x_i	x_{i+1}	$e_i = \lvert x_{i+1} - x_i \rvert$
0	2.5	2.7647	0.24706
1	2.7647	2.86974	0.105035
2	2.86974	2.93575	0.660

Let $\quad f(x) = (x-a)^n g(x)$

i.e. there exist multiple roots of order n

$$f^{n-1}(a) = f^{n-2}(a) = \dots = f'(a) = f(a) = 0$$

$$f'(x) = n(x-a)^{n-1} g(a) + (x-a)^n g'(x)$$

$$(x-a) f'(x) = \cancel{n(x-a)^n} \; n f(x) + \underbrace{(x-a)^{n+1} g'(x)}_{\to 0}$$

$$x - a = \frac{n f(x)}{f'(x)}$$

if $\eta_{o.c.} < 2$ always then modify

i	x_i	$x_{i+1} = x_i - \dfrac{2f(x_i)}{f'(x_i)}$	$e_i = \lvert x_{i+1} - x_i \rvert$	η
0	2.5	3.0294	0.5294	
				5.55
1	3.0294	3.00009	0.0293	
				2.639
2	3.00009	3.00000	0.00009	

VAIO

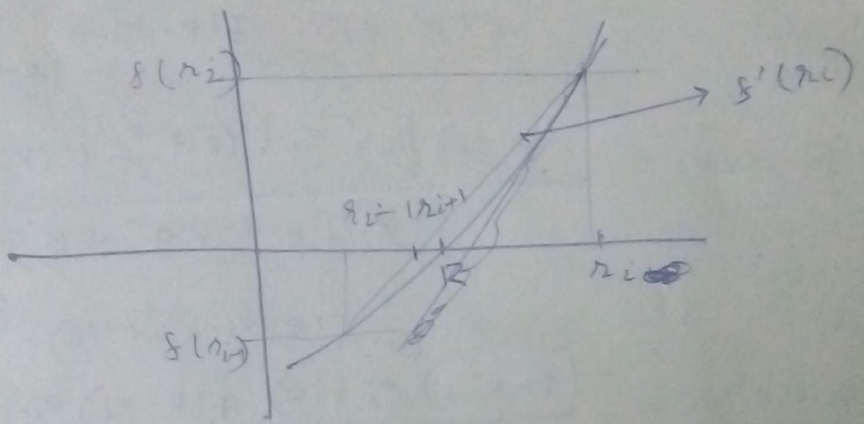
Secant Method

For N-R we have to provide the functional representation of both $f(x)$, $f'(x)$.

$f(x) = x^2 e^x \sin x$ Slope of AB = $\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$

$f'(x) =$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$



$$\text{Slope of AB} = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$= x_i - \frac{f(x_i)}{\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}}$$

$$= x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

$$= \frac{x_{i-1} f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}$$

There is a limit of no of iterations

R is the interval root of $f(x)=0$.

$\therefore f(R)=0$.

Let $x_{i-1} = R - e_{i-1}$.

$x_i = R - e_i$

$x_{i+1} = R - e_{i+1}$

$$x_{i+1} = \frac{x_{i-1} f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}$$

$$R - e_{i+1} = \frac{(R - e_{i-1}) f(R - e_i) - (R - e_i) f(R - e_{i-1})}{f(R - e_i) - f(R - e_{i-1})}$$

$$R - e_{i+1} = \frac{(R - e_{i-1})\left[\cancel{f(R)} - e_i f'(R) + \frac{e_i^2}{2!} f''(R) + \dots\right] - (R - e_i)\left[\cancel{f(R)} - e_{i-1} f'(R) + \frac{e_{i-1}^2}{2!} f''(R) - \dots\right]}{\cancel{f(R)} - e_i f'(R) - \cancel{f(R)} + e_{i-1} f'(R)}$$

$$R - e_{i+1} = \frac{-(R - e_{i-1}) e_i f'(R) + (R - e_i) e_{i-1} f'(R) + (R - e_{i-1}) \frac{e_i^2}{2} f''(R) - (R - e_i) \frac{e_{i-1}^2}{2} f''(R)}{(e_{i-1} - e_i) f'(R)}$$

$$R - e_{i+1} = \frac{(-R e_i + \cancel{e_i e_{i-1}} + R e_{i-1} - \cancel{e_i e_{i-1}}) f'(R) + \frac{(R e_i^2 - e_{i-1} e_i^2 - R e_{i-1}^2 + e_i e_{i-1}^2)}{2} f''(R)}{(e_{i-1} - e_i) f'(R)}$$

(Neglect e_i^2, e_{i-1}^2)

$$R - e_{i+1} = \frac{R(e_{i-1} - e_i) f'(R) + \frac{[R(e_i + e_{i-1})(e_i - e_{i-1}) - e_{i-1} e_i (e_i - e_{i-1})] f''(R)}{2}}{(e_{i-1} - e_i) f'(R)}$$

$$\cancel{R} - e_{i+1} = \cancel{R} + \frac{(\cancel{R e_i + R e_{i-1}} - e_i e_{i-1}) f''(R)}{2 f'(R)}$$

$$e_{i+1} = \cancel{\frac{(Re_i + Re_{i-1} - e_i e_{i-1})\, f''(R)}{2f'(R)}}$$

$$e_{i+1} = \frac{e_i e_{i-1}}{2} \frac{f''(R)}{f'(R)} \longrightarrow (1)$$

Let us assume the order of convergence be n

$$e_{i+1} = k e_i^n \text{ ——— } (2)$$

$$e_i = k e_{i-1}^n \text{ ——— } (3)$$

$$e_{i-1} = \left(\frac{e_i}{k}\right)^{\frac{1}{n}} \text{ ——— } (4)$$

(1), (2), (4) we can write

$$k e_i^n = \frac{e_i \left(\frac{e_i}{k}\right)^{\frac{1}{n}}}{2} \frac{f''(R)}{f'(R)}$$

$$\Rightarrow \quad k e_i^n \approx e_i^{1+\frac{1}{n}} \frac{f''(R)}{2k^{\frac{1}{n}} f'(R)}$$

$$\Rightarrow \quad e_i^n \approx e_i^{1+\frac{1}{n}} \frac{f''(R)}{2k^{1+\frac{1}{n}} f'(R)}$$

Now $\frac{f''(R)}{2k^{1+\frac{1}{n}} f'(R)}$ is ~~constant~~ approximated as constant.

Equating power of e_i from both sides

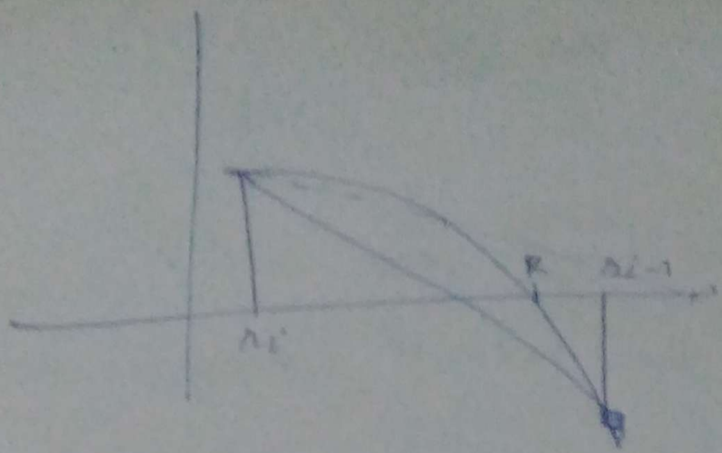
$$n = 1 + \frac{1}{n}$$

$$n^2 - n - 1 = 0$$

$$n = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2} = \frac{1 \pm 2.236}{2}$$

$$n = 1.618, \ -0.61$$

We have to choose positive n



$$|f'(x_i)| = \left| \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \right|$$

Multipoint iteration method:

Let R be the actual root of $f(x)=0$ and x_i be an approximation of the root. $x_i = R - e_i$

or, $R = x_i + e_i$

or, $f(x_i + e_i) = f(R)$

Now, $f(R) = f(x_i + e_i) = 0$.

Expanding by Taylor's Formula:

$$f(x_i) + e_i f'(x_i) + \frac{e_i^2}{2!} f''(x_i) + \frac{e_i^3}{3!} f'''(x_i) + \dots = 0$$

$$\Rightarrow f(x_i) + e_i \{ \underbrace{f'(x_i) + \frac{e_i}{2} f''(x_i) + \frac{e_i^2}{3!} f'''(x_i) + \dots}_{\approx f'(x_i + e_i/2)} \} = 0$$

$$\Rightarrow -f(x_i) = e_i f'\left(x_i + \frac{e_i}{2}\right)$$

$$\Rightarrow e_i = -\frac{f(x_i)}{f'\left(x_i + \frac{e_i}{2}\right)}$$

$$= -\frac{f(x_i)}{f'\left(x_i - \frac{f(x_i)}{2f'(x_i)}\right)}$$

From N-R method

$e_i = -\frac{f(x_i)}{f'(x_i)}$

$$x_{i+1} = x_i + e_i$$

$$= x_i - \frac{f(x_i)}{f'\left\{x_i - \dfrac{f(x_i)}{2f'(x_i)}\right\}}$$

$$= x_i - \frac{f(x_i)}{f'(x_{i+1}^*)} \qquad x_{i+1}^* = x_i - \frac{f(x_i)}{f'(x_{i+1})}$$

$$x_{i+1}^* = x_i - \frac{f(x_i)}{2f'(x_i)}$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_{i+1}^*)}.$$

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30.8.18

Multipoint iteration method

$$x_{k+1}^{*} = x_k - \frac{1}{2}\frac{f(x_k)}{f'(x_k)}.$$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_{k+1}^{*})}.$$

$R = x_k + e_k$.

$R = x_{k+1} + e_{k+1}$.

$$x_{k+1}^{*} = R - e_k - \frac{\frac{1}{2} f(R-e_k)}{f'(R-e_k)}$$

$$\frac{f(R-e_k)}{f'(R-e_k)} = \frac{f(R)^{\to 0} - e_k f'(R) + \frac{e_k^2}{2!} f''(R) - \frac{e_k^3}{3!} f'''(R) + \dots}{f'(R) - e_k f''(R) + \frac{e_k^2}{2!} f'''(R) - }$$

$$= \frac{-e_k + \frac{e_k^2}{2!}\frac{f''(R)}{f'(R)} - \frac{e_k^3}{3!}\frac{f'''(R)}{f'(R)} + \dots}{1 - e_k\frac{f''(R)}{f'(R)} + \frac{e_k^2}{2!}\frac{f'''(R)}{f'(R)} - \dots}$$

$$\frac{f''(R)}{f'(R)} = a_1$$

$$\frac{f'''(R)}{f'(R)} = a_2$$

$$= \left(-e_k + a_1\frac{e_k^2}{2!} - a_2\frac{e_k^3}{3!} + \right)\left(1 - a_1 e_k + a_2\frac{e_k^2}{2!} - \quad\right)^{-1}$$

$$= \left(-e_k + \frac{a_1}{2}e_k^2 - \frac{a_2}{3!}e_k^3\right)\left(1 + a_1 e_k - a_2\frac{e_k^2}{2!}\right.$$

$$-e_k - \frac{a_1}{2}e_k^2 + \left(\frac{a_1^2}{2} + \frac{a_2}{3}\right)e_k^3$$

$$r_{k+1}^* = (R - e_k) - \frac{1}{2}\left\{-e_k - \frac{a_1}{2}e_k^2 + \left(\frac{a_1^2}{2} + \frac{a_2}{3}\right)e_k^3\right\}$$

$$= R - \frac{e_k}{2} + \frac{a_1}{4}e_k^2 - \left(\frac{a_1^2}{4} + \frac{a_2}{6}\right)e_k^3$$

$$r_{k+1} = R - e_k - \frac{f(R-e_k)}{f'\left[R - \left\{\frac{e_k}{2} + \frac{a_1}{4}e_k^2 - \left(\frac{a_1^2}{4} + \frac{a_2}{6}\right)e_k^3\right\}\right]}$$

$$r_{k+1} = R - e_k - \frac{f(R)^{\nearrow 0} - e_k f'(R) + \frac{e_k^2}{2}f''(R) - \frac{e_k^3}{3!}f'''(R)}{f'(R) + \left[\frac{e_k}{2} + \frac{a_1}{4}e_k^2 - \left(\frac{a_1^2}{4} + \frac{a_2}{6}\right)e_k^3\right]f''(R) + \left[\frac{e_k}{2} + \frac{a_1}{4}e_k^2 - \left(\frac{a_1^2}{4} + \frac{a_2}{6}\right)e_k^3\right]^2 f'''(R) - \ldots}$$

$$r_{k+1} = R - e_k - \frac{-e_k + \frac{a_1}{2}e_k^2 - \frac{a_2}{6}e_k^3}{1 - \left\{\frac{e_k}{2} + \frac{a_1}{4}e_k^2 - \left(\frac{a_1^2}{4} + \frac{a_2}{6}\right)e_k^3\right\}a_1}$$

$$r_{k+1} = R - e_k - \left(-e_k + \frac{a_1}{2}e_k^2 - \frac{a_2}{6}e_k^3\right)\left[1 + \left\{\frac{e_k}{2} - \frac{a_1}{4}e_k^2 - \left(\frac{a_1^2}{4} + \frac{a_2}{6}\right)e_k^3\right\}a_1\right].$$

$$r_{k+1} = R - e_k - \left\{-e_k + \cancel{\frac{a_1}{2}e_k^2} - \frac{a_2}{6}e_k^3 - \cancel{\frac{a_1 e_k^2}{2}} + \frac{a_1^2 e_k^2}{4} + \frac{a_1^3}{8}e_k^4 - \frac{a_1 a_2}{12}e_k^4 \ldots\right\}$$

$$= R - \cancel{e_k} + \cancel{e_k} + \left(\frac{a_2}{6} - \frac{a_1^2}{4}\right)e_k^3 - \left(\frac{a_1 a_2}{12} - \frac{a_1^3}{8}\right)\ldots$$

$$e_{k+1} = \left(\frac{a_2}{6} - \frac{a_1^2}{4}\right) e_k^3 - \left(\frac{a_1 a_2}{12} - \frac{a_1^3}{8}\right) e_k + \ldots$$

$$\therefore \quad e_{k+1} \approx \left(\frac{a_2}{6} - \frac{a_1^2}{4}\right) e_k^3$$

Chebyshev Method

Let x_k be the inital approximation of the root R of $f(x)=0$.

$$R = x_k + e_k$$

$$f(x_k + e_k) = 0.$$

$$\Rightarrow \quad f(x_k) + e_k f'(x_k) + \frac{e_k^2}{2!} f''(x_k) + \frac{e_k^3}{3!} f'''(x_k) + \ldots = 0.$$

~~rejecti~~ neglecting ~~regularity~~ e_k^3 and higher order terms

$$-e_k f'(x_k) = \cancel{\frac{e_k^2}{2!} f''(e_k)} \; \cancel{e_k^2}$$
$$f(x_k) + \frac{e_k^2}{2!} f''(x_k)$$

$$-e_k = \frac{f(x_k)}{f'(x_k)} + \frac{e_k^2}{2!} \frac{f''(x_k)}{f'(x_k)}.$$

From N-R method

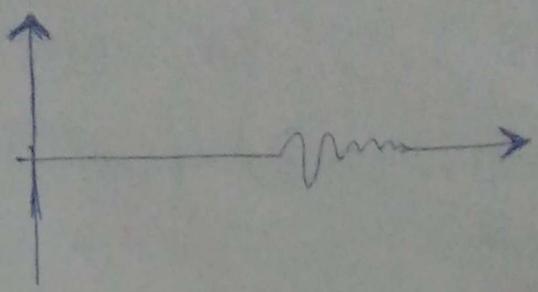
$$e_k = -\frac{f(x_k)}{f'(x_k)}.$$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} - \frac{\{f(x_k)\}^2 f''(x_2)}{2\{f'(x_k)\}^3}.$$

Divergence ~~check~~ checking: $|f'(x_k)| < \epsilon$??

19/8/18

Finding complex roots of a polynomial eq^n.



$$x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1} x + a_n = 0$$

$$(x^2+px+q)(x^{n-2} + b_1 x^{n-3} + b_2 x^{n-4} + \dots + b_{n-3} x + b_{n-2}) + \underbrace{Rx + S}_{\text{remainder}} = 0.$$

p_0, q_0 $\quad (x^2 + p x_0 + q_0)$

Linear remainder term

$\alpha \rightarrow R = p b_{n-2}$ and $= q b_{n-3}$

This should be equal to 0 of the original polynomial eqn exactly dividisle by (x^2+px+q).

$$x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1} x + a_n$$

$$= (x^2+px+q)(x^{n-2} + b_1 x^{n-3} + b_2 x^{n-4} + \dots + b_{n-3} x + b_{n-2}) + Rx + S.$$

$$= x^n + (p+b_1) x^{n-1} + (q + b_1 p + b_2) x^{n-2} + \dots + (q b_1 + p b_2 + b_3) x^{n-3}$$

Equating the co-efficients of
eqn.

$a_1 = b_1 + p \longrightarrow (i)$

$a_2 = b_2 + pb_1 + q \longrightarrow (ii)$

$a_3 = b_3 + pb_2 + q^2 \longrightarrow (iii)$

$a_j = b_j + pb_{j-1} + qb_{j-2} \longrightarrow (iv)$

$b_1 = a_1 - p$

$b_2 = a_2 - pb_1 - q$

(3) $b_3 = a_3 - pb_2 - qb_1$

$b_j = a_j - pb_{j-1} - qb_{j-2}$

$j = 3, 4, \ldots,$

$R = a_{n-1} - pb_{n-2} - qb_{n-3} \longrightarrow (4)$

$S = a_n - qb_{n-2} \longrightarrow (5)$.

$\frac{\partial R}{\partial p} = -b_{n-2} - p\frac{\partial b_{n-2}}{\partial p} - q\frac{\partial b_{n-3}}{\partial p}$

$\frac{\partial S}{\partial p} = -q\frac{\partial b_{n-2}}{\partial p}$.

~~Series~~ Since the a_j, b_j values are known

Solving (4), (5) we get R, S

$\frac{\partial S}{\partial p} = -q\frac{\partial b_{n-2}}{\partial p}$.

$\max(|p_0 - p_1|, |q_0 - q_1|) < \epsilon$.

$\min(|p_{i+1} - p_i|, |q_{i+1} - q_i|) < \epsilon$.

$\frac{\partial S}{\partial q} = -q\frac{\partial b_{n-2}}{\partial q} - b_{n-2}$.

Bairstow's method:

$R(p, q) = 0$

$S(p, q) = 0$.

Let, (p_0, q_0) be the approximations to (p, q).

and $\Delta p, \Delta q$ be the correction method to get the values of (p, q).

$$R(p_0 + \Delta p, q_0 + \Delta q) = 0 \Rightarrow R(p_0, q_0) + \Delta p \left.\frac{\partial R}{\partial p}\right|_{(p_0, q_0)} + \Delta q \left.\frac{\partial R}{\partial q}\right|_{(p_0, q_0)} + \cdots = 0$$

$$S(p_0 + \Delta p, q_0 + \Delta q) = 0$$

$\Downarrow$

$$S(p_0, q_0) + \Delta p \left.\frac{\partial S}{\partial p}\right|_{(p_0, q_0)} + \Delta q \left.\frac{\partial S}{\partial q}\right|_{(p_0, q_0)} + \cdots = 0$$

Neglect ~~the~~ the higher order terms

$$R(p_0, q_0) + \Delta p \left.\frac{\partial R}{\partial p}\right|_{(p_0, q_0)} + \Delta q \left.\frac{\partial R}{\partial q}\right|_{(p_0, q_0)} = 0$$

$$S(p_0, q_0) + \Delta p \left.\frac{\partial S}{\partial p}\right|_{(p_0, q_0)} + \Delta q \left.\frac{\partial S}{\partial q}\right|_{(p_0, q_0)} = 0$$

$$\Delta p \left.\frac{\partial R}{\partial p}\right|_{(p_0, q_0)} + \Delta q \left.\frac{\partial R}{\partial q}\right|_{(p_0, q_0)} = -R(p_0, q_0) \longrightarrow (i)$$

$$\Delta p \left.\frac{\partial S}{\partial p}\right|_{(p_0, q_0)} + \Delta q \left.\frac{\partial S}{\partial q}\right|_{(p_0, q_0)} = -S(p_0, q_0) \longrightarrow (ii)$$

$$\begin{array}{ccc} \frac{\partial R}{\partial p} & \frac{\partial R}{\partial q} & R \\ \frac{\partial S}{\partial p} & \frac{\partial S}{\partial q} & S \end{array}$$

$S = b[5]$

$R = b[4]$

$$\Delta p = \frac{S \frac{\partial R}{\partial q} - R \frac{\partial S}{\partial q}}{\left.\frac{\partial R}{\partial p}\right|_{(p_0, q_0)} \left.\frac{\partial S}{\partial q}\right|_{(p_0, q_0)} - \left.\frac{\partial S}{\partial p}\right|_{(p_0, q_0)} \left.\frac{\partial R}{\partial q}\right|_{(p_0, q_0)}}$$

$$\Delta q = \frac{R \frac{\partial S}{\partial p} - S \frac{\partial R}{\partial p}}{\left.\frac{\partial R}{\partial p}\right|_{(p_0, q_0)} \left.\frac{\partial S}{\partial q}\right|_{(p_0, q_0)} - \left.\frac{\partial S}{\partial p}\right|_{(p_0, q_0)} \left.\frac{\partial R}{\partial q}\right|_{(p_0, q_0)}}$$

$$c_1 = \frac{\partial b_1}{\partial p} = -1$$

$$c_2 = \frac{\partial b_2}{\partial p} = -p\frac{\partial b_1}{\partial p} - b_1 = p - b_1$$

$$c_3 = \frac{\partial b_3}{\partial p} = -p\frac{\partial b_2}{\partial p} - b_2 - q\frac{\partial b_1}{\partial p}$$

$$= -pc_2 - b_2 - qc_1$$

$$c_j = -b_{j-1} - p\,c_{j-1} - q\,c_{j-2}$$

$$d_1 = 0$$

$$d_2 = -1$$

$$d_j = -p\,d_{j-1} - q\,d_{j-2} - b_{j-2}.$$

Solution of linear simalteneous eq^n

$$x_1 + 5x_2 + 3x_3 = 10 \quad \text{—— (1)}$$

$$x_1 + 3x_2 + 2x_3 = 5 \quad \text{—— (2)}$$

$$2x_1 + 4x_2 - 6x_3 = -4. \quad \text{—— (3)}$$

if $\begin{bmatrix} 1 & 5 & 3 \\ 1 & 3 & 2 \\ 2 & 4 & -6 \end{bmatrix} = 0$ no sol^n.

$\neq 0$ unique ".

① - ②

$$2x_2 + x_3 = 5. \longrightarrow ④$$

2 × ② - ③

$$2x_2 + 10x_3 = 14 \longrightarrow ⑤$$

$$-9x_3 = -9$$

$x_3 = 1$. $\quad 2x_2 + 1 = 5$

$x_2 = 2$

Gaussian elimination method:

① - ② ⇒ $-2x_2 - x_3 = -5$. —— ④

2① - ③ ⇒ $-6x_2 - 12x_3 = -24$. ←—— ⑤

⑤ - ④ × 3

$$-6x_2 - 12x_3 = -24$$

$$-6x_2 - 3x_3 = -15$$

$$-9x_3 = -9 \qquad x_3 = 1$$

$$x_1 + 5x_2 + 3x_3 = 10 \quad \text{— ①} \qquad x_1$$

$$-2x_2 - x_3 = -5 \quad \text{— ④} \qquad x_2$$

$$-9x_3 = -9 \quad \text{— ⑤} \qquad x_3$$

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1(n-1)}x_{n-1} + a_{1n}x_n = a_{1(n+1)}$$

$$① \times \frac{a_{21}}{a_{11}} \Rightarrow a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2(n-1)}x_{(n-1)} + a_{2n}x_n = a_{2(n+1)}$$

$$① \times \frac{a_{31}}{a_{11}} \Rightarrow a_{31}x_1 + \text{— —} = \text{—}$$

(n-1) different eqn.

$$a_{n1}x_1 + a_{n2}x_2 + \text{— —} + a_{nn}x_n = a_{n(n+1)}$$

$$m_{i1} = \frac{a_{i1}}{a_{11}} \qquad i = 2, \dots, n.$$

$$a_{ij} = a_{ij} - m_{i1} * a_{1j} \qquad i = 2, \dots, n. \quad j = 1, 2, \dots, n+1.$$

$$m_{i2} = \frac{a_{i2}}{a_{22}}$$

$$a_{ij} = a_{ij} - m_{i2} * a_{2j} \qquad i = 3, \dots, n. \quad j = 2, \dots, n+1.$$

at any stage k

$$a_{kk}x_k + a_{k(k+1)}x_{k+1} + \dots + a_{kn}x_n = a_{k(n+1)}$$

$$a_{(k+1)k}x_k + a_{(k+1)(k+1)}x_{k+1} + \dots + a_{(k+1)n}x_n = a_{(k+1)(n+1)}$$

$$a_{nk}x_k + a_{n(k+1)}x_{k+1} + \dots + a_{nn}x_n = a_{n(n+1)}$$

$$m_{ik} = \frac{a_{ik}}{a_{kk}}$$

$$a_{ij} = a_{ij} - m_{ik} * a_{kj}$$

if, $|a_{11}| << |a_{21}|$. Overflow

$|a_{kk}| << |a_{ik}|$

To avoid it a_{kk} taken max by swaping max a_{ki} with a_{kk}.

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & a_{1(n+1)} \\ a_{21} & a_{22} & \dots & a_{2n} & a_{2(n+1)} \\ \vdots & & & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} & a_{n(n+1)} \end{array}\right]$$

$\Downarrow$

$n \times (n+1)$

$$\left[\begin{array}{ccccc} a_{11} & a_{12} & \dots & a_{1n} & a_{1(n+1)} \\ & a_{22} & \dots & a_{2n} & a_{2(n+1)} \\ & & \ddots & \vdots & \vdots \\ & & & a_{nn} & a_{n(n+1)} \end{array}\right]$$

$$x_n = \frac{a_{n(n+1)}}{a_{nn}}$$

$$x_{n-1} = \frac{a_{n-1(n+1)} - a_{n-1\,n}\,x_n}{a_{n-1\,n-1}}$$

$$x_i = \frac{a_{i(n+1)} - \sum_{j=i+1}^{n} a_{ij}\,x_j}{a_{ii}}$$

Input the co-efficient matrix A[n][n+1]

```
for (k=1; k<n; k++)
{
    big=0
    for(i=k; i<=n; i++)
    {
     if (fabs a[i][k] > big)   // finding coeff with largest magnitude along the column A,
       {
         big = a[i][k];
         p = i
       }
    }
    for (j=k; j<=(n+1); j++)   // swapping the row containing the largest magnitude coeff in the k th column with in the k th column with k th row
     { temp = a[k][j];
       a[k][j] = a[p][j];
         a[p][j] = temp;
     }
```

```
for(i=k+1; i<=n; i++)
{
    m[i][k] = a[i][k]/a[k][k];

    for(j=k; j<=(n+1); j++)

        a[i][j] = a[i][j] - m[i][k] * a[k][j];

    }
}  // elimination process is complete.

for(i=(n-1); i>=1; i--)
{
    sum=0;
    for(j=n; j<i; j--)
        sum += a[i][j] * x[j];

    x[i] = (a[i][n+1] - sum) / a[i][i];
}
```

Numerical Analysis

$x_1 + 5x_2 + 3x_3 = 10$

$x_1 + 3x_2 + 2x_3 = 5$

$2x_1 + 4x_2 - 6x_3 = -4$

Gaussian Elimination

augmented co-eff matrix = $\left[\begin{array}{ccc|c} 1 & 5 & 3 & 10 \\ 1 & 3 & 2 & 5 \\ 2 & 4 & -6 & -4 \end{array}\right]$

$\downarrow$

$\left[\begin{array}{ccc|c} 2 & 4 & -6 & -4 \\ 1 & 3 & 2 & 5 \\ 1 & 5 & 3 & 10 \end{array}\right]$ $R_1 \leftrightarrow R_3$

$\downarrow$

$\left[\begin{array}{ccc|c} 2 & 4 & -6 & -4 \\ 0 & 1 & 5 & 7 \\ 0 & 3 & 6 & 12 \end{array}\right]$ $R_2 \rightarrow R_2 - R_1/2$

$\downarrow$

$\left[\begin{array}{ccc|c} 2 & 4 & -6 & -4 \\ 0 & 3 & 6 & 12 \\ 0 & 1 & 5 & 7 \end{array}\right]$ $R_2 \leftrightarrow R_3$

$\downarrow$

$\left[\begin{array}{ccc|c} 2 & 4 & -6 & -4 \\ 0 & 3 & 6 & 12 \\ 0 & 0 & 3 & 3 \end{array}\right]$ $R_3 \rightarrow R_3 - R_2/3$

$3x_3 = 3 \quad \therefore x_3 = 1$

$3x_2 + 6 \cdot 1 = 12 \Rightarrow 3x_2 = 6 \Rightarrow x_2 = 2$.

$2x_1 + 4 \cdot 2 + (-6) \cdot 1 = -4$.

$\Rightarrow \quad 2x_1 + 2 = -4$.

$\Rightarrow \quad x_1 = -3$.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = a_{1(n+1)}$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = a_{2(n+1)}.$$

$$\vdots$$

$$a_{(n-1)1}x_1 + a_{(n-1)2}x_2 + \dots + a_{(n-1)n}x_n = a_{(n-1)(n+1)}$$

$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = a_{n(n+1)}.$$

$$m_{ik} = \frac{a_{ik}}{a_{kk}} \qquad i = k+1, \dots, n.$$

Total no of divisions

$$(n-1) + (n-2) + \dots + 1 = \frac{n(n-1)}{2}.$$

no of multiplications required for the elimination process for eliminating, the co-eff of x from 2nd row to n th row

no of multiplications $= (n+1)(n-1)$

$= n^2 - 1$.

~~Suffers~~

Suffixes at any stage of elimination we are left with the k- eqns with unknown for eliminating the co-efficients along first column, ~~____~~ of

Total no of multiplications required

$$\sum_{k=n}^{2} k^2 - 1$$

$$= \left[n^2 + (n-1)^2 + \cdots + 2^2\right] - \left[n + (n-1) + \cdots + 2\right].$$

Back substitution : $\frac{n(n+1)}{2}$.

Gauss Jordan elimination

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = a_{1(n+1)}$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = a_{2(n+1)}$$

$$\vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = a_{n(n+1)}$$

$$\left[\begin{array}{cccc|c} a_{11} & 0 & \cdots & 0 & a_{1(n+1)} \\ 0 & a_{22} & \cdots & 0 & a_{2(n+1)} \\ \vdots & & \ddots & 0 & \vdots \\ & & & a_{nn} & a_{n(n+1)} \end{array}\right]$$

$$\Downarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & \frac{a_{1(n+1)}}{a_{11}} \\ 0 & 1 & & 0 & \frac{a_{2(n+1)}}{a_{22}} \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & & 1 & \frac{a_{n(n+1)}}{a_{nn}} \end{array}\right]$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \\ \vdots & \vdots & & \vdots \\ & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Let the sol^n of system of linear eq^n: $\begin{bmatrix} c_{11} \\ c_{21} \\ \vdots \\ c_{n1} \end{bmatrix}$.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} c_{12} \\ c_{22} \\ \vdots \\ c_{n2} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}.$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & \cdots & \cdots & c_{2n} \\ \vdots & & & \\ c_{n1} & \cdots & \cdots & c_{nn} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ & \vdots & & \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Inverse of a matrix

$$\left[\begin{array}{cccc|cccc} a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 & 0 & \cdots & 1 \end{array}\right]$$

$$\Downarrow$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & c_{11} & c_{12} & \cdots & c_{1n} \\ 0 & 1 & \cdots & 0 & c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & & & \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & 1 & c_{n1} & & \cdots & c_{nn} \end{array}\right]$$

LU decomposition method:

$A = [a_{ij}]$ $\quad X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ $\quad B = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$A = LU$

$$L = \begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & & \ddots & 0 \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & u_{12} & \cdots & u_{1n} \\ 0 & 1 & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$\begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & & & \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & 1 & & & \\ 0 & 0 & 1 & & \\ \vdots & & & & \\ 0 & \cdots & & & 1 \end{bmatrix}$$

$$\cancel{Ux = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}$$

$Ux =$

$$\begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & & \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}.$$

$$\begin{bmatrix} l_{11} & & \\ & l_{22} & \\ & & \ddots \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

$i = 1, 2, \ldots, n$

$l_{ij} = a_{ij}$

NUMERICAL ANALYSIS

Iterative method for linear simultaneous eq^n

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n.$$

$$\left[x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)}\right] = \left[X^{(0)}\right]^T$$

$$X^{(1)} = \left[x_1^{(1)}\, x_2^{(1)} \dots x_n^{(1)}\right]^T.$$

$$\left.\begin{aligned}
a_{11}x_1^{(1)} &= b_1 - \left(a_{12}x_2^{(0)} + \dots + a_{1n}x_n^{(0)}\right) \\
a_{22}x_2^{(1)} &= b_2 - \left(a_{21}x_1^{(0)} + \dots + a_{2n}x_n^{(0)}\right) \\
&\vdots \\
a_{nn}x_n^{(1)} &= b_n - \left(a_{n1}x_1^{(0)} + \dots + a_{nn}x_n^{(0)}\right)
\end{aligned}\right\} \quad (1)$$

check if

$$\max_{1 \le i \le n} \left[\left\{x_i^{(1)} - x_i^{(0)}\right\}\right] \le \epsilon.$$

$$a_{11}x_1^{(k+1)} = b_1 - \left(a_{12}x_2^{(k)} + \dots + a_{1n}x_n^{(k)}\right)$$

$$a_{nn}x_n^{(k+1)} = b_n - \left(a_{n1}x_1^{(k)} + \dots + a_{nn}x_n^{(k)}\right)$$

Jacobi's method

In matrix notation

$$x^{(k+1)} = Hx^{k} + c$$

where $x^{(k+1)}$, x^{k} are $n \times 1$ (i.e. column) vector.

H is a $n \times n$ matrix depending on A

and c is again a column vector depending on A, B.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

$$L = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ a_{21} & 0 & \cdots & 0 & 0 \\ a_{31} & a_{32} & \cdots & 0 & 0 \\ \vdots & & & & \\ a_{n1} & a_{n2} & \cdots & a_{n-1} & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 \\ 0 & a_{22} & & & \\ & & \ddots & & \\ 0 & & & & a_{nn} \end{bmatrix}$$

$$U = \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & \cdots & a_{2n} \\ \vdots & & & \vdots \\ 0 & 0 & 0 & 0 \cdots 0 \end{bmatrix}$$

In matrix notation eq^n (1) can be written as

$$\boxed{Dx^{(k+1)} = B - (L+U)X^{(k)}}$$

or, $X^{(k+1)} = D^{-1}\left[B - (L+U)X^{(k)}\right]$

$$\Rightarrow x^{(k+1)} - x^{(k)} = D^{-1}\{B - (L+U)x^k\} - x^k$$

$$\Rightarrow x^{(k+1)} - x^k = D^{-1}\{B - (L+U)x^k - Dx^k\}$$

$$\Rightarrow x^{k+1} - x^k = D^{-1}\left[B - (L+U+D)x^k\right]$$

$$\Rightarrow \boxed{x^{k+1} - x^k = D^{-1}(B - Ax^k)}$$

$$x^{(k+1)} = D^{-1}\left[B - (L+U)x^k\right]$$

$$x^{k+1} = X + E^{k+1}$$

$$x^k = X + E^k$$

E^{k+1}, E^k are the error vectors associated with x^{k+1}, x^k resp.

$$X + E^{(k+1)} = D^{-1}\left[B - (L+U)(X+E^k)\right]$$

$$\Rightarrow E^{k+1} = D^{-1}\left[B - (L+U)X - (L+U)E^k\right] - X$$

$$\Rightarrow E^{k+1} = D^{-1}\left[B - \underbrace{(L+U+D)}_{A}X - (L+U)E^k\right]$$

$$\Rightarrow E^{k+1} = D^{-1}\left[B - AX - (L+U)E^{(k)}\right]$$

$$\therefore E^{k+1} = -D^{-1}(L+U)E^{(k)}$$

for convergence:

$$\|-D^{-1}(L+U)\| < 1$$

$\Rightarrow \quad \| -D^{-1}(A-D) \| < 1$

$\Rightarrow \quad \| I - D^{-1}A \| < 1$.

Let, $(I - D^{-1}A)Y = \lambda Y$ where λ is eigenvalue and Y is the corresponding eigenvector of the non zero column vector.

or, $(D-A)Y = \lambda D Y$.

or, $-\sum_{\substack{i=1 \\ i \neq j}}^{n} a_{ij} y_j = \lambda a_{ii} y_i \qquad i = 1, \ldots, n \quad j = 1, \ldots, n$.

$\therefore \lambda = \left| \dfrac{\sum_{\substack{i=1 \\ i \neq j}}^{n} a_{ij}}{|a_{ii}|} \right| \qquad i = 1, \ldots, n$

for convergence $\lambda < 1$.

$$\frac{\sum_{\substack{i=1 \\ i \neq j}}^{n} |a_{ij}|}{|a_{ii}|} < 1.$$

$$\Rightarrow \sum_{\substack{i=1 \\ i \neq j}}^{n} |a_{ij}| < |a_{ii}| \qquad i = 1, \ldots, n.$$

$x + 6y = 4$

$x - 2y - 6z = 14$

$9x + 4y + z = -17$

$$A = \begin{bmatrix} 1 & 6 & 0 \\ 1 & -2 & -6 \\ 9 & 4 & 1 \end{bmatrix}$$

can not be done iterative

$1 < 6 + 0$

$|-2| < |1| + |-6|$

$1 < 9 + 4$

$$\left[\begin{array}{ccc|c} 1 & 6 & 0 & 4 \\ 1 & -2 & -6 & 14 \\ 9 & 4 & 1 & -17 \end{array}\right].$$

$$\Rightarrow \left[\begin{array}{ccc|c} 9 & 4 & 1 & -17 \\ 1 & -2 & -6 & 14 \\ 1 & 6 & 0 & 4 \end{array}\right] \quad R_1 \leftrightarrow R_3$$

$$\Rightarrow \left[\begin{array}{ccc|c} 9 & 4 & 1 & -17 \\ 1 & 6 & 0 & 4 \\ 1 & -2 & -6 & 14 \end{array}\right] \quad R_2 \leftrightarrow R_3$$

$$A = \begin{bmatrix} 9 & 4 & 1 \\ 1 & 6 & 0 \\ 1 & -2 & -6 \end{bmatrix} \quad B = \begin{bmatrix} -17 \\ 4 \\ 14 \end{bmatrix}.$$

$$D = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -6 \end{bmatrix}. \quad D^{-1} = \begin{bmatrix} \frac{1}{9} & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & -\frac{1}{6} \end{bmatrix},$$

$x^{(0)} = y^{(0)} = z^{(0)} = 0$

$x^{(1)} = -1.9 \quad y^{(1)} = 0.67 \quad z^{(1)} = -2.33$

$x^{(2)} = 1.93, \quad y^{(2)} = 0.98 \quad z^{(2)} = -2.87$

$x^{(3)} = -2.006 \quad y^{(3)} = 0.99 \quad z^{(3)} = -2.98$

Gauss-Seidel method:

$$a_{11} x_1^{(k+1)} = b_1 - (a_{12} x_2^k + \dots + a_{1n} x^k)$$

$$a_{22} x_2^{(k+1)} = b_2 - (a_{21} x_1^{(k+1)} + \dots + a_{2n} x^k)$$

$$a_{n-1} x_{n-1} + a_n x_n = c_{n-1}$$

$$a_{nn} x_n^{(k+1)} = b_n - (a_{n1} x_1^{(k+1)} + \dots + a_{n,n-1} x_{n-1}^{(k+1)})$$

$$a_{11} x_1^{(k+1)} = b_1 - (a_{12} x_2^{(k)} + \dots + a_{1n} x_n^{(k)})$$

$$a_{21} x_1^{(k+1)} + a_{22} x_2^{(k+1)} = b_1 - (a_{23} x_2^{(k)} + \dots + a_{1n} x_n^{(k)})$$

$$a_{31} x_1^{(k+1)} + a_{32} x_2^{(k+1)} + a_{33} x_3^{(k+1)}$$

$$= b_1 - (a_{34} x_4^{(k)} + \dots + a_{3n} x_n^{(k)})$$

$$a_{n1} x_1^{(k+1)} + a_{n2} x_2^{(k+1)} + \dots + a_{nn} x_n^{(k+1)} = b_n$$

$$\boxed{(L+D)\, x^{(k+1)} = B - Ux^{(k)}}$$

$$\Rightarrow X^{(k+1)} = (L+D)^{-1} (B - UX^{(k)}).$$

$$\Rightarrow X^{(k+1)} - X^{k} = (L+D)^{-1} [B - UX^{(k)} - (L+D)X^{(k)}].$$

$$\Rightarrow X^{k+1} - X^{k} = (L+D)^{-1} [B - AX^{(k)}].$$

$$X^{k+1} = X + E^{(k+1)}.$$

$$X^{k} = X + E^{(k)}.$$

$$\cancel{X^{k+1} = (L+D)}$$

$$X - E^{(k+1)} = (L+D)^{-1} [B - UX - UE^{(k)}],$$

$$\Rightarrow E^{(k+1)} = -(L+D)^{-1} UE^{(k)}$$

Iterative method to determine A^{-1}

Let B be an approx inverse of A

$$AB \neq I$$

Let,

$$AB = I + E$$

premultiplying both sides with A^{-1}

$$A^{-1}AB = A^{-1}(I+E)$$

$$B = A^{-1}(I+E)$$

Post multiplying both sides with $(I+E)^{-1}$.

$$A^{-1} = B(I+E)^{-1} \approx B(I - E + E^2 - E^3 + \dots)$$

If $\|E\| < 1$ we can write:

$$A^{-1} = B(I-E) \approx B(I - AB + I) \approx B(\cancel{2*}I - AB)$$

Let $B^{(k+1)}$, $B^{(k)}$ be the approximate inverse

at $(k+1), k$ at iteration then

$$B^{(k+1)} = B^{(k)}\left(2 * I - AB^{(k)}\right).$$

$$k = 1, 2, 3, \dots$$

continuing iteration method

$$\max_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \left| b_{ij}^{(k+1)} - b_{ij}^{(k)} \right| < \epsilon.$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.$$

$$B^{(0)} = \begin{bmatrix} 1.8 & -0.9 \\ -0.9 & 0.9 \end{bmatrix}$$

$$AB^{(k+1)} = AB^{(k)}\left(2I - AB^{(k)}\right).$$

$$AB^{(k+1)} = 2\,AB^{(k)} - \left\{AB^{(k)}\right\}^2$$

$$AB^{(k+1)} - I = 2\,AB^{(k)} - \left[AB^{(k)}\right]^2 - I$$

$$AB^{(k+1)} - I = -\left(AB^{k} - I\right)^2$$

$$E^{(k+1)} = \left\{E^{(k)}\right\}^2.$$

convergence is of order 2.

Sol^n of non-linear simalteneous eq^n

$$x - \sin(x+y) = 0$$

$$y - \cos(x+y) = 0.$$

$f_1(x_1, \ldots, x_n) = 0 \rightarrow x_1 = F_1(x_1, x_2, \ldots, x_n)$

$f_2(x_1, \ldots, x_n) = 0 \rightarrow x_2 = F_2(x_1, x_2, \ldots, x_n)$

$\vdots$

$f_n(x_1, x_2, \ldots, x_n) = 0 \rightarrow x_n = F_n(x_1, x_2, \ldots, x_n)$

$$x_1^{(k+1)} = F_1\left(x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}\right)$$

$$x_2^{(k+1)} = F_2\left(x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}\right)$$

$$\vdots$$

$$x_n^{(k+1)} = F_n\left(x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}\right).$$

$$\max_{1 \le i < n} \{ x_i^{(k+1)} - x_i^{(k)} \} < \epsilon$$

For convergence

$$\sum_{i=1}^{n} \left| \frac{\partial F_i}{\partial x_i} \right| < 1 .$$

Newton - Raphson Method

Let $x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}$ be the initial approximination and $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$

from

$$f_1(x_1^{(0)} + \Delta x_1, x_2^{(0)} + \Delta x_2, \ldots, x_n^{(0)} + \Delta x_n) = 0$$

$$f_2(x_1^{(0)} + \Delta x_1, x_2^{(0)} + \Delta x_2, \ldots, x_n^{(0)} + \Delta x_n) = 0$$

$$\vdots$$

$$f_n(x_1^{(0)} + \Delta x_1, x_2^{(0)} + \Delta x_2, \ldots, x_n^{(0)} + \Delta x_n) = 0.$$

$$f_1(x_1^{(0)}) + \Delta x_1 \left.\frac{\partial f_1}{\partial x_1}\right| + \Delta x_2 \left.\frac{\partial f_1}{\partial x_2}\right| + \text{———} + \Delta x_n \frac{\partial f_1}{\partial x_n} = 0$$

$$\max(|\Delta x_i|) < \epsilon.$$

Transform method for finding all the eigenvalues and corresponding eigen vectors

$\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of a $n \times n$ matrix A

and $x_1, x_2, \ldots, x_n$ are the corresponding eigen ~~matrix~~ vectors.

$$Ax_1 = \lambda_1 x_1$$

$$Ax_2 = \lambda_2 x_2$$

$$\vdots$$

$$Ax_n = \lambda_n x_n.$$

Let, $x_1 = \begin{bmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{bmatrix}$, $x_2 = \begin{bmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{bmatrix}$

$x_n = \begin{bmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{bmatrix}$

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$$A = \begin{bmatrix} \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix} & \begin{pmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{pmatrix} & \cdots & \begin{pmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{pmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$$\begin{bmatrix} \lambda_1 x_{11} & \lambda_2 x_{12} & \cdots & \lambda_n x_{1n} \\ \lambda_1 x_{21} & \lambda_2 x_{12} & \cdots & \lambda_n x_{2n} \\ \lambda_1 x_{n1} & \lambda_2 x_{n2} & \cdots & \lambda_n x_{nn} \end{bmatrix}$$

$$V = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix}$$

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$$AV = VD.$$

$$V^{-1}AV = V^{-1}VD = D.$$

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \qquad A^T = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

$$V = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$V^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = V^T$$

$V^{-1} A V$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} a\cos\theta + b\sin\theta & b\cos\theta + c\sin\theta \\ -a\sin\theta + b\cos\theta & -b\sin\theta + c\cos\theta \end{bmatrix}$$

$$\times \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$= [\; a\cos^2\theta + b\sin\theta\cos\theta + b\sin\theta\cos\theta$

$+ c\sin^2\theta$

$- a\sin\theta\cos\theta$

$- b\sin^2\theta$

$+ b\cos^2\theta + c\sin\theta\,\cdots$

$+ c\cos\theta\sin$

$-a\sin\theta\cos\theta + b\cos^2\theta - b\sin\theta\cos\theta$

$$= \begin{bmatrix} a\cos^2\theta + b\sin\theta\cos\theta + b\cos\theta\sin\theta + c\sin^2\theta & -a\sin\theta\cos\theta - b\sin^2\theta + b\cos^2\theta + c\sin\theta\cos\theta \\ -a\sin\theta\cos\theta + b\cos^2\theta - b\sin^2\theta + c\sin\theta\cos\theta & a\sin^2\theta - b\sin\theta\cos\theta - b\sin\theta\cos\theta + c\cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} a\cos^2\theta + 2b\sin\theta\cos\theta + c\sin^2\theta & (c-a)\sin\theta\cos\theta + b(\cos^2\theta - \sin^2\theta) \\ (c-a)\sin\theta\cos\theta + b(\cos^2\theta - \sin^2\theta) & a\sin^2\theta - 2b\sin\theta\cos\theta + c\cos^2\theta \end{bmatrix}$$

Since, Diagonal matrix

$$(c-a)\sin\theta\cos\theta + b(\cos^2\theta - \sin^2\theta) = 0$$

$$\Rightarrow \quad b\cos 2\theta = \frac{a-c}{2}\sin 2\theta$$

$$\Rightarrow \quad \tan 2\theta = \frac{2b}{a-c}$$

$$\therefore \quad \theta = \frac{1}{2}\tan^{-1}\left(\frac{2b}{a-c}\right)$$

one eigen value $= a\cos^2\theta + 2b\sin\theta\cos\theta + c\sin^2\theta$. one eigen vector $= \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$

another eigen value $= a\sin^2\theta - 2b\sin\theta\cos\theta + c\cos^2\theta$

$$= \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$|a_{rs}|$ $|a_{sr}|$

$P_1^{-1} A =$

$P_1^T A P_1 = A_1$

A B

$P^{-1} \cancel{BB} = A_1$

$A_1 = P_1^T A P_1$

$A_2 = P_2^T A_1 P_2$

$= P_2^T P_1^T A P_1 P_2$

$= (P_1 P_2)^T A (P_1 P_2)$

$\vdots$

$\boxed{A_z} = P_z^T P_{z-1}^T \cdots P_2^T P_1^T A P_1 P_2 \cdots P_{z-1} P_z$

Purely

diagonal matrix

$P = P_1 P_2 \cdots P_{z-1} P_z$

Let, $|a_{rs}^{(k-1)}|$ $(s>r)$ be the largest magnitude off-diagonal element of the intermediate simillar matrix A_{k-1}.

then the P_k is formed as,

$$\left.\begin{aligned} p_{rr}^{(k)} &= p_{ss}^{(k)} = \cos\theta \\ -p_{rs}^{(k)} &= p_{rs}^{(k)} = \sin\theta \end{aligned}\right\}$$

$P_{ii} = 1$ for $i \neq r, s$,

$P_{ij} = 0$ otherwise

where $\theta = \frac{1}{2}\tan^{-1}\left(\dfrac{2a_{rs}^{(k-1)}}{a_{rr} - a_{ss}}\right)$

$$A_K = P_K^T A_{K-1} P_K.$$

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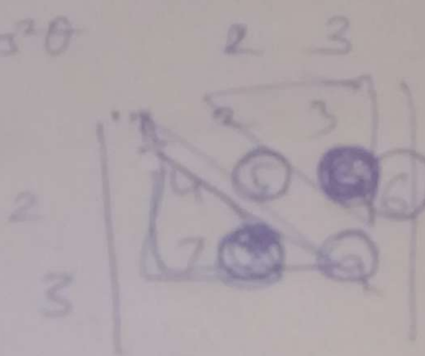
$$a_{ir}^{(k)} = a_{ri}^{(k)} = a_{ir}^{(k-1)}\cos\theta + a_{is}^{(k-1)}\sin\theta$$

$$a_{is}^{(k)} = a_{si}^{(k)} = \cancel{a_{is}^{(k-1)}\sin\theta\cos\theta} - a_{ir}^{(k-1)}\sin\theta + a_{is}^{(k-1)}\cos\theta$$

$$a_{rr}^{(k)} = a_{rr}^{(k-1)}\cos^2\theta + 2a_{rs}^{(k-1)}\sin\theta\cos\theta + a_{ss}^{(k-1)}\sin^2\theta$$

$$a_{ss}^{(k)} = a_{rr}^{(k-1)}\sin^2\theta - 2a_{rs}^{(k-1)}\cos\theta\sin\theta + a_{ss}^{(k-1)}\cos^2\theta$$



$$a_{rs}^{(k)} = a_{sr}^{(k)} = 0$$

$$a_{ij}^{(k)} = a_{ij}^{(k-1)}$$

otherwise.

Jacobi's method for finding the eigenvalues and eigenvectors for symmetric matrices.

$$A = \begin{vmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{vmatrix}$$

largest magnitude off-~~dimension~~diagonal element $a_{13} = a_{31} = 2$.

$p_{11} = p_{33} = \cos\theta$.

$-p_{13} = p_{31} = \sin\theta$

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2a_{13}}{a_{11} - a_{33}}\right)$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{2a_{13}}{0}\right)$$

$$= \frac{1}{2}\tan^{-1}(\infty)$$

$$= \frac{1}{2} \times \frac{\pi}{2}$$

$$= \frac{\pi}{4}$$

$$P_1 = \begin{bmatrix} \cos\frac{\pi}{4} & 0 & -\sin\frac{\pi}{4} \\ 0 & 1 & 0 \\ \sin\frac{\pi}{4} & 0 & \cos\frac{\pi}{4} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

$$\Lambda = P_1^T A P_1$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{\sqrt{2}} & 2 & \frac{3}{\sqrt{2}} \\ \sqrt{2} & 3 & \sqrt{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$A_2 = P_2^T A_1 P_2$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$a_{12} = 2$.

$a_{11} = 3$

$a_{22} = 3$.

$a_{21} = 0$.

$$V = P_1 P_2$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Rutishauser method for arbitrary matrix :

$$A = LU$$

where $U = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{bmatrix}$

$$L = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ l_{12} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \cdots & 1 \end{bmatrix}$$

$$AU^{-1} = LUU^{-1} = L$$

$$A = LU$$

$$= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ l_{12} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \cdots & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{bmatrix}$$

$$A_1 = UAU^{-1} = UL$$

Let $A_1 = L_1 U_1$ where $(l_{ii}^{(1)} = 1)$,

$$A_2 = U_1 A_1 U_1^{-1} = U_1 L_1$$

This is continued until matrix is reduced to an upper triangular ~~This is co~~ ~~matri~~ form.

Then the diagonal

$$A_1 = UAU^{-1} = UL$$

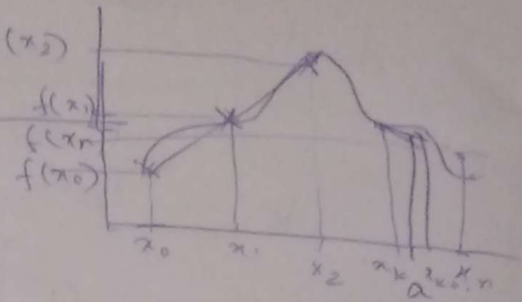
Let $A_1 = L_1 U_1$ (where $L_{ii}^{(1)} = 1$)

$$A_2 = U_1 A_1 U_1^{-1} = U_1 L_1$$

This is continued untill A_2 matrix is reduced to an upper ~~~~ triangular form. Then the diagonal ~~~~ element untill represent the eigenvalue of A.

Interpolation

x_i	$y_i = f(x_i)$
x_0	y_0
x_1	y_1
$\vdots$	$\vdots$
x_n	y_n



$f(x) = ?$ when $x = a$

Assumed line btwn x_k, x_{k+1} is linear

$$x_k \leq a \leq x_{k+1}$$

$$\frac{y - y_k}{x - x_k} = \frac{y_{k+1} - y_k}{x_{k+1} - x_k}$$

$$y = y_k + \left(\frac{y_{k+1} - y_k}{x_{k+1} - x_k}\right)(x - x_k)$$

$$y\Big|_{at\ x = a} = y_k + \left(\frac{y_{k+1} - y_k}{x_{k+1} - x_k}\right)(a - x_k)$$

$(n+1)$ tabular data.

(i) The independent variable (ie x values) are equispaced. $[x_i = x_0 + ih]$

$\boxed{\text{Newton Forward / Backward diff / Central diff / Gauss method}}$

(ii) are not equispaced.

divided difference method

Lagrange interpolation "

Iteration " "

Finite difference operators.

Forward diff (Δ)

First order diff is defined as $\Delta y_i = y_{i+1} - y_i$

$\Delta^2 y_{i+1} = \Delta^2 y_{i+1} - \Delta y_i$

x_i	y_i	Δy_i	$\Delta^2 y_i$	$\Delta^3 y_i$	$\Delta^4 y_i$
x_0	y_0	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_i$
x_1	y_1	Δy_1	$\Delta^2 y_1$	$\Delta^2 y_1$	
x_2	x_2	Δy_2	$\Delta^2 y_2$		
x_3	y_3	Δy_3			
x_4	y_4				

k th order forward diff

$\Delta^k y_i = \Delta^{k-1} y_{i+1} - \Delta^{k-1} y_i$

$$\Delta^2 y_{i+1} = (y_{i+2} - y_{i+1}) - (y_{i+1} - y_i)$$

$$= y_{i+2} - 2y_{i+1} + y_i$$

$$\Delta^3 y_i = \Delta^2 y_{i+1} - \Delta^2 y_i$$

$$= (y_{i+3} - 2y_{i+2} + y_{i+1}) - (y_{i+2} - 2y_{i+1} + y_i)$$

$$= y_{i+3} - 3y_{i+2} + 3y_{i+1} - y_i$$

Backward diff (∇)

$$\nabla y_{i+1} = y_{i+1} - y_i.$$

Second order backward diff:

$$\nabla^2 y_{i+1} = \nabla y_{i+1} - \nabla y_i$$

$\nabla^4 y_i = y_{i+2} - 2y_{i+1} + y_i$

x_i	y_i	∇y_i	$\nabla^2 y_i$	$\nabla^3 y_i$	$\nabla^4 y_i$
x_0	y_0				
x_1	y_1	∇y_1			
x_2	y_2	∇y_2	$\nabla^2 y_2$		
x_3	y_3	∇y_3	$\nabla^2 y_3$	$\nabla^3 y_3$	
x_4	y_4	∇y_4	$\nabla^2 y_4$	$\nabla^3 y_4$	$\nabla^4 y_4$

Shift operator

This operator shifts a functional value y_i to higher index

$$E\,y_i = y_{i+1}$$

Inverse shift operator $\quad y_i = E^{-1} y_{i+1}$

Equivalent current finite difference operator

$$\Delta y_i = y_{i+1} - y_i = E y_i - y_i \equiv (E-1) y_i$$

or, $\Delta \equiv (E-1)$

$$E \equiv 1 + \Delta$$

Similarly $\quad \nabla y_{i+1} = y_{i+1} - y_i = y_{i+1} - E^{-1} y_{i+1}$

$$\equiv (1 - E^{-1}) y_{i+1}$$

$$\nabla \equiv 1 - E^{-1}$$

$$\therefore E = (1-\nabla)^{-1} \equiv \frac{1}{1-\nabla}$$

1. $\Delta = E\nabla$

$$\Delta y_i = y_{i+1} - y_i$$

$$= E y_i - E y_{i-1}$$

$$= E \nabla y_i$$

$$\Delta = E\nabla$$

$$\nabla - \Delta = -\Delta\nabla$$

$$\Delta + \nabla = \quad \Delta/\nabla - \nabla/\Delta$$

2

$$\nabla y_i - \Delta y_i = \cancel{(y_i - y_{i-1})} - (y_{i+1} - y_i)$$

$$\cancel{- (y_{i+1} - y_i)}$$

$$= y_i - E^{-1} y_i - E y_i + y_i$$

$$= \cancel{(2 - E^{-1} - E) y_i}$$

$$= \cancel{2 - 1 - \Delta}$$

$$\cancel{= \frac{1}{1-E}}$$

$$\cancel{\Delta \nabla y_i}$$

$$= \cancel{-(1-E)}$$

$$= (\nabla - \Delta) y_i$$

$$= \left[\frac{1}{1-E^{-1}} - E + 1\right] y_i$$

$$= \left[\frac{1 - (1-E)(1-E^{-1})}{1-E^{-1}}\right] y_i$$

$$= \left[\frac{1 - [1 - E - E^{-1} + 1]}{1 - E^{-1}}\right] y_i$$

$$- \Delta \nabla =$$

$$= - (E - 1)\left(1 - \frac{1}{E}\right).$$

$$= - \left[E - 1 - 1 + \frac{1}{E}\right].$$

$$= \left(1 - \frac{1}{E}\right) - (E - 1)$$

3.

$$\Delta/\nabla - \nabla/\Delta$$

$$= \frac{\Delta^2 - \nabla^2}{\nabla \Delta}$$

$$= \frac{(\Delta + \nabla)(\Delta - \nabla)}{\nabla \Delta}$$

$$= \frac{\Delta + \nabla \times \nabla \Delta}{\nabla \Delta}$$

$$= \Delta + \nabla.$$

$$\boxed{y_p = E^p y_0}$$

$$y_p = (1 + \Delta)^p y_0$$

$$y_p = \left[1 + p\Delta + \frac{p(p-1)}{2}\Delta^2 + \frac{p(p-1)(p-2)}{3!}\Delta^3 + \ldots \right]$$

Newton's Forward Interpolation: near top end of the tabular value.

Newtons backward difference interpolation formula for interpolation near the ending point

$$y_p = E^{-p} y_n = (1-\nabla)^p y_n$$

$$= \left[1 - p\nabla + \frac{p(p-1)}{2!}\nabla^2 - \frac{p(p-1)(p-2)}{3!}\nabla^3 + \dots\right] y_n.$$

$$= y_n - p\nabla y_n + \frac{p(p-1)}{2!}\nabla^2 y_n - \frac{p(p-1)(p-2)}{3!}\nabla^3 y_n + \dots$$

Gauss Central difference method:

x_i	y_i	Δy_i	$\Delta^2 y_i$	$\Delta^3 y_i$	$\Delta^4 y_i$
x_{-3}	y_{-3}	Δ			
x_{-2}	y_{-2}				
x_{-1}	y_{-1}				
x_0	y_0				
x_1	y_1				
x_2	y_2				
x_3	y_3				

Let the interpolating polynomial be

$$y_p = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_{-1}) + a_4(x-x_0)(x-x_1)(x-x_{-1})(x-x_2)$$

at $x = x_i$ $\quad y_p = y_i; \; i = -n, (n-1) \ldots -1, 0, 1, 2, \ldots n$

$x = x_i$

at $x = x_0$ $\quad a_0 = y_0$

at $x = x_{-1}$ $\quad y_p = y_{-1}$

$$y_{-1} = y_0 + a_1(x_{-1} - x_0)$$

$$a_1 = \frac{y_{-1} - y_0}{x_{-1} - x_0}$$

x_i	y_i	Δy_i	$\Delta^2 y_i$	$\Delta^3 y_i$	$\Delta^4 y_i$	$\Delta^5 y_i$	$\Delta^6 y_i$
			$\Delta^2 y_{-3}$	$\Delta^3 y_{-3}$	$\Delta^4 y_{-3}$	$\Delta^5 y_{-3}$	$\Delta^6 y_{-3}$
x_{-3}	y_{-3}	Δy_{-3}		$\Delta^3 y_{-2}$	$\Delta^4 y_{-2}$	$\Delta^5 y_{-2}$	
			$\Delta^2 y_{-2}$				
x_{-2}	y_{-2}	Δy_{-2}		$\Delta^3 y_{-1}$	$\Delta^4 y_{-1}$		
x_{-1}	y_{-1}	Δy_{-1}	$\Delta^2 y_{-1}$				
				$\Delta^3 y_0$			
x_0	y_0	Δy_0	$\Delta^2 y_0$				
x_1	y_1	Δy_1	$\Delta^2 y_1$				
		Δy_2					
x_2	y_2						
x_3	y_3						

$$y_1 = a_0 - a_1(x_{-2} - x_0) + a_2(x_{-2} - x_0)(x_0 - x_1)$$

$$a_2 = \frac{\Delta^2 y_{-1}}{2h^2}$$

$$a_3 = \frac{\Delta^3 y_{-2}}{3! h^2}$$

$$a_4 = \frac{\Delta^4 y_{-4}}{4! h^4}$$

Forward difference version:

$$y_p = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_{-1}) + a_4(x - x_0)(x - x_1)(x - x_{-1})(x - x_2)$$

$$a_0 = y_0$$

$$a_1 = \frac{\Delta y_0}{h}$$

$$a_2 = \frac{\Delta^2 y_{-1}}{h^2}$$

$$a_3 = \frac{\Delta^3 y_{-1}}{h^3}$$

Let the temp v/s sp heat

of ethyl alcohol

x_i	y_i	Δx_i	$\Delta^2 y_i$	$\Delta^3 y$
1	8	9	-2	0
2	17	7	-2	0
3	24	5	-2	0
4	29	3	-2	
5	52	1		
6	53			

Numerical analysis

Divided Difference method:

x_i	y_i	$y[x_i, x_{i+1}]$	$y[x_i, x_{i+1}, x_{i+2}]$
x_0	y_0	$y[x_0, x_1]$	$y[x_0, x_1, x_2]$
		$y[x_1, x_2]$	$y[x_1, x_2, x_3]$
x_1	y_1		$\vdots$
x_2	y_2	$y[x_2, x_3]$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
		$y[x_{n-1}, x_n]$	
x_{n-1}	y_{n-1}		
x_n	y_n		

n $n-1$ division

First order devided difference is defined as

$$y[x_i, x_{i+1}] = \frac{y_{i+1} - y_i}{x_{i+1} - x_i} = \frac{y_{i+1}}{x_{i+1} - x_i} - \frac{y_i}{x_{i+1} - x_i}$$

2nd " " "

$$y[x_i, x_{i+1}, x_{i+2}]$$

$$= \frac{y[x_{i+1}, x_{i+2}] - y[x_i, x_{i+1}]}{x_{i+2} - x_i}$$

$$= \frac{\dfrac{y_{i+2}}{x_{i+2} - x_{i+1}} - \dfrac{y_{i+1}}{x_{i+2} - x_{i+1}} - \dfrac{y_{i+1}}{x_{i+1} - x_i} + \dfrac{y_i}{x_{i+1} - x_i}}{x_{i+2} - x_i}$$

$$= \frac{y_{i+2}}{(x_{i+2} - x_{i+1})(x_{i+2} - x_i)} - \frac{y_{i+1}(\cancel{x_{i+2} - x_i})}{(x_{i+2} - x_{i+1})(x_{i+1} - x_i)(\cancel{x_{i+2} - x_i})} + \frac{y_i}{(x_{i+2} - x_i)(x_{i+1} - x_i)}$$

$$= \frac{y_{i+2}}{(x_{i+2}-x_{i+1})(x_{i+2}-x_i)}$$

$$- \frac{y_{i+1}}{(x_{i+2}-x_{i+1})(x_{i+1}-x_i)}$$

$$+ \frac{y_i}{(x_{i+2}-x_i)(x_{i+1}-x_i)}.$$

$\therefore y[x_i, x_{i+1}, \ldots, x_{i+k}]$ [By induction].

$$= \frac{y[x_{i+1}, x_{i+2}, \ldots, x_{i+k}] - y[x_i, \ldots, x_{i+k-1}]}{x_{i+k} - x_i}$$

$$= \frac{y_{i+k}}{(x_{i+k}-x_{i+k-1}) \ldots\ldots (x_{i+k}-x_i)}$$

$$+ \frac{y_{i+k-1}}{(x_{i+k-1}-x_{i+k}) \ldots (x_{i+k-1}-x_i)}$$

$$+ \frac{y_i}{(x_i-x_{i+1}) \ldots\ldots (x_i-x_{i+k-1})}$$

$$y[x, x_0] = \frac{y-y_0}{x-x_0}.$$

$$\Rightarrow (y-y_0) = y[x, x_0](x-x_0)$$

$$\therefore y = y_0 + y[x, x_0](x-x_0) \quad \text{—— (1)}$$

$$y[x, x_0, x_1] = \frac{y[x_0, x_1] - y[x, x_0]}{x_1 - x}$$

$$\Rightarrow y[x_0, x_1] - y[x, x_0] = y[x, x_0, x_1](x_1 - x)$$

$$\Rightarrow y[x, x_0] = y[x_0, x_1] + (x - x_1)\, y[x, x_0, x_1] \quad \text{--- } (2)$$

From (1), (2):

$$\Rightarrow y = y_0 + \{y[x_0, x_1] + (x - x_1)\, y[x, x_0, x_1]\}(x - x_0)$$

$$\Rightarrow y = y_0 + (x - x_0)\, y[x_0, x_1] + (x - x_0)(x - x_1)\, y[x, x_0, x_1]$$

$$y = y_0 + (x - x_0)\, y[x_0, x_1] + (x - x_0)(x - x_1)\, y[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)\, y[x, x_0, x_1, x_2]$$

$\vdots$

$$y = y_0 + (x - x_0)\, y[x_0, x_1] + (x - x_0)(x - x_1)\, y[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)\, y[x_0, x_1, x_2, x_3] + \dots + (x - x_0)(x - x_1) \dots (x - x_{n-1})\, y[x_0, x_1, \dots, x_n] + (x - x_0)(x - x_1) \dots (x - x_n)\, y[x, x_0, x_1, \dots, x_n]$$

Total division

$$n + (n-1) + \dots + 1 = n(n+1)/2$$

Total multiplication $1 + 2 + \dots + n = n\frac{(n+1)}{2}$

" " division $= n(n+1)$.

Lagrange's method

Here we assume the interpolation as a weighted sum of $(n+1)$ nth degree polynomial.

$$y = y_0 b_0(x) + y_1 b_1(x) + \dots + y_n b_n(x)$$

Each $b_i(x)$ is a polynomial of degree n.

Sum this polynomial as

$$y_0 = y_0 b_0(x_0) + y_1 b_1(x_0) + \dots + y_n b_n(x_0)$$

$$y_1 = y_0 b_0(x_1) + y_1 b_1(x_1) + \dots + y_n b_n(x_1)$$

$$\vdots$$

$$y_n = y_0 b_0(x_n) + y_1 b_1(x_n) + \dots + y_n b_n(x_n)$$

$$b_i(x_j) = \left.\begin{matrix} 1 & i = j \\ = 0 & \text{otherwise} \end{matrix}\right\}.$$

$$b_i(x) = c_i (x - x_0)(x - x_1) \dots (x - x_{i-1})(x - x_{i+1}) \dots (x - x_n)$$

$$x = x_i \Rightarrow 1 = c_i (x_i - x_0)(x_i - x_1) \dots (x_i - x_{i-1})(x_i - x_{i+1}) \dots (x_i - x_n)$$

$$\therefore c_i = \frac{1}{(x_i - x_0)(x_i - x_1) \dots (x_i - x_{i-1})(x_i - x_{i+1}) \dots (x_i - x_n)}$$

$$L_i = \frac{1}{\prod_{\substack{j=0 \\ j \neq i}}^{n} (x_i - x_j)}$$

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \left\{ \frac{x - x_j}{x_i - x_j} \right\}$$

$$y = \sum_{i=0}^{n} y_i \prod_{\substack{j=0 \\ j \neq i}}^{n} \left(\frac{x - x_j}{x_i - x_j} \right) \quad 2n+2$$

No of multiplication / division $2n+2$ for each term in sum

$\therefore$ Total no of $*$, $/$ $= (2n+2)(n+1)$

$= 2(n+1)^2$

Find y at $x=3$

x_i	y_i	$y[x_i, x_{i+1}]$	$y[x_i, x_{i+1}, x_{i+2}]$	$y[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$	$y[\cdot]$
0	1	$\frac{14-1}{1-0} = 13$	$\frac{1-13}{2-0} = (-6)$	$\frac{-2+6}{4-0} = 1$	0
1	14	$\frac{15-14}{2-1} = 1$	$\frac{(-5)-1}{4-1} = -2$	$\frac{2+2}{5-1} = 1$	0
2	15	$\frac{5-15}{4-2} = -(5)$	$\frac{1+5}{5-2} = 2$	$\frac{6-2}{6-2} = 1$	
4	5	$\frac{6-5}{5-4} = 1$	$\frac{13-1}{6-4} = 6$		
5	6	$\frac{19-6}{6-5} = 13$			
6	19				

$$y = y_0 + (3 - x_0)\, y[x_0, x_1]$$

$$+ (3 - x_0)(3 - x_1)\, y[x_0, x_1, x_2]$$

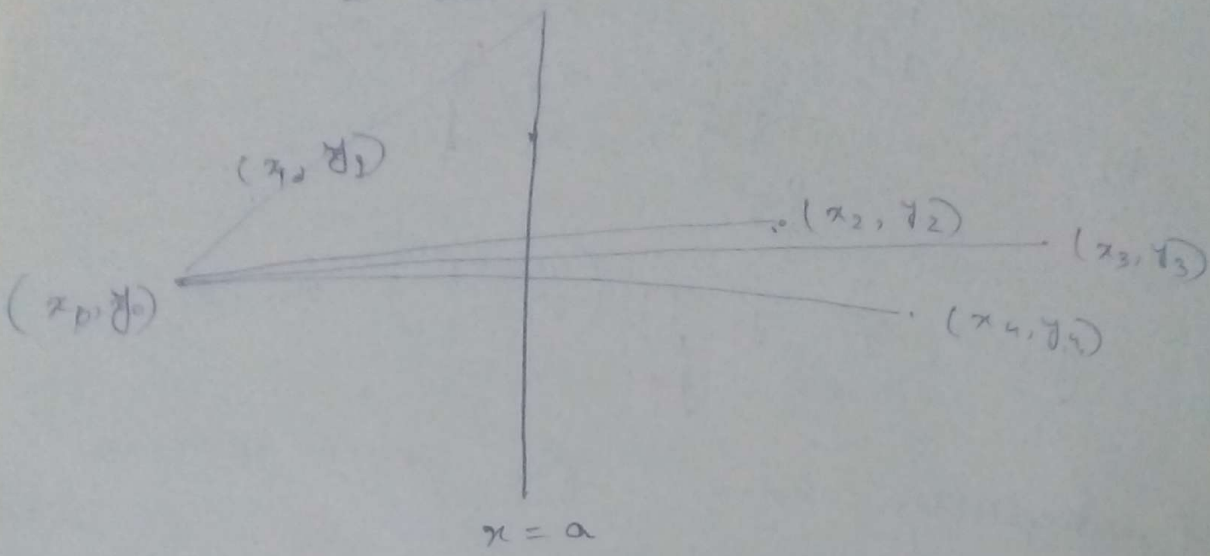
$$+ (3 - x_0)(3 - x_1)(3 - x_2)\, y[x_0, x_1, x_2, x_3]$$

$$+ (3-0)(3-1)(3-2) \cdot 1$$

$$= 1 + (3-0)\,13 + (3-0)(3-1)(-6)$$

$= 1 + 39 - 36 + 6$

$= 10$



Eq^n of each st line:

$$\frac{y - y_i}{x - x_i} = \frac{y_i - y_0}{x_i - x_0}$$

$$\Rightarrow \quad \frac{y(a) - y_i}{a - x_i} = \frac{y_i - y_0}{x_i - x_0}$$

$$\Rightarrow \quad y_{ai}(x) = y_i + \frac{(a - x_i)}{(x_i - x_0)}(y_i - y_0).$$

$$= \frac{1}{x_i - x_0}\left[\cancel{y_i x_i} - y_i x_0 + a y_i - a y_0 \cancel{- x_i y_i} + x_i y_0\right].$$

$$= \frac{1}{x_i - x_0}\left[a(y_i - y_0) + x_i y_0 - x_0 y_i\right].$$

x_i	y_i	$y_{0,i}^{(3)}$	$y_{0,1,i}^{(3)}$	$y_{0,1,2,i}^{(3)}$
0	1			
1				
	14	40		
2			4	
	15	22		10
			16	
4	5	4		10
			22	
5	6	4		10
			28	
6	19	10		

Curve fitting using ~~very least~~ least square order method

Let the eq^n of the curve to be fitted be

$$y = a_1 b_1(x) + a_2 b_2(x) + \dots + a_n b_n(x)$$

where each $b_1(x)$ is polynomial or ~~logarithmic~~ transcendal eq^n (sinx, cosx, logx, e^x)

$$pv^\gamma = c$$

$$\log x + \gamma \log y = \log c$$

$$\Rightarrow \log x + \gamma \log y - \log c = 0$$

$$\Rightarrow Y = \frac{1}{\gamma}(c - x)$$

$$Y = \log y \quad x = \log x$$

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$$E = \sum e_i^2 = \sum_{i=0}^{n} \{y(x_i) - y_i\}^2$$

$$E = \sum_{i=0}^{n} \{ a_1 b_1(x_i) + a_2 b_2(x_i) + \dots + a_k b_k(x_i) - y_i \}^2$$

$y = mx + c$

For minimum E

$$\frac{\partial E}{\partial a_1} = \frac{\partial E}{\partial a_2} = \dots = \frac{\partial E}{\partial a_k} = 0.$$

$$\frac{\partial E}{\partial a_1} = 2 \left[a_1 \sum b_1(x_i) + a_2 \sum b_2(x_i) + \dots + a_k \sum b_k(x_i) - \sum y_i \right] \sum b_1(x_i) = 0.$$

or,

$$a_1 \{\sum b_1(x_i)\}^2 + a_2 \sum b_1(x_i) b_2(x_i) + \dots a_k \sum b_k(x_i) b_1(x_i) - \sum y_i b_1(x_i)\} = 0.$$

$\therefore$

$$a_1 \{\sum b_1(x_i)\}^2 + a_2 \sum b_1(x_i) b_2(x_i) + \dots + a_k \sum b_k(x_i)\, b_1(x_i) = \sum y_i b(x_i)$$

$\frac{\partial E}{\partial a_2}$

$$a_1 \sum b_1(x_i) \sum b_2(x_i) + a_2 \{\sum b_2(x_i)\}^2 + \dots = \sum y_i b_2(x_i)$$

$$\frac{\partial E}{\partial a_k} \rightarrow a_1 \sum b_1 b_2 + a_2 \sum b_2 b_k + \ldots + a_k \sum b_k^2 = \sum y_i b_k$$

09/10/18

NM

$x_i \quad y_i$

Let n th degree polynomial containing tabular data.

$$y = y_0 + p \Delta x_0 + \frac{p(p-1)}{2!} \Delta^2 x_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 x_0 + \ldots$$

Let $x = x_0 + ph$ $\qquad p = \frac{x - x_0}{h}$

$$\frac{dy}{dx} = \frac{dy}{dp} \cdot \frac{dp}{dx} = \frac{1}{h} \cdot \frac{dy}{dp}$$

$$= \frac{1}{h} \frac{d}{dp} \left[y_0 + p \Delta x_0 + \frac{p(p-1)}{2!} \Delta^2 x_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 x_0 + \ldots \right]$$

$$= \frac{1}{h} \left[\Delta x_0 + \frac{2p-1}{2!} \Delta^2 x_0 + \frac{3p^2 - 6p + 2}{3!} \Delta^3 x_0 + \ldots \right]$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dp}\left(\frac{dy}{dx}\right) \frac{dp}{dx}$$

$$= \frac{1}{h} \frac{d}{dp}\left(\frac{dy}{dx}\right)$$

$$= \frac{1}{h^2} \frac{d}{dp} \left[\Delta x_0 + \frac{2p-1}{2} \Delta^2 x_0 + \frac{3p^2 - 6p\ldots}{3!} \Delta^3 x_0 + \ldots \right.$$

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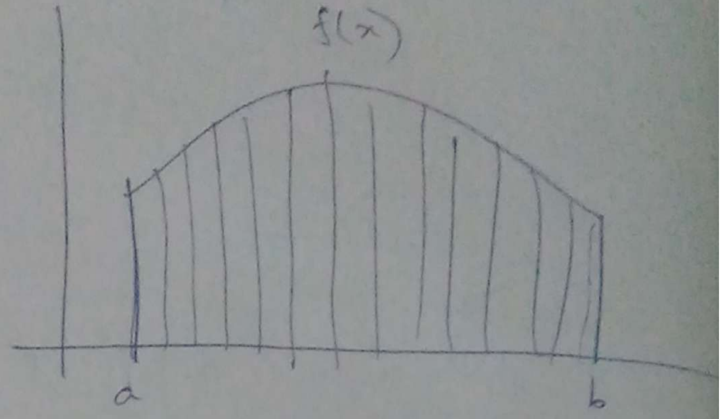
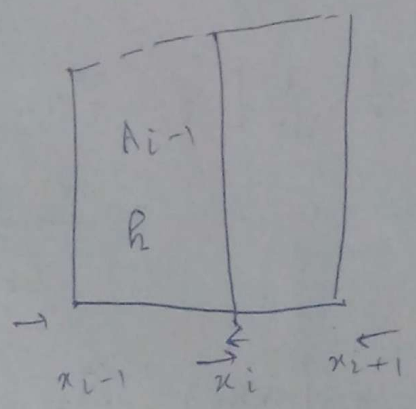
Numerical Integration

$$\int_a^b f(x)\,dx$$



When the strips are of equal width the corresponding integration formula are called Newton-Cotes formula, otherwise the formula are called Gaussian formulae.



for the interval $[x_{i-1}, x_i]$ is ~~called~~ the function $f(x)$ is approximated as st. line.

The area under the interval $[x_{i-1}, x_i]$ will be equal the area of the trapezeum ABCD

$$A_{i-1} = h_i\,[f(x_i) + f(x_{i-1})]/2$$

$$= \frac{h_i}{2}\,[f(x_{i-1}) + f(x_i)]$$

Nu

$$I_{trapezoid} = \sum_{i=1}^{n} A_{i-1}$$

$$= \sum_{i=1}^{n} \frac{h}{2}\left[f(x_{i-1}) + f(x_i)\right]$$

$$I_{actual} = \int_{x_{i-1}}^{x_i} f(x)\,dx = F(x)\Big]_{x_{i-1}}^{x_i}$$

$$= F(x_i) - F(x_{i-1})$$

$$I_{actual} = F(x_0 + h) - F(x_0)$$

$$= \cancel{F(x_0)} + hF'(x_0) + \frac{h^2}{2!}F''(x_0) + \dots - \cancel{F(x_0)}$$

$$= h f(x_0) + \frac{h^2}{2!} f'(x_0) + \frac{h^3}{3!} f''(x_0) + \dots$$

$$I_{trapeziod} = \frac{h}{2}\left[f(x_{i-1}) + f(x_i)\right]$$

$$= \frac{h}{2}\left[f(x_{i-1}) + f(x_{i-1} + h)\right]$$

$$= \frac{h}{2}\left[f(x_{i-1}) + f(x_{i-1}) + h f'(x_{i-1}) + \frac{h^2}{2!} f''(x_{i-1}) + \dots\right]$$

$$= h f(x_{i-1}) + \frac{h^2}{2} f'(x_{i-1}) + \frac{h^3}{4} f''(x_{i-1}) + \dots$$

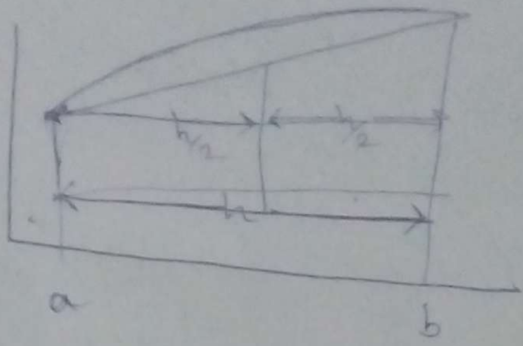
$I_{actual} - I_{trapezoid} = \left(\frac{h^3}{6} - \frac{h^3}{4}\right) f''(x_{i+1})$

$= -\frac{h^3}{12} f''(x_{i+1})$

If $h < 1$

$h_2 = h/2 < \frac{1}{2}$

$h_2^2 < \frac{1}{4}$



$I_0 = \frac{h}{2}\left[f(a) + f(b)\right]$

$I_1 = \frac{h}{4}\left[f(a) + 2f(c) + f(b)\right]$

$I_0 < I_1$

$h = b - a$

$n = 1$

$S0 = f(a) + f(b)$

$I1 = (A1 + S0)/2$

Repeat

{

s1 = 0; I0 = I1;

h = h/2;

for{i = 1; i <= n; i = i + 2i}

s1 = s1 + f(xi + i * h);

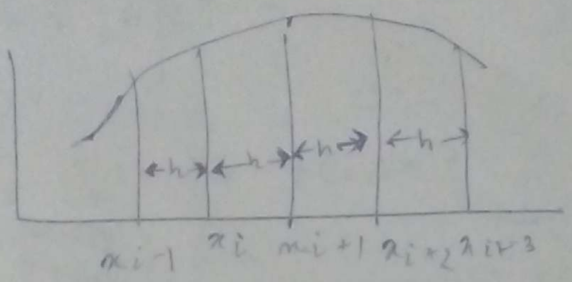
n = 2 * n;

} S0 = S0 + 2 * s1 ~~If~~ I1 = h * S0

untill (fabs (I1 - I0) < E)

~~until~~

Simpson's $\frac{1}{3}$ rule :



~~x_{i-1} $f(x_i)$~~

x_{i-1} $\quad f(x_{i-1})$ $\quad \Delta f(x_{i-1})$ $\quad \Delta^2 f(x_{i-1})$

$\quad\quad\quad\quad\quad\quad\quad \Delta f(x_i)$

x_i $\quad f(x_i)$

x_{i+1} $\quad f(x_{i+1})$

Let us put a Newton's forward difference polynomial for 3 points :-

$$y = f(x_{i-1}) + p\,\Delta f(x_{i-1}) + \frac{p(p-1)}{2!}\Delta^2 f(x_{i-1})$$

$$p = \frac{x - x_{i-1}}{h}$$

If $x = x_{i-1}$ $\quad p = 0$

$x = x_{i+1}$ $\quad p = 2$.

$$\int_{x_i}^{x_{i+1}} y\,dx$$

$\Delta f(x_i) = f(x_i) - f(x_{i-1})$

$\Delta^2 f(x_{i-1}) = f(x_{i+1}) - 2f(x_i) \to f(x_{i-1})$

~~$= h \int_{x_{i-1}}^{x_{i+1}} y\,dp$~~ $= h \int_0^2 y\,dp$

$= h \int_0^2 \{ f(x_{i-1}) + p\,\Delta f(x_{i-1})$

$+ \frac{p(p-1)}{2!}\Delta^2 f(x_{i-1})$

$\}\,dp$

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$$= h\left[\left[p f(x_{i-1})\right]_0^2 + \left[\frac{p^2}{2}\Delta f(x_{i-1})\right]_0^2 + \frac{1}{2}\left[\left(\frac{p^3}{3}-\frac{p^2}{2}\right)\Delta^2 f(x_{i-1})\right]_0^2\right]$$

$$= h\left[2f(x_{i-1}) + 2\Delta f(x_{i-1}) + \frac{1}{2}\left(\frac{8}{3}-\frac{4}{2}\right)\Delta^2 f(x_{i-1})\right]$$

$$= h\left[2f(x_{i-1}) + 2\Delta f(x_{i-1}) + \frac{1}{3}\Delta^2 f(x_{i-1})\right]$$

$$= \frac{h}{3}\left[\cancel{6f(x_{i-1})} + 6f(x_i) - \cancel{6f(x_{i-1})} + f(x_{i+1}) - 2f(x_i) + f(x_{i-1})\right]$$

$$= \frac{h}{3}\left[f(x_{i+1}) + 4f(x_i) + f(x_{i-1})\right].$$

$$\sum^{n} \frac{h}{3}\left[f(x_{i+1}) + 4f(x_i) + f(x_{i-1})\right]$$

$$I_{actual} = \int_{x_{i-1}}^{x_{i+1}} f(x)\,dx$$

$$= F(x_{i+1}) - F(x_{i-1})$$

$$= F(x_{i-1} + 2h) - F(x_{i-1})$$

$$= \cancel{F(x_{i-1})} + 2h F'(x_{i-1}) + \frac{(2h)^2}{2!} F''(x_{i-1}) + \frac{(2h)^3}{3!} F'''(x_{i-1})$$

$$+ \frac{(2h)^4}{4!} F''''(x_{i-1}) + \ldots - \cancel{F(x_{i-1})}$$

$$\frac{16}{24}\frac{2}{3}$$

$$+ \frac{(2h)^5}{5!} F'''''(x_{i-1})$$

$$= 2h\, f(x_{i-1}) + 2h^2 f'(x_{i-1}) + \frac{4h^3}{3} f''(x_{i-1}) + \frac{2h^4}{3} f'''(x_{i-1}) + \frac{4}{15} h^5 f''''(x_{i-1})$$

$$A_{i-1} = \frac{h}{3}\left[f(x_{i-1}) + 4 f(x_{i-1}+h) + f(x_{i-1}+2h)\right]$$

$$= \frac{h}{3}\Big[f(x_{i-1}) + 4\Big\{ f(x_{i-1}) + h f'(x_{i-1}) + \frac{h^2}{2!} f''(x_{i-1}) + \frac{h^3}{3!} f'''(x_{i-1}) + \frac{h^4}{4!} f^{4}(x_{i-1}) + \ldots$$

$$+ \Big[f(x_{i-1}) + 2h f'(x_{i-1}) + \frac{(2h)^2}{2!} f''(x_{i-1}) + \frac{(2h)^3}{3!} f'''(x_{i-1}) + \frac{(2h)^4}{4!} f^4(x_{i-1}) \ldots \Big]$$

$$= \frac{h}{3}\Big[\cancel{f(x_{i-1}) + 4 f(x_{i-1}) + f(x_i}$$

$$6 f(x_{i-1}) + 6h f'(x_{i-1}) + 4h^2 f''(x_{i-1}) + 2h^3 f'''(x_{i-1}) + \frac{20}{24} h^4 f^4(x_{i-1}) + \ldots \Big]$$

$$= 2h\, f(x_{i-1}) + 2h^2 f'(x_{i-1}) + \frac{4h^3}{3} f''(x_{i-1})$$

I actual − A_{i+1}

$$= \left(\frac{4}{15} - \frac{5}{18}\right) h^5 f^4(x_{i-1})$$

$$= \frac{20 - 30}{90} h^5 f^4(x_{i-1})$$

$$= -\frac{h^5}{9} f'''''(x_{i-1})$$

Total error = $\sum_{i=1}^{n} e_i$

$$= -\sum_{i=1}^{n} \frac{h^5}{9} f''''(x_{i-1})$$

Boole's method

This method fits a n th degree polynomial through consecutive 5 points of function

$$I_{incremental} = \frac{2h}{45}\left(7y_{i-1} + 32y_i + 12y_{i+1} + 32y_{i+2} + 7y_{i+3}\right)$$

trancational error $= -\dfrac{2(b-a)h^6}{945}$

$$\int_a^b f(x)dx = I_{trapezoidal}(h) + \underbrace{a_1 h^2}_{f''(x)} + \underbrace{a_2 h^4}_{f^{iv}(x)} + \underbrace{a_3 h^6}_{f^{vi}(x)} + \dots \longrightarrow (1)$$

function such that odd order derivative does not exist.

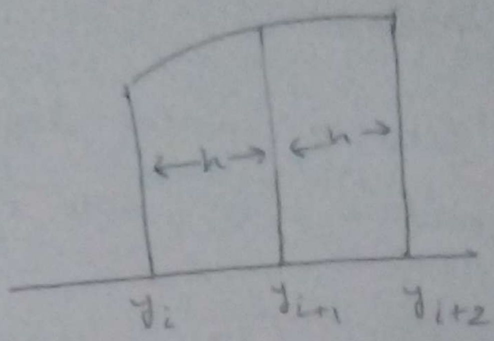
$$\int_a^b f(x)\,dx = I^{(2h)}_{trapezoidal} + a_1(2h)^2 + a_2(2h)^4 + a_3(2h)^6 + \dots$$

$$\longrightarrow (2)$$

$(1) \times (4) - (2)$

$$(4-1)\int_a^b f(x)dx = 4I^{(h)}_{trapezoidal} - I^{(2h)}_{trapezoidal} + b_1' h^4 + b_2' h^6 + \dots$$

$$\int_a^b f(x)\,dx = \frac{4I^{(h)}_{trapezoidal} - I^{(2h)}_{trapezoidal}}{3} + \frac{b_1'}{3} h^4 + b_2 h^6 + \dots$$



$$4 \cdot \frac{I_{trapezoidal}(h) - I_{trapezoidal}(2h)}{3}$$

$$\frac{\cancel{4}^{2} \times \frac{h}{\cancel{2}}\left(y_i + 2y_{i+1} + y_{i+2}\right) - \cancel{2}h\left(\frac{y_i + y_{i+2}}{\cancel{2}}\right)}{3}$$

$$= \frac{h}{3}\left(y_i + 4y_{i+1} + y_{i+2}\right).$$

$$\int_a^b f(x)\,dx = I_{simpson\,\frac{1}{3}} \cancel{2(2h) + b_1(2h)^4 + b_2(2}$$

$$I_{simpson\,\frac{1}{3}\,of}(h) + b_1 h^4 + b_2 h^6 + b_3 h^8, \ldots \quad (3)$$

$$\int_a^b f(x)\,dx = I_{simpson\,\frac{1}{3}\,of}(2h) + b_1(2h)^4 + b_2(2h)^6 + b_3(2h)^8 + \ldots \quad (4)$$

(3) × 16 − (4) / I.

$$\int_a^b f(x)\,dx = \frac{16\,I_{simpson}(h) - I_{simpson}(2h)}{15} + a_1 h^6 + a_2 h^8 + \ldots$$

$I_{trapezoidal}(h)$

$I_{simpson \frac{1}{3}}(h/2)$

$I_{trapezoidal}\left(\frac{h}{2}\right)$

$I_{bole}(h/4)$

$I_{simpson}\left(\frac{h}{4}\right)$

$I_{trapezoidal}\left(\frac{h}{4}\right)$

Let us represent the integral values computed by trapezoidal method as a Romberg term $R(h,0)$.

we can generate higher order Romberg terms as

$$R(h,1) = \frac{4R(h,0) - R(2h,0)}{3}$$

$$R(h,2) = \frac{16R(h,0) - R(2h,0)}{15} \quad \text{Rich}$$

$$\boxed{R(h,k) = \frac{4^k R(h,k-1) - R(2h,k-1)}{4^k - 1}}$$

Let at $(k-1)$ th ~~iteration~~

$$\int_a^b f(x)\,dx = R(h,k-1) + d_1 h^{2k} + d_2 h^{2k+2} + d_3 h^{2k+4} + \dots \quad \text{—⑤}$$

$$\int_a^b f(x)\,dx = R(2h,k-1) + d_1(2h)^{2k} + d_2(2h)^{2k+2} + d_3(2h)^{2k+4} + \dots \quad \text{—⑥}$$

$⑤ \times 4^k - ⑥ \,/\, 4^k - 1$

$$\int_a^b f(x)\,dx = \frac{4^k R(h,k-1) - R(2h,k-1)}{4^k - 1} + e_1 h^{2k+2} + \dots$$

i	$R\left(\frac{h}{2^i}, 0\right)$	$R\left(\frac{h}{2^i}, 1\right)$	$R\left(\frac{h}{2^i}, 2\right)$	$R\left(\frac{h}{2^i}, 3\right)$
0	$R(h, 0)$	$R(h, 1)$	$R(h, 2)$	$R(h, 3)$
1	$R\left(\frac{h}{2}, 0\right)$	$R\left(\frac{h}{2}, 1\right)$	$R\left(\frac{h}{2}, 2\right)$	
2	$R\left(\frac{h}{4}, 0\right)$	$R\left(\frac{h}{4}, 1\right)$		
3	$R\left(\frac{h}{8}, 0\right)$	-	-	-

$$\int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4} = 0.785398.$$

Trapezoid rule

$h = 0.5$ $\quad I = 0.775$

$h = 0.25$ $\quad I = 0.782794$

$h = 0.125$ $\quad I = 0.784747$

$I = 0.0625$ $\quad I = 0.785235$

Simpson $\frac{1}{3}$ rule.

$h = 0.5$ $\quad I = 0.78493$

$h = 0.25$ $\quad I = 0.785392$

$h = 0.125$ $\quad I = 0.7853\ 98$

$h = 0.0625$ $\quad I = 0.785398$

$$I = \sum_{i=0}^{n} w_i f(x_i) + E.$$

$n+1$ no of w_i's

$(2n+2)$ variables / unknowns.

Gauss Quadrature formula:

To choose $(2n+2)$ conditions to find out $(2n+2)$ unknown Gauss assesment that the integration formula will be such that it will integrate the polynomial terms x^k, $k = 0, \dots, 2n+1$ with zero error

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n} w_i f(x_i) + E.$$

Let $n=1$

$$\int_{-1}^{1} f(x)\,dx = w_0 f(x_0) + w_1 f(x_1)$$

for x^0

$$\int_{-1}^{1} x^0\,dx = w_0 + w_1 = 2$$

x^1: $\int_{-1}^{1} x^1\,dx = w_0x_0 + w_1x_1 = 0$

x^2: $\int_{-1}^{1} x^2\,dx = w_0x_0^2 + w_1x_1^2 = \frac{2}{3}$.

x^3: $\int_{-1}^{1} x^3\,dx = w_0x_0^3 + w_1x_1^3 = 0$.

It can be proved that x_i values are the root of Legendre polynomial $P_{n+1}(x) = 0$.

n	x_i	w_i	
1	± 0.57735	1.000.	$P_2(x)=0$.
2	$\pm 0.77459, 0000$	0.55555, ,0.88888.	$3x^2-1=0$

$x = \pm\frac{1}{\sqrt{3}}$

$x_0 = \frac{1}{\sqrt{3}}$ $x_1 = -\frac{1}{\sqrt{3}}$.

$w_0 = w_1 = 1$.

$$\int_a^b f(x)\,dx = \frac{b-a}{2}\int_{-1}^{1} f\left\{\frac{(b-a)t + b+a}{2}\right\} dt.$$

$$\int_a^b f(x)\,dx = \frac{b-a}{2}\sum_{i=0}^{n} w_i f\left[\frac{(b-a)t_i + (b+a)}{2}\right]$$

$$\int_0^1 \frac{dx}{1+x^2} = \frac{1-0}{2}\int_{-1}^{1} f\left[\frac{(1-0)t_i + (1+0)}{2}\right] dt$$

$$= \frac{1}{2}\int_{-1}^{1} f\left(\frac{t+1}{2}\right) dt$$

for $n=1$, $$= \frac{1}{2}\left[\sum_{i=0}^{n} w_i f\left(\frac{t_i+1}{2}\right)\right]$$

$t_0 = -0.57735$, $t_1 = 0.57735$

$$\int_0^1 \frac{dx}{1+x^2} = 0.73248$$

$n=2$;

~~$w_0 = 0.5$~~

$t_0 = -0.77459$ $\quad t_1 = 0$ $\quad t_2 = 0.7459$

$w_0 = 0.55555$ $\quad w_1 = 0.88888$ $\quad w_2 = 0.5555$

$$\int_0^1 \frac{dx}{1+x^2} = 0.735207$$

Solution of differential eqⁿ

$$\frac{dy}{dx} = f(x, y).$$

with some specified initial conditions.

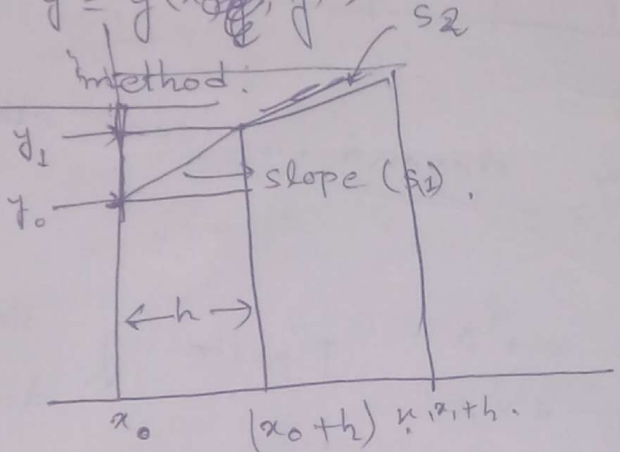
$$\frac{\partial y}{\partial x} = f(x, y, z)$$

$$\frac{\partial y}{\partial z} = g(x, y, z)$$

$$y(x_0) = y_0$$

$$y = g(x, y).$$

Euler's method.



$$s_1 = f(x_0, y_0)$$

$$s_1 = \tan\theta = \frac{y_1 - y_0}{x_1 - x_0} = \frac{y_1 - y_0}{h}.$$

$$y_1 = y_0 + h s_1$$

$$y_2 = y_1 + h s_2.$$

at any instance pt. i solution curve (x_{k-1}, y_{k-1})
$h f(x_{k-1}, y_{k-1})$

$\therefore$ next pt $= (x_{k-1} + h,\ y_{k-1} + h s_k).$

Error estimate in euler's method

$$x_k = x_{k-1} + h.$$

$$y_k = y_{k-1} + h s_k = y(x_{k-1} + h).$$

$y'(x_{k-1}) = f(x_{k-1}, y_{k-1})$

$y''(x_{k-1}) = f'(x_{k-1}, y_{k-1})$

$$y_k = y(x_{k-1} + h)$$
$$= y(x_{k-1}) + h y'(x_{k-1}) + \frac{h^2}{2} y''(x_{k-1}) + \dots$$

$$= \underbrace{y(x_{k-1}) + h f(x_{k-1}, y_{k-1})} + \frac{h^2}{2} f'(x_{k-1}, y_{k-1}) + \dots$$

truncation error of order h^2 $= \frac{h^2}{2} f'(x_{k-1}, y_{k-1})$

Let the errors associated with y_{k-1}, y_k and their true values are be e_{k-1}, e_k

y'_{k-1}, y'_k

$$y_{k-1} = y'_{k-1} + e_{k-1}$$
$$y_k = y'_k + e_k$$

$$y'_k + e_k = y'_{k-1} + e_{k-1} + h f(x_{k-1}, y'_{k-1} + e_{k-1})$$

$$y'_k + e_k = y'_{k-1} + e_{k-1} + h\left[f(x_{k-1}, y_{k-1}) + e_{k-1} f'(x_{k-1}, y_{k-1}) + \dots\right]$$

$$y'_k + e_k = y'_{k-1} + h f(x_{k-1}, y_{k-1})$$

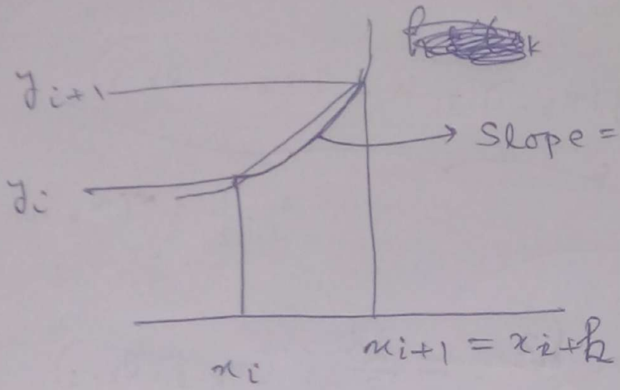
$e_k\left(+1+hf'(x_{k-1}, y_{k-1})\right)$

$$e_k \simeq e_{k-1}\left[1 + h\, f'(x_{k-1}, y_{k-1})\right]$$

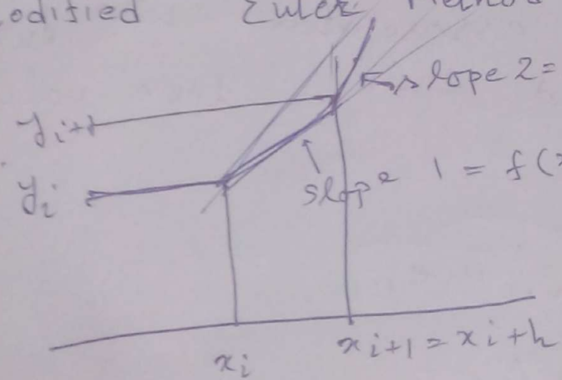
$$|e_k| \approx |e_{k-1}|\, |1 + hf'(x_{k-1}, y_{k-1})|$$

for stability:

$$|1 + hf'(x_{k-1}, y_{k-1})| < 1$$



Modified Euler Method:



$$\text{Slope} = \frac{\text{slope 1} + \text{slope 2}}{2}$$

$$y_{i+1} = y_i + h\left(\frac{\text{slope 1} + \text{slope 2}}{2}\right)$$

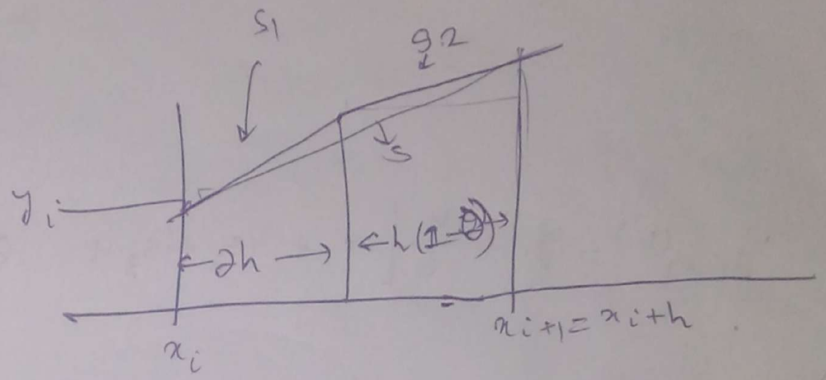
$$y_{i+1}^{(k+1)} = y_i + h\left[\frac{f(x_i, y_i) + f(x_{i+1}, y_{i+1})}{2}\right]$$

until $\left| y_{i+1}^{(k+1)} - y_{i+1}^{(k)} \right| < \epsilon$

order $= h^3$

~~Modified~~

Range Kutta Method (2^{nd} order):



$s = w_1 s_1 + w_2 s_2$

where $w_1, w_2 < 1$.

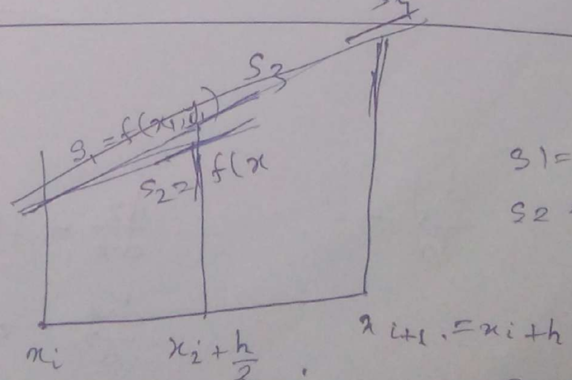
and $w_1 + w_2 = 1$

if $\partial = \frac{1}{2}$

$w_1 = \frac{1}{3}$ $\quad w_2 = \frac{2}{3}$.

$$\frac{w_2}{w_1} = \frac{1+\partial}{1-\partial}.$$

Range Kutta 4^{th} order formula:



$s_1 = f(x_i, y_i)$

$s_2 = f\left(x_i + \frac{h}{2}, y_i + \frac{h}{2} s_1\right)$

$s_3 = f\left(x_i + \frac{h}{2}, y_i + \frac{h}{2} s_2\right)$

$$s_4 = f(x_i + h,\ y_i + h s_3)$$

$$s = \frac{1}{6}\left[s_1 + 4s_2 + 4s_3 + s_4\right]$$

$$y_{i+1} = y_i + hs$$

$$y_{i+1}^{(1)} = y_i + \frac{h}{6}\left[s_1 + 4s_2 + 4s_3 + f(x_i + h,\ y_i + h\right.$$

$$f(x_{i+1}, y_{i+1})$$

$$y_{i+1}^{(k)} = y_i + \frac{h}{6}\left[s_1 + 4s_2 + 4s_3 + f(x_i + h, y_i + h\right.$$

$$(x_{i+1}, y_{i+1}^{(k-1)})\Big]$$

Solve of 2^{nd} order ordinary differential eqn

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right)$$

with initial condn $\quad y(x_0) = y_0$

$$\left.\frac{dy}{dx}\right|_{x_0} = z_0.$$

$$\frac{dy}{dx} = z \qquad \frac{dz}{dx} = f(x, y, z)$$

$$\frac{dy}{dx} = z$$

next point can be series as,

$$x_{i+1} = x_i + h$$

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$z_{i+1} = z_i + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4).$$

where, $k_1 = h\,g(x_i, y_i, z_i)$

$$l_1 = hf(x_i, y_i, z_i)$$

$$k_2 = h\,g\left(x_i + \frac{h}{2}, y_i + \frac{k_1}{2}, z_i + \frac{l_1}{2}\right).$$

$$l_2 = hf\left(x_i + \frac{h}{2}, y_i + \frac{k_1}{2}, z_i + \frac{l_1}{2}\right)$$

$$k_3 = h\,g\left(x_i + \frac{h}{2}, y_i + \frac{k_2}{2}, z_i + \frac{l_2}{2}\right).$$

$$l_3 = hf\left(x_i + \frac{h}{2}, y_i + \frac{k_2}{2}, z_i + \frac{l_3}{2}\right)$$

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