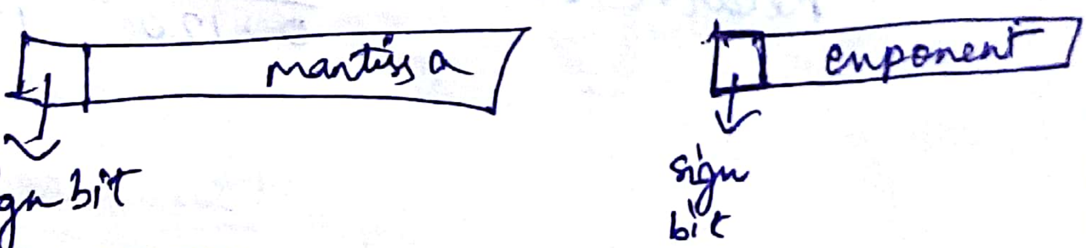
- Computer can perform four basic arithmetic operations like addition, subtraction, multiplication, division and a logical operation which is comparison.

- Computer has a finite word size, it can represent a number with finite precision.

$$123456.789 \Rightarrow 0.\underbrace{123456789}_{\text{mantissa}} \times 10^{\overset{\text{exponent}}{6}}$$

Any number with infinite precision has to be rounded off after a finite number of digits after decimal point.



Round Off Error

acc. to . 127 is added so as to remove the sign bit and express

or a factor

- Computation has to finished within a finite time, we cannot leave the computer doing the same process for an infinite period of time.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

Truncation error: since truncating the series after a finite nos. of terms.

formulation of problem and unavailability of accurate data.

24/7/18

$x = \bar{x} + e_x$ — local error

$y = \bar{y} + e_y$

$p = xy = \bar{x}\bar{y} + (\bar{x}e_y + \bar{y}e_x + e_x e_y)$

absolute error $\doteq$ | computed value − exact value |

12345.678 = computed value
12345.675 = exact value
absolute error = 0.003

Relative error $= \frac{0.003}{12345.675}$ (insignificant).

0.123 — computed
0.120 — exact
abs. err = 0.003

Relative error $= \frac{0.003}{0.120} = \frac{3}{120} = 0.025$
$= 2.5\%$ (significant)

Rel. Error $= \frac{\text{absolute Err}}{\text{Exact Value}}$

Iterative method ($x_{n+1} = \phi(x_n)$)

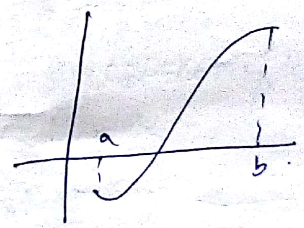
Let $x = \xi$ be a root of $f(x) = 0$ and let I be an Interval containing the point $x = \xi$. Let $\phi(x)$ and $\phi'(x)$ be continuous in I where $\phi(x)$ is defined by the equation $x = \phi(x)$ which is equivalent to $f(x) = 0$. Then if $|\phi'(x)| < 1$ for all x in I, the seq. of approx. $x_0, x_1, x_2, \ldots x_n$ defined by $x_{n+1} = \phi(x_n)$ converges to the root ξ, provided that the initial approx. x_0 is chosen in I.

If $y = f(x)$ cont. over $[a, b]$
if $f(a)$ & $f(b)$ are of opp signs, then there $\exists$ a number in $[a, b]$ such that $f(\xi) = 0$

(There exists odd no. of roots).

Eq^n involving trig, hyperbolic or logarithmic terms are transcendental eq^n.

Bisection Method



initial a and b are so chosen that $f(a)$ & $f(b)$ are of opp sign.

$f(a) * f(b) < 0$

① $m = \frac{a+b}{2}$

② if $f(m) == 0$ → m is a root → go to 6.

③ if $f(a) * f(m) < 0$ then $b = m$
else $a = m$.

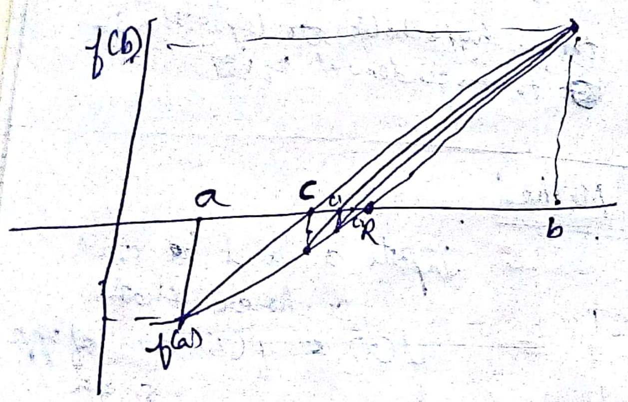
④ Repeat steps 1), 2), 3) until the size of the interval $|a-b|$ is less than the required precision.

⑤ $m = (a+b)/2$

⑥ Print m ⑦ Exit

The method fails only when the curve touches the x-axis. It converges very slowly.

Method of false position or Regala falsi method



$$\frac{y-f(a)}{x-a} = \frac{f(b)-f(a)}{(b-a)}$$

$$\Rightarrow (x-a) = \frac{(b-a)\{y-f(a)\}}{f(b)-f(a)}$$

$$\Rightarrow c-a = \frac{(b-a)\{-f(a)\}}{f(b)-f(a)}$$

$$\Rightarrow c = \frac{(a-b)f(a)}{f(b)-f(a)} + a$$

$$= \frac{af(b) - bf(a)}{f(b)-f(a)}$$

1) $c = \frac{af(b) - bf(a)}{f(b)-f(a)}$

2) if $f(c) == 0$ then c is a root goto

3) if $(f(a) * f(c)) < 0$ then $b = c$
else $a = c$

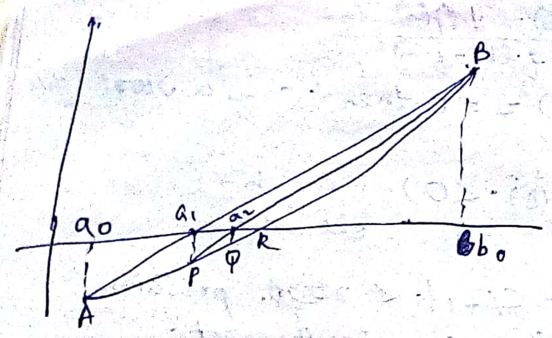
4) if $|c_i - c_{i+1}| <$ reqd. precision
then c_{i+1} is the root.

for results corrected upto 5 decimal places, precision value will be 0.000005

Over bisection, regula falsi method takes less number of iterations to converge to the root.

Analysis of any numerical algorithm considers the following facts:

1) Whether the method is guaranteed to provide solution.
2) How fast will it provide solution.
3) Amount of error associated with the solution.



From Δs $a_1 a_0 A$ and $a_1 b_0 B$,

$$\frac{a_1 - a_0}{|f(a_0)|} = \frac{b_0 - a_1}{|f(b_0)|} \quad \text{—①}$$

From Δs $a_2 a_1 P$ and $a_2 b_0 B$.

$$\frac{a_2 - a_1}{|f(a_1)|} = \frac{b_0 - a_2}{|f(b_0)|} \quad \text{—②}$$

$(a_2 - a_1)|f(b_0)| = f(a_1)|(b_0 - a_2)$

$\Rightarrow (a_2 - a_1)|f(b_0)| + a_2|f(a_1)| = b_0|f(a_1)|$

$\Rightarrow (a_2 - a_1)|f(b_0)| + (a_2 - a_1)|f(a_1)| = (b_0 - a_1)|f(a_1)|$

$\Rightarrow (a_2 - a_1)(|f(b_0)| + |f(a_1)|) = (b_0 - a_1)|f(a_1)|$

$$= \frac{(a_1 - a_0)|f(b_0)||f(a_1)|}{|f(a_0)|}$$

$$\therefore a_2 - a_1 = \frac{(a_1 - a_0)|f(b_0)||f(a_1)|}{|f(a_0)|(|f(b_0)| + |f(a_1)|)}$$

$$= (a_1 - a_0)\left[\left\{|f(a_1)|/|f(a_0)|\right\} \Big/ \left(1 + \frac{|f(a_1)|}{|f(b_0)|}\right)\right\} <$$

$\because f(b_0) > f(a_1)$ always

& $\frac{|f(a_1)|}{|f(a_0)|} < 1$ always.

Solve $e^{-x} - x = 0$ using bisection method.

iteration no(i)	a_i	b_i	m_i	$f(m_i)$	$f(a_i) * f(m_i)$
1	0	1	0.5	0.10653	+ve
2	0.5	1	0.75	-0.2776	-ve
3	0.5	0.75	0.625	-0.89736	-ve
4	0.5	0.625	0.5625	0.00728	+ve
5	0.5625	0.675	0.51375	-0.0415	-ve
6	0.5625	0.59375	0.57875	-0.0177	-ve

~~#include <iostream.h>~~

```
#include <stdio.h>
# include <math.h>
double func(double x)
{
    return (exp(-x)-x);
}
int main()
{
    float low, up, root, diff;
    printf("Lower limit : ");
    scanf("%f", &low);
    printf("Upper limit : ");
    scanf("%f", &up);
```

```
if (f(low) * f(up) > 0)
{   printf ("No root exist by RR method");
    return 0;
}
else
{
```

How fast does the method converge to the root ??

Order of convergence is n if $e_{i+1} = K e_i^n$

K is const. of proportionality

Let $e_i = 0.01$ & $n = 2$

$e_{i+1} \propto (0.01)^n = 0.0001$

$\therefore |e_{i+1}| = |K| |e_i|^n$

$\log|e_{i+1}| = \log|K| + n \log|e_i|$

$\therefore n = \dfrac{\log_e |e_{i+1}| - \log|K|}{\log|e_i|}$

$$n \doteq \left| \frac{\log |e_{i+1}| - \log|K|}{\log|e_i|} \right|$$

neglect

Fixed Point iteration method / Iterative method using repeated substitution.

given $f(x) = 0$, whose root has to be computed.

We rewrite this equ^n as $x = g(x)$

Let us assume the initial approx. to the root of $f(x) = 0$ be r_0, we generate a sequence of approx to the root as follows:

$r_1 = g(r_0)$

$r_2 = g(r_1)$

⋮

$r_{n+1} = g(r_n)$

continue until $|r_{n+1} - r_n| <$ reqd. precision.

$x = \pm\sqrt{x+6}$ — (1)

$x^2 = x+6$

$x = 1 + 6/x$ — (2)

or $x = x^2 - 6$ — (3)

Let $x = \sqrt{x+6}$

$g(x) = \sqrt{x+6}$ $g'(x) = \frac{1}{2\sqrt{x+6}}$

at $x = 3$ $|g'(x)| < 1$.

Let $x_0 = 1$

$x_1 = \sqrt{1+6} = \sqrt{7} = 2.645$

$x_2 = \sqrt{2.645+6} = 2.94$

$x_3 = \sqrt{2.94+6} = \sqrt{2.98} = 2.989$

$x_4 = \sqrt{2.989+6} = 2.998$ $\to 3$

$x = -\sqrt{x+6}$.

$g(x) = -\sqrt{x+6}$, $g'(x) = \frac{-1}{2\sqrt{x+6}}$

at $x=-2$ $g'(x) = \frac{-1}{4}$

$\therefore |g'(x)| < 1$

$x_0 = 1$

$x_1 = -\sqrt{1+6} = -2.645$

$x_2 = -\sqrt{-2.645+6} = -1.831$

$x_3 = -\sqrt{1.831+6} = -2.04$

$x_4 = \sqrt{-2.04+6} = -1.989$

$x_5 = -\sqrt{-1.989+6} = -2.002$, $\to -2$

$x = 1+6/x$

$g(x) = 1 + 6/x$.

$g'(x) = -6/x^2$

at $x = 3$ $g'(x) = -\frac{6}{9} = -\frac{2}{3}$,

$|g'(x)| < 1$

$x_0 = 1$

$x_1 = 1 + 6/1 = 7$

$x_2 = 1 + 6/7 = 1.857$

$x_3 = 1 + 6/1.857 = 4.23$

$x_4 = 1 + 6/4.23 = 2.418$

$x_5 = 1+6/2.418 = 3.481$.

$x = x^2 - 6$

$g(x) = x^2 - 6$ $g'(x) = 2x$

$|g'(x)| > 1$

Hence diverges

$x_0 = 1$ $\therefore x_1 = 1 - 6 = -5$

$x_2 = 19$.

$x_3 = 355$ ↳ diverges.

Theorem: Let $x = R$ be a root of the eqn f(x)=0 which is rewritten as $x = g(x)$. Let both $g(x)$ & $g'(x)$ exist and continuous over an interval (a, b) containing $x = R$. If $|g'(x)| < 1$ over the interval (a, b) & the initial approx. to the root x_0 is also in (a, b), then the sequence of approx. $x_1, x_2, x_3 \ldots$ will converge to the root $x = R$.

Proof: Let $x = R$ be a root of $x = g(x)$

$\therefore R = g(R)$. — (1)

Let x_0 be the initial approx. to the root, then the seq. of approx. can be generated as

$$\left.\begin{aligned} r_1 &= g(r_0) \\ r_2 &= g(r_1) \\ r_3 &= g(r_2) \\ r_4 &= g(r_3) \\ &\vdots \\ r_{n+1} &= g(r_n) \end{aligned}\right\} \text{(2)}$$

Let $e_0, e_1, e_2, \ldots, e_{n+1}$ be the errors associated with $r_0, r_1, \ldots, r_{n+1}$.

$$\left.\begin{aligned} e_0 &= R - r_0 \\ e_1 &= R - r_1 = g(R) - g(r_0) \\ e_2 &= R - r_2 = g(R) - g(r_1) \\ &\vdots \\ e_{n+1} &= R - r_{n+1} = g(R) - g(r_n) \end{aligned}\right\} \text{(3)}$$

Using mean value theorem, we can rewrite the eqn in (3) as

$$\left.\begin{aligned} e_0 &= R - r_0 \\ e_1 &= (R - r_0)\, g'(c_1) \;;\; r_0 < c_1 < R \\ e_2 &= (R - r_1)\, g'(c_2) \;;\; r_1 < c_2 < R \\ &\vdots \\ e_{n+1} &= (R - r_n)\, g'(c_{n+1}) \;;\; r_{n+1} < c_{n+1} < R \end{aligned}\right\} \text{(4)}$$

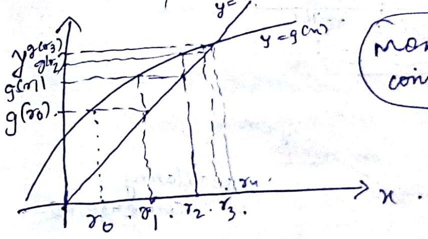
Now from (4)

$$e_1 = e_0\, g'(c_1)$$
$$e_2 = e_1\, g'(c_2)$$
$$\vdots$$
$$e_{n+1} = e_n\, g'(c_{n+1})$$

For convergence $|e_{n+1}| < |e_n| < \ldots < |e_2| < |e_1| < |e_0|$

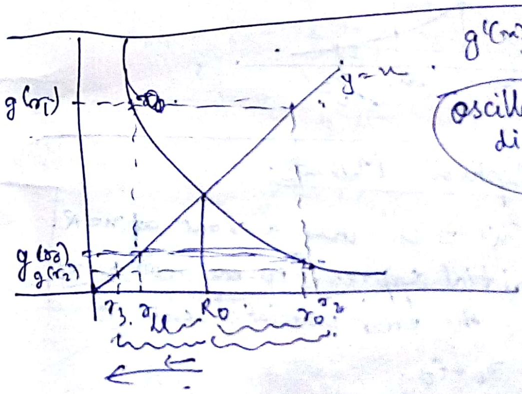
This will be possible if $|g'(c_i)| < 1$.

We represent eqn $x = g(x)$ as a combination of two func $y = x$ & $y = g(x)$ intersecting at $x = R$.



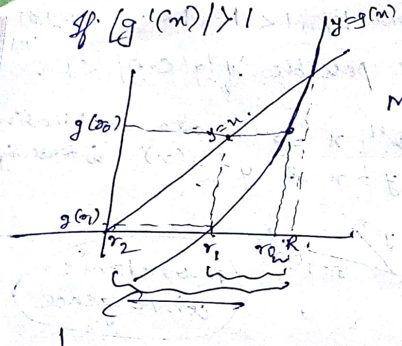
Monotonic convergence

$g'(x) > 1$

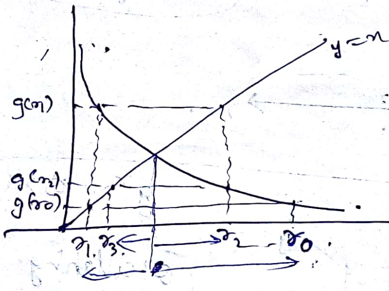


Oscillating divergence

If $|g'(x)| > 1$



Monotonic divergence



oscillating convergence.

Newton Raphson Method.

Let $f(x) = 0$ has a root at $x=R$.

Let the initial approxn. to the root be r_0

Let e_0 be the error associated with r_0

$$R = r_0 + e_0.$$

now $f(R) = 0$

or $f(r_0 + e_0) = 0$

Using Taylor series expansion,

$$f(r_0) + e_0 f'(r_0) + \frac{e_0^2}{2!} f''(r_0) + \cdots = 0$$

we assume $|e_0| << 1$

$\therefore$ we neglect e_0^2 and higher order terms in the series.

$$f(r_0) + e_0 f'(r_0) = 0$$

or $e_0 = -\frac{f(r_0)}{f'(r_0)}$

We can generate the next approximation to the root as $r_1 = r_0 + e_0 = r_0 - \frac{f(r_0)}{f'(r_0)}$

continuing this way, we generate a sequence of approximations as -

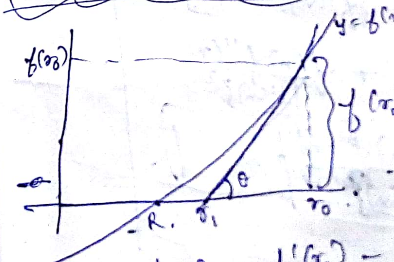
$$r_1 = r_0 - \frac{f(r_0)}{f'(r_0)}$$

$$r_2 = r_1 - \frac{f(r_1)}{f'(r_1)}$$

$$\vdots$$

$$r_{n+1} = r_n - \frac{f(r_n)}{f'(r_n)}.$$

continue until $|r_{n+1} - r_n| <$ required precision.



$$\tan\theta = f'(r_0) = \frac{f(r_0)}{r_0 - r_1}.$$

$$r_0 - r_1 = \frac{f(r_0)}{f'(r_0)} \Rightarrow r_1 = r_0 - \frac{f(r_0)}{f'(r_0)}$$

If $|f'(r_1)| << 1$, the method fails.

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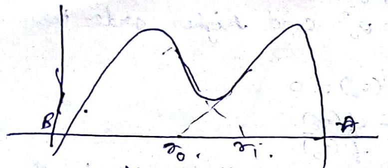

the line never reaches A or B

NR method: $r_{n+1} = r_n - \frac{f(r_n)}{f'(r_n)}$

Iterative method: $r_{n+1} = g(r_n)$

$\therefore g(n) = \cancel{x} - \frac{f(n)}{f'(n)}$

For convergence,

$|g'(n)| < 1$

$g'(n) = 1 - \left[\frac{\{f'(n)\}^2 - f(n) f''(n)}{\{f'(n)\}^2}\right]$

$= 1 - 1 + \frac{f(n) f''(n)}{\{f'(n)\}^2}$

$\Rightarrow \boxed{\left|\frac{f(n) f''(n)}{\{f'(n)\}^2}\right| < 1}$

$f(n) = n^3 - n - 3$. $\quad f'(n) = 3n^2 - 1$, $f''(n) = 6n$

Solve $f(n) = 0$ using Newton Raphson method.

Take $r_0 = 0$

i	r_i	r_{i+1}	$\lvert r_{i+1} - r_i \rvert$	$f(r_i) \cdot f''$	$\left\lvert\frac{f(r_i) f''(r_i)}{\{f'(r_i)\}^2}\right\rvert$
0	0	-3	3	-3	0 < 1
1	-3	-1.96	1.04		0.718 < 1
2	-1.96	-1.146	0.82	-8.56	0.91 < 1
3	-1.146	~~0.004~~ -0.274		-3.359	2.02 > 1 (Stop)
4	-0.274	-3.198			

Order of convergence for N-R method.

$r_{i+1} = R - e_{i+1}$

$r_i = R - e_i \Rightarrow R = r_i + e_i$

$f(r_i + e_i) = 0$

Expanding by Taylor's formula

$$f(r_i) + e_i f'(r_i) + \frac{e_i^2}{2!} f''(r_i) + \frac{e_i^3}{3!} f'''(r_i) + \dots = 0$$

we assume $|e_i| << 1$, so we neglect e_i^3 and higher order terms,

$$f(r_i) + e_i f'(r_i) + \frac{e_i^2}{2!} f''(r_i) = 0$$

or $$\frac{f(r_i)}{f'(r_i)} + e_i + e_i^2 \frac{f''(r_i)}{2f'(r_i)} = 0$$

$$-\frac{f(r_i)}{f'(r_i)} = (R - r_i) + \frac{e_i^2 f''(r_i)}{2f'(r_i)}$$

$$\Rightarrow r_i - \frac{f(r_i)}{f'(r_i)} = R + \frac{e_i^2 f''(r_i)}{2f'(r_i)}$$

$$\Rightarrow r_{i+1} = R + \frac{e_i^2 f''(r_i)}{2f'(r_i)}$$

$$\Rightarrow r_{i+1} - R = \frac{e_i^2 f''(r_i)}{2f'(r_i)} \Rightarrow -e_{i+1} = \frac{e_i^2 f''(r_i)}{2f'(r_i)}$$

or $$|e_{i+1}| = |e_i|^2 \left| \frac{f''(r_i)}{2f'(r_i)} \right|$$

$\therefore$ order of convergence $= 2$.

$\rightarrow \log|e_{i+1}| = \log|K_p| + 2\log|e_i|$

or $$\frac{\log|e_{i+1}|}{\log|e_i|} = \frac{\log|K_p|}{\log|e_i|} + 2$$

For e.g.

$f(x) = x^2 - x - 6$.

$f'(x) = 2x - 1$

$r_0 = 2.5$.

i	r_i	r_{i+1}	$\lvert r_{i+1} - r_i \rvert$	$\lvert g'(x) \rvert$	Order of convergence n
0	2.5	3.0625	0.5625	<1	—
1	3.0625	3.000762	0.0617	<1	4.84
2	3.000762	2.99695	0.0038	<1	2.00
3	2.997	3.000001	0.00305	<1	1.00
4	3.000001	2.999996	0.000005	<1	2.107
5	2.999996	3.000000	0.0000041	<1	1.02

For fixed point iteration method :

$$(R-r_{i+1}) = (R-r_i)\, g'(c_i) \quad , r_i < c_i < R$$

Let $|g'(c_i)| < 1$ for all c_i's in the interval $[r_0, R]$

and let $|g'(c_i)| = k$.

We can write,

$$R - r_{i+1} = k(R - r_i) \quad -①$$

$$R - r_{i+2} = k(R - r_{i+1}) \quad -②$$

① ÷ ② .

$$\frac{(R-r_{i+1})}{(R-r_{i+2})} = \frac{(R-r_i)}{(R-r_{i+1})}$$

$$\Rightarrow (R-r_{i+1})^2 = (R-r_i)(R-r_{i+2})$$

$$\Rightarrow R^2 - 2Rr_{i+1} + (r_{i+1})^2 = R^2 + r_i r_{i+2} - R(r_i + r_{i+2})$$

$$\Rightarrow R(r_i - 2r_{i+1} + r_{i+2}) = r_i r_{i+2} - (r_{i+1})^2$$

$$\Rightarrow R = \frac{r_i r_{i+2} - (r_{i+1})^2}{r_i + r_{i+2} - 2r_{i+1}} = \boxed{\frac{(r_{i+1})^2 - r_i r_{i+2}}{2r_{i+1} - r_i - r_{i+2}}}$$

i	r_i	r_{i+1}	r_{i+2}
0	1	2.64575	
1	2.6	2.94037	

} r_0, r_1, r_2 (3.0046)

2	3.0046	3.00076	
3	3.00076	3.00013	} 3.000006
4	3.00006	3.000001	

$$f(x) = x^3 - 4x^2 - 3x + 18$$
$$= x^2(x+2) - 6x(x+2) + 9(x+2)$$
$$= (x+2)(x^2 - 6x + 9) = (x+2)(x-3)^2$$

$$f'(x) = 3x^2 - 8x - 3.$$

$r_0 = 2.5$.

i	r_i	r_{i+1}	$e_i = \lvert r_{i+1} - r_i \rvert$
0	2.5	2.7647	0.264706
1	2.7647	2.86974	0.105035
2	2.8674	2.93575	0.0660

The value converges very fast to the root as at $x=3$ $f(x) \to 0$ also $f'(x) \to 0$

Let $f(x) = (x-a)^n g(x)$

i.e. there exists multiple roots of order n at $x = a$.

$$\therefore f^{n-1}(a) = f^{(n-2)}(a) = \cdots = f''(a) = f'(a) = f(a) = 0.$$

$$f'(x) = n(x-a)^{n-1} g(x) + (x-a)^n g'(x)$$

$$\Rightarrow (x-a) f'(x) = n(x-a)^n g(x) + (x-a)^{n+1} g'(x)$$

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$= n f(x) + \underbrace{(x-a)^{n+1} g'(x)}_{\text{tends to } 0}$

As $x \to a$,

$(x-a) = n f(x) / f'(x)$

or $\boxed{a = x - n \dfrac{f(x)}{f'(x)}}$

For multiple roots

i	r_i	$r_{i+1} = r_i - 2\dfrac{f(r_i)}{f'(r_i)}$	$e_i = \lvert r_{i+1} - r_i \rvert$
0	2.5	3.0294	0.5294
1	3.0294	3.00009	0.0293
2	3.00009	3.000000	0.00009

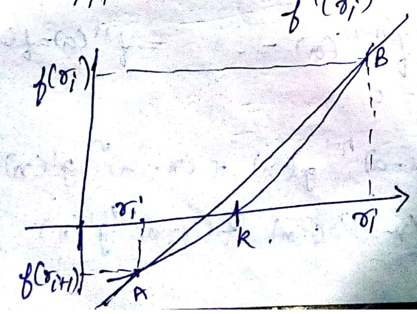
n
5.55
2.839

SECANT METHOD:

For N-R ~~mthd~~ we have to provide the functional representation of both $f(x)$ & $f'(x)$.

$f'(x)$ may not be easy to compute

$$r_{i+1} = r_i - \frac{f(r_i)}{f'(r_i)}$$



Slope of AB

$= \dfrac{f(r_i) - f(\ldots)}{r_i - \ldots}$

$\approx f'(r_i)$

(only if $r_i \approx R$)

it is an off

$$\therefore r_{i+1} = r_i - \frac{f(r_i)(r_i - r_{i-1})}{f(r_i) - f(r_{i-1})}$$

$$= \frac{f(r_i) r_{i-1} - f(r_{i-1}) r_i}{f(r_i) - f(r_{i-1})}$$

It differs from Regula falsi in the respect of interval checking hence it has the possibility of divergence.

R is the actual root of $f(x) = 0$

$\therefore f(R) = 0$

Let $r_{i-1} = R - e_{i-1}$

$r_i = R - e_i$

$r_{i+1} = R - e_{i+1}$

$$r_{i+1} = \frac{r_{i-1} \cdot f(r_i) - r_i f(r_{i-1})}{f(r_i) - f(r_{i-1})}.$$

$$\Rightarrow R - e_{i+1} = \frac{(R-e_{i-1}) f(R-e_i) - (R-e_i) f(R-e_{i-1})}{f(R-e_i) - f(R-e_{i-1})}$$

$$\Rightarrow R - e_{i+1} = \frac{(R-e_{i-1})\{f(R) - e_i f'(R) + \frac{e_i^2}{2} f''(R) \ldots\} - (R-e_i)\{f(R) - e_{i-1} f'(R) + \frac{e_{i-1}^2}{2} f''(R)\}}{\{f(R) - e_i f'(R) \ldots\} - \{f(R) - e_{i-1} f'(R)\}}$$

$$= \frac{(R-e_{i-1})\{-e_i f'(R) + \frac{e_i^2}{2} f''(R)\} - (R-e_i)\{-e_{i-1} f'(R) + \frac{e_{i-1}^2}{2} f''(R)\}}{f'(R)(e_{i-1} - e_i)}$$

$$+Re_{i-1}f'(R) - R\frac{e_{i-1}^2}{2}f''(R) + e_i\frac{e_{i-1}^2}{2}f''(R)$$
$$\overline{f'(R)(e_{i-1}-e_i)}$$

$$\approx R\frac{\cancel{\frac{f''(R)}{2}(-e_i^2+e_{i-1}^2)}+Rf'(R)(e_{i-1}-e_i)}{f'(R)(e_{i-1}-e_i)}$$

$$:= \frac{Rf'(R) - \cancel{\frac{Rf''(R)}{2}(e_{i-1}+e_i)}}{f'(R)}$$

$$\cancel{\approx R -}$$

$$\approx \frac{Rf'(R)(e_{i-1}-e_i) - \frac{e_{i-1}e_i}{2}f''(R)(e_{i-1}-e_i)}{f'(R)(e_{i-1}-e_i)}$$

$$\approx R - \frac{e_{i-1}e_i f''(R)}{2f'(R)}$$

$$\therefore R - e_{i+1} \approx R - \frac{e_{i-1}e_i f''(R)}{2f'(R)}$$

$$-e_{i+1} \approx -\frac{e_{i-1}e_i f''(R)}{2f'(R)}$$

$$\therefore e_{i+1} \approx \frac{e_{i-1}e_i}{2}\frac{f''(R)}{f'(R)} \quad \text{—(1)}$$

Let us assume the order of convergence be n.

$$\therefore e_{i+1} = ke_i^n \quad \text{—(2)}$$

$$\& \quad e_i = k e_{i-1}^n \quad \text{—(3)}$$

$$\therefore e_{i-1} = \left(\frac{e_i}{k}\right)^{1/n} \quad \text{—(4)}$$

with (1), (2) & (4), we can write.

$$ke_i^n \approx \left(e_i/k\right)^{1/n} e_i \frac{f''(R)}{2f'(R)}$$

$$\approx e_i^{1+1/n}\frac{f''(R)}{2k^{1/n}f'(R)}.$$

$$\therefore e_i^n \approx e_i^{1+1/n}\frac{f''(R)}{2k^{1+1/n}f'(R)}.$$

Approx. equating powers of e_i from both sides.

$$n = 1 + 1/n.$$

$$\Rightarrow n^2 - n - 1 = 0$$

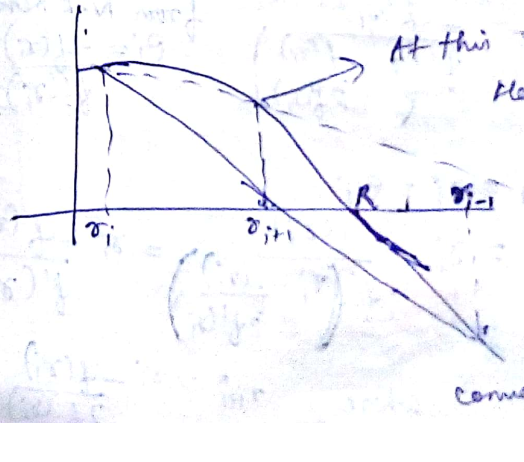
$$\Rightarrow n = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1\pm\sqrt{5}}{2} = \frac{1\pm 2.236}{2}$$

$$n = 1.618, \; -0.61$$

1.618 → method converges. → better than regular falsi.

−0.61 → method diverges.

Spcl. Case



At this $f'(x_{i+1}) \to 0$ Hence root diverges

$|f'(x_{i+1})|$ can be used to check if the method converges or diverges

Multipoint Iteration Method.

Let R be the actual root of $f(x)=0$ & r_i be an approxi. to the root.

$\therefore r_i = R - e_i$

or $R = r_i + e_i$

Now $f(R) = f(r_i + e_i) = 0$

Expanding by taylor's formula

$f(r_i) + e_i f'(r_i) + \frac{e_i^2}{2} f''(r_i) + \frac{e_i^3}{3!} f'''(r_i) + \dots = 0$

$\Rightarrow f(r_i) + e_i \underbrace{\left\{ f'(r_i) + \frac{e_i}{2} f''(r_i) + \frac{e_i^2}{3!} f'''(r_i) + \dots \right\}}_{\approx f'(r_i + e_i/2)} = 0$

$\Rightarrow f(r_i) + e_i f'(r_i + e_i/2) = 0$

$\Rightarrow e_i = \frac{-f(r_i)}{f'(r_i + e_i/2)}$

$= \frac{-f(r_i)}{f'\left(r_i - \frac{f(r_i)}{2f'(r_i)}\right)}$ from N-R method $e_i = \frac{-f(r_i)}{f'(r_i)}$

$\therefore r_{i+1} = r_i + e_i$

$= r_i - \frac{f(r_i)}{f'\left(r_i - \frac{f(r_i)}{2f'(r_i)}\right)} = r_i - \frac{f(r_i)}{f'(r_{i+1}^*)}$

where $r_{i+1}^* = r_i - \frac{f(r_i)}{2f'(r_i)}$

for a given r_i

$r_{i+1}^* = r_i - \frac{f(r_i)}{2f'(r_i)}$

$r_{i+1} = r_i - \frac{f(r_i)}{f'(r_{i+1}^*)}$

$R = r_k + e_k$

$R = r_{k+1} + e_{k+1}$

$r_{k+1}^* = r_k - \frac{f(r_k)}{2f'(r_k)} = R - e_k - \frac{f(r_k)}{2f'(r_k)}$

$= (R - e_k) - \boxed{\frac{f(R-e_k)}{2f'(R-e_k)}}$

$\frac{f(R-e_k)}{f'(R-e_k)} = \frac{f(R) - e_k f'(R) + \frac{e_k^2}{2} f''(R) - \frac{e_k^3}{3!} f'''(R) + \dots}{f'(R) - e_k f''(R) + \frac{e_k^2}{2} f'''(R) - \dots}$

$\boxed{f(R)=0}$ $= \frac{-e_k + \frac{e_k^2}{2} \frac{f''(R)}{f'(R)} - \frac{e_k^3}{3!} \frac{f'''(R)}{f'(R)}}{1 - e_k \frac{f''(R)}{f'(R)} + \frac{e_k^2}{2} \frac{f'''(R)}{f'(R)}}$

Let $\frac{f''(R)}{f'(R)} = a_1$ and $\frac{f'''(R)}{f'(R)} = a_2$

$= \frac{\left(-e_k + \frac{e_k^2}{2} a_1 - \frac{e_k^3}{3!} a_2\right)}{\left(1 - e_k a_1 + \frac{e_k^2}{2} a_2\right)}$

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$$= \left(-e_k + \frac{e_k^2}{2}a_1 - \frac{e_k^3}{3!}a_2\right)\left(1 - e_k a_1 + \frac{e_k^2}{2}a_2\right)^{-1}$$

$$\simeq \left(-e_k + \frac{e_k^2}{2}a_1 - \frac{e_k^3}{3!}a_2\right)\left(1 + e_k a_1 - \frac{e_k^2}{2}a_2\right)$$

$$= -e_k + \frac{e_k^2}{2}a_1 - \frac{e_k^3}{3!}a_2 + \frac{e_k^3}{2}a_1 - e_k^2 a_1 - \frac{e_k^4 a_1 a_2}{3!} + \frac{e_k^3}{2}a_2 - \frac{e_k^4}{4}a_1 a_2 + \frac{e_k^5}{2 \times 3!}a_2^2$$

$$= -e_k + \frac{e_k^2}{2}a_1 - \frac{e_k^3}{3!}a_2 + \frac{e_k^3 a_1^2}{2} - e_k^2 a_1 + \frac{e_k^3}{2}a_2$$

$$= -e_k - \frac{e_k^2 a_1}{2} + e_k^3\left(\frac{a_1^2}{2} + \frac{a_2}{3}\right).$$

$$\therefore r^*_{k+1} = (R - e_k) - \frac{1}{2}\left\{-e_k - \frac{a_1 e_k^2}{2} + \left(\frac{a_1^2}{2} + \frac{a_2}{3!}\right)\right.$$

$$= R - \frac{e_k}{2} + \frac{a_1}{4}e_k^2 - \left(\frac{a_1^2}{4} + \frac{a_2}{6}\right)e_k^3.$$

$$r_{k+1} = r_k - \frac{f(r_k)}{f'(r^*_{k+1})}$$

$$R - e_{k+1} = R - e_k - \frac{f(R - e_k)}{f'\left(R - \left[\frac{e_k}{2} - \frac{a_1}{4}e_k^2 + \left(\frac{a_1^2}{4} + \frac{a_2}{6}\right)e_k^3\right]\right)}$$

$$\frac{f(R - e_k)}{f'\left(R - \left\{\frac{e_k}{2} - \frac{a_1}{4}e_k^2 + \left(\frac{a_1^2}{4} + \frac{a_2}{6}\right)e_k^3\right\}\right)}$$

$$= \frac{f(R) - e_k f'(R) + \frac{e_k^2}{2}f''(R) - \frac{e_k^3}{3!}f'''(R)}{f'(R) - E f''(R) + \cdots}$$ (By taylor series)

$$= \frac{-e_k + \frac{e_k^2}{2}a_1 - \frac{e_k^3}{3!}a_2}{1 - \left\{\frac{e_k}{2} - \frac{a_1}{4}e_k^2 + \left(\frac{a_1^2}{4} + \frac{a_2}{6}\right)e_k^3\right\}a_1}$$ $\boxed{a_1 = \frac{f''(R)}{f'(R)}}$

$$= \left[-e_k + \frac{e_k^2}{2}a_1 - \frac{e_k^3 a_2}{6}\right]\left[1 + \frac{e_k}{2}a_1 - \frac{a_1^2 e_k^2}{4} + \left(\frac{a_1^3}{4} + \frac{a_1 a_2}{6}\right)\ldots\right.$$

$$= -e_k - \cancel{\frac{e_k^2}{2}a_1} + \cancel{\frac{e_k^2}{2}a_1} + \frac{a_1^2 e_k^3}{4} + \frac{e_k^3 a_1^2}{4} - \frac{e_k^3 a_2}{6}$$

$$\therefore R - e_{k+1} = R - e_k - \left(-e_k + \frac{a_1^2}{2}e_k^3 - \frac{e_k^3}{6}a_2\right)$$

$$\Rightarrow -e_{k+1} = e_k^3\left(\frac{a_2}{6} - \frac{a_1^2}{2}\right)$$

$$\Rightarrow e_{k+1} = e_k^3\left(\frac{a_1^2}{2} - \frac{a_2}{6}\right)$$ $\boxed{\text{find up to } e_k^4}$

$\therefore$ order of convergence = 3

For same multiple roots, order of convergence becomes < 3. (In my case $r = 1.1$)

$(x+2)(x-3)(x+1)$

$(x^2 - x - 6)(x+1)$

$x^3 - x^2 - 6x + x^2 - x - 6$

$x^3 - 7x - 6$

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Chebyshev Method.

Let r_k be the initial approx to the root R of $f(x) = 0$.

$$\therefore R = r_k + e_k.$$

$$f(R) = f(r_k + e_k) = 0$$

$$\Rightarrow f(r_k) + e_k f'(r_k) + \frac{e_k^2}{2} f''(r_k) + \frac{e_k^3}{6} f'''(r_k) \dots = 0$$

neglecting e_k^3 and higher order term

$$-e_k f'(r_k) = f(r_k) + \frac{e_k^2}{2} f''(r_k).$$

$$\Rightarrow -e_k = \frac{f(r_k)}{f'(r_k)} + \frac{e_k^2}{2} \frac{f''(r_k)}{f'(r_k)}.$$

$$\therefore R = r_k + e_k$$

$$= r_k - \frac{f(r_k)}{f'(r_k)} - \frac{e_k^2 \cdot f''(r_k)}{2 \cdot f'(r_k)}.$$

$$\boxed{\text{order of convergence} = 3}$$

From N-R method.

$$e_k = -\frac{f(r_k)}{f'(r_k)}$$

$$\therefore R = r_{k+1} = r_k - \frac{f(r_k)}{f'(r_k)} - \frac{f(r_k)^2 f''(r_k)}{2\{f'(r_k)\}^3}$$

$|f'(r_k)| < \epsilon$ [condition for checking the method converges].

Newton, Chebyshev method have a very high chance of diverging from the actual root. They are computation intensive.

<u>Find The order of convergence.</u>

(Lin's method)

Finding the complex roots of a polynomial eqn.

$$x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1}x + a_n = 0$$

let x^2+px+q be a quad. factor of ~~eqn~~ poly.

$$(x^2+px+q)(x^{n-2} + b_1 x^{n-3} + b_2 x^{n-4} + \cdots + b_{n-3}x + b_{n-2}) = 0$$

let p_0, q_0 be the approximated values

then

$$(x^2+p_0x+q_0)(x^{n-2} + b_1 x^{n-3} + b_2 x^{n-4} + \cdots + b_{n-3}x + b_{n-2}) + \boxed{Rx+S} = 0$$

linear Remainder term.

This should be equal to zero if the original polynomial is exactly divisible by (x^2+px+q)

$$x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} + a_n = (x^2+px+q)(x^{n-2} + b_1 x^{n-3} + b_2 x^{n-4} + b_3 x^{n-5} + \cdots + b_{n-3} x + b_{n-2}) + Rx + S$$

$$\Rightarrow \quad = x^n + (b_1+p)x^{n-1} + (b_2+pb_1+q)x^{n-2} + (b_3+pb_2+qb_1)x^{n-3} + \cdots$$

Equating the coefficients from both sides we get –

$$a_1 = b_1 + p \Rightarrow b_1 = a_1 - p$$

$$a_2 = b_2 + pb_1 + q \Rightarrow b_2 = a_2 - pb_1 - q.$$

$$a_3 = b_3 + pb_2 + qb_1 \Rightarrow b_3 = a_3 - pb_2 - qb_1$$

$$\vdots$$

$$a_j = b_j + pb_{j-1} + qb_{j-2} \Rightarrow b_j = a_j - pb_{j-1} - qb_{j-2} \quad j = 3, 4, \ldots, n-2 \quad (3)$$

$$\Rightarrow R = a_{n-1} - pb_{n-2} - qb_{n-3} \quad (4)$$

$$\Rightarrow S = a_n - qb_{n-2} \quad (5)$$

$$a_{n-1} - qb_{n-3} = pb_{n-2}$$

Let the initial approx value for p & q be p_0, q_0.

By equating ④ & ⑤ to 0
we generate new value for p & q (say) (p_1, q_1)

$\max(|p_0 - p_1|, |q_0 - q_1|) < \epsilon$
cond
$\max(|p_{i-1} - p_i|, |q_{i-1} - q_i|) < \epsilon$.

$\boxed{\text{Bairstow's method}}$

$\boxed{R(p,q)}$, $\boxed{S(p,q)}$.

R & S are f^n of p & q.

Let, (p_0, q_0) be the approx. to (p,q).
and Δp & Δq be the corrections needed to get the true values of (p,q).

$\therefore R(p_0 + \Delta p, q_0 + \Delta q) = 0$

$\Rightarrow R(p_0, q_0) + \Delta p \frac{\partial R}{\partial p}\Big|_{(p_0,q_0)} + \Delta q \frac{\partial R}{\partial q}\Big|_{(p_0,q_0)} + \dots$

$S(p_0 + \Delta p, q_0 + \Delta q) = 0$

$\Rightarrow S(p_0, q_0) + \Delta p \frac{\partial S}{\partial p}\Big|_{(p_0,q_0)} + \Delta q \frac{\partial S}{\partial q}\Big|_{(p_0,q_0)} + \dots = 0$.

neglecting the higher order terms.

$$\Delta p \frac{\partial R}{\partial p}\Big|_{(p_0,q_0)} + \Delta q \frac{\partial R}{\partial q}\Big|_{q_0,p_0} = -R(p_0,q_0)$$

$$\Delta p \frac{\partial S}{\partial p}\Big|_{(p_0,q_0)} + \Delta q \frac{\partial S}{\partial q}\Big|_{(p_0,q_0)} = -S(p_0,q_0)$$

$$\Delta p = \frac{S(p_0,q_0) \frac{\partial R}{\partial q}\Big| - R(p_0,q_0) \frac{\partial S}{\partial q}\Big|}{\frac{\partial R}{\partial p}\Big| \frac{\partial S}{\partial q}\Big| - \frac{\partial S}{\partial p}\Big| \cdot \frac{\partial R}{\partial q}\Big|}$$

$$\Delta q = \frac{\cancel{S(p_0,q_0) \frac{\partial R}{\partial p}\Big| + R(p_0,q_0) \frac{\partial S}{\partial p}\Big|}}{\cancel{\frac{\partial R}{\partial p}\Big| \frac{\partial S}{\partial q}\Big| - \frac{\partial S}{\partial p}\Big| \frac{\partial R}{\partial q}\Big|}}$$

$$\frac{\partial R}{\partial p} = -p \frac{\partial b_{n-2}}{\partial p} - b_{n-2} - q \frac{\partial b_{n-3}}{\partial p}$$

Partial derivatives of equation ③ are:

w.r.t p.

$c_1 = \frac{\partial b_1}{\partial p} = -1$

$c_2 = \frac{\partial b_2}{\partial p} = -p \frac{\partial b_1}{\partial p} - b_1 = -p - b_1$

$c_3 = -pc_2 - b_2 - qc_1$

$\vdots$

$\boxed{c_j = -b_{j-1} - pc_{j-1} - qc_{j-2}}$

$\frac{\partial R}{\partial p}\Big| = -b_{n-2} - pc_{n-2} - qc_{n-3}$

$\frac{\partial S}{\partial p}\Big| = -q c_{n-2}$.

w.r.t q.

$d_1 = \frac{\partial b_1}{\partial q} = 0$

$d_2 = \frac{\partial b_2}{\partial q} = -1$

$d_3 = -pd_2 - b_1 - qd_1$

$d_j = -pd_{j-1} - b_{j-2} - qd_{j-3}$

$\frac{\partial R}{\partial q}\Big| = -pd_{n-2} - b_{n-3} - q d_{n-3}$

$\frac{\partial S}{\partial q}\Big| = -b_{n-2} - q d_{n-2}$

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Express Δp & Δq in terms a_i, b_i, c_i,

$$\Delta p = \frac{(a_n - q_0 b_{n-2})(-p_0 d_{n-2} - b_{n-3} - q_0 d_{n-3}) - (a_{n-1} - p_0 b_{n-2} - q_0 b_{n-3})(-b_{n-2} - q_0 d_{n-2})}{(-b_{n-2} - p_0 c_{n-3} - q_0 d_{n-3})(-b_{n-2} - q d_{n-2}) - (-q c_{n-2})(-p d_{n-2} - b_{n-3} - q d_{n-3})}$$

$$= a_n p d_{n-2} - a_n b_{n-3} - a_n d_{n-3} q_0 + q_0^2 b_{n-2} d_{n-1} + a_{n-1} b_{n-2} +$$

$=$

~~[illegible]~~

Solution of linear simultaneous equ^ns:

$$x_1 + 5x_2 + 3x_3 = 10 \quad -1$$
$$x_1 + 3x_2 + 2x_3 = 5 \quad -2$$
$$2x_1 + 4x_2 - 6x_3 = -4. \quad -3.$$

1 − 2 →

$$2x_2 + x_3 = 5 \quad —4$$

2 × 2 − 3 →.

$$2x_2 + 10x_3 = 14. \quad -5.$$

5 − 4 → $9x_3 = 9$

$\boxed{\text{Back substitution}}$

Gaussian elimination method.

$$x_1 + 5x_2 + 3x_3 = 10$$
$$\boxed{x_1} + 3x_2 + 2x_3 = 5 \xrightarrow{2-1} -2x_2 - x_3 = -5$$
$$\boxed{2x_1} + 4x_2 - 6x_3 = -4 \xrightarrow{3-2\times1} -6x_2 - 12x_3 = -24$$

$$\Rightarrow \quad x_1 + 5x_2 + 3x_3 = 10 \quad -1$$
$$-2x_2 - x_3 = -5 \quad -4$$
$$\boxed{-6x_2} - 12x_3 = -24 \quad -5$$

$$\Rightarrow \quad x_1 + 5x_2 + 3x_3 = 10 \quad -1$$
$$-2x_2 - x_3 = -5 \quad -4$$
$$-9x_3 = -9 \quad -6 \quad (5 - 3\times4)$$

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = a_{1(n+1)}$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = a_{2(n+1)}$

$a_{31}x_1 + a_{32}x_2 + a_{33}x_3$

$\vdots$

$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n = a_{n(n+1)}$

Steps

$(-① \times m_{i1} + (i)) \Rightarrow$ generating new $(n-1)$ equations such that x_1 is eliminated

$$\boxed{m_{i1} = \frac{a_{i1}}{a_{11}} \quad i = 2, \ldots, n}$$

Coefficients change acc. to: $a_{ij}' = a_{ij} - m_{i1} a_{1j}$

$i = 2, \ldots, n$

$j = 1, 2, \ldots, n$

$m_{i2} = \dfrac{a_{i2}}{a_{22}}$

$a_{ij}' = a_{ij} - m_{i2} * a_{2j} \quad i = 3, \ldots, n$

$j = 2, \ldots, n$

At k stage:

$a_{kk}x_k + a_{k(k+1)}x_{k+1} + \cdots + a_{k(n-1)}x_{n-1} + a_{kn}x_n = a_{k(n+1)}$

$(k+1) \rightarrow a_{(k+1)k}x_k + a_{(k+1)(k+1)}x_{k+1} + \cdots + a_{(k+1)(n-1)}x_{n-1} + a_{(k+1)n}x_n = a_{(k+1)(n+1)}$

$\rightarrow (n)\ a_{nk}x_k + a_{n(k+1)}x_{k+1} + \cdots + a_{n(n-1)}x_{n-1} + a_{nn}x_n = a_{n(n+1)}$

$m_{ik} = \dfrac{a_{ik}}{a_{kk}} \quad i = (k+1), \ldots, n$

$a_{ij}' = a_{ij} - m_{ik} * a_{kj}$

$i = (k+1), \ldots, n$

$j = k, \ldots, (n+1)$

Problem: if $|a_{kk}| \ll |a_{ik}|$ then m_{ik} is very large (overflow may occur)

To avoid this,

partial pivoting is done which means row having largest first non-zero element is swapped with the first row of concerned $n \times k$ matrix

Ans:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & a_{1(n+1)} \\ a_{21} & a_{22} & \cdots & a_{2n} & a_{2(n+1)} \\ \vdots & & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & a_{n(n+1)} \end{array}\right]$$

$n \times (n+1)$

$\Downarrow$

$$\left[\begin{array}{cccccc} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & a_{1(n+1)} \\ & a_{22} & a_{23} & \cdots & a_{2n} & a_{2(n+1)} \\ & & a_{33} & & & a_{3(n+1)} \\ & & & & a_{nn} & a_{n(n+1)} \end{array}\right]$$

$$x_n = \frac{a_{n(n+1)}}{a_{nn}}$$

$$x_{n-1} = \frac{a_{(n-1)(n+1)} - a_{(n-1)n}\, x_n}{a_{(n-1)(n-1)}}$$

In general, $x_i = \dfrac{a_{i(n+1)} - \sum_{j=i+1}^{n} a_{ij} * x_j}{a_{ii}}$

$i = (n-1), (n-2), \ldots, 2, 1)$

Input the coefficient matrix $A[n][n+1]$

```
for (k=1; k<n; k++)
{
    big = 0;
    for (i=k; k<=n; i++)            //finding the coeff. of largest magn. along the column k.
    {
        if (fabs(a[i][k]) > big)
        {   big = a[i][k];
            p = i;
        }
    }
    for (j=k; j<=(n+1); j++)        //swapping the row containing the largest magnitude coeff. in the kth column with kth row.
    {   temp = a[k][j];
        a[k][j] = a[p][j];
        a[p][j] = temp;
    }
    for (i=k+1; i<=n; i++)
    {   m[i][k] = a[i][k]/a[k][k];
        for (j=k; j<=(n+1); j++)
            a[i][j] = a[i][j] - m[i][k]* a[k][j];
    }
}   //elimination process.
```

```
x[n] = a[n][n+1]/a[n][n];
for (i=(n-1); i>=1; i--)
{
    sum = 0;
    for (j=n; j>i; j--)
    sum = sum + a[i][j] * x[j];
    x[i] = (a[i][n+1] - sum)/a[i][i];
}
```

$$x^2+14x+1 \overline{\big) 4x^3 + 7x^2 + 8x + 1} \quad 4x - 49$$

$4x^3 + 56x^2 + 4x$

$0 \quad -49x^2 + 4x + 1$

$-49x^2 - 14\times49x - 49$

$$x-2 \overline{\big) x^3 - x - 6} \quad x^2 + 2x + 3$$

$x^3 - 2x^2$

$2x^2 - 3, +1 \; 2x^2 - x$

x^3

Equ^n:

$x_1 + 5x_2 + 3x_3 = 10$

$x_1 + 3x_2 + 2x_3 = 5$

$2x_1 + 4x_2 - 6x_3 = -4$.

Gaussian elimination

Augmented coefficient matrix

$$\left[\begin{array}{ccc|c} 1 & 5 & 3 & 10 \\ 1 & 3 & 2 & 5 \\ 2 & 4 & -6 & -4 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 2 & 4 & -6 & 4 \\ 1 & 3 & 2 & 5 \\ 1 & 5 & 3 & 10 \end{array}\right] R_1 \leftrightarrow R_3$$

$$\Rightarrow \left[\begin{array}{ccc|c} 2 & 4 & -6 & -4 \\ 1-1 & 3-2 & 2+3 & 5+2 \\ 1-1 & 5-2 & 3+3 & 10+2 \end{array}\right] \begin{array}{l} R_2 \to R_2 - R_1/2 \\ R_3 \to R_3 - R_1/2 \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 2 & 4 & -6 & -4 \\ 0 & 1 & 5 & 7 \\ 0 & 3 & 6 & 12 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 2 & 4 & -6 & -4 \\ 0 & 3 & 6 & 12 \\ 0 & 1 & 5 & 7 \end{array}\right] R_2 \leftrightarrow R_3$$

$$\Rightarrow \left[\begin{array}{ccc|c} 2 & 4 & -6 & -4 \\ 0 & 3 & 6 & 12 \\ 0 & 1-1 & 5-2 & 7-4 \end{array}\right] R_3 \to R_3 - R_2/3,$$

$$\Rightarrow \left[\begin{array}{ccc|c} 2 & 4 & -6 & -4 \\ 0 & 3 & 6 & 12 \\ 0 & 0 & 3 & 3 \end{array}\right]$$

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1(n-1)}x_{n-1} + a_{1n}x_n = a_{1(n+1)}$

$\vdots$

$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{n(n-1)}x_{n-1} + a_{nn}x_n = a_{n(n+1)}$

$m_{ik} = a_{ik}/a_{kk}$

→ Total no. of divisions reqd. for getting multipliers

$= (n-1)+(n-2)+\cdots+1$

$= \frac{n(n-1)}{2}$

→ No. of multipliers reqd. for the elimination process for eliminating the coeff of x_1 from 2^{nd} row to n^{th} row.

$\therefore$ no. of multiplications $= (n+1)(n-1) = n^2-1$

→ Suppose at any stage of elimination we are left with k eqns with k unknowns for eliminating the coefficient along first column
• no. of multiplications required will be (k^2-1).

→ Total no. of multiplications reqd. for entire elimination process to reduce the system of eqns into upper triangular form

$$\sum_{k=n}^{2}(k^2-1) = (n^2+(n-1)^2+\cdots+2^2)-(n-1)$$
$$= (n^2+(n-1)^2+\cdots+2^2+1^2)-n$$
$$= \frac{1}{6}n(n+1)(2n+1) - n.$$
$$= \frac{2n^3+3n^2-5n}{6} = O(n^3).$$

Calculate same for back propagation

$= 1+2+3 - - - - +n$

$= n(n+1)$

Gauss Jordan elimination

Similar to abv:

No. of multipliers req. : $(n-1)(k+1)$

$$= \sum_{k=n}^{2}(n-1)(k+1) = (n-1)\,((n+1)+n+(n-1)+\cdots+2)$$
$$= (n-1)\left[\frac{n(n+1)}{2} - 1\right]$$

$$\left[\begin{array}{cccccc|c} a_{11} & 0 & 0 & \cdots & 0 & 0 & a_{1(n+1)} \\ 0 & a_{22} & 0 & \cdots & 0 & 0 & a_{2(n+1)} \\ \vdots & & & & & & \vdots \\ 0 & 0 & 0 & \cdots & a_{(n-1)(n-1)} & 0 & a_{(n-1)(n+1)} \\ 0 & 0 & 0 & \cdots & 0 & a_{nn} & a_{n(n+1)} \end{array}\right]$$

$$\Downarrow$$

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & \cdots & 0 & 0 & a_{1(n+1)}/a_{11} \\ 0 & 1 & 0 & \cdots & 0 & 0 & a_{2(n+1)}/a_{22} \\ \vdots & & & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & a_{(n-1)(n+1)}/a_{(n-1)(n-1)} \\ 0 & 0 & 0 & \cdots & 0 & 1 & a_{n(n+1)}/a_{nn} \end{array}\right]$$

$m_{ik} = \frac{a_{ik}}{a_{kk}}$

$\boxed{a_{ij} = a_{ij} - m_{ik}a_{kj}}$

$x_n = \frac{a_{n(n+1)}}{a_{nn}}$

$x_i = \frac{a_{i(n+1)} - \sum_{j=i+1}^{n} a_{ij}x_j}{a_{ii}}$

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$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Let solution of this system be $[C_{11}\ C_{21} \cdots C_{n1}]$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & \cdots & & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} \\ C_{21} \\ \vdots \\ C_{n1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\overset{A}{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Let solⁿ of it be $[C_{12}\ C_{22} \cdots C_{2n}]$

We can write

$$A \begin{bmatrix} C_{12} \\ C_{22} \\ \vdots \\ C_{2n} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$ Let solution be

$[C_{1n}\ C_{2n} \cdots C_{nn}]$

We can write : $$A \begin{bmatrix} C_{1n} \\ C_{2n} \\ \vdots \\ C_{nn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Combining all :

$$A \begin{bmatrix} C_{11} & C_{12} & C_{13} & \cdots & C_{1n} \\ C_{21} & C_{22} & C_{23} & \cdots & C_{2n} \\ \vdots & & & & \\ C_{n1} & C_{n2} & C_{n3} & \cdots & C_{nn} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & 1 \cdots & 0 \\ \vdots & & & \vdots \\ & \cdots & 0 & 1\ 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Inverse of a matrix :

$$\left[\begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{matrix}\ \right|\left. \begin{matrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{matrix}\right]$$

$n \times n$ $\qquad$ $n \times n$ Identity matrix

$\Downarrow$ By applying Gauss Jordan elimination

$$\left[\begin{matrix} 1 & 0 & \cdots & & 0 \\ 0 & 1 & \cdots & & 0 \\ \vdots & & & & \vdots \\ & & & 0\ 1 & 0 \\ 0 & 0 & \cdots & 0\ 0 & 1 \end{matrix}\ \right|\left. \begin{matrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & & C_{2n} \\ C & & & \\ \vdots & & & \vdots \\ C_{n1} & C_{n2} & & C_{nn} \end{matrix}\right]$$

1
2
3
4
5
6
7
⋮
N

L-U Decomposition Method

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad B = \begin{bmatrix} \vdots \end{bmatrix}$$

$$AX = B.$$

Let $\boxed{A = LU}$

$$L = \begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & & \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix} \quad U = \begin{bmatrix} 1 & u_{12} & u_{13} & \cdots \\ 0 & 1 & u_{23} & \cdots \\ \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots \end{bmatrix}$$

$$AX = B$$

$$\Rightarrow \begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & & 0 \\ \vdots & & & \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix} \underbrace{\begin{bmatrix} 1 & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & 1 & u_{23} & \cdots & u_{2n} \\ \vdots & \vdots & & & \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}}$$

$$UX = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}.$$

$$\Rightarrow \begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & & 0 \\ \vdots & & & \vdots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Iterative method for soln of linear simultaneous eqn:

$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$

$\vdots$

$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$

$[x_1^{(0)}, x_2^{(0)} \cdots x_n^{(0)}]^T = X^0$

$X^{(1)} = [x_1^{(1)}\ x_2^{(1)} \cdots x_n^{(1)}]^T$

$a_{11}x_1^{(1)} = b_1 - (a_{12}x_2^{(0)} + \cdots + a_{1n}x_n^{(0)})$

$a_{22}x_2^{(1)} = b_2 - (a_{21}x_1^{(0)} + \cdots + a_{2n}x_n^{(0)})$

$\vdots$

$a_{nn}x_n^{(1)} = b_n - (a_{n1}x_1^{(0)} + a_{n2}x_2^{(0)} + \cdots + a_{n(n-1)}x_{n-1}^{(0)})$

Check if

$\max\limits_{1 \le i \le n} \{ |x_i^{(1)} - x_i^{(0)}| \} < \epsilon$

then stop.

else continue the iterations.

Jacobi's method.

$$a_{11} x_1^{(k+1)} = b_1 - (a_{12} x_2^{(k)} + \cdots + a_{1n} x_n^{(k)})$$

$$a_{22} x_2^{(k+1)} = b_2 - (a_{21} x_1^{(k)} + \cdots + a_{2n} x_n^{(k)})$$

$\vdots$

$$a_{nn} x_n^{(k+1)} = b_n - (a_{n1} x_1^{(k)} + a_{n2} x_2^{(k)} + \cdots + a_{n(n-1)} x_{n-1}^{(k)})$$

$$\rightarrow \boxed{x_1^{(0)} = x_2^{(0)} = \cdots = x_n^{(0)} = 0}$$

In matrix notation: $AX = B$

$$X^{k+1} = HX^{(k)} + C$$

where $X^{(k+1)}$ and $X^{(k)}$ are $n \times 1$ (i.e. a column vector)

H is a $n \times n$ matrix depending on A and C is again a column vector depending on A & B.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{(n-1)1} & a_{(n-1)2} & a_{(n-1)3} & \cdots & a_{(n-1)n} \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \qquad A = L + D + U$$

$$L = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ a_{21} & 0 & \cdots & 0 & 0 \\ a_{31} & a_{32} & 0 \cdots & 0 & 0 \\ \vdots & & & & \\ a_{n1} & a_{n2} & \cdots & a_{n(n-1)} & 0 \end{bmatrix} \qquad D = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & & \cdots & a_{nn} \end{bmatrix}$$

$$U = \begin{bmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & 0 & a_{23} & & \\ \vdots & & & & \\ 0 & 0 & \cdots & 0 & a_{(n-1)n} \\ 0 & 0 & & 0 & 0 \end{bmatrix}$$

In matrix notation, the set of equ^{n}s in ① can be written as.

$$DX^{(k+1)} = B - (L+U)X^{(k)}$$

or $X^{(k+1)} = D^{-1}\{B - (L+U)X^{(k)}\}$.

or $X^{(k+1)} - X^{(k)} = D^{-1}\{B - (L+U)X^{k}\} - X^{(k)}$ $\quad I = D^{-1}D$.

$$= D^{-1}\{B - (L+U+D)X^{(k)}\}$$

$$= D^{-1}\{B - AX^{(k)}\}.$$

$$= D^{-1}B - D^{-1}AX^{(k)}$$

$$\therefore X^{k+1} = (I - D^{-1}A)X^{(k)} + D^{-1}B$$

$X^{(k+1)} = D^{-1}\{B - (L+U)X^{(k)}\}$

Let $X^{(k+1)} = X + E^{(k+1)}$

& $X^{(k)} = X + E^{(k)}$. $E^{(k+1)}$ & $E^{(k)}$ are the error vectors associated with $X^{(k+1)}$ & $X^{(k)}$

$X + E^{(k+1)} = D^{-1}\{B - (L+U)(X + E^{(k)})\}$

$\Rightarrow E^{(k+1)} = D^{-1}\{B - (L+U)X - (L+U)E^{(k)}\} - X$

$\Rightarrow E^{(k+1)} = D^{-1}\{B - AX - (L+U)E^{(k)}\}$

$= -D^{-1}(L+U)E^{(k)}$

For convergence

$\|(-D^{-1}(L+U))\| < 1$

$\Rightarrow \|(-D^{-1}(A-D))\| < 1$

$\Rightarrow \|(I - D^{-1}A)\| < 1$

Let $(I - D^{-1}A)Y = \lambda Y$ where λ is eigen value

or $(D-A)Y = \lambda DY$ & Y = corr. eigen vector. non zero

~~Let f_{ij} ... ~~

or $-\sum_{\substack{j=1 \\ i \neq j}}^{n} a_{ij} y_j = \lambda a_{ii} y_i$ $i = 1, \dots n$, $j = 1, \dots, n$

Since Y is a non zero vector, $|y_i| \neq 0$

$\therefore |\lambda| = \left\{\sum_{\substack{j=1 \\ i \neq j}}^{n} |a_{ij}|\right\} / |a_{ii}|$ $i = 1, \dots, n$

for convergence $|\lambda| < 1$

$\left\{\sum_{\substack{j=1 \\ i \neq j}}^{n} |a_{ij}|\right\} / |a_{ii}| < 1$

or $\boxed{\sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| < |a_{ii}|}$ $i = 1, \dots, n$

1 6 0
1 −2 −6
9 4 1

Partial pivoting can be used to convert a non convergent

$x + 6y = 4$

$x - 2y - 6z = 14$

$9x + 4y + z = -17$

$A = \begin{bmatrix} 1 & 6 & 0 \\ 1 & -2 & -6 \\ 9 & 4 & 1 \end{bmatrix}$

$\left[\begin{array}{ccc|c} 1 & 6 & 0 & 4 \\ 1 & -2 & -6 & 14 \\ 9 & 4 & 1 & -17 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 9 & 4 & 1 & -17 \\ 1 & -2 & -6 & 14 \\ 1 & 6 & 0 & 4 \end{array}\right]$ $R_1 \leftrightarrow R_3$

$\Rightarrow \left[\begin{array}{ccc|c} 9 & 4 & 1 & -17 \\ 1 & 6 & 0 & 4 \\ 1 & -2 & -6 & 14 \end{array}\right]$ $R_2 \leftrightarrow R_3$

$A = \begin{bmatrix} 9 & 4 & 1 \\ 1 & 6 & 0 \\ 1 & -2 & -6 \end{bmatrix}$ $B = \begin{bmatrix} -17 \\ 4 \\ 14 \end{bmatrix}$ $D = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -6 \end{bmatrix}$

$D^{-1} = \begin{bmatrix} 1/9 & 0 & 0 \\ 0 & 1/6 & 0 \\ 0 & 0 & -1/6 \end{bmatrix}$

$x^{(0)} = y^{(0)} = z^{(0)} = 0$

$x^{(1)} = -1.9$, $y^{(1)} = 0.67$, $z^{(1)} = -2.33$

$x^{(2)} = 1.93$, $y^{(2)} = 0.98$, $z^{(2)} = -2.87$

$x^{(3)} = -2.0061$, $y^{(3)} = 0.99$, $z^{(3)} = -2.98$

$x^{(4)} = -1.996$, $y^{(4)} = 1.001$, $z^{(4)} = -3.0$

$x^{(5)} = -2.000$, $y^{(5)} = 0.999$, $z^{(5)} = -3.0$

$x^{(6)} = -2.000$, $y^{(6)} = 1.0$, $z^{(6)} = -3.0$

Gauss – Seidel Method

$a_{11} x_1^{(k+1)} = b_1 - (a_{12} x_2^{(k)} + \dots + a_{1n} x_n^{(k)})$

$a_{22} x_2^{(k+1)} = b_2 - (a_{21} x_1^{(k+1)} + \dots + a_{2n} x_n^{(k)})$

$a_{33} x_3^{(k+1)} = b_3 - (a_{31} x_1^{(k+1)} + a_{32} x_2^{(k+1)} + \dots + a_{3n} x_n^{(k)})$

$\vdots$

$a_{nn} x_n^{(k+1)} = b_n - (a_{n1} x_1^{(k+1)} + a_{n2} x_2^{(k+1)} + \dots + a_{n(n-1)} x_{(n-1)}^{(k+1)})$

$a_{11} x_1^{(k+1)} = ①$

$a_{21} x_1^{(k+1)} + a_{22} x_2^{(k+1)} = ②$

$a_{31} x_1^{(k+1)} + a_{32} x_2^{(k+1)} + a_{33} x_3^{(k+1)} = ③$

$\vdots$

$a_{n1} x_1^{(k+1)} + a_{n2} x_2^{(k+1)} + \dots + a_{nn} x_n^{(k+1)} = b_n$

$① \to b_1 - (a_{12} x_2^{(k)} + a_{13} x_3^{(k)} + \dots + a_{1n} x_n^{(k)})$

$② \to b_2 - (a_{23} x_3^{(k)} + a_{24} x_4^{(k)} + \dots + a_{2n} x_n^{(k)})$

$③ \to b_3 - (a_{34} x_4^{(k)} + \dots + a_{3n} x_n^{(k)})$

$(L+D) x^{(k+1)} = B - U x^{(k)}$

$\Rightarrow X^{(k+1)} = (L+D)^{-1} (B - U X^{(k)})$

$\Rightarrow X^{(k+1)} - X^{(k)} = (L+D)^{-1} (B - U X^{k}) - X^{k}$

$\Rightarrow X^{(k+1)} - X^{(k)} = (L+D)^{-1} \{B - U X^{(k)} - (L+D) X^{(k)}\}$

$= (L+D)^{-1} \{B - A X^{(k)}\}$

$= (L+D)^{-1} \{ \dots - A X^{(k)} \}$

$\Rightarrow X^{(k+1)} = (L+D)^{-1} B - (L+D)^{-1} U X^{(k)}$

$E^{(k+1)} = -(L+D)^{-1} U E^{(k)}$

To do

Calc. conv. criteria of this method as earlier.

$\|(L+D)^{-1} U\| < 1$

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Iterative method to determine A^{-1}

Let B be an approximate inverse of A

$\therefore AB \neq I$.

Let $AB = I + E \Rightarrow E = AB - I$ — (1)

premultiplying both sides with A^{-1}

$A^{-1}AB = A^{-1}(I+E)$

$\Rightarrow$ ~~$B = A^{-1} + A^{-1}$~~ or $B = A^{-1}(I+E)$

Post multiplying both sides with $(I+E)^{-1}$

$A^{-1} = B(I+E)^{-1} \approx B(I - E + E^2 - E^3 \dots)$

if $\|E\| << 1$ we can write

$A^{-1} \approx B(I-E)$

$\approx B(I - AB + I) \approx B(2*I - AB)$

Let $B^{(k+1)}$ & $B^{(k)}$ be the approx. inverse at $(k+1)$ & kth iter.

$B^{(k+1)} = B^{(k)} * (2I - AB^{(k)})$.

$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad A^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$.

Let $B^{(0)} = \begin{bmatrix} 1.8 & -0.9 \\ -0.9 & 0.9 \end{bmatrix}$

$B^{(1)} = \begin{bmatrix} 1.98 & -0.99 \\ -0.99 & 0.99 \end{bmatrix}$

$B^{(2)} = \begin{bmatrix} 1.998 & -0.9999 \\ -0.9999 & 0.9999 \end{bmatrix}$

Premultiplying both sides with A

$AB^{(k+1)} = AB^{(k)}(2I - AB^{(k)}) = 2I\{AB^{(k)}\} - \{AB^{(k)}\}^2$

or $AB^{(k+1)} - I = 2I\{AB^{(k)}\} - \{AB^{(k)}\}^2 - I$

$= -\{AB^{(k)} - I\}^2$ or (By (1))

$\Rightarrow E^{(k+1)} = -(E^{(k)})^2 \quad \therefore \boxed{\text{Order} = 2}$

$x - \sin(x+y) = 0$

$y - \cos(x+y) = 0$ (0.951701, 0.307078)

If changed to $y - \cos(x-y) = 0$ (0.935082, 0.998020)

Sol^n of non linear simultaneous eq^n

$f_1(x_1, x_2, \ldots, x_n) = 0$

$f_2(x_1, x_2, \ldots, x_n) = 0$

$\vdots$

$f_n(x_1, x_2, \ldots, x_n) = 0$

$$\to \begin{cases} x_1^{(k+1)} = F_1(x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}) \\ x_2^{(k+1)} = F_2(x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}) \\ \vdots \\ x_n^{(k+1)} = F_n(x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}) \end{cases}$$

$$\max_{1 \le i \le n} \{ |x_i^{(k+1)} - x_i^{(k)}| \} < \epsilon$$

for convergence

$$\text{I} \quad \sum_{j=1}^{n} \left| \frac{\partial F_i}{\partial x_j} \right| < 1 \quad i = 1, 2, \ldots, n$$

Newton – Raphson method

Let $x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}$ be the initial approx.

& $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$ be the corrections needed.

$\therefore f_1(x_1^{(0)} + \Delta x_1, x_2^{(0)} + \Delta x_2, \ldots, x_n^{(0)} + \Delta x_n) = 0$

$f_2(x_1^{(0)} + \Delta x_1, x_2^{(0)} + \Delta x_2, \ldots, x_n^{(0)} + \Delta x_n) = 0$

$\vdots$

$f_n(x_1^{(0)} + \Delta x, x_2^{(0)} + \Delta x_2, \ldots, x_n^{(0)} + \Delta x_n) = 0$

Expanding by taylor series

$$f_1(X^{(0)}) + \Delta x_1 \left.\frac{\partial f_1}{\partial x_1}\right|_{X^{(0)}} + \Delta x_2 \left.\frac{\partial f_1}{\partial x_2}\right|_{X^{(0)}} + \ldots + \Delta x_n \frac{\partial f_1}{\partial x_n} = 0$$

$$\max_{1 \le i \le n} (|\Delta x_i|) < \epsilon$$

$$f_2(X^0) + \Delta x_1 \left.\frac{\partial f_2}{\partial x_1}\right|_{X^{(0)}} + \Delta x_2 \left.\frac{\partial f_2}{\partial x_2}\right|_{X^{(0)}} = 0$$

$\Delta x_1 f_x + \Delta x_2 f_y = -f$

$\Delta x_1 g_x + \Delta x_2 g_y = -g$

$$\Delta x_1 = \frac{\begin{vmatrix} -f & f_y \\ -g & g_y \end{vmatrix}}{\begin{vmatrix} f_x & f_y \\ g_x & g_y \end{vmatrix}} = \frac{(g f_y - f g_y)}{f_x g_y - f_y g_x}$$

$$\Delta x_2 = \begin{vmatrix} f_x & -f \\ g_x & -g \end{vmatrix} \Big/ \begin{vmatrix} f_x & f_y \\ g_x & g_y \end{vmatrix} = f g_x - g f_x$$

Eigenvalue and Eigenvector of a square matrix

$AX = \lambda X$

λ is called eigen value

X is called eigenvector.

λ is a non-zero scalar

X is a non-zero column matrix or vector.

λ is called eigen value of A & X is the corresponding eigenvector.

λ & X represents magnitude & dir. resp.

$AX = \lambda I X$ where I is an identity matrix of dimension same as A.

$AX - \lambda I X = 0 \Rightarrow (A - \lambda I) X = 0$

$\det(A - \lambda I) = 0$

characteristic eq^n for computing eigen value

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad \lambda I = \begin{bmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \lambda & & \\ \vdots & & & \\ 0 & 0 & \cdots & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix}$$

$\det(A - \lambda I)$ is a poly of λ of degree n.

1) For a $n \times n$ matrix, there are n different eigen values & corr. n diff eigen vectors.

2) If the eigen values are distinct, the corresponding eigenvectors are orthogonal.

$X^T = X^{-1}$

3) Two matrices A & B are called similar if there exist a non-singular matrix P such that $B = P^{-1}AP$.

If two matrices A & B are similar then they have the same set of eigenvalues

Let $B = P^{-1}AP$

$AX = \lambda X$

Premultiplying with P^{-1}.

$P^{-1}AX = \lambda P^{-1}X$

Let $P^{-1}X = Y$ or $PP^{-1}X = PY$ or $X = P$

$\because P^{-1}AX = \lambda Y$

or $P^{-1}APY = \lambda Y$

$\Rightarrow BY = \lambda Y$

not only A & B have same set of eigenvalues corr. eigenvectors are related as $X = PY$

Find all eigenvalues of the following matrix

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & 1 \end{bmatrix}$$

$$\det \begin{vmatrix} 1-\lambda & 2 & -2 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)\{(\lambda-1)(\lambda+1) - 3\} - 2\{-1-\lambda-1\}$
$\quad - 2[3 - 1 + \lambda]$

$\Rightarrow (1-\lambda)(\lambda^2-4) + 2(\lambda+2) - 2(2+\lambda) = 0$

$\lambda = -2$

$\Rightarrow (\lambda+2)((\lambda-2)(1-\lambda) + 2 - 2) = 0$

$\therefore \boxed{\lambda = -2, \lambda = 2, 1}$

$AX = \lambda X$

$$\Rightarrow \begin{bmatrix} x_1 + 2x_2 - 2x_3 \\ x_1 + x_2 + x_3 \\ x_1 + 3x_2 - x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$\Rightarrow 2x_2 = 2x_3$

$x_1 = -x_3$

$x_1 = -3x_2$

$x_3 = y \quad \therefore \quad x_2 = y$

$x_1 = -y$

$$X = \begin{bmatrix} -y \\ y \\ y \end{bmatrix}$$

$X^T X = 1$

$$\Rightarrow [-y \;\; y \;\; y] \begin{bmatrix} -y \\ y \\ y \end{bmatrix} = 1$$

$\Rightarrow y^2 + y^2 + y^2 = 1$

$\Rightarrow 3y^2 = 1 \quad \boxed{y = \pm\frac{1}{\sqrt{3}}}$

Power method for finding the largest eigenvalue of a sq. matrix.

$$AX = \lambda X$$

Let us start the iterations with a trial eigenvector $X^{(0)}$

$$AX^{(0)} = y^{(1)} = c^{(1)} x^{(1)}$$

$$Ax^{(1)} = y^{(2)} = c^{(2)} x^{(2)}$$

max. value in x is 1

$$\vdots$$

$$Ax^{(k)} = y^{(k+1)} = c^{(k+1)} x^{(k+1)}$$

Continue until $|c^{(k)} - c^{(k+1)}| < \epsilon$.

Then $c^{(k+1)}$ is the largest magn. eigenvalue of A and $x^{(k+1)}$ is the corr. eigenvector

$$A = \begin{bmatrix} 10 & 5 & 6 \\ 5 & 20 & 4 \\ 6 & 4 & 30 \end{bmatrix} \quad x^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$Ax^{(0)} = \begin{bmatrix} 21 \\ 29 \\ 40 \end{bmatrix} = 40 \begin{bmatrix} 21/40 \\ 29/40 \\ 1 \end{bmatrix} = 40 \begin{bmatrix} 0.525 \\ 0.725 \\ 1.0 \end{bmatrix}$$

$$= c^{(1)} x^{(1)}$$

$$Ax^{(1)} = \begin{bmatrix} 14.875 \\ 21.125 \\ 36.050 \end{bmatrix} = 36.050 \begin{bmatrix} 0.413 \\ 0.586 \\ 1.000 \end{bmatrix} = c^{(2)} x^{(2)}$$

$$c^{(11)} = 33.718, \quad x^{(11)} = \begin{bmatrix} 0.341 \\ 0.417 \\ 1.000 \end{bmatrix}$$

$$c^{(12)} = 33.714, \quad x^{(12)} = \begin{bmatrix} 0.341 \\ 0.417 \\ 1.000 \end{bmatrix}$$

corrected upto 2 dec. places.

Proof of Power method:

Let us assume $\lambda_1, \lambda_2, \dots, \lambda_n$ be the n-eigenvalues of a $n \times n$ matrix A and

$$|\lambda_1| > |\lambda_2| > \dots > |\lambda_n|$$

Let $v_1, v_2, \dots, v_n$ be the corr. eigen vectors.

Let us assume another vector V in the eigenvectors space which is represented as a linear combination of $v_1, v_2, \dots, v_n$.

$$\therefore V = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$$

$$AV = A\{c_1 v_1 + c_2 v_2 + c_3 v_3 + \dots + c_n v_n\}$$

$$= \{c_1 A v_1 + c_2 A v_2 + \dots + c_n A v_n\}$$

$$= \{c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + \dots + c_n \lambda_n v_n\}$$

$$= \lambda_1 \{c_1 v_1 + c_2 \left(\frac{\lambda_2}{\lambda_1}\right) v_2 + \dots + c_n \left(\frac{\lambda_n}{\lambda_1}\right) v_n\}$$

$= \lambda_1 V_1$

$AV_1 = A\left\{c_1 v_1 + c_2\left(\frac{\lambda_2}{\lambda_1}\right)v_2 + \cdots + c_n\left(\frac{\lambda_n}{\lambda_1}\right)v_n\right\}$

$= c_1 A v_1 + c_2 A\left(\frac{\lambda_2}{\lambda_1}\right)v_2 + \cdots + c_n A\left(\frac{\lambda_n}{\lambda_1}\right)v_n$

$\cancel{c_1 A v_1 + c_2}$

$= \lambda_1\left\{c_1 v_1 + c_2\left(\frac{\lambda_2}{\lambda_1}\right)^2 v_2 + \cdots + c_n\left(\frac{\lambda_n}{\lambda_1}\right)^2 v_n\right\}$

$AV_k = A\left\{c_1 v_1 + c_2\left(\frac{\lambda_2}{\lambda_1}\right)^k v_2 + \cdots + c_n\left(\frac{\lambda_n}{\lambda_1}\right)^k v_n\right\}$

$\doteq \lambda_1\left\{c_1 v_1 + c_2\left(\frac{\lambda_2}{\lambda_1}\right)^{k+1} v_2 + \cdots + c_n\left(\frac{\lambda_n}{\lambda_1}\right)^{k+1} v_n\right\}$

As $k \to \infty$

$V_k = c_1 V_1$

$V_{k+1} = c_1 V_1$

$A c_1 v_1 = \lambda_1 c_1 v_1$

$\Rightarrow A v_1 = \lambda_1 v_1$

Premultiplying with A^{-1}

$A^{-1} A X = \lambda A^{-1} X$

$\Rightarrow X = \lambda A^{-1} X$

$\Rightarrow A^{-1} X = \frac{1}{\lambda} X = \beta X$

minimum eigen value

The largest eigenvalue is removed from the matrix using a method called deflation method to generate a new matrix A_1 where

$$A_1 = A - \frac{\lambda_1 \overbrace{X_1 X_1^T}^{n\times n \text{ matrix}}}{\underbrace{X_1^T X_1}_{\text{scalar}}}$$

To find the eigenvalue of A_1 corr. to X_1, we calculate,

$$A_1 X_1 = A X_1 - \frac{\lambda_1 X_1 X_1^T}{X_1^T X_1} X_1$$

$$= A X_1 - \lambda_1 X_1 = 0$$

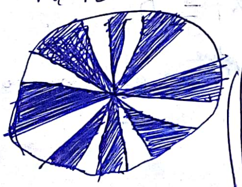
This proves that eigenvalue corresponding to X_1, ~~we calc.~~ is 0.

Transform method for finding all the eigenvalues and corresponding eigenvectors

$\lambda_1, \lambda_2, \ldots, \lambda_n$ be the n eigenvalues of a $n \times n$ matrix A and $X_1, X_2, \ldots, X_n$ be the corresponding eigenvectors.

$$AX_1 = \lambda_1 X_1$$
$$AX_2 = \lambda_2 X_2$$
$$\vdots$$
$$AX_n = \lambda_n X_n$$

Let $X_1 = \begin{bmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{bmatrix}, X_2 = \begin{bmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{bmatrix} \ldots X_n = \begin{bmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{bmatrix}$



Important

$$A \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & & x_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots \\ 0 & \lambda_2 & \cdots \\ \vdots & \vdots & \\ 0 & 0 & \cdots \end{bmatrix}$$

$V \quad D$

$\therefore AV = VD$

$V^{-1}AV = V^{-1}VD = D.$

Transform method for finding all the eigenvalues & corr. eigenvectors for symmetric matrices.

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

$$V = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$V^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Why??

$$V^{-1}AV = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} a\cos\theta + b\sin\theta & b\cos\theta + c\sin\theta \\ -a\sin\theta + b\cos\theta & -b\sin\theta + c\cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} a\cos^2\theta + 2b\sin\theta\cos\theta + c\sin^2\theta & b\cos 2\theta + c\sin\theta\cos\theta - a\sin\theta\cos\theta \\ -a\sin\theta\cos\theta + b\cos 2\theta + c\cos\theta\sin\theta & a\sin^2\theta - 2b\sin\theta\cos\theta + c\cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} a\cos^2\theta + b\sin 2\theta + c\sin^2\theta & b\cos 2\theta + \frac{(c-a)\sin 2\theta}{2} \\ b\cos 2\theta + \frac{(c-a)\sin 2\theta}{2} & a\sin^2\theta - b\sin 2\theta + c\cos^2\theta \end{bmatrix}$$

$b\cos 2\theta + (c-a)\frac{\sin 2\theta}{2} = 0$. (Non-diagonal elements are zero)

$b + \frac{(c-a)}{2}\tan 2\theta = 0$

$\tan 2\theta = \frac{2b}{a-c}$

$2\theta = \tan^{-1}\left(\frac{2b}{a-c}\right)$.

$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2b}{a-c}\right)$. $a \neq c$.

Jacobi's method for finding all the eigenvalues and corresponding eigenvectors for symmetric matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ & & a_{(n-1)(n-1)} & a_{(n-1)n} \\ a_{n1} & a_{n2} & \cdots \; a_{n(n-1)} & a_{nn} \end{bmatrix}$$

$A_1 = P_1^T A P_1$

$A_2 = P_2^T A_1 P_2 = P_2^T P_1^T A P_1 P_2$

$\vdots$

$A_Z = \underbrace{P_Z^T P_{Z-1}^T \cdots P_2 P_1}_{V^{-1}} A \underbrace{P_1 P_2 \cdots P_{Z-1} P_Z}_{V}$

until Purely diagonal matrix

Let $a_{rs}^{(k-1)}$ $(s>r)$ be the largest magnitude off diagonal element of the intermediate similar matrix A_{k-1} then the P_k is formed as

$$\left.\begin{aligned} p_{rr}^{(k)} &= p_{ss}^{(k)} = \cos\theta \\ -p_{rs}^{(k)} &= p_{sr}^{(k)} = \sin\theta \end{aligned}\right\} \text{where}$$

$p_{ii} = 1$ for $i \neq r, s$

$p_{ij} = 0$; otherwise.

$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2a_{rs}^{(k-1)}}{a_{rr}^{(k-1)} - a_{ss}^{(k-1)}}\right)$

$A_k = P_k^T A_{k-1} P_k$

$$\left.\begin{aligned} a_{ir}^{(k)} &= a_{ri}^{(k)} = a_{ir}^{(k-1)}\cos\theta + a_{is}^{(k-1)}\sin\theta \\ a_{is}^{(k)} &= a_{si}^{(k)} = -a_{ir}^{(k-1)}\sin\theta + a_{is}^{(k-1)}\cos\theta \end{aligned}\right\} \begin{aligned} &i = 1, 2, \ldots n \\ &i \neq r, s \end{aligned}$$

$a_{rr}^{(k)} = a_{rr}^{(k-1)}\cos^2\theta + 2a_{rs}^{(k-1)}\cos\theta\sin\theta + a_{ss}^{(k-1)}\sin^2\theta$

$a_{ss}^{(k)} = a_{rr}^{(k-1)}\sin^2\theta - 2a_{rs}^{(k-1)}\cos\theta\sin\theta + a_{ss}^{(k-1)}\cos^2\theta$

$a_{rs}^{(k)} = a_{sr}^{(k)} = 0$

$a_{ij}^{(k)} = a_{ij}^{(k-1)}$, otherwise

$$V_0 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$V_1 = V_0 P_1$

$V_2 = V_1 P_2$ $\rightarrow$ $V_k = V_{k-1} P_k$

$v_{ir}^{(k)} = v_{ir}^{(k-1)}\cos\theta + v_{is}^{(k-1)}\sin\theta$

$$v_{js} = -v_{ir}^{(n-1)} \sin\theta + v_{is}^{(n-1)} \cos\theta$$

$$i = 1, 2, \ldots, n$$

$$A = \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix}$$

Largest magnitude off ~~displace~~ diag element $= a_{13} = a_{31} = 2$

$$\left.\begin{aligned} p_{11} = p_{33} &= \cos\theta \\ -p_{13} = p_{31} &= \sin\theta \end{aligned}\right\} \text{where } \theta = \frac{1}{2}\tan^{-1}\left(\frac{2a_{13}}{a_{11}-a_{33}}\right)$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{4}{1-1}\right)$$

$$= \frac{1}{2} \times \frac{\pi}{2} = \frac{\pi}{4}$$

$$P_1 = \begin{bmatrix} \cos \pi/4 & 0 & -\sin \pi/4 \\ 0 & 1 & 0 \\ \sin \pi/4 & 0 & \cos \pi/4 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$A_1 = P_1^T A P_1 = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & \ldots \\ 0 & 1 & \ldots \\ 1/\sqrt{2} & 0 & \ldots \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$A_2 = P_2^T A_1 P_2 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$V = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/2 & -1/2 & 1/\sqrt{2} \end{bmatrix}$$

Rutehauser method for arbitrary matrix.

$A = LU$ where $U = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{bmatrix}$

$L = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ l_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ l_{n1} & l_{n2} & \cdots & 1 \end{bmatrix}$

Let us apply the transformation

$A_1 = UAU^{-1} = UL$

again let $A_1 = L_1 U_1$ where $(l_{ii}^{(1)} = 1)$

and $A_2 = U_1 A_1 U_1^{-1} = U_1 L_1$

This is continued until A_k matrix is reduced to an upper triangular form, then the diagonal elements will represent the eigenvalues of A.