# TODAY'S LECTURE

- Simplex Method
  - Recap of algebraic form
  - Simplex Method in Tabular form
- Simplex Method for other forms
  - Equality Constraints
  - Minimization Problems

  (Big M and Twophase methods)
- Sensitivity / Shadow Prices
- Simplex Method in Matrix form
  - Basics of Matrix Algebra
  - Revised Simplex
- Dual Simplex
- R-resources / demonstration

# SIMPLEX METHOD (MATRIX FORM)
# MATRIX ALGEBRA BASICS
# REVISED SIMPLEX

## Matrices and Matrix Operations

A matrix is a rectangular array of numbers. For example

$$\mathbf{A} = \begin{bmatrix} 2 & 5 \\ 3 & 0 \\ 1 & 1 \end{bmatrix}$$

is a 3**x**2 matrix (matrices are denoted by **Boldface Capital Letters**)

In more general terms,

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \|a_{ij}\|$$

is an $m \times n$ matrix, where $a_{11}, \ldots, a_{mn}$ represent the numbers that are the elements of this matrix; $\|a_{ij}\|$ is shorthand notation for identifying the matrix whose element in row $i$ and column $j$ is $a_{ij}$ for every $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

## Matrix Operations

Let $\mathbf{A} = \|a_{ij}\|$ and $\mathbf{B} = \|b_{ij}\|$ be two matrices having the same number of rows and the same number of columns.

Matrices $\mathbf{A}$ and $\mathbf{B}$ are said to be *equal* ($\mathbf{A} = \mathbf{B}$) if and only if *all* the corresponding elements are equal ($a_{ij} = b_{ij}$ for all $i$ and $j$).

<u>To multiply a matrix by a number (denote this number by *k* )</u>

$$k\mathbf{A} = \|ka_{ij}\|$$

For example,

$$3\begin{bmatrix} 1 & \frac{1}{3} & 2 \\ 5 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 6 \\ 15 & 0 & -9 \end{bmatrix}$$

<u>To add two matrices **A** and **B**</u>

$$\mathbf{A} + \mathbf{B} = \|a_{ij} + b_{ij}\|$$

For example,

$$\begin{bmatrix} 5 & 3 \\ 1 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 3 \\ 4 & 7 \end{bmatrix}$$

## Subtraction of two matrices

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-1)\mathbf{B},$$

so that

$$\mathbf{A} - \mathbf{B} = \|a_{ij} - b_{ij}\|.$$

For example,

$$\begin{bmatrix} 5 & 3 \\ 1 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ -2 & 5 \end{bmatrix}$$

## Matrix multiplication

Matrix multiplication **AB** is defined if and only if the *number of columns* of **A** equals the *number of rows* of **B**.

Thus, if **A** is an $m \times n$ matrix and **B** is an $n \times s$ matrix, then their product is

$$\mathbf{AB} = \left\| \sum_{k=1}^{n} a_{ik} b_{kj} \right\|,$$

where this product is an $m \times s$ matrix. However, if **A** is an $m \times n$ matrix and **B** is an $r \times s$ matrix, where $n \neq r$, then **AB** is not defined.

To illustrate matrix multiplication,

$$\begin{bmatrix} 1 & 2 \\ 4 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1(3)+2(2) & 1(1)+2(5) \\ 4(3)+0(2) & 4(1)+0(5) \\ 2(3)+3(2) & 2(1)+3(5) \end{bmatrix} = \begin{bmatrix} 7 & 11 \\ 12 & 4 \\ 12 & 17 \end{bmatrix}.$$

On the other hand, if one attempts to mutliply these matrices in the reverse order, the resulting product

$$\begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & 0 \\ 2 & 3 \end{bmatrix}$$

is not even defined.

Even when both **AB** and **BA** are defined,

$$\mathbf{AB} \neq \mathbf{BA}$$

in general.

<u>Matrix division is not defined</u>

## Matrix operations satisfy the following laws.

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A},$$

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}),$$

$$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC},$$

$$\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C},$$

The relative sizes of these matrices are such that the indicated operations are defined.

## Transpose operations

This operation involves nothing more than interchanging the rows and columns of the matrix.

Thus, for any matrix $\mathbf{A} = \|a_{ij}\|$, its transpose $\mathbf{A}^T$ is

$$\mathbf{A}^T = \|a_{ji}\|.$$

For example, if

$$\mathbf{A} = \begin{bmatrix} 2 & 5 \\ 1 & 3 \\ 4 & 0 \end{bmatrix}, \quad \text{then} \quad \mathbf{A}^T = \begin{bmatrix} 2 & 1 & 4 \\ 5 & 3 & 0 \end{bmatrix}.$$

# Special kinds of matrices

## Identity matrix

The identity matrix **I** is a square matrix whose elements are 0s except for 1s along the main diagonal. ( A square matrix is one in which the number of rows is equal to the number of columns).

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$\mathbf{IA} = \mathbf{A} = \mathbf{AI},$$

where **I** is assigned the appropriate number of rows and columns in each case for the multiplication operation to be defined.

## Null matrix

The null matrix **0** is a matrix of any size whose elements are all 0s .

$$\mathbf{0} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

Therefore, for any matrix **A**,

$$\mathbf{A} + \mathbf{0} = \mathbf{A}, \qquad \mathbf{A} - \mathbf{A} = \mathbf{0}, \qquad \text{and} \qquad \mathbf{0A} = \mathbf{0} = \mathbf{A0},$$

where **0** is the appropriate size in each case for the operations to be defined.

## Nonsingular matrices

Nonsingular matrices are sometimes also called regular matrices. A square matrix is nonsingular iff its determinant is nonzero (Lipschutz 1991, p. 45). For example, there are 6 nonsingular (0,1)-matrices:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

## Inverse of matrix

(*a*) If $\mathbf{A}$ is nonsingular, there is a unique nonsingular matrix $\mathbf{A}^{-1}$, called the **inverse** of $\mathbf{A}$, such that $\mathbf{AA}^{-1} = \mathbf{I} = \mathbf{A}^{-1}\mathbf{A}$.

(*b*) If $\mathbf{A}$ is nonsingular and $\mathbf{B}$ is a matrix for which either $\mathbf{AB} = \mathbf{I}$ or $\mathbf{BA} = \mathbf{I}$, then $\mathbf{B} = \mathbf{A}^{-1}$.

(*c*) Only nonsingular matrices have inverses.

To illustrate matrix inverses, consider the matrix

$$\mathbf{A} = \begin{bmatrix} 5 & -4 \\ 1 & -1 \end{bmatrix}.$$

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & -4 \\ 1 & -5 \end{bmatrix}$$

$$\mathbf{AA}^{-1} = \begin{bmatrix} 5 & -4 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -4 \\ 1 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix};$$

$$\mathbf{A}^{-1}\mathbf{A} = \begin{bmatrix} 1 & -4 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 5 & -4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

# INVERSE MATRIX

- 2x2 Matrix

how do we calculate the Inverse?
Well, for a 2x2 Matrix the Inverse is:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \underbrace{\frac{1}{ad-bc}}_{\text{determinant}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

In other words: **swap** the positions of a and d, put **negatives** in front of b and c, and **divide** everything by the determinant (ad-bc).Let us try an example:

$$\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}^{-1} = \frac{1}{4x6-7x2} \begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix}$$

$$= \frac{1}{10} \begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix}$$

## Example: find the Inverse of "A":

$$A = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

We start with the matrix A, and write it down with an Identity Matrix I next to it:

$$\begin{array}{ccc|ccc} & A & & & I & \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 3 & 0 & 2 & 1 & 0 & 0 \\ 2 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

(This is called the "Augmented Matrix")

Now we do our best to turn "A" (the Matrix on the left) into an Identity Matrix. The goal is to make Matrix A have **1**s on the diagonal and **0**s elsewhere (an Identity Matrix) ... and the right hand side comes along for the ride, with every operation being done on it as well.

But we can only do these **"Elementary Row Operations"**:

- **swap** rows
- **multiply** or divide each element in a a row by a constant
- replace a row by **adding** or subtracting a multiple of another row to it

And we must do it to the **whole row**, like this:

$$\begin{array}{cc} A & I \\ \left[\begin{array}{ccc} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{array}\right. & \left|\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \end{array}$$

Start with **A** next to **I**

$$\left[\begin{array}{ccc|ccc} 5 & 0 & 0 & 1 & 1 & 0 \\ 2 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \text{ Add}$$

Add row 2 to row 1,

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.2 & 0.2 & 0 \\ 2 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \text{ Divide by 5}$$

then divide row 1 by 5,

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.2 & 0.2 & 0 \\ 0 & 0 & -2 & -0.4 & 0.6 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \text{ Subtract} \times 2$$

Then take 2 times the first row, and subtract it from the second row,

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.2 & 0.2 & 0 \\ 0 & 0 & 1 & 0.2 & -0.3 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \text{ Multiply by } -\tfrac{1}{2}$$

Multiply second row by -1/2,

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.2 & 0.2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0.2 & -0.3 & 0 \end{array}\right] \text{ Swap}$$

Now swap the second and third row,

$$\begin{array}{cc} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right. & \left|\begin{array}{ccc} 0.2 & 0.2 & 0 \\ -0.2 & 0.3 & 1 \\ 0.2 & -0.3 & 0 \end{array}\right] \text{ Subtract} \\ I & A^{-1} \end{array}$$

Last, subtract the third row from the second row,

And we are done!

And matrix **A** has been made into an Identity Matrix ...

... and at the same time an Identity Matrix got made into $\mathbf{A}^{-1}$

$$A^{-1} = \begin{bmatrix} 0.2 & 0.2 & 0 \\ -0.2 & 0.3 & 1 \\ 0.2 & -0.3 & 0 \end{bmatrix}$$

- Step 1: calculating the Matrix of Minors,
  - (A "minor" is the determinant of the square matrix formed by deleting one row and one column from some larger square matrix. Since there are lots of rows and columns in the original matrix, you can make lots of minors from it.)
- Step 2: then turn that into the Matrix of Cofactors,
  - Once you find a minor $M_{i,j}$, you take the subscript on the name of the minor (the "i, j" part) and add the two numbers i and j. Whatever result you get from this addition, make this value the power on –1, so you get "+1" or "–1", depending on whether i + j is even or odd. Then multiply this on the minor $M_{i,j}$. This gives you the "cofactor" $A_{i,j}$. That is: $(-1)^{i+j} M_{i,j} = A_{i,j}$
- Step 3: then the Adjugate, and
  - the **adjugate**, classical **adjoint**, or adjunct of a square **matrix** is the transpose of its cofactor **matrix**
- Step 4: multiply that by 1/Determinant.

# Find the Inverse of A:

It needs 4 steps. It is all simple arithmetic but there is a lot of it, so try not to make a mistake!

$$A = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

# Step 1: Matrix of Minors

$$\begin{bmatrix} \bullet & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} \quad 0\times1 - (-2)\times1 = 2$$

$$\begin{bmatrix} 3 & \bullet & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} \quad 2\times1 - (-2)\times0 = 2$$

. . .

$$\begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & \bullet & 1 \end{bmatrix} \quad 3\times-2 - 2\times2 = -10$$

$$\begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & \bullet \end{bmatrix} \quad 3\times0 - 0\times2 = 0$$

And here is the calculation for the whole matrix:

$$\begin{bmatrix} 0\times1 - (-2)\times1 & 2\times1 - (-2)\times0 & 2\times1 - 0\times0 \\ 0\times1 - 2\times1 & 3\times1 - 2\times0 & 3\times1 - 0\times0 \\ 0\times(-2) - 2\times0 & 3\times(-2) - 2\times2 & 3\times0 - 0\times2 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ -2 & 3 & 3 \\ 0 & -10 & 0 \end{bmatrix}$$

*Matrix of Minors*

## Step 2: Matrix of Cofactors

This is easy! Just apply a "checkerboard" of minuses to the "Matrix of Minors". In other words, we need to change the sign of alternate cells, like this:

$$\underbrace{\begin{bmatrix} 2 & 2 & 2 \\ -2 & 3 & 3 \\ 0 & -10 & 0 \end{bmatrix}}_{\text{Matrix of Minors}} \rightarrow \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \rightarrow \underbrace{\begin{bmatrix} 2 & -2 & 2 \\ +2 & 3 & -3 \\ 0 & +10 & 0 \end{bmatrix}}_{\text{Matrix of CoFactors}}$$

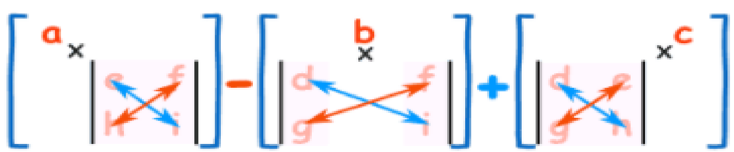
## Step 3: Adjugate (also called Adjoint)

Now "Transpose" all elements of the previous matrix... in other words swap their positions over the diagonal (the diagonal stays the same):

$$\begin{bmatrix} 2 & 2 & 0 \\ -2 & 3 & 10 \\ 2 & -3 & 0 \end{bmatrix}$$

## Step 4: Multiply by 1/Determinant

Now find the determinant of the original matrix. This isn't too hard, because we already calculated the determinants of the smaller parts when we did "Matrix of Minors".




So: multiply the top row elements by their matching "minor" determinants:

$$\text{Determinant} = 3\times2 - 0\times2 + 2\times2 = \mathbf{10}$$

And now multiply the Adjugate by 1/Determinant:

$$A^{-1} = \frac{1}{10}\underbrace{\begin{bmatrix} 2 & 2 & 0 \\ -2 & 3 & 10 \\ 2 & -3 & 0 \end{bmatrix}}_{\text{Adjugate}} = \underbrace{\begin{bmatrix} 0.2 & 0.2 & 0 \\ -0.2 & 0.3 & 1 \\ 0.2 & -0.3 & 0 \end{bmatrix}}_{\text{Inverse}}$$

## Vectors

A special kind of matrix that plays an important role in matrix theory is the kind that has either a *single row* or a *single column*. Such matrices are often referred to as **vectors**. Thus

$$\mathbf{x} = [x_1, x_2, \ldots, x_n]$$

is a **row vector**, and

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

(We use **boldface lowercase** letters to represent vectors**)**

is a **column vector**.

A **null vector 0** is either a row vector or a column vector whose elements are *all* 0s, that is,

$$\mathbf{0} = [0, 0, \ldots, 0] \qquad \text{or} \qquad \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

One reason vectors play an important role in matrix theory is that any $m \times n$ matrix can be partitioned into either $m$ row vectors or $n$ column vectors, and important properties of the matrix can be analyzed in terms of these vectors.

## Partitioning of matrices

Up to this point, matrices have been rectangular arrays of elements, each of which is a number. However, the notation and results are also valid if each element is itself a matrix.

For example, the matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

may be written as

$$\mathbf{A} = [\mathbf{C}_1 \quad \mathbf{C}_2 \quad \mathbf{C}_3] \quad \text{where} \quad \mathbf{C}_1 = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} \quad \mathbf{C}_2 = \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$$

$$\text{and} \quad \mathbf{C}_3 = \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix}$$

or as

$$\mathbf{A} = \begin{bmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \end{bmatrix} \quad \text{where} \quad \mathbf{R}_1 = [a_{11} \quad a_{12} \quad a_{13}] \quad \text{and} \quad \mathbf{R}_2 = [a_{21} \quad a_{22} \quad a_{23}]$$

or as

$$\mathbf{A} = [\mathbf{A}_1 \quad \mathbf{A}_2]$$

where

$$\mathbf{A}_1 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad \mathbf{A}_2 = \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix}$$

or where

$$\mathbf{A}_1 = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} \quad \text{and} \quad \mathbf{A}_2 = \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}$$

The process of dividing a matrix into smaller matrices, or submatrices, is called partitioning and is usually denoted by a dotted line. The four partitions described would be denoted respectively as follows:

$$\left[\begin{array}{c:c:c} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{array}\right]; \quad \left[\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ \hdashline a_{21} & a_{22} & a_{23} \end{array}\right]$$

$$\left[\begin{array}{cc:c} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{array}\right] \quad \text{and} \quad \left[\begin{array}{c:cc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{array}\right]$$

Matrix operations can then be performed with matrices whose elements are matrices, provided the rules of operation are valid for the given matrix and for the resulting submatrices.

Example : Calculate **AB**, given

$$\mathbf{A} = \begin{bmatrix} 6 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 4 & 3 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

***Solution.*** Partition the two matrices:

$$\mathbf{A} = \left[\begin{array}{c:cc} 6 & 0 & 1 \\ 2 & 1 & 0 \end{array}\right] = [\mathbf{A}_1 \quad \mathbf{A}_2] \quad \text{and} \quad \mathbf{B} = \left[\begin{array}{cc} 4 & 3 \\ \hdashline 0 & 1 \\ 1 & 0 \end{array}\right] = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}$$

where

$$\mathbf{A}_1 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{B}_1 = [4 \quad 3], \quad \text{and} \quad \mathbf{B}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

then

$$\mathbf{AB} = [\mathbf{A}_1 \quad \mathbf{A}_2] \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix} = \mathbf{A}_1\mathbf{B}_1 + \mathbf{A}_2\mathbf{B}_2$$

and

$$\mathbf{AB} = \left[ \begin{bmatrix} 6 \\ 2 \end{bmatrix} [4 \quad 3] + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right] = \left[ \begin{bmatrix} 24 & 18 \\ 8 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right]$$

$$\mathbf{AB} = \begin{bmatrix} 25 & 18 \\ 8 & 7 \end{bmatrix}$$

# Matrix Form of Linear Programming

## Original Form of the Model

Maximize $Z = 3x_1 + 5x_2,$

subject to

$$x_1 \leq 4$$
$$2x_2 \leq 12$$
$$3x_1 + 2x_2 \leq 18$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0.$$

Maximize $Z = \mathbf{cx},$

subject to

$$\mathbf{Ax} \leq \mathbf{b} \qquad \text{and} \qquad \mathbf{x} \geq \mathbf{0},$$

where $\mathbf{c}$ is the row vector $\mathbf{c} = [c_1, c_2, \ldots, c_n]$,

$\mathbf{x}$, $\mathbf{b}$, and $\mathbf{0}$ are the column vectors and $\mathbf{A}$ is the matrix

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \quad \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

## Augmented Form of the Model

Maximize $Z = 3x_1 + 5x_2,$

subject to

$$\begin{array}{llllll} (1) & x_1 & & + x_3 & & = 4 \\ (2) & & 2x_2 & & + x_4 & = 12 \\ (3) & 3x_1 & + 2x_2 & & & + x_5 = 18 \end{array}$$

and

$$x_j \geq 0, \qquad \text{for } j = 1, 2, 3, 4, 5.$$

Maximize $Z = \mathbf{cx},$

subject to

$$[\mathbf{A}, \mathbf{I}] \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} = \mathbf{b} \quad \text{and} \quad \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} \geq \mathbf{0}$$

where $\mathbf{I}$ is the $m \times m$ identity matrix

$$\mathbf{x}_s = \begin{bmatrix} x_{n+1} \\ x_{n+2} \\ \vdots \\ x_{n+m} \end{bmatrix}$$

Maximize $Z = \mathbf{cx}$,

subject to

$\mathbf{Ax} \le \mathbf{b}$ and $\mathbf{x} \ge \mathbf{0}$,

$\Rightarrow$

Maximize $Z = \mathbf{cx}$,

subject to

$$[\mathbf{A}, \mathbf{I}]\begin{bmatrix}\mathbf{x} \\ \mathbf{x}_s\end{bmatrix} = \mathbf{b} \text{ and } \begin{bmatrix}\mathbf{x} \\ \mathbf{x}_s\end{bmatrix} \ge \mathbf{0}$$

where $\mathbf{c}$ is the row vector $\mathbf{c} = [c_1, c_2, \ldots, c_n]$, $\mathbf{x}$, $\mathbf{b}$, and $\mathbf{0}$ are the column vectors and $\mathbf{A}$ is the matrix

where $\mathbf{I}$ is the $m \times m$ identity matrix

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \quad \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$\mathbf{x}_s = \begin{bmatrix} x_{n+1} \\ x_{n+2} \\ \vdots \\ x_{n+m} \end{bmatrix}$$

Maximize $Z = 3x_1 + 5x_2$,

subject to

$$\begin{aligned} (1) \quad & x_1 + x_3 = 4 \\ (2) \quad & 2x_2 + x_4 = 12 \\ (3) \quad & 3x_1 + 2x_2 + x_5 = 18 \end{aligned}$$

and

$x_j \ge 0$, for $j = 1, 2, 3, 4, 5$.

$$\mathbf{c} = [3, 5], \quad [\mathbf{A}, \mathbf{I}] = \left[\begin{array}{cc:ccc} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\mathbf{b} = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{x}_s = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

# Solving for a Basic Feasible Solution

For initialization,

Maximize $Z = \mathbf{cx}$,

subject to

$$[\mathbf{A}, \mathbf{I}]\begin{bmatrix}\mathbf{x}\\ \mathbf{x}_s\end{bmatrix} = \mathbf{b} \quad \text{and} \quad \begin{bmatrix}\mathbf{x}\\ \mathbf{x}_s\end{bmatrix} \geq \mathbf{0}$$

For any iteration,

Maximize $Z = c_B \mathbf{x}_B + c_N \mathbf{x}_N$

subject to

$$[\mathbf{B}, \mathbf{N}]\begin{bmatrix}\mathbf{x}_B\\ \mathbf{x}_N\end{bmatrix} = \mathbf{b} \quad \text{and} \quad \begin{bmatrix}\mathbf{x}_B\\ \mathbf{x}_N\end{bmatrix} \geq \mathbf{0}$$

**TABLE 4.8** Complete set of simplex tableaux for the Wyndor Glass Co. problem

| Iteration | Basic Variable | Eq. | $Z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | Right Side |
|---|---|---|---|---|---|---|---|---|---|
| 0 | $Z$ | (0) | 1 | −3 | −5 | 0 | 0 | 0 | 0 |
| | $x_3$ | (1) | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| | $x_4$ | (2) | 0 | 0 | 2 | 0 | 1 | 0 | 12 |
| | $x_5$ | (3) | 0 | 3 | 2 | 0 | 0 | 1 | 18 |
| 1 | $Z$ | (0) | 1 | −3 | 0 | 0 | $\frac{5}{2}$ | 0 | 30 |
| | $x_3$ | (1) | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| | $x_2$ | (2) | 0 | 0 | 1 | 0 | $\frac{1}{2}$ | 0 | 6 |
| | $x_5$ | (3) | 0 | 3 | 0 | 0 | −1 | 1 | 6 |
| 2 | $Z$ | (0) | 1 | 0 | 0 | 0 | $\frac{3}{2}$ | 1 | 36 |
| | $x_3$ | (1) | 0 | 0 | 0 | 1 | $\frac{1}{3}$ | $-\frac{1}{3}$ | 2 |
| | $x_2$ | (2) | 0 | 0 | 1 | 0 | $\frac{1}{2}$ | 0 | 6 |
| | $x_1$ | (3) | 0 | 1 | 0 | 0 | $-\frac{1}{3}$ | $\frac{1}{3}$ | 2 |

$$\mathbf{B} = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1m} \\ B_{21} & B_{22} & \cdots & B_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ B_{m1} & B_{m2} & \cdots & B_{mm} \end{bmatrix} \quad \mathbf{N} = \begin{bmatrix} N_{11} & N_{12} & \cdots & N_{1n} \\ N_{21} & N_{22} & \cdots & N_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ N_{m1} & N_{m2} & \cdots & N_{mn} \end{bmatrix} \quad \mathbf{x}_B = \begin{bmatrix} x_{B1} \\ x_{B2} \\ \vdots \\ x_{Bm} \end{bmatrix} \quad \mathbf{x}_N = \begin{bmatrix} x_{N1} \\ x_{N2} \\ \vdots \\ x_{Nn} \end{bmatrix}$$

# Solving for a Basic Feasible Solution

For initialization,

$$\text{Maximize} \quad Z = \mathbf{cx},$$

subject to

$$[\mathbf{A}, \mathbf{I}]\begin{bmatrix}\mathbf{x}\\ \mathbf{x}_s\end{bmatrix} = \mathbf{b} \ \text{ and } \begin{bmatrix}\mathbf{x}\\ \mathbf{x}_s\end{bmatrix} \geq \mathbf{0}$$

$$\mathbf{x}_B = \mathbf{x} = \mathbf{I}^{-1}\mathbf{b} = \mathbf{b}$$
$$Z = \mathbf{c}_B\,\mathbf{I}^{-1}\mathbf{b} = \mathbf{c}_B\,\mathbf{b}$$

For any iteration,

$$\text{Maximize} \quad Z = \mathbf{c}_B\,\mathbf{x}_B + \mathbf{c}_N\,\mathbf{x}_N$$

subject to

$$[\mathbf{B}, \mathbf{N}]\begin{bmatrix}\mathbf{x}_B\\ \mathbf{x}_N\end{bmatrix} = \mathbf{b} \ \text{ and } \begin{bmatrix}\mathbf{x}_B\\ \mathbf{x}_N\end{bmatrix} \geq \mathbf{0}$$

$$Z = \mathbf{c}_B\,\mathbf{x}_B + \mathbf{c}_N\mathbf{x}_N = \mathbf{c}_B\,\mathbf{x}_B = \mathbf{c}_B\,\mathbf{B}^{-1}\mathbf{b}$$

$$\mathbf{B}\mathbf{x}_B + \mathbf{N}\mathbf{x}_N = \mathbf{b}$$
$$\mathbf{B}\mathbf{x}_B = \mathbf{b} - \mathbf{N}\mathbf{x}_N$$
$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N$$
$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$$

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$$
$$Z = \mathbf{c}_B\,\mathbf{B}^{-1}\mathbf{b}$$

$$\mathbf{B} = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1m} \\ B_{21} & B_{22} & \cdots & B_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ B_{m1} & B_{m2} & \cdots & B_{mm} \end{bmatrix} \quad \mathbf{N} = \begin{bmatrix} N_{11} & N_{12} & \cdots & N_{1n} \\ N_{21} & N_{22} & \cdots & N_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ N_{m1} & N_{m2} & \cdots & N_{mn} \end{bmatrix} \quad \mathbf{x}_B = \begin{bmatrix} x_{B1} \\ x_{B2} \\ \vdots \\ x_{Bm} \end{bmatrix} \quad \mathbf{x}_N = \begin{bmatrix} x_{N1} \\ x_{N2} \\ \vdots \\ x_{Nn} \end{bmatrix}$$

# Example

**TABLE 4.8** Complete set of simplex tableaux for the Wyndor Glass Co. problem

| Iteration | Basic Variable | Eq. | Coefficient of: $Z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | Right Side |
|---|---|---|---|---|---|---|---|---|---|
| 0 | $Z$ | (0) | 1 | −3 | −5 | 0 | 0 | 0 | 0 |
| | $x_3$ | (1) | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| | $x_4$ | (2) | 0 | 0 | 2 | 0 | 1 | 0 | 12 |
| | $x_5$ | (3) | 0 | 3 | 2 | 0 | 0 | 1 | 18 |
| 1 | $Z$ | (0) | 1 | −3 | 0 | 0 | $\frac{5}{2}$ | 0 | 30 |
| | $x_3$ | (1) | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| | $x_2$ | (2) | 0 | 0 | 1 | 0 | $\frac{1}{2}$ | 0 | 6 |
| | $x_5$ | (3) | 0 | 3 | 0 | 0 | −1 | 1 | 6 |
| 2 | $Z$ | (0) | 1 | 0 | 0 | 0 | $\frac{3}{2}$ | 1 | 36 |
| | $x_3$ | (1) | 0 | 0 | 0 | 1 | $\frac{1}{3}$ | $-\frac{1}{3}$ | 2 |
| | $x_2$ | (2) | 0 | 0 | 1 | 0 | $\frac{1}{2}$ | 0 | 6 |
| | $x_1$ | (3) | 0 | 1 | 0 | 0 | $-\frac{1}{3}$ | $\frac{1}{3}$ | 2 |

$$\mathbf{c} = [3, 5], \quad [\mathbf{A}, \mathbf{I}] = \left[\begin{array}{cc|ccc} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\mathbf{b} = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{x}_s = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$$\mathbf{x}_B = \mathbf{x} = \mathbf{I}^{-1}\mathbf{b} = \mathbf{b}$$
$$Z = \mathbf{c}_B \mathbf{I}^{-1}\mathbf{b} = \mathbf{c}_B \mathbf{b}$$

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$$
$$Z = \mathbf{c}_B \mathbf{B}^{-1}\mathbf{b}$$

*Iteration 0*

$$\mathbf{x}_B = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{B}^{-1}, \quad \text{so} \quad \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}$$

$$\mathbf{c}_B = [0, 0, 0], \quad \text{so} \quad Z = [0, 0, 0] \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = 0$$

*Iteration 1*

$$\mathbf{x}_B = \begin{bmatrix} x_3 \\ x_2 \\ x_5 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{bmatrix}, \quad \mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix},$$

$$\text{so} \quad \begin{bmatrix} x_3 \\ x_2 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 6 \end{bmatrix}, \quad \mathbf{c}_B = [0, 5, 0], \quad \text{so } Z = [0, 5, 0] \begin{bmatrix} 4 \\ 6 \\ 6 \end{bmatrix} = 30.$$

*Iteration 2*

$$\mathbf{x}_B = \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 2 & 3 \end{bmatrix}, \quad \mathbf{B}^{-1} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix},$$

$$\begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix}, \quad \mathbf{c}_B = [0, 5, 3], \quad Z = [0, 5, 3] \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix} = 36.$$

# Matrix Form of the Set of Equations in the Simplex Tableau

**TABLE 4.8** Complete set of simplex tableaux for the Wyndor Glass Co. problem

| Iteration | Basic Variable | Eq. | Coefficient of: $Z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | Right Side |
|---|---|---|---|---|---|---|---|---|---|
| 0 | $Z$ | (0) | 1 | −3 | −5 | 0 | 0 | 0 | 0 |
| | $x_3$ | (1) | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| | $x_4$ | (2) | 0 | 0 | 2 | 0 | 1 | 0 | 12 |
| | $x_5$ | (3) | 0 | 3 | 2 | 0 | 0 | 1 | 18 |
| 1 | $Z$ | (0) | 1 | −3 | 0 | 0 | $\frac{5}{2}$ | 0 | 30 |
| | $x_3$ | (1) | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| | $x_2$ | (2) | 0 | 0 | 1 | 0 | $\frac{1}{2}$ | 0 | 6 |
| | $x_5$ | (3) | 0 | 3 | 0 | 0 | −1 | 1 | 6 |
| 2 | $Z$ | (0) | 1 | 0 | 0 | 0 | $\frac{3}{2}$ | 1 | 36 |
| | $x_3$ | (1) | 0 | 0 | 0 | 1 | $\frac{1}{3}$ | $-\frac{1}{3}$ | 2 |
| | $x_2$ | (2) | 0 | 0 | 1 | 0 | $\frac{1}{2}$ | 0 | 6 |
| | $x_1$ | (3) | 0 | 1 | 0 | 0 | $-\frac{1}{3}$ | $\frac{1}{3}$ | 2 |

For the original set of equations, the matrix form is

$$\begin{bmatrix} 1 & -\mathbf{c} & \mathbf{0} \\ \mathbf{0} & \mathbf{A} & \mathbf{I} \end{bmatrix} \begin{bmatrix} Z \\ \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{b} \end{bmatrix}.$$

For any iteration,

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$$
$$Z = \mathbf{c}_B \mathbf{B}^{-1}\mathbf{b}$$

$$\begin{bmatrix} 1 & \mathbf{c}_B\mathbf{B}^{-1} \\ 0 & \mathbf{B}^{-1} \end{bmatrix} \begin{bmatrix} 1 & -\mathbf{c} & \mathbf{0} \\ \mathbf{0} & \mathbf{A} & \mathbf{I} \end{bmatrix} \begin{bmatrix} Z \\ \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{c}_B\mathbf{B}^{-1} \\ 0 & \mathbf{B}^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ \mathbf{b} \end{bmatrix}$$

$$\begin{bmatrix} 1 & \mathbf{c}_B\mathbf{B}^{-1} \\ 0 & \mathbf{B}^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b} \\ \mathbf{B}^{-1}\mathbf{b} \end{bmatrix}$$

$$\begin{bmatrix} 1 & \mathbf{c}_B\mathbf{B}^{-1} \\ 0 & \mathbf{B}^{-1} \end{bmatrix} \begin{bmatrix} 1 & -\mathbf{c} & \mathbf{0} \\ \mathbf{0} & \mathbf{A} & \mathbf{I} \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c} & \mathbf{c}_B\mathbf{B}^{-1} \\ 0 & \mathbf{B}^{-1}\mathbf{A} & \mathbf{B}^{-1} \end{bmatrix}$$

**TABLE 5.8** Initial and later simplex tableaux in matrix form

| Iteration | Basic Variable | Eq. | Coefficient of: $Z$ | Original Variables | Slack Variables | Right Side |
|---|---|---|---|---|---|---|
| 0 | $Z$ | (0) | 1 | $-\mathbf{c}$ | $\mathbf{0}$ | 0 |
| | $\mathbf{x}_B$ | (1, 2, . . . , $m$) | $\mathbf{0}$ | $\mathbf{A}$ | $\mathbf{I}$ | $\mathbf{b}$ |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| Any | $Z$ | (0) | 1 | $\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c}$ | $\mathbf{c}_B\mathbf{B}^{-1}$ | $\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$ |
| | $\mathbf{x}_B$ | (1, 2, . . . , $m$) | $\mathbf{0}$ | $\mathbf{B}^{-1}\mathbf{A}$ | $\mathbf{B}^{-1}$ | $\mathbf{B}^{-1}\mathbf{b}$ |

$$\begin{bmatrix} 1 & \mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c} & \mathbf{c}_B\mathbf{B}^{-1} \\ \mathbf{0} & \mathbf{B}^{-1}\mathbf{A} & \mathbf{B}^{-1} \end{bmatrix} \begin{bmatrix} Z \\ \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} = \begin{bmatrix} \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b} \\ \mathbf{B}^{-1}\mathbf{b} \end{bmatrix}.$$

# Example

**TABLE 4.8** Complete set of simplex tableaux for the Wyndor Glass Co. problem

| Iteration | Basic Variable | Eq. | Coefficient of: $Z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | Right Side |
|---|---|---|---|---|---|---|---|---|---|
| 0 | $Z$ | (0) | 1 | −3 | −5 | 0 | 0 | 0 | 0 |
| | $x_3$ | (1) | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| | $x_4$ | (2) | 0 | 0 | 2 | 0 | 1 | 0 | 12 |
| | $x_5$ | (3) | 0 | 3 | 2 | 0 | 0 | 1 | 18 |
| 1 | $Z$ | (0) | 1 | −3 | 0 | 0 | $\frac{5}{2}$ | 0 | 30 |
| | $x_3$ | (1) | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| | $x_2$ | (2) | 0 | 0 | 1 | 0 | $\frac{1}{2}$ | 0 | 6 |
| | $x_5$ | (3) | 0 | 3 | 0 | 0 | −1 | 1 | 6 |
| 2 | $Z$ | (0) | 1 | 0 | 0 | 0 | $\frac{3}{2}$ | 1 | 36 |
| | $x_3$ | (1) | 0 | 0 | 0 | 1 | $\frac{1}{3}$ | $-\frac{1}{3}$ | 2 |
| | $x_2$ | (2) | 0 | 0 | 1 | 0 | $\frac{1}{2}$ | 0 | 6 |
| | $x_1$ | (3) | 0 | 1 | 0 | 0 | $-\frac{1}{3}$ | $\frac{1}{3}$ | 2 |

$$\mathbf{c} = [3, 5], \quad [\mathbf{A}, \mathbf{I}] = \left[\begin{array}{cc:ccc} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 & 1 \end{array}\right], \quad \mathbf{b} = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}$$

**For Iteration 2**

$$\mathbf{x}_B = \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 2 & 3 \end{bmatrix}, \quad \boxed{\mathbf{B}^{-1} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}},$$

$$\mathbf{B}^{-1}\mathbf{A} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}},$$

$$\mathbf{c}_B\mathbf{B}^{-1} = [0, 5, 3] \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} = \boxed{[0, \tfrac{3}{2}, 1]},$$

$$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c} = [0, 5, 3] \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} - [3, 5] = \boxed{[0, 0]} \quad \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = \boxed{\begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix}}, \quad \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b} = [0, 5, 3] \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix} = \boxed{36}.$$

$$\begin{bmatrix} 1 & \boxed{\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c}} & \boxed{\mathbf{c}_B\mathbf{B}^{-1}} \\ 0 & \boxed{\mathbf{B}^{-1}\mathbf{A}} & \boxed{\mathbf{B}^{-1}} \end{bmatrix} \begin{bmatrix} Z \\ \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} = \begin{bmatrix} \boxed{\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}} \\ \boxed{\mathbf{B}^{-1}\mathbf{b}} \end{bmatrix}.$$

$$\left[\begin{array}{c|cc|ccc} 1 & 0 & 0 & 0 & \frac{3}{2} & 1 \\ \hline 0 & 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} \end{array}\right] \begin{bmatrix} Z \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 36 \\ 2 \\ 6 \\ 2 \end{bmatrix}$$

# Summary of the Revised Simplex Method

**TABLE 4.8** Complete set of simplex tableaux for the Wyndor Glass Co. problem

| Iteration | Basic Variable | Eq. | $Z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | Right Side |
|---|---|---|---|---|---|---|---|---|---|
| 0 | $Z$ | (0) | 1 | −3 | −5 | 0 | 0 | 0 | 0 |
| | $x_3$ | (1) | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| | $x_4$ | (2) | 0 | 0 | 2 | 0 | 1 | 0 | 12 |
| | $x_5$ | (3) | 0 | 3 | 2 | 0 | 0 | 1 | 18 |
| 1 | $Z$ | (0) | 1 | −3 | 0 | 0 | $\frac{5}{2}$ | 0 | 30 |
| | $x_3$ | (1) | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| | $x_2$ | (2) | 0 | 0 | 1 | 0 | $\frac{1}{2}$ | 0 | 6 |
| | $x_5$ | (3) | 0 | 3 | 0 | 0 | −1 | 1 | 6 |
| 2 | $Z$ | (0) | 1 | 0 | 0 | 0 | $\frac{3}{2}$ | 1 | 36 |
| | $x_3$ | (1) | 0 | 0 | 0 | 1 | $\frac{1}{3}$ | $-\frac{1}{3}$ | 2 |
| | $x_2$ | (2) | 0 | 0 | 1 | 0 | $\frac{1}{2}$ | 0 | 6 |
| | $x_1$ | (3) | 0 | 1 | 0 | 0 | $-\frac{1}{3}$ | $\frac{1}{3}$ | 2 |

**TABLE 5.8** Initial and later simplex tableaux in matrix form

| Iteration | Basic Variable | Eq. | $Z$ | Original Variables | Slack Variables | Right Side |
|---|---|---|---|---|---|---|
| 0 | $Z$<br>$\mathbf{x}_B$ | (0)<br>(1, 2, . . . , $m$) | 1<br>**0** | $-\mathbf{c}$<br>$\mathbf{A}$ | **0**<br>$\mathbf{I}$ | 0<br>$\mathbf{b}$ |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| Any | $Z$<br>$\mathbf{x}_B$ | (0)<br>(1, 2, . . . , $m$) | 1<br>**0** | $\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A}-\mathbf{c}$<br>$\mathbf{B}^{-1}\mathbf{A}$ | $\mathbf{c}_B\mathbf{B}^{-1}$<br>$\mathbf{B}^{-1}$ | $\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$<br>$\mathbf{B}^{-1}\mathbf{b}$ |

1. Initialization (Iteration 0)

$$\mathbf{c}=[3,5],\quad [\mathbf{A},\mathbf{I}]=\left[\begin{array}{cc:ccc}1&0&1&0&0\\0&2&0&1&0\\3&2&0&0&1\end{array}\right]$$

$$\mathbf{b}=\begin{bmatrix}4\\12\\18\end{bmatrix},\quad \mathbf{x}=\begin{bmatrix}x_1\\x_2\end{bmatrix},\quad \mathbf{x}_s=\begin{bmatrix}x_3\\x_4\\x_5\end{bmatrix}$$

$$\mathbf{x}_B=\begin{bmatrix}x_3\\x_4\\x_5\end{bmatrix},\ \mathbf{B}=\mathbf{I}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\mathbf{B}^{-1}$$

$$\mathbf{x}_B=\begin{bmatrix}x_3\\x_4\\x_5\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}4\\12\\18\end{bmatrix}=\begin{bmatrix}4\\12\\18\end{bmatrix}$$

$$\mathbf{c}_B=[0,0,0],\qquad \text{so}\qquad Z=[0,0,0]\begin{bmatrix}4\\12\\18\end{bmatrix}=0$$

Optimality test:

$$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A}-\mathbf{c}=[0,0,0]\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0\\0&2\\3&2\end{bmatrix}-[3,5]=[-3,-5]$$

TABLE 4.8 Complete set of simplex tableaux for the Wyndor Glass Co. problem

| Iteration | Basic Variable | Eq. | $Z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | Right Side |
|---|---|---|---|---|---|---|---|---|---|
| 0 | $Z$ | (0) | 1 | −3 | −5 | 0 | 0 | 0 | 0 |
| | $x_3$ | (1) | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| | $x_4$ | (2) | 0 | 0 | 2 | 0 | 1 | 0 | 12 |
| | $x_5$ | (3) | 0 | 3 | 2 | 0 | 0 | 1 | 18 |
| 1 | $Z$ | (0) | 1 | −3 | 0 | 0 | $\frac{5}{2}$ | 0 | 30 |
| | $x_3$ | (1) | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| | $x_2$ | (2) | 0 | 0 | 1 | 0 | $\frac{1}{2}$ | 0 | 6 |
| | $x_5$ | (3) | 0 | 3 | 0 | 0 | −1 | 1 | 6 |
| 2 | $Z$ | (0) | 1 | 0 | 0 | 0 | $\frac{3}{2}$ | 1 | 36 |
| | $x_3$ | (1) | 0 | 0 | 0 | 1 | $\frac{1}{3}$ | $-\frac{1}{3}$ | 2 |
| | $x_2$ | (2) | 0 | 0 | 1 | 0 | $\frac{1}{2}$ | 0 | 6 |
| | $x_1$ | (3) | 0 | 1 | 0 | 0 | $-\frac{1}{3}$ | $\frac{1}{3}$ | 2 |

TABLE 5.8 Initial and later simplex tableaux in matrix form

| Iteration | Basic Variable | Eq. | $Z$ | Original Variables | Slack Variables | Right Side |
|---|---|---|---|---|---|---|
| 0 | $Z$ | (0) | 1 | $-\mathbf{c}$ | $\mathbf{0}$ | 0 |
| | $\mathbf{x}_B$ | $(1, 2, \ldots, m)$ | $\mathbf{0}$ | $\mathbf{A}$ | $\mathbf{I}$ | $\mathbf{b}$ |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| Any | $Z$ | (0) | 1 | $\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A}-\mathbf{c}$ | $\mathbf{c}_B\mathbf{B}^{-1}$ | $\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$ |
| | $\mathbf{x}_B$ | $(1, 2, \ldots, m)$ | $\mathbf{0}$ | $\mathbf{B}^{-1}\mathbf{A}$ | $\mathbf{B}^{-1}$ | $\mathbf{B}^{-1}\mathbf{b}$ |

## 2. Iteration 1

### Step 1: Determine the entering basic variable

$$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A}-\mathbf{c}=[0,0,0]\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0\\0&2\\3&2\end{bmatrix}-[3,5]=[-3,-5]$$

$$-c_2=-5<-3=-c_1$$

So $x_2$ is chosen to be the entering variable.

### Step 2: Determine the leaving basic variable

$$\mathbf{B}^{-1}\mathbf{A}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}\text{—}&0\\\text{—}&2\\\text{—}&2\end{bmatrix}=\begin{bmatrix}\text{—}&0\\\text{—}&2\\\text{—}&2\end{bmatrix}$$

$$a_{12}=0 \qquad a_{22}=2 \qquad a_{32}=2$$

$$b_2/a_{22}=\frac{12}{2} \qquad b_3/a_{32}=\frac{18}{2}$$

So the number of the pivot row r =2

Thus, $x_4$ is chosen to be the leaving variable.

## Step 3: Determine the new BF solution

The new set of basic variables is $\mathbf{x}_B = \begin{bmatrix} x_3 \\ x_2 \\ x_5 \end{bmatrix}$

To obtain the new $\mathbf{B}^{-1}$, $\boldsymbol{\eta} = \begin{bmatrix} -\frac{a_{12}}{a_{22}} \\ \frac{1}{a_{22}} \\ -\frac{a_{32}}{a_{22}} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ -1 \end{bmatrix}$, $\quad \mathbf{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix}$

$$\mathbf{B}_{\text{new}}^{-1} = \mathbf{E}\mathbf{B}_{\text{old}}^{-1},$$

So the new $\mathbf{B}^{-1}$ is

$$\mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\mathbf{x}_B = \begin{bmatrix} x_3 \\ x_2 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 6 \end{bmatrix}$$

**TABLE 5.8** Initial and later simplex tableaux in matrix form

| Iteration | Basic Variable | Eq. | Coefficient of: Z | Original Variables | Slack Variables | Right Side |
|---|---|---|---|---|---|---|
| 0 | $Z$<br>$\mathbf{x}_B$ | (0)<br>(1, 2, . . . , $m$) | 1<br>**0** | $-\mathbf{c}$<br>$\mathbf{A}$ | **0**<br>$\mathbf{I}$ | 0<br>**b** |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| Any | $Z$<br>$\mathbf{x}_B$ | (0)<br>(1, 2, . . . , $m$) | 1<br>**0** | $\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c}$<br>$\mathbf{B}^{-1}\mathbf{A}$ | $\mathbf{c}_B\mathbf{B}^{-1}$<br>$\mathbf{B}^{-1}$ | $\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$<br>$\mathbf{B}^{-1}\mathbf{b}$ |

Optimality test:

The nonbasic variables are $x_1$ and $x_4$.

$$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c} = [0, 5, 0]\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & — \\ 0 & — \\ 3 & — \end{bmatrix} - [3, —] = [-3, —],$$

$$\mathbf{c}_B\mathbf{B}^{-1} = [0, 5, 0]\begin{bmatrix} — & 0 & — \\ — & \frac{1}{2} & — \\ — & -1 & — \end{bmatrix} = [—, \tfrac{5}{2}, —],$$

**TABLE 5.8** Initial and later simplex tableaux in matrix form

| Iteration | Basic Variable | Eq. | Coefficient of: $Z$ | Coefficient of: Original Variables | Coefficient of: Slack Variables | Right Side |
|---|---|---|---|---|---|---|
| 0 | $Z$<br>$\mathbf{x}_B$ | (0)<br>(1, 2, . . . , $m$) | 1<br>**0** | $-\mathbf{c}$<br>$\mathbf{A}$ | **0**<br>$\mathbf{I}$ | 0<br>$\mathbf{b}$ |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| Any | $Z$<br>$\mathbf{x}_B$ | (0)<br>(1, 2, . . . , $m$) | 1<br>**0** | $\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c}$<br>$\mathbf{B}^{-1}\mathbf{A}$ | $\mathbf{c}_B\mathbf{B}^{-1}$<br>$\mathbf{B}^{-1}$ | $\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$<br>$\mathbf{B}^{-1}\mathbf{b}$ |

3. Iteration 2

Step 1: Determine the entering basic variable

$\boldsymbol{x_1}$ is chosen to be the entering variable.

Step 2: Determine the leaving basic variable

$$\mathbf{B}^{-1}\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & — \\ 0 & — \\ 3 & — \end{bmatrix} = \begin{bmatrix} 1 & — \\ 0 & — \\ 3 & — \end{bmatrix}. \qquad \mathbf{x}_B = \begin{bmatrix} x_3 \\ x_2 \\ x_5 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 6 \end{bmatrix}$$

**TABLE 5.8** Initial and later simplex tableaux in matrix form

| Iteration | Basic Variable | Eq. | Coefficient of: $Z$ | Original Variables | Slack Variables | Right Side |
|---|---|---|---|---|---|---|
| 0 | $Z$<br>$\mathbf{x}_B$ | (0)<br>$(1, 2, \ldots, m)$ | 1<br>**0** | $-\mathbf{c}$<br>$\mathbf{A}$ | **0**<br>$\mathbf{I}$ | 0<br>$\mathbf{b}$ |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| Any | $Z$<br>$\mathbf{x}_B$ | (0)<br>$(1, 2, \ldots, m)$ | 1<br>**0** | $\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c}$<br>$\mathbf{B}^{-1}\mathbf{A}$ | $\mathbf{c}_B\mathbf{B}^{-1}$<br>$\mathbf{B}^{-1}$ | $\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$<br>$\mathbf{B}^{-1}\mathbf{b}$ |

The ratio 4/1 > 6/3 indicate that x5 is the leaving basic variable

Step 3: Determine the new BF solution

The new set of basic variables is

$$\mathbf{x}_B = \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} \quad \text{with} \quad \boldsymbol{\eta} = \begin{bmatrix} -\dfrac{a'_{11}}{a'_{31}} \\ -\dfrac{a'_{21}}{a'_{31}} \\ \dfrac{1}{a'_{31}} \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{3} \\ 0 \\ \dfrac{1}{3} \end{bmatrix}. \qquad \mathbf{E} = \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

Therefore, the new $\mathbf{B}^{-1}$ is

$$\mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}, \quad \mathbf{x}_B = \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix}.$$

Optimality test:

The nonbasic variables are $x_4$ and $x_5$.

$$\mathbf{c}_B\mathbf{B}^{-1} = [0, 5, 3] \begin{bmatrix} — & \frac{1}{3} & -\frac{1}{3} \\ — & \frac{1}{2} & 0 \\ — & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} = [—, \tfrac{3}{2}, 1].$$

# Relationship between the initial and final simplex tableaux

**TABLE 4.8** Complete set of simplex tableaux for the Wyndor Glass Co. problem

| Iteration | Basic Variable | Eq. | Coefficient of: $Z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | Right Side |
|---|---|---|---|---|---|---|---|---|---|
| 0 | $Z$ | (0) | 1 | −3 | −5 | 0 | 0 | 0 | 0 |
| | $x_3$ | (1) | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| | $x_4$ | (2) | 0 | 0 | 2 | 0 | 1 | 0 | 12 |
| | $x_5$ | (3) | 0 | 3 | 2 | 0 | 0 | 1 | 18 |
| 1 | $Z$ | (0) | 1 | −3 | 0 | 0 | $\frac{5}{2}$ | 0 | 30 |
| | $x_3$ | (1) | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| | $x_2$ | (2) | 0 | 0 | 1 | 0 | $\frac{1}{2}$ | 0 | 6 |
| | $x_5$ | (3) | 0 | 3 | 0 | 0 | −1 | 1 | 6 |
| 2 | $Z$ | (0) | 1 | 0 | 0 | 0 | $\frac{3}{2}$ | 1 | 36 |
| | $x_3$ | (1) | 0 | 0 | 0 | 1 | $\frac{1}{3}$ | $-\frac{1}{3}$ | 2 |
| | $x_2$ | (2) | 0 | 0 | 1 | 0 | $\frac{1}{2}$ | 0 | 6 |
| | $x_1$ | (3) | 0 | 1 | 0 | 0 | $-\frac{1}{3}$ | $\frac{1}{3}$ | 2 |

**TABLE 5.8** Initial and later simplex tableaux in matrix form

| Iteration | Basic Variable | Eq. | Coefficient of: $Z$ | Original Variables | Slack Variables | Right Side |
|---|---|---|---|---|---|---|
| 0 | $Z$ | (0) | 1 | $-\mathbf{c}$ | $\mathbf{0}$ | 0 |
| | $\mathbf{x}_B$ | $(1, 2, \ldots, m)$ | $\mathbf{0}$ | $\mathbf{A}$ | $\mathbf{I}$ | $\mathbf{b}$ |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| Any | $Z$ | (0) | 1 | $\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A}-\mathbf{c}$ | $\mathbf{c}_B\mathbf{B}^{-1}$ | $\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$ |
| | $\mathbf{x}_B$ | $(1, 2, \ldots, m)$ | $\mathbf{0}$ | $\mathbf{B}^{-1}\mathbf{A}$ | $\mathbf{B}^{-1}$ | $\mathbf{B}^{-1}\mathbf{b}$ |

**Initial Tableau**

Row 0: $\mathbf{t} = [-3, -5 \mid 0, 0, 0 \mid 0] = [-\mathbf{c} \mid \mathbf{0} \mid 0]$.

Other rows:
$$\mathbf{T} = \left[\begin{array}{cc|ccc|c} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 2 & 0 & 1 & 0 & 12 \\ 3 & 2 & 0 & 0 & 1 & 18 \end{array}\right] = [\mathbf{A} \mid \mathbf{I} \mid \mathbf{b}].$$

Combined:
$$\begin{bmatrix} \mathbf{t} \\ \mathbf{T} \end{bmatrix} = \left[\begin{array}{c|c|c} -\mathbf{c} & \mathbf{0} & 0 \\ \mathbf{A} & \mathbf{I} & \mathbf{b} \end{array}\right].$$

**Final Tableau**

Row 0: $\mathbf{t}^* = [0, 0 \mid 0, \frac{3}{2}, 1 \mid 36] = [\mathbf{z}^* - \mathbf{c} \mid \mathbf{y}^* \mid Z^*]$.

Other rows:
$$\mathbf{T}^* = \left[\begin{array}{cc|ccc|c} 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & 2 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & 6 \\ 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & 2 \end{array}\right] = [\mathbf{A}^* \mid \mathbf{S}^* \mid \mathbf{b}^*].$$

Combined:
$$\begin{bmatrix} \mathbf{t}^* \\ \mathbf{T}^* \end{bmatrix} = \left[\begin{array}{c|c|c} \mathbf{z}^* - \mathbf{c} & \mathbf{y}^* & Z^* \\ \mathbf{A}^* & \mathbf{S}^* & \mathbf{b}^* \end{array}\right].$$

$$\mathbf{z}^* = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} \qquad \mathbf{y}^* = \mathbf{c}_B\mathbf{B}^{-1} \qquad Z^* = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$$

$$\mathbf{A}^* = \mathbf{B}^{-1}\mathbf{A} \qquad \mathbf{S}^* = \mathbf{B}^{-1} \qquad \mathbf{b}^* = \mathbf{B}^{-1}\mathbf{b}$$

$$(1)\quad \mathbf{t}^* = \mathbf{t} + \mathbf{y}^*\mathbf{T} = [\mathbf{y}^*\mathbf{A} - \mathbf{c} \mid \mathbf{y}^* \mid \mathbf{y}^*\mathbf{b}] = [\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c} \mid \mathbf{c}_B\mathbf{B}^{-1} \mid \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}]$$

$$(2)\quad \mathbf{T}^* = \mathbf{S}^*\mathbf{T} = [\mathbf{S}^*\mathbf{A} \mid \mathbf{S}^* \mid \mathbf{S}^*\mathbf{b}] = [\mathbf{B}^{-1}\mathbf{A} \mid \mathbf{B}^{-1} \mid \mathbf{B}^{-1}\mathbf{b}]$$

■ **TABLE 4.8** Complete set of simplex tableaux for the Wyndor Glass Co. problem

| Iteration | Basic Variable | Eq. | $Z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | Right Side |
|---|---|---|---|---|---|---|---|---|---|
| 0 | $Z$ | (0) | 1 | −3 | −5 | 0 | 0 | 0 | 0 |
| | $x_3$ | (1) | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| | $x_4$ | (2) | 0 | 0 | 2 | 0 | 1 | 0 | 12 |
| | $x_5$ | (3) | 0 | 3 | 2 | 0 | 0 | 1 | 18 |
| 1 | $Z$ | (0) | 1 | −3 | 0 | 0 | $\frac{5}{2}$ | 0 | 30 |
| | $x_3$ | (1) | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| | $x_2$ | (2) | 0 | 0 | 1 | 0 | $\frac{1}{2}$ | 0 | 6 |
| | $x_5$ | (3) | 0 | 3 | 0 | 0 | −1 | 1 | 6 |
| 2 | $Z$ | (0) | 1 | 0 | 0 | 0 | $\frac{3}{2}$ | 1 | 36 |
| | $x_3$ | (1) | 0 | 0 | 0 | 1 | $\frac{1}{3}$ | $-\frac{1}{3}$ | 2 |
| | $x_2$ | (2) | 0 | 0 | 1 | 0 | $\frac{1}{2}$ | 0 | 6 |
| | $x_1$ | (3) | 0 | 1 | 0 | 0 | $-\frac{1}{3}$ | $\frac{1}{3}$ | 2 |

**Initial Tableau**

Row 0: $\mathbf{t} = [-3, \ -5 \mid 0, 0, 0 \mid 0] = [-\mathbf{c} \mid \mathbf{0} \mid 0].$

Other rows: $\mathbf{T} = \left[\begin{array}{cc|ccc|c} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 2 & 0 & 1 & 0 & 12 \\ 3 & 2 & 0 & 0 & 1 & 18 \end{array}\right] = [\mathbf{A} \mid \mathbf{I} \mid \mathbf{b}].$

Combined: $\begin{bmatrix} \mathbf{t} \\ \mathbf{T} \end{bmatrix} = \left[\begin{array}{c|c|c} -\mathbf{c} & \mathbf{0} & 0 \\ \mathbf{A} & \mathbf{I} & \mathbf{b} \end{array}\right].$

For iteration 1:

$$\mathbf{S}^* = \mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} \qquad \mathbf{y}^* = [0, \ \tfrac{5}{2}, \ 0]$$

$$\mathbf{t}^* = \mathbf{t} + \mathbf{y}^*\mathbf{T} = [-3, \ -5 \mid 0, \ 0, \ 0 \mid 0] + [0, \ \tfrac{5}{2}, \ 0]\left[\begin{array}{cc|ccc|c} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 2 & 0 & 1 & 0 & 12 \\ 3 & 2 & 0 & 0 & 1 & 18 \end{array}\right] = [-3, \ 0, \ \boxed{0, \ \tfrac{5}{2}, \ 0}, \ 30]$$

$$\mathbf{T}^* = \mathbf{S}^*\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix}\left[\begin{array}{cc|ccc|c} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 2 & 0 & 1 & 0 & 12 \\ 3 & 2 & 0 & 0 & 1 & 18 \end{array}\right] = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & 6 \\ 3 & 0 & 0 & -1 & 1 & 6 \end{bmatrix}$$

**TABLE 4.8** Complete set of simplex tableaux for the Wyndor Glass Co. problem

| Iteration | Basic Variable | Eq. | $Z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | Right Side |
|---|---|---|---|---|---|---|---|---|---|
| 0 | $Z$ | (0) | 1 | −3 | −5 | 0 | 0 | 0 | 0 |
| | $x_3$ | (1) | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| | $x_4$ | (2) | 0 | 0 | 2 | 0 | 1 | 0 | 12 |
| | $x_5$ | (3) | 0 | 3 | 2 | 0 | 0 | 1 | 18 |
| 1 | $Z$ | (0) | 1 | −3 | 0 | 0 | $\frac{5}{2}$ | 0 | 30 |
| | $x_3$ | (1) | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| | $x_2$ | (2) | 0 | 0 | 1 | 0 | $\frac{1}{2}$ | 0 | 6 |
| | $x_5$ | (3) | 0 | 3 | 0 | 0 | −1 | 1 | 6 |
| 2 | $Z$ | (0) | 1 | 0 | 0 | 0 | $\frac{3}{2}$ | 1 | 36 |
| | $x_3$ | (1) | 0 | 0 | 0 | 1 | $\frac{1}{3}$ | $-\frac{1}{3}$ | 2 |
| | $x_2$ | (2) | 0 | 0 | 1 | 0 | $\frac{1}{2}$ | 0 | 6 |
| | $x_1$ | (3) | 0 | 1 | 0 | 0 | $-\frac{1}{3}$ | $\frac{1}{3}$ | 2 |

**Initial Tableau**

Row 0: $\mathbf{t} = [-3, -5 \mid 0, 0, 0 \mid 0] = [-\mathbf{c} \mid \mathbf{0} \mid 0].$

Other rows: $\mathbf{T} = \left[\begin{array}{cc|ccc|c} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 2 & 0 & 1 & 0 & 12 \\ 3 & 2 & 0 & 0 & 1 & 18 \end{array}\right] = [\mathbf{A} \mid \mathbf{I} \mid \mathbf{b}].$

Combined: $\begin{bmatrix} \mathbf{t} \\ \mathbf{T} \end{bmatrix} = \left[\begin{array}{c|c|c} -\mathbf{c} & \mathbf{0} & 0 \\ \mathbf{A} & \mathbf{I} & \mathbf{b} \end{array}\right].$

For iteration 2:

$$\mathbf{S}^* = \mathbf{B}^{-1} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \qquad \mathbf{y}^* = [0, \ \tfrac{3}{2}, \ 1]$$

$$\mathbf{t}^* = \mathbf{t} + \mathbf{y}^*\mathbf{T} = [-3, \ -5 \mid 0, \ 0, \ 0 \mid 0] + [0, \ \tfrac{3}{2}, \ 1] \left[\begin{array}{cc|ccc|c} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 2 & 0 & 1 & 0 & 12 \\ 3 & 2 & 0 & 0 & 1 & 18 \end{array}\right] = [0, \ 0, \ \boxed{0, \ \tfrac{3}{2}, \ 1}, \ 36]$$

$$\mathbf{T}^* = \mathbf{S}^*\mathbf{T} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \left[\begin{array}{cc|ccc|c} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 2 & 0 & 1 & 0 & 12 \\ 3 & 2 & 0 & 0 & 1 & 18 \end{array}\right] = \begin{bmatrix} 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & 2 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & 6 \\ 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & 2 \end{bmatrix}$$

# REVISED SIMPLEX METHOD

- Computationally effective modification of the standard Simplex Method.

- Unfortunately our examples are too small to make the computational savings significant. You may even get an impression that it is more time consuming.

- However for large problems, particularly where $n$ is much greater than $m$, the saving could be significant.

# DUAL SIMPLEX

# Duality Theory

*Primal Problem*

Maximize $Z = \sum_{j=1}^{n} c_j x_j,$

subject to

$\sum_{j=1}^{n} a_{ij} x_j \le b_i$ for $i = 1, 2, \ldots, m$

and

$x_j \ge 0,$ for $j = 1, 2, \ldots, n.$

*Dual Problem*

Minimize $y_0 = \sum_{i=1}^{m} b_i y_i,$

subject to

$\sum_{i=1}^{m} a_{ij} y_i \ge c_j,$ for $j = 1, 2, \ldots, n$

and

$y_i \ge 0,$ for $i = 1, 2, \ldots, m.$

*Primal Problem*

Maximize $Z = \mathbf{cx},$

subject to

$$\mathbf{Ax} \le \mathbf{b}$$

and

$$\mathbf{x} \ge \mathbf{0}.$$

*Dual Problem*

Minimize $y_0 = \mathbf{yb},$

subject to

$$\mathbf{yA} \ge \mathbf{c}$$

and

$$\mathbf{y} \ge \mathbf{0}.$$

**TABLE 5.8** Initial and later simplex tableaux in matrix form

| Iteration | Basic Variable | Eq. | Coefficient of: $Z$ | Original Variables | Slack Variables | Right Side |
|---|---|---|---|---|---|---|
| 0 | $Z$<br>$\mathbf{x}_B$ | (0)<br>(1, 2, . . . , $m$) | 1<br>**0** | $-\mathbf{c}$<br>$\mathbf{A}$ | $\mathbf{0}$<br>$\mathbf{I}$ | 0<br>$\mathbf{b}$ |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| Any | $Z$<br>$\mathbf{x}_B$ | (0)<br>(1, 2, . . . , $m$) | 1<br>**0** | $\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c}$<br>$\mathbf{B}^{-1}\mathbf{A}$ | $\mathbf{c}_B\mathbf{B}^{-1}$<br>$\mathbf{B}^{-1}$ | $\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$<br>$\mathbf{B}^{-1}\mathbf{b}$ |

$$\mathbf{y} = \mathbf{c}_B\mathbf{B}^{-1}$$

**TABLE 6.1** Primal and dual problems for the Wyndor Glass Co. example

*Primal Problem in Algebraic Form*

Maximize $Z = 3x_1 + 5x_2,$

subject to

$$x_1 \le 4$$
$$2x_2 \le 12$$
$$3x_1 + 2x_2 \le 18$$

and $x_1 \ge 0, \quad x_2 \ge 0.$

*Dual Problem*

*Dual Problem in Algebraic Form*

Minimize $W = 4y_1 + 12y_2 + 18y_3,$

subject to

$$y_1 + 3y_3 \ge 3$$
$$2y_2 + 2y_3 \ge 5$$

and

$y_1 \ge 0, \quad y_2 \ge 0, \quad y_3 \ge 0.$

*Primal Problem in Matrix Form*

Maximize $Z = [3, 5]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$

subject to

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}$$

and

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \ge \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

*Dual Problem in Matrix Form*

Minimize $W = [y_1, y_2, y_3]\begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}$

subject to

$$[y_1, y_2, y_3]\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{bmatrix} \ge [3, 5]$$

and

$$[y_1, y_2, y_3] \ge [0, 0, 0].$$

**TABLE 6.2** Primal-dual table for linear programming, illustrated by the Wyndor Glass Co. example

***(a) General Case***

| | | | Primal Problem: Coefficient of: $x_1$ | $x_2$ | $\cdots$ | $x_n$ | Right Side | |
|---|---|---|---|---|---|---|---|---|
| Dual Problem | Coefficient of: | $y_1$ | $a_{11}$ | $a_{12}$ | $\cdots$ | $a_{1n}$ | $\le b_1$ | Coefficients for Objective Function (Minimize) |
| | | $y_2$ | $a_{21}$ | $a_{22}$ | $\cdots$ | $a_{2n}$ | $\le b_2$ | |
| | | $\vdots$ | | | | | $\vdots$ | |
| | | $y_m$ | $a_{m1}$ | $a_{m2}$ | $\cdots$ | $a_{mn}$ | $\le b_m$ | |
| | Right Side | | VI<br>$c_1$ | VI<br>$c_2$ | $\cdots$<br>$\cdots$ | VI<br>$c_n$ | | |
| | | | Coefficients for Objective Function (Maximize) | | | | | |

***(b) Wyndor Glass Co. Example***

| | $x_1$ | $x_2$ | |
|---|---|---|---|
| $y_1$ | 1 | 0 | $\le 4$ |
| $y_2$ | 0 | 2 | $\le 12$ |
| $y_3$ | 3 | 2 | $\le 18$ |
| | VI<br>3 | VI<br>5 | |

**Initial Tableau**

Row 0: $\mathbf{t} = [-3, -5 \mid 0, 0, 0 \mid 0] = [-\mathbf{c} \mid \mathbf{0} \mid 0]$.

Other rows: $\mathbf{T} = \left[\begin{array}{cc|ccc|c} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 2 & 0 & 1 & 0 & 12 \\ 3 & 2 & 0 & 0 & 1 & 18 \end{array}\right] = [\mathbf{A} \mid \mathbf{I} \mid \mathbf{b}]$.

Combined: $\begin{bmatrix} \mathbf{t} \\ \mathbf{T} \end{bmatrix} = \left[\begin{array}{c|c|c} -\mathbf{c} & \mathbf{0} & 0 \\ \mathbf{A} & \mathbf{I} & \mathbf{b} \end{array}\right]$.

**Final Tableau**

Row 0: $\mathbf{t}^* = [0, 0 \mid 0, \frac{3}{2}, 1 \mid 36] = [\mathbf{z}^* - \mathbf{c} \mid \mathbf{y}^* \mid Z^*]$.

Other rows: $\mathbf{T}^* = \left[\begin{array}{cc|ccc|c} 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & 2 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & 6 \\ 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & 2 \end{array}\right] = [\mathbf{A}^* \mid \mathbf{S}^* \mid \mathbf{b}^*]$.

Combined: $\begin{bmatrix} \mathbf{t}^* \\ \mathbf{T}^* \end{bmatrix} = \left[\begin{array}{c|c|c} \mathbf{z}^* - \mathbf{c} & \mathbf{y}^* & Z^* \\ \mathbf{A}^* & \mathbf{S}^* & \mathbf{b}^* \end{array}\right]$.

**TABLE 5.8** Initial and later simplex tableaux in matrix form

| Iteration | Basic Variable | Eq. | Coefficient of: Z | Coefficient of: Original Variables | Coefficient of: Slack Variables | Right Side |
|---|---|---|---|---|---|---|
| 0 | $Z$<br>$\mathbf{x}_B$ | (0)<br>(1, 2, . . . , $m$) | 1<br>**0** | $-\mathbf{c}$<br>$\mathbf{A}$ | $\mathbf{0}$<br>$\mathbf{I}$ | 0<br>$\mathbf{b}$ |
| | | | | | | |
| Any | $Z$<br>$\mathbf{x}_B$ | (0)<br>(1, 2, . . . , $m$) | 1<br>**0** | $\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c}$<br>$\mathbf{B}^{-1}\mathbf{A}$ | $\mathbf{c}_B\mathbf{B}^{-1}$<br>$\mathbf{B}^{-1}$ | $\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$<br>$\mathbf{B}^{-1}\mathbf{b}$ |

$$\mathbf{y}^* = \mathbf{c}_B\mathbf{B}^{-1}$$

$$\mathbf{z}^* = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{A}$$

$$\mathbf{z} = \mathbf{y}\mathbf{A},$$

so $$z_j = \sum_{i=1}^{m} a_{ij} y_i,$$

$$z_j - c_j = \sum_{i=1}^{m} a_{ij} y_i - c_j$$

for $j = 1, 2, \dots, n$.

*Dual Problem*

Minimize $\quad y_0 = \sum_{i=1}^{m} b_i y_i,$

subject to

$$\sum_{i=1}^{m} a_{ij} y_i \geq c_j, \qquad \text{for } j = 1, 2, \dots, n$$

and

$$y_i \geq 0, \qquad \text{for } i = 1, 2, \dots, m.$$

$z_j - c_j$ **is the surplus variable for the functional constraints in the dual problem.**

**TABLE 6.4** Notation for entries in row 0 of a simplex tableau

| Iteration | Basic Variable | Eq. | Coefficient of: $Z$ | $x_1$ | $x_2$ | $\cdots$ | $x_n$ | $x_{n+1}$ | $x_{n+2}$ | $\cdots$ | $x_{n+m}$ | Right Side |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Any | $Z$ | (0) | 1 | $z_1 - c_1$ | $z_2 - c_2$ | $\cdots$ | $z_n - c_n$ | $y_1$ | $y_2$ | $\cdots$ | $y_m$ | $W$ |

**TABLE 4.8** Complete set of simplex tableaux for the Wyndor Glass Co. problem

| Iteration | Basic Variable | Eq. | Coefficient of: $Z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | Right Side |
|---|---|---|---|---|---|---|---|---|---|
| 0 | $Z$ | (0) | 1 | −3 | −5 | 0 | 0 | 0 | 0 |
| | $x_3$ | (1) | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| | $x_4$ | (2) | 0 | 0 | 2 | 0 | 1 | 0 | 12 |
| | $x_5$ | (3) | 0 | 3 | 2 | 0 | 0 | 1 | 18 |
| 1 | $Z$ | (0) | 1 | −3 | 0 | 0 | $\frac{5}{2}$ | 0 | 30 |
| | $x_3$ | (1) | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| | $x_2$ | (2) | 0 | 0 | 1 | 0 | $\frac{1}{2}$ | 0 | 6 |
| | $x_5$ | (3) | 0 | 3 | 0 | 0 | −1 | 1 | 6 |
| 2 | $Z$ | (0) | 1 | 0 | 0 | 0 | $\frac{3}{2}$ | 1 | 36 |
| | $x_3$ | (1) | 0 | 0 | 0 | 1 | $\frac{1}{3}$ | $-\frac{1}{3}$ | 2 |
| | $x_2$ | (2) | 0 | 0 | 1 | 0 | $\frac{1}{2}$ | 0 | 6 |
| | $x_1$ | (3) | 0 | 1 | 0 | 0 | $-\frac{1}{3}$ | $\frac{1}{3}$ | 2 |

**TABLE 6.5** Row 0 and corresponding dual solution for each iteration for the Wyndor Glass Co. example

| Iteration | Primal Problem: Row 0 | Dual Problem: $y_1$ | $y_2$ | $y_3$ | $z_1 - c_1$ | $z_2 - c_2$ | $W$ |
|---|---|---|---|---|---|---|---|
| 0 | $[-3, \; -5 \;\vdots\; 0, \; 0, \; 0 \;\vdots\; 0]$ | 0 | 0 | 0 | −3 | −5 | 0 |
| 1 | $[-3, \; 0 \;\vdots\; 0, \; \frac{5}{2}, \; 0 \;\vdots\; 30]$ | 0 | $\frac{5}{2}$ | 0 | −3 | 0 | 30 |
| 2 | $[0, \; 0 \;\vdots\; 0, \; \frac{3}{2}, \; 1 \;\vdots\; 36]$ | 0 | $\frac{3}{2}$ | 1 | 0 | 0 | 36 |

**If a solution for the primal problem and its corresponding solution for the dual problem are both feasible, the value of the objective function is optimal.**

**If a solution for the primal problem is feasible and the value of the objective function is not optimal (for this example, not maximum), the corresponding dual solution is infeasible.**

# Summary of Primal-Dual Relationships

**Weak duality property:** If $\mathbf{x}$ is a feasible solution for the primal problem and $\mathbf{y}$ is a feasible solution for the dual problem, then

$$\mathbf{cx} \leq \mathbf{yb}.$$

**Strong duality property:** If $\mathbf{x}^*$ is an optimal solution for the primal problem and $\mathbf{y}^*$ is an optimal solution for the dual problem, then

$$\mathbf{cx}^* = \mathbf{y}^*\mathbf{b}.$$

**Complementary-solutions property:** At each iteration, the simplex method simultaneously identifies a CPF solution $\mathbf{x}$ for the primal problem and a **complementary solution** $\mathbf{y}$ for the dual problem (found in row 0, the coefficients of the slack variables), where

$$\mathbf{cx} = \mathbf{yb}.$$

If $\mathbf{x}$ is *not optimal* for the primal problem, then $\mathbf{y}$ is *not feasible* for the dual problem.

# Summary of Primal-Dual Relationships

**Complementary optimal solutions property:** At the final iteration, the simplex method simultaneously identifies an optimal solution $\mathbf{x}^*$ for the primal problem and a **complementary optimal solution** $\mathbf{y}^*$ for the dual problem (found in row 0, the coefficients of the slack variables), where

$$\mathbf{c}\mathbf{x}^* = \mathbf{y}^*\mathbf{b}.$$

The $y_i^*$ are the shadow prices for the primal problem.

**Symmetry property:** For *any* primal problem and its dual problem, all relationships between them must be *symmetric* because the dual of this dual problem is this primal problem.

**Duality theorem:** The following are the only possible relationships between the primal and dual problems.

1. If one problem has *feasible solutions* and a *bounded* objective function (and so has an optimal solution), then so does the other problem, so both the weak and strong duality properties are applicable.
2. If one problem has *feasible solutions* and an *unbounded* objective function (and so *no optimal solution*), then the other problem has *no feasible solutions*.
3. If one problem has *no feasible solutions*, then the other problem has either *no feasible solutions* or an *unbounded* objective function.

**Complementary basic solutions property:** Each *basic* solution in the *primal problem* has a **complementary basic solution** in the *dual problem*, where their respective objective function values ($Z$ and $y_0$) are equal. Given row 0 of the simplex tableau for the primal basic solution, the complementary dual basic

**TABLE 6.4** Notation for entries in row 0 of a simplex tableau

| Iteration | Basic Variable | Eq. | Coefficient of: $Z$ | $x_1$ | $x_2$ | ... | $x_n$ | $x_{n+1}$ | $x_{n+2}$ | ... | $x_{n+m}$ | Right Side |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Any | $Z$ | (0) | 1 | $z_1 - c_1$ | $z_2 - c_2$ | ... | $z_n - c_n$ | $y_1$ | $y_2$ | ... | $y_m$ | $W$ |

**Complementary slackness property:** Given the association between variables in Table 6.7, the variables in the primal basic solution and the complementary dual basic solution satisfy the **complementary slackness** relationship shown in Table 6.8. Furthermore, this relationship is a symmetric one, so that these two basic solutions are complementary to each other.

**TABLE 6.7** Association between variables in primal and dual problems

| | Primal Variable | Associated Dual Variable |
|---|---|---|
| Any problem | (Decision variable) $x_j$<br>(Slack variable) $x_{n+i}$ | $z_j - c_j$ (surplus variable) $j = 1, 2, \ldots, n$<br>$y_i$ (decision variable) $i = 1, 2, \ldots, m$ |
| Wyndor problem | Decision variables: $x_1$<br>$x_2$<br>Slack variables: $x_3$<br>$x_4$<br>$x_5$ | $z_1 - c_1$ (surplus variables)<br>$z_2 - c_2$<br>$y_1$ (decision variables)<br>$y_2$<br>$y_3$ |

**TABLE 6.8** Complementary slackness relationship for complementary basic solutions

| Primal Variable | Associated Dual Variable | |
|---|---|---|
| Basic | Nonbasic | ($m$ variables) |
| Nonbasic | Basic | ($n$ variables) |

Primal Problem

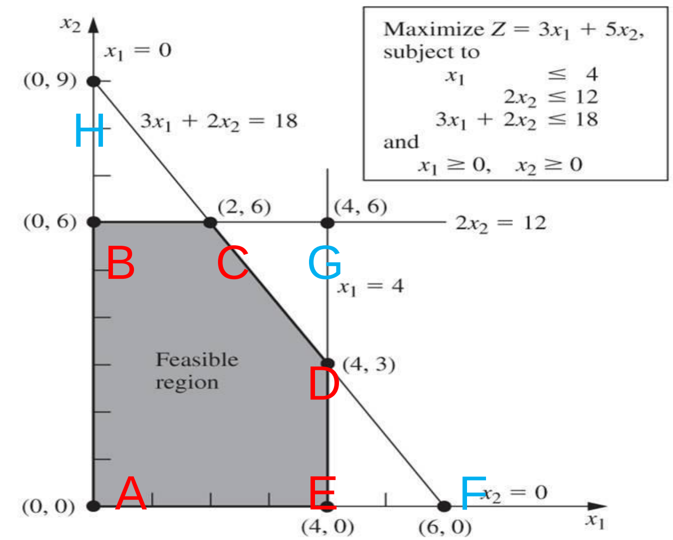
Maximize $Z = 3x_1 + 5x_2,$

subject to

$$
\begin{aligned}
&(1) \quad x_1 + x_3 = 4\\
&(2) \quad 2x_2 + x_4 = 12\\
&(3) \quad 3x_1 + 2x_2 + x_5 = 18
\end{aligned}
$$

and

$x_j \geq 0, \quad$ for $j = 1, 2, 3, 4, 5.$




Dual Problem

Minimize $W = 4y_1 + 12y_2 + 18y_3,$

subject to

$$
\begin{aligned}
y_1 + 3y_3 - 3 - z_1 - c_1 &= 0\\
2y_2 + 2y_3 - 5 - z_2 - c_2 &= 0
\end{aligned}
$$

and

$y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0,\ z_1 - c_1 \geq 0,\ z_2 - c_2 \geq 0$

*Table 6.9* **Complementary Basic Solutions for the Wyndor Glass Co. Example**

| | No. | *Primal Problem* Basic Solution | Feasible? | $Z = y_0$ | *Dual Problem* Feasible? | Basic Solution |
|---|---|---|---|---|---|---|
| A | 1 | (0, 0, 4, 12, 18) | Yes | 0 | No | (0, 0, 0, −3, −5) |
| E | 2 | (4, 0, 0, 12, 6) | Yes | 12 | No | (3, 0, 0, 0, −5) |
| F | 3 | (6, 0, −2, 12, 0) | No | 18 | No | (0, 0, 1, 0, −3) |
| D | 4 | (4, 3, 0, 6, 0) | Yes | 27 | No | $(-\frac{9}{2}, 0, \frac{5}{2}, 0, 0)$ |
| B | 5 | (0, 6, 4, 0, 6) | Yes | 30 | No | $(0, \frac{5}{2}, 0, -3, 0)$ |
| C | 6 | (2, 6, 2, 0, 0) | Yes | 36 | Yes | $(0, \frac{3}{2}, 1, 0, 0)$ |
| G | 7 | (4, 6, 0, 0, −6) | No | 42 | Yes | $(3, \frac{5}{2}, 0, 0, 0)$ |
| H | 8 | (0, 9, 4, −6, 0) | No | 45 | Yes | $(0, 0, \frac{5}{2}, \frac{9}{2}, 0)$ |

**Complementary optimal basic solutions property:** Each *optimal* basic solution in the *primal problem* has a **complementary optimal basic solution** in the dual problem, where their respective objective function values ($Z$ and $y_0$) are equal. Given row 0 of the simplex tableau for the optimal primal solution, the complementary optimal dual solution $(\mathbf{y}^*, \mathbf{z}^* - \mathbf{c})$ is found as shown in Table 6.4.

**TABLE 6.4** Notation for entries in row 0 of a simplex tableau

| Iteration | Basic Variable | Eq. | Coefficient of: $Z$ | $x_1$ | $x_2$ | $\cdots$ | $x_n$ | $x_{n+1}$ | $x_{n+2}$ | $\cdots$ | $x_{n+m}$ | Right Side |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Any | $Z$ | (0) | 1 | $z_1 - c_1$ | $z_2 - c_2$ | $\cdots$ | $z_n - c_n$ | $y_1$ | $y_2$ | $\cdots$ | $y_m$ | $W$ |

**TABLE 6.10 Classification of basic solutions**

| | | Satisfies Condition for Optimality? Yes | No |
|---|---|---|---|
| Feasible? | Yes | Optimal | Suboptimal |
| | No | Superoptimal | Neither feasible nor superoptimal |

Neither feasible nor superoptimal

Neither feasible nor superoptimal

*Table 6.9* Complementary Basic Solutions for the Wyndor Glass Co. Example

| | *Primal Problem* | | | *Dual Problem* | |
|---|---|---|---|---|---|
| No. | Basic Solution | Feasible? | $Z = y_0$ | Feasible? | Basic Solution |
| 1 | (0, 0, 4, 12, 18) | Yes | 0 | No | (0, 0, 0, −3, −5) |
| 2 | (4, 0, 0, 12, 6) | Yes | 12 | No | (3, 0, 0, 0, −5) |
| 3 | (6, 0, −2, 12, 0) | No | 18 | No | (0, 0, 1, 0, −3) |
| 4 | (4, 3, 0, 6, 0) | Yes | 27 | No | $(-\frac{9}{2}, 0, \frac{5}{2}, 0, 0)$ |
| 5 | (0, 6, 4, 0, 6) | Yes | 30 | No | $(0, \frac{5}{2}, 0, -3, 0)$ |
| 6 | (2, 6, 2, 0, 0) | Yes | 36 | Yes | $(0, \frac{3}{2}, 1, 0, 0)$ |
| 7 | (4, 6, 0, 0, −6) | No | 42 | Yes | $(3, \frac{3}{2}, 0, 0, 0)$ |
| 8 | (0, 9, 4, −6, 0) | No | 45 | Yes | $(0, 0, \frac{5}{2}, \frac{9}{2}, 0)$ |

Suboptimal

Optimal

Superoptimal

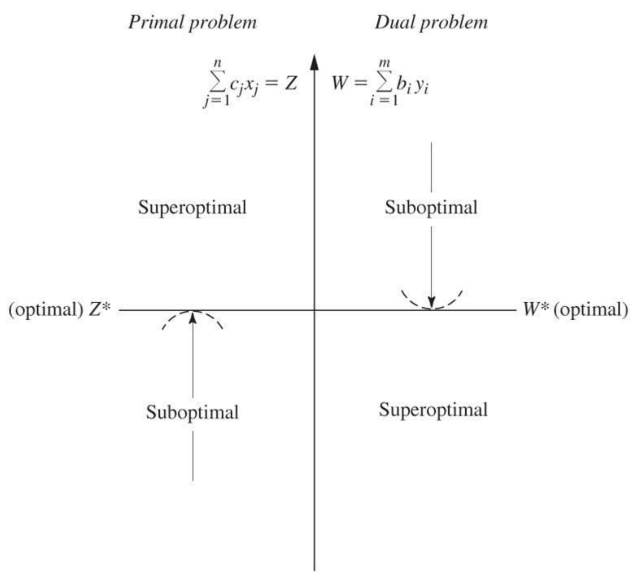
Superptimal

Optimal

Suboptimal

**TABLE 6.11** Relationships between complementary basic solutions

| Primal Basic Solution | Complementary Dual Basic Solution | Both Basic Solutions: Primal Feasible? | Both Basic Solutions: Dual Feasible? |
|---|---|---|---|
| Suboptimal | Superoptimal | Yes | No |
| Optimal | Optimal | Yes | Yes |
| Superoptimal | Suboptimal | No | Yes |
| Neither feasible nor superoptimal | Neither feasible nor superoptimal | No | No |



# Adapting to Other Primal Forms

**Primal Problem**

Maximize $Z = \sum_{j=1}^{n} c_j x_j,$

subject to

$\sum_{j=1}^{n} a_{ij} x_j \le b_i \quad$ for $i = 1, 2, \ldots, m$

and

$x_j \ge 0, \quad$ for $j = 1, 2, \ldots, n.$

**Dual Problem**

Minimize $y_0 = \sum_{i=1}^{m} b_i y_i,$

subject to

$\sum_{i=1}^{m} a_{ij} y_i \ge c_j, \quad$ for $j = 1, 2, \ldots, n$

and

$y_i \ge 0, \quad$ for $i = 1, 2, \ldots, m.$

**TABLE 6.12** Conversions to standard form for linear programming models

| Nonstandard Form | Equivalent Standard Form |
|---|---|
| Minimize $Z$ | Maximize $(-Z)$ |
| $\sum_{j=1}^{n} a_{ij} x_j \ge b_i$ | $-\sum_{j=1}^{n} a_{ij} x_j \le -b_i$ |
| $\sum_{j=1}^{n} a_{ij} x_j = b_i$ | $\sum_{j=1}^{n} a_{ij} x_j \le b_i$ and $-\sum_{j=1}^{n} a_{ij} x_j \le -b_i$ |
| $x_j$ unconstrained in sign | $x_j^+ - x_j^-,\quad x_j^+ \ge 0,\quad x_j^- \ge 0$ |

**TABLE 6.13 Constructing the dual of the dual problem**

*Dual Problem*

Minimize $W = \mathbf{yb}$,

subject to

$$\mathbf{yA} \geq \mathbf{c}$$

and

$$\mathbf{y} \geq \mathbf{0}.$$

$\longrightarrow$

*Converted to Standard Form*

Maximize $(-W) = -\mathbf{yb}$,

subject to

$$-\mathbf{yA} \leq -\mathbf{c}$$

and

$$\mathbf{y} \geq \mathbf{0}.$$

$\downarrow$

*Its Dual Problem*

Minimize $(-Z) = -\mathbf{cx}$,

subject to

$$-\mathbf{Ax} \geq -\mathbf{b}$$

and

$$\mathbf{x} \geq \mathbf{0}.$$

$\longleftarrow$

*Converted to Standard Form*

Maximize $Z = \mathbf{cx}$,

subject to

$$\mathbf{Ax} \leq \mathbf{b}$$

and

$$\mathbf{x} \geq \mathbf{0}.$$

## TABLE 6.14 Corresponding primal-dual forms

| Primal Problem (or Dual Problem) | | Dual Problem (or Primal Problem) |
|---|---|---|
| Maximize $Z$ (or $W$) | | Minimize $W$ (or $Z$) |
| Constraint $i$: | | Variable $y_i$ (or $x_i$): |
| $\leq$ form | $\longleftrightarrow$ | $y_i \geq 0$ |
| $=$ form | $\longleftrightarrow$ | Unconstrained |
| $\geq$ form | $\longleftrightarrow$ | $y_i' \leq 0$ |
| Variable $x_j$ (or $y_j$): | | Constraint $j$: |
| $x_j \geq 0$ | $\longleftrightarrow$ | $\geq$ form |
| Unconstrained | $\longleftrightarrow$ | $=$ form |
| $x_j' \leq 0$ | $\longleftrightarrow$ | $\leq$ form |

*Primal Problem*

Maximize $\quad Z = \sum_{j=1}^{n} c_j x_j,$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad \text{for } i = 1, 2, \ldots, m$$

and

$$x_j \geq 0, \qquad \text{for } j = 1, 2, \ldots, n.$$

*Dual Problem*

Minimize $\quad y_0 = \sum_{i=1}^{m} b_i y_i,$

subject to

$$\sum_{i=1}^{m} a_{ij} y_i \geq c_j, \qquad \text{for } j = 1, 2, \ldots, n$$

and

$$y_i \geq 0, \qquad \text{for } i = 1, 2, \ldots, m.$$

$$\max z = \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i = 1, 2, \cdots, m$$

$$x_j \geq 0, \quad j = 1, 2, \cdots, n$$

$$\max z = \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, 2, \cdots, m \qquad (1)$$

$$-\sum_{j=1}^{n} a_{ij} x_j \leq -b_i, \quad i = 1, 2, \cdots, m \qquad (2)$$

$$x_j \geq 0, \quad j = 1, 2, \cdots, n$$

$y_i'$ : *dual* variables corresponding to (1)

$y_i''$ : dual variables corresponding to (2)

$$\min \omega = \sum_{i=1}^{m} b_i y_i' + \sum_{i=1}^{m} (-b_i y_i'')$$

subject to

$$\sum_{i=1}^{m} a_{ij} y_i' + \sum_{i=1}^{m} (-a_{ij} y_i'') \geq c_j, \quad j = 1, 2, \cdots, n$$

$$y_i', y_i'' \geq 0, \quad i = 1, 2, \cdots, m$$

$$\min \omega = \sum_{i=1}^{m} b_i (y_i' - y_i'')$$

$$\sum_{i=1}^{m} a_{ij} (y_i' - y_i'') \geq c_j, \quad j = 1, 2, \cdots, n$$

$$y_i = y_i' - y_i''$$

$y_i$: unconstrained in sign

$$\min \omega = \sum_{i=1}^{m} b_i y_i$$

$$\sum_{i=1}^{m} a_{ij} y_i \geq c_j, \quad j = 1, 2, \cdots, n$$

$$\max z = \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i, \quad i = 1, 2, \cdots, m$$

$$x_j \geq 0, \quad j = 1, 2, \cdots, n$$

$$\max z = \sum_{j=1}^{n} c_j x_j$$

subject to

$$-\sum_{j=1}^{n} a_{ij} x_j \leq -b_i, \quad i = 1, 2, \cdots, m$$

$$x_j \geq 0, \quad j = 1, 2, \cdots, n$$

$y_i''$ : dual variables corresponding to (2)

$$\min \omega = \sum_{i=1}^{m} (-b_i y_i'')$$

subject to

$$\sum_{i=1}^{m} (-a_{ij} y_i'') \geq c_j, \quad j = 1, 2, \cdots, n$$

$$y_i'' \geq 0, \quad i = 1, 2, \cdots, m$$

$$y_i = -y_i''$$

$$\min \omega = \sum_{i=1}^{m} b_i y_i$$

$$\sum_{i=1}^{m} a_{ij} y_i \geq c_j, \quad j = 1, 2, \cdots, n$$

$$y_i \leq 0, \quad i = 1, 2, \cdots, m$$

## TABLE 6.14 Corresponding primal-dual forms

| Primal Problem (or Dual Problem) | | Dual Problem (or Primal Problem) |
|---|---|---|
| Maximize $Z$ (or $W$) | | Minimize $W$ (or $Z$) |
| Constraint $i$: | | Variable $y_i$ (or $x_i$): |
| $\leq$ form | $\longleftrightarrow$ | $y_i \geq 0$ |
| $=$ form | $\longleftrightarrow$ | Unconstrained |
| $\geq$ form | $\longleftrightarrow$ | $y_i' \leq 0$ |
| Variable $x_j$ (or $y_j$): | | Constraint $j$: |
| $x_j \geq 0$ | $\longleftrightarrow$ | $\geq$ form |
| Unconstrained | $\longleftrightarrow$ | $=$ form |
| $x_j' \leq 0$ | $\longleftrightarrow$ | $\leq$ form |

*Primal Problem*

Maximize $\quad Z = \sum_{j=1}^{n} c_j x_j,$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad \text{for } i = 1, 2, \ldots, m$$

and

$$x_j \geq 0, \qquad \text{for } j = 1, 2, \ldots, n.$$

*Dual Problem*

Minimize $\quad y_0 = \sum_{i=1}^{m} b_i y_i,$

subject to

$$\sum_{i=1}^{m} a_{ij} y_i \geq c_j, \qquad \text{for } j = 1, 2, \ldots, n$$

and

$$y_i \geq 0, \qquad \text{for } i = 1, 2, \ldots, m.$$

**TABLE 6.15** One primal-dual form for the radiation therapy example

| *Primal Problem* | | *Dual Problem* | |
|---|---|---|---|
| Maximize $-Z = -0.4x_1 - 0.5x_2,$ | | Minimize $W = 2.7y_1 + 6y_2 + 6y_3',$ | |
| subject to | | subject to | |
| $0.3x_1 + 0.1x_2 \leq 2.7$ | $\longleftrightarrow$ | $y_1 \geq 0$ | (S) |
| $0.5x_1 + 0.5x_2 = 6$ | $\longleftrightarrow$ | $y_2$ unconstrained in sign | (O) |
| $0.6x_1 + 0.4x_2 \geq 6$ | $\longleftrightarrow$ | $y_3' \leq 0$ | (B) |
| and | | and | |
| $x_1 \geq 0$ | $\longleftrightarrow$ | $0.3y_1 + 0.5y_2 + 0.6y_3' \geq -0.4$ | (S) |
| $x_2 \geq 0$ | $\longleftrightarrow$ | $0.1y_1 + 0.5y_2 + 0.4y_3' \geq -0.5$ | (S) |

**TABLE 6.16** The other primal-dual form for the radiation therapy example

| *Primal Problem* | | *Dual Problem* |
|---|---|---|
| Minimize $Z = 0.4x_1 + 0.5x_2,$ | | Maximize $W = 2.7y_1' + 6y_2' + 6y_3,$ |
| subject to | | subject to |
| $0.3x_1 + 0.1x_2 \leq 2.7$ | $\longleftrightarrow$ | $y_1' \leq 0$ |
| $0.5x_1 + 0.5x_2 = 6$ | $\longleftrightarrow$ | $y_2'$ unconstrained in sign |
| $0.6x_1 + 0.4x_2 \geq 6$ | $\longleftrightarrow$ | $y_3 \geq 0$ |
| and | | and |
| $x_1 \geq 0$ | $\longleftrightarrow$ | $0.3y_1' + 0.5y_2' + 0.6y_3 \leq 0.4$ |
| $x_2 \geq 0$ | $\longleftrightarrow$ | $0.1y_1' + 0.5y_2' + 0.4y_3 \leq 0.6$ |

# Dual Simplex Method



**TABLE 6.10** Classification of basic solutions

| | | Satisfies Condition for Optimality? Yes | No |
|---|---|---|---|
| Feasible? | Yes | Optimal | Suboptimal |
| | No | Superoptimal | Neither feasible nor superoptimal |

**TABLE 4.8** Complete set of simplex tableaux for the Wyndor Glass Co. problem

| Iteration | Basic Variable | Eq. | Coefficient of: $Z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | Right Side |
|---|---|---|---|---|---|---|---|---|---|
| 0 | $Z$ | (0) | 1 | −3 | −5 | 0 | 0 | 0 | 0 |
| | $x_3$ | (1) | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| | $x_4$ | (2) | 0 | 0 | 2 | 0 | 1 | 0 | 12 |
| | $x_5$ | (3) | 0 | 3 | 2 | 0 | 0 | 1 | 18 |
| 1 | $Z$ | (0) | 1 | −3 | 0 | 0 | $\frac{5}{2}$ | 0 | 30 |
| | $x_3$ | (1) | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| | $x_2$ | (2) | 0 | 0 | 1 | 0 | $\frac{1}{2}$ | 0 | 6 |
| | $x_5$ | (3) | 0 | 3 | 0 | 0 | −1 | 1 | 6 |
| 2 | $Z$ | (0) | 1 | 0 | 0 | 0 | $\frac{3}{2}$ | 1 | 36 |
| | $x_3$ | (1) | 0 | 0 | 0 | 1 | $\frac{1}{3}$ | $-\frac{1}{3}$ | 2 |
| | $x_2$ | (2) | 0 | 0 | 1 | 0 | $\frac{1}{2}$ | 0 | 6 |
| | $x_1$ | (3) | 0 | 1 | 0 | 0 | $-\frac{1}{3}$ | $\frac{1}{3}$ | 2 |

*Table 6.9* Complementary Basic Solutions for the Wyndor Glass Co. Example

| No. | *Primal Problem* Basic Solution | Feasible? | $Z = y_0$ | *Dual Problem* Feasible? | Basic Solution |
|---|---|---|---|---|---|
| 1 | (0, 0, 4, 12, 18) | Yes | 0 | No | (0, 0, 0, −3, −5) |
| 2 | (4, 0, 0, 12, 6) | Yes | 12 | No | (3, 0, 0, 0, −5) |
| 3 | (6, 0, −2, 12, 0) | No | 18 | No | (0, 0, 1, 0, −3) |
| 4 | (4, 3, 0, 6, 0) | Yes | 27 | No | $(-\frac{9}{2}, 0, \frac{5}{2}, 0, 0)$ |
| 5 | (0, 6, 4, 0, 6) | Yes | 30 | No | $(0, \frac{5}{2}, 0, -3, 0)$ |
| 6 | (2, 6, 2, 0, 0) | Yes | 36 | Yes | $(0, \frac{3}{2}, 1, 0, 0)$ |
| 7 | (4, 6, 0, 0, −6) | No | 42 | Yes | $(3, \frac{5}{2}, 0, 0, 0)$ |
| 8 | (0, 9, 4, −6, 0) | No | 45 | Yes | $(0, 0, \frac{5}{2}, \frac{9}{2}, 0)$ |

**Suboptimal** **Superptimal**

**Optimal** **Optimal**

**Superoptimal** **Suboptimal**

Simplex Method : Keep the solution in any iteration suboptimal ( not satisfying the condition for optimality, but the condition for feasibility).

Dual Simplex Method : Keep the solution in any iteration superoptimal ( not satisfying the condition for feasibility, but the condition for optimality).

If a solution satisfies the condition for optimality, the coefficients in row (0) of the simplex tableau must nonnegative.

If a solution does not satisfy the condition for feasibility, one or more of the values of b in the right-side of simplex tableau must be negative.

# Summary of Dual Simplex Method

Maximize $Z = -4y_1 - 12y_2 - 18y_3$

subject to

$$y_1 + 3y_3 \geq 3$$
$$2y_2 + 2y_3 \geq 5$$
$$y_1 \geq 0, \quad y_2 \geq 0, \quad y_3 \geq 0.$$

| Iteration | Basic Variable | Eq. | Coefficient of: $Z$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ | Right Side |
|---|---|---|---|---|---|---|---|---|---|
| 0 | $Z$ | (0) | 1 | 4 | 12 | 18 | 0 | 0 | 0 |
| | $y_4$ | (1) | 0 | −1 | 0 | −3 | 1 | 0 | −3 |
| | $y_5$ | (2) | 0 | 0 | −2 | −2 | 0 | 1 | −5 |
| 1 | $Z$ | (0) | 1 | 4 | 0 | 6 | 0 | 6 | −30 |
| | $y_4$ | (1) | 0 | −1 | 0 | −3 | 1 | 0 | −3 |
| | $y_2$ | (2) | 0 | 0 | 1 | 1 | 0 | $-\frac{1}{2}$ | $\frac{5}{2}$ |
| 2 | $Z$ | (0) | 1 | 2 | 0 | 0 | 2 | 6 | −36 |
| | $y_3$ | (1) | 0 | $\frac{1}{3}$ | 0 | 1 | $-\frac{1}{3}$ | 0 | 1 |
| | $y_2$ | (2) | 0 | $-\frac{1}{3}$ | 1 | 0 | $\frac{1}{3}$ | $-\frac{1}{2}$ | $\frac{3}{2}$ |

1. *Initialization:* After converting any functional constraints in $\geq$ form to $\leq$ form (by multiplying through both sides by $-1$), introduce slack variables as needed to construct a set of equations describing the problem. Find a basic solution such that the coefficients in Eq. (0) are zero for basic variables and nonnegative for nonbasic variables (so the solution is optimal if it is feasible). Go to the feasibility test.

2. *Feasibility test:* Check to see whether all the basic variables are *nonnegative*. If they are, then this solution is feasible, and therefore optimal, so stop. Otherwise, go to an iteration.

3. *Iteration:*

*Step 1* Determine the *leaving basic variable:* Select the *negative* basic variable that has the largest absolute value.

*Step 2* Determine the *entering basic variable:* Select the nonbasic variable whose coefficient in Eq. (0) reaches zero first as an increasing multiple of the equation containing the leaving basic variable is added to Eq. (0). This selection is made by checking the nonbasic variables with *negative coefficients* in that equation (the one containing the leaving basic variable) and selecting the one with the smallest absolute value of the ratio of the Eq. (0) coefficient to the coefficient in that equation.

*Step 3* Determine the *new basic solution:* Starting from the current set of equations, solve for the basic variables in terms of the nonbasic variables by Gaussian elimination. When we set the nonbasic variables equal to zero, each basic variable (and $Z$) equals the new right-hand side of the one equation in which it appears (with a coefficient of $+1$). Return to the feasibility test.

# Sensitivity Analysis

Maximize $Z = \sum_{j=1}^{n} c_j x_j,$

subject to

$\sum_{j=1}^{n} a_{ij} x_j \le b_i$ for $i = 1, 2, \ldots, m$

and

$x_j \ge 0,$ for $j = 1, 2, \ldots, n.$

Maximize $Z = \mathbf{cx},$

subject to

$$\mathbf{Ax} \le \mathbf{b}$$

and

$$\mathbf{x} \ge \mathbf{0}.$$

The simplex method already has been used to obtain an optimal solution to a linear programming model with specified values for the $b_i$, $c_j$, and $a_{ij}$ parameters. To initiate sensitivity analysis, one or more of the parameters is changed. After the changes are made, let $\bar{b}_i$, $\bar{c}_j$, and $\bar{a}_{ij}$ denote the values of the various parameters. Thus, in matrix notation,

$$\mathbf{b} \rightarrow \bar{\mathbf{b}}, \qquad \mathbf{c} \rightarrow \bar{\mathbf{c}}, \qquad \mathbf{A} \rightarrow \bar{\mathbf{A}},$$

for the revised model.

**Initial Tableau**

Row 0: $\mathbf{t} = [-3, -5 \mid 0, 0, 0 \mid 0] = [-\mathbf{c} \mid \mathbf{0} \mid 0].$

Other rows: $\mathbf{T} = \left[\begin{array}{cc|ccc|c} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 2 & 0 & 1 & 0 & 12 \\ 3 & 2 & 0 & 0 & 1 & 18 \end{array}\right] = [\mathbf{A} \mid \mathbf{I} \mid \mathbf{b}].$

Combined: $\begin{bmatrix} \mathbf{t} \\ \mathbf{T} \end{bmatrix} = \left[\begin{array}{c|c|c} -\mathbf{c} & \mathbf{0} & 0 \\ \mathbf{A} & \mathbf{I} & \mathbf{b} \end{array}\right].$

**Final Tableau**

Row 0: $\mathbf{t}^* = [0, 0 \mid 0, \frac{3}{2}, 1 \mid 36] = [\mathbf{z}^* - \mathbf{c} \mid \mathbf{y}^* \mid Z^*].$

Other rows: $\mathbf{T}^* = \left[\begin{array}{cc|ccc|c} 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & 2 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & 6 \\ 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & 2 \end{array}\right] = [\mathbf{A}^* \mid \mathbf{S}^* \mid \mathbf{b}^*].$

Combined: $\begin{bmatrix} \mathbf{t}^* \\ \mathbf{T}^* \end{bmatrix} = \left[\begin{array}{c|c|c} \mathbf{z}^* - \mathbf{c} & \mathbf{y}^* & Z^* \\ \mathbf{A}^* & \mathbf{S}^* & \mathbf{b}^* \end{array}\right].$

$$(1)\quad \mathbf{t}^* = \mathbf{t} + \mathbf{y}^*\mathbf{T} = [\mathbf{y}^*\mathbf{A} - \mathbf{c} \mid \mathbf{y}^* \mid \mathbf{y}^*\mathbf{b}] = [\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c} \mid \mathbf{c}_B\mathbf{B}^{-1} \mid \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}]$$

$$(2)\quad \mathbf{T}^* = \mathbf{S}^*\mathbf{T} = [\mathbf{S}^*\mathbf{A} \mid \mathbf{S}^* \mid \mathbf{S}^*\mathbf{b}] = [\mathbf{B}^{-1}\mathbf{A} \mid \mathbf{B}^{-1} \mid \mathbf{B}^{-1}\mathbf{b}]$$

$$\mathbf{S}^* = \mathbf{B}^{-1} \qquad \mathbf{y}^* = \mathbf{c}_B\mathbf{B}^{-1}$$

*Original Model*

Maximize $\quad Z = [3, 5]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$

subject to

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}$$

and

$$\mathbf{x} \geq \mathbf{0}.$$

*Revised Model*

Maximize $\quad Z = [4, 5]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$

subject to

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 4 \\ 24 \\ 18 \end{bmatrix}$$

and

$$\mathbf{x} \geq \mathbf{0}.$$

$$\bar{\mathbf{c}} = [4, 5], \qquad \bar{\mathbf{A}} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 2 & 2 \end{bmatrix}, \qquad \bar{\mathbf{b}} = \begin{bmatrix} 4 \\ 24 \\ 18 \end{bmatrix}$$

**TABLE 4.8** Complete set of simplex tableaux for the Wyndor Glass Co. problem

| Iteration | Basic Variable | Eq. | Coefficient of: $Z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | Right Side |
|---|---|---|---|---|---|---|---|---|---|
| 0 | $Z$ | (0) | 1 | −3 | −5 | 0 | 0 | 0 | 0 |
| | $x_3$ | (1) | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| | $x_4$ | (2) | 0 | 0 | 2 | 0 | 1 | 0 | 12 |
| | $x_5$ | (3) | 0 | 3 | 2 | 0 | 0 | 1 | 18 |
| 1 | $Z$ | (0) | 1 | −3 | 0 | 0 | $\frac{5}{2}$ | 0 | 30 |
| | $x_3$ | (1) | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| | $x_2$ | (2) | 0 | 0 | 1 | 0 | $\frac{1}{2}$ | 0 | 6 |
| | $x_5$ | (3) | 0 | 3 | 0 | 0 | −1 | 1 | 6 |
| 2 | $Z$ | (0) | 1 | 0 | 0 | 0 | $\frac{3}{2}$ | 1 | 36 |
| | $x_3$ | (1) | 0 | 0 | 0 | 1 | $\frac{1}{3}$ | $-\frac{1}{3}$ | 2 |
| | $x_2$ | (2) | 0 | 0 | 1 | 0 | $\frac{1}{2}$ | 0 | 6 |
| | $x_1$ | (3) | 0 | 1 | 0 | 0 | $-\frac{1}{3}$ | $\frac{1}{3}$ | 2 |

$$\bar{\mathbf{c}} = [4, 5], \qquad \bar{\mathbf{A}} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 2 & 2 \end{bmatrix}, \qquad \bar{\mathbf{b}} = \begin{bmatrix} 4 \\ 24 \\ 18 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 2 & 2 \end{bmatrix} \quad \mathbf{c}_B = [0, 5, 4] \quad \mathbf{B}^{-1} = \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\mathbf{B}^{-1}\bar{\mathbf{A}} = \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

| | Basic Variable | Eq. | *Coefficient of:* $Z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | Right Side |
|---|---|---|---|---|---|---|---|---|---|
| New initial tableau | $Z$ | (0) | 1 | −4 | −5 | 0 | 0 | 0 | 0 |
| | $x_3$ | (1) | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| | $x_4$ | (2) | 0 | 0 | 2 | 0 | 1 | 0 | 24 |
| | $x_5$ | (3) | 0 | 2 | 2 | 0 | 0 | 1 | 18 |
| Revised final tableau | $Z$ | (0) | 1 | 0 | 0 | 0 | $\frac{1}{2}$ | 2 | 48 |
| | $x_3$ | (1) | 0 | 0 | 0 | 1 | $\frac{1}{2}$ | $-\frac{1}{2}$ | 7 |
| | $x_2$ | (2) | 0 | 0 | 1 | 0 | $\frac{1}{2}$ | 0 | 12 |
| | $x_1$ | (3) | 0 | 1 | 0 | 0 | $-\frac{1}{2}$ | $\frac{1}{2}$ | −3 |

$$\mathbf{c}_B\mathbf{B}^{-1} = [0, 5, 4] \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = [0, \tfrac{1}{2}, 2]$$

$$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \bar{\mathbf{c}} = [0, 5, 4] \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} - [4, 5] = [0, 0]$$

$$\mathbf{B}^{-1}\bar{\mathbf{b}} = \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 4 \\ 24 \\ 18 \end{bmatrix} = \begin{bmatrix} 7 \\ 12 \\ -3 \end{bmatrix} \qquad \mathbf{c}_B\mathbf{B}^{-1}\bar{\mathbf{b}} = [0, 5, 4] \begin{bmatrix} 7 \\ 12 \\ -3 \end{bmatrix} = 48$$

# Case 1 : Changes in $b_i$

Suppose that the only changes in the current model are that one or more of the $b_i$ parameters ($i = 1, 2, \ldots, m$) has been changed. In this case, the *only* resulting changes in the final simplex tableau are in the *right side* column.

**TABLE 5.8** Initial and later simplex tableaux in matrix form

| Iteration | Basic Variable | Eq. | Coefficient of: $Z$ | Original Variables | Slack Variables | Right Side |
|---|---|---|---|---|---|---|
| 0 | $Z$<br>$\mathbf{x}_B$ | (0)<br>$(1, 2, \ldots, m)$ | 1<br>**0** | $-\mathbf{c}$<br>$\mathbf{A}$ | $\mathbf{0}$<br>$\mathbf{I}$ | 0<br>$\mathbf{b}$ |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| Any | $Z$<br>$\mathbf{x}_B$ | (0)<br>$(1, 2, \ldots, m)$ | 1<br>**0** | $\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A}-\mathbf{c}$<br>$\mathbf{B}^{-1}\mathbf{A}$ | $\mathbf{c}_B\mathbf{B}^{-1}$<br>$\mathbf{B}^{-1}$ | $\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$<br>$\mathbf{B}^{-1}\mathbf{b}$ |

**Final Tableau**

Row 0: $\mathbf{t}^* = [0, 0 \mid 0, \tfrac{3}{2}, 1 \mid 36] = [\mathbf{z}^* - \mathbf{c} \mid \mathbf{y}^* \mid Z^*]$.

Other rows: $\mathbf{T}^* = \left[\begin{array}{cc|ccc|c} 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & 2 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & 6 \\ 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & 2 \end{array}\right] = [\mathbf{A}^* \mid \mathbf{S}^* \mid \mathbf{b}^*]$.

Combined: $\begin{bmatrix} \mathbf{t}^* \\ \mathbf{T}^* \end{bmatrix} = \left[\begin{array}{c|c|c} \mathbf{z}^* - \mathbf{c} & \mathbf{y}^* & Z^* \\ \mathbf{A}^* & \mathbf{S}^* & \mathbf{b}^* \end{array}\right]$.

$$\mathbf{S}^* = \mathbf{B}^{-1} \qquad \mathbf{y}^* = \mathbf{c}_B\mathbf{B}^{-1}$$

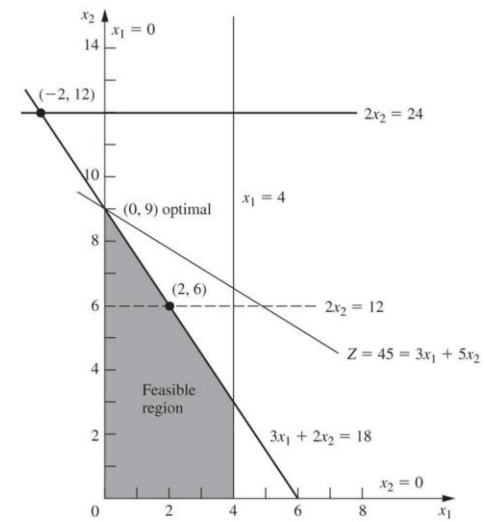
$$\mathbf{b} = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} \longrightarrow \overline{\mathbf{b}} = \begin{bmatrix} 4 \\ 24 \\ 18 \end{bmatrix}$$

Right side of final row 0: $Z^* = \mathbf{y}^*\overline{\mathbf{b}}$

Right side of final rows $1, 2, \ldots, m$: $\mathbf{b}^* = \mathbf{S}^*\overline{\mathbf{b}}$

**TABLE 4.8** Complete set of simplex tableaux for the Wyndor Glass Co. problem

| Iteration | Basic Variable | Eq. | Coefficient of: $Z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | Right Side |
|---|---|---|---|---|---|---|---|---|---|
| 0 | $Z$ | (0) | 1 | −3 | −5 | 0 | 0 | 0 | 0 |
| | $x_3$ | (1) | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| | $x_4$ | (2) | 0 | 0 | 2 | 0 | 1 | 0 | 12 |
| | $x_5$ | (3) | 0 | 3 | 2 | 0 | 0 | 1 | 18 |
| 1 | $Z$ | (0) | 1 | −3 | 0 | 0 | $\frac{5}{2}$ | 0 | 30 |
| | $x_3$ | (1) | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| | $x_2$ | (2) | 0 | 0 | 1 | 0 | $\frac{1}{2}$ | 0 | 6 |
| | $x_5$ | (3) | 0 | 3 | 0 | 0 | −1 | 1 | 6 |
| 2 | $Z$ | (0) | 1 | 0 | 0 | 0 | $\frac{3}{2}$ | 1 | 36 |
| | $x_3$ | (1) | 0 | 0 | 0 | 1 | $\frac{1}{3}$ | $-\frac{1}{3}$ | 2 |
| | $x_2$ | (2) | 0 | 0 | 1 | 0 | $\frac{1}{2}$ | 0 | 6 |
| | $x_1$ | (3) | 0 | 1 | 0 | 0 | $-\frac{1}{3}$ | $\frac{1}{3}$ | 2 |




$$\mathbf{b} = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix} \longrightarrow \bar{\mathbf{b}} = \begin{bmatrix} 4 \\ 24 \\ 18 \end{bmatrix}$$

$$Z^* = \mathbf{y}^*\bar{\mathbf{b}} = [0, \tfrac{3}{2}, 1] \begin{bmatrix} 4 \\ 24 \\ 18 \end{bmatrix} = 54$$

$$\mathbf{b}^* = \mathbf{S}^*\bar{\mathbf{b}} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 4 \\ 24 \\ 18 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \\ -2 \end{bmatrix}, \qquad \text{so} \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \\ -2 \end{bmatrix}$$

# Incremental analysis

Equivalently, because the only change in the original model is $\Delta b_2 = 24 - 12 = 12$, incremental analysis can be used to calculate these same values more quickly.

$$\Delta Z^* = \mathbf{y}^* \, \Delta\bar{\mathbf{b}} = \mathbf{y}^* \begin{bmatrix} \Delta b_1 \\ \Delta b_2 \\ \Delta b_3 \end{bmatrix} = \mathbf{y}^* \begin{bmatrix} 0 \\ 12 \\ 0 \end{bmatrix}$$

$$\Delta \mathbf{b}^* = \mathbf{S}^* \, \Delta\bar{b} = \mathbf{S}^* \begin{bmatrix} \Delta b_1 \\ \Delta b_2 \\ \Delta b_3 \end{bmatrix} = \mathbf{S}^* \begin{bmatrix} 0 \\ 12 \\ 0 \end{bmatrix}$$

$$\mathbf{y}^* = [0, \tfrac{3}{2}, 1] \qquad \mathbf{S}^* = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

**TABLE 4.8** Complete set of simplex tableaux for the Wyndor Glass Co. problem

| Iteration | Basic Variable | Eq. | Coefficient of: $Z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | Right Side |
|---|---|---|---|---|---|---|---|---|---|
| 0 | $Z$ | (0) | 1 | −3 | −5 | 0 | 0 | 0 | 0 |
| | $x_3$ | (1) | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| | $x_4$ | (2) | 0 | 0 | 2 | 0 | 1 | 0 | 12 |
| | $x_5$ | (3) | 0 | 3 | 2 | 0 | 0 | 1 | 18 |
| 1 | $Z$ | (0) | 1 | −3 | 0 | 0 | $\frac{5}{2}$ | 0 | 30 |
| | $x_3$ | (1) | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| | $x_2$ | (2) | 0 | 0 | 1 | 0 | $\frac{1}{2}$ | 0 | 6 |
| | $x_5$ | (3) | 0 | 3 | 0 | 0 | −1 | 1 | 6 |
| 2 | $Z$ | (0) | 1 | 0 | 0 | 0 | $\frac{3}{2}$ | 1 | 36 |
| | $x_3$ | (1) | 0 | 0 | 0 | 1 | $\frac{1}{3}$ | $-\frac{1}{3}$ | 2 |
| | $x_2$ | (2) | 0 | 0 | 1 | 0 | $\frac{1}{2}$ | 0 | 6 |
| | $x_1$ | (3) | 0 | 1 | 0 | 0 | $-\frac{1}{3}$ | $\frac{1}{3}$ | 2 |

$\Delta Z^* = \frac{3}{2}(12) = 18,$ so $Z^* = 36 + 18 = 54$

$\Delta b_1^* = \frac{1}{3}(12) = 4,$ so $b_1^* = 2 + 4 = 6$

$\Delta b_2^* = \frac{1}{2}(12) = 6,$ so $b_2^* = 6 + 6 = 12$

$\Delta b_3^* = -\frac{1}{3}(12) = -4,$ so $b_3^* = 2 - 4 = -2$

Therefore, the current (previously optimal) basic solution has become

$$(x_1, x_2, x_3, x_4, x_5) = (-2, 12, 6, 0, 0)$$

The dual simplex method now can be applied to find the new optimal solution.

**TABLE 4.8** Complete set of simplex tableaux for the Wyndor Glass Co. problem

| Iteration | Basic Variable | Eq. | $Z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | Right Side |
|---|---|---|---|---|---|---|---|---|---|
| 0 | $Z$ | (0) | 1 | −3 | −5 | 0 | 0 | 0 | 0 |
| | $x_3$ | (1) | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| | $x_4$ | (2) | 0 | 0 | 2 | 0 | 1 | 0 | 12 |
| | $x_5$ | (3) | 0 | 3 | 2 | 0 | 0 | 1 | 18 |
| 1 | $Z$ | (0) | 1 | −3 | 0 | 0 | $\frac{5}{2}$ | 0 | 30 |
| | $x_3$ | (1) | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| | $x_2$ | (2) | 0 | 0 | 1 | 0 | $\frac{1}{2}$ | 0 | 6 |
| | $x_5$ | (3) | 0 | 3 | 0 | 0 | −1 | 1 | 6 |
| 2 | $Z$ | (0) | 1 | 0 | 0 | 0 | $\frac{3}{2}$ | 1 | 36 |
| | $x_3$ | (1) | 0 | 0 | 0 | 1 | $\frac{1}{3}$ | $-\frac{1}{3}$ | 2 |
| | $x_2$ | (2) | 0 | 0 | 1 | 0 | $\frac{1}{2}$ | 0 | 6 |
| | $x_1$ | (3) | 0 | 1 | 0 | 0 | $-\frac{1}{3}$ | $\frac{1}{3}$ | 2 |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $Z$ | (0) | 1 | 0 | 0 | 0 | $\frac{3}{2}$ | 1 | 36 |
| $x_3$ | (1) | 0 | 0 | 0 | 1 | $\frac{1}{3}$ | $-\frac{1}{3}$ | 6 |
| $x_2$ | (2) | 0 | 0 | 1 | 0 | $\frac{1}{2}$ | 0 | 12 |
| $x_1$ | (3) | 0 | 1 | 0 | 0 | $-\frac{1}{3}$ | $\frac{1}{3}$ | −2 |

**TABLE 6.21** Data for Variation 2 of the Wyndor Glass Co. model

*Model Parameters*

| | | |
|---|---|---|
| $c_1 = 3,$ | $c_2 = 5$ | $(n = 2)$ |
| $a_{11} = 1,$ | $a_{12} = 0,$ | $b_1 = 4$ |
| $a_{21} = 0,$ | $a_{22} = 2,$ | $b_2 = 24$ |
| $a_{31} = 3,$ | $a_{32} = 2,$ | $b_3 = 18$ |

**Final Simplex Tableau after Reoptimization**

| Basic Variable | Eq. | $Z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | Right Side |
|---|---|---|---|---|---|---|---|---|
| $Z$ | (0) | 1 | $\frac{9}{2}$ | 0 | 0 | 0 | $\frac{5}{2}$ | 45 |
| $x_3$ | (1) | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| $x_2$ | (2) | 0 | $\frac{3}{2}$ | 1 | 0 | 0 | $\frac{1}{2}$ | 9 |
| $x_4$ | (3) | 0 | −3 | 0 | 0 | 1 | −1 | 6 |

$$\Delta b_1^* = \tfrac{1}{3}(12) = 4, \qquad \text{so } b_1^* = 2 + 4 = 6$$

$$\Delta b_2^* = \tfrac{1}{2}(12) = 6, \qquad \text{so } b_2^* = 6 + 6 = 12$$

$$\Delta b_3^* = -\tfrac{1}{3}(12) = -4, \qquad \text{so } b_3^* = 2 - 4 = -2$$

$$b_1^* = 2 + \tfrac{1}{3}\Delta b_2$$

$$b_2^* = 6 + \tfrac{1}{2}\Delta b_2$$

$$b_3^* = 2 - \tfrac{1}{3}\Delta b_2$$

# The allowable range of $b_i$ to stay feasible

$$b_1^* = 2 + \tfrac{1}{3}\Delta b_2 \geq 0 \Rightarrow \tfrac{1}{3}\Delta b_2 \geq -2 \Rightarrow \Delta b_2 \geq -6,$$

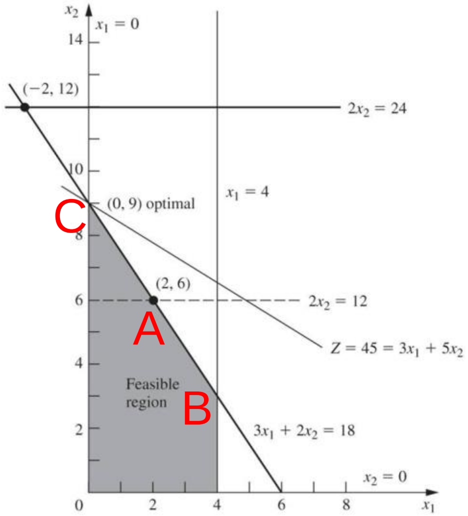
$$b_2^* = 6 + \tfrac{1}{2}\Delta b_2 \geq 0 \Rightarrow \tfrac{1}{2}\Delta b_2 \geq -6 \Rightarrow \Delta b_2 \geq -12,$$

$$b_3^* = 2 - \tfrac{1}{3}\Delta b_2 \geq 0 \Rightarrow 2 \geq \tfrac{1}{3}\Delta b_2 \Rightarrow \Delta b_2 \leq 6.$$

The solution remains feasible only if

$$-6 \leq \Delta b_2 \leq 6$$

$$b_2 = 12 + \Delta b_2 \implies 6 \leq b_2 \leq 18$$




For any $b_i$, its **allowable range to stay feasible** is the range of values over which the optimal BF solution (with adjusted values for the basic variables) remains feasible. (It is assumed that the change in this one $b_i$ value is the only change in the model.) The adjusted values for the basic variables are obtained from the formula $\mathbf{b}^* = \mathbf{S}^*\bar{\mathbf{b}}$. The calculation of the allowable range to stay feasible then is based on finding the range of values of $b_i$ such that $\mathbf{b}^* \geq \mathbf{0}$.

# Case 2a : Changes in the coefficients of a nonbasic variable

Consider a particular variable $x_j$ (fixed $j$) that is a nonbaic variable in the optimal solution shown by the final simplex tableau.

$\mathbf{A}_j$ is the vector of column j in the matrix **A** .

We have $c_j \longrightarrow \bar{c}_j, \qquad \mathbf{A}_j \longrightarrow \bar{\mathbf{A}}_j$ for the revised model.

$$c_1 = 3 \longrightarrow \bar{c}_1 = 4$$

$$\mathbf{A}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \longrightarrow \bar{\mathbf{A}}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

We can observe that the changes lead to a single revised constraint for the dual problem.

$$y_1 + 3y_3 \geq 3 \longrightarrow y_1 + 2y_3 \geq 4$$

$$y_1^* = 0, \qquad y_2^* = 0, \qquad y_3^* = \tfrac{5}{2}.$$

$$0 + 2(\tfrac{5}{2}) \geq 4$$

**TABLE 6.21** Data for Variation 2 of the Wyndor Glass Co. model

*Model Parameters*

| | | |
|---|---|---|
| $c_1 = 3,$ | $c_2 = 5$ | $(n = 2)$ |
| $a_{11} = 1,$ | $a_{12} = 0,$ | $b_1 = 4$ |
| $a_{21} = 0,$ | $a_{22} = 2,$ | $b_2 = 24$ |
| $a_{31} = 3,$ | $a_{32} = 2,$ | $b_3 = 18$ |

**Final Simplex Tableau after Reoptimization**

| Basic Variable | Eq. | Coefficient of: $Z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | Right Side |
|---|---|---|---|---|---|---|---|---|
| $Z$ | (0) | 1 | $\frac{9}{2}$ | 0 | 0 | 0 | $\frac{5}{2}$ | 45 |
| $x_3$ | (1) | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| $x_2$ | (2) | 0 | $\frac{3}{2}$ | 1 | 0 | 0 | $\frac{1}{2}$ | 9 |
| $x_4$ | (3) | 0 | $-3$ | 0 | 0 | 1 | $-1$ | 6 |

# The allowable range of the coefficient $c_i$ of a nonbasic variable

For any $c_j$, its **allowable range to stay optimal** is the range of values over which the current optimal solution (as obtained by the simplex method for the current model before $c_j$ is changed) remains optimal. (It is assumed that the change in this one $c_j$ is the only change in the current model.) When $x_j$ is a nonbasic variable for this solution, the solution remains optimal as long as $z_j^* - c_j \geq 0$, where $z_j^* = \mathbf{y}^*\mathbf{A}_j$ is a constant unaffected by any change in the value of $c_j$. Therefore, the allowable range to stay optimal for $c_j$ can be calculated as $c_j \leq \mathbf{y}^*\mathbf{A}_j$.

**TABLE 6.21** Data for Variation 2 of the Wyndor Glass Co. model

*Model Parameters*

| | | |
|---|---|---|
| $c_1 = 3,$ | $c_2 = 5$ | $(n = 2)$ |
| $\boxed{a_{11} = 1,}$ | $a_{12} = 0,$ | $b_1 = 4$ |
| $\boxed{a_{21} = 0,}$ | $a_{22} = 2,$ | $b_2 = 24$ |
| $\boxed{a_{31} = 3,}$ | $a_{32} = 2,$ | $b_3 = 18$ |

$$c_1 \leq \mathbf{y}^*\mathbf{A}_1 = [0, \quad 0, \quad \tfrac{5}{2}] \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = 7\tfrac{1}{2},$$

**Final Simplex Tableau after Reoptimization**

| Basic Variable | Eq. | Coefficient of: $Z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | Right Side |
|---|---|---|---|---|---|---|---|---|
| $Z$ | (0) | 1 | $\frac{9}{2}$ | 0 | 0 | 0 | $\frac{5}{2}$ | 45 |
| $x_3$ | (1) | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| $x_2$ | (2) | 0 | $\frac{3}{2}$ | 1 | 0 | 0 | $\frac{1}{2}$ | 9 |
| $x_4$ | (3) | 0 | −3 | 0 | 0 | 1 | −1 | 6 |

$c_1 \leq 7\frac{1}{2}$ is the allowable range to stay optimal.

# Case 2b : Introduction of a new variable

Maximize $Z = 3x_1 + 5x_2 + 4x_{new}$,

subject to

$$x_1 + 2x_{new} \le 4$$

$$2x_2 + 3x_{new} \le 12$$

$$3x_1 + 2x_2 + x_{new} \le 18$$

$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_{new} \ge 0.$$

After we introduced slack variables, the original optimal solution for this problem without $x_{new}$ was $(x_1, x_2, x_3, x_4, x_5) = (2, 6, 2, 0, 0)$. Is this solution, along with $x_{new} = 0$, still optimal? $(y_1, y_2, y_3, z_1 - c_1, z_2 - c_2) = (0, \frac{3}{2}, 1, 0, 0)$

*Table 6.9* **Complementary Basic Solutions for the Wyndor Glass Co. Example**

| No. | Primal Problem: Basic Solution | Primal Problem: Feasible? | $Z = y_0$ | Dual Problem: Feasible? | Dual Problem: Basic Solution |
|---|---|---|---|---|---|
| 1 | (0, 0, 4, 12, 18) | Yes | 0 | No | (0, 0, 0, −3, −5) |
| 2 | (4, 0, 0, 12, 6) | Yes | 12 | No | (3, 0, 0, 0, −5) |
| 3 | (6, 0, −2, 12, 0) | No | 18 | No | (0, 0, 1, 0, −3) |
| 4 | (4, 3, 0, 6, 0) | Yes | 27 | No | $(-\frac{9}{2}, 0, \frac{5}{2}, 0, 0)$ |
| 5 | (0, 6, 4, 0, 6) | Yes | 30 | No | $(0, \frac{5}{2}, 0, -3, 0)$ |
| 6 | (2, 6, 2, 0, 0) | Yes | 36 | Yes | $(0, \frac{3}{2}, 1, 0, 0)$ |
| 7 | (4, 6, 0, 0, −6) | No | 42 | Yes | $(3, \frac{5}{2}, 0, 0, 0)$ |
| 8 | (0, 9, 4, −6, 0) | No | 45 | Yes | $(0, 0, \frac{5}{2}, \frac{9}{2}, 0)$ |

$$2y_1 + 3y_2 + y_3 \ge 4$$

$$2(0) + 3(\tfrac{3}{2}) + (1) \ge 4$$

# Case 3 : Changes in the coefficients of a basic variable

$$c_2 = 5 \longrightarrow \bar{c}_2 = 3, \qquad \mathbf{A}_2 = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \longrightarrow \bar{\mathbf{A}}_2 = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$$

**TABLE 6.24** Sensitivity analysis procedure applied to Variation 5 of the Wyndor Glass Co. model

| | Basic Variable | Eq. | $Z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | Right Side |
|---|---|---|---|---|---|---|---|---|---|
| | $Z$ | (0) | 1 | $-\frac{3}{4}$ | 0 | 0 | 0 | $\frac{3}{4}$ | $\frac{27}{2}$ |
| | $x_3$ | (1) | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| | $x_2$ | (2) | 0 | $\frac{3}{4}$ | 1 | 0 | 0 | $\frac{1}{4}$ | $\frac{9}{2}$ |
| | $x_4$ | (3) | 0 | $-\frac{9}{4}$ | 0 | 0 | 1 | $-\frac{3}{4}$ | $\frac{21}{2}$ |
| New final tableau after reoptimization (only one iteration of the simplex method needed in this case) | $Z$ | (0) | 1 | 0 | 0 | $\frac{3}{4}$ | 0 | $\frac{3}{4}$ | $\frac{33}{2}$ |
| | $x_1$ | (1) | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| | $x_2$ | (2) | 0 | 0 | 1 | $-\frac{3}{4}$ | 0 | $\frac{1}{4}$ | $\frac{3}{2}$ |
| | $x_4$ | (3) | 0 | 0 | 0 | $\frac{9}{4}$ | 1 | $-\frac{3}{4}$ | $\frac{39}{2}$ |

**TABLE 6.21** Data for Variation 2 of the Wyndor Glass Co. model

*Model Parameters*

| | | |
|---|---|---|
| $c_1 = 3,$ | $c_2 = 5$ | $(n = 2)$ |
| $a_{11} = 1,$ | $a_{12} = 0,$ | $b_1 = 4$ |
| $a_{21} = 0,$ | $a_{22} = 2,$ | $b_2 = 24$ |
| $a_{31} = 3,$ | $a_{32} = 2,$ | $b_3 = 18$ |

**Final Simplex Tableau after Reoptimization**

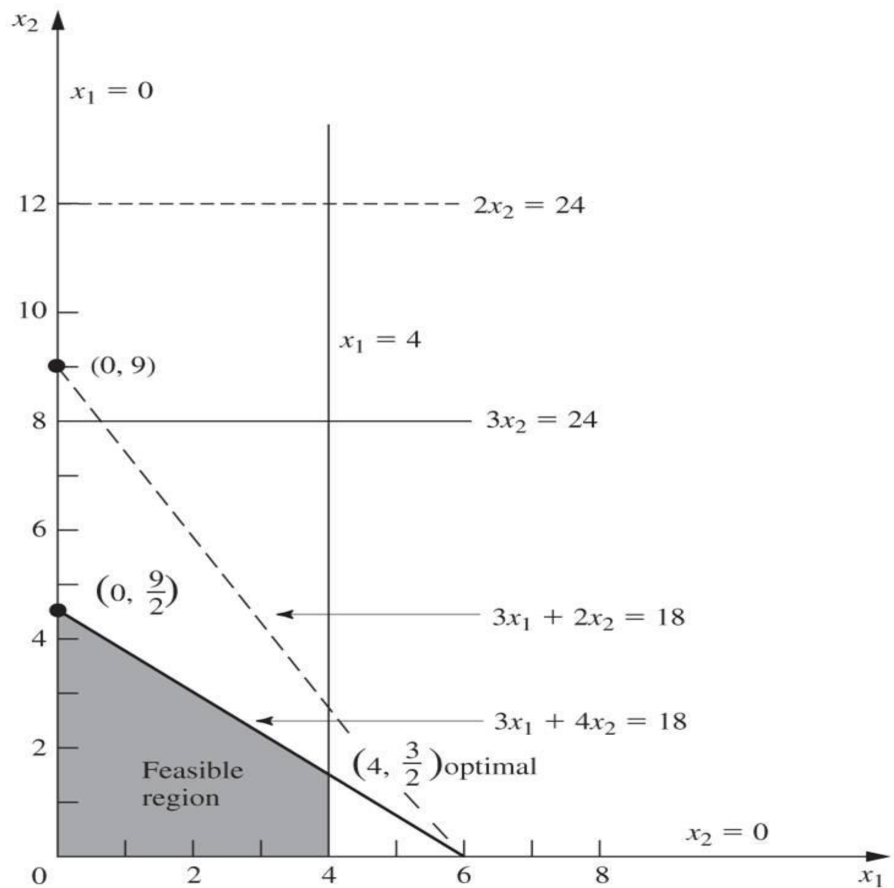
| Basic Variable | Eq. | $Z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | Right Side |
|---|---|---|---|---|---|---|---|---|
| $Z$ | (0) | 1 | $\frac{9}{2}$ | 0 | 0 | 0 | $\frac{5}{2}$ | 45 |
| $x_3$ | (1) | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| $x_2$ | (2) | 0 | $\frac{3}{2}$ | 1 | 0 | 0 | $\frac{1}{2}$ | 9 |
| $x_4$ | (3) | 0 | $-3$ | 0 | 0 | 1 | $-1$ | 6 |

**TABLE 5.8** Initial and later simplex tableaux in matrix form

| Iteration | Basic Variable | Eq. | $Z$ | Original Variables | Slack Variables | Right Side |
|---|---|---|---|---|---|---|
| 0 | $Z$ | (0) | 1 | $-\mathbf{c}$ | $\mathbf{0}$ | 0 |
| | $\mathbf{x}_B$ | $(1, 2, \ldots, m)$ | $\mathbf{0}$ | $\mathbf{A}$ | $\mathbf{I}$ | $\mathbf{b}$ |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| Any | $Z$ | (0) | 1 | $\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c}$ | $\mathbf{c}_B\mathbf{B}^{-1}$ | $\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$ |
| | $\mathbf{x}_B$ | $(1, 2, \ldots, m)$ | $\mathbf{0}$ | $\mathbf{B}^{-1}\mathbf{A}$ | $\mathbf{B}^{-1}$ | $\mathbf{B}^{-1}\mathbf{b}$ |

**TABLE 6.24** Sensitivity analysis procedure applied to Variation 5 of the Wyndor Glass Co. model

| | Basic Variable | Eq. | Coefficient of: $Z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | Right Side |
|---|---|---|---|---|---|---|---|---|---|
| | $Z$ | (0) | 1 | $-\frac{3}{4}$ | 0 | 0 | 0 | $\frac{3}{4}$ | $\frac{27}{2}$ |
| | $x_3$ | (1) | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| | $x_2$ | (2) | 0 | $\frac{3}{4}$ | 1 | 0 | 0 | $\frac{1}{4}$ | $\frac{9}{2}$ |
| | $x_4$ | (3) | 0 | $-\frac{9}{4}$ | 0 | 0 | 1 | $-\frac{3}{4}$ | $\frac{21}{2}$ |
| New final tableau after reoptimization (only one iteration of the simplex method needed in this case) | $Z$ | (0) | 1 | 0 | 0 | $\frac{3}{4}$ | 0 | $\frac{3}{4}$ | $\frac{33}{2}$ |
| | $x_1$ | (1) | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| | $x_2$ | (2) | 0 | 0 | 1 | $-\frac{3}{4}$ | 0 | $\frac{1}{4}$ | $\frac{3}{2}$ |
| | $x_4$ | (3) | 0 | 0 | 0 | $\frac{9}{4}$ | 1 | $-\frac{3}{4}$ | $\frac{39}{2}$ |



# The allowable range of the coefficient $c_i$ of a basic variable

$$c_2 = 3 + \Delta c_2$$

$$\left[0, 0, \tfrac{3}{4} - \tfrac{3}{4}\Delta c_2, 0, \tfrac{3}{4} + \tfrac{1}{4}\Delta c_2 \;\vdots\; \tfrac{33}{2} + \tfrac{3}{2}\Delta c_2\right]$$

$$\tfrac{3}{4} - \tfrac{3}{4}\Delta c_2 \geq 0 \quad \Rightarrow \quad \tfrac{3}{4} \geq \tfrac{3}{4}\Delta c_2 \quad \Rightarrow \quad \Delta c_2 \leq 1.$$
$$\tfrac{3}{4} + \tfrac{1}{4}\Delta c_2 \geq 0 \quad \Rightarrow \quad \tfrac{1}{4}\Delta c_2 \geq -\tfrac{3}{4} \quad \Rightarrow \quad \Delta c_2 \geq -3$$

Thus, the range of values is $-3 \leq \Delta c_2 \leq 1$.

Since $c_2 = 3 + \Delta c_2$, add 3 to this range of values, which yields

$$0 \leq c_2 \leq 4$$

as the allowable range to stay optimal for $c_2$.

*Model Parameters*

| | | |
|---|---|---|
| $c_1 = 3,$ | $\boxed{c_2 = 3}$ | $(n = 2)$ |
| $a_{11} = 1,$ | $a_{12} = 0,$ | $b_1 = 4$ |
| $a_{21} = 0,$ | $a_{22} = 3,$ | $b_2 = 24$ |
| $a_{31} = 3,$ | $a_{32} = 4,$ | $b_3 = 18$ |

**Final Simplex Tableau after Reoptimization**

| Basic Variable | Eq. | Coefficient of: $Z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | Right Side |
|---|---|---|---|---|---|---|---|---|
| $Z$ | (0) | 1 | 0 | 0 | $\frac{3}{4}$ | 0 | $\frac{3}{4}$ | $\frac{33}{2}$ |
| $x_1$ | (1) | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| $x_2$ | (2) | 0 | 0 | 1 | $-\frac{3}{4}$ | 0 | $\frac{1}{4}$ | $\frac{3}{2}$ |
| $x_4$ | (3) | 0 | 0 | 0 | $\frac{9}{4}$ | 1 | $-\frac{3}{4}$ | $\frac{39}{2}$ |

**TABLE 5.8** Initial and later simplex tableaux in matrix form

| Iteration | Basic Variable | Eq. | Coefficient of: $Z$ | Original Variables | Slack Variables | Right Side |
|---|---|---|---|---|---|---|
| 0 | $Z$<br>$\mathbf{x}_B$ | (0)<br>(1, 2, . . . , $m$) | 1<br>**0** | $-\mathbf{c}$<br>$\mathbf{A}$ | $\mathbf{0}$<br>$\mathbf{I}$ | 0<br>$\mathbf{b}$ |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| Any | $Z$<br>$\mathbf{x}_B$ | (0)<br>(1, 2, . . . , $m$) | 1<br>**0** | $\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c}$<br>$\mathbf{B}^{-1}\mathbf{A}$ | $\mathbf{c}_B\mathbf{B}^{-1}$<br>$\mathbf{B}^{-1}$ | $\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$<br>$\mathbf{B}^{-1}\mathbf{b}$ |

# Case 4 : Introduction of a new constraint

**Example:** To illustrate this case, suppose that the new constraint

$$2x_1 + 3x_2 \leq 24$$

is introduced into the model

The previous optimal solution (0, 9) violates the new constraint

**TABLE 6.21** Data for Variation 2 of the Wyndor Glass Co. model

*Model Parameters*

| | | |
|---|---|---|
| $c_1 = 3,$ | $c_2 = 5$ | $(n = 2)$ |
| $a_{11} = 1,$ | $a_{12} = 0,$ | $b_1 = 4$ |
| $a_{21} = 0,$ | $a_{22} = 2,$ | $b_2 = 24$ |
| $a_{31} = 3,$ | $a_{32} = 2,$ | $b_3 = 18$ |

**Final Simplex Tableau after Reoptimization**

| Basic Variable | Eq. | Coefficient of: $Z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | Right Side |
|---|---|---|---|---|---|---|---|---|
| $Z$ | (0) | 1 | $\frac{9}{2}$ | 0 | 0 | 0 | $\frac{5}{2}$ | 45 |
| $x_3$ | (1) | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| $x_2$ | (2) | 0 | $\frac{3}{2}$ | 1 | 0 | 0 | $\frac{1}{2}$ | 9 |
| $x_4$ | (3) | 0 | $-3$ | 0 | 0 | 1 | $-1$ | 6 |

**TABLE 6.25** Sensitivity analysis procedure applied to Variation 6 of the Wyndor Glass Co. model

| | Basic Variable | Eq. | Coefficient of: $Z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | Right Side |
|---|---|---|---|---|---|---|---|---|---|---|
| Revised final tableau | $Z$ | (0) | 1 | $\frac{9}{2}$ | 0 | 0 | 0 | $\frac{5}{2}$ | 0 | 45 |
| | $x_3$ | (1) | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 4 |
| | $x_2$ | (2) | 0 | $\frac{3}{2}$ | 1 | 0 | 0 | $\frac{1}{2}$ | 0 | 9 |
| | $x_4$ | (3) | 0 | $-3$ | 0 | 0 | 1 | $-1$ | 0 | 6 |
| | $x_6$ | New | 0 | 2 | 3 | 0 | 0 | 0 | 1 | 24 |
| Converted to proper form | $Z$ | (0) | 1 | $\frac{9}{2}$ | 0 | 0 | 0 | $\frac{5}{2}$ | 0 | 45 |
| | $x_3$ | (1) | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 4 |
| | $x_2$ | (2) | 0 | $\frac{3}{2}$ | 1 | 0 | 0 | $\frac{1}{2}$ | 0 | 9 |
| | $x_4$ | (3) | 0 | $-3$ | 0 | 0 | 1 | $-1$ | 0 | 6 |
| | $x_6$ | New | 0 | $-\frac{5}{2}$ | 0 | 0 | 0 | $-\frac{3}{2}$ | 1 | $-3$ |
| New final tableau after reoptimization (only one iteration of dual simplex method needed in this case) | $Z$ | (0) | 1 | $\frac{1}{3}$ | 0 | 0 | 0 | 0 | $\frac{5}{3}$ | 40 |
| | $x_3$ | (1) | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 4 |
| | $x_2$ | (2) | 0 | $\frac{2}{3}$ | 1 | 0 | 0 | 0 | $\frac{1}{3}$ | 8 |
| | $x_4$ | (3) | 0 | $-\frac{4}{3}$ | 0 | 0 | 1 | 0 | $-\frac{2}{3}$ | 8 |
| | $x_5$ | New | 0 | $\frac{5}{3}$ | 0 | 0 | 0 | 1 | $-\frac{2}{3}$ | 2 |

# Parametric Linear Programming

## Systematic Changes in the $c_j$ Parameters

The objective function of the ordinary linear programming model is

$$Z = \sum_{j=1}^{n} c_j x_j$$

For the case where the  parameters are being changed, the objective function of the ordinary linear programming model is replaced by

$$Z(\theta) = \sum_{j=1}^{n} (c_j + \alpha_j \theta) x_j$$

where θ is a parameter and $\alpha_j$ are given input constants representing the relative rates at which the coefficients are to be changed.

# Example:

To illustrate the solution procedure, suppose $\alpha_1$=2 and $\alpha_2$ = -1 for the original Wyndor Glass Co. problem, so that

$$Z(\theta) = (3 + 2\theta)x_1 + (5 - \theta)x_2$$

Beginning with the final simplex tableau for θ = 0, we see that its Eq. (0)

$$(0) \qquad Z + \tfrac{3}{2}x_4 + x_5 = 36$$

**TABLE 5.8** Initial and later simplex tableaux in matrix form

| Iteration | Basic Variable | Eq. | Coefficient of: $Z$ | Original Variables | Slack Variables | Right Side |
|---|---|---|---|---|---|---|
| 0 | $Z$ | (0) | 1 | $-\mathbf{c}$ | $\mathbf{0}$ | 0 |
| | $\mathbf{x}_B$ | $(1, 2, \ldots, m)$ | $\mathbf{0}$ | $\mathbf{A}$ | $\mathbf{I}$ | $\mathbf{b}$ |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| Any | $Z$ | (0) | 1 | $\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c}$ | $\mathbf{c}_B\mathbf{B}^{-1}$ | $\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$ |
| | $\mathbf{x}_B$ | $(1, 2, \ldots, m)$ | $\mathbf{0}$ | $\mathbf{B}^{-1}\mathbf{A}$ | $\mathbf{B}^{-1}$ | $\mathbf{B}^{-1}\mathbf{b}$ |

**TABLE 4.8** Complete set of simplex tableaux for the Wyndor Glass Co. problem

| Iteration | Basic Variable | Eq. | Coefficient of: $Z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | Right Side |
|---|---|---|---|---|---|---|---|---|---|
| 0 | $Z$ | (0) | 1 | $-3$ | $-5$ | 0 | 0 | 0 | 0 |
| | $x_3$ | (1) | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| | $x_4$ | (2) | 0 | 0 | 2 | 0 | 1 | 0 | 12 |
| | $x_5$ | (3) | 0 | 3 | 2 | 0 | 0 | 1 | 18 |
| 1 | $Z$ | (0) | 1 | $-3$ | 0 | 0 | $\frac{5}{2}$ | 0 | 30 |
| | $x_3$ | (1) | 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| | $x_2$ | (2) | 0 | 0 | 1 | 0 | $\frac{1}{2}$ | 0 | 6 |
| | $x_5$ | (3) | 0 | 3 | 0 | 0 | $-1$ | 1 | 6 |
| 2 | $Z$ | (0) | 1 | $-2\theta$ | $\theta$ | 0 | $\frac{3}{2}$ | 1 | 36 |
| | $x_3$ | (1) | 0 | 0 | 0 | 1 | $\frac{1}{3}$ | $-\frac{1}{3}$ | 2 |
| | $x_2$ | (2) | 0 | 0 | 1 | 0 | $\frac{1}{2}$ | 0 | 6 |
| | $x_1$ | (3) | 0 | 1 | 0 | 0 | $-\frac{1}{3}$ | $\frac{1}{3}$ | 2 |

If θ ≠ 0, we have

$$(0) \qquad Z - 2\theta x_1 + \theta x_2 + \tfrac{3}{2}x_4 + x_5 = 36.$$

Because both $x_1$ and $x_2$ are basic variables, they both need to be eliminated algebraically from Eq. (0)

$$Z - 2\theta x_1 + \theta x_2 + \tfrac{3}{2}x_4 + x_5 = 36$$

$$+2\theta \text{ times Eq. (3)}$$

$$-\ \theta \text{ times Eq. (2)}$$

$$(0) \quad Z + \left(\tfrac{3}{2} - \tfrac{7}{6}\theta\right)x_4 + \left(1 + \tfrac{2}{3}\theta\right)x_5 = 36 - 2\theta.$$

The optimality test says that the current BF solution will remain optimal as long as these coefficients of the nonbasic variables remain nonnegative:

$$\tfrac{3}{2} - \tfrac{7}{6}\theta \geq 0, \qquad \text{for } 0 \leq \theta \leq \tfrac{9}{7},$$

$$1 + \tfrac{2}{3}\theta \geq 0, \qquad \text{for all } \theta \geq 0.$$

| Range of $\theta$ | Basic Variable | Eq. | $Z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | Right Side | Optimal Solution |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | Coefficient of: | | | | | | | |
| $0 \leq \theta \leq \frac{9}{7}$ | $Z(\theta)$ | (0) | 1 | 0 | 0 | 0 | $\frac{9-7\theta}{6}$ | $\frac{3+2\theta}{3}$ | $36-2\theta$ | $x_4 = 0$, $x_5 = 0$ |
| | $x_3$ | (1) | 0 | 0 | 0 | 1 | $\frac{1}{3}$ | $-\frac{1}{3}$ | 2 | $x_3 = 2$ |
| | $x_2$ | (2) | 0 | 0 | 1 | 0 | $\frac{1}{2}$ | 0 | 6 | $x_2 = 6$ |
| | $x_1$ | (3) | 0 | 1 | 0 | 0 | $-\frac{1}{3}$ | $\frac{1}{3}$ | 2 | $x_1 = 2$ |

Summary of the Parametric Linear Programming Procedure for Systematic Changes the $c_j$ Parameters

**TABLE 7.2** The $c_j$ parametric linear programming procedure applied to the Wyndor Glass Co. example

| Range of $\theta$ | Basic Variable | Eq. | $Z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | Right Side | Optimal Solution |
|---|---|---|---|---|---|---|---|---|---|---|
| | $Z(\theta)$ | (0) | 1 | 0 | 0 | 0 | $\frac{9-7\theta}{6}$ | $\frac{3+2\theta}{3}$ | $36-2\theta$ | $x_4 = 0$<br>$x_5 = 0$ |
| $0 \le \theta \le \frac{9}{7}$ | $x_3$ | (1) | 0 | 0 | 0 | 1 | $\frac{1}{3}$ | $-\frac{1}{3}$ | 2 | $x_3 = 2$ |
| | $x_2$ | (2) | 0 | 0 | 1 | 0 | $\frac{1}{2}$ | 0 | 6 | $x_2 = 6$ |
| | $x_1$ | (3) | 0 | 1 | 0 | 0 | $-\frac{1}{3}$ | $\frac{1}{3}$ | 2 | $x_1 = 2$ |
| | $Z(\theta)$ | (0) | 1 | 0 | 0 | $\frac{-9+7\theta}{2}$ | 0 | $\frac{5-\theta}{2}$ | $27+5\theta$ | $x_3 = 0$<br>$x_5 = 0$ |
| $\frac{9}{7} \le \theta \le 5$ | $x_4$ | (1) | 0 | 0 | 0 | 3 | 1 | $-1$ | 6 | $x_4 = 6$ |
| | $x_2$ | (2) | 0 | 0 | 1 | $-\frac{3}{2}$ | 0 | $\frac{1}{2}$ | 3 | $x_2 = 3$ |
| | $x_1$ | (3) | 0 | 1 | 0 | 1 | 0 | 0 | 4 | $x_1 = 4$ |
| | $Z(\theta)$ | (0) | 1 | 0 | $-5+\theta$ | $3+2\theta$ | 0 | 0 | $12+8\theta$ | $x_2 = 0$<br>$x_3 = 0$ |
| $\theta \ge 5$ | $x_4$ | (1) | 0 | 0 | 2 | 0 | 1 | 0 | 12 | $x_4 = 12$ |
| | $x_5$ | (2) | 0 | 0 | 2 | $-3$ | 0 | 1 | 6 | $x_5 = 6$ |
| | $x_1$ | (3) | 0 | 1 | 0 | 1 | 0 | 0 | 4 | $x_1 = 4$ |

1. Solve the problem with $\theta = 0$ by the simplex method.
2. Use the sensitivity analysis procedure (Cases 2$a$ and 3, Sec. 6.7) to introduce the $\Delta c_j = \alpha_j \theta$ changes into Eq. (0).
3. Increase $\theta$ until one of the nonbasic variables has its coefficient in Eq. (0) go negative (or until $\theta$ has been increased as far as desired).
4. Use this variable as the entering basic variable for an iteration of the simplex method to find the new optimal solution. Return to step 3.

## Systematic Changes in the $b_i$ Parameters

For the case where the $b_i$ parameters change systematically, the one modification made in the original linear programming model is that is replaced by, for $i = 1, 2, \ldots, m$, where the $\alpha_i$ are given input constants. Thus the problem becomes

$$\text{Maximize} \qquad Z(\theta) = \sum_{j=1}^{n} c_j x_j,$$

subject to

$$\sum_{j=1}^{n} a_{ij} \le b_i + \alpha_i \theta \qquad \text{for } i = 1, 2, \ldots, m$$

$$x_j \ge 0 \qquad \text{for } j = 1, 2, \ldots, n.$$

The goal is to identify the optimal solution as a function of $\theta$

# Example:

Maximize $\quad Z = -4y_1 - 12y_2 - 18y_3$

subject to

$$y_1 + 3y_3 \geq 3$$
$$2y_2 + 2y_3 \geq 5$$
$$y_1 \geq 0, \quad y_2 \geq 0, \quad y_3 \geq 0.$$

| Iteration | Basic Variable | Eq. | Coefficient of: Z | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ | Right Side |
|---|---|---|---|---|---|---|---|---|---|
| 0 | Z | (0) | 1 | 4 | $\boxed{12}$ | 18 | 0 | 0 | 0 |
| | $y_4$ | (1) | 0 | −1 | 0 | −3 | 1 | 0 | −3 |
| | $y_5$ | (2) | 0 | 0 | −2 | −2 | 0 | 1 | −5 |
| 1 | Z | (0) | 1 | 4 | 0 | $\boxed{6}$ | 0 | 6 | −30 |
| | $y_4$ | (1) | 0 | −1 | 0 | −3 | 1 | 0 | −3 |
| | $y_2$ | (2) | 0 | 0 | 1 | 1 | 0 | $-\frac{1}{2}$ | $\frac{5}{2}$ |
| 2 | Z | (0) | 1 | 2 | 0 | 0 | 2 | 6 | −36 |
| | $y_3$ | (1) | 0 | $\frac{1}{3}$ | 0 | 1 | $-\frac{1}{3}$ | 0 | 1 |
| | $y_2$ | (2) | 0 | $-\frac{1}{3}$ | 1 | 0 | $\frac{1}{3}$ | $-\frac{1}{2}$ | $\frac{3}{2}$ |

Suppose that $\alpha_1$=2 and $\alpha_2$ = -1 so that the functional constraints become

$$y_1 + 3y_3 \geq 3 + 2\theta \quad \text{or} \quad -y_1 - 3y_3 \leq -3 - 2\theta$$
$$2y_2 + 2y_3 \geq 5 - \theta \quad \text{or} \quad -2y_2 - 2y_3 \leq -5 + \theta.$$

This problem with θ = 0 has already been solved in the table, so we begin with the final tableau given there.

Using the sensitivity procedure, we find that the entries in the right side column of the tableau change to the values given below

$$y_0^* = \mathbf{y}^*\bar{\mathbf{b}} = [2, 6]\begin{bmatrix} -3 - 2\theta \\ -5 + \theta \end{bmatrix} = -36 + 2\theta,$$

$$\mathbf{b}^* = \mathbf{S}^*\bar{\mathbf{b}} = \begin{bmatrix} -\frac{1}{3} & 0 \\ \frac{1}{3} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} -3 - 2\theta \\ -5 + \theta \end{bmatrix} = \begin{bmatrix} 1 + \frac{2\theta}{3} \\ \frac{3}{2} - \frac{7\theta}{6} \end{bmatrix}$$

Therefore, the two basic variables in this tableau

$$y_3 = \frac{3 + 2\theta}{3} \quad \text{and} \quad y_2 = \frac{9 - 7\theta}{6}$$

Remain nonnegative for $0 \le \theta \le \frac{9}{7}$.

## Summary of the Parametric Linear Programming Procedure for Systematic Changes the $b_i$ Parameters

**TABLE 7.3** The $b_i$ parametric linear programming procedure applied to the dual of the Wyndor Glass Co. example

| Range of $\theta$ | Basic Variable | Eq. | $Z$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ | Right Side | Optimal Solution |
|---|---|---|---|---|---|---|---|---|---|---|
| $0 \le \theta \le \frac{9}{7}$ | $Z(\theta)$ | (0) | 1 | 2 | 0 | 0 | 2 | 6 | $-36 + 2\theta$ | $y_1 = y_4 = y_5 = 0$ |
| | $y_3$ | (1) | 0 | $\frac{1}{3}$ | 0 | 1 | $-\frac{1}{3}$ | 0 | $\frac{3 + 2\theta}{3}$ | $y_3 = \frac{3 + 2\theta}{3}$ |
| | $y_2$ | (2) | 0 | $-\frac{1}{3}$ | 1 | 0 | $\frac{1}{3}$ | $-\frac{1}{2}$ | $\frac{9 - 7\theta}{6}$ | $y_2 = \frac{9 - 7\theta}{6}$ |
| $\frac{9}{7} \le \theta \le 5$ | $Z(\theta)$ | (0) | 1 | 0 | 6 | 0 | 4 | 3 | $-27 - 5\theta$ | $y_2 = y_4 = y_5 = 0$ |
| | $y_3$ | (1) | 0 | 0 | 1 | 1 | 0 | $-\frac{1}{2}$ | $\frac{5 - \theta}{2}$ | $y_3 = \frac{5 - \theta}{2}$ |
| | $y_1$ | (2) | 0 | 1 | $-3$ | 0 | $-1$ | $\frac{3}{2}$ | $\frac{-9 + 7\theta}{2}$ | $y_1 = \frac{-9 + 7\theta}{2}$ |
| $\theta \ge 5$ | $Z(\theta)$ | (0) | 1 | 0 | 12 | 6 | 4 | 0 | $-12 - 8\theta$ | $y_2 = y_3 = y_4 = 0$ |
| | $y_5$ | (1) | 0 | 0 | $-2$ | $-2$ | 0 | 1 | $-5 + \theta$ | $y_5 = -5 + \theta$ |
| | $y_1$ | (2) | 0 | 1 | 0 | 3 | $-1$ | 0 | $3 + 2\theta$ | $y_1 = 3 + 2\theta$ |

1. Solve the problem with $\theta = 0$ by the simplex method.
2. Use the sensitivity analysis procedure (Case 1, Sec. 6.7) to introduce the $\Delta b_i = \alpha_i \theta$ changes to the *right side* column.
3. Increase $\theta$ until one of the basic variables has its value in the *right side* column go negative (or until $\theta$ has been increased as far as desired).
4. Use this variable as the leaving basic variable for an iteration of the dual simplex method to find the new optimal solution. Return to step 3.

- **Acknowledgements**


- Dr. Yicheng Wang (Visiting Researcher, CADSWES during Fall 2009 – early Spring 2010) for slides from his Optimization course during Fall 2009
- Introduction to Operations Research by Hillier and Lieberman, McGraw Hill
- Hughes-McMakee-notes\chapter-05.pdf