

$$\int f(\sin \theta, \cos \theta) d\theta = \int f\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right) \left(\frac{2 dt}{1+t^2}\right).$$

$$\begin{aligned} \int \sec \theta d\theta &= \int \frac{1}{\cos \theta} d\theta \\ &= \int \frac{2}{1+t^2} \left( \frac{1+t^2}{1-t^2} \right) dt \\ &= \int \frac{2}{1-t^2} dt \\ &= \int \left( \frac{1}{1-t} + \frac{1}{1+t} \right) dt \\ &= \left[ \ln \left| \frac{1+t}{1-t} \right| + C \right] \\ &= \ln \left| \frac{1+\tan \frac{\theta}{2}}{1-\tan \frac{\theta}{2}} \right| + C. \end{aligned}$$

$$\begin{aligned} &\ln |\sec \theta + \tan \theta| + C' \\ &= \ln \left| \frac{1+t^2}{1-t^2} + \frac{2t}{1-t^2} \right| + C' = \ln \left| \frac{(t+1)^2}{1-t^2} \right| + C' \\ &= \ln \left| \frac{1+t}{1-t} \right| + C' \end{aligned}$$

$$\begin{aligned} \boxed{\int \sin^3 \theta \cos \theta d\theta} &= \boxed{\frac{1}{4} \sin^4 \theta + C} \\ &= \int \left( \frac{2t}{1+t^2} \right)^3 \left( \frac{1-t^2}{1+t^2} \right) \left( \frac{2}{1+t^2} \right) dt \\ &= \int \frac{16t^3(1-t^2)}{(1+t^2)^3} dt \\ &= \int \frac{8u(1-u)}{(1+u)^3} du \quad u=t^2, \quad du=2t dt \quad \text{orange arrow} \end{aligned}$$

$$\begin{aligned} &\frac{1}{4} \left( \frac{2t}{1+t^2} \right)^4 = \frac{4t^4}{(1+t^2)^4} = \frac{4u^2}{(1+u)^4} \\ &\frac{8u(1+u)^4 - 4u^2(4)(1+u)^3}{(1+u)^6} \\ &= \frac{8u(1+u) - 16u^2}{(1+u)^3} \\ &= \frac{8u(1-u)}{(1+u)^3} \quad \text{orange bracket} \end{aligned}$$