Abstract

The paper aims at the introduction of integer partitions as a concept of Number Theory and Combinatorics into forming graphs by systematically eliminating invalid permutations of the partitions of the sum of degrees of the vertices for a given number of vertices.

Introduction

This paper explores an approach to generating caterpillars based on the integer partitions of the sum of the degrees of its vertices, each of whose permutations result in a graph provided they satisfy a specified number of conditions. Going through the permutations of the partitions and eliminating the caterpillars that do not meet the specifications, all of the possible unlabelled caterpillars are ultimately deduced. The proposed algorithm goes by all possible degree distributions, the margin of error is minimized.

The results obtained are consistent with Harary's and Schwenk's proposed formula[c], 1973.

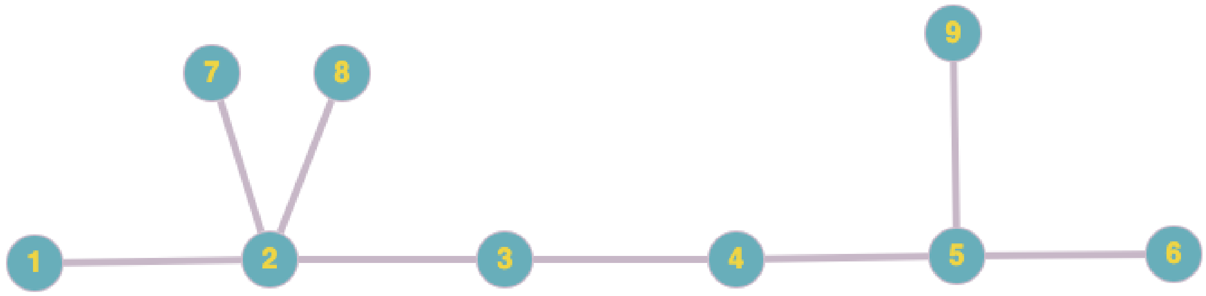
With the list thus obtained, the total number of non-isomorphic caterpillars that can be formed on n vertices is computed by strategically labeling the vertices.

Definitions

An integer partition is a way of splitting a number into integer parts. By definition, the partition stays however we order the parts, so we may choose the convention of listing the parts from the largest part down to the smallest. [1]

Let be any arbitrary tree on edges. If its vertices can be distinctly labeled using integers so that all the induced edge labels (vertices labeled and induce label on edge) are also distinct, then the labeling is called graceful. It was conjectured that all trees are graceful.

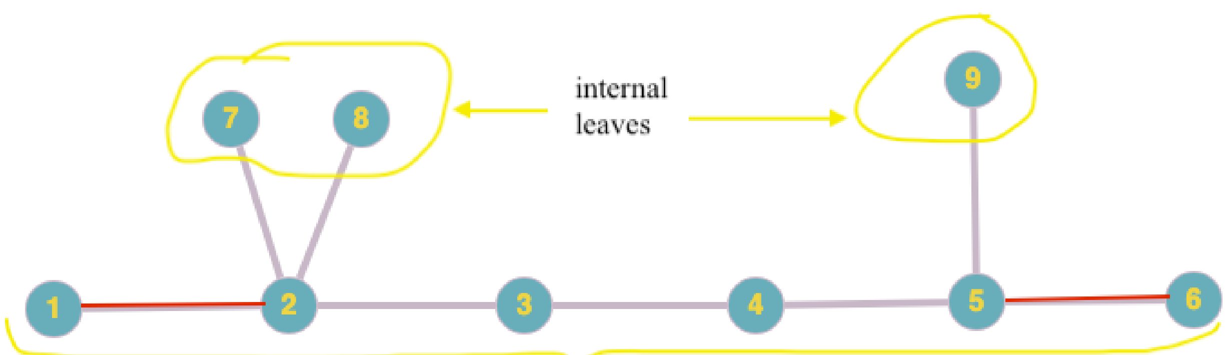
A *path* is a tree with only two leaves. A *caterpillar* is a tree such that if all leaves are removed, the remaining graph is a path. This path can be termed as *backbone* of the caterpillar. [2]



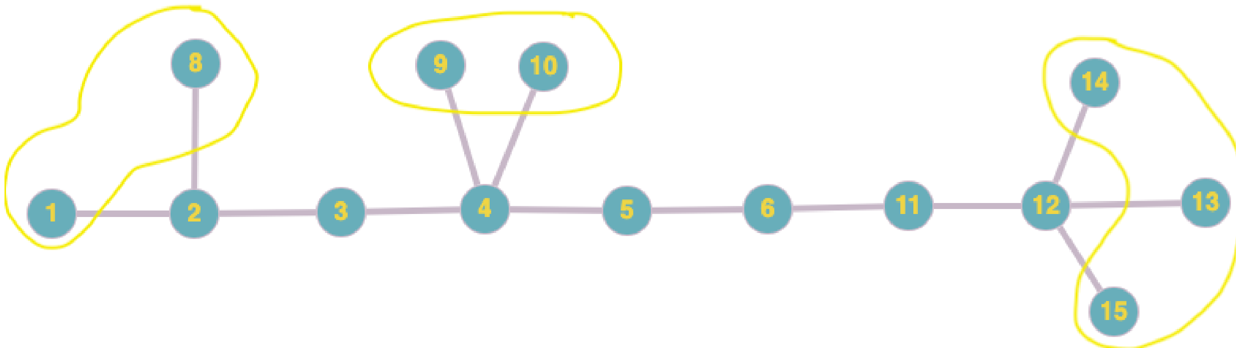
Some terms we will be using

Spine - Main chain of the caterpillar which is the longest possible path that can be formed.
Vertices 1 to 6 form the spine or the main chain of the caterpillar.

Internal leaves - the leaves that are incident on the vertices of the main chain. The terminal leaves of the spine are NOT considered as internal leaves.



Equivalent Leaves - The leaves which can have equivalent positions and when the labeling, if their labels are interchanged, it won't make a difference.



e_1 is one set of equivalent leaves whose elements are leaves 1 and 8.

e_2 is one set of equivalent leaves whose elements are leaves 9 and 10.

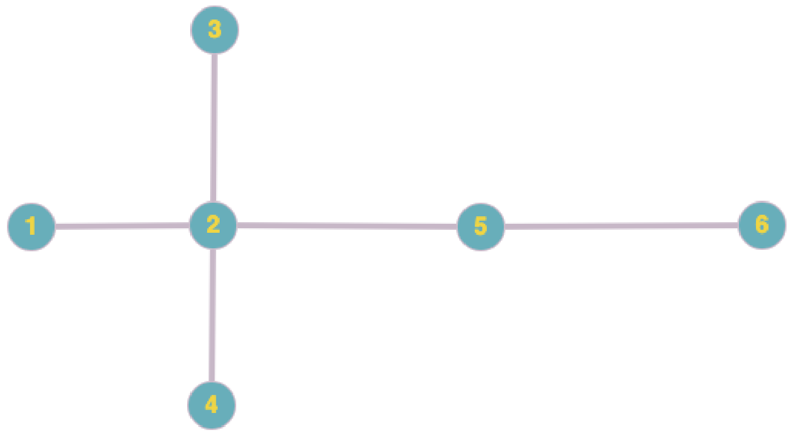
e_3 is one set of equivalent leaves whose elements are leaves 13, 14 and 15.

Symmetric Caterpillars - The caterpillars whose mirror images are superimposable on them are referred to as symmetric graphs. Similarly, the caterpillars whose mirror images are non-superimposable on them are referred to as asymmetric graphs.

Examples:
The following is an example of a symmetric caterpillar.

Symmetric Caterpillar on 6 Vertices

The following is an example of an asymmetric caterpillar.



Asymmetric Caterpillar on 6 Vertices

Theorem

The number of non-isomorphic caterpillars is given as follows.

Let $\gamma(n)$ be a function that generates all the possible unlabelled caterpillars possible for a given number of vertices n.

Let N be the total number of non-isomorphic labeled caterpillars which is possible for a given number of vertices n.

$$N = g(K_1) + g(K_2) + ... + g(K_i),$$

where K_1, K_2,... K_i are non-isomorphic graphs generated using the function $\gamma(n)$.

$$g(K) = {}^{n}C_{e_1} * {}^{n-e_1}C_{e_2} * {}^{n-e_1-e_2}C_{e_3} * \ldots * {}^{n-e_1-e_2-e_3-.....-e_{p-1}}C_{e_p} * (n - e_1 - e_2 - e_3 .... - e_p)!$$

if K is non-symmetric.

$$g(K) = \frac{{}^{n}C_{e_1} * {}^{n-e_1}C_{e_2} * {}^{n-e_1-e_2}C_{e_3} * \ldots * {}^{n-e_1-e_2-e_3-.....-e_{p-1}}C_{e_p} * (n-e_1-e_2-e_3....-e_p)!}{2},$$

if K is symmetric.

Method/Algorithm

This approach involves approaching the problem in two stages.

Part 1 - Structures
Given n vertices, the number of unlabelled non-isomorphic graphs that can be formed is determined.

Part 2 - Labeling
Once the unlabelled graphs have been determined, they are labeled.

PART - 1

GENERATING THE UNLABELED CATERPILLARS

1. Let n be the number of vertices. The maximum number of edges that can be present in it is $n - 1$.
2. The maximum degree a vertex can have is $n - 1$, in which case the caterpillar will be a $K_{1,n-1}$ bipartite graph.
3. According to the Handshaking Lemma, the sum of the degrees of all the vertices of a graph is equal to twice the number of vertices in the graph.

   Using this lemma, the sum of the degrees of all the vertices is deduced to be $2(n - 1)$, where $n - 1$ is the number of edges in the caterpillar.
1. Introducing the concept of Integer Partitions to partition the sum of the degrees of the vertices: By obtaining the list of partitions for a given sum of degrees of the vertices of a graph, all of the possible distribution of degrees per vertex is covered.

   However, there are some invalid partitions in the list of all possible integer partitions for a particular number. We filter them out using the following principle.
   a. The number of parts in the partition must exactly be equal to the number of vertices in the caterpillar. No isolated vertex is allowed in a caterpillar. Every part of a partition will correspond to the degree of one vertex in the caterpillar.
   b. The maximum degree that a vertex can have is $n - 1$. The partitions for which the maximum degree is greater than $n - 1$ are eliminated.

      Thus, a filtered list of partitions is obtained from which the algorithm can proceed.

How many non-isomorphic caterpillars can one partition have?

4. Which vertex will have what degree?
   Now that we have a particular array of degrees, we generate all possible permutations of the partition. This ensures that every vertex has all the degrees in the partition. If there are permutations which are mirror images of one another, one is preserved and the other is eliminated, which is because they are isomorphic.
5. This list still has invalid arrangements and they are determined using the following two conditions.
   a. The first and the last terms of the permutation, which corresponds to the terminal leaves of the spine of the caterpillar must be of degree 1.
   b. The internal leaves in the caterpillar will arrange themselves so as to satisfy the degree of the internal vertices on the spine of the caterpillar.

      Main chain - this will be the sequence of degrees of the vertices of the spine of the caterpillar with the internal leaves removed.

      Two permutations which have the same main chain will form the same structure/caterpillar. One of them is preserved and the rest are eliminated.

      The degree distribution for all the valid and possible non-isomorphic caterpillars for a particular number of vertices is thus obtained from which all unlabelled non-isomorphic caterpillars are obtained.

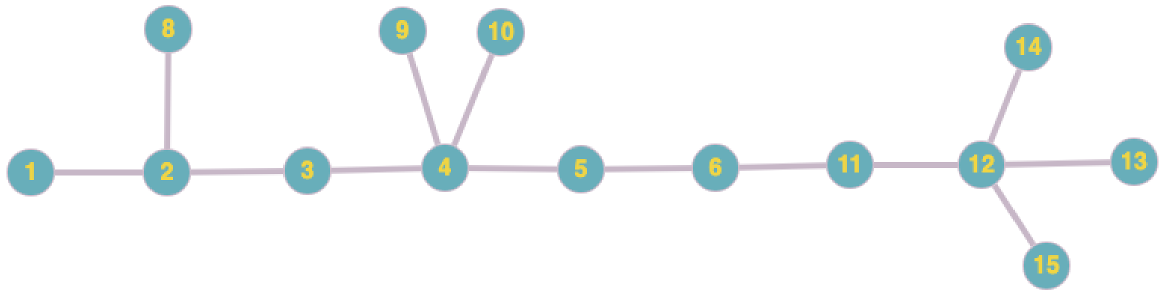
Let this be given by a function $\gamma(n)$.

PART - 2

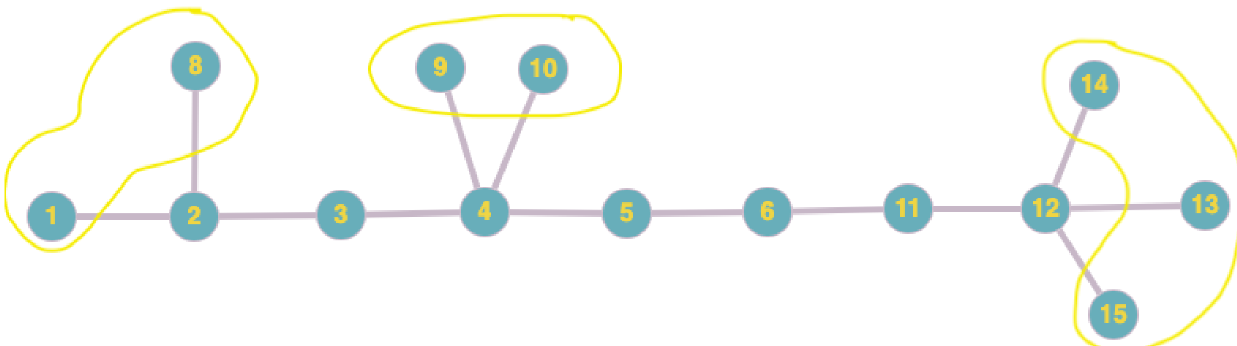
LABELING

1. Introducing the concept of equivalent leaves and defining $e_1, e_2, ..., e_p$.
   All the leaves on a particular vertex of the caterpillar form a set of equivalent leaves.

   Example.
   Consider the following caterpillar.



The equivalent sets of leaves are circled as shown below.



e_1 is one set of equivalent leaves whose elements are leaves 1 and 8.

e_2 is one set of equivalent leaves whose elements are leaves 9 and 10.

e_3 is one set of equivalent leaves whose elements are leaves 13, 14 and 15.

Why are they equivalent?

Consider the set e_1. Fix labels $a, b, \ldots, k, l$ to the remaining 13 vertices of the caterpillar. If vertices 1 and 8 are to be named with the letters m and n, the possible combinations are 1 being m and 8 being n or 1 being n and 8 being m. Both of these combinations, however, result in the same graph. Therefore, we consider them equivalent vertices.

Labeling

Let the vertices be named with alphabets a to n.

Start with e_1. Since the number of equivalent vertices in e_1 is 2, which are vertices 1 and 8,

$e_1 = 2$

The combinations of the alphabets a to n taken two at a time is ${}^{n}C_{e_1} = {}^{15}C_2$.

Moving to e_2. Since the number of equivalent vertices in e_2 is 2, which are vertices 9 and 10,

$e_2 = 2$

The combinations of the alphabets, excluding the ones that are taken by vertices 1 and 8 for a particular combination, taken two at a time is ${}^{n-e_1}C_{e_2} = {}^{13}C_2$.

Lastly, in e_3, there are 3 equivalent leaves, therefore,

$e_3 = 3$

The combinations of the alphabets, excluding the ones that are taken by sets e_1 and e_2 for particular sets of combinations, taken two at a time is ${}^{n-e_1-e_2}C_{e_3} = {}^{10}C_2$.

Labeling of the spine

Excluding the leaves which are parts of equivalent sets, there will be $(n - e_1 - e_2 - e_3)! = (15 - 2 - 2 - 3)!$ number of permutations for the labels of the vertices of the spine of the caterpillar.

Therefore, the total number of labeled non-isomorphic caterpillars possible is:

$${}^{15}C_2 * {}^{13}C_2 * {}^{10}C_3 * (10)!$$

References

[1]. Andrews, G., & Eriksson, K. (2004). *Integer partitions*. Cambridge University Press.

[2]. Hossain, M., Aziz, M., & Kaykobad, M. (2014). New Classes of Graceful Trees. *Journal Of Discrete Mathematics*, *2014*, 1-6. https://doi.org/10.1155/2014/194759