Homework 7

Problem 1:

a. $Mv_i = \sigma_i u_i$
b. $M^T u_i = \sigma_i v_i$
c. $M^T M v_i = \sigma_i M^T u_i \ (from\ a.) = \sigma_i^2 v_i \ (from\ b.)$
   $MM^T u_i = \sigma_i M v_i \ (from\ b.) = \sigma_i^2 u_i \ (from\ a.)$
d. $MM^T = U\Lambda V^T \ V\Lambda U^T = U\Lambda^2 U^T$
   $\sigma_1^2, \sigma_2^2, \dots \ \sigma_p^2$ are the eigenvalues of MM^T and $u_1, u_2, \dots u_p$ are the corresponding p eigenvectors
e. The matrix $M^T M$ can be written as: $M^T M = \sum_{i=1}^{p} \sigma_i^2 v_i . v_i^T$. From this equation and from part c., we can see that $M^T M$ has p eigenvalues $\sigma_1^2, \sigma_2^2, \dots \ \sigma_p^2$ and corresponding eigenvectors $v_{1,} \ v_2, \dots v_p$. The rest of the eigenvalues are 0.
   Thus $M^T M$ and MM^T have the same p eigenvalues
f. If M has rank k, then $\sigma_{k+1} = \sigma_{k+2} \dots = \sigma_p = 0$

Problem 2:

a. The best rank-1 approximation to M is: $\begin{pmatrix} 1.57454629 & 2.08011388 & 2.58568148 \\ 3.75936076 & 4.96644562 & 6.17353048 \end{pmatrix}$
b. The decomposition is not unique. We can find orthonormal vectors a and b such that $uv^T = ab^T$. This is when $a = -u \ and \ b = -v$
c. Best k-rank approximation of M can be written as:
   $\widehat{M} = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \dots + \sigma_k u_k v_k^T$

Problem 3:

The Gram matrix for the given data points is as follows: $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 3 \end{pmatrix}$

The other set of points with exactly the same Gram matrix are:

(-1, 0, 0), (-1, 0, -1), (-1, -1, 0), (-1, -1, -1). This is because, the Gram matrix B can be written as $B_{ij} = y_i . y_j$ and $y_i . y_j = -y_i . -y_j$

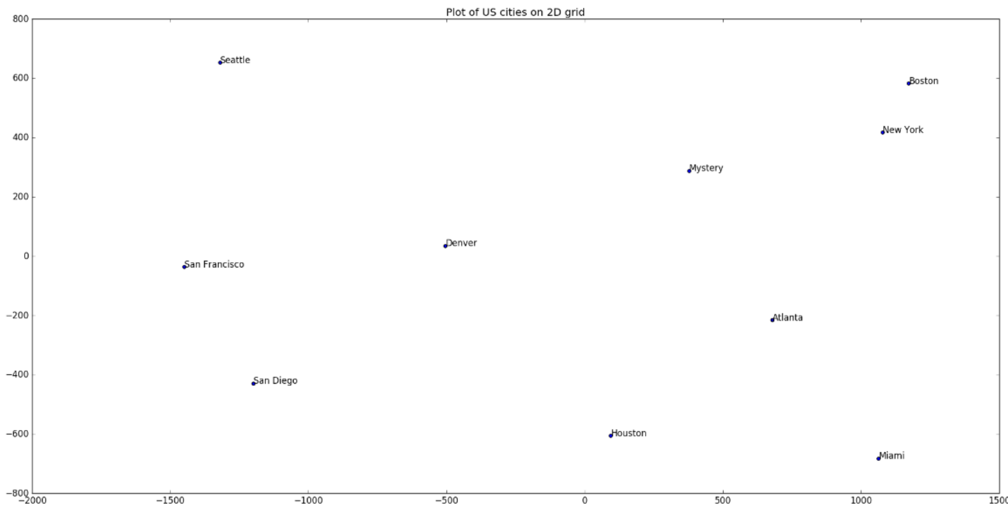
Thus the second solution is negative of the first solution.

Problem 4:

The plot of the cities was inverted with respect to the actual map of U.S cities. This was fixed by multiplying the 2D points by (-1). Thus we plot (-x, -y) instead of (x, y).

The two set of points have the same Gram matrix, since we can decompose the matrix B to get two solutions for calculating the square root of the Eigen value matrix (Λ): $\Lambda^{1/2}$ and $-\Lambda^{1/2}$

The following is the corrected plot of U.S. cities:



The mystery city appears to be **Chicago**.