SYNOPSIS

ON APPROXIMATE SOLUTIONS OF FRACTIONAL CONVECTION-DIFFUSION EQUATION AND ITS APPLICATION

Submitted to

SAVITRIBAI PHULE PUNE UNIVERSITY, PUNE.

For the Partial Fulfilment of the degree of Doctor of Philosophy in Mathematics under the Faculty of Science

By

Research Student

DILIP BAPURAO GHULE
Eknath Sitaram Divekar College of Arts,
Science and Commerce, Varvand,
Tal.-Daund, Dist.- Pune 412215.

Under the Guidance of

Dr. KALYANRAO C. TAKALE
Department of Mathematics,
R.N.C. Arts, J.D.B. Commerce and
N.S.C. Science College, Nashik-Road,
Dist.- Nashik 422101

January-2016

1 Rationale and Significance

Now a days, fractional calculus is a new field of mathematical study that deals with the investigation and applications of derivatives and integrals of non integer orders. Fractional differential equations arise as a mathematical modeling of systems and processes in the fields of physics, chemistry, biology, economics, control theory, signal and image processing, bio-physics, polymer rheology, aerodynamics etc. The model problems are very difficult to handle and to obtain its analytical solutions. Recently, researchers develop some fractional order finite difference schemes and iterative methods for fractional partial differential equations and obtain its solution [7, 10].

The convection-diffusion equation is a combination of the diffusion and convection (advection) equations, and describes physical phenomena where particles, energy, or other physical quantities are transferred inside a physical system due to two processes such as diffusion and convection. Because of this, the same equation can be called the advection-diffusion equation, drift-diffusion equation, [1] or (generic) scalar transport equation [4]. The convection-diffusion equation (with no sources or drains, $R = 0$) can be viewed as a stochastic differential equation, describing random motion with diffusivity D and bias. The convection-diffusion equation and these type of equations occur in finance, science and technology etc. Therefore, the convection-diffusion equation is an active part of research.

2 Survey of Work

Many problems in science and engineering when formulated mathematically are readily express in terms of partial differential equations with initial and boundary conditions (IBVP's). For example, heat and moment transport problems, heat transfer and fluid flow problems, etc are governed by partial differential equations. These mathematical models is a representation or an abstract interpretation of physical reality that is amenable to analysis and calculation. Numerical simulation allows us to calculate the solutions of these models on a computer and therefore to simulate physical reality. It is not always possible to obtain analytical solution of every modeled problem. Also, sometimes it is very difficult to solve these modeled problems because of its nonlinearity and complex geometry. Therefore, there are some finite difference methods (FDM's) and decomposition methods exist to solve such modeled problems. For solving modeled problems in to partial differential equations interdisciplinary teamwork is essential. Modeling represents a considerable part of the work of an applied mathematics and requires a through knowledge, not only of mathematicians, but also scientific discipline to which it is applied. The differential equations occurs in science and engineering, because the laws of nature typically result in equations relating spatial and temporial changes in one or more variables. Due to complexity of this differential equation it is very difficult to find exact solution of these equations. Therefore, researchers preferring finite difference schemes and iterative methods to obtain the approximate the solutions of fractional partial differential equations in physical sciences and engineering.

The following definitions are useful for the further developments of fractional order finite difference schemes.

Definition 2.1 ***(Grunvwald-Letnikov derivative):*** *Functional operater ${}_cD_t^\alpha$,for a constant $\alpha \in R$, is defined by*

$$ {}_cD_t^\alpha u(t) = \lim_{h \to 0} \frac{\sum_{k=0}^{[\frac{t-c}{h}]} [(-1)^k]^\alpha C_k f(t-kh)}{h^\alpha} $$

where [x] is integer part of x, is called Grunvwald-Letnikov derivative.

Definition 2.2 ***(Riemann-Liouville fractional integral):*** *[6, 7]*
Let $u \in C_\alpha$ and $\alpha \geq -1$ then Riemann-Liouville fractional integral of $u(x,t)$ of order α is denoted by $I^\alpha u(x,t)$ and is defined as

$$ I^\alpha u(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(x,\tau)}{(t-\tau)^{1-\alpha}}\, d\tau, \quad t > 0, \quad \alpha > 0. $$

Definition 2.3 ***(Riemann-Liouville fractional derivative):****[6, 7]*
Riemann-Liouville fractional derivative with respect to the time variable t, defined by

$$ {}_a^RD_t^\alpha u(t,x) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial x} \int_a^t \frac{u(t,\tau)}{(t-\tau)^\alpha}\, d\tau, \quad 0 < \alpha < 1. $$

is called Riemann-Liouville fractional derivative of order α.

In the year 1967, Caputo, introduced new and useful definition of fractional derivative popularly known as Caputo fractional derivative, we define it as follows

Definition 2.4 *The Caputo fractional derivative of $u(x,t)$is denoted by ${}_a^CD_x^\beta u(x,t)$ and is defined as [6, 7]*

$$ {}_a^CD_x^\beta u(x,t) = \begin{cases} \frac{1}{\Gamma(n-\beta)} \int_a^t \frac{1}{(t-s)^{\beta-n+1}} \frac{\partial^n u(x,s)}{\partial s^n}\, ds, & n-1 < \beta < n \\ \frac{\partial^n u(x,t)}{\partial t^n}, & \beta = n \in N. \end{cases} $$

Furthermore, we shall study convection-diffusion equation and these type of partial differential equations and fractional partial differential equations. The general convection-diffusion equation is

$$ -\frac{\partial U}{\partial t} + \sigma(x,t)\frac{\partial^2 U}{\partial x^2} + \mu(x,t)\frac{\partial U}{\partial x} + b(x,t)U = f(x,t) \; in \, D $$

initial and boundary conditions

$$ U(x,0) = \psi(x),\ x \in \Omega $$
$$ U(A,t) = g_0(t),\ U(B,t) = g_1(t),\ t \in (0,T) $$

where $\Omega = (A, B) and D = \Omega \times (0, T)$.
The convection-diffusion equation is a relatively simple equation describing flows, or alternatively, describing a stochastically-changing system. Therefore, the same or similar equation arises in many contexts unrelated to flows through space are as follows:
(i) It is formally identical to the Fokker-Planck equation for the velocity of a particle.
(ii) It is closely related to the Black-Scholes equation and other equations in financial mathematics.
(iii) It is closely related to the Navier-Stokes equations.
The following Black-Scholes European call option initial (final) boundary value problem (BSECOP-FBVP) studied by A. S. Shinde and K. C. Takale (2012). This is the similar type of convection-diffusion equation.

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - rC = 0,\ (S, t)\ \varepsilon\ R^+ \times (0, T)$$
$$\text{initial condition:} C(S, T) = max(S - E, 0),\ \text{S}\ \varepsilon\ R^+$$
$$\text{boundary conditions:} C(0, t) = 0,$$
$$C(S, T) = S - Ee^{-r(T-t)},\ \text{as}\ S \to \infty,\ \text{t}\ \varepsilon\ [0, T]$$

A. S. Shinde and K. C. Takale: Study of Black-Scholes Model and its Applications, Proceidia Engineering, ScienceDirct, Vol.38, (2012).

3 Aims and Objectives

(i) To study the Convection-Diffusion equation and similar type of partial differential equations.

(ii) To study the Fractional Convection-Diffusion equation and similar type of fractional partial differential equations.

(iii) To obtain the approximate solutions of Fractional Convection-Diffusion equation and similar type of fractional partial differential equations by fractional order finite difference schemes and iterative methods. Also, we study the stability and convergence of the schemes.

(iv) To simulate the solution using mathematical software Mathematica.

4 Methodology and Technique

Methodology consists of the following steps

(i) To study the fractional partial differential equations and to derive the finite difference formulae.

(ii) To study the Convection-Diffusion equation and similar type of partial differential equations. Also, we obtain the numerical solutions of Convection-Diffusion equation and similar type of partial differential equations by finite difference schemes and iterative methods.

(iii) To study the Fractional Convection-Diffusion equation and similar type of fractional partial differential equations.

(iv) To develop the fractional order finite difference schemes and fractional iterative methods for Fractional Convection-Diffusion equation and similar type of fractional partial differential equations.

(v) To study the recent applications of Fractional Convection-Diffusion equation which occurs in science and technology.

(vi) The numerical solutions of these problems are simulated by Mathematica software.

(vii) We will publish the work in reputed Journals.

Conclusions expected: We shall develop the fractional iterative methods and fractional order finite difference schemes for fractional convection-diffusion equations and similar type of fractional partial differential equations. Also, we will discuss the stability and convergence of the schemes. Furthermore, we obtain the numerical solutions of fractional convection-diffusion equations and similar type of fractional partial differential equations and their solutions shall be represented graphically by mathematica.

5 Plan of Research

First Year

(i) To study partial differential equations and to obtain their analytical solution.

(ii) To study the Convection-Diffusion equations occurs in science and technology.

(iii) To develop the finite difference schemes and iterative methods for partial differential equations and Convection-Diffusion equation. Also, we obtain the numerical solutions of these equations.

(iv) To study the Mathematica software.

Second Year

(i) To study the fractional Convection-Diffusion equations occurs in science and technology.

(ii) To study the fractional order finite difference schemes and iterative methods available in the literature.

(iii) To develop the fractional order finite difference schemes and algorithms for fractional Convection-Diffusion equations. Also, we obtain its numerical solutions. Furthermore, we discuss the stability and convergence of the schemes upto the length.

Third Year

(i) To study the recent applications of fractional Convection-Diffusion equations.

(ii) To obtain its numerical solution and simulate these solutions by Mathematica software.

(iii) We will publish the work in reputed Journals.

Bibliography: In this section, we shall provide the detailed of the references which are used in the development of our work.

References

[1] **Ben Q. Li**, *Discontinuous Finite in Fluid Dynamics and Heat Transfer*, (2006).

[2] **Dean G. Duffy**, *Solutions of Partial Differential Equations*, CBS Publishers and Distributors 485, Jain Bhawan, Bholanath nagar, Shahdara, Delhi, India, (1988).

[3] **Duchateau P. and Zachmann D.**, *Applied Partial Differential Equations*, *Harper and Row*, New York, (1989).

[4] **Ewa Majchrzak, Lukasz Turchan**, *The Finite Difference Method for Transiant Convection Diffusion*, (2012).

[5] **HangGuang Sen, Wen Chen, YangQuang Chen**, *Variable-order fractional differential operators in Anomalous Diffusion Modeling*, (2009).

[6] **K. B. Oldham and J. Spanier**, *The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order*, Academic Press, (1974).

[7] **I. Podlubny**, *Fractional Differential equations*, Academic Press, San Diago, (1999).

[8] **Roberto Ayala and Alfredo Tuesta**, *Introduction to The Concepts and Applications of Fractional and Variable order differential Calculus*, May 13, (2007).

[9] **B. Ross (Ed)**, *Fractional Calculus and Its Applications, Lecture Notes in Mathematics*, Vol. 457, Springer-Verlag, New York, (1975).

[10] **S.G.Samko, B.Ross**, *Integration and differentiation to a variable fractional order*, Integr.Transf. Spec. F. 1, pp. 277-300, (1993).

DILIP BAPURAO GHULE
Research Student

Dr. KALYANRAO C. TAKALE
Supervisor