The LNM Institute of Information Technology, Jaipur

Mid Term Examination Solutions (Semester-I, 2012-13)

Mathematics-I

Date: 29^{th} September 2012 Maximum Marks:40 Duration: 1.5 hours

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1. (i) Use the Archimedean property of real numbers to show that [02 Marks]

$$\bigcap_{n\in\mathbb{N}} \left(0, \frac{1}{n}\right] = \Phi.$$

Sol. Suppose if possible $x \in \bigcap_{n\in\mathbb{N}} \left(0, \frac{1}{n}\right]$. Then for all $n \in \mathbb{N}$, $x \in \left(0, \frac{1}{n}\right]$, that is, for all $n \in \mathbb{N}$, $0 < x \leq \frac{1}{n}$. [01 Mark]

But by the Archimedean Property, if $x > 0$, there exists $n \in \mathbb{N}$ with $\frac{1}{n} < x$, this is a contradiction. Therefore, $\bigcap_{n\in\mathbb{N}} \left(0, \frac{1}{n}\right] = \Phi$. [01 Marks]

(ii) Let $x_1 = 1$ and $x_{n+1} = \left(\frac{n}{n+1}\right) x_n$ for all $n \geq 1$. Find x_2, x_3 and x_4. Show that the sequence (x_n) converges. [01+03 Marks]

Sol. given $x_1 = 1$, $x_{n+1} = \left(\frac{n}{n+1}\right) x_n$. Therefore,

$$x_2 = \frac{2}{3}, \quad x_3 = \frac{3}{4} \cdot \frac{2}{3} = \frac{1}{2}, \quad x_4 = \frac{4}{5} \cdot \frac{1}{2} = \frac{2}{5}. \qquad \text{[01 Mark]}$$

Clearly, $0 < x_n < 1$ for all $n \geq 1$. Moreover, $x_{n+1} - x_n = -\frac{1}{n+1}x_n < 0$. Therefore, (x_n) is decreasing. Hence it converges being monotonically decreasing and bounded below. [02 Marks]

2. (i) Determine the points of continuity/discontinuity for the function $f : \mathbb{R} \to \mathbb{R}$ defined by [03 Marks]

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Sol. If $x \neq 0$, the function f is continuous being the composition and product of continuous functions. [01 Mark]

If $x = 0$ then consider a sequence (x_n) such that $x_n \to 0$. Now $f(x_n) = x_n^2 \sin\left(\frac{1}{x_n}\right)$. [01 Mark]

Therefore,

$$\lim_{x\to\infty} f(x_n) = \lim_{x\to\infty} x_n \cdot \frac{\sin\left(\frac{1}{x_n}\right)}{\left(\frac{1}{x_n}\right)} \to 0 = f(0).$$

Hence f is continuous at 0. [01 Mark]

(ii) If $f(x)$ and $g(x)$ be continuous on $[0,1]$, and suppose that $f(0) < g(0)$, $f(1) > g(1)$. Show that there exists some $x \in (0,1)$ such that $f(x) = g(x)$. From this deduce that the equation

$$\frac{x+1}{3} = \sin\frac{\pi x}{2}$$

has a solution in $(0,1)$. [02+01 Marks]

Sol. Consider the function $h(x) = f(x) - g(x)$. Note that $h(x)$ is continuous being the difference of two continuous functions. Moreover, $h(0) = f(0) - g(0) < 0$ and $h(1) = f(1) - g(1) > 0$. Therefore, by using IVT there exists $x \in (0,1)$ such that $h(x) = 0$ i.e. $f(x) = g(x)$. [02 Marks]

The second part follows by taking $f(x) = \sin\frac{\pi x}{2}$ and $g(x) = \frac{x+1}{3}$ in the function $h(x)$ defined above. [01 Mark]

3. (i) Let $f, g : [1,2] \to \mathbb{R}$ be continuous on $[1,2]$ and differentiable on $(1,2)$. Suppose that $f(1) = 1$, $f(2) = 4$, $g(1) = 1$, $g(2) = 2$ and $g'(x) \geq 1$ for all $x \in (1,2)$. Show that $\exists\, c \in (1,2)$ such that $f'(c) \geq 3$. Further, if $g(\frac{3}{2}) = \frac{3}{2}$ show that there are distinct c_1, $c_2 \in (1,2)$ such that $f'(c_1)g'(c_2) + f'(c_2)g'(c_1) \geq 6$. [05 Marks]

(***Hint:-*** *You may think of using one of the Mean Value Theorems*)

Sol. Using Cauchy man value theorem there exists $c \in (1,2)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(2) - f(1)}{g(2) - g(1)} = 3,$$

Therefore, $f'(c) = 3g'(c)$ but $g'(c) \geq 1 \implies f'(c) \geq 3$. [02 Marks]

Given that $g(\frac{3}{2}) = \frac{3}{2}$. Applying CMVT to f and g on $[1, \frac{3}{2}]$ and $[\frac{3}{2}, 2]$, we see that there exists $c_1 \in (1, \frac{3}{2})$ and $c_2 \in (\frac{3}{2}, 2)$ such that

$$\frac{f'(c_1)}{g'(c_1)} = \frac{f(\frac{3}{2}) - f(1)}{g(\frac{3}{2}) - g(1)} = \frac{f(\frac{3}{2}) - 1}{\frac{3}{2} - 1} = 2f\left(\frac{3}{2}\right) - 2. \tag{1}$$

and

$$\frac{f'(c_2)}{g'(c_2)} = \frac{f(2) - f(\frac{3}{2})}{g(2) - g(\frac{3}{2})} = \frac{4 - f(\frac{3}{2})}{2 - \frac{3}{2}} = 8 - 2f\left(\frac{3}{2}\right). \tag{2}$$

Adding (1) and (2), we get

$$\frac{f'(c_1)}{g'(c_1)} + \frac{f'(c_2)}{g'(c_2)} = 6.$$

$$f'(c_1)g'(c_2) + f'(c_2)g'(c_1) = 6g'(c_1)g'(c_2) \geq 6.$$

Clearly c_1 and c_2 are distinct. [03 Marks]

4. (i) Let f be a differentiable function such that $f(0) = 0$ and $f'(0) = 1$. For a positive integer k show that

$$\lim_{x\to 0}\frac{1}{x}\left\{f(x)+f\left(\frac{x}{2}\right)+f\left(\frac{x}{3}\right)+\ldots+f\left(\frac{x}{k}\right)\right\}=1+\frac{1}{2}+\ldots+\frac{1}{k}.$$

[03 Marks]

Sol. We have

$$\begin{aligned}
&\lim_{x\to 0}\frac{1}{x}\left\{f(x)+f\left(\frac{x}{2}\right)+\ldots+f\left(\frac{x}{k}\right)\right\}\\
&=\lim_{x\to 0}\left\{\frac{f(x)-f(0)}{x}+\frac{1}{2}\frac{f\left(\frac{x}{2}\right)-f(0)}{\frac{x}{2}}+\ldots+\frac{1}{k}\frac{f\left(\frac{x}{k}\right)-f(0)}{\frac{x}{k}}\right\}\\
&=f'(0)+\frac{1}{2}f'(0)+\ldots+\frac{1}{k}f'(0)\\
&=1+\frac{1}{2}+\ldots+\frac{1}{k}.
\end{aligned}$$

(ii) Compute the limit $\lim\limits_{x\to\infty}\left(x^2-x^3\sin\left(\frac{1}{x}\right)\right)$. [03 Marks]

Sol. $x^2-x^3\sin\left(\frac{1}{x}\right)=\frac{1-x\sin\left(\frac{1}{x}\right)}{\frac{1}{x^2}}$. [01 Mark]

Therefore we can write $\lim\limits_{x\to\infty}\dfrac{1-x\sin\left(\frac{1}{x}\right)}{\frac{1}{x^2}}=\lim\limits_{y\to 0}\dfrac{1-\frac{\sin y}{y}}{y^2}$. [01 Mark]

This is equal to $\lim\limits_{y\to 0}\dfrac{y-\sin y}{y^3}=\lim\limits_{y\to 0}\dfrac{\sin y}{6y}=\dfrac{1}{6}$. [01 Mark]

5. Determine all values of x for which the series

$$\sum_{n=2}^{\infty}\frac{x^n}{n(\ln n)^2}$$

converges. Justify your answer with proper arguments. [08 Marks]

Sol. Here $a_n=\frac{x^n}{n(\ln n)^2}$. Therefore $\frac{a_{n+1}}{a_n}=\frac{n(\ln n)^2}{(n+1)(\ln(n+1))^2}x$. [01 Mark]

Now $\lim\limits_{x\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|=|x|$. [02 Marks]

Therefore, the given series is convergent for all $x : |x| < 1$ by ratio test. [01 Mark]

When $x = 1$ then $a_n \geq 0$, (a_n) is decreasing. Therefore by Cauchy's condensation test, the series $\sum\limits_{n=2}^{\infty}\dfrac{1}{n(\ln n)^2}$ converges iff the series $\sum\limits_{n}2^k a_{2k}=\sum\limits_{k}2^k\dfrac{1}{2^k(\ln 2^k)^2}=\sum\limits_{k}\dfrac{1}{k^2(\ln 2)^2}$, which is convergent. [02 Marks]

When $x=-1$ then the series converges since it converges absolutely by previous case.

Alternately, the Leibniz test also can be used here to conclude that the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^2}$ converges as $\frac{(1}{n(\ln n)^2} \geq 0$ and $\frac{1}{n(\ln n)^2} \downarrow 0$.
Hence the given series converges for all $x : |x| \leq 1$. [02 Marks]

6. (i) Let $f : [a, b] \longrightarrow \mathbb{R}$, $f, f', f'', \ldots, f^{(n-1)}$ be continuous on $[a, b]$ and suppose $f^{(n)}$ exists on (a, b). Then there prove that exists $c \in (a, b)$ such that

$$f(b) = f(a)+f'(a)(b-a)+\frac{f''(a)}{2!}(b-a)^2+\ldots+\frac{f^{(n-1)}(a)}{(n-1)!}(b-a)^{n-1}+\frac{f^{(n)}(c)}{n!}(b-a)^n.$$

[05 Marks]

Sol. Define

$$F(x) = f(b) - f(x) - f'(x)(b-x) - \frac{f''(x)}{2!}(b-x)^2 + \ldots + \frac{f^{(n-1)}(x)}{(n-1)!}(b-x)^{n-1}.$$

It is enough to show that $F(a) = \frac{(b-a)^n}{n!} f^{(n)}(c)$ for some $c \in (a, b)$, which will prove the theorem. Note that

$$F'(x) = -\frac{f^{(n)}(x)}{(n-1)!}(b-x)^{n-1}. \qquad \text{[02 Marks]} \qquad (3)$$

Define

$$G(x) = F(x) - \left(\frac{b-x}{b-a}\right)^n F(a). \qquad \text{[01 Mark]}$$

Then $G(a) = G(b) = 0$. Moreover, $G(x)$ is continuous and differentiable on (a, b) and hence by Rolle's theorem there exists some $c \in (a, b)$ such that

$$G'(c) = F'(c) + \frac{n(b-c)^{n-1}}{(b-a)^n} F(a) = 0. \qquad \text{[01 Mark]} \qquad (4)$$

From (3) and (4) we obtain that $\frac{f^{(n)}(c)}{(n-1)!}(b-c)^{n-1} = \frac{n(b-c)^{n-1}}{(b-a)^n} F(a)$. This implies that $F(a) = \frac{(b-a)^n}{n!} f^{(n)}(c)$. This completes the proof the theorem. [01 Mark]

(ii) For what values of x, can we replace $\sin x$ by $x - \frac{x^3}{6}$ with an error of magnitude less than or equal to 5×10^{-4}. Give reasons for your answer. [04 Marks]

Sol. By Taylor's theorem

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{5!} \cos c, \quad \text{where} \quad c \in (0, x).$$

[02 Marks]

Therefore,

$$\left| \sin x - \left(x - \frac{x^3}{6} \right) \right| = \left| \frac{x^5}{5!} \cos c \right| \leq \left| \frac{x^5}{5!} \right|.$$

[01 Mark]

This is less than $\frac{5}{10^4}$ if $|x| < \frac{5 \times 5!}{10^4} = \frac{6}{100} = 0.06$. [01 Mark]

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