The LNM Institute of Information Technology
Jaipur, Rajasthan

End Semester Exam : Solution Manual 19^{th} November 2016
Mathematics-I Time: 3 Hours, Maximum Marks: 100

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1. (a) Prove that every absolutely convergent series is convergent but converse is not true, give an example. **[6]**

Ans. Let $\sum a_n$ be a absolutely convergent series so $\sum |a_n|$ is convergent.
For each n, we have

$$-|a_n| \le a_n \le |a_n|$$

$$0 \le a_n + |a_n| \le 2|a_n|.$$

Given $\sum |a_n|$ is convergent so by comparison test $\Sigma(a_n + |a_n|)$ is convergent. **[2]**

Now

$$\Sigma a_n = \Sigma(a_n + |a_n|) - \Sigma|a_n|$$

Clearly, $\Sigma(a_n + |a_n|)$ and $\Sigma|a_n|$ are convergent. We know that the difference of two convergent series is again convergent. Hence $\sum a_n$ is convergent. **[2]**

Conversely, Consider the series $\Sigma_{n=1}^{n=\infty} \frac{(-1)^{n-1}}{n}$. This series converges by Leibnitz test, however the series $\Sigma_{n=1}^{n=\infty} \frac{1}{n}$ diverges. **[2]**

(b) Examine the following series for convergence: **[6]**

$$\frac{1}{2^3} - \frac{1}{3^3}(1+2) + \frac{1}{4^3}(1+2+3) - \frac{1}{5^3}(1+2+3+4) + ...\infty.$$

Ans. Here

$$x_n = \frac{(-1)^{n-1}n}{2(n+1)^2} = (-1)^{n-1}a_n.$$

Then

$$a_n - a_{n+1} = \frac{n^2+n-1}{2(n+1)^2(n+2)^2} > 0.$$

So $a_n > a_{n+1}$. **[3]**
Also $\lim_{n\to\infty} a_n = 0$. Thus by Leibnitz test Σa_n is convergent hence Σx_n is convergent. **[3]**

2. (a) Show that the equation $x^{13} + 7x^3 - 5 = 0$ has exactly one (real) root. **[6]**

Ans. Let$f(x) = x^{13} + 7x^3 - 5$. Then $f(0) < 0$ and $f(1) > 0$. By the IVP there is at least one root of $f(x) = 0$ in the interval $(0, 1)$. **[2]**

If there are two distinct roots in the interval $(0, 1)$, then by Rolle's theorem there is some $0 < x_0 < 1$ such that $f'(x_0) = 0$ which is not true since $f'(x) = 13x^{12} + 21x^2 > 0$ for all $x \in (0, 1)$. **[2]**

Moreover, observe that $f(x) < 0$ for all $x < 0$. Also $f(x) > 0$ for all $x > 1$. So equation $x^{13}+7x^3-5 = 0$ has exactly one (real) root. **[2]**

(b) Find the interval of concavity and convexity for the following function: **[6]**

$$f(x) = \frac{x^2 - 4}{x - 1}, x \in R - \{1\}$$

Ans. Let $f(x) = \frac{x^2-4}{x-1} = x + 1 - \frac{3}{x-1}$.

$f'(x) = \frac{3}{(x-1)^2}$ and $f''(x) = \frac{-6}{(x-1)^3}$. **[2]**

Since $f''(x) > 0$ for all $x < 1$. the function is convex on (Convex)$(-\infty, 1)$. **[2]**

Since $f''(x) < 0$ for all $x > 1$. the function is concave on (Concave)$(1, \infty)$ **[2]**.

3. (a) Using Taylor's theorem, for any $k \in \mathbb{N}$ and for all $x > 0$, show that

$$x - \frac{1}{2}x^2 + \ldots - \frac{1}{2k}x^{2k} < \log(1+x) < x - \frac{1}{2}x^2 + \ldots + \frac{1}{2k+1}x^{2k+1}.$$

[7]

Ans. By Taylor's theorem, $\exists\, c \in (0, x)$ s.t.

$$\log(1+x) = x - \frac{x^2}{2} + \ldots + \frac{(-1)^{n-1}}{n}x^n + \frac{(-1)^n}{n+1}\frac{x^{n+1}}{(1+c)^{n+1}}.$$

Note that, for any $x > 0$, $\frac{(-1)^n}{n+1}\frac{x^{n+1}}{(1+c)^{n+1}} > 0$ if $n = 2k$. Hence

$$x - \frac{1}{2}x^2 + \ldots - \frac{1}{2k}x^{2k} < \log(1+x). \tag{1}$$

Also for any $x > 0$ and $\frac{(-1)^n}{n+1}\frac{x^{n+1}}{(1+c)^{n+1}} < 0$ if $n = 2k+1$. Hence

$$\log(1+x) < x - \frac{1}{2}x^2 + \ldots + \frac{1}{2k+1}x^{2k+1}. \tag{2}$$

From equation (1) and (2), we have

$$x - \frac{1}{2}x^2 + \ldots - \frac{1}{2k}x^{2k} < \log(1+x) < x - \frac{1}{2}x^2 + \ldots + \frac{1}{2k+1}x^{2k+1}.$$

(b) Using Sandwich theorem, discuss the convergence of the following sequence: **[7]**

$$x_n = \frac{1}{n^2+1} + \frac{2}{n^2+2} + \cdots + \frac{n}{n^2+n}.$$

Ans. Note that

$$\frac{1+2+\ldots+n}{n+n^2} \le x_n \le \frac{1+2+\ldots+n}{1+n^2}.$$

[2]

$$\frac{n(n+1)}{2(n+n^2)} \le x_n \le \frac{n(n+1)}{2(1+n^2)}.$$

$$\frac{1}{2} \le x_n \le \frac{n(n+1)}{2(1+n^2)}.$$

[2]

By Sandwich theorem $x_n \to \frac{1}{2}$. **[2]**

4. (a) Prove or disprove that the every continuous function defined on a closed interval $[a, b]$ is Riemann integrable. **[6 marks]**

Ans. Since f is continuous on $[a, b]$, so f must be uniformly continuous on $[a, b]$. [1 marks]

Therefore, for every $\varepsilon > 0$, $\exists$ a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta \ \forall x, y \in [a, b]$.

. [1 marks]

Let P be a partition of $[a, b]$ such that $\Delta x_i < \delta$, $\forall i = 1, 2, \ldots, n$. Then

$$M_i - m_i \leq \varepsilon, \qquad \forall i = 1, 2, \ldots, n.$$

[2 marks]

Hence

$$U(P, f) - L(P, f) = \sum_{i=1}^{n} (M_i - m_i)\Delta x_i \leq \varepsilon(b - a).$$

Hence f is Riemann integrable on $[a, b]$. [2 marks]

(b) The Dirichlet function $f : [0, 1] \longrightarrow \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational,} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

Is the function f Riemann-integrable on $[0, 1]$? Justify your answer. **[6 marks]**

Ans. This function is not Riemann integrable. If $P = \{a = x_0 < x_1 < \ldots < x_n = b\}$ is a partition of $[0, 1]$, then

$$M_i = \sup_{x \in [x_{i-1}, x_i]} f = 1 \quad \text{and} \quad m_i = \inf_{x \in [x_{i-1}, x_i]} f = 0,$$

[2 marks]

since every interval of non-zero length contains both rational and irrational numbers. It follows that

$$U(P, f) = 1, \qquad L(P, f) = 0,$$

for every partition P of $[0, 1]$. [2 marks]

So $\sup L(P, f) = \underline{\int_{0}^{1}} f(x)dx = 0$ and $\inf U(P, f) = \overline{\int_{0}^{1}} f(x)dx = 1$ are not equal. [2 marks]

5. (*a*) If f and g are two continuous functions on $[a,b]$ and if $g(x) \geq 0$ for $x \in [a,b]$ then show that there exist $c \in [a,b]$ such that

$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx.$$

. **[6 marks]**

Ans. If $\int_a^b g(x)dx = 0$, the result follows.

Let $\int_a^b g(x)dx \neq 0$ and $m = \inf\{f(x) : a \leq x \leq b\}$ and $M = \sup\{f(x) : a \leq x \leq b\}$. Then $m \leq f(x) \leq M \, \forall x \in [a,b]$. [2 marks]

Now

$$m\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq M\int_a^b g(x)dx$$

[1 marks]

i.e.

$$m \leq \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} \leq M \, .$$

[1 marks]

Since, f is continuous, so by the intermediate value property, the result follows. [2 marks]

(*b*) Determine all those values of p for which $\int_1^\infty |x^p \cos x|dx$ converges. **[6 marks]**

Ans. Note that $\int_1^\infty |x^p \cos x|dx = \int_1^\infty \left|\frac{\cos x}{x^q}\right| dx$ for $q = -p$. [1 marks]

For $q \leq 0$, $|\cos x| \leq \left|\frac{\cos x}{x^q}\right|$, hence by comparison test $\int_1^\infty \left|\frac{\cos x}{x^q}\right| dx$ diverges. [1 marks]

Let $q > 1$. Since $\left|\frac{\cos x}{x^q}\right| \leq \frac{1}{x^q}$, by comparison test $\int_1^\infty \left|\frac{\cos x}{x^q}\right| dx$ converges. [1 marks]

Now consider $0 < q \leq 1$. Note that $\left|\frac{\cos x}{x^q}\right| \geq \left|\frac{\cos^2 x}{x^q}\right| \geq \frac{1 + \cos 2x}{2x^q}$. [1 marks]

By the Dirichlet test $\int_1^\infty \frac{\cos 2x}{2x^q}dx$ converges but $\int_1^\infty \frac{1}{2x^q}dx$ diverges, therefore $\int_1^\infty \left|\frac{\cos x}{x^q}\right| dx$ diverges. [2 marks]

6. (a) Consider the function

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Show that f_x is not bounded on closed unit disk centered at $(0,0)$. **[6 marks]**

Solution: For $(x,y) \neq (0,0)$, we have

$$f_x(x,y) = \frac{y^3 - x^2 y}{(x^2+y^2)^2} \qquad \textbf{[1 marks]}$$

To compute $f_x(0,0)$, note that $f(x,0) = 0$ for all $x \in \mathbb{R}$, therefore $f_x(0,0) = 0$. **[2 marks]**

$$f_x(x,y) = \begin{cases} \dfrac{y^3 - x^2 y}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Note that $f_x\left(0, \dfrac{1}{n}\right) = n \to \infty$. Therefore, f_x is unbounded on the closed unit disk. There are other possible paths approaching $(0,0)$, such that value of f becomes arbitrarily large or small along the path. One need to check carefully what students are writing. **[3 marks]**

Alternate Sol: Recall the theorem, "If the partial derivatives of $f(x,y)$ exist throughout $B_r(x_0,y_0)$ for some $r > 0$ and if either f_x or f_y is bounded on the disk $B_r(x_0,y_0)$ then f is continuous at (x_0,y_0)." From above theorem, if f is not continuous at $(0,0)$, then none of the partial derivatives is bounded on any disk centered at $(0,0)$. To prove discontinuity of f at $(0,0)$, consider the sequence $\left(\left(\dfrac{1}{n}, \dfrac{1}{n}\right)\right)$. Clearly it converge to $(0,0)$. But

$$f\left(\frac{1}{n}, \frac{1}{n}\right) = \frac{1}{2} \nrightarrow 0 = f(0,0)$$

Therefore, f is not continuous at origin.

(b) Consider the function

$$f(x,y) = \begin{cases} \dfrac{x^2y + xy^2}{|x|+|y|} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Check differentiability of f at origin. **[6 marks]**

Solution: $f(x,0) = 0$ for all $x \in \mathbb{R}$, therefore $f_x(0,0) = 0$. Since function is symmetric in $f(x,y) = f(y,x)$ therefore $f_y(0,0) = 0$. one can obtain the same thing by definition also. **[2 marks]**

Now consider the double limit. For $(h,k) \neq (0,0)$, consider

$$\begin{aligned} \frac{|f(0+h,0+k) - f(0,0) - hf_x(0,0) - kf_y(0,0)|}{\sqrt{h^2+k^2}} &= \frac{|f(h,k)|}{\sqrt{h^2+k^2}} \\ &= \frac{|h^2k + hk^2|}{(|h|+|k|)\sqrt{h^2+k^2}} \\ &\leq \frac{|h|^2|k|}{(|h|+|k|)\sqrt{h^2+k^2}} + \frac{|h||k|^2}{(|h|+|k|)\sqrt{h^2+k^2}} \\ &\leq |k| + |h| \qquad \textbf{[3 marks]} \end{aligned}$$

Therefore

$$\lim_{(h,k)\to(0,0)} \frac{|f(0+h,0+k) - f(0,0) - hf_x(0,0) - kf_y(0,0)|}{\sqrt{h^2+k^2}} = 0$$

Hence f is differentiable. **[1 marks]**

Increment Approach:

$$f(0+h,0+k) - f(0,0) = f(h,k) = \frac{h^2k + hk^2}{|h|+|k|} \quad = hf_x(0,0) + kf_y(0,0) + \epsilon_1(h,k)h + \epsilon_2(h,k)k$$

where

$$\epsilon_1(h,k) = \frac{hk}{|h|+|k|} \quad \epsilon_2(h,k) = \frac{hk}{|h|+|k|}$$

Note that there are other possible choices for $\epsilon_1(h,k)$ and $\epsilon_2(h,k)$. For example,

$$\epsilon_1(h,k) = \frac{hk+k^2}{|h|+|k|} \quad \epsilon_2(h,k) = 0$$

[2 marks]

Now we show that $\epsilon_1(h,k) \to 0$ as $(h,k) \to (0,0)$.

$$|\epsilon_1(h,k)| = \frac{|hk|}{|h|+|k|} \leq |h| \text{ or} |k| \implies \lim_{(h,k)\to(0,0)} \epsilon_1(h,k) = 0$$

$$|\epsilon_1(h,k)| = \frac{|hk+k^2|}{|h|+|k|} = \frac{|h+k||k|}{|h|+|k|} \leq |k| \implies \lim_{(h,k)\to(0,0)} \epsilon_1(h,k) = 0$$

Hence f is differentiable. **[2 marks]**.

7. Consider the function

$$f(x,y) = \begin{cases} \dfrac{x+y}{x-y} & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases}$$

(a) Find the iterated limits (if these exist)

$$\lim_{x\to 0}\left[\lim_{y\to 0} f(x,y)\right] \quad \text{and} \quad \lim_{y\to 0}\left[\lim_{x\to 0} f(x,y)\right]$$

[6 marks]

Solution: For $x_0 \neq 0$,

$$f(x_0,y) = \begin{cases} \dfrac{x_0+y}{x_0-y} & \text{if } y \neq x_0 \\ 0 & \text{if } y = x_0. \end{cases}$$

Therefore, $\lim_{y\to 0} f(x_0,y) = 1$ for all $x_0 \neq 0$. **[2 marks]**

Hence $\lim_{x\to 0}\left[\lim_{y\to 0} f(x,y)\right] = 1.$ **[1 marks]**.

For $y_0 \neq 0$,

$$f(x,y_0) = \begin{cases} \dfrac{x+y_0}{x-y_0} & \text{if } x \neq y_0 \\ 0 & \text{if } x = y_0. \end{cases}$$

Therefore, $\lim_{x\to 0} f(x,y_0) = -1$ for all $y_0 \neq 0$. **[2 marks]**

Hence $\lim_{y\to 0}\left[\lim_{x\to 0} f(x,y)\right] = -1.$ **[1 marks]**

(b) Find the directional derivative of f (if it exists) at $(0,0)$ in the direction of the vector $v = (1,2)$. **[6 marks]**

Solution: The given vector is not unit vector hence in order find directional derivative in the direction of the vector $(1,2)$ we find its unit vector. $|v| = \sqrt{5}$ Hence unit vector in direction of v would be the vector $u = \left(\dfrac{1}{\sqrt{5}}, \dfrac{2}{\sqrt{5}}\right)$. **[2 marks]**

For $t \neq 0$, consider

$$\frac{f\left(0+\frac{t}{\sqrt{5}}, 0+\frac{2t}{\sqrt{5}}\right) - f(0,0)}{t} = \frac{f\left(\frac{t}{\sqrt{5}}, \frac{2t}{\sqrt{5}}\right)}{t} = \frac{-3}{t}$$

As $t \to 0$, limit does not exists. Therefore directional derivative does not exists. **[4 marks]**

8. (a) Find the absolute minimum and the absolute maximum of the function f given by $f(x, y) := 2x^2 - 4x + y^2 - 4y + 1$ on the closed triangular region bounded by the lines given by $x = 0, y = 2$, and $y = 2x$. **[10 marks]**

Solution: Vertices's of the triangle D are $(0, 0), (0, 2), (1, 2)$. Following are the steps.

i. To locate points of absolute extrema, we first determine the critical points of f. Solve $\nabla f(x, y) = (0, 0)$. We get two simultaneous equation $4x - 4 = 0$ and $2y - 4 = 0$. Hence simultaneous solution is $(1, 2)$, which is not an interior point of the triangle. Therefore there are not critical points of f inside the triangle. So absolute maximum and minimum will be attained on the boundary of the triangular region. **[2 marks]**

ii. Now we identify the points on the boundary of D, where f possibly can have absolute extreme values.

A. We consider $f(0, y) = y^2 - 4y + 1, y \in [0, 2]$. So $f'(0, y) = 2y - 4$. Hence the there is no critical point, and the boundary points are $y = 0, 2$. Hence possible candidate of absolute extrema of f are $(0, 0), (0, 2)$. **[2 marks]**

B. We consider $f(x, 2) = 2x^2 - 4x + 5, x \in [0, 1]$. So $f'(x, 2) = 4x - 4$. Hence the there is no critical point, and the boundary points are $x = 0, 1$. Hence possible candidate of absolute extrema of f are $(1, 2)$. We already got the point $(0, 2)$ in previous step. **[2 marks]**

C. We consider $f(x, 2x) = 2x^2 - 4x + 4x^2 - 8x + 1 = 6x^2 - 12x + 1, x \in [0, 1]$. So $f'(x, 2) = 12x - 12$. Hence the there is no critical point, and the boundary points are $x = 0, 1$. Both points are already identified in previous step. **[2 marks]**

iii. We can now tabulate all the relevant values as follows.

(x, y)	$(0, 0)$	$(1, 2)$	$(0, 2)$
$f(x, y)$	1	-5	5

It follows that the absolute maximum of f on D is 5, which is attained at $(0, 2)$, and the absolute minimum of f on D is -5, which is attained at $(1, 2)$. **[2 marks]**

(b) Write the following iterated integral with the order of integration reversed.

$$\int_0^1 \left(\int_1^{e^x} dy \right) dx.$$

[4 marks]

Solution:

$$\int_1^e \left(\int_{\ln y}^1 dx \right) dy.$$