Post-Newtonian Binary Inspiral Waveform

NM Term Paper

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1 Introduction

[1]

The gravitational waveform for a binary system can be written in terms of the modes h_{lm} as

$$h(t) = h_+ - ih = \sum_{l=2}^{\infty} \sum_{m=-l}^{l} Y_{lm}^{-2}(\iota, \phi) h_{lm} \tag{1}$$

where Y_{lm}^{-2} are the spin-2 weighted sperical harmonics, which satisfy

$$Y_{lm}^{-2}(\theta, \phi) = (-1)^l Y_{l,-m}^{-2}(\pi - \theta, \phi) \tag{2}$$

and the complex modes satisfy

$$h_{l,m} = (-1)^l h_{l,-m}^* \tag{3}$$

The dominant mode is the $l = 2, m = \pm 2$ mode, i.e $h_{2,\pm 2}$, which is given in 3 post-Newtonian order by

$$\begin{split} h_{22} = -8\sqrt{\frac{5}{\pi}}\mu \exp(-2\iota\varphi)x\{1 - (\frac{107}{42} - \frac{55}{42}\eta)x \\ + (2\pi + 6\iota \ln(x/x_0)))x^{3/2} - (\frac{2173}{1512} + \frac{1096}{216}\eta - \frac{2047}{1512}\eta^2)x^2 \\ - ((\frac{107}{21} - \frac{34}{21}\eta)\pi + 24\iota\eta + \iota(\frac{107}{7} - \frac{34}{7}\eta)\ln(x/x_0))x^{5/2} + (\frac{27027409}{646800} - \frac{856}{105}\gamma_E \\ + 2/3\pi^2 - \frac{1712}{105}\ln(2) - \frac{428}{105}\ln(x) - 18(\ln(x/x_0))^2 - (\frac{278185}{33264} - \frac{41}{96}\pi^2)\eta - \frac{20261}{2772}\eta^2 \\ + \frac{114635}{99792}\eta^3 + \iota\frac{428}{105}\pi + 12\iota\pi\ln(x/x_0))x^3\} \end{split} \tag{4}$$

and the spin-2 weighted spherical harmonics for $l = 2, m = \pm 2$ are

$$Y_{2,\pm 2}^{-2}(\theta, \phi) = (1/8)(\sqrt{5/\pi})(1 \pm cos\theta)e^{\pm 2\iota\phi} \tag{5}$$

For the binary systems with component masses m_1 and m_2 in the point-particle post-Newtonian approximation, the total mass $M := m_1 + m_2$, the reduced mass $\mu := m_1 m_2/M$ and the symmetric mass ratio $\eta := \mu/M$, the post-Newtonian parameter $x := (GM\omega/c^3)^{2/3}$ where ω is the orbital angular momentum.

To get the phase evolution, the energy and flux functions are

$$\mathcal{E}(x) = -\frac{1}{2}\eta x\{1 - (\frac{3}{4} + \frac{1}{12}\eta)x - (\frac{27}{8} - \frac{19}{8}\eta + \frac{1}{24}\eta^2)x^2 - [\frac{675}{64} - (\frac{34445}{576} - \frac{205}{96}\pi^2)\eta + \frac{155}{96}\eta^2 + \frac{35}{5184}\eta^3]x^3\} \tag{6}$$

and

$$\mathcal{F}(x) = \frac{32}{5}\eta^2 x^5\{1 - (\frac{1247}{336} + \frac{35}{12}\eta)x + 4\pi x^{3/2} - (\frac{44711}{9072} - \frac{9271}{504}\eta - \frac{65}{18}\eta^2)x^2 - (\frac{8191}{672} + \frac{583}{24}\eta)\pi x^{5/2}$$
$$+ [\frac{6643739519}{69854400} + \frac{16}{3}\pi^2 - \frac{1712}{105}\gamma_E - \frac{856}{105}\ln 16x + (\frac{41}{48}\pi^2 - \frac{134543}{7776})\eta - \frac{94403}{3024}\eta^2 - \frac{775}{324}\eta^3]x^3$$
$$- (\frac{16285}{504} - \frac{214745}{1728}\eta - \frac{193385}{3024}\eta^2)\pi x^{7/2}\} \quad (7)$$

where $\gamma_E \approx 0.577216$ is the euler constant. These functions are known to 3 post-Newtonian order and to 3.5 post-Newtonian order respectively. The energy and flux equations are used to obtain the orbital phase as a function of time or frequency by different orbital evolution. Here I will do TaylorT4 orbital evolution. Here we use geometrized units, in which G = c = 1.

1.1 TaylorT1 Orbital Evolution:

In the TaylorT1 orbital evolution, we obtain the orbital phase $\varphi(t)$ by numerically integrating the system of ordinary differential equations

$$\frac{dx}{dt} = -\frac{c^3}{GM}\frac{\mathcal{F}}{d\mathcal{E}/dx} \quad (8)$$

$$\frac{d\varphi}{dt} = \frac{c^3}{GM}x^{3/2} \quad (9)$$

1.2 TaylorT4 Orbital Evolution:

TaylorT4 evolution scheme is the same as TaylorT1 except that the ratio $\mathcal{F}/(d\mathcal{E}/dx)$ is expanded in a power series, so we get two coupled set of ordinary differential equations:

$$\frac{dx}{dt} = \frac{64}{5}\frac{c^3}{GM}\eta x^5\{1 - (\frac{743}{336} + \frac{11}{4}\eta)x + 4\pi x^{3/2}$$
$$- (\frac{34103}{18144} - \frac{13661}{2016}\eta - \frac{59}{18}\eta^2)x^2 - (\frac{4159}{672} + \frac{189}{8}\eta)\pi x^{5/2}$$
$$+ (\frac{16447322263}{139708800} + \frac{16}{3}\pi^2 - \frac{1712}{105}\gamma_E - \frac{856}{105}\ln(16x)$$
$$+ (\frac{41}{48}\pi^2 - \frac{134543}{7776})\eta - \frac{94403}{3024}\eta^2 - \frac{775}{324}\eta^3)x^3$$
$$- (\frac{4415}{4032} - \frac{358675}{6048}\eta - \frac{91495}{1512}\eta^2)\pi x^{7/2}\} \quad (10)$$

and

$$\frac{d\varphi}{dt} = \frac{c^3}{GM}\frac{x^{3/2}}{M} \quad (11)$$

2 Problem Statement

So the problem statement is that

I will solve the coupled ODEs for TaylorT4 i.e equation (10) and (11), and will get the post-Newtonian parameter $x(t)$ and orbital phase $\varphi(t)$. And then I will use this solution to get the dominant mode $h_{2,\pm 2}$ upto 3 post-Newtonian order and finally will get the plus and cross polarisation.

Also I did the same for 1 PN order.

3 Numerical Method Used

So the numerical tools I used to solve the coupled ODEs is

- numpy
- scipy
- matplotlib.pyplot
- sympy
- from scipy.misc derivative

and the method I used to solve is **integrate.solveivp**

4 Results

4.1 Parameters

The values of the parameters I took

- mass1= mass2= 5(in solar mass)
- γ_E= 0.577216
- G= c= 1
- initial conditions for x and φ =[0.1, 0]
- θ= $\pi/3$ and ϕ= -$\pi/8$

4.2 Waveform using TaylorT4

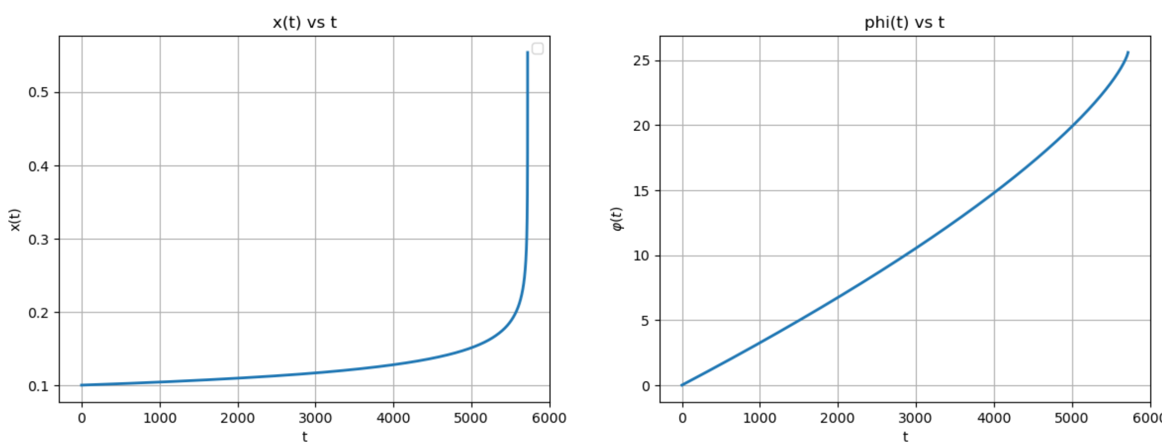


Figure 1: $x(t)$ and $\varphi(t)$ from TaylorT4

Now after getting the post-Newtonian parameter $x(t)$ and the orbital phase $\varphi(t)$, now I put these solution to the dominant mode $h_{2,\pm 2}$ and multiplying this with the spin -2 weighted spherical harmonics $Y_{2,\pm 2}^{-2}$. So we got $h(t)$ and the real part of $h(t)$ is our h_+ and the imaginary part of it is $h_\times$. Then I plot h_+ and $h_\times$ with time.

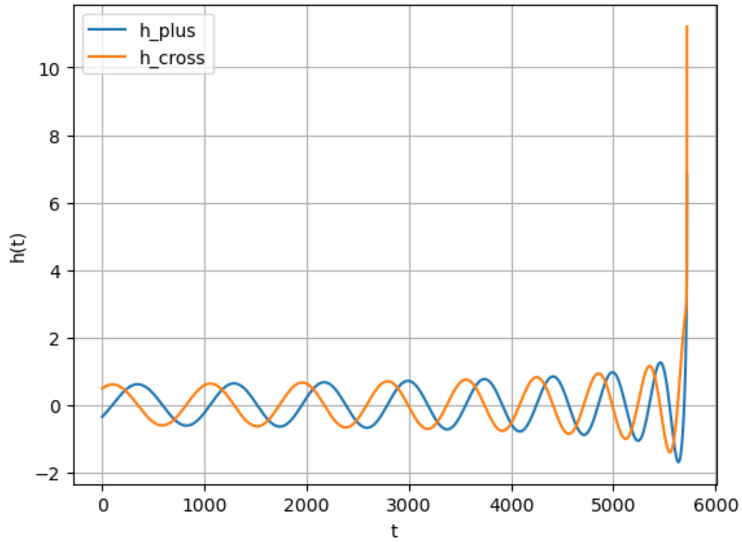


Figure 2: h_{plus} and h_{cross} from TaylorT4

4.3 Waveform for 1 PN order

For 1 PN order, the ODEs are

$$\frac{dv}{dt} = -\frac{\mathcal{F}(v)}{d\mathcal{E}/dv} \tag{12}$$

$$\frac{d\varphi}{dt} = \frac{v^3}{m} \tag{13}$$

here $v = x^{1/2}$. And the energy and flux are given in 1 PN order as

$$\mathcal{E}(v) = -\frac{1}{2}\eta v^2, \mathcal{F}(v) = \frac{32}{5}\eta^2 v^{10} \tag{14}$$

and the two gravitational wave polarizations can be written as

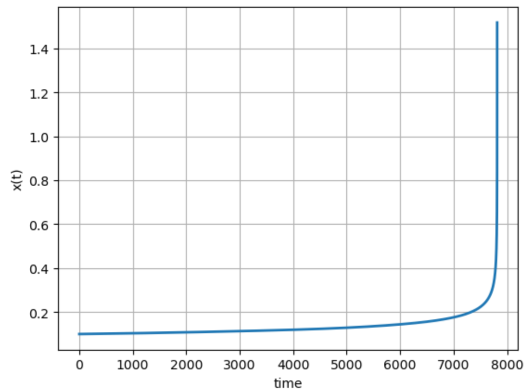
$$h_+ = 4\eta v^2 \cos\varphi, h_\times = 4\eta v^2 \sin\varphi \tag{15}$$

and here also I took the mass as m1=m2= 5(solar mass) and the initial conditions for v and φ is [0.3, 0].

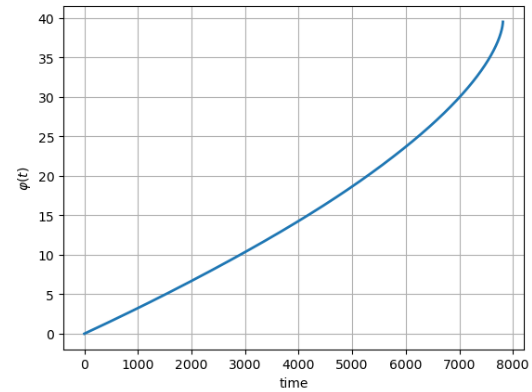
So from here we get the post-Newtonian parameter x and orbital phase φ also the polarizations and the plots are showing below.

4.4 Compare 1PN and 3PN order

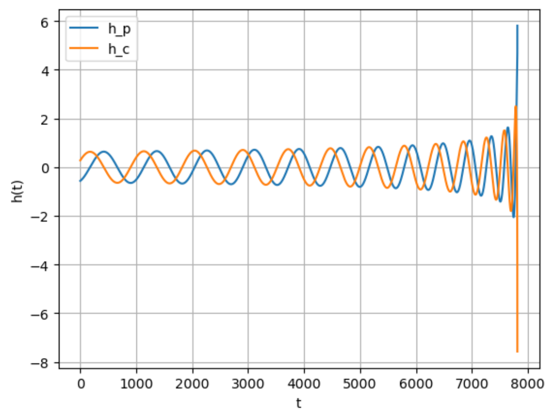
After getting waveform for both, I compare them. Although the time scale is different, but it can be shown that just near to merger the amplitude of the waveform are not matching, but long before to the merger the amplitude of the waveform are matching.



(a) $x(t)$ vs t



(b) $\varphi(t)$ vs t



(c) plus and cross polarizations

Figure 3: Waveform for 1PN order

5 Conclusion

So the main purpose to do higher PN order expansion is that we can get some physical information about the source. Let see the physical effects of PN phasing

- For 0PN, the got information about chirp mass.
- For 1PN, the information about possibility to measure component masses and Periastron advance
- For 1.5PN, its about tails of GWs, spin-orbit interaction
- For 2PN, spin-spin interaction, spin-induced quadropole
- For 2.5PN, Blackhole horizon flux
- For 3PN, tails of tails

and so on...
So the higher the PN order, the more information we can get for the source.

References

[1] Jolien DE Creighton and Warren G Anderson. *Gravitational-wave physics and astronomy: An introduction to theory, experiment and data analysis*. John Wiley & Sons, 2012.

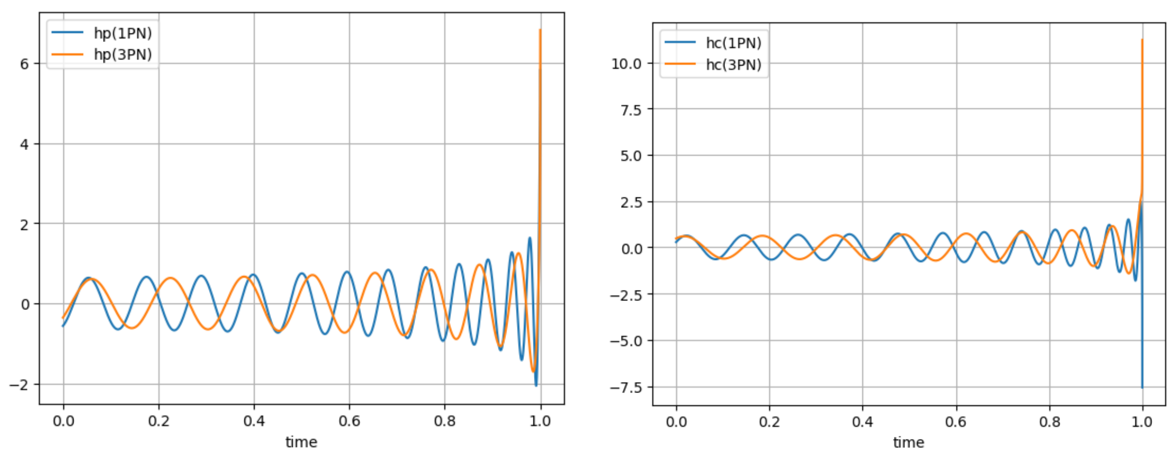


Figure 4: h_p and h_c for both 3PN and 1PN order