Unit-2

Design of FIR Filters:

i. Symmetric and anti-symmetric FIR Filters,

ii. FIR Filter structure: Direct form structure, cascade form structures, frequency sampling structures, lattice structure.

iii. Design of Linear phase FIR Filters using Windows,

iv. Design of Linear phase FIR filters by frequency Sampling method.

2.1 LINEAR PHASE FIR FILTER

A system is referred to as a generalized-linear-phase system if

$$H(e^{j\omega}) = A(e^{j\omega})e^{-j\alpha\omega + j\beta} \qquad 1$$

$A(e^{j\omega})$ is real function of ω. The phase is

$$\theta(\omega) = -\alpha\omega + \beta$$

Note the phase could be called affine phase since the phase is an affine transformation of ω.

The group delay is

$$grp[H(e^{j\omega})] = -\frac{d\theta(\omega)}{d\omega} = \alpha$$

Recall (O&S Problem 5.51) if 2α is an integer, we might have impulse response symmetry about α.

Since

$$H(e^{j\omega}) = A(e^{j\omega})[\cos(\beta - \omega\alpha) + j\sin(\beta - \omega\alpha)]$$

and

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]\cos(\omega n) - j\sum_{n=-\infty}^{\infty} h[n]\sin(\omega n)$$

we have

$$\tan(\beta - \omega\alpha) = \frac{\sin(\beta - \omega\alpha)}{\cos(\beta - \omega\alpha)} = -\frac{\sum_{n=-\infty}^{\infty} h[n]\sin(\omega n)}{\sum_{n=-\infty}^{\infty} h[n]\cos(\omega n)}$$

Therefore

$$\sum_{n=-\infty}^{\infty} h[n]\sin[\omega(n-\alpha) + \beta] = 0 \qquad \forall\omega \qquad 2$$

That is constant group delay implies Equation 1 so Equation 2 is a necessary condition for generalized linear phase. However Equation 2 is not a sufficient condition for generalized linear phase.

As examples, two types of generalized linear phase systems are given below:

Causal Generalized Linear-Phase Systems

Now we look at causal systems whose impulse response $h[n]$ is real.

If

$$h[n] = \begin{cases} h[M-n], & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases} \qquad 3)$$

then

$$H(e^{j\omega}) = A_e(e^{j\omega})e^{-j\omega M/2})$$

where $A_e(e^{j\omega})$ is a real, even and periodic function of ω.

If

$$h[n] = \begin{cases} -h[M-n], & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases} \qquad 4)$$

then

$$H(e^{j\omega}) = A_o(e^{j\omega})e^{-j\omega M/2})$$

where $A_o(e^{j\omega})$ is a real, odd and periodic function of ω.

Equations 3 and 4 are sufficient conditions for generalized linear-phase but they are not necessary conditions.

Type I FIR Linear-Phase Systems

Type I FIR's are symmetric about an integer.

A Type I FIR is characterized by

$$h[n] = h[M-n], \qquad 0 \le n \le M$$

and

$$M \text{ is an even integer}$$

Symmetric and anti-symmetric FIR Filters

We can show Type I FIR's have linear-phase by checking its Fourier Transform.

$$
\begin{aligned}
H(e^{j\omega}) &= \sum_{n=0}^{M} h[n]e^{-j\omega n} \\
&= \sum_{n=0}^{M/2-1} h[n]e^{-j\omega n} + h[M/2]e^{-j\omega M/2} + \sum_{n=M/2+1}^{M} h[n]e^{-j\omega n} \\
&= \sum_{n=0}^{M/2-1} h[n]e^{-j\omega n} + h[M/2]e^{-j\omega M/2} + \sum_{k=M/2+1}^{M} h[M-k]e^{-j\omega(M-k)} \qquad (k = M-n) \\
&= \sum_{n=0}^{M/2-1} h[n]e^{-j\omega n} + h[M/2]e^{-j\omega M/2} + \sum_{k=M/2+1}^{M} h[k]e^{-j\omega(M-k)} \qquad (h[k] = h[M-k]) \\
&= \sum_{n=0}^{M/2-1} h[n](e^{-j\omega n} + e^{-j\omega(M-n)}) + h[M/2]e^{-j\omega M/2} \\
&= e^{-j\omega M/2}\left[\sum_{n=0}^{M/2-1} h[n]2\cos(n-\frac{M}{2})\omega + h[M/2]\right]
\end{aligned}
$$

The first term $e^{-j\omega M/2}$ gives a phase of $-\omega M/2$ to $H(e^{j\omega})$. Since $h[n]$ is real, the second term in the product above contribute a phase of 0 or π to $H(e^{j\omega})$. So the overall phase of $H(e^{j\omega})$ is

$$-\omega\frac{M}{2} \quad \text{or} \quad -\omega\frac{M}{2}+\pi$$

The phase of $H(e^{j\omega})$ is linear by definition of linear-phase $-j\alpha+\beta$, where

$$\alpha = \frac{M}{2}, \quad \beta = 0 \text{ or } \pi$$

Symmetric and anti-symmetric FIR Filters

We can show Type II FIR's have linear-phase by checking its Fourier Transform.

$$
\begin{aligned}
H(e^{j\omega}) &= \sum_{n=0}^{M} h[n]e^{-j\omega n} \\
&= \sum_{n=0}^{(M-1)/2} h[n]e^{-j\omega n} + \sum_{n=(M+1)/2}^{M} h[n]e^{-j\omega n} \\
&= \sum_{n=0}^{(M-1)/2} h[n]e^{-j\omega n} + \sum_{k=(M+1)/2}^{M} h[M-k]e^{-j\omega(M-k)} \qquad (k = M-n) \\
&= \sum_{n=0}^{(M-1)/2} h[n]e^{-j\omega n} + \sum_{k=(M+1)/2}^{M} h[k]e^{-j\omega(M-k)} \qquad (h[k] = h[M-k]) \\
&= \sum_{n=0}^{(M-1)/2} h[n](e^{-j\omega n} + e^{-j\omega(M-n)}) \\
&= e^{-j\omega M/2}\left[\sum_{n=0}^{(M-1)/2} h[n]2\cos(n-\frac{M}{2})\omega\right]
\end{aligned}
$$

The first term $e^{-j\omega M/2}$ gives a phase of $-\omega M/2$ to $H(e^{j\omega})$. Since $h[n]$ is real, the second term in the product above contributes a phase of 0 or π to $H(e^{j\omega})$. So the overall phase of $H(e^{j\omega})$ is

$$-\omega\frac{M}{2} \quad \text{or} \quad -\omega\frac{M}{2}+\pi$$

The phase of $H(e^{j\omega})$ is linear by definition of linear-phase $-j\alpha+\beta$, where

$$\alpha = \frac{M}{2}, \quad \beta = 0 \text{ or } \pi$$

Type III FIR Linear-Phase Systems

Type III FIR's are anti-symmetric about an integer.

A Type III FIR is characterized by

$$h[n] = -h[M-n], \qquad 0 \le n \le M$$

and

$$M \text{ is an even integer}$$

Note that at $n = M/2$,

$$h[M/2] = -h[M-(M/2)] = -h[M/2]$$

So we have $h[M/2] = 0$.

We can show Type III FIR's have linear-phase by checking its Fourier Transform.

$$\begin{aligned}
H(e^{j\omega}) &= \sum_{n=0}^{M} h[n]e^{-j\omega n} \\
&= \sum_{n=0}^{M/2-1} h[n]e^{-j\omega n} + h[M/2]e^{-j\omega M/2} + \sum_{n=M/2+1}^{M} h[n]e^{-j\omega n} \\
&= \sum_{n=0}^{M/2-1} h[n]e^{-j\omega n} + \sum_{k=M/2+1}^{M} h[M-k]e^{-j\omega(M-k)} \qquad (h[M/2]=0, k=M-n) \\
&= \sum_{n=0}^{M/2-1} h[n]e^{-j\omega n} - \sum_{k=M/2+1}^{M} h[k]e^{-j\omega(M-k)} \qquad (h[k]=-h[M-k]) \\
&= \sum_{n=0}^{M/2-1} h[n](e^{-j\omega n} - e^{-j\omega(M-n)}) \\
&= e^{-j\omega M/2}\left[(-j)\sum_{n=0}^{M/2-1} h[n]2\sin(n-\frac{M}{2})\omega\right]
\end{aligned}$$

The first term $e^{-j\omega M/2}$ gives a phase of $-\omega M/2$ to $H(e^{j\omega})$. Since $h[n]$ is real, the second term in the product above contribute a phase of $\frac{\pi}{2}$ or $\frac{3\pi}{2}$ to $H(e^{j\omega})$. So the overall phase of $H(e^{j\omega})$ is

$$-\omega\frac{M}{2} + \frac{\pi}{2} \quad \text{or} \quad -\omega\frac{M}{2} + \frac{3\pi}{2}$$

The phase of $H(e^{j\omega})$ is linear by definition of linear-phase $-j\alpha + \beta$, where

$$\alpha = \frac{M}{2}, \quad \beta = 0 \text{ or } \pi$$

Type IV FIR Linear-Phase Systems

Type IV FIR's are anti-symmetric about the half of an integer.

A Type IV FIR is characterized by

$$h[n] = -h[M-n], \qquad 0 \le n \le M$$

and

$$M \text{ is an odd integer}$$

We can show Type IV FIR's have linear-phase by checking its Fourier Transform.

$$
\begin{aligned}
H(e^{j\omega}) &= \sum_{n=0}^{M} h[n]e^{-j\omega n} \\
&= \sum_{n=0}^{(M-1)/2} h[n]e^{-j\omega n} + \sum_{n=(M+1)/2}^{M} h[n]e^{-j\omega n} \\
&= \sum_{n=0}^{(M-1)/2} h[n]e^{-j\omega n} + \sum_{k=(M+1)/2}^{M} h[M-k]e^{-j\omega(M-k)} \qquad (k = M-n) \\
&= \sum_{n=0}^{(M-1)/2} h[n]e^{-j\omega n} - \sum_{k=(M+1)/2}^{M} h[k]e^{-j\omega(M-k)} \qquad (h[k] = -h[M-k]) \\
&= \sum_{n=0}^{(M-1)/2} h[n](e^{-j\omega n} - e^{-j\omega(M-n)}) \\
&= e^{-j\omega M/2}\left[(-j)\sum_{n=0}^{(M-1)/2} h[n]2\sin(n-\frac{M}{2})\omega\right]
\end{aligned}
$$

The first term $e^{-j\omega M/2}$ gives a phase of $-\omega M/2$ to $H(e^{j\omega})$. Since $h[n]$ is real, the second term in the product above contributes a phase of $\frac{\pi}{2}$ or $\frac{3\pi}{2}$ to $H(e^{j\omega})$. So the overall phase of $H(e^{j\omega})$ is

$$-\omega\frac{M}{2} + \frac{\pi}{2} \quad \text{or} \quad -\omega\frac{M}{2} + \frac{3\pi}{2}$$

The phase of $H(e^{j\omega})$ is linear by definition of linear-phase $-j\alpha + \beta$, where

$$\alpha = \frac{M}{2}, \quad \beta = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}$$