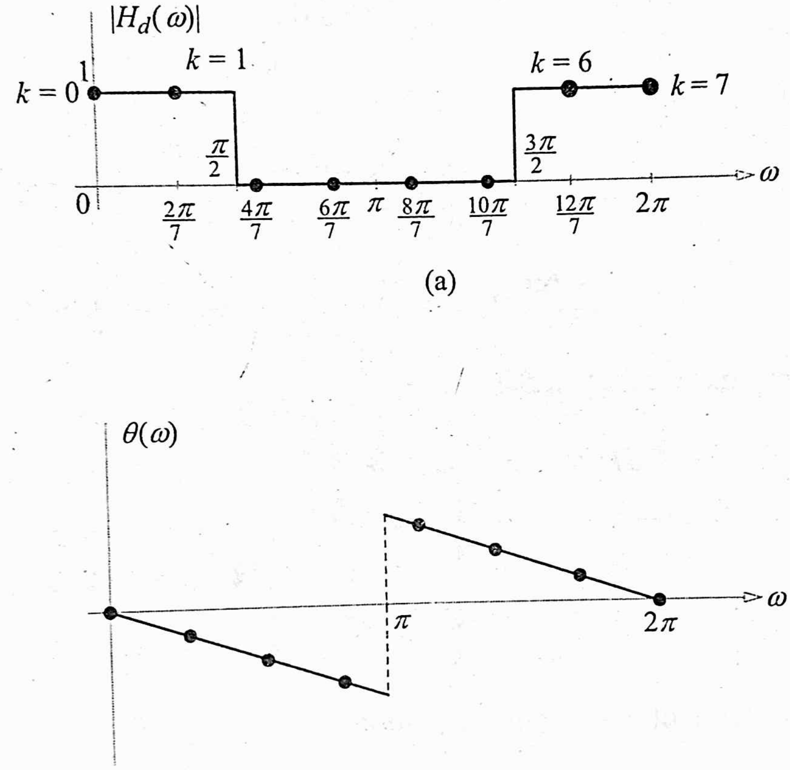
Example 5.19 A lowpass filter has the desired frequency response

$$H_d(\omega) = H_d(e^{j\omega}) = \begin{cases} e^{-j3\omega}, & 0 < \omega < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < \omega < \pi \end{cases}$$

Determine $h(n)$ based on frequency-sampling technique. Take $N = 7$.

☐ Solution



(b)

Fig. Ex.5.19 Magnitude and phase responses of the desired lowpass filter.

Let the ideal response of a linear-phase lowpass filter be

$$H_d(\omega) = \begin{cases} e^{-j\frac{(N-1)\omega}{2}}, & 0 < \omega < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < \omega < \pi \end{cases}$$

In the present context,

$$H_d(\omega) = \begin{cases} e^{-j3\omega}, & 0 < \omega < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < \omega < \pi \end{cases}$$

The ideal magnitude frequency response is taken to be symmetric about π while the ideal phase response is taken to be antisymmetric about π. The ideal magnitude and phase responses with samples taken for $N = 7$ are shown in Fig. Ex.5.19.

The samples of $|H_d(\omega)|$ and $\theta(\omega)$ are taken at $\omega = \omega_k = \frac{2\pi k}{N}$, $k = 0, \cdots N-1$. The range of k is found as follows:

i. for $0 \leqslant \omega < \frac{\pi}{2}$, the values of k are 0, 1

ii. for $\frac{\pi}{2} < \omega < \frac{3\pi}{2}$, the values of k are 2, 3, 4, 5

iii. for $\frac{3\pi}{2} < \omega < 2\pi$, the value of k is 6.

From Fig Ex.5.19(a), we find that

$$|H(k)| = \begin{cases} 1, & k = 0, 1 \\ 0, & k = 2, 3, 4, 5 \\ 1, & k = 6 \end{cases}$$

Also, from Fig. Ex.5.19(b), we find that

$$\theta_k = -3\omega_k \quad = \quad -3 \times \frac{2\pi}{N}k = \frac{-6\pi}{7}k \quad \text{for} \quad k = 0, 1, 2, 3$$

and

$$\theta_k = \frac{-6\pi}{7}(k-7) \qquad \text{for} \quad k = 4, 5, 6$$

Since $H(k)$ is complex, we may write

$$H(k) = |H(k)|e^{j\theta_k}$$

Hence,

$$H(k) = \begin{cases} e^{-j\frac{6\pi k}{7}}, & k = 0, 1 \\ 0, & k = 2, 3, 4, 5 \\ e^{-j\frac{6\pi}{7}(k-7)}, & k = 6 \end{cases}$$

We find the inverse DFT of $H(k)$ using equation (5.29). That is,

$$h(n) = \frac{1}{N}\left[H(0) + 2\sum_{k=1}^{\frac{(N-1)}{2}} \text{Re}\left\{H(k)e^{j\frac{2\pi nk}{N}}\right\}\right]$$

Hence,

$$\begin{aligned} h(n) &= \frac{1}{7}\left[H(0) + 2\sum_{k=1}^{3} \text{Re}\left\{H(k)e^{j\frac{2\pi nk}{7}}\right\}\right] \\ &= \frac{1}{7}\left[H(0) + 2\,\text{Re}\left\{H(1)e^{j\frac{2\pi n}{7}}\right\}\right] \\ &= \frac{1}{7}\left[1 + 2\,\text{Re}\left\{e^{-j\frac{6\pi}{7}}e^{j\frac{2\pi n}{7}}\right\}\right] \\ &= \frac{1}{7}\left[1 + 2\cos\left(\frac{2\pi}{7}(n-3)\right)\right], \quad 0 \leqslant n \leqslant 6 \end{aligned}$$

The filter coefficients are tabulator below:

n	$h(n)$	n	$h(n)$
0	−0.11456	4	0.320997
1	0.07928	5	0.07928
2	0.320997	6	−0.11456
3	0.42857		

Example 5.20 Design a 17-tap linear-phase FIR filter with a cutoff frequency $\omega_C = \frac{\pi}{2}$. The design is to be done based on frequency sampling technique.

☐ Solution

The ideal lowpass frequency response with a linear-phase is

$$H_d(e^{j\omega}) = H_d(\omega) = \begin{cases} e^{-j\frac{(N-1)\omega}{2}}, & 0 < \omega < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < \omega < \pi \end{cases}$$

The ideal magnitude response is taken to be even symmetric about π while the phase response is taken to be odd symmetric about π .

The ideal magnitude and phase responses with samples for $N = 17$ are shown in Fig. Ex.5.20. The heavy dots denote the frequency samples of $H_d(\omega)$ taken at $\omega = \omega_k = \frac{2\pi k}{N}$ for $k = 0, \cdots N-1$.

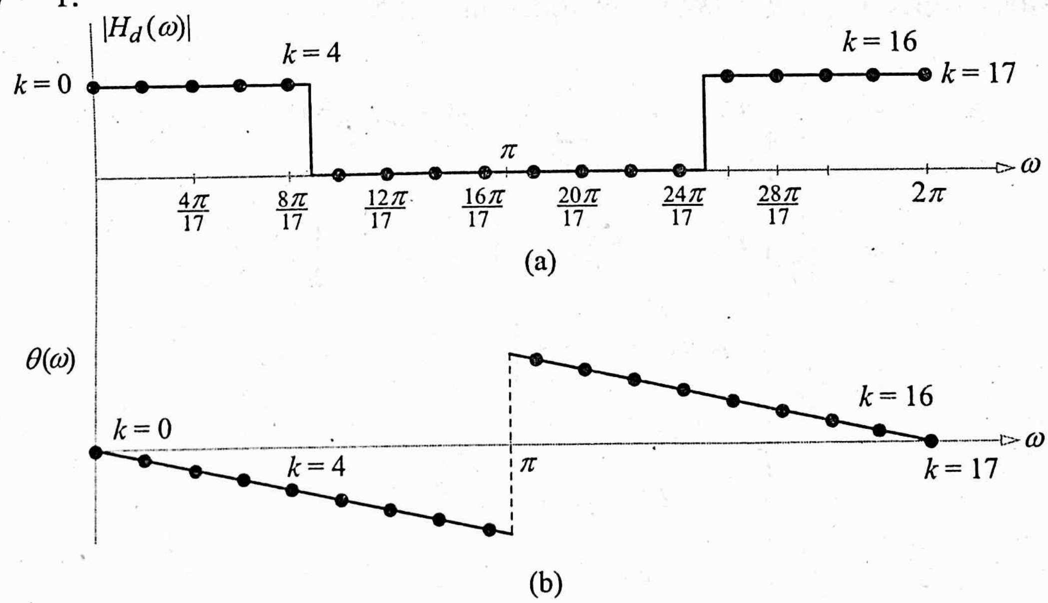


Fig. Ex.5.20 Magnitude and phase responses for $H_d(\omega)$.

In the present context,

$$H_d(\omega) = \begin{cases} e^{-j8\omega}, & 0 < \omega < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < \omega < \pi \end{cases}$$

The samples of $|H_d(\omega)|$ and $\theta(\omega)$ are taken at $\omega = \omega_k = \frac{2\pi k}{N}$, $k = 0, 1, \cdots N-1$. From Fig. Ex.5.20(a), we find that

$$|H(k)| = \begin{cases} 1, & 0 \leqslant k \leqslant 4 \\ 0, & 5 \leqslant k \leqslant 12 \\ 1, & 13 \leqslant k \leqslant 16 \end{cases}$$

Also, from Fig. Ex.5.20(b), we find that

$$\begin{aligned} \theta_k = -8\omega_k &= -8 \times \frac{2\pi k}{N} \\ &= \frac{-16\pi k}{17}, \quad 0 \leqslant k \leqslant 8 \end{aligned}$$

and $$\theta_k = \frac{-16\pi}{17}(k-17), \quad 9 \leqslant k \leqslant 16.$$

Since, $H(k)$ is complex, we may write

$$H(k) = |H(k)|e^{j\theta_k}$$

$$\Rightarrow \quad H(k) = \begin{cases} e^{-j\frac{16\pi k}{17}}, & 0 \leqslant k \leqslant 4 \\ 0, & 5 \leqslant k \leqslant 12 \\ e^{-j\frac{16\pi}{17}(k-17)}, & 13 \leqslant k \leqslant 16 \end{cases}$$

We find the inverse DFT of $H(k)$ using equation (5.29).

That is,
$$\begin{aligned} h(n) &= \frac{1}{N}\left[H(0) + 2\sum_{k=1}^{\frac{(N-1)}{2}} \text{Re}\left\{H(k)e^{j\frac{2\pi nk}{N}}\right\}\right] \\ &= \frac{1}{17}\left[H(0) + 2\sum_{k=1}^{8} \text{Re}\left\{H(k)e^{j\frac{2\pi nk}{17}}\right\}\right] \\ &= \frac{1}{17}\left[\begin{array}{c} H(0) + 2\left[\text{Re}\left\{H(1)e^{j\frac{2\pi n}{17}} + H(2)e^{j\frac{4\pi n}{17}}\right.\right. \\ \left.\left. + H(3)e^{j\frac{6\pi n}{17}} + H(4)e^{j\frac{8\pi n}{17}}\right\}\right] \end{array}\right] \\ &= \frac{1}{17}\left[\begin{array}{c} 1 + 2\,\text{Re}\left\{e^{-j\frac{16\pi}{17}}e^{j\frac{2\pi n}{17}} + e^{-j\frac{32\pi}{17}}e^{j\frac{4\pi n}{17}}\right. \\ \left. + e^{-j\frac{48\pi}{17}}e^{j\frac{6\pi n}{17}} + e^{-j\frac{64\pi}{17}}e^{j\frac{8\pi n}{17}}\right\} \end{array}\right] \\ &= \frac{1}{17}\left[\begin{array}{c} 1 + 2\cos\left[\frac{2\pi}{17}(n-8)\right] + 2\cos\left[\frac{4\pi}{17}(n-8)\right] \\ + 2\cos\left[\frac{6\pi}{17}(n-8)\right] + 2\cos\left[\frac{8\pi}{17}(n-8)\right] \end{array}\right], \quad 0 \leqslant n \leqslant 16 \end{aligned}$$

The FIR filter coefficients are tabulated below:

n	$h(n)$	n	$h(n)$	n	$h(n)$
0	0.0398	6	−0.0299	12	0.03154
1	−0.0488	7	0.31876	13	0.06598
2	−0.03459	8	0.5294	14	−0.03459
3	0.06598	9	0.31876	15	−0.0488
4	0.03154	10	−0.0299	16	0.0398
5	−0.10747	11	−0.10747		

Example 5.21 Determine the filter coefficients $h(n)$ obtained by sampling $H_d(\omega)$ given by

$$H_d(\omega) = \begin{cases} e^{-j3\omega}, & 0 < \omega < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < \omega < \pi \end{cases}$$

Also, obtain the frequency response $H(\omega)$. Take $N = 7$.

□ Solution

The computation of $h(n)$ is done in Example 5.19. The result is given below:

$$h(n) = \frac{1}{7}\left[1 + 2\cos\left(\frac{2\pi}{7}(n-3)\right)\right]$$

Let us now, proceed to find the frequency response $H(\omega)$.

We have, $$h(n) = \frac{1}{7}\left[1 + 2\cos\left(\frac{2\pi}{7}(n-3)\right)\right]$$

$$\Rightarrow \quad h(n) = \frac{1}{7}\left[1 + e^{j\frac{2\pi}{7}(n-3)} + e^{-j\frac{2\pi}{7}(n-3)}\right]$$

We know that $$H(\omega) = \sum_{n=0}^{N-1} h(n)e^{-j\omega n}$$

$$\Rightarrow \quad H(\omega) = \frac{1}{7}\sum_{n=0}^{6}\left[1 + e^{j\frac{2\pi}{7}(n-3)} + e^{-j\frac{2\pi}{7}(n-3)}\right]e^{-j\omega n}$$

We know that $$\sum_{n=0}^{N-1} e^{-j\omega n} = e^{-j\frac{(N-1)\omega}{2}}\frac{\sin\left(\frac{N\omega}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}$$

$$\Rightarrow \quad \sum_{n=0}^{6} e^{-j\omega n} = e^{-j3\omega}\frac{\sin\left(\frac{7\omega}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}$$

Hence,

$$
\begin{aligned}
7\,H(\omega) &= \sum_{n=0}^{6} e^{-jn\omega} + e^{-j\frac{6\pi}{7}} \sum_{n=0}^{6} e^{-jn\left(\omega-\frac{2\pi}{7}\right)} \\
&\quad + e^{j\frac{6\pi}{7}} \sum_{n=0}^{6} e^{-jn\left(\omega+\frac{2\pi}{7}\right)} \\
&= e^{-j3\omega} \left[\begin{array}{c} \frac{\sin\left(\frac{7\omega}{2}\right)}{\sin\left(\frac{\omega}{2}\right)} + e^{-j\frac{6\pi}{7}} e^{-j3\left(\frac{-2\pi}{7}\right)} \frac{\sin\left[\frac{7}{2}\left(\omega-\frac{2\pi}{7}\right)\right]}{\sin\left[\frac{1}{2}\left(\omega-\frac{2\pi}{7}\right)\right]} \\ + e^{j\frac{6\pi}{7}} e^{-j3\left(\frac{2\pi}{7}\right)} \frac{\sin\left[\frac{7}{2}\left(\omega+\frac{2\pi}{7}\right)\right]}{\sin\left[\frac{1}{2}\left(\omega+\frac{2\pi}{7}\right)\right]} \end{array} \right]
\end{aligned}
$$

$$
\Rightarrow \qquad H(\omega) = \frac{1}{7} e^{-j3\omega} \left[\frac{\sin\left(\frac{7\omega}{2}\right)}{\sin\left(\frac{\omega}{2}\right)} + \frac{\sin\left[\frac{7}{2}\left(\omega-\frac{2\pi}{7}\right)\right]}{\sin\left[\frac{1}{2}\left(\omega-\frac{2\pi}{7}\right)\right]} + \frac{\sin\left[\frac{7}{2}\left(\omega+\frac{2\pi}{7}\right)\right]}{\sin\left[\frac{1}{2}\left(\omega+\frac{2\pi}{7}\right)\right]} \right]
$$

5.9.1 Advantages and Disadvantages of frequency sampling design

- The realization of FIR filter given in Fig. 5.16 is suitable for implementation in parallel processors, where each processor performs a maximum of two complex multiplications per sample.
- Ad The filter realization does not require an inverse DFT, which may be computationally tedious for long filters.
- For single-processor implementation, the FIR realization given in Fig. 5.16 requires $(2N-1)$ complex multiplications and 1 real multiplication per sample, whereas a Direct form-I FIR realization needs only N real multiplications.
- The pole-zero cancellation may not actually happen when the filter is implemented in a finite wordlength processor. This may lead to instability.

5.10 Equiripple Filters

The design of an FIR lowpass filter using the window design technique is simple and generally results in a filter with relatively good performance. However, the window design does not yield optimal filters. The reasons for not being optimal are:

1. The passband and stopband deviations, δ_P and δ_S, are approximately equal. Eventhough, it is common to require δ_S to be much smaller than δ_P, these parameters cannot be independently controlled in the window design method. Hence, with the window design method, it is necessary to over design the filter in the passband in order to meet the stricter requirements in the stopband.
2. For most windows, the ripple is not uniform either in the passband or in the stopband and generally decreases when moving away from the transistion band. Allowing the ripple to be uniformly spread over the entire band would result in a smaller peak ripple.