Obtain the coefficients of an FIR filter to meet the specifications given belo
using the window method.

Passband edge frequency: 1.5 KHz

Stopband edge frequency: 2 KHz

Minimum stopband attenuation: 50 dB

Sampling frequency: 8 KHz

Solution

We are required to design a linear-phase FIR filter $H(z)$ to be used in the A/D – H(z) – D/A structure shown in Fig. Ex.5.7 so that the cascade combination behaves like an equivalent analog filter having the following specifications.

$$\Omega_P = 2\pi \times 1.5 \times 10^3 \text{ rad/sec}; \quad \Omega_S = 2\pi \times 2 \times 10^3 \text{ rad/sec}$$

$$K_S = -A_S = -50 \text{ dB}$$

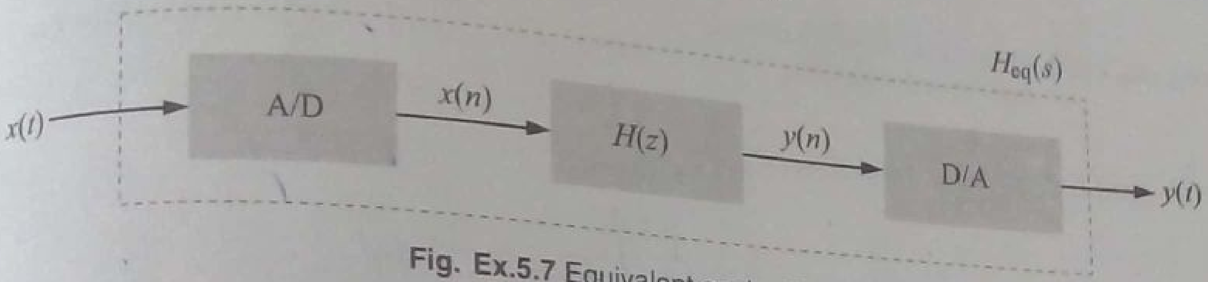

Fig. Ex.5.7 Equivalent analog filter.

Converting the above analog specifications into equivalent digital specifications using the formula, $\omega = \Omega T$ with $T = \frac{1}{8\times10^3}$ sec, we get

$$\omega_P = \Omega_P T = \frac{2\pi \times 1.5 \times 10^3}{8 \times 10^3} = 0.375\,\pi \text{ rad}$$

$$\omega_S = \Omega_S T = \frac{2\pi \times 2 \times 10^3}{8 \times 10^3} = 0.5\,\pi \text{ rad}$$

Step 1: Table 5.1 indicates that the Hamming or Blackman window will satisfy the stopband attenuation requirement. Hamming window is selected since it has a lower transistion width than the Blackman window and thus giving the smallest value of N.

Step 2: The length of window is computed using the expression

$$\omega_S - \omega_P \geqslant k\frac{2\pi}{N}$$

Since, $k = 4$ for the Hamming window, we get

$$0.5\pi - 0.375\pi \geqslant \frac{8\pi}{N}$$

$$\Rightarrow \quad N \geqslant \frac{8\pi}{0.5\pi - 0.375\pi}$$

$$\Rightarrow \quad N \geqslant 64$$

Let $N = 65$, so that $\alpha = \frac{N-1}{2}$ is an integer.

Step 3: Let the magnitude and phase responses of the desired lowpass filter be as shown in Fig. Ex.5.7(a) and Fig. Ex.5.7(b) respectively.

Then,
$$H_d(\omega) = \begin{cases} e^{-j\omega\alpha}, & |\omega| < \omega_C \\ 0, & \omega_C < |\omega| < \pi \end{cases}$$

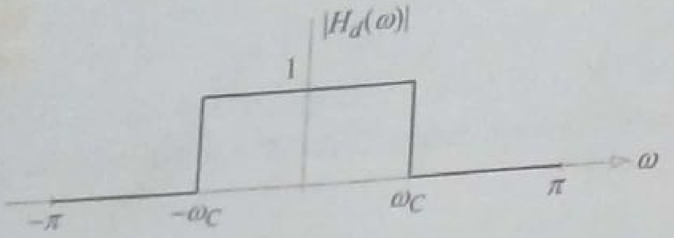

Fig. Ex.5.7(a) Magnitude response of the desired lowpass filter.

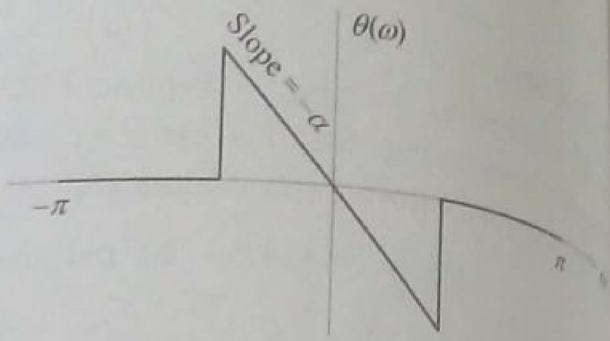

Fig. Ex.5.7(b) Phase response of the desired lowpass filter.

Step 4: To find the impulse response, $h_d(n)$.

$$h_d(n) \triangleq \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)e^{j\omega n}\,d\omega$$

$$\Rightarrow \quad h_d(n) = \frac{1}{2\pi}\int_{-\omega_C}^{\omega_C} e^{-j\omega\alpha}\,e^{j\omega n}\,d\omega$$

$$= \frac{\sin[\omega_C(n-\alpha)]}{\pi(n-\alpha)}, \quad n \neq \alpha$$

Also, $$h_d(\alpha) = \frac{1}{2\pi}\int_{-\omega_C}^{\omega_C} e^0\,d\omega = \frac{\omega_C}{\pi}$$

Step 5:

Select $$\omega_C = \omega_P + \frac{\Delta\omega}{2}$$

$$= 0.375\pi + \frac{(0.5\pi - 0.375\pi)}{2}$$

$$= 0.4375\pi \text{ rad}$$

Also, select $$\alpha = \frac{N-1}{2} = \frac{65-1}{2} = 32$$

Step 6: Finally, the impulse response $h(n)$ of the FIR filter is

$$h(n) = h_d(n)\,w_{\text{Ham}}(n), \quad 0 \leqslant n \leqslant N-1$$

where $$w_{\text{Ham}}(n) = 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right), \quad 0 \leqslant n \leqslant N-1$$

$$\Rightarrow \quad w_{\text{Ham}}(n) = 0.54 - 0.46\cos\left(\frac{2\pi n}{64}\right), \quad 0 \leqslant n \leqslant 64$$

5.1 A lowpass filter is to be designed with the following desired frequency response:

$$H_d\left(e^{j\omega}\right) = H_d(\omega) = \begin{cases} e^{-j2\omega}, & |\omega| < \frac{\pi}{4} \\ 0, & \frac{\pi}{4} < |\omega| < \pi \end{cases}$$

Determine the filter coefficients $h_d(n)$ and $h(n)$ if $w(n)$ is a rectangular window defined as follows:

$$w_R(n) = \begin{cases} 1, & 0 \leqslant n \leqslant 4 \\ 0, & \text{otherwise} \end{cases}$$

Also, find the frequency response, $H(\omega)$ of the resulting FIR filter.

The magnitude response and phase response of the desired lowpass IIR filter is sketched below:

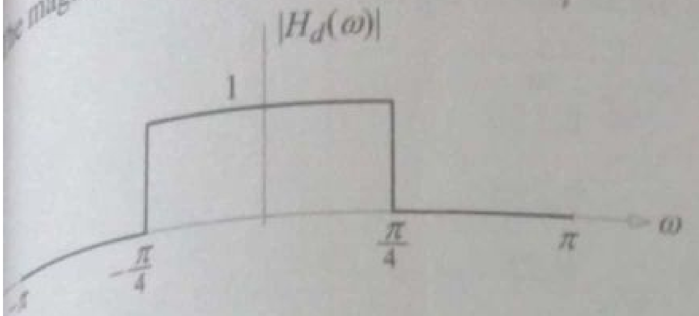

Fig. Ex.5.1(a) Magnitude response of the desired lowpass filter.

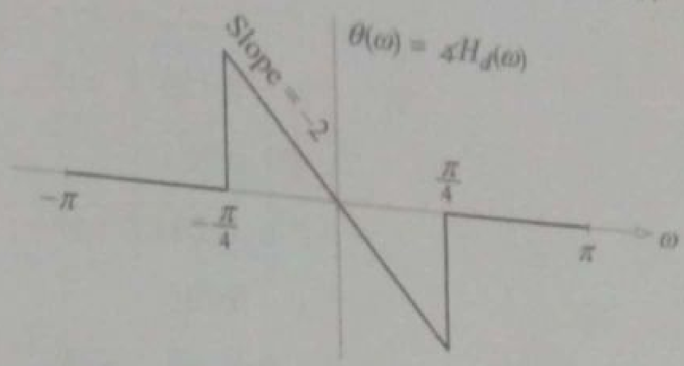

Fig. Ex.5.1 (b) Phase response of the desired lowpass filter.

Since, the magnitude response is zero over the frequency band: $\frac{\pi}{4} < |\omega| < \pi$, we infer from Paley-Wiener theorem that $h_d(n)$ is noncausal.

We know that,
$$h_d(n) \triangleq \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)e^{j\omega n}\,d\omega$$

$$\Rightarrow \quad h_d(n) = \frac{1}{2\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} e^{-j2\omega}e^{j\omega n}\,d\omega$$

$$= \frac{\sin\left[\frac{\pi}{4}(n-2)\right]}{\pi(n-2)}, \; n \neq 2$$

Also,
$$h_d(2) = \frac{1}{2\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} e^0\,d\omega = \frac{1}{2\pi}\times\frac{\pi}{2} = \frac{1}{4}$$

Given
$$w_R(n) = \begin{cases} 1, & 0 \leqslant n \leqslant 4 \\ 0, & \text{otherwise} \end{cases}$$

Hence,
$$h(n) = h_d(n)\,w_R(n)$$
$$= \begin{cases} \frac{\sin\left[\frac{\pi}{4}(n-2)\right]}{\pi(n-2)} \times 1, & n = 0, 1, 3, 4 \\ \frac{1}{4}\times 1, & n = 2 \end{cases}$$

The filter coefficients $h_d(n)$ and $h(n)$ are tabulated below:

n	$h_d(n)$	$w_R(n)$	$h(n) = h_d(n)w_R(n)$
0	0.159	1	0.159
1	0.225	1	0.225
2	0.25	1	0.25
3	0.225	1	0.225
4	0.159	1	0.159

Since, N is odd, the frequency response of the [4]centre symmetric FIR filter is computed a ows:

$$H(\omega) = e^{-j\omega\left(\frac{N-1}{2}\right)}\left[h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{\left(\frac{N-3}{2}\right)} 2h(n)\cos\left[\omega\left(n-\left(\frac{N-1}{2}\right)\right)\right]\right]$$

Here, $N = 5$.

Hence,

$$\begin{aligned} H(\omega) &= e^{-j2\omega}\left(h(2) + \sum_{n=0}^{1} 2h(n)\cos[\omega(n-2)]\right) \\ &= e^{-j2\omega}[h(2) + 2h(0)\cos 2\omega + 2h(1)\cos\omega] \\ &= e^{-j2\omega}(0.25 + 0.318\cos 2\omega + 0.45\cos\omega) \end{aligned}$$

Example 5.4 The desired frequency response of a lowpass filter is given by

$$H_d(e^{j\omega}) = H_d(\omega) = \begin{cases} e^{-j3\omega}, & |\omega| < \frac{3\pi}{4} \\ 0, & \frac{3\pi}{4} < |\omega| < \pi \end{cases}$$

Determine the frequency response of the FIR filter if Hamming window is used with $N = 7$.

Solution

...definition, the inverse DTFT of $H_d(\omega)$ is

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)e^{j\omega n}\, d\omega$$

$$\Rightarrow \quad h_d(n) = \frac{1}{2\pi}\int_{-\frac{3\pi}{4}}^{\frac{3\pi}{4}} e^{-j3\omega}\, e^{j\omega n}\, d\omega$$

$$= \frac{\sin\left[\frac{3\pi}{4}(n-3)\right]}{\pi(n-3)}, \quad n \neq 3$$

Also,

$$h_d(3) = \frac{1}{2\pi}\int_{-\frac{3\pi}{4}}^{\frac{3\pi}{4}} e^{0}\, d\omega = \frac{3}{4}$$

The impulse response of the FIR filter is

$$h(n) = h_d(n)\, w_{\text{Ham}}(n), \qquad 0 \leqslant n \leqslant 6$$

where

$$w_{\text{Ham}}(n) = 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right), \quad 0 \leqslant n \leqslant N-1$$

$$= 0.54 - 0.46\cos\left(\frac{2\pi n}{6}\right), \quad 0 \leqslant n \leqslant 6$$

Hence, the impulse response of the FIR filter for $0 \leqslant n \leqslant 6$ is

$$h(n) = \begin{cases} \frac{\sin\left[\frac{3\pi}{4}(n-3)\right]}{\pi(n-3)} \times \left[0.54 - 0.46\cos\left(\frac{2\pi n}{6}\right)\right], & n \neq 3 \\ \frac{3}{4} \times \left[0.54 - 0.46\cos\left(\frac{2\pi n}{6}\right)\right], & n = 3 \end{cases}$$

The table below shows the details of the computations for $h(n)$.

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n	$h_d(n)$	$w_{\text{Ham}}(n)$	$h(n) = h_d(n)w_{\text{Ham}}(n)$
0	0.075	0.08	0.006
1	−0.159	0.31	−0.049
2	0.225	0.77	0.173
3	0.75	1	0.75
4	0.225	0.77	0.173
5	−0.159	0.31	−0.049
6	0.075	0.08	0.006

Since N is odd, the frequency response of the center symmetric FIR filter is given by

$$
\begin{aligned}
H(e^{j\omega}) &= H(\omega) \\
&= e^{-j\omega\left(\frac{N-1}{2}\right)}\left(h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{\left(\frac{N-3}{2}\right)} 2h(n)\cos\left[\omega\left(n-\left(\frac{N-1}{2}\right)\right)\right]\right) \\
&= e^{-j3\omega}\left[h(3) + 2h(0)\cos 3\omega + 2h(1)\cos 2\omega + 2h(2)\cos\omega\right] \\
&= e^{-j3\omega}\left(0.75 + 2\times 0.006\cos 3\omega + 2\times -0.049\cos 2\omega + 2\times 0.173\cos\omega\right) \\
&= e^{-j3\omega}\left(0.75 + 0.012\cos 3\omega - 0.098\cos 2\omega + 0.346\cos\omega\right)
\end{aligned}
$$

2 A filter is to be designed with the following desired frequency response:

$$H_d(\omega) = \begin{cases} 0, & \frac{-\pi}{4} < \omega < \frac{\pi}{4} \\ e^{-j2\omega}, & \frac{\pi}{4} < |\omega| < \pi \end{cases}$$

Find the frequency response of the FIR filter designed using a rectangular dow defined below:

$$w_R(n) = \begin{cases} 1, & 0 \leqslant n \leqslant 4 \\ 0, & \text{otherwise} \end{cases}$$

FIR filter is one whose impul... ...t to its mi...

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solution

...inverse DTFT of $H_d(\omega)$ is

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n} d\omega$$

$$\Rightarrow \quad h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\frac{-\pi}{4}} e^{-j2\omega}\, e^{j\omega n}\, d\omega + \frac{1}{2\pi}\int_{\frac{\pi}{4}}^{\pi} e^{-j2\omega}\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{\pi(n-2)}\left[\sin[\pi(n-2)] - \sin\left(\frac{\pi}{4}(n-2)\right)\right],\ n \neq 2$$

lso,

$$h_d(2) = \frac{1}{2\pi}\int_{-\pi}^{\frac{-\pi}{4}} e^{0}\, d\omega + \frac{1}{2\pi}\int_{\frac{\pi}{4}}^{\pi} e^{0}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{3\pi}{4} + \frac{3\pi}{4}\right] = \frac{3}{4}$$

he impulse response of the FIR filter is given by

$$h(n) = h_d(n)\, w_R(n),\ 0 \leqslant n \leqslant 4$$

here

$$w_R(n) = \begin{cases} 1, & 0 \leqslant n \leqslant 4 \\ 0, & \text{otherwise} \end{cases}$$

he coefficients $h_d(n)$ and $h(n)$ are tabulated below for $0 \leqslant n \leqslant 4$.

n	$h_d(n)$	$w_R(n)$	$h(n) = h_d(n)w_R(n)$
0	−0.159	1	−0.159
1	−0.225	1	−0.225
2	0.75	1	0.75
3	−0.225	1	−0.225
4	−0.159	1	−0.159

ce N is odd, the frequency response of the center symmetric FIR filter is given by

$$H(e^{j\omega}) = H(\omega)$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)}\left(h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{\frac{(N-3)}{2}} 2h(n)\cos\left[\omega\left(n - \left(\frac{N-1}{2}\right)\right)\right]\right)$$

Example 5.19 A lowpass filter has the desired frequency response

$$H_d(\omega) = H_d(e^{j\omega}) = \begin{cases} e^{-j3\omega}, & 0 < \omega < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < \omega < \pi \end{cases}$$

Determine $h(n)$ based on frequency-sampling technique. Take $N = 7$.

Solution

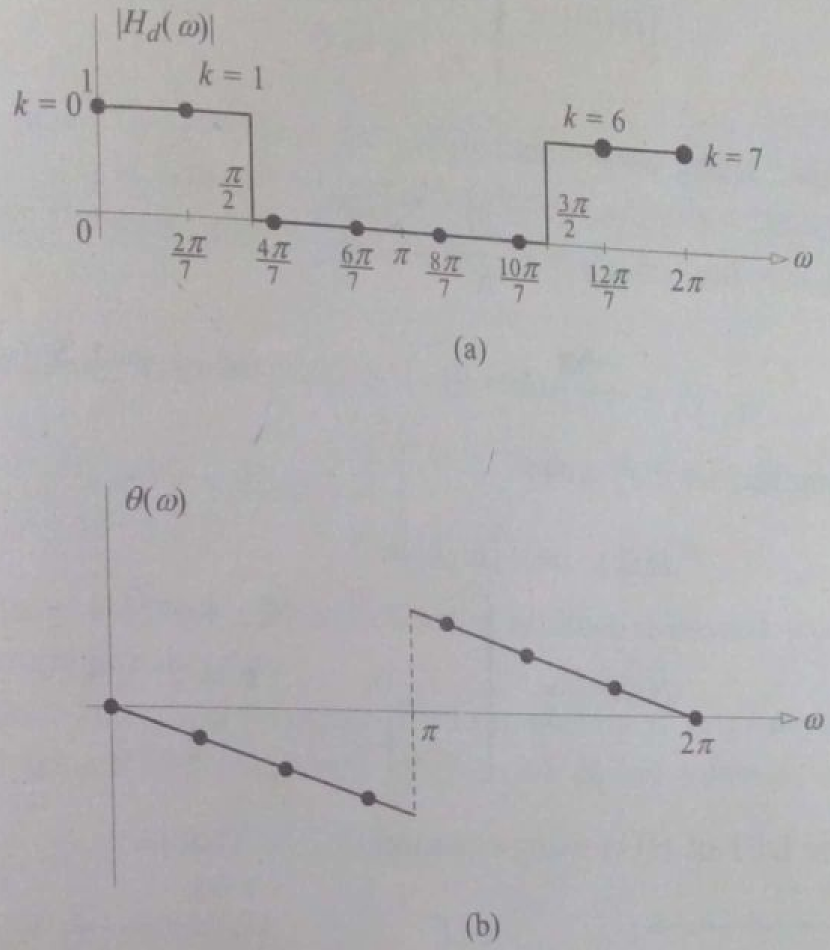

Fig. Ex.5.19 Magnitude and phase responses of the desired lowpass filter.

Let the ideal response of a linear-phase lowpass filter be

$$H_d(\omega) = \begin{cases} e^{-j\frac{(N-1)\omega}{2}}, & 0 < \omega < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < \omega < \pi \end{cases}$$

In the present context,

$$H_d(\omega) = \begin{cases} e^{-j3\omega}, & 0 < \omega < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < \omega < \pi \end{cases}$$

The ideal magnitude frequency response is taken to be symmetric about π while the ideal phase response is taken to be antisymmetric about π. The ideal magnitude and phase responses with samples taken for $N = 7$ are shown in Fig. Ex.5.19.

The samples of $|H_d(\omega)|$ and $\theta(\omega)$ are taken at $\omega = \omega_k = \frac{2\pi k}{N}$, $k = 0, \cdots N-1$.
The range of k is found as follows:

i. for $0 \leqslant \omega < \frac{\pi}{2}$, the values of k are 0, 1
ii. for $\frac{\pi}{2} < \omega < \frac{3\pi}{2}$, the values of k are 2, 3, 4, 5
iii. for $\frac{3\pi}{2} < \omega < 2\pi$, the value of k is 6.

From Fig Ex.5.19(a), we find that

$$|H(k)| = \begin{cases} 1, & k = 0, 1 \\ 0, & k = 2, 3, 4, 5 \\ 1, & k = 6 \end{cases}$$

Also, from Fig. Ex.5.19(b), we find that

$$\theta_k = -3\omega_k \quad = \quad -3 \times \frac{2\pi}{N} k = \frac{-6\pi}{7} k \quad \text{for} \quad k = 0, 1, 2, 3$$

and

$$\theta_k = \frac{-6\pi}{7}(k-7) \qquad \text{for} \quad k = 4, 5, 6$$

Since $H(k)$ is complex, we may write

$$H(k) \quad = \quad |H(k)| e^{j\theta_k}$$

Hence,

$$H(k) \quad = \quad \begin{cases} e^{-j\frac{6\pi k}{7}}, & k = 0, 1 \\ 0, & k = 2, 3, 4, 5 \\ e^{-j\frac{6\pi}{7}(k-7)}, & k = 6 \end{cases}$$

We find the inverse DFT of $H(k)$ using equation (5.29). That is,

$$h(n) \quad = \quad \frac{1}{N}\left[H(0) + 2\sum_{k=1}^{\frac{(N-1)}{2}} \text{Re}\left\{ H(k) e^{j\frac{2\pi nk}{N}} \right\} \right]$$

Hence,

$$\begin{aligned} h(n) \quad &= \quad \frac{1}{7}\left[H(0) + 2\sum_{k=1}^{3} \text{Re}\left\{ H(k) e^{j\frac{2\pi nk}{7}} \right\} \right] \\ &= \frac{1}{7}\left[H(0) + 2\,\text{Re}\left\{ H(1) e^{j\frac{2\pi n}{7}} \right\} \right] \\ &= \frac{1}{7}\left[1 + 2\,\text{Re}\left\{ e^{-j\frac{6\pi}{7}} e^{j\frac{2\pi n}{7}} \right\} \right] \\ &= \frac{1}{7}\left[1 + 2\cos\left(\frac{2\pi}{7}(n-3) \right) \right], \quad 0 \leqslant n \leqslant 6 \end{aligned}$$

The filter coefficients are tabulator below:

n	$h(n)$	n	$h(n)$
0	−0.11456	4	0.320997
1	0.07928	5	0.07928
2	0.320997	6	−0.11456
3	0.42857		

ample 5.20 Design a 17-tap linear-phase FIR filter with a cutoff frequency $\omega_C = \frac{\pi}{2}$. The design is to be done based on frequency sampling technique.

Solution

ideal lowpass frequency response with a linear-phase is

$$H_d(e^{j\omega}) = H_d(\omega) = \begin{cases} e^{-j\frac{(N-1)\omega}{2}}, & 0 < \omega < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < \omega < \pi \end{cases}$$

The ideal magnitude response is taken to be even symmetric about π while the phase response aken to be odd symmetric about π .

The ideal magnitude and phase responses with samples for $N = 17$ are shown in Fig. Ex.5.20. The heavy dots denote the frequency samples of $H_d(\omega)$ taken at $\omega = \omega_k = \frac{2\pi k}{N}$ for $= 0, \cdots N-1$.

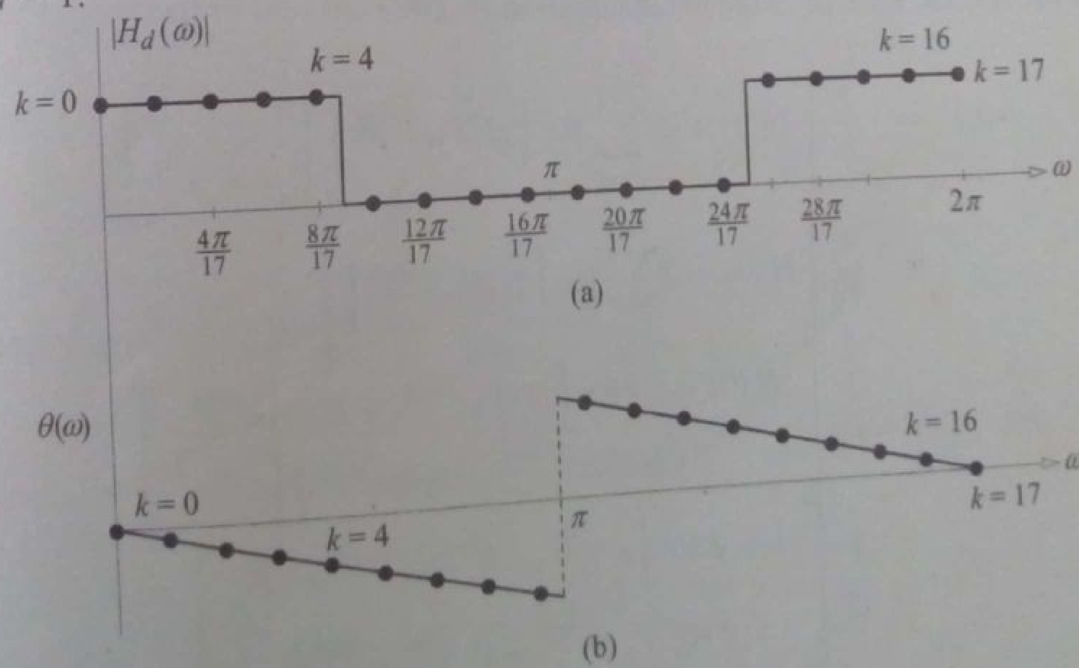

Fig. Ex.5.20 Magnitude and phase responses for $H_d(\omega)$.

In the present context,

$$H_d(\omega) = \begin{cases} e^{-j8\omega}, & 0 < \omega < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < \omega < \pi \end{cases}$$

The samples of $|H_d(\omega)|$ and $\theta(\omega)$ are taken at $\omega = \omega_k = \frac{2\pi k}{N}, k = 0, 1, \cdots N-1$.
From Fig. Ex.5.20(a), we find that

$$|H(k)| = \begin{cases} 1, & 0 \leqslant k \leqslant 4 \\ 0, & 5 \leqslant k \leqslant 12 \\ 1, & 13 \leqslant k \leqslant 16 \end{cases}$$

Also, from Fig. Ex.5.20(b), we find that

$$\begin{aligned} \theta_k = -8\omega_k &= -8 \times \frac{2\pi k}{N} \\ &= \frac{-16\pi k}{17}, \quad 0 \leqslant k \leqslant 8 \end{aligned}$$

and
$$\theta_k = \frac{-16\pi}{17}(k-17), \quad 9 \leqslant k \leqslant 16.$$

Since, $H(k)$ is complex, we may write

$$H(k) = |H(k)|e^{j\theta_k}$$

$$\Rightarrow \quad H(k) = \begin{cases} e^{-j\frac{16\pi k}{17}}, & 0 \leqslant k \leqslant 4 \\ 0, & 5 \leqslant k \leqslant 12 \\ e^{-j\frac{16\pi}{17}(k-17)}, & 13 \leqslant k \leqslant 16 \end{cases}$$

We find the inverse DFT of $H(k)$ using equation (5.29).

That is,
$$\begin{aligned} h(n) &= \frac{1}{N}\left[H(0) + 2\sum_{k=1}^{\frac{(N-1)}{2}} \text{Re}\left\{H(k)e^{j\frac{2\pi nk}{N}}\right\}\right] \\ &= \frac{1}{17}\left[H(0) + 2\sum_{k=1}^{8} \text{Re}\left\{H(k)e^{j\frac{2\pi nk}{17}}\right\}\right] \\ &= \frac{1}{17}\left[\begin{array}{c} H(0) + 2\Big[\text{Re}\Big\{H(1)e^{j\frac{2\pi n}{17}} + H(2)e^{j\frac{4\pi n}{17}} \\ +H(3)e^{j\frac{6\pi n}{17}} + H(4)e^{j\frac{8\pi n}{17}}\Big\}\Big] \end{array}\right] \\ &= \frac{1}{17}\left[\begin{array}{c} 1 + 2\,\text{Re}\Big\{e^{-j\frac{16\pi}{17}}e^{j\frac{2\pi n}{17}} + e^{-j\frac{32\pi}{17}}e^{j\frac{4\pi n}{17}} \\ +e^{-j\frac{48\pi}{17}}e^{j\frac{6\pi n}{17}} + e^{-j\frac{64\pi}{17}}e^{j\frac{8\pi n}{17}}\Big\} \end{array}\right] \\ &= \frac{1}{17}\left[\begin{array}{c} 1 + 2\cos\left[\frac{2\pi}{17}(n-8)\right] + 2\cos\left[\frac{4\pi}{17}(n-8)\right] \\ +2\cos\left[\frac{6\pi}{17}(n-8)\right] + 2\cos\left[\frac{8\pi}{17}(n-8)\right] \end{array}\right], \quad 0 \leqslant n \leqslant 16 \end{aligned}$$

$$\Rightarrow \quad H(z) = \frac{1}{N}\sum_{k=0}^{N-1} H(k) \sum_{n=0}^{N-1}\left[e^{j\frac{2\pi k}{N}} z^{-1}\right]^n$$

We know that
$$\sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a}, \quad a \neq 1$$

Hence,
$$H(z) = \frac{1}{N}\sum_{k=0}^{N-1} H(k) \frac{1-\left[e^{j\frac{2\pi k}{N}} z^{-1}\right]^N}{1-e^{j\frac{2\pi k}{N}} z^{-1}}$$

$$\Rightarrow \quad H(z) = \frac{1}{N}\sum_{k=0}^{N-1} H(k)\left[\frac{1-z^{-N}}{1-e^{j\frac{2\pi K}{N}} z^{-1}}\right]$$

The term $1 - z^{-N}$ has a factor $1 - e^{j\frac{2\pi k}{N}} z^{-1}$. Hence, pole-zero cancellation takes place a $H(z)$ contains only zeros, a necessary and sufficient condition for an FIR filter.

The realization of an FIR filter based on frequency sampling design is given in Fig. 5.16.

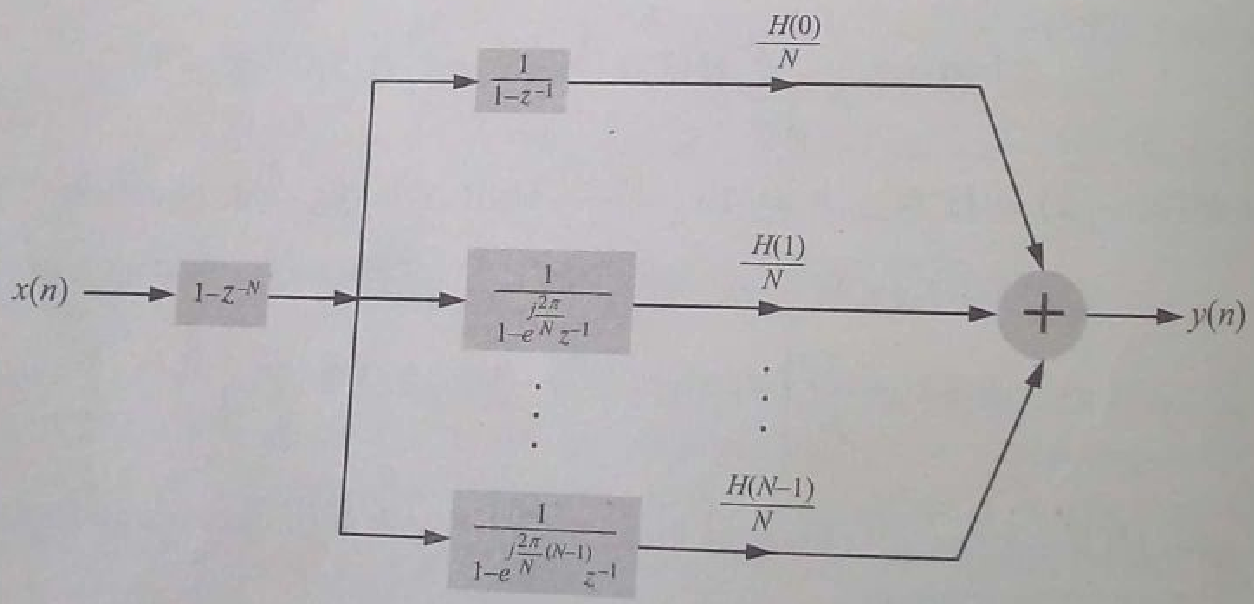

Fig. 5.16 Realization of an FIR filter based on frequency sampling design.