FinSearch End-Term Report

Topic: Option pricing models and their accuracy

Group: E 10

Sr No	Name	Roll No
1	Dattaraj Salunkhe	22B1296
2	Kshitij Bahadarpurkar	22B1802
3	Swayam Patel	22B1816
4	Priyansh Singh	22B1856

The report consists of 3 sections:

1. The Black-Scholes Model: Working principles, assumptions for the model, advantages and limitations
2. The Binomial Model: Framework, working, assumptions, advantages and disadvantages
3. Comparison of both models on a dataset: The dataset consists of stock price and the corresponding option price on different days and both the models have been used to predict option prices and the predicted values are compared with the actual values

A. The Black Scholes Model

The Black–Scholes or Black–Scholes–Merton model is a mathematical model for the dynamics of a financial market containing derivative investment instruments. From the parabolic partial differential equation in the model, known as the Black–Scholes equation, one can deduce the Black–Scholes formula, which gives a theoretical estimate of the price of European-style options and shows that the option has a unique price given the risk of the security and its expected return.

1. **The Differential Equation**

   The Black-Scholes Model is based on the following differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

   Where:

   V: Price of the option

   S: Price of the Stock

   r: Risk free interest rate

   σ: Volatility of the underlying asset (standard deviation of stock price's return)

After Solving the Differential Equation, the price of a European Style option is given by

Call Option: $SN(d_1) - Ke^{-rT}N(d_2)$

Put Option: $Ke^{-rT}N(-d_2) - SN(-d_1)$

In these Equations:

K: Strike Price of the option

T: Time for option Expiry

d_1: $\frac{\ln(\frac{S}{K})+(r+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$

d_2: $d_1 - \sigma\sqrt{T}$

N: Cumulative Standard Normal Distribution

It is defined as $N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{\frac{-t^2}{2}}\,dt$

2. **Assumptions for the Model to work**
    - No dividends are paid out during the life of the option
    - There are no transaction costs in buying the option
    - The returns of the underlying asset are normally distributed
    - The risk-free rate and volatility of the underlying asset are known and constant
    - The asset prices follow a log-normal distribution

3. **Advantages of the Model**
    - It provides a method to price options. Calculating option prices using modern computers is fairly simple. It provides a structured, defined method that is tried and tested
    - By understanding the theoretical value of an option using the Black-Scholes model, investors can develop hedging strategies. This involves taking opposite positions in other instruments (like the underlying asset or other options) to offset potential risks associated with holding the initial option.
    - It enhances efficiency in trading. Due to the Black-Scholes model, there is greater transparency in the market and traders have a better understanding of the factors affecting the pricing of options
    - Since, it is a widely accepted model, it enhances consistency and compatibility across different markets and jurisdictions

4. **Disadvantages of the Model**
    - Works only on European options. For American options that can be exercised before the expiry date, this simplistic model will not work
    - The assumption that the stock provides no dividend makes it difficult to apply uniformly on all options

- The model also assumes that stock prices follow a log normal distribution and the volatility stays constant, which is not true especially when certain strong events tend to influence the markets

B. The Binomial Model

1. Overview

The Binomial Option Pricing Model is a popular method used to evaluate options. Unlike the Black-Scholes model, the binomial model provides a numerical method for the valuation of options, making it versatile for American options and other scenarios where early exercise might be optimal.

2. Model Framework

The binomial model divides the time to expiration into potentially a large number of time intervals or steps. At each step, the stock price can move up or down by a specific factor.

3. Parameters and variables

- S: Current stock price
- K: Strike price of the option
- T: Time to expiration
- N: Number of steps
- r: Risk-free interest rate
- σ: Volatility of the underlying asset
- u: Factor by which the price increases (up factor)
- d: Factor by which the price decreases (down factor)
- p: Risk-neutral probability of the price moving up

4. Key Formulas

- Up and down factors
  - $u = e^{\sigma\sqrt{\frac{T}{N}}}$
  - $d = \frac{1}{u}$
- Risk Neutral Probability
  - $p = \frac{e^{r\left(\frac{T}{N}\right)} - d}{u - d}$

5. Steps of the Binomial Model

- Initialisation
  - Set up a binomial tree representing possible paths the stock price could take at each step.
- Recursive Calculation
  - $S_j = S_0 u^{2j-N}$

    Where N is the step number and j vary from 0 to N
- Option Valuation at Maturity
  - $C(S_T) = \max(S_T - K, 0)$ for call option
  - $P(S_T) = \max(K - S_T, 0)$ for put option

- Backward Induction
  - Calculate the option price at each node by discounting the expected payoff from the next step
  - $C_{i,j} = e^{-r\left(\frac{T}{N}\right)}(pC_{i+1,j+1} + (1-p)C_{i+1,j})$
  - $P_{i,j} = e^{-r\left(\frac{T}{N}\right)}(pP_{i+1,j+1} + (1-p)P_{i+1,j})$

6. **Assumptions**
   - The market is frictionless (no transaction costs or taxes).
   - The risk-free rate and volatility are constant over the option's life.
   - The underlying asset does not pay dividends.
   - The option can only be exercised at discrete points in time.
7. **Advantages**
   - Flexibility
     - Can be used for both European and American options.
     - Suitable for options with complex features, like barriers and varying interest rates.
   - Early Exercise Feature
     - Handles American options, allowing for early exercise decisions.
   - Intuitive
     - Provides a clear and structured approach to option pricing, facilitating understanding and implementation.

8. **Disadvantages**
   - Computational Intensity
     - As the number of steps increases, the computations become more intensive, potentially requiring significant processing power.
   - Simplifying Assumptions
     - Assumes constant volatility and interest rates, which may not reflect real market conditions.
   - Model Sensitivity
     - Highly sensitive to input parameters such as volatility and interest rates, leading to potential inaccuracies in valuation.

9. **Conclusion**
   - The Binomial Option Pricing Model is a powerful tool for valuing options, particularly those that allow for early exercise. Its step-by-step approach offers flexibility and clarity, though it requires careful consideration of input parameters and computational resources.

C. Comparison of the Models

Both of the models have been compared using the dataset of Reliance obtained from the NSE website.
Link: https://www.nseindia.com/get-quotes/derivatives?symbol=RELIANCE

It consists of both put and call options of the following types:

- **Reliance Dataset 1**: K constant @ 2400, expiry date as 28 dec 2023, for the year 2023
- **Reliance Dataset 2**: K constant @ 2500, expiry date as 28 dec 2023, for the year 2023
- **Reliance Dataset 3**: K constant @ 2600, expiry date as 28 dec 2023, for the year 2023

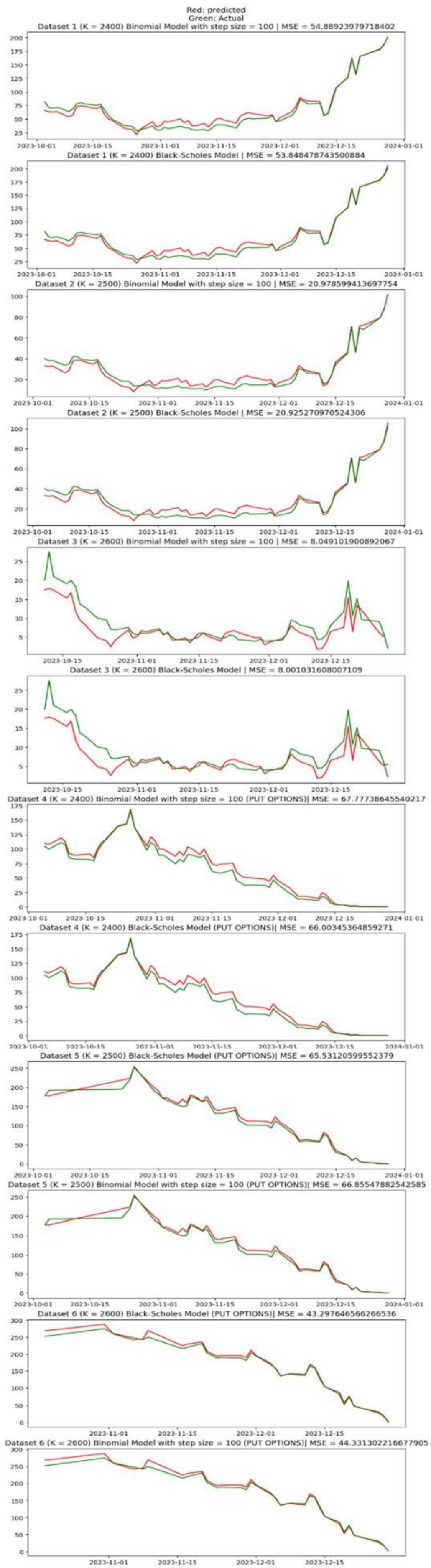
To calculated the volatility the logarithm of stock price ratio between consecutive days was calculated. The standard deviation of the ratios was multiplied by the square root of the number of days to obtain the volatility of the stock.
The risk-free rate was assumed to be $7\%\ p.a$

1. Observations

   The following graphs were obtained as a result of using the models
   - Plot of predicted and actual price along with Mean Square Error (figure 1)
   - Plot of the MSE error of the Binomial Model with respect to number of steps (figure 2)
   - Considering the predicted and actual prices as vectors, the cosine between them gets closer to 1 as we increase the number of steps in the binomial model (figure 3)

   (These graphs are also available on the github page)

2. Conclusion
   - The Binomial Model and Black-Scholes model are fairly accurate and the Mean Square Error with respect to the actual prices are close
   - As we increase the number of steps, the binomial MSE gets closer and closer to the black-scholes MSE with some oscillations with damping amplitudes as seen in figure 2
   - The MSE also depends largely on moneyness of the option. The dataset 3 remains out of the money/ in the money for call/put options and thus shows low error. Whereas the dataset 1 which keeps on changing its moneyness shows slightly large error
3. Link to the Code:
   https://github.com/priyanshsingh1765/Options-pricing-models-and-their-accuracy



Red: predicted
Green: Actual
Dataset 1 (K = 2400) Binomial Model with step size = 100 | MSE = 54.88923979718402
Dataset 1 (K = 2400) Black-Scholes Model | MSE = 53.848478743500884
Dataset 2 (K = 2500) Binomial Model with step size = 100 | MSE = 20.978599413697754
Dataset 2 (K = 2500) Black-Scholes Model | MSE = 20.925270970524306
Dataset 3 (K = 2600) Binomial Model with step size = 100 | MSE = 8.049101900892067
Dataset 3 (K = 2600) Black-Scholes Model | MSE = 8.001031608007109
Dataset 4 (K = 2400) Binomial Model with step size = 100 (PUT OPTIONS)| MSE = 67.77738645540217
Dataset 4 (K = 2400) Black-Scholes Model (PUT OPTIONS)| MSE = 66.00345364859271
Dataset 5 (K = 2500) Black-Scholes Model (PUT OPTIONS)| MSE = 65.53120599552379
Dataset 5 (K = 2500) Binomial Model with step size = 100 (PUT OPTIONS)| MSE = 66.85547882542585
Dataset 6 (K = 2600) Black-Scholes Model (PUT OPTIONS)| MSE = 43.297646566266536
Dataset 6 (K = 2600) Binomial Model with step size = 100 (PUT OPTIONS)| MSE = 44.331302216677905

Figure 1

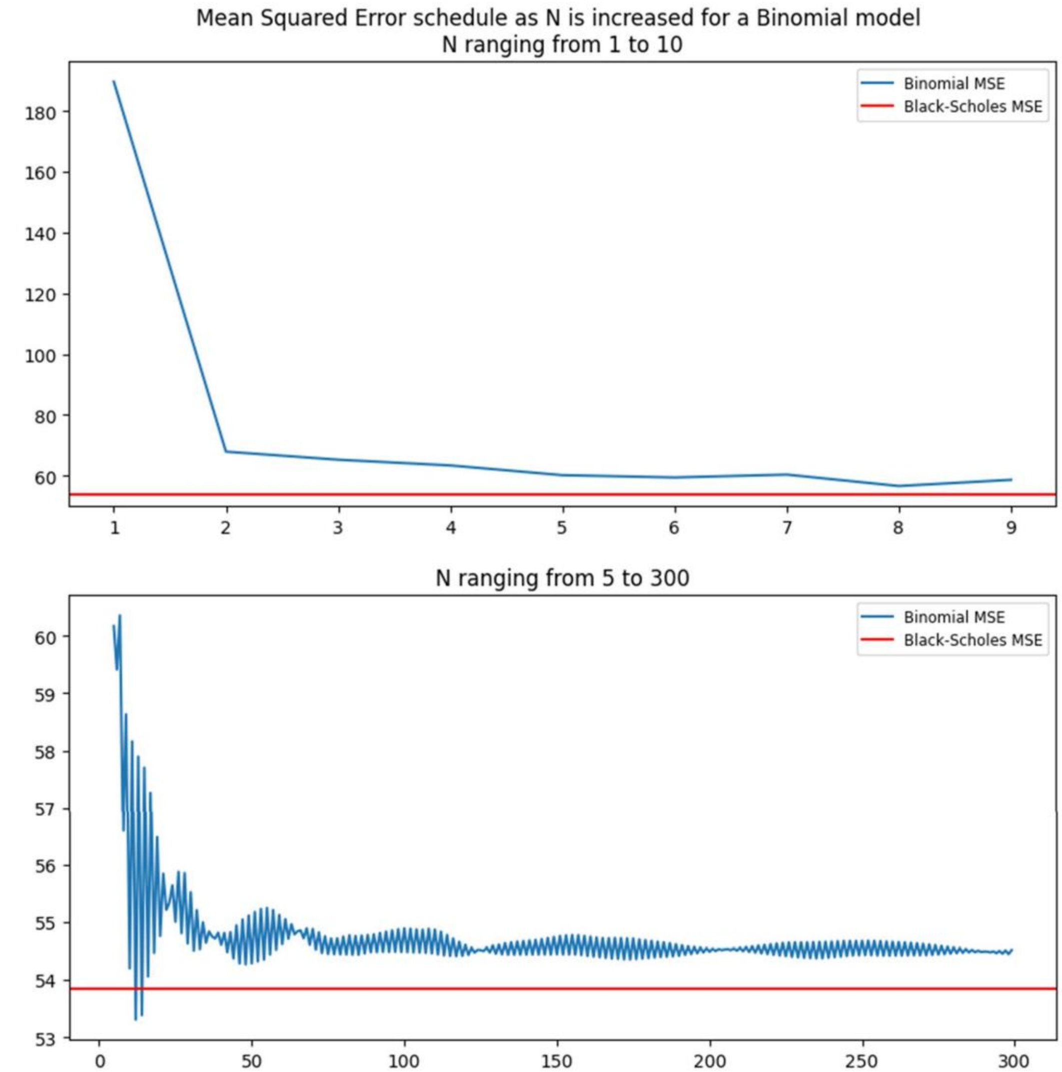

Mean Squared Error schedule as N is increased for a Binomial model
N ranging from 1 to 10
Binomial MSE
Black-Scholes MSE
180
160
140
120
100
80
60
1
2
3
4
5
6
7
8
9
N ranging from 5 to 300
Binomial MSE
Black-Scholes MSE
60
59
58
57
56
55
54
53
0
50
100
150
200
250
300

Figure 2

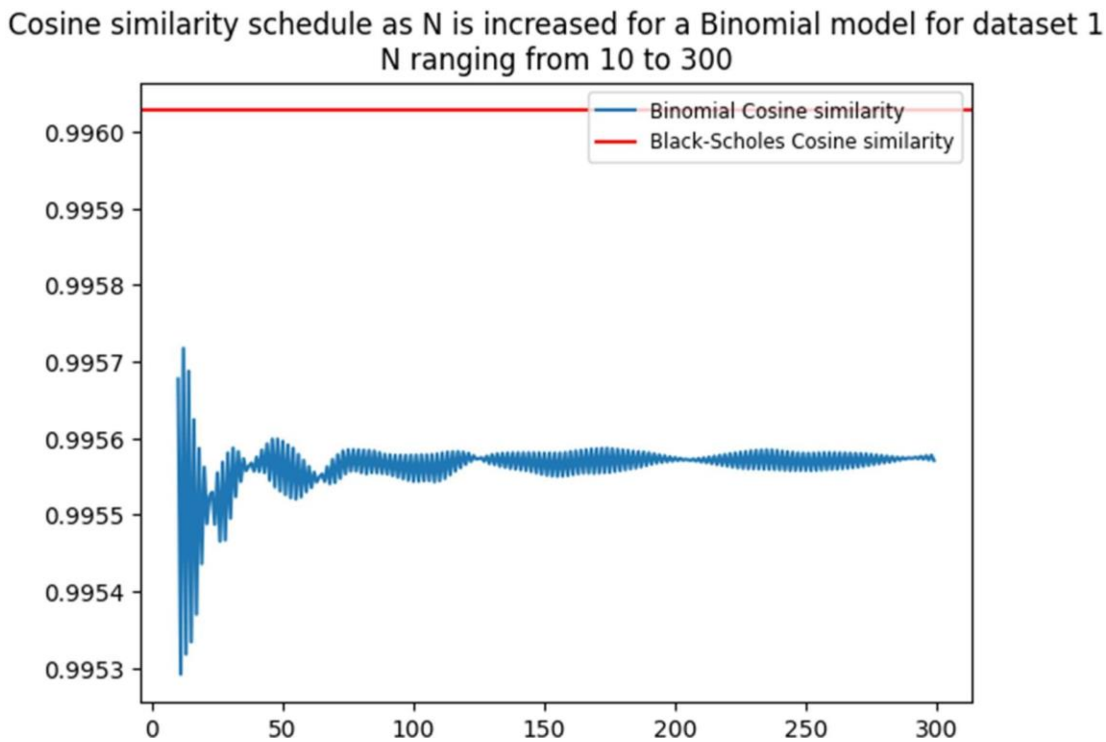

Cosine similarity schedule as N is increased for a Binomial model for dataset 1
N ranging from 10 to 300
Binomial Cosine similarity
Black-Scholes Cosine similarity
0.9960
0.9959
0.9958
0.9957
0.9956
0.9955
0.9954
0.9953
0
50
100
150
200
250
300

Figure 3

The Monte Carlo method is a computational algorithm that uses randomness to solve problems that might be deterministic in principle. It is particularly useful in scenarios where it is difficult or impossible to calculate an exact solution analytically. By generating a large number of random samples, the method allows for the approximation of complex mathematical or financial models. In the context of option pricing, Monte Carlo simulations are used to estimate the fair value of options by simulating the future paths of the underlying asset's price.

Monte Carlo Simulations for European Options

European options, which can only be exercised at expiration, are one of the simplest types of options to price using Monte Carlo simulations. The process begins by modeling the price dynamics of the underlying asset, typically assuming that the asset follows a geometric Brownian motion. This model incorporates factors such as the asset's current price, volatility, risk-free interest rate, and the time to maturity.

The core idea is to simulate thousands or even millions of potential future price paths for the underlying asset. For each path, the option's payoff at expiration is calculated based on whether the option is in-the-money (i.e., if exercising the option is profitable). The average of these payoffs, discounted back to the present value using the risk-free rate, provides an estimate of the option's fair value.

The accuracy of the Monte Carlo method improves with the number of simulated paths, making it particularly powerful for complex scenarios where closed-form solutions, like the Black-Scholes formula, are not easily applicable. However, it is computationally intensive, as a large number of simulations are required to achieve a high level of accuracy.

Monte Carlo Simulations for American Options

American options, which can be exercised at any time before expiration, present a greater challenge for pricing. The flexibility of early exercise makes it difficult to use traditional analytical methods like the Black-Scholes model. The Monte Carlo method, however, can be adapted to price American options through techniques such as the Longstaff-Schwartz algorithm.

In this approach, the simulation begins similarly to the European option, with the generation of multiple price paths. However, at each potential exercise point before expiration, the method evaluates whether exercising the option would be more profitable than holding it. This decision is made using regression techniques to estimate the continuation value, which represents the expected payoff if the option is held rather than exercised. By comparing this continuation value to the immediate exercise payoff, the simulation can determine the optimal exercise strategy for the option holder.

The result is a robust estimate of the American option's value, taking into account the possibility of early exercise. This flexibility is essential for accurately pricing options on assets that exhibit significant price volatility or when the holder expects large dividends before expiration.

Advantages of the Monte Carlo Method in Option Pricing

The Monte Carlo method offers several advantages in option pricing:

1. **Flexibility**: The method can handle a wide range of option types and complex payoffs, including path-dependent options like Asian or barrier options.
2. **Accuracy**: By increasing the number of simulations, the accuracy of the option price estimate can be improved, making it suitable for high-stakes financial decisions.
3. **Adaptability**: The method can incorporate various market dynamics, such as stochastic volatility or interest rates, which are difficult to model using traditional closed-form solutions.

Limitations and Challenges

Despite its strengths, the Monte Carlo method is not without limitations. It is computationally expensive, requiring significant processing power to achieve high accuracy. Additionally, the method's reliance on randomness means that results can vary slightly between simulations, although this variability decreases with more simulations.

Another challenge is the proper calibration of input parameters, such as volatility and interest rates, which can significantly impact the accuracy of the results. In practice, these parameters are often estimated from historical data, but incorrect estimates can lead to biased option prices.