1 Convergence of Binomial Model to Black-Scholes Model

The convergence of the binomial model to the Black-Scholes model can be understood by considering the mathematical logic and the principles underlying both models.

1.1 Binomial Model Parameters

In the binomial model, the price of the underlying asset can move up or down at each time step Δt, with probabilities p and $1-p$. The parameters for the up and down movements are:

$$u = e^{\sigma\sqrt{\Delta t}}$$

$$d = e^{-\sigma\sqrt{\Delta t}}$$

where σ is the volatility of the underlying asset.

The probability of an upward movement in the risk-neutral world is given by:

$$p = \frac{e^{(r-q)\Delta t} - d}{u - d}$$

where r is the risk-free rate and q is the dividend yield.

1.2 Central Limit Theorem and Lognormal Distribution

As the number of time steps n increases, the time step size Δt decreases ($\Delta t = \frac{T}{n}$). According to the Central Limit Theorem, the sum of a large number of small, independent, identically distributed random variables tends to follow a normal distribution.

In the binomial model, the log of the asset price after n steps is approximately:

$$\log(S_T) = \log(S_0) + n\left(\frac{r - q - \frac{\sigma^2}{2}}{n}\right) + \sigma\sqrt{n\Delta t}\cdot Z$$

where Z is a standard normal variable.

As $n \to \infty$ and $\Delta t \to 0$, this expression approximates a lognormal distribution:

$$S_T \sim S_0 e^{(r-q-\frac{\sigma^2}{2})T + \sigma\sqrt{T}Z}$$

1.3 Black-Scholes Model

The Black-Scholes model is based on the assumption that the underlying asset price follows a geometric Brownian motion:

$$dS = (r-q)S\,dt + \sigma S\,dW$$

where dW is a Wiener process (standard Brownian motion).

The Black-Scholes formula for a European call option is:

$$C = S_0 e^{-qT} N(d_1) - K e^{-rT} N(d_2)$$

where

$$d_1 = \frac{\log(S_0/K) + (r - q + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

1.4 Convergence

The binomial model's discrete-time framework approximates the continuous-time framework of the Black-Scholes model as follows:

- **Time Step Size:** As $\Delta t \to 0$, the binomial model's time steps become infinitesimally small, closely approximating the continuous-time changes in the Black-Scholes model.
- **Lognormal Distribution:** The binomial model's asset price distribution, due to the Central Limit Theorem, approaches the lognormal distribution assumed in the Black-Scholes model.
- **Risk-Neutral Valuation:** Both models use the risk-neutral valuation principle to price options, ensuring consistency in their approach.

Therefore, as the number of time steps n increases, the binomial model's option prices converge to those predicted by the Black-Scholes model due to the alignment of their underlying stochastic processes and risk-neutral valuation principles.

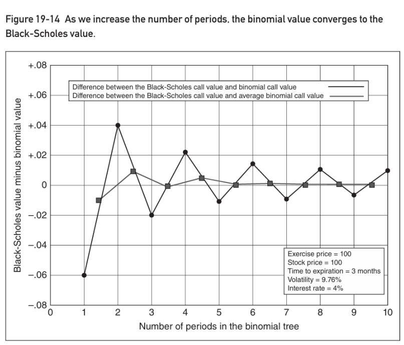

Figure 1: This is a snip fron Sheldon Natenberg's Option Volatility & Pricing