Solving the Fibonacci recurrence with generating functions

Let $a_n = \hat{F}_n$ for $n \geq 0$. Then $a_0 = a_1 = 1$ and $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$. Let $A(x) = \sum_{n\geq 0} a_n x^n$ be the generating function for $\langle a \rangle_{n\geq 0}$. Using the recurrence, we have

$$\begin{aligned}
& & a_n &= a_{n-1} + a_{n-2} \quad \text{for } n \geq 2\,, \\
&\text{multiply by } x^n: & a_n x^n &= a_{n-1}x^n + a_{n-2}x^n \\
&\text{sum over the range of validity:} & \sum_{n\geq 2} a_n x^n &= \sum_{n\geq 2} a_{n-1}x^n + \sum_{n\geq 2} a_{n-2}x^n \\
& & A(x) - a_1 x - a_0 &= x\,[A(x) - a_0] + x^2\,[A(x)] \\
&\text{solve for } A(x): & A(x)\left[1 - x - x^2\right] &= a_0 + (a_1 - a_0)x = 1 \\
& & A(x) &= \frac{1}{1 - x - x^2}
\end{aligned}$$

We now use partial fractions to obtain the power series for $A(x)$. Note that

$$1 - x - x^2 = -\left(\left[\frac{1+\sqrt{5}}{2}\right] + x\right)\left(\left[\frac{1-\sqrt{5}}{2}\right] + x\right) = -(x+\alpha_1)(x+\alpha_2)\,,$$

where $\alpha_1 = \frac{1+\sqrt{5}}{2}$ is the Golden Ratio and $\alpha_2 = \frac{1-\sqrt{5}}{2}$. Observe that $\alpha_1 = -\frac{1}{\alpha_2}$. Let

$$\begin{aligned}
& & \frac{1}{1-x-x^2} &= \frac{B}{x+\alpha_1} + \frac{C}{x+\alpha_2} \\
&\text{cross-multiplying} & -1 &= B(x+\alpha_2) + C(x+\alpha_1) \\
&\text{when } x = -\alpha_1: & -1 &= B(-\alpha_1+\alpha_2) \quad \Longrightarrow \quad B = \frac{1}{\sqrt{5}} \\
&\text{when } x = -\alpha_2: & -1 &= C(-\alpha_2+\alpha_1) \quad \Longrightarrow \quad C = -\frac{1}{\sqrt{5}}
\end{aligned}$$

Thus,

$$\begin{aligned}
A(x) &= \left(\frac{1}{\sqrt{5}}\right)\frac{1}{x+\alpha_1} + \left(-\frac{1}{\sqrt{5}}\right)\frac{1}{x+\alpha_2} \\
&= \left(\frac{1}{\sqrt{5}}\right)\frac{-\frac{1}{\alpha_1}}{\left(-\frac{x}{\alpha_1}\right)-1} + \left(-\frac{1}{\sqrt{5}}\right)\frac{-\frac{1}{\alpha_2}}{\left(-\frac{x}{\alpha_2}\right)-1} \\
&= -\frac{\alpha_2}{\sqrt{5}}\left[\frac{1}{1-\alpha_2 x}\right] + \frac{\alpha_1}{\sqrt{5}}\left[\frac{1}{1-\alpha_1 x}\right] \quad \text{since } \alpha_1 = -\frac{1}{\alpha_2} \\
&= -\frac{\alpha_2}{\sqrt{5}}\sum_{n\geq 0}\alpha_2^n x^n + \frac{\alpha_1}{\sqrt{5}}\sum_{n\geq 0}\alpha_1^n x^n \\
&= \sum_{n\geq 0}\left[\frac{\alpha_1^{n+1}}{\sqrt{5}} - \frac{\alpha_2^{n+1}}{\sqrt{5}}\right]x^n.
\end{aligned}$$

Thus, $a_n = \hat{F}_n = \frac{\alpha_1^{n+1}}{\sqrt{5}} - \frac{\alpha_2^{n+1}}{\sqrt{5}}$ for $n \geq 0$. This is the same formula obtained using the characteristic polynomial, as shown in Example 2.2.4.