Computer Algorithms: Karatsuba Fast Multiplication

May 15, 2012 algorithms Anatolii Alexeevitch Karatsuba, Andrey Kolmogorov, Cohen-Sutherland, Divide and conquer algorithm, Karatsuba algorithm, Mathematics, Multiplication, Multiplication algorithm, PHP, structured algorithm Stoimen

Introduction

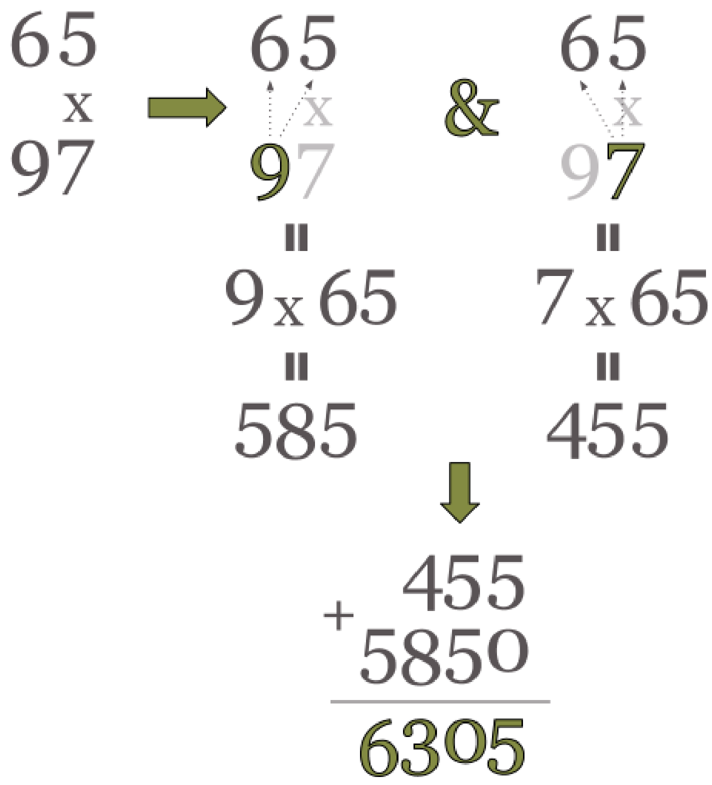
Typically multiplying two n-digit numbers require n^2 multiplications. That is actually how we, humans, multiply numbers. Let's take a look of an example in case we've to multiply two 2-digit numbers.

```
12 x 15 = ?
```

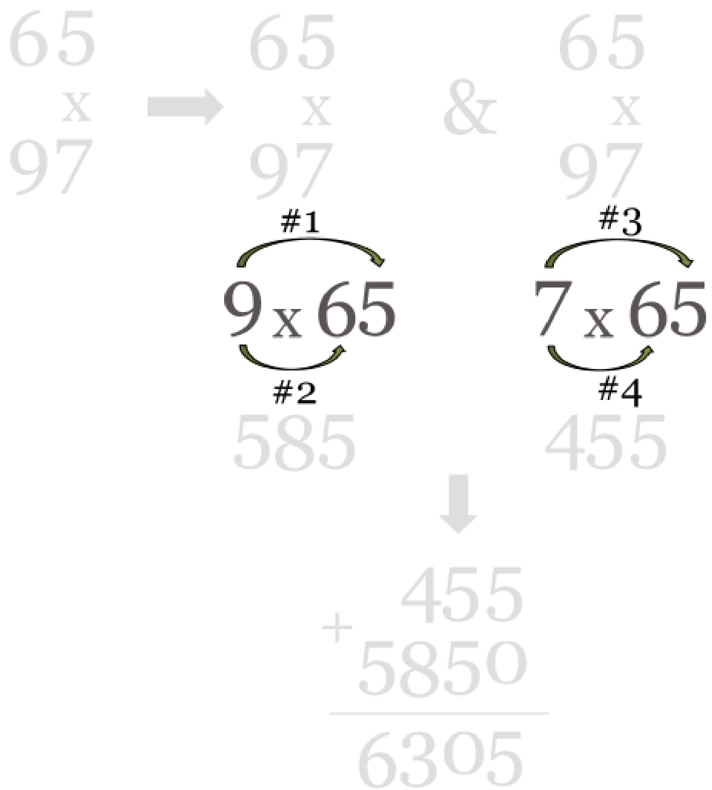
OK, we know that the answer is 180 and there are lots of intuitive methods that help us get the right answer. Indeed 12 x 15 it's just a bit more difficult to calculate than 10 x 15, because multiplying by 10 it really easy – we just add one 0 at the end of the number. Thus 15 x 10 equals 150. But now again on 12 x 15 – we know that this equals 10 x 15 (which is 150) and 2 x 15, which is also very easy to calculate and it is 30. The result of 12×15 will be 150 + 30, which fortunately isn't difficult to get and equals to 180.

That was easy but in some cases the calculations are a bit more difficult and we need a structured algorithm to get the right answer. What about 65 x 97? That is not so easy as 12 x 15, right?

The algorithm we know from the primary school, described on the diagram below, is well structured and help us multiply two numbers.



We see that even for two-digit numbers this is quite difficult – we have 4 multiplications and some additions.



We need 4 multiplications in order to calculate the product of two 2-digit numbers!

However so far we know how to multiply numbers, the only problem is that our task becomes very difficult as the numbers grow. If multiplying 65 by 97 was somehow easy, what about

```
374773294776321
x
222384759707982
```

It seems almost impossible.

History

Andrey Kolmogorov is one of the brightest russian mathematicians of the 20th century. In 1960, during a seminar, Kolmogorov stated that two n-digit numbers can't be multiplied with less than n^2 multiplications!
Only a week later a 23-year young student called Anatolii Alexeevitch Karatsuba proved that the multiplication of two n-digit numbers can be computed with n ^ lg(3) multiplications with an ingenious divide and conquer approach.

Overview

Basically Karatsuba stated that if we have to multiply two n-digit numbers x and y, this can be done with the following operations, assuming that B is the base of and m < n.

First both numbers x and y can be represented as x1,x2 and y1,y2 with the following formula.

```
x = x1 * B^m + x2
y = y1 * B^m + y2
```

Obviously now xy will become as the following product.

```
xy = (x1 * B^m + x2)(y1 * B^m + y2) =>

a = x1 * y1
b = x1 * y2 + x2 * y1
c = x2 * y2
```

Finally xy will become:

```
xy = a * B^2m + b * B^m + c
```

However a, b and c can be computed at least with four multiplication, which isn't a big optimization. That is why Karatsuba came up with the brilliant idea to calculate b with the following formula:

```
b = (x1 + x2)(y1 + y2) - a - c
```

That make use of only three multiplications to get xy.

Let's see this formula by example.

```
47 x 78

x = 47
x = 4 * 10 + 7

x1 = 4
x2 = 7

y = 78
y = 7 * 10 + 8

y1 = 7
y2 = 8

a = x1 * y1 = 4 * 7 = 28
c = x2 * y2 = 7 * 8 = 56
b = (x1 + x2)(y1 + y2) - a - c = 11 * 15 - 28 - 56
```

Now the thing is that 11 * 15 it's again a multiplication between 2-digit numbers, but fortunately we can apply the same rules two them. This makes the algorithm of Karatsuba a perfect example of the “divide and conquer” algorithm.

Implementation

Standard Multiplication

Typically the standard implementation of multiplication of n-digit numbers require n^2 multiplications as you can see from the following PHP implementation.

```
$x = array(1,2,3,4);
$y = array(5,6,7,8);

function multiply($x, $y)
{
 $len_x = count($x);
 $len_y = count($y);
 $half_x = ceil($len_x / 2);
 $half_y = ceil($len_y / 2);
 $base = 10;

 // bottom of the recursion
 if ($len_x == 1 && $len_y == 1) {
  return $x[0] * $y[0];
 }

 $x_chunks = array_chunk($x, $half_x);
 $y_chunks = array_chunk($y, $half_y);

 // predefine aliases
 $x1 = $x_chunks[0];
 $x2 = $x_chunks[1];
 $y1 = $y_chunks[0];
 $y2 = $y_chunks[1];

 return  multiply($x1, $y1) * pow($base, $half_x * 2)      // a
    + (multiply($x1, $y2) + multiply($x2, $y1)) * pow($base, $half_x)   // b
    + multiply($x2, $y2);       // c
}

// 7 006 652
echo multiply($x, $y);
```

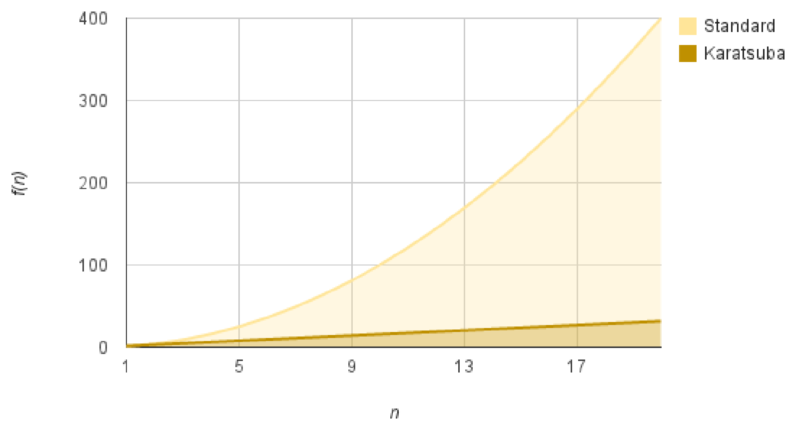
Karatsuba Multiplication

Karatsuba replaces two of the multiplications – this of x1 * y2 + x2 * y1 with only one – (x1 + x2)(y1 + y2) and this makes the algorithm faster.

```
$x = array(1,2,3,4);
$y = array(5,6,7,8);

function karatsuba($x, $y)
{
 $len_x = count($x);
 $len_y = count($y);

 // bottom of the recursion
 if ($len_x == 1 && $len_y == 1) {
  return $x[0] * $y[0];
 }
 if ($len_x == 1 || $len_y == 1) {
  $t1 = implode('', $x);
  $t2 = implode('', $y);
  return (int)$t1 * $t2;
 }

 $a = array_chunk($x, ceil($len_x/2));
 $b = array_chunk($y, ceil($len_y/2));

 $deg = floor($len_x/2);

 $x1 = $a[0]; // 1
 $x2 = $a[1]; // 2
 $y1 = $b[0]; // 1
 $y2 = $b[1]; // 2

 return  ($a = karatsuba($x1, $y1)) * pow(10, 2 * $deg)
   + ($c = karatsuba($x2, $y2))
   + (karatsuba(sum($x1, $x2), sum($y1, $y2)) - $a - $c) * pow(10, $deg);
}

// 7 006 652
echo karatsuba($x, $y);
```

Complexity

Assuming that we replace two of the multiplications with only one makes the program faster. The question is how fast. Karatsuba improves the multiplication process by replacing the initial complexity of $O(n^2)$ by $O(n^{lg3})$, which as you can see on the diagram below is much faster for big n.



O(n^2) grows much faster than O(n^lg3)

Application

It's obvious where the Karatsuba algorithm can be used. It is very efficient when it comes to integer multiplication, but that isn't its only advantage. It is often used for polynomial multiplications.

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12 thoughts on "Computer Algorithms: Karatsuba Fast Multiplication"

192709

May 21, 2012 at 12:21 pm

Hi,
I think that
b = (x1 + x2)(y1 + y2) – a – b = 11 * 15 – 28 – 56
should be
b = (x1 + x2)(y1 + y2) – a – c = 11 * 15 – 28 – 56
.
Good Article – thank you!
Marcus

★ Stoimen

May 21, 2012 at 12:24 pm

@192709 – Right! Fixed!

Joshua

September 15, 2012 at 9:34 pm

Thanks a million for this article; it is very simple, understandable and well written. I Needed to learn this as part of my Algorithms course and you really simplified the Karatsuba method for me. Thank you very much

shrinath

December 8, 2012 at 11:28 am

hi. very nice article. you made it too simple to understand.. great work.

OptiMiser

January 16, 2013 at 11:14 pm

Good Article. True. But I was looking for fast algorithm of multiplication for big integers that may be 1522456*4^100 000 000… in size. So this algorithm multiplies the job of CPU instead of simplifying 🙂

gerardo

February 9, 2013 at 11:42 am

sorry. I've got a question: Could I represent a number in three parts and then apply this algorithm, or y only can apply this with two parts.

★ Stoimen

February 11, 2013 at 12:08 pm

Of course, you can use the associative rule.

Priyanka Kumari

May 9, 2013 at 10:10 am

very neatly made us understand the algorithm…. gud wrk really helpful!!!

Niko

May 24, 2013 at 1:12 pm

Hey that's a very good article which you wrote there.
Thanks a lot.
I habe only a question, to say it better, a problem executing your code on PHP.
I dont't get the correct result and i can not figure it out why. First of i got a fatal error because the function "sum()" does not exists. I implemented it but i dont know how exactly, becuase the parameters which are loaded in the sum() function are arrays.
If i got this right, we have to sum alla array-values and the the 2 sums again, and return it.
I hope this is still reading someone and could help me.

Thank you.

Nezza

February 11, 2014 at 2:39 am

Hi there, this is very simple and easy to understand, thank you and you have done a grate job.

Dakota

April 5, 2014 at 11:30 pm

What will be the no of single digits multiplications to compute the product of two 6 digits numbers?

Himanshu

October 31, 2014 at 11:02 am

Hi,

Thanks for the article, it is really well explained with great examples.
But I tried implementing in java based on your code for PHP but you are considering different length for X and Y variable while dividing into two parts.
It was not working for me with different size(xlen/2, ylen/2) of X and Y so I tried with the min value of (xlen/2, ylen/2), and then its working fine.
Could you please tell me why this is not working in case of your logic.

Thanks for the great article

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