The Probabilistic Method · Probability (statistics) · Statistics (academic discipline)

[illegible]

https://www.quora.com/What-are-the-best-applications-of-linearity-of-expectation

6 Answers

Ray Li, MOPper

1.9k Views · Upvoted by Vladimir Novakovski, Competed in the USA Math Olympiad and Putnam and Ben Golub, Ph.D. probability courses at Stanford; do research in probability theory

Another cute example: Al plays a solitaire game. He starts with a rooted tree. On a single move, he selects one of the remaining vertices in the tree uniformly at random, and removes it along with its entire subtree. The game ends when he removes the root vertex. Determine, as a function of the tree, the expected length of the game.

Solution: It's quite tedious to do this using the basic definition of expected value. Instead, note that a vertex is removed when either it is selected or one of its ancestors is selected, and all these happen with equal probability. Thus, the probability that a node is selected is $\frac{1}{\text{depth}+1}$, so the expected number of selected nodes (i.e. the expected number of moves) is $\sum_{\text{vertex } v} \frac{1}{\text{depth}_v+1}$

Source: Problem - 280C - Codeforces

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Written 20 Mar 2014 · View Upvotes · Answer requested by Eugene Chen

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Qiaochu Yuan, PhD student in Mathematics at UC Berkeley

9.7k Views · Upvoted by Alon Amit, Math Ph.D. with lots of practical experience in statistics. and Ben Golub, Ph.D. probability courses at Stanford; do research in probability theory

Buffon's needle is the following classic problem: suppose you drop a needle of length 1 on a floor made of parallel strips, each of which has width 1. What's the probability that the needle lands on the boundary between two strips?

There is a straightforward solution that involves computing a certain integral. But you can use linearity of expectation to solve this problem without computing any integrals, by solving a harder-looking problem called **Buffon's noodle**. In Buffon's noodle, the needle is... well, it's a noodle now. It still has length 1, but it can curve. The new question is: what's the *expected number of times* that the noodle intersects the boundary between two strips? Note that if the noodle is a needle, then it can intersect a boundary at most once (with probability 1), so this reduces to the probability in the needle case.

But rephrasing the problem as a problem about expectation allows us to cleverly leverage linearity of expectation, as follows.

1. Generalize the problem to a noodle of arbitrary length L.
2. Note that if you cut the noodle into two smaller noodles, then the expected number of intersections for the big noodle is the sum of the expected number of intersections for the two smaller noodles. It follows that the answer is linear with respect to cutting up noodles.
3. This part requires a bit of handwaving; it can be done more rigorously but I think that detracts from the charm of the argument. Cut the noodle up into a bunch of really small noodles. If the noodles are reasonable (e.g. not space-filling curves), then the really small noodles will be

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approximately really small needles, and if you make sure they all have the same really small length then the answer should be approximately the same for all of them. It follows that the answer is linear with respect to the *length* of noodles. That is, it must be a constant multiple cL of L.

4. It remains to determine the constant c, and to do that it suffices to solve the problem for any noodle shape. Let's solve it for a circle of diameter 1, where it's obvious: the number of intersections is always 2. Since a circle of diameter 1 has circumference π, it follows that $c = \frac{2}{\pi}$.

Hence the answer to Buffon's noodle, and in particular Buffon's needle, is $\frac{2}{\pi}$.

Written 22 May 2015 · View Upvotes

Eugene Chen, MIT, Mathematics '17
4k Views · Upvoted by David Joyce, Professor of Mathematics at Clark University and Vladimir Novakovski, Competed in the USA Math Olympiad and Putnam

Here's a few cute examples involving permutations.

What is the expected number of fixed points in a permutation π of $(1, 2, \ldots, n)$?

Let I_k denote the indicator random variable that is 1 if $\pi(k) = k$, and 0 otherwise. It is clear that $\mathbb{E}[I_k] = \frac{1}{n}$. By linearity of expectation,

$$\mathbb{E}[I_1 + I_2 + \cdots + I_n] = \sum_{k=1}^{n} \mathbb{E}[I_k] = \sum_{k=1}^{n} \frac{1}{n} = 1.$$

So the expected number of fixed points in a permutation is independent of the number of elements in the permutation.

What is the expected number of cycles in a permutation π of $(1, 2, \ldots, n)$?

Let $X_k = \frac{1}{l}$, where the length of the cycle including k is l. Let $X = X_1 + X_2 + \cdots + X_n$. We want to compute $\mathbb{E}[X]$, since each cycle of length l contributes a total of $l \cdot \frac{1}{l} = 1$ to the sum.

We'll compute X_k for some k. For each l, the probability that k is in a cycle of length l is

$$\frac{n-1}{n} \cdot \frac{n-2}{n-1} \cdot \cdots \cdot \frac{n-l+1}{n-l+2} \cdot \frac{1}{n-l+1} = \frac{1}{n}.$$

It follows that

$$\mathbb{E}[X_k] = \sum_{l=1}^{n} \frac{1}{nl} = \frac{1}{n} H_n,$$

where H_n denotes the nth harmonic number. By linearity of expectation,

$$\mathbb{E}[X] = \sum_{k=1}^{n} \mathbb{E}[X_k] = H_n,$$

which is roughly $\ln n$.

Let $(a_1, a_2, \ldots, a_n)$ be a random permutation of $(1, 2, \ldots, n)$. What is the expected number of distinct values in $\{\max_{1 \le i \le n}(a_1, \ldots, a_i)\}$?

For each $1 \le k \le n$ let I_k denote the indicator random variable that is 1 if $\max_{1 \le i \le k}(a_1, \ldots, a_i) \ne \max_{1 \le i \le k-1}(a_1, \ldots, a_i)$, and 0 otherwise. Then I_k is 1 if a_k is larger than each of $a_{k-1}, \ldots, a_1$, which occurs with probability $\frac{1}{k}$. By linearity of expectation,

$$\mathbb{E}[I_1 + I_2 + \cdots + I_n] = \sum_{k=1}^{n} \mathbb{E}[I_k] = \sum_{k=1}^{n} \frac{1}{k} = H_n,$$

which is roughly $\ln n$.

Steven Hao

Let G be a directed graph, initially empty, on n nodes. For each node x, we independently choose a node y uniformly at random from the n nodes of the graph. With probability p, we draw a directed edge from node x to y.
Then the probability that there is a cycle in G is p.

To prove this, we use strong induction on n.
We label the nodes 1...n.
When n=1, the probability of a cycle is the probability that node 1 has an edge to itself, which is indeed p.

For the inductive step, we consider the set S of nodes who have an edge coming out them. Let G' be the subgraph of G consisting of the nodes in S and the edges connecting them, and let $k = |S|$.

Each node in G' has a probability $\frac{k}{n}$ of having an out-edge.

If $k < n$, by the inductive hypothesis, G' has a cycle with probability $\frac{k}{n}$.

If $k = n$, then the probability of a cycle is $\frac{k}{n} = 1$ (any graph with n nodes and n edges has a cycle)

In any case, the probability of a cycle existing is $\frac{k}{n}$, which is on average (this is where we apply linearity of expectation) $\frac{np}{n} = p$.

Obinna Okechukwu, Knows a little about mathematics

One clean and simple application of the linearity of expectation (and probabilistic method) is in the proof of the following theorem by Szele (1943) :

> There exists a tournament, T with n players and

> $\frac{n!}{2^{n-1}}$ hamiltonian paths.

It can be proven using a very simple application of the probabilistic method.

Proof
Take a random tournament
T' where we determine the outcome of each game by a flip of a fair coin (independently).
Let the expected number of hamiltonian paths in this tournament be
X.
Now for each permutation
σ of the vertices (players), define an indicator random variable
X_σ such that
$X_\sigma = 1$ if it forms a hamiltonian path and
$X_\sigma = 0$ otherwise.

Note that the probability that a hamiltonian path is formed is simply
$\frac{1}{2^{n-1}}$. So we can conclude that
$E[X_\sigma] = \frac{1}{2^{n-1}}$

Also,
$X = \sum X_\sigma$ over all
$n!$ permutations.
By the linearity of expectation,
$E[X] = \sum E[X_\sigma]$

So
$E[X] = \frac{n!}{2^{n-1}}$

By the pigeonhole principle, there exists a tournament
T which has at least
$E[X]$ hamiltonian paths. And we are done.

Written 8 Mar 2014 · View Upvotes

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Diane Burrell, Yeah, I've had experiences. I actually learned from a few, too.
81.7k Views

Something that happened to me was pretty bad. I was once mugged, violently, in plain view on a street in the evening. I was preparing to get on my scooter, and was just placing my backpack on the seat . A young man came up, said "Excuse me,", and then grabbed my backpack - which had a strap still around my wrist. He dragged me across the street past a stopped jeep and into an alley, where his fr...

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