Discrete Mathematics - Recurrence Relation

Previous Page Next Page

In this chapter, we will discuss how recursive techniques can derive sequences and be used for solving counting problems. The procedure for finding the terms of a sequence in a recursive manner is called **recurrence relation**. We study the theory of linear recurrence relations and their solutions. Finally, we introduce generating functions for solving recurrence relations.

Definition

A recurrence relation is an equation that recursively defines a sequence where the next term is a function of the previous terms (Expressing F_n as some combination of F_i with $i<n$).

Example − Fibonacci series: $F_n = F_{n-1} + F_{n-2}$, Tower of Hanoi: $F_n = 2F_{n-1} + 1$

Linear Recurrence Relations

A linear recurrence equation of degree k is a recurrence equation which is in the format $x_n = A_1 x_{n-1} + A_2 x_{n-1} + A_3 x_{n-1} + \dots A_k x_{n-k}$ (A_n is a constant and $A_k \neq 0$) on a sequence of numbers as a first-degree polynomial.

These are some examples of linear recurrence equations −

Recurrence relations	Initial values	Solutions
$F_n = F_{n-1} + F_{n-2}$	$a_1 = a_2 = 1$	**Fibonacci number**
$F_n = F_{n-1} + F_{n-2}$	$a_1 = 1, a_2 = 3$	**Lucas number**
$F_n = F_{n-2} + F_{n-3}$	$a_1 = a_2 = a_3 = 1$	**Padovan sequence**
$F_n = 2F_{n-1} + F_{n-2}$	$a_1 = 0, a_2 = 1$	**Pell number**

How to solve linear recurrence relation

Suppose, a two ordered linear recurrence relation is: $F_n = AF_{n-1} + BF_{n-2}$ where A and B are real numbers.

The characteristic equation for the above recurrence relation is −

$$x^2 - Ax - B = 0$$

Three cases may occur while finding the roots −

Case 1 − If this equation factors as $(x - x_1)(x - x_1) = 0$ and it produces two distinct real roots x_1 and x_2, then $F_n = ax_1^n + bx_2^n$ is the solution. [Here, a and b are constants]

Case 2 − If this equation factors as $(x - x_1)^2 = 0$ and it produces single real root x_1, then $Fn = a\, x_1^n + bn\, x_1^n$ is the solution.

Case 3 − If the equation produces two distinct real roots x_1 and x_2 in polar form $x_1 = r \angle \theta$ and $x_2 = r \angle(-\theta)$, then $F_n = r^n (a \cos(n\theta) + b \sin(n\theta))$ is the solution.

Problem 1

Solve the recurrence relation $F_n = 5F_{n-1} - 6F_{n-2}$ where $F_0 = 1$ and $F_1 = 4$

Solution

The characteristic equation of the recurrence relation is −

$$x^2 - 5x + 6 = 0,$$

So, $(x - 3)(x - 2) = 0$

Hence, the roots are −

$$x_1 = 3 \text{ and } x_2 = 2$$

The roots are real and distinct. So, this is in the form of case 1

Hence, the solution is −

$$F_n = ax_1^n + bx_2^n$$

Here, $F_n = a3^n + b2^n$ (As $x_1 = 3$ and $x_2 = 2$)

Therefore,

$1 = F_0 = a3^0 + b2^0 = a+b$

$4 = F_1 = a3^1 + b2^1 = 3a+2b$

Solving these two equations, we get $a = 2$ and $b = -1$

Hence, the final solution is −

$$F_n = 2.3^n + (-1) . 2^n = 2.3^n - 2^n$$

Problem 2

Solve the recurrence relation $F_n = 10F_{n-1} - 25F_{n-2}$ where $F_0 = 3$ and $F_1 = 17$

Solution

The characteristic equation of the recurrence relation is −

$x^2 - 10x - 25 = 0,$

So, $(x - 5)^2 = 0$

Hence, there is single real root $x_1 = 5$

As there is single real valued root, this is in the form of case 2

Hence, the solution is −

$$F_n = ax_1^n + bnx_1^n$$

$$3 = F_0 = a.5^0 + b.0.5^0 = a$$

$$17 = F_1 = a.5^1 + b.1.5^1 = 5a+5b$$

Solving these two equations, we get a = 3 and b = 2/5

Hence, the final solution is −

$$F_n = 3.5^n + (2/5) .n.2^n$$

Problem 3

Solve the recurrence relation $F_n = 2F_{n-1} - 2F_{n-2}$ where $F_0 = 1$ and $F_1 = 3$

Solution

The characteristic equation of the recurrence relation is −

$$x^2 -2x -2 = 0$$

Hence, the roots are −

$x_1 = 1 + i$ and $x_2 = 1 - i$

In polar form,

$x_1 = r \angle \theta$ and $x_2 = r \angle(- \theta)$, where $r = \sqrt{2}$ and $\theta = \pi / 4$

The roots are imaginary. So, this is in the form of case 3.

Hence, the solution is −

$$F_n = (\sqrt{2})^n (a \cos(n. \pi / 4) + b \sin(n. \pi / 4))$$

$$1 = F_0 = (\sqrt{2})^0 (a \cos(0. \pi / 4) + b \sin(0. \pi / 4)) = a$$

$$3 = F_1 = (\sqrt{2})^1 (a \cos(1. \pi / 4) + b \sin(1. \pi / 4)) = \sqrt{2} (a/\sqrt{2} + b/\sqrt{2})$$

Solving these two equations we get a = 1 and b = 2

Hence, the final solution is −

$$F_n = (\sqrt{2})n (\cos(n. \pi / 4) + 2 \sin(n. \pi / 4))$$

Particular Solutions

A recurrence relation is called non-homogeneous if it is in the form

$F_n = AF_{n-1} + BF_{n-2} + F(n)$ where $F(n) \neq 0$

The solution (a_n) of a non-homogeneous recurrence relation has two parts. First part is the solution (a_h) of the associated homogeneous recurrence relation and the second part is the particular solution (at). So, $a_n = a_h + a_t$.

Let $F(n) = cx^n$ and x_1 and x_2 are the roots of the characteristic equation −

$x^2 = Ax + B$ which is the characteristic equation of the associated homogeneous recurrence relation −

- If $x \neq x_1$ and $x \neq x_2$, then $a_t = Ax^n$
- If $x = x_1$, $x \neq x_2$, then $a_t = Anx^n$
- If $x = x_1 = x_2$, then $a_t = An^2x^n$

Problem

Solve the recurrence relation $F_n = 3F_{n-1} + 10F_{n-2} + 7.5^n$ where $F_0 = 4$ and $F_1 = 3$

Solution

The characteristic equation is −

$$x^2 - 3x -10 = 0$$

Or, $(x-5)(x+2) = 0$

Or, $x_1 = 5$ and $x_2 = -2$

Since, $x = x_1$ and $x \neq x_2$, the solution is −

$$a_t = Anx^n = An5^n$$

After putting the solution into the non-homogeneous relation, we get −

$$An5^n = 3A(n-1)5^{n-1} + 10A(n-2)5^{n-2} + 7.5^n$$

Dividing both sides by 5^{n-2}, we get −

$$An5^2 = 3A(n-1)5 + 10A(n-2)5^0 + 7.5^2$$

Or, $25An = 15An - 15A + 10An - 20A + 175$

Or, $35A = 175$

Or, $A = 5$

So, $F_n = n5^{n+1}$

Hence, the solution is −

$$F_n = n5^{n+1} + 6.(-2)^n - 2.5^n$$

Generating Functions

Generating Functions represents sequences where each term of a sequence is expressed as a coefficient of a variable *x* in a formal power series.

Mathematically, for an infinite sequence, say $a_0, a_1, a_2, \ldots\ldots\ldots, a_k, \ldots\ldots$, the generating function will be −

$$G_x = a_0 + a_1x + a_2x^2 + \ldots\ldots\ldots + a_kx^k + \ldots\ldots\ldots = \sum_{k=0}^{\infty} a_kx^k$$

Some Areas of Application

Generating functions can be used for the following purposes −

- For solving a variety of counting problems. For example, the number of ways to make change for a Rs. 100 note with the notes of denominations Rs.1, Rs.2, Rs.5, Rs.10, Rs.20 and Rs.50
- For solving recurrence relations
- For proving some of the combinatorial identities
- For finding asymptotic formulae for terms of sequences

Problem 1

What are the generating functions for the sequences $\{a_k\}$ with $a_k = 2$ and $= 3k$?

Solution

When $a_k = 2$, generating function,

$G(x) = \sum_{k=0}^{\infty} 2x^k = 2 + 2x + 2x^2 + 2x^3 + \ldots\ldots\ldots$

When $a_k = 3k$, $G(x) = \sum_{k=0}^{\infty} 3kx^k = 0 + 3x + 6x^2 + 9x^3 + \ldots\ldots$

Problem 2

What is the generating function of the infinite series; 1, 1, 1, 1, ……….?

Solution

Here, $a_k = 1$, *for* $0 \leq k \leq \infty$.

Hence, $G(x) = 1 + x + x^2 + x^3 + \ldots\ldots\ldots = \frac{1}{(1-x)}$

Some Useful Generating Functions

- For $a_k = a^k$, $G(x) = \sum_{k=0}^{\infty} a^kx^k = 1 + ax + a^2x^2 + \ldots\ldots\ldots = \frac{1}{(1-ax)}$
- For $a_k = (k+1)$, $G(x) = \sum_{k=0}^{\infty} (k+1)x^k = 1 + 2x + 3x^2 \ldots\ldots\ldots = \frac{1}{(1-x)^2}$
- For $a_k = c_k^n$, $G(x) = \sum_{k=0}^{\infty} c_k^nx^k = 1 + c_1^nx + c_2^nx^2 + \ldots\ldots\ldots + x^2 = (1+x)^n$
- For $a_k = \frac{1}{k!}$, $G(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \ldots\ldots\ldots = e^x$