Discrete Mathematics

GO Classes

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$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \qquad \sum \frac{x^i}{i!}$$

Imp formulae

$$1+2+3+4+\ldots\ldots n = \frac{n(n+1)}{2}$$

$$1^2+2^2+3^2+4^2+\ldots\ldots n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$1^3+2^3+3^3+4^3+\ldots\ldots n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Sequence and series

1. Arithematic Series

n^{th} term, $a_n = a+(n-1)d$

Sum of n terms, $S_n = \frac{n}{2}\left[2a+(n-1)d\right]$

2. Geometric Series

n^{th} term, $a_n = ar^{n-1}$

Sum of n terms, $S_n = a\left(\frac{1-r^n}{1-r}\right)$

$S_\infty = \frac{a}{1-r}$

3. Harmonic series.

$\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d} + \dots \frac{1}{a+(n-1)d}$

n^{th} term, $a_n = \frac{1}{a+(n-1)d}$

Sum of n terms, $S_n = \frac{1}{d} \ln\left\{\frac{2a+(2n-1)d}{2a-d}\right\}$

ARITHEMATIC - GEOMETRIC PROGRESSION

A.P. :- $1, 2, 3, \ldots\ldots n$

G.P. :- $x, x^2, x^3, \ldots\ldots n$

$1x + 2x^2 + 3x^3 + \ldots\ldots nx^n$ A.G.P.

series obtained by multiplying corresponding terms of A.P. & G.P.

Ques:- $1 + 2x + 3x^2 + 4x^3 + \ldots\ldots$

$$S_n = 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots nx^{n-1}$$

$$- \; xS_n = \quad x + 2x^2 + 3x^3 + 4x^4 \ldots\ldots\ldots nx^n$$

$$(1-x)S_n = 1 + x + x^2 + x^3 + \ldots\ldots\ldots x^{n-1} - nx^n$$

$$= 1\left(\frac{1-x^n}{1-x}\right) - nx^n$$

For infinite terms —

$$\therefore (1-x)S_\infty = \frac{1}{1-x} \qquad \therefore S_\infty = \frac{1}{(1-x)^2}$$

Ques:- $\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \ldots$

A.P. :- $1, 2, 3, 4 \ldots$

G.P. :- $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16} \ldots$

$$S_\infty = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \ldots$$

$$\frac{1}{2}S_\infty = \quad \frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \frac{4}{32} + \ldots$$

$$S_\infty - \frac{1}{2}S_\infty = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$$

$$\left(1 - \frac{1}{2}\right)S_\infty = \frac{1}{2} \cdot \frac{1}{1-\frac{1}{2}}$$

$$\therefore S_\infty = \cancel{\frac{1}{2}} \times \frac{1}{\cancel{1/2}} \times \frac{1}{1/2} = 2$$

Ques:- $2 + 5x + 8x^2 + 11x^3 + \dots \infty$

$$S_\infty = 2 + 5x + 8x^2 + 11x^3 + \dots \infty$$

$$xS_\infty = 2x + 5x^2 + 8x^3 + 11x^4 + \dots \infty$$

$$S_\infty - xS_\infty = 2 + 3x + 3x^2 + 3x^3 + \dots \infty$$

$$(1-x) S_\infty = \cancel{2 + 3(x +} \; 2 + 3x(1 + x + x^2 + \dots \infty)$$

$$(1-x) S_\infty = 2 + 3x \cdot 1 \cdot \frac{1}{1-x} \Rightarrow (1-x) S_\infty = 2 + \frac{3x}{1-x} = \frac{2+x}{1-x}$$

$$\therefore S_\infty = \frac{2+x}{(1-x)^2}$$

Ques:- $1 - 3x + 5x^2 - 7x^3 + \dots \infty$

$$S_\infty = 1 - 3x + 5x^2 - 7x^3 + \dots \infty$$

$$-xS_\infty = -x + 3x^2 - 5x^3 + 7x^4 + \dots \infty$$

$$(1+x) S_\infty = 1 - 2x + 2x^2 - 2x^3 + \dots \infty$$

$$(1+x) S_\infty = 1 - 2x(1 - x + x^2 + \dots \infty)$$

$$(1+x) S_\infty = 1 - 2x \cdot 1 \cdot \frac{1}{1+x} = \frac{1+x-2x}{1+x} = \frac{1-x}{1+x}$$

$$\Rightarrow S_\infty = \frac{1-x}{(1+x)^2}$$

Priyanshu (GATE 2024 AIR 379, GATE 2023 AIR 384) @ https://x.com/Priyanshu__2109

Modular Arithematic

Divisibility -

If a and b are integers, a divides b if there exists an integer c such that

$$ac = b$$

$a|b \longrightarrow$ a divides b?

Division algorithm.

Let a be an integer and d be a positive inte[ger]. Then, there are unique integers q and r with $0 \le r < d$ such that

$$a = dq + r.$$

$$r = a \bmod d$$

* Remainder should be greater than 0

$-11 \div 3 \rightarrow q = -4$, $r = 2$

* If 2 numbers a and b give the same remainder when divided by m, then, we say a is congruent to b modulo m

$$a \equiv b \pmod{m}$$

$$a \% m = b \bmod m.$$

$$10 \equiv 15 \pmod 5$$

$$\boxed{\begin{array}{l} a \bmod n = r \\ a \pm nk \bmod n = r \end{array}}$$

$a \equiv b \pmod{n}$ $\qquad \begin{aligned} a &= nk_1 + r \\ b &= nk_2 + r \end{aligned}$

$\Rightarrow \quad a - b = n(k_1 - k_2)$

$\boxed{\Rightarrow \quad a-b \text{ is multiple of } n.}$ $\qquad n \mid a-b \to T$

$a \equiv b \pmod{n} \iff n \mid (a-b) \iff a \bmod n = b \bmod n$

$a \equiv b \pmod{n}$

$\Rightarrow ((((a \bmod n) \bmod n) \bmod n) \bmod n) \equiv b \pmod{n}$

Important theorems

If $a \equiv b \pmod{m}$ and
$c \equiv d \pmod{m}$ then,

$a + c \equiv b + d \pmod{m}$

$a * c \equiv b * d \pmod{m}$

$a - c \equiv b - d \pmod{m}$

★ Division does not hold !!

Ques

$994 \cdot 996 \cdot 997 \cdot 998 \pmod{1000}$

$= (994 \bmod 1000)(996 \bmod 1000)(997 \bmod 1000)(998 \bmod 100$

$= (-6 \bmod 1000)(-4 \bmod 1000)(-3 \bmod 1000)(-2 \bmod 1000)$

$= (-6) * (-4)(-3)(-2) \bmod 1000$

$= 144 \bmod 1000$

$= \underline{144}$

Ques

$17^{753} \bmod 9$

$= 8^{753} \bmod 9 = (8 \bmod 9)^{753} \bmod 9$

$= (-1 \bmod 9)^{753} \bmod 9 = (-1)^{753} \bmod 9$

$= -1 \bmod 9 = \underline{\underline{8}}$

PROOF TECHNIQUES

- Disproof by counterexample.
- Exhaustive proof
- Direct proof
- Proof by ~~count~~ contraposition
- Proof by contradiction

1. Direct Proof

To proove :- If P then Q

Start with P (Assume)

$\downarrow$ Apply facts that are already known.

Derive Q

If n is even, n^2 is even

2. Proof by ~~count~~ contraposition

To proove :- If P then Q,

$P \Rightarrow Q$

$\overline{Q} \rightarrow \overline{P}$

Start with $\overline{Q}$.

$\downarrow$ Apply facts

Ex $\rightarrow$ If n^2 is even, n is even

Derive $\overline{P}$

3. Proof by contradiction

If $P \Rightarrow Q$ $\longrightarrow$ We need to proove that $\overline{P}$ is impossible

Proove $\sqrt{2}$ is irrational

let $\sqrt{2} = \frac{a}{b}$ a, b = no common factor ge other than 1

$2 = \frac{a^2}{b^2} \Rightarrow 2b^2 = a^2$

$\Rightarrow a^2$ is even $\Rightarrow a$ is even

let $a = 2k$

$2b^2 = (2k)^2 \Rightarrow 4k^2 = 2b^2 \Rightarrow b^2 = 2k^2$

b^2 is even $\Rightarrow b$ is even

a, b have 2 as factor

i.e. $GCD(a, b) \neq 1$

$\therefore$ assumption is wrong!

$\therefore \sqrt{2}$ is not rational

$\therefore \sqrt{2}$ is irrational.

4. Disproof by counter example

Find an example where the assumption is false.

Logarithm

$$\log_b a^m = m \log_b a$$

$$\log_b mn = \log_b m + \log_b n$$

$$\log_b\left(\frac{m}{n}\right) = \log_b m - \log_b n$$

$$a = b^{\log_b a}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^{\log_b c} = c^{\log_b a}$$

Discrete Mathematics

- Mathematical Logic (37)+(39)
- Set Theory (62)
- Combinatorics (27+)
- Graph Theory (20+)
- Group Theory (40+)

⊙ field of mathematics where we study discrete objects.

1. Mathematical Logic

⊙ Language for Computers / AI / Automated Reasoning

⊙ Mathematical logic gives precise, unambiguous meaning to mathematical statements / theorems, etc.

Propositional Logic

First Order Logic.

→ Propositional logic

⊙ simplest logic

⊙ variable can be either $T_{(1)}$ or $F_{(0)}$

Proposition :-

declarative sentence that can be either true or false (cannot be both)

commands
questions
paradoxes
free variable
} not propositions

Propositional variable :-

- each proposition is represented by a propositional variable.
- denoted by lower case letters
- each variable can be either T or F (not both)

Atomic Proposition :- proposition whose truth or falsity does not depend on the truth or falsity of any other proposition

- atomic proposition is represented by proposition variable.

Compound Proposition :- propositions formed by combination of one or more atomic propositions using logical connectives

Standard Logical Connectives

- NOT ($\neg P \equiv \sim p \equiv \overline{P} \equiv p'$)
- AND ($\cdot$, $\wedge$)
- OR ($+$, $\vee$)
- Exclusive OR ($\oplus$)
  - OR :... but not both
- Implication ($\rightarrow$)
- Biimplication ($\leftrightarrow$)
- NAND ($\uparrow$)
- NOR ($\downarrow$)

(And = But = Although = however = moreover = yet
still = even though = though = nevertheless,)

→ translate to conjunction

$p \rightarrow q$

- p → Condition / Antecedent / Hypothesis / Premise
- q → Conclusion / Consequence

P is necessary for Q = $Q \rightarrow P$

P is sufficient for Q = $P \rightarrow Q$

P is sufficient for Q = Q is necessary for P

$A \leftrightarrow B = A \to B$ AND $B \to A$

★ $P \leftrightarrow Q = \overline{P \oplus Q}$

1. Implication = Property
2. Biimplication = Definition

. $P \Leftrightarrow Q$
- P biimplication Q
- P implies Q & Q implies P
- P implies Q & vice versa
- Q if and only if P.
- P is sufficient and necessary for Q

Propositional Logic
collection of ~~p~~ all propositional formulae.

Precedence -

$\neg > \wedge > \vee > \to > \leftrightarrow$

$T \to Q = Q$

$F \to Q = T$

$P \to T = T$

$P \to F = \neg P$

P	Q	P→Q
F	F	T
F	T	T
T	F	F
T	T	T

Propositional formula –

- Symbols T and F are propositional formulae.
- Every propositional variable is propositional formula.
- Propositional formulae connected by logical connectives are propositional formula.

Truth Table

Truth Table tells about the truth value of a compound proposition for each combination of truth value of atomic propositions.

Tautology :- compound proposition which is always TRUE for all truth values of atomic propositions.

Contradiction :- compound proposition which is always
(Fallacy) FALSE for all truth values of atomic proposition

Unsatisfiable

Contingency :- compound proposition which is true for some truth values and false for some.

Satisfiable :- compound statement can be made true for atleast one combination of truth values.

~~Unsatisfiable :- compound statement can be made false for atleast one~~ combination of truth value.

Valid = Tautology = always true

Invalid :- Sometimes false = contradiction / contingency.

(consistent)
Satisfiable = sometimes true = tautology / contingency.

Unsatisfiable = contradiction = always false.
(inconsistent)

Falsifiable = sometimes false

not falsifiable = always true = tautology = valid.

Imp. –

1. always True
   - → Tautology
   - → Valid
   - → Logically true
   - → Unfalsifiable.

   satisfiable

2. sometimes true, sometimes false
   - → contingency
   - → satisfiable
   - → falsifiable
   - → invalid.

3. always false
   - → unsatisfiable
   - → logically false

By Case method –

Create 2 cases –
(I) P = True (II) P = False

Other variables can be either T or F.
simplify the expression for each case.

Ques
Show that $p \wedge \neg(q \vee p)$ is contradiction

Case I :- P is True

$T \wedge \neg(q \vee T)$

$T \wedge \neg T$

$T \wedge F$

F

Case II :- P is False

$F \wedge \neg(q \vee F)$

F

∴ The proposition is false contradiction

$(\neg(p \wedge q)) \leftrightarrow (\neg p \vee \neg q)$

Case I :- P is True

$(\neg(T \wedge q)) \leftrightarrow (\neg T \vee \neg q)$

$\neg q \leftrightarrow \neg q$

True

Case II :- P is False

$(\neg(F \wedge q)) \leftrightarrow (\neg F \vee \neg q)$

$(\neg F) \leftrightarrow (T \vee \neg q)$

$T \leftrightarrow T$

$= T$

∴ tautology

$(p \vee q) \leftrightarrow (q \vee p)$

Case I:- p is T

$(T \vee q) \leftrightarrow (q \vee T)$

$T \leftrightarrow T$

$\underline{\underline{T}}$

Case II:- P is F

$(F \vee q) \leftrightarrow (q \vee F)$

$q \leftrightarrow q$

$\underline{\underline{T}}$

<u>tautology</u>

Propositional expressions : α, β

$\alpha \equiv \beta$ (α is equivalent to β) iff

- They have same truth table
- always equal truth values
- $\alpha \leftrightarrow \beta$ is tautology
- $\alpha \to \beta$ and $\beta \to \alpha$ are tautologies

$P \to Q$
Converse :- $Q \to P$
Contrapositive :- $\neg Q \to \neg P$
Inverse :- $\neg P \to \neg Q$

AND = But = Although = Though = Even Though = However = Yet = Still = Moreover = Nevertheless = Nonetheless = comma (12)

$P \to Q$

- If P then Q
- Q if P
- Q <u>whenever</u> P
- Q is necessary for P
- Q <u>provided that</u> P
- Whenever P then also Q
- P is sufficient for Q
- P only if Q.
  Q only when

$P \leftrightarrow Q$

- P is necessary and sufficient for q.
- If P then q and conversely
- p iff q.

* P <u>unless</u> Q = If not Q, then, P

$\neg Q \to P$

$\neg P \to Q$

unless = if not

provided that = if

P unless Q = $\neg Q \to P$

= $Q + P$ = OR

$\therefore$ unless = OR

If p then q unless r

$\equiv (p \to q)$ unless r

$\equiv \neg r \to (p \to q)$

Logical Laws / Logical Identities

$\neg(\neg p) = p$

Domination ~~True~~ Laws

$p \vee T = T$
$p \wedge F = F$

Negation Laws

$A \vee \bar{A} = T$
$A \wedge \bar{A} = F$

Identity Law

$p \wedge T = p$
$p \vee F = p$

Idempotent law

$p \vee p = p$
$p \wedge p = p$

(only $\vee, \wedge$)

De Morgan's law

$\overline{p \vee q} = \bar{p} \wedge \bar{q}$

$\overline{p \wedge q} = \bar{p} \vee \bar{q}$

Absorption law

$p \vee pq = \cancel{p \vee q}\ p$

$p \wedge (p \vee q) = p$

Implication laws

$p \to q = \bar{p} \vee q$

$p \to q = \bar{q} \to \bar{p}$

Commutative law

$p \vee q = q \vee p$
$p \wedge q = q \wedge p$

$[\wedge, \vee, \leftrightarrow, \oplus, \uparrow, \downarrow]$ commutative operators

Associative law

$(p \vee q) \vee r = p \vee (q \vee r)$
$(p \wedge q) \wedge r = p \wedge (q \wedge r)$

$[\wedge, \vee, \leftrightarrow, \oplus]$ associative $[\to, \uparrow, \downarrow]$ not associative

Distributive law

$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$
$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$

[$\vee$ is distributive over $\wedge$
$\wedge$ is distributive over $\vee$]

$p \to (q \wedge r) = (p \to q) \wedge (p \to r)$
[$\to$ is distributive over $\wedge$]

$p \wedge (q \to r) = (p \wedge q) \to (p \wedge r)$
[$\wedge$ is distributive over $\to$]

$p \to (q \vee r) = (p \to q) \vee (p \to r)$
$p \vee (q \to r) = (p \vee q) \to (p \vee r)$

[$\to$ is distributive over $\vee$]
[$\vee$ is distributive over $\to$]

Formal definition of valid argument

$$\text{Argument}\begin{cases} \left.\begin{array}{c} P_1 \\ P_2 \\ \vdots \\ P_n \end{array}\right\} \text{Premises} \\ \overline{\quad C \quad} \; \} \text{ Conclusion} \end{cases}$$

The argument is true iff $P_1 \wedge P_2 \wedge P_3 \wedge \dots P_n \rightarrow C$ is tautology

To check if argument is valid or not –

↳ Make conclusion false and try to make all the premises true.

If possible, argument is invalid

If not, argument is valid.

$$\begin{array}{c} A \rightarrow B \\ A \\ \hline B \end{array}$$

argument is very popular known as <u>modus ponens</u>

(1) $\begin{array}{c} P \rightarrow Q \\ P \\ \hline Q \end{array}$ modus ponens

(2) $\begin{array}{c} P \rightarrow Q \\ \neg Q \\ \hline \neg P \end{array}$ modus Tollens

(3) $\begin{array}{c} p \\ q \\ \hline p \end{array}$ Conjunctive simplification.

(4) $\begin{array}{c} p \vee q \\ \neg p \\ \hline q \end{array}$ Disjunctive syllogism.

(5) $\begin{array}{c} A \rightarrow B \\ B \rightarrow C \\ \hline A \rightarrow C \end{array}$ Transitivity of implications Hypothetical syllogism.

(6) $\begin{array}{c} p \\ \hline p \vee q \end{array}$

(7) $\begin{array}{c} p \wedge q \\ \hline p \end{array}$

(8) $\begin{array}{c} p \vee q \\ \neg p \vee r \\ \hline q \vee r \end{array}$ Resolution

Inference symbol

KB = Knowledge Base (set of premises)

KB $\models$ Y   means Knowledge base infers Y

$\models$ logically infers   means $\dfrac{K.B.}{Y}$

$P_1, P_2, P_3 \models C$ ie. $P_1 \wedge P_2 \wedge P_3 \rightarrow C$

$\not\models$ → does not infer $\models$ infers

infers = entails = implies

C is consequence of K.B. $KB \models C$

First Order Logic / Predicate calculus

All men are mortals. Socrate is a man
$\therefore$ Socrate is mortal.

No way to express the above argument with propositional logic.

Propositional logic ~~have~~ has ~~very~~ limited expressive power. ☹

$\therefore$ first order logic is needed

First Order Logic –
A world of objects, their properties, their relationships, their transformation (function)

~~In~~ ~~FOL~~,

Properties

→ Each variable refers to some objects in a set called the domain of discourse
set of all possible values.

Properties
↙ Domain ↓ Predicate ↘ Quantifier

→ FOL is also concerned with properties of these objects.

→ In FOL, we also have relations over/ between/among objects (called Predicates)

* unary relations/predicate are called properties.

$P(x,y) : x<y.$

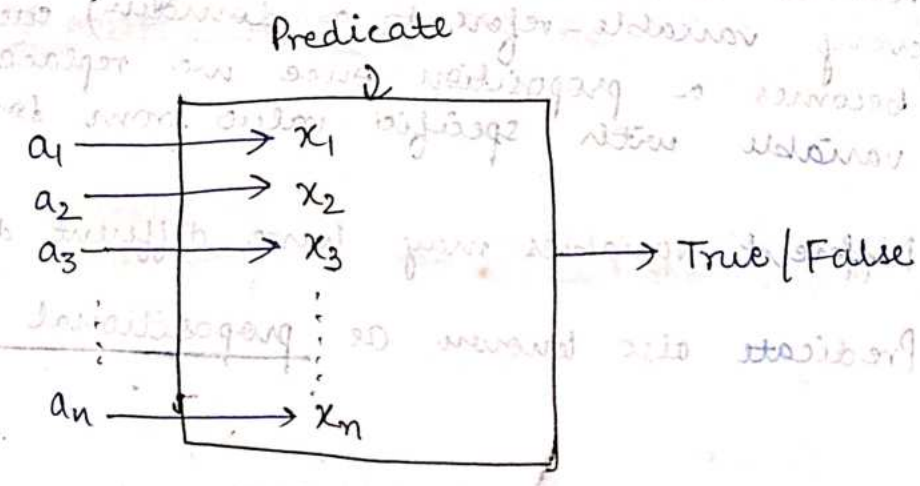
★→ Predicate tells about properties of object / relation between objects.

* Once we replace every variable with some value from their domain, then, it becomes <u>propositional</u>

* Predicate with 0 variable = 0-ary predicate = proposition

Predicate with 1 variable = unary predicate

Predicate with 2 variables = binary predicate



n-ary predicate.

→ Quantification words can be represented in FOL. *

all, some

for all → $\forall$

for all x, $P(x)$ is true = $\forall_x P(x)$

$\forall_x P(x) = P(a) \wedge P(b) \wedge P(c) \wedge P(d)$ $\quad D(x) = \{a, b, c, d\}$.

there exists → $\exists$

$\exists_x P(x)$ → There is atleast one/some x for which $P(x)$ is true.

Predicate -

Predicate tells us about the properties of objects and relationship among objects.

$S(x)$: x is a student (unary predicate)

$F(x, y)$: x and y are friends (binary predicate)

- Predicate is a sentence containing variables (where every variable refers to a domain) such that it becomes a proposition once we replace each variable with specific value from domain.
- Different variables may have different domains.
- Predicate also known as <u>propositional function</u>.

★→ Domain in FOL is ALWAYS non empty; unless explicitly stated.

★→ By default, domain of every variable is same

★ When predicate is quantified, it becomes proposition.

$E(x)$ = x is even $\longrightarrow$ predicate

There exists x in the domain, $E(x)$ is true $\longrightarrow$ proposition.

(Quantification words → there exists, some, forall)

Quantification words in english —

few, all, many, some, any ...

Quantification words in FOL —

All ≡ every

some ≡ atleast one

↳ 2 quantifiers —

① for all ≡ every ≡ universal quantifier

② there exists ≡ atleast one ≡ existential quantifier.

* Quantifier is a way of creating proposition from ~~som~~ predicate.

<u>Quantification of a property</u> :-

Quantification of a property is talking about if a property is satisfied by multiple objects.

Universal quantification

- Universal quantification means saying that a property P is satisfied by all elements in the domain.
- i.e. all elements in the domain satisfy property P.

$\forall$: universal quantifier symbol
(read as 'for all')

- For all elements x in the domain, P(x) is true

$$\forall_{x \in D}(P(x)) \equiv \forall_x(P(x))$$ [$x \in D$ is implicit]

- For finite set with domain, $D = \{a, b, c, d\}$

$$\forall_x P(x) = P(a) \wedge P(b) \wedge P(c) \wedge P(d)$$

universal quantifier is conjunction of all elements.

Existential quantification

- Existential quantification means saying that there exists atleast one element in domain for which the property P is true.
- some element in the domain satisfies the property.

$\exists$: existential quantifier symbol
(read as 'there exists')

There exists ~~an~~ atleast one element for which P(x) is true

$$\exists_x P(x) \equiv \exists_{x \in D} P(x)$$

Example - Domain : N

$$\exists_x \left(even(x) \wedge prime(x) \right) \rightarrow \text{True} \ (x=2)$$

⊙ Finite set with domain $D = \{a, b, c, d\}$

$$\exists_x P(x) = P(a) \vee P(b) \vee P(c) \vee P(d)$$

existential quantification is a disjunction of all elements.

1. When domain is empty,

→ ⊙ universal quantifier statement is always true. (bcoz no counterexample)

$$\forall_x \alpha(x) = T$$

→ ⊙ existential quantifier statement is always false (bcoz no witness)

$$\exists_x \beta(x) = F.$$

2. When there is no free variable.

→ ⊙ $\forall_x P(x) = P(x)$ [bcoz ∵ free variable is not present, ∴ P(x) is proposition]

Ex. $\boxed{\begin{array}{l} P: 2+2=6 \\ \forall_x P = \text{false} \end{array}}$

no free variable

→ ⊙ $\exists_x P(x) = P$

3. Predicate with no variable is proposition

IMP ★ Domain : set of all animals

1. Every rabit is cute

$\forall_x(\text{Rabbit}(x) \rightarrow \text{cute}(x))$ ✓

$\forall_x(\text{Rabbit}(x) \wedge \text{cute}(x))$ ✗

(bcoz it means ~~for~~ ~~at~~ every element is rabbit & cute)

2. Some rabbit is cute.

$\exists_x(\text{Rabbit}(x) \wedge \text{cute}(x))$ ✓

$\exists_x(\text{Rabbit}(x) \rightarrow \text{cute}(x))$ ✗

(bcoz it is true for those animals which are not rabbit)

Useful intuition

All P's are Q's $\equiv \forall_x(P(x) \rightarrow Q(x))$

Some P's are Q's $\equiv \exists_x(P(x) \wedge Q(x))$

[with universal quantifier, we use implication. with existential quantifier we use conjunction]

$\exists_x(R(x) \rightarrow H(x)) = \exists_x(\overline{R(x)} \vee H(x))$

There is some animal which is either not a rabbit or it hops.

1. All P's are Q's

$\forall_x (P(x) \rightarrow Q(x))$

P = Rabbit
Q = Cute

2. Some Ps are Qs

$\exists_x (P(x) \wedge Q(x))$

3. No Ps are Qs

$\equiv$ All p's are not Qs

$\equiv$ It is not the case that some Ps are Qs

$\forall_x (P(x) \rightarrow \neg Q(x))$

$\neg \exists_x (P(x) \wedge Q(x))$

4. Some Ps are not Qs

$\exists_x (P(x) \wedge \neg Q(x))$

5. Only A's are B's / Every A is B

$\forall_x (\neg A(x) \rightarrow \neg B(x))$

$\forall_x (B(x) \rightarrow A(x))$

7. Not all As are B's

$\exists_x (A(x) \wedge \neg B(x))$

6. All and only A's are B's

$\forall_x (A(x) \Leftrightarrow B(x))$

7. $\forall_{P(x)} Q(x) \equiv \forall_x (P(x) \rightarrow Q(x))$

8. $\exists_{P(x)} (Q(x)) \equiv \exists_x (P(x) \wedge Q(x))$

IMPORTANT

No A is B

Not all A's are B's

★ $\neg \forall_x \left(A(x) \rightarrow B(x) \right) \equiv \exists_x \left(A(x) \wedge \neg B(x) \right)$

★ Every student loves someone

Love(x,y) : x loves y.

$\forall_x \left(\text{student}(x) \rightarrow \exists_y \left(\cancel{\text{stu}} \ \text{Love}(x,y) \right) \right)$

Bounded Variable

$\forall_x$ Even(x) [x is bounded by ~~domain~~ quantifier

proposition ← $\forall_x P(x)$

↳ Bounded variable / Quantified variable / Dummy variable

$\forall_3 P(3)$ doesn't make sense

We cannot replace x by any value from domain

Free Variable

$E(x)$ = x is even — not a proposition

↳ free variable

x is free to take any value from domain.

★ also known as real variable

$\forall_x P(x)$ — proposition

Bounded

$\exists_x Q(x)$ — proposition

Bounded

$E(x)$ — not proposition

free.

$N(x) \longrightarrow$ free variable

$\forall_x N(x) \longrightarrow$ bounded variable

Free variable
- → Not bounded by any quantifier
- → free to take any value from domain

If any expression does not contain free variable, it is proposition ↔

Domain : N

S: $\forall_x (x > y)$ $\quad S(y) = \forall_x (x>y)$

$x \rightarrow$ Bounded variable

$y \rightarrow$ Free variable

[Not a proposition]

$\forall_y \left(\underbrace{\forall_x (x > y)}_{\text{every natural no. is greater than } y} \right)$ } proposition.

false proposition.

$\exists_y \left(\underbrace{\forall_x (x > y)}_{S(y)} \right) = \underbrace{S(1)}_{F} \lor \underbrace{S(2)}_{F} \lor \underbrace{S(3)}_{F} \lor \underbrace{S(4)}_{F} \ldots$

False proposition.

Convert predicate to proposition —
- → Replace free variable with some value from domain
- → quantify free variable.

Bounded v/s Free variable

↦ A bounded variable is a variable that is subject to a quantifier. A variable that is not bound is called free variable.

↦ A proposition can only contain bounded variables, no free variables.

Important NOTE

If [A doesn't have any free variable x, then,

$\forall_x A \equiv A$

$\exists_x A \equiv A$

$\exists_x P(x) : A$ → no free variable → proposition

$\forall_x A \equiv A$

$\exists_x A = A$

i.e. applying quantifier on a proposition gives same value as that of the proposition

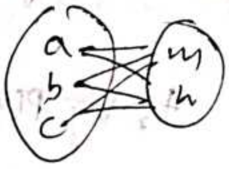
Scope of quantifier

The part of a logical expression to which a quantifier is applicable (applied) is called scope of the quantifier

$\forall_x [\ .\ x\ \ldots\ x\ ..\ ]$

Types of nested quantifiers

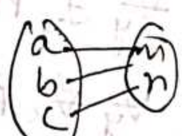
① $\forall_x \forall_y \; P(x,y)$



$x \in \{a, b, c\}$ $\quad\quad$ $y \in \{p, q, r\}$

$$\forall_x \forall_y P(x,y) = P(a,p) \land P(a,q) \land P(a,r) \land P(b,p) \land P(b,q) \land P(b,r) \land P(c,p) \land P(c,q) \land P(c,r)$$

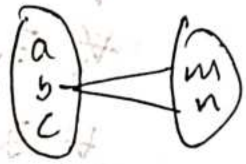
② $\forall_x \exists_y \; P(x,y)$



$x \in \{a,b,c\}$ $\quad\quad$ $y \in (m,n)$

$$(P(a,m) \lor P(a,n)) \land (P(b,m) \lor P(b,n)) \land (P(c,m) \lor P(c,n))$$

③ $\exists_x \forall_y \; (P(x,y))$

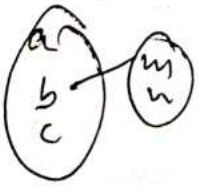


$x \in \{a,b,c\}$ $\quad\quad$ $y \in (m,n)$

? $\cancel{P(a,m) \land P(b,m) \land P(c,m}$

$$(P(a,m) \land P(a,n)) \lor (P(b,m) \land P(b,n)) \lor (P(c,m) \land P(c,n))$$

④ $\exists_x \exists_y \; P(x,y)$



$x \in \{a,b,c\}$ $\quad\quad$ $y = \{m,n\}$

$$P(a,m) \lor P(a,n) \lor P(b,m) \lor P(b,n) \lor P(c,m) \lor P(c,n)$$

$x \rightarrow$ set of students $\qquad P(x,y) \rightarrow x$ has taken course y

$y \rightarrow$ set of courses

$\forall_x \forall_y P(x,y) \rightarrow$ Every student has taken every course

$\forall_x \exists_y P(x,y) \rightarrow$ Every student has taken atleast one cou

$\exists_x \forall_y P(x,y) \rightarrow$ Some student has taken all the courses

$\exists_x \exists_y P(x,y) \rightarrow$ Some student has taken some course.

$\forall_y \exists_x P(x,y) \rightarrow$ Every course is taken by atleast one stude

$\forall_y \forall_x P(x,y) \rightarrow$ Every course is taken by every student

$\exists_y \forall_x P(x,y) \rightarrow$ Some course is taken by ~~ever~~ every stud

$\exists_x \exists_x P(x,y) \rightarrow$ Some course is taken by some student.

* Order of quantifiers matters for different quantifiers

$$\forall_x \exists_y \alpha \neq \exists_x \forall_y \alpha$$
$$\forall_x \exists_y \alpha \neq \exists_y \forall_x \alpha$$

* In case of same quantifiers, order does not matter

$$\forall_x \forall_y \alpha = \forall_y \forall_x \alpha$$
$$\exists_x \exists_y \alpha = \exists_y \exists_x \alpha$$

English to FOL translations –

① ~~There~~ All large cubes are nice $\forall_x \left(large(x) \wedge cube(x) \rightarrow nice(x) \right)$

② There is atleast one cube $\exists_x \left(cube(x) \right)$

③ There are atleast 2 cubes $\exists_x \exists_y \left(cube(x) \wedge cube(y) \wedge (x \neq y) \right)$

④ There are atleast 3 cubes

$$\exists_x \exists_y \exists_z \left(cube(x) \wedge cube(y) \wedge cube(z) \wedge x \neq y \wedge y \neq z \wedge x \neq z \right)$$

⑤ There is atmost one cube

$\cancel{\exists_x \exists_y (cube(x)}$ [no cube / 1 cube]

$$\forall_x \forall_y \left((cube(x) \wedge cube(y)) \rightarrow x = y \right)$$

⑥ There is exactly one cube

atleast one cube $\wedge$ atmost one cube

① $\exists_x \left(cube(x) \right) \wedge \forall_x \forall_y \left((cube(x) \wedge cube(y)) \rightarrow x = y \right)$

OR

$$\exists_x \left(cube(x) \wedge \forall_y \left(cube(y) \rightarrow x = y \right) \right)$$

② $\exists_x \forall_y \left(cube(x) \wedge (cube(y) \rightarrow x = y) \right)$

③ $\exists_x \forall_y \left(cube(y) \longleftrightarrow x = y \right)$

⑦ There are atmost 2 cubes

$$\forall_x \forall_y \forall_z \big(cube(x) \land cube(y) \land cube(z) \rightarrow (x=y) \lor (y=z) \lor (x=z) \big)$$

⑧ There are exactly 2 cubes

atmost 2 cubes $\land$ atleast 2 cubes

$$\Big(\forall_x \forall_y \forall_z \big(cube(x) \land cube(y) \land cube(z) \rightarrow x=y \lor y=z \lor x=z \big) \land$$

$$\exists_x \exists_y \big(cube(x) \land cube(y) \land x \neq y \big) \Big)$$

OR

$$\exists_x \exists_y \big(cube(x) \land cube(y) \land x \neq y \land \forall_z \big(cube(z) \rightarrow (x=z) \lor (y=z) \big) \big)$$

x is prime –

$$x>1 \land (\forall_y (y|x \rightarrow (y=1) \lor (y=x)))$$

Negation of quantifier

$S = \forall_x P(x) \qquad \neg S = \neg\forall_x P(x) = \exists_x(\neg P(x))$

$S = \exists_x P(x) \qquad \neg S = \neg\exists_x P(x) = \forall_x(\neg P(x))$

Ex.1.

$$\neg\forall_x \exists_y (P(x) \rightarrow Q(y))$$

$$= \exists_x(\neg\exists_y(P(x) \rightarrow Q(y)))$$

$$= \exists_x \forall_y (\neg(P(x) \rightarrow Q(y)))$$

$$= \exists_x \forall_y (P(x) \land \neg Q(y))$$

$\left[\neg(P \rightarrow Q) = \overline{\overline{P} \lor Q} = \overline{\overline{P}} \land \overline{Q} = P \land \overline{Q}\right]$

Ex.2.

$$\neg\exists_x \forall_y (P(x) \land Q(y))$$

$$= \forall_x(\neg\forall_y (P(x) \land Q(y)))$$

$$= \forall_x \exists_y (\neg(P(x) \land Q(y)))$$

$$= \forall_x \exists_y (\neg P(x) \lor \neg Q(y))$$

Valid FOL Expression

$\forall_x P(x) \longrightarrow \exists_x P(x)$

$\forall_x (P(x) \land Q(x)) \longrightarrow \forall_x P(x) \land \forall_x Q(x)$

$\forall_x P(x) \land \forall_x Q(x) \longrightarrow \forall_x (P(x) \land Q(x))$

$\forall_x (P(x) \land Q(x)) \longrightarrow \forall_y P(y)$

$\forall_x (P(x)) \lor \forall_x Q(x) \longrightarrow \forall_x (P(x) \lor Q(x))$

Distribution of quantifiers over logical connectives

$$\underbrace{\Pi_x (P(x) \# Q(x))}_{\text{Compact expression}} \underset{\text{valid?}}{\overset{\text{valid?}}{\rightleftarrows}} \underbrace{\Pi_x (P(x)) \# \Pi_x Q(x)}_{\text{Expanded expression}}$$

Π_x: quantifier $(\exists, \forall, \exists!)$

$\exists!$: uniqueness quantifier (there exists one & only one)

$\#$: logical connective

	$\forall$	$\exists$
$\land$	$\leftrightarrow$	$\rightarrow$
$\lor$	$\leftarrow$	$\leftrightarrow$
$\rightarrow$	$\rightarrow$	$\leftarrow$
$\leftrightarrow$	$\rightarrow$	x
$\oplus$	x	$\leftarrow$
$\uparrow$	$\rightarrow$	$\leftarrow$
$\downarrow$	$\rightarrow$	

$\rightarrow$: Compact to expanded

$\leftarrow$: Expanded to compact

x : both aren't valid

$\leftrightarrow$: both sides are valid.

Null quantification rule

Distribution of quantifiers over logical connectives, when some expression isn't affected by quantifiers.

$\equiv$ Some part has no free variable.

A: has no free variable

$\forall_x A = A$, $\exists_x A = A$

[By case method]

① $\forall_x (P(x) \vee A) \equiv \forall_x P(x) \vee A$

② $\exists_x (P(x) \vee A) \equiv \exists_x P(x) \vee A$

③ $\forall_x (P(x) \wedge A) \equiv \forall_x P(x) \wedge A$

④ $\exists_x (P(x) \wedge A) \equiv \exists_x P(x) \wedge A$

⑤ $\forall_x (A \to P(x)) \equiv A \to \forall_x P(x)$

⑥ $\exists_x (A \to P(x)) \equiv A \to \exists_x P(x)$

* ⑦ $\forall_x (P(x) \to A) \equiv \exists_x P(x) \to A$

* ⑧ $\exists_x (P(x) \to A) \equiv \forall_x P(x) \to A$

$$\frac{\forall_x P(x)}{P(b)}$$ universal instantiation

$$\frac{\text{for arbitrary } n \in D,\ P(n)}{\forall_x P(x)}$$ universal generalization

$$\frac{P(a)}{\exists_x P(x)}$$ Existential generalization

$$\frac{\exists_x P(x)}{\therefore P(c) \text{ for some element } c}$$ Existential instantiation

Interpretation –

The values of variables (true/false) is called interpretation.

For n variables,

no of interpretations $= 2^N$

Model –

Interpretations for which the formula evaluates to true are called models.

Co model –

Interpretations for which formula evaluates to false

Set Theory

Set – collection of objects is called set.
Set is represented by { }.

Ex:- $\{a, b, c, d, e\}$ = English vowels

- order of ~~coll~~ elements does not matter
- no duplicate elements

→ unordered collection of distinct elements.

$S = \{\{a\}, \{a, b\}, c, d\}$ 4 elements in S.

P: ϕ = empty set $|P| = 0$

Q: $\{\phi\}$ = set containing 1 element $|Q| = 1$

$\phi \neq \{\phi\}$

$|S| = \infty$ if S is infinite set (set of all integers)

Set Representations –
- → Verbal representation
- → Roster (list) representation
- → Venn diagram representation
- → Set Builder Representation

Subset

ϕ is the subset of every set
Every set is subset of itself
$\phi \subseteq \phi$

If n elements, no. of subsets = 2^n

If n elements, no. of proper subsets = $2^n - 1$.

Power set -

The set of all subsets of a set S is called power set of

$S = \{a, b, c\}$

$P(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,$

$|P(S)| = 2^n = 2^{|S|}$

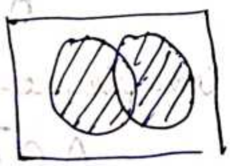
$|P(|P(S)|)| = 2^{|P(S)|} = 2^{2^{|S|}}$

$U =$ Universal set (contains everything)
$\phi =$ Empty set (contains nothing)

Operations on set

Union, Intersection, Difference, Complement

$\Delta =$ symmetric difference
(exclusive OR)



Disjoint sets :- Intersection of sets is ϕ.

$A \,\Delta\, B = (A-B) \cup (B-A)$

$A \,\Delta\, B = (A \cup B) - (A \cap B)$

$A - B = A \cap \overline{B}$

$\overline{A} \cup B = \overline{A-B}$

$\overline{A} - B = \overline{A \cup B}$

Equal sets -

$A = B$ iff $A \subseteq B$ and $B \subseteq A$

Set Identities

① Identity law – $A \cup \phi = A$
$A \cap U = A$

② Domination law –
$A \cup U = U$
$A \cap \phi = \phi$

③ Idempotent law
$A \cup A = A$
$A \cap A = A$

④ Complementation law
$\overline{\overline{A}} = A$

⑤ Complement law
$A \cap \overline{A} = \phi$ $\quad A \cup \overline{A} = U$

If $S \subseteq A$ and $S \subseteq B$ then,
$S \subseteq A \cap B$.

Ordered pair –
pairs of elements in which order ~~does~~ ~~not~~ matters

$(a, b) \neq (b, a)$

$(a, b) = (c, d)$
iff $a = c$ & $b = d$.

$A \times B = \{(x, y) \mid x \in A \ \& \ y \in B\}$

Cross product with empty set $= \phi$

$$A \times \phi = \phi \times A = \phi \times \phi = \phi$$

$A \times B = B \times A$ iff $A = B$ or $A = \phi$ or $B = \phi$

Cross product
- → Not associative
- → Not commutative

Relation

Subset of $A \times B$ is relation

$A =$ m elements
$B =$ n elements

No. of relations from set $A \to B =$

$$R \subseteq A \times B$$

No. of elements in $A \times B = m \times n$

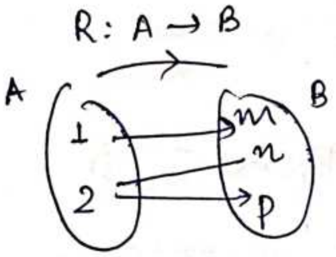
No. of subsets in $A \times B = 2^{m \times n}$

$\therefore$ No. of relations possible from $A \to B = 2^{mn}$

R: $A \to A$ = Relation R is on set A

$R \subseteq N \times N$ = Relation R is on set $(N \times N) \times (N \times N)$

Representations of relations



$R = \{(1,m)\ (2,n)\ (2,p)\}$

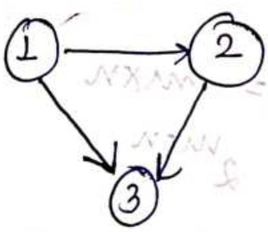
Arrow diagram representation

Set Representation

$A = \{1, 2, 3\}$
↳ Base set

xRy iff $x<y$

$R = \{(1,2)\ (1,3)\ (2,3)\}$



Graph Representation

↳ arrow if $a \to b$
node for each element of base set.

	1	2	3
1	✗	✓	✓
2	✗	✗	✓
3	✗	✗	✗

Matrix Representation

Types of Relations

valid if relation exists on a single set $A \to A$

1. Reflexive Relation

every element is related to itself.

$A = \{a, b, c\}$ R is defined on set A

R contains $\{(a,a)\ (b,b)\ (c,c),\ (a,c)\}$ → other elements can be present

Relation R is reflexive iff

$\forall_{x \in A} (xRx)$

Not Reflexive → $\exists_{x \in A} (x \not R x)$ main diagonal contains atleast one 1 & 1 zero

Irreflexive → $\forall_{x \in A} (x \not R x)$ main diagonal contains all 0s

Reflexive → $\forall_{x \in A} (xRx)$ main diagonal contains all 1s

2. Symmetric Relation

$\forall_{a,b \in A} (aRb \to bRa)$

bidirectional edges

$M = M^T$

no unidirectional edges in graph

Antisymmetric –

unidirectional edges

if x & y are different,

$xRy \longrightarrow y \not R x$

$\forall_{x,y} \left[(x \neq y \wedge xRy) \to y \not R x \right]$ $\cancel{\forall_{x,y \in A} [x \neq y \to (xRy \to y \not R x)]}$

$\forall_{a,b \in A} \left[(aRb \land bRa) \rightarrow a=b \right]$

Asymmetric :- antisymmetric and irreflexive

unidirectional edges
no self loop

$\forall_{a,b \in A} \left(aRb \rightarrow b \not{R} a \right)$

3. Transitive Relation

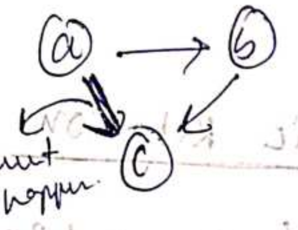
If aRb and bRc then aRc

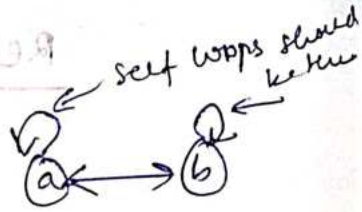
$\forall_{a,b,c \in A} \left((aRb \land bRc) \rightarrow aRc \right)$

Violation of transitivity

$\exists_{a,b,c \in A} \left(aRb \land bRc \land a \not{R} c \right)$

★ solve ques. using graph representation





parity of integer → no. is even or odd

parity of 2 = even
3 = odd
0 = even

Equivalence Relation

A relation R is equivalence relation iff it is Reflexive, Transitive & symmetric

Partition of set

A set S is partitioned into k non empty subsets $A_1 \; A_2 \; A_3 \ldots \; A_k$ if

- every pair of subsets is disjoint i.e. $A_i \cap A_j = \phi \quad i \neq j$
- $A_1 \cup A_2 \cup A_3 \cup \ldots A_k = S$
- $A_i \neq \phi$
- $A_i \subseteq S$

① Every part is a set
② Partition is a set.

$S = \{1, 2\}$

equivalence relation always creates partition of base set.

Partitions of $S = \{\{1\}, \{2\}\}, \{\{1,2\}\}$

$[x]_R = \{ y \mid xRy \} = \{ y \mid yRx \}$

equivalence relation

$[0]$ equivalence class containing zero.

If there are a^4 equivalence classes of size n. Then,

$|R| = |E_1|^2 + |E_2|^2 + |E_3|^2 + \ldots$

$= n^2 + n^2 + n^2 + \ldots$

Equivalence relation has unique equivalence class

$R = (E_1 \times E_1) \cup (E_2 \times E_2)$

fiction

Base set = A

Largest equivalence relation that can be created = $R = A \times A$

No. of equivalence classes = 1

Cardinality = $|A|^2$

(RST)

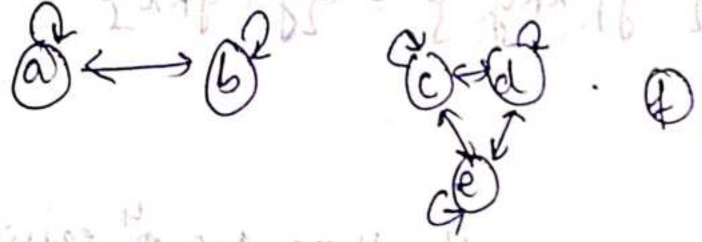
Smallest equivalence relation = identity relation

No of equivalence classes = $|A|$

Cardinality = $|A|$

Graph of equivalence relation –

$\{a, b\}$ $\{c, d, e\}$ $\{f\}$ are equivalence classes



Complete directed graph for each equivalence ~~relat~~ class.

$[a]_R$ → equivalence class of element a

{set of all elements to which a is related}

No. of equivalence relations on a set of 3 elements.

No. of equivalence relations = No. of partitions possib

for 3 elements, partitions possible are

{abc} {a, bc} {ab, c} {ac, b} {a,b,c}.

5 partitions

$\therefore$ 5 equivalence relations

Partial Order Relations

atleast 2 elements b/w which no order is present } partial order

order exists b/w every pair of elements in the set } total order

Partial Order Relations :-

- A relation R on a set A is a partial order for A if R is reflexive, antisymmetric and transitive

- Set A with a partial order is called partially ordered set (Poset).

(Base set, POR) $\longrightarrow$ Poset
(A, R)

Standard Poset ~~POR~~ —

↦ $(N, \leq)$ is Poset

bcoz reflexive, antisymmetric & transitive

↦ $(N, \geq)$

↦ $(P(A), \subseteq)$ subset relation on powerset of set.

↦ $(P(A), \supseteq)$ superset relation...

↦ divisibility relation

aRb → a|b

Partial Order Relations

① Relation on non empty base set

RefX

Symmetric ✓

Antisymmetric ✓

Transitive ✓

Irreflexive ✓

ϕ relation on empty base set.

Ref ✓

Symmetric ✓

Antisymmetric ✓

Transitive ✓

Irreflexive ✓

② Divides relation on integers is not antisymmetric

$2 | -2 \qquad -2 | 2$

③ Divides relation on natural numbers is antisymmetric place holder for POR

* "$\leq$" → used for partial order relation
  "$\sim$" → used for equivalence relation

Total order → partial order + every pair of elements should be comparable.

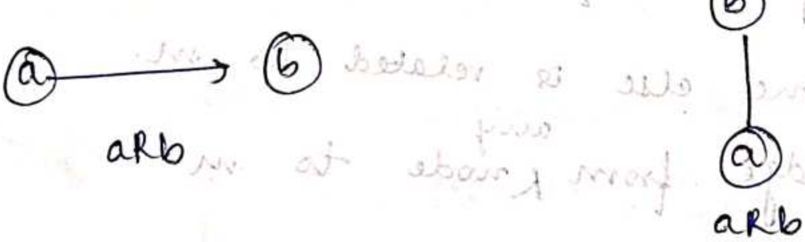
Hasse Diagram (HD)

used for representation of partial order relations (property & definition of partial order relation)

- self loops are not shown (coz POR is reflexive, so, no need to explicitly mention)
- transitive edges are not shown. (remove arrow heads)

⊙ Graph like representation of POR.

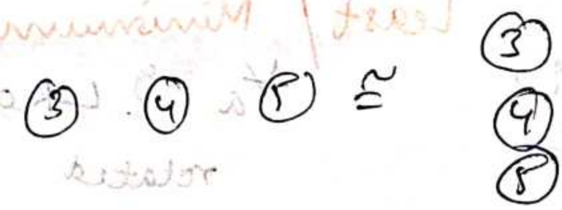
⊙ all arrows are in upward direction.



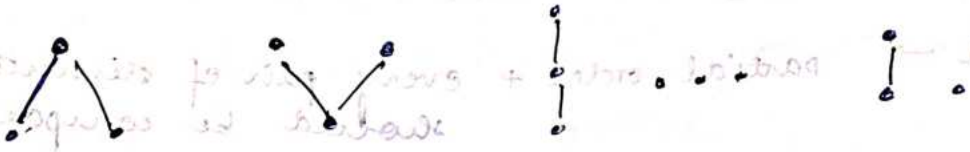
aRb iff there is some upward path (not edge) from a to b.

⊙ No horizontal edges

⊙ no concept of levels



(e) For 3 elements (unlabelled), following structures are possible for HD —



$3+1+1$ $= 5$	$3+1+1$ 5	$3+1+1+1$ 6	3 3	$3+1$ 4	cardinality no. of relations

Maximal element 'M'

$\forall_{a \neq M}$; $M \not R a$

M is not related to anyone else.

no edge from M to other node.

Minimal element 'm'

$\forall_{a \neq M}$ $a \not R m$

no one else is related to m.

no edge from any node to m.

Greatest / Maximum element (G)

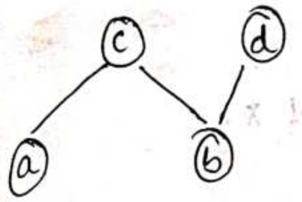
$\forall_a$ aRG

every element is related to G

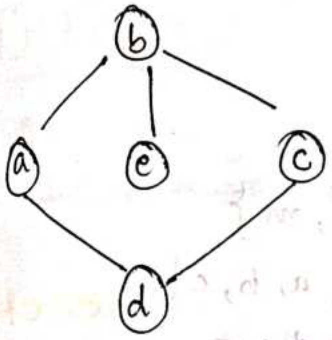
Least / Minimum element (L)

$\forall_a$ LRa.

related to every element



Maximal element = c, d
Minimal element = a, b
Maximum element = DNE (Does not exist)
Minimum element = DNE



Maximal element = b
Minimal element = d, e
Maximum element = b
Minimum element = DNE

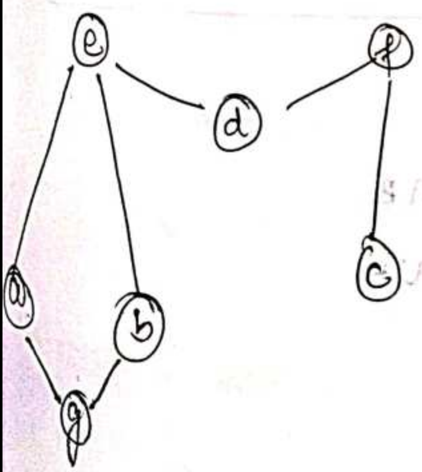
Upper bound (U.B.)

upper bound ~~of a set of~~ of a set S is

$\forall_{x \in S} (x \, R \, (U.B.))$

lower bound (LB)

$\forall_{x \in S} (L.B. \, R \, x)$



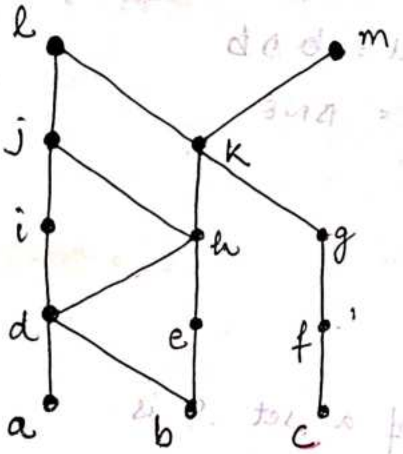
Maximal = {e, f}
Minimal = {g, d, c}
UB {a, g, b} = {e}
UB {a, g} = {a, e}
LB {a, g} = {g}
LB {e, f} = {d}
UB {e, f} = ϕ
UB {a} = {a, e}

Least upper bound of a set X: (LBU)

The least element in the upper bound of X.

Greatest lower bound of a set X: (GLB)

The greatest element in the lower bound of X.



Maximal element = {l, m}

Minimal element = {a, b, c}

Greatest element = ϕ DNE

Least element = ϕ DNE

L.B{d, k, f} = ϕ

GLB{d, k, f} = DNE

U.B. {d, k, f} = {k, l, m}

LUB{d, k, f} = k

LB{l, m} = {k, h, d, a, e, b, g, f, c}

G.L.B.{l, m} = k

U.B.{l, m} = ϕ

L.U.B.{l, m} = DNE

UB{d, e} = {h, j, k, l, m}

LUB{d, e} = h

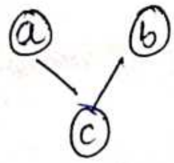
If a poset has more than one maximal elements, greatest element does not exist

* GLB{a, b} = $a \wedge b$ or $a \cap B$

LUB{a, b} = $a \vee b$ or $a \cup B$

LUB{a, b} = $a \vee b$ = $a \cup B$ = Join{a, b}

GLB{a, b} = $a \wedge b$ = $a \cap B$ = meet{a, b}

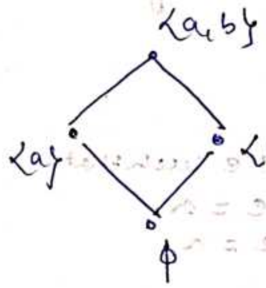


$GLB(a,b,c) = a \wedge b \wedge c = c$

$LUB(a,b,c) = a \vee b \vee c = LUB(a,b)$

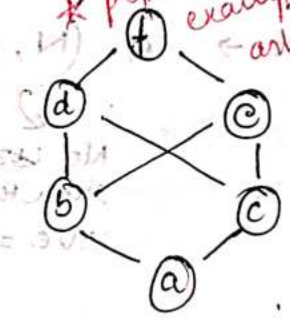
Lattices

A lattice is a poset $(A, \leq)$ in which any 2 elements a, b, have an $LUB(a,b)$ & a $GLB(a,b)$



$(P(A), \subseteq)$

* popular example ← asked in Gate

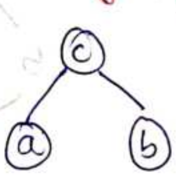


$b \vee c$ = DNE (not unique join point)

$d \wedge e$ = DNE

∴ not lattice

Cardinality of partial order relation–



No. of relations of type (a, x) = 2 (aRa, aRc)

" " " type (b, x) = 2 (bRb, bRc)

" " " " " (c, x) = 1 (cRc)

∴ Cardinality = $2+2+1 = 5$

★ On 'n' elements, the largest possible POR is a chain (Total order relation)

Cardinality = $3+2+1 = 6$ (aRx, bRx, cRx)

Cardinality $= \frac{n(n+1)}{2}$

smallest por → a b c Cardinality = n

★ $A \times A$ is POR if and only if $|A| \leq 1$

$(N, |)$ is a lattice

becoz for every pair of elements a, b

$LUB(a,b) = a \vee b = LCM(a,b)$

$GLB(a,b) = a \wedge b = GCD(a,b)$

★ Every total order is a lattice

★ Identity property is not satisfied by all Lattices

ex→ $(N, \le)$ $(R, \le)$

$(N, \le)$: No e such that $a \vee e = a$

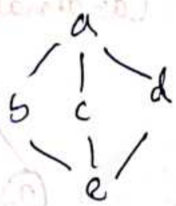
$(R, \le)$: No e such that $a \wedge e = a$, $a \vee e = a$

★ Distributive property is not satisfied by some lattice

$b \wedge (c \vee d) \neq (b \wedge c) \vee (b \wedge d)$

$(b \wedge a) \quad e \wedge e$

$b \neq e$



→ If an element is deleted from base set of POSET, then, the resulting relation is also POSET

If (A, R) is a POSET, then, (B, R) is also a POSET $\forall B \subseteq A$.

If (A, R) is equivalence relation, then, (B, R) is also an equivalence relation $\forall B \subseteq A$.

Sublattice

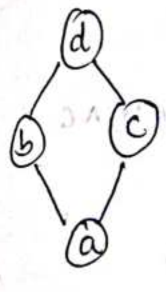
Given a lattice L

S is sublattice of L if

⊢ S is subset of L

⊢ S is a lattice

⊢ ~~GUB~~, LUB, GLB of $^{(a,b) \in L}$ L, S should be same



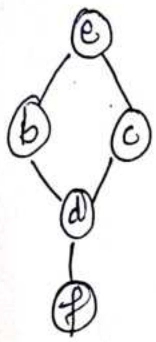
$\xrightarrow{\text{Delete c}}$

Subset of L ✓

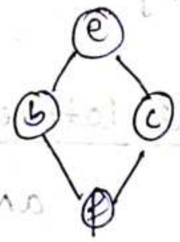
lattice ✓ ∴ sublattice ✓

	L	S
$a \vee b$	b	b
$b \vee d$	d	d
$a \vee d$	d	d
$a \wedge b$	a	a
⋮		

✓



$\xrightarrow{\text{delete d}}$

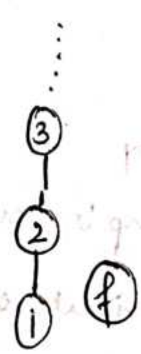


Subset of L ✓

lattice ✓ ∴ not sublattice ✓

	L	S
$b \wedge c$	d	f ✗

★ In the subset of L, i.e. base set of S, if a, b are present, their GLB and LUB should also be present.



maximal element = f

greatest element = DNE

★ In finite poset,

unique maximal → greatest

unique minimal → least

In infinite poset,

no guarantee.

Properties satisfied by all lattices.

- Commutative $a \vee b = b \vee a$ $\quad a \wedge b = b \wedge a$
- Associative $(a \vee b) \vee c = a \vee (b \vee c)$ $\quad a \wedge (b \wedge c) = (a \wedge b) \wedge c$
- Idempotence $a \wedge a = a$ $\quad a \vee a = a$
- Absorption $a \vee (a \wedge b) = a$ $\quad a \wedge (a \vee b) = a$
- $\forall_{x,y \in L}$ $\{x R (x \vee y)$, $(x \wedge y) R x$, $(x \wedge y) R (x \vee y)\}$

Properties do not satisfied by all lattices

- Identity $a \vee 0 = a$ $\quad a \wedge 1 = a$
- Distributive $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
- Complement $a \vee \neg a = 1$ $\quad a \wedge \neg a = 0$

Types of lattices

1. Bounded lattice — identity
2. Complemented Lattice — complement
3. Distributive lattice — distributive
4. Boolean lattice — all

1. Bounded Lattice

- A lattice is bounded if it has minimum & maximum element.
- These are denoted by 0 & 1.

→ Infinite lattice that is bounded — $([0,1], \leq)$, $(P(N), \subseteq)$ power set of natural nos

$(W, |)$ — least = 1, Greatest = 0 (whole nos)

→ Every finite lattice is bounded.

Minimum element → 0
Maximum element → 1

$\forall_x \quad x \vee 1 = 1$
$\quad\;\; x \wedge 1 = x$

$\forall_x \quad x \vee 0 = x$
$\quad\;\; x \wedge 0 = 0$

- Bounded lattice is a lattice with identity element for both GLB and LUB.

Identity element for GLB = Greatest element (1)
Identity element for LUB = Least element (0)

Domination law = $S \cup U = M$ $\quad$ U = universal set.
$\qquad S \cap \phi = \phi$ $\quad$ ϕ = empty set

U is dominator for union
ϕ is ~~donim~~ dominator for intersection

Dominator for ~~GLB~~ LUB = $a \vee G = G$ (greatest element)
Dominator for ~~LUB~~ GLB = $a \wedge L = L$ (least element)
$\qquad\qquad\qquad\qquad\qquad\qquad\quad$ 0

- Bounded lattice is a lattice with dominator for both GLB and LUB.

Greatest element
- Identity element for GLB
- Dominator for LUB

Least element
- Identity element for LUB
- Dominator for GLB.

2. Complemented Lattice

- Lattice which follows complementary property.

Complement of an element 'a':

'b' is complement of a iff

$a \vee b = 1$ (greatest element)

$a \wedge b = 0$ (least element)

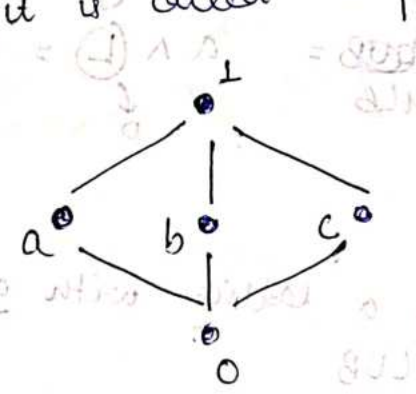
- If lattice is not bounded, then, complement cannot be defined.

Complement of an element 'a' $= a^{-1}$

$(\text{least})^{-1} = \text{greatest}$

$(\text{greatest})^{-1} = \text{least}$

- If every element in the lattice has atleast one complement, then, it is called complemented lattice.



$a^{-1} = b, c$

$b^{-1} = a, c$

$c^{-1} = a, b$

$0^{-1} = 1$

$1^{-1} = 0$

- A TOSET can be complemented lattice iff it has ≤ 2 elements.

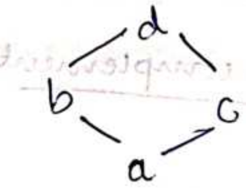
3. Distributive ~~Pro~~ Lattice

Distributive property:

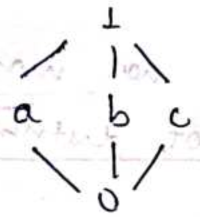
$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

- Lattices which satisfy distributive property are called distributive lattice. $\forall_{x,y,z}$



distributive lattice ✓

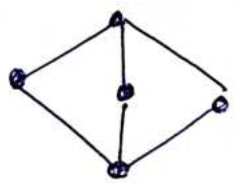


$a \vee (b \wedge c) \neq (a \vee b) \wedge (a \vee c)$

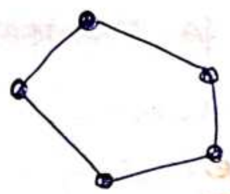
$= a \vee 0 \qquad = 1 \wedge 1$

$= a \qquad\quad = 1$

$\therefore$ distributive property not satisfied



Kite lattice
Diamond lattice
M_3



Pentagon lattice
N_5

Popular non distributive lattice

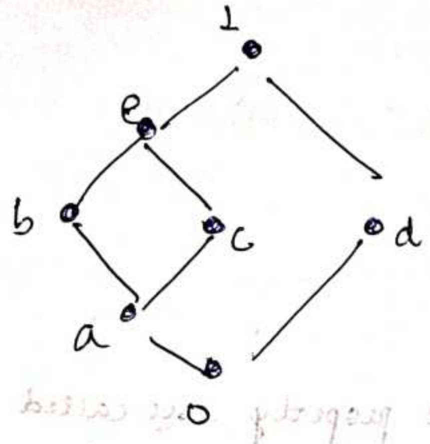
Theorem :-

- A lattice L is distributive iff there is no sublattice of L which is (kite or pentagon).

- If a lattice has ≤ 4 elements, it is definitely distributive.

Theorem:-

- A distributive lattice can have <u>atmost one complement</u>



$b^{-1} = d$

$c^{-1} = d$

$d^{-1} = b, c$ ∴ not distributive

[distributive lattices can have atmost one complement)

★ If some element has <u>more than one complement</u>, then the lattice is <u>not distributive</u>

If ≤ 1 complements, lattice may / may not be distributive.

4. Boolean Lattice

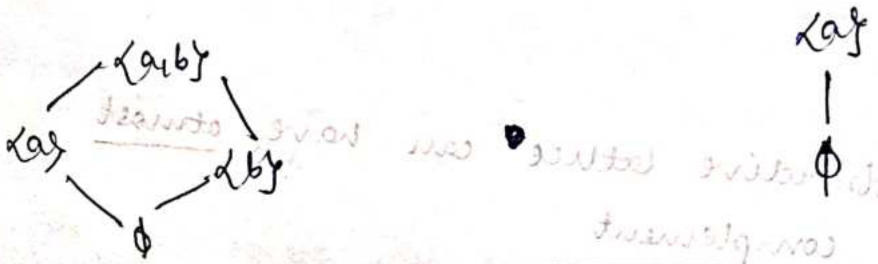
Boolean lattice is <u>complemented distributive</u> lattice.

- → complemented lattice (hence bounded)
- → distributive lattice

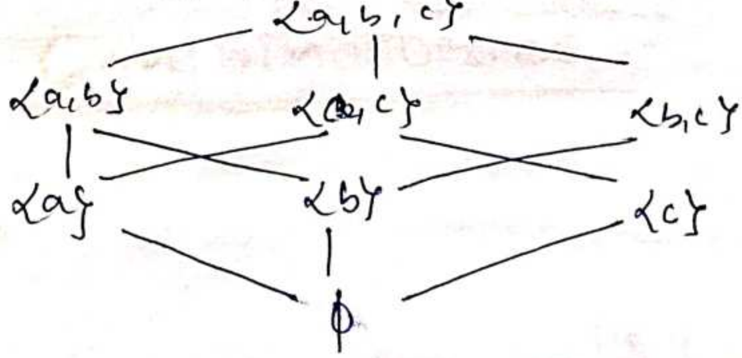
→ Every boolean algebra has the same structure as $(P(A), \subseteq)$ structure.

→ 2^n elements in boolean algebra

$n=2$: $(\{a,b\}, \subseteq)$ $n=0$: $(\phi, \subseteq)$ $n=1$: $(\{a\}, \subseteq)$

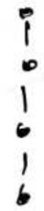


$(\{a,b,c\}, \phi)$



→ Not every lattice with 2^n elements is boolean lattice.

ex →

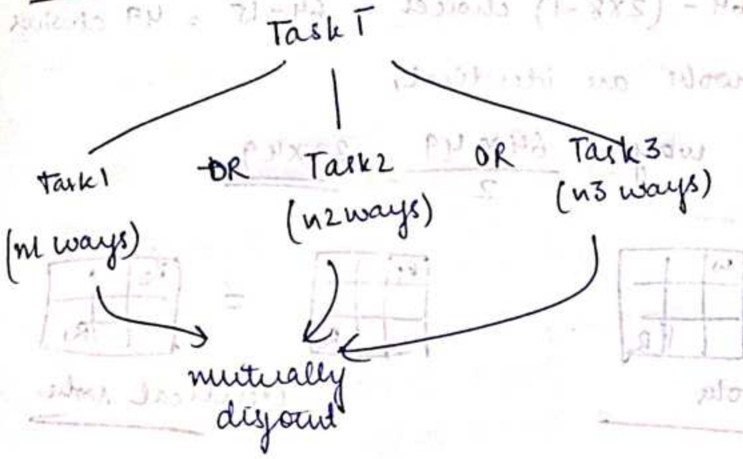


⊙ Boolean lattice is also called boolean algebra because it satisfies all properties of boolean algebra

- → Commutative
- → Idempotent
- → Associative
- → Absorption
- → Consistency
- → Identity
- → Complement
- → Distributive

Combinatorics

Sum Rule



No. of ways of completing task $= (n_1 + n_2 + n_3)$ ways.

2 imp. points —
- subtasks must be mutually disjoint
- OR

If there are $n(A)$ ways to do task A and distinct from them, $n(B)$ ways to do B, then, the no of ways to do A or B is $n(A) + n(B)$.

How many k long palindromes can be formed from an n-set?

The first $\lceil k/2 \rceil$ elements are to be selected arbitrarily from n ~~len~~ set.

$\therefore$ Ans $\rightarrow n^{\lceil k/2 \rceil}$

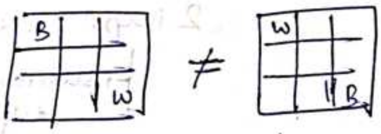
Ques :- In how many ways can we place 2 identical rooks on a 8X8 chessboard so that they occupy different rows & different columns ?

1st rook $\longrightarrow$ 64 choices

2nd rook $\longrightarrow$ $64 - (2\times 8 - 1)$ choices $= 64 - 15 = 49$ choices

Since the 2 rooks are identical,

$\therefore$ No of ways $= \dfrac{64 \times 49}{2} = \underline{32 \times 49}$



different rooks

identical rooks

Ques How many subsets of exactly 2 elements are there for a set of n elements ?

1st element $\rightarrow$ n choices

2nd element $\rightarrow$ $(n-1)$ choices

$n(n-1)$ choices of ordered pairs

For a set, $\{a, b\} = \{b, a\}$ (2-1 correspondence)

$\therefore$ No. of ways $= \dfrac{n(n-1)}{2}$

Ques :- How many linear orders of 6 elements a, b, c, d, e, f are there such that 'a' comes before 'b' (not necessarily immediatly)

No. of ~~ways~~ orders of 6 elements $= 6!$

Half of them contain a before b

$\therefore$ No. of order $= \dfrac{6!}{2}$

Ques – How many linear orders of 6 elements a, b, c, d, e, f are there such that 'a' comes before 'b' and 'b' comes before 'c' (not necessarily immediately) ('abc' subsequence is there.)

No. of possible orders = $6! = 720$

No. of possible orders of 'abc' : $3! = 6$

$\therefore$ Ans = $\frac{6!}{3!} = \frac{720}{6} = \underline{\underline{120}}$

$\boxed{e}$ a b $\boxed{d}$ $\boxed{e}$ c
$\boxed{e}$ b a $\boxed{d}$ $\boxed{e}$ c
$\boxed{e}$ c b $\boxed{d}$ $\boxed{e}$ a
$\boxed{e}$ b c $\boxed{d}$ $\boxed{e}$ a
$\boxed{e}$ a c $\boxed{d}$ $\boxed{e}$ b
$\boxed{e}$ c a $\boxed{d}$ $\boxed{e}$ b

e a b d e c

for every six possible arrangements, 1 is chosen

mutation
ically
duct
rule.

$$^nP_r = \frac{\lfloor n}{\lfloor n-r} = \frac{n!}{(n-r)!}$$

$$^nC_r = \frac{\lfloor n}{\lfloor r \lfloor n-r} = \frac{n!}{r!\,(n-r)!}$$

$$^nP_r = (r!)(^nC_r)$$

$$^nC_r = {}^nC_{r-1}$$

Ques – No. of orderings / permutations / arrangements of r distinct elements from a set of n elements?

^nP_r

			...	
n	$n-1$	$n-2$	...	$n-r+1$

$= n(n-1)(n-2)(n-3) \cdots (n-r+1)$

$= \frac{n!}{(n-r)!} = \underline{\underline{^nP_r}}$

Ques - No. of ways to select r people from a set of n people.

^nC_r

(apply division rule to permutation)

Why 0! is 1?

Ans → By convention / definition

↳ no. of subsets of size r of a set of size n $= {}^nC_r$

not proof
one of the reasons behind this convention

no. of subsets of size 0 of a set of size n = $\underset{(\phi)}{1}$

$\therefore {}^nC_0 = 1 \Rightarrow \frac{n!}{n!\,0!} = 1$

$\Rightarrow \underline{0! = 1}$

^nC_r → binomial coefficient

$^nC_r = \frac{n!}{(n-r)!\,r!}$ algebraic meaning

^nC_r = selecting r people from n people — combinatorial meaning

If repeatation is allowed ⇒ ^nP_r, ^nC_r cannot be used.

Poker hand → (poker hand consists of 5 randomly chosen cards out of 52 cards.

Ques How many bit strings of length n contain exactly r 1s?

^nC_r

Templates

1. Some elements are never together –
First arrange remaining elements and then in gaps put these elements.

Ques – 8 men 5 women; stand in line; no 2 women stand next to each other

8 men can stand in 8! ways

| M_1 | M_2 | M_3 | M_4 | M_5 | M_6 | M_7 | M_8 |

9 gaps are there.
5 to be occupied by women

$\therefore$ Ans $\rightarrow$ $8! \times {}^9C_5 \times 5!$

Ques – 7 women & 9 men are on the faculty.
1) ways to select a committee of five members if atleast one woman must be in the committee?

${}^7C_1 \times {}^{15}C_4$ X wrong (overcounting)

Ans – 1 – no woman
$= {}^{16}C_5 - {}^9C_5$

2) ways to select a committee of five members if atleast 1 man & atleast one woman must be in the committee.

${}^7C_1 \times {}^9C_1 \times {}^{14}C_3$ X wrong

Ans – ${}^{16}C_5$ – no men – no women $= {}^{16}C_5 - {}^7C_5 - {}^9C_5$

Pascal's identity

can be used in DP of c program

① $$\boxed{{}^{n+1}C_k = \underbrace{{}^{n}C_{k-1}} + \underbrace{{}^{n}C_k}}$$

Recursive definition of binomial coefficient.

- ${}^{n+1}C_k$: from $(n+1)$ students, selects k students.
- ${}^{n}C_{k-1}$: 1 particular student is taken. Select k-1 students from n students.
- ${}^{n}C_k$: 1 particular student is not taken. select k students from n student

take 2 students x, y

② $$\boxed{{}^{n}C_k = \underbrace{{}^{n-2}C_{k-2}} + \underbrace{{}^{n-2}C_k} + \underbrace{2 \cdot {}^{n-2}C_{k-1}}}$$

- ${}^{n}C_k$: Select k students from n students
- ${}^{n-2}C_{k-2}$: ~~take 2 stude~~ both are selected ∴ select k-2 from n-2 student
- ${}^{n-2}C_k$: both rejected. ∴ select k from n-2
- $2 \cdot {}^{n-2}C_{k-1}$: one select one rejected ∴ select k-1 from n-2

③ $$\boxed{{}^{n+m}C_r = {}^{n}C_0\,{}^{m}C_r + {}^{n}C_1\,{}^{m}C_{r-1} + \dots - {}^{n}C_r \cdot {}^{m}C_0}$$

$$\sum_{k=0}^{r} {}^{n}C_k\,{}^{m}C_{r-k}$$

$$\cancel{\frac{n+m}{r}} \binom{n+m}{r} = \sum_{k=0}^{r} \binom{n}{k}\binom{m}{r-k}$$

Vandermonde's identity ← $[{}^{n}C_r = {}^{n}C_{n-r}]$

$$= \sum_{k=0}^{r} \binom{n}{n-k}\binom{m}{m-r+k} \dots$$

$$\binom{2n}{n} = \sum_{j=0}^{n} \binom{n}{j}^2$$

$$\binom{2n}{2} = 2\binom{n}{1} + n^2$$

n boys n girls select 2 = both boys + both girls + 1 boy 1 girl

$$\binom{n}{2} + \binom{n}{2} \qquad \binom{n}{1}\cdot\binom{n}{1}$$

$$= 2\binom{n}{2} + n^2$$

(4)

$$\sum_{r=0}^{n} {}^{n}C_r = \underbrace{2^n}_{\text{total subsets}}$$

i.e. $\underbrace{{}^{n}C_0}_{\text{subsets of length 0}} + \underbrace{{}^{n}C_1}_{\text{subsets of length 1}} + {}^{n}C_2 + \dots {}^{n}C_n = 2^n$

$\binom{n}{r} = \left(\frac{n}{r}\right)\binom{n-1}{r-1}$

(5)

$$\sum_{k=0}^{n} k\binom{n}{k} = n\,2^{n-1}$$

$k \cdot {}^{n}C_k = n \cdot {}^{n-1}C_{k-1}$

select k subset from n people
choose 1 president among them.

choose one person as president.
Select subset of (n-1) people.

Hockey Stick Identity

(6)

$${}^{n+1}C_{r+1} = {}^{n}C_r + {}^{n-1}C_r + {}^{n-2}C_r + \dots\dots {}^{r}C_r = \sum_{k=r}^{n} {}^{k}C_r$$

select max. number

from n+1 numbers, select r+1 numbers

max. no is (n+1) selected
from remaining n elements, select r elements

max no. n is selected
from remaining (n-1) elements select r elements.

max. no. selected = r+1
remaining r elements (smaller than r)
select r elements.

Binomial Theorem

$$(x+y)^n = {}^nC_0 x^n y^0 + {}^nC_1 (x)^{n-1} y + {}^nC_2 x^{n-2} y^2 + \ldots\ldots {}^nC_n x^0 y^n$$

$$\Rightarrow (1+x)^n = {}^nC_0 x^0 + {}^nC_1 x^1 + {}^nC_2 x^2 + \ldots\ldots {}^nC_n x^n$$

$$= \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots \binom{n}{n}x^n$$

$$\Rightarrow (1+1)^n = 2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots\ldots + \binom{n}{n}.$$

Putting $x = -1$,

$$(1-1)^n = 0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \ldots \binom{n}{n}$$

$$\Rightarrow \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots$$

4

i.e. sum of even binomial coefficients = sum of odd binomial coefficients

Permutations with ~~Repeat~~ Repetitions

Repetition allowed ∴ ${}^nP_r$ cannot be used.

Q. n distinct objects
permute r of them with repetition allowed.

$$\underbrace{\boxed{\;}}_{n \text{ choices}}\underbrace{\boxed{\;}}_{n \text{ choi}}\boxed{\;}\boxed{\;\cdots\;}\underbrace{\boxed{\;}}_{n \text{ choices}} = (n)^r$$

Q Total n elements
n_1 of type 1, n_2 of type 2, ... n_k of type k

No. of permutations $= \dfrac{n!}{n_1!\,(n_2)!\ldots(n_k)!}$ $\quad \boxed{n_1+n_2+n_3\ldots n_k = n}$

No. of permutations of word 'MISSISSIPPI' $= \dfrac{11!}{4!\,4!\,2!\,1!}$

$$= \underbrace{{}^{11}C_4}_{S} \cdot \underbrace{{}^{7}C_4}_{I} \cdot \underbrace{{}^{3}C_2}_{P} \cdot \underbrace{{}^{1}C_1}_{M}$$

Ques No. of ways in which 'PERMUTATIONS' can be arranged if there are always 4 letters b/w P & S.

$$\boxed{\;}\boxed{\;}\boxed{\;}\boxed{\;}\boxed{\;}\boxed{\;}\boxed{\;}\boxed{\;}\boxed{\;}\boxed{\;}\boxed{\;}\boxed{\;}$$

7 choices for P & S.

$$7 \times 2! \times \frac{10!}{2!}$$

Ques In how many ways can the letters in 'WONDERING' be arranged with ~~ex~~ exactly 2 consecutive vowels?

WONDERING — 9 letters
3 vowels
N repeating 2 times

$$\underbrace{{}^3C_2 \cdot 2!}_{\substack{\text{no. of ways of arranging}\\ \text{2 vowels from 3}}} \cdot \underbrace{\frac{6!}{2!}}_{\substack{\text{arranging all the}\\ \text{consonants}}} \quad \underbrace{{}^7C_2 \cdot 2!}_{\substack{\text{group of 2 vowels \& 3rd vowel cannot}\\ \text{be together cor exactly 2 should be together}}}$$

7 gaps

Ques No. of permutations of the word 'ABCDEF' in which A comes before B and C comes before D. (not necessarily immediately)

$$\underbrace{{}^6C_2 \times 1}_{\text{A comes before B}} \cdot \underbrace{{}^4C_2 \times 1}_{\text{C comes before D}} \cdot \underbrace{2!}_{\text{permute remaining}}$$

Ques How many permutations of [5 2 1 8 9 7] exist that follow the rule

- 5 must come first
- 8 must come before both 9 & 7
- 2 must come before 1.

$$\underbrace{1}_{\text{1st cond}} \times \underbrace{{}^5C_3 \times 2!}_{\text{2nd cond}} \quad \underbrace{{}^2C_2 \cdot 1}_{\text{3rd cond}}$$

Ques How many sequences of A B C D E F G H contain the subsequences of <C, A, B> or <B, E, D>

$= \ ^8C_3 \cdot 5! + \ ^8C_3 \cdot 5! - \ ^8C_5 \cdot 3!$

$= \underbrace{\frac{8!}{3!}}_{CAB} + \underbrace{\frac{8!}{3!}}_{BED} - \underbrace{\frac{8!}{5!}}_{CABED}$

Distributing Objects Into Boxes

DODB → Different objects different Boxes.

- order of elements does not matter in distributing objects into boxes.

1. DODB Template

Distinguishable objects into distinguishable boxes.

$Obj_1 \ Obj_2 \ Obj_3$ into $B_1, B_2 \dots B_n$

[Product rule with simple combination]

Ques:- distribute hands of 5 cards to each of 4 players from deck of 52 cards

$^{52}C_5 \cdot \ ^{47}C_5 \cdot \ ^{42}C_5 \cdot \ ^{37}C_5$ ways

$= \frac{52!}{5!\,5!\,5!\,5! \cdot 32!}$

Ques – n distinct objects

k different people.

$A_1 \quad A_2 \quad A_3 \; \dots \quad A_k$

n_1 obj $\quad n_2$ obj $\quad n_3$ obj $\quad n_k$ obj

$(n_1 + n_2 + n_3 + \dots n_k) = n$

No. of ways $\cancel{= {}^{n}C_{n_1} * {}^{n}C_{n_2} \times {}^{n}C_{n_3} \dots {}^{n}C_{n_k}}$

$\boxed{n_1 + n_2 + \dots n_k = n}$

No. of ways $= {}^{n}C_{n_1} \cdot {}^{n-n_1}C_{n_2} \cdot {}^{n-(n_1+n_2)}C_{n_3} \dots {}^{n-\dots}C_{n_k}$

$$= \frac{n!}{n_1!\, n_2! \dots n_k!}$$

Ques – 8 distinct objects; 3 boys;

everyone gets atleast 2 object.

2 cases: 2+2 2 2 ; 2 2+1 2+1

$${}^{3}C_1 \cdot {}^{8}C_4 \cdot {}^{4}C_2 \cdot {}^{2}C_2 + {}^{3}C_1 \cdot {}^{8}C_2 \cdot {}^{6}C_3 \cdot {}^{3}C_3$$

Ques – 52 cards to 4 people

everyone gets 13 cards.

oldest player gets queen of spades.

$${}^{51}C_{13} \cdot {}^{39}C_{13} \cdot {}^{26}C_{13} \cdot {}^{13}C_{13}$$

$$\binom{n}{n_1\ n_2\ n_3 \ldots n_k} = \frac{n!}{n_1!\, n_2!\, n_3! \ldots n_k!}$$ $\longleftarrow$ notation

$n_1 + n_2 + \ldots + n_k = n$

Ques – 15 objects 5 boxes

~~one box~~ boxes have 5, 4, 3, 2, 1 objects

$$\frac{15!}{1!\,2!\,3!\,4!\,5!}\ 5! \longrightarrow$$ because it is not given which box contain how many items

Ques – n distinguishable balls into n boxes
no. of balls in each box can vary

Ball$_1$	Ball$_2$	---	Ball$_n$
n choice	n	---	n

Ans → $\underline{\underline{n^n}}$

n balls m boxes. Then?

$$m \times m \times \ldots n \text{ times} = \underline{\underline{m^n}}$$

Analogy — Santa has identical chocolates to give to children.

2. IODB Template

Identical/Indistinguishable objects into distinguishable boxes.

IODB problem is same as Star - Bar problem

where star → no. of chocolates to be distributed

bar → no. of children − 1

$xx \mid xx \mid x$

ways of ~~distributing~~ permuting n stars and ~~n~~ to r bars $= \dfrac{(n+r)!}{n!\,r!}$

[n identical objects
r identical object]

Ques distributing 30 identical objects into 3 distinct boxes each box must have ~~o~~ atleast 5 items

$5+x \mid 5+y \mid 5+z \qquad x+y+z = 30-15 = 15$

problem narrows down to permutation 15 stars ~~a~~ 2 bars

$\cancel{\dfrac{15!}{2!}} \quad \dfrac{17!}{15!\,2!} = {}^{17}C_2 = {}^{17}C_{15}$

→ Combination with repetition

== Starbars

Selection from items with repetition.

3 fruits — mango, ~~org~~ orange, apples

ways of selecting 4 fruits.

≡ 4 stars 2 bar problem: ${}^{6}C_2$

In how many ways can we choose r objects from n objects → ${}^{n}C_r$

In how many ways can we choose r objects from n kinds of objects → ${}^{n+r-1}C_r$

→ Non negative Integer solutions of a equations:

① $x_1 + x_2 + x_3 = 11$

where $x_1\ x_2\ x_3$ are non negative integers

How many solutions?

$\equiv$ 11 stars 2 bar $= \underline{\underline{{}^{13}C_2}}$

② $a+b+c = 10$

$a \geq 1,\ b \geq 2,\ c \geq 3$

Convert to standard temp. by putting min value of a, b, c

$x + y + z = 4$ $[x = a-1 \quad y = b-2 \quad z = c-3]$

$x, y, z \geq 0$

2 bars 4 stars $= \underline{\underline{{}^{6}C_2}}$

IODB Template is equivalent to—

- no. of combinations of n objects taken r at a time with repetition.
- no. of ways r identical objects can be distributed among n distinct container.
- non negative integer soln of eqn

$$\boxed{x_1 + x_2 + \dots + x_n = r}$$

- k element multiset
- non decreasing subsequence.

→ Multiset problem—

k element multiset from n element set

$$\binom{n+k-1}{k} = {}^{n+k-1}C_k$$

(n-1) bars

k stars.

Buy 3 hats and there are 5 colors.

4 bars. 3 stars

$^7C_3$

Buy 5 hats and there are 3 colors

2 bars 5 star

$^7C_2$

→ Non decreasing integer sequence problem.

[GATE 2015] No. of non decreasing subsequences of length n from the set {1, 2, 3 ... m}

n stars (m-1) bars.

$^{n+m-1}C_n$

Ques - No. of n digit natural numbers in which digits are in non decreasing ~~to~~ order

0 will not be a digit

$\therefore \{1, 2, 3 \ldots 9\} \Rightarrow$ 8 bars
n stars.

$\therefore$ Ans — $^{n+8}C_8$

Ques - No. of n digit natural numbers in which digits are in non increasing order

0 can be there

$\therefore \{0, 1, 2 \ldots 9\} \Rightarrow$ 9 bars
n star

Ans — $^{n+9}C_9 - 1$ → for all zeros

Ques - No. of n digit natural numbers in which digits are ~~always~~ strictly increasing / ~~decreasing~~

$^{10}C_n$

$n \leq 10$
coz if $n > 10$ repetition will occur.

for strictly decreasing —
0 cannot be selected

$\therefore$ ^9C_n

(order matters)

→ Integer composition of a number
composition of an integer n is a way of writing n as the sum of a sequence of positive integer.

$4 = 4+0 = 1+3 = 3+1 = 2+2 = 1+1+2 = 1+2+1$

<u>Ques</u> - set of compositions of 6 into 3 parts

$a+b+c = 6$ $\quad a \geq 1$, $b \geq 1$, $c \geq 1$

$\therefore x+y+z = 3$

$\therefore$ Ans - $^{3+2}C_2$

Compositions of n into k parts

$x_1 + x_2 + \ldots + x_k = n-k$ (cos positive can't be zero)

$\therefore$ Ans → $^{n-k+k-1}C_{k-1} = {}^{n-1}C_{k-1}$

Total no. of composition $= \sum_{r=1}^{n} \binom{n-1}{r-1}$

let $x = r-1$
$r=1 \Rightarrow x=0$
$r=n \Rightarrow x=n-1$

$= \sum_{x=0}^{n-1} \binom{n-1}{x} = 2^{n-1}$

(No. of equivalence relations on a set = No. of partition)

No. of partitions / equivalence relations of set $\{1,2,3,4\}$

GATE 1997

1 part partition = 4 → 1 way

2 part partition = (1,3) & (2,2) → $^4C_1 \cdot {^3C_3} + \frac{^4C_2 \cdot {^2C_2}}{2!}$

$= 4 + \frac{4\times3}{2\times2} = 7$

3 part partition → (1, 1, 2) → $\frac{^4C_1 \cdot {^3C_1} \cdot {^2C_2}}{2!} = 6$

4 part partition → (1, 1, 1, 1) → 1 way

Total no of ways = 1+7+6+1

$\boxed{= \underline{\underline{15}} \text{ partitions}}$

Partition of a set → DOIB template (order matter)
Different object identical boxes.

Partition of integer (number) → IOIB template

Composition of number → IODB template $\sum^{n-1}C_{k-1}$

DOIB
- Partition of a set
- No. of equivalence relations on a set.

IOIB
- partition of integer (partition no of n)

Generating Functions

The generating function on a sequence $a_0, a_1, a_2 \ldots a_k$ of real numbers is the infinite series

$$\boxed{G(x) = a_0 + a_1 x + a_2 x^2 + \ldots = \sum_{k=0}^{\infty} a_k x^k}$$

Generating functions are defined only for infinite sequences.

AGP function / sequence

$$S = a + (a+d)r + (a+2d)r^2 + (a+3d)r^3 + \ldots$$

$$rS = ar + (a+d)r^2 + (a+2d)r^3 + \ldots$$

$$(S - rS) = a + dr + dr^2 + dr^3 + \ldots$$

$$(S - rS) = a + dr(1 + r + r^2 + \ldots)$$

$$S(1-r) = a + dr \cdot \frac{1}{1-r}$$

$$\Rightarrow \boxed{S = \frac{a}{1-r} + \frac{dr}{(1-r)^2}}$$ imp. remember the formula.

(AGP series) $1 + 2x + 3x^2 + 4x^3 + \ldots$

$$GF(s) = 1 + 2x + 3x^2 + 4x^3 + \ldots$$

$$= \frac{1}{1-x} + \frac{1 \cdot x}{(1-x)^2} = \frac{1}{1-x} + \frac{x}{(1-x)^2}$$

$$= \frac{1-x+x}{(1-x)^2} = \frac{1}{(1-x)^2}$$

imp If generating function for the sequence $a_0\ a_1\ a_2 \dots \Leftrightarrow G(x)$

Then,

GF for seq. $a_1, 2a_2, 3a_3 + 4a_4 + \dots \Leftrightarrow G'(x)$

Interesting obs —

$G(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$

$\boxed{a_0 = G(0)}$ but $G(1) = a_0 + a_1 + a_2 + \dots$

$\therefore$ To find a_1 —

$G'(x) = a_1 + 2a_2x + 3a_3x^2 + \dots$

$\boxed{G'(0) = a_1}$

$\boxed{G''(0) = 2a_2}$ $\boxed{G'''(0) = 3!\, a_3}$

imp $\boxed{G^n(0) = n!\, a_n}$

Extended Binomial Theorem —

$^{1/2}C_2 = \dfrac{(1/2)(1/2 - 1)}{2!} = \dfrac{-1}{4\times 2} = -1/8$

$^{-3}C_2 = \dfrac{(-3)(-3-1)}{2!} = \dfrac{(-3)(-4)}{2} = 6$

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots \binom{n}{n}x^n + \underbrace{\binom{n}{n+1}x^{n+1} + \dots}$$

These terms are 0 if $n \in \mathbb{N}$, else non zero

$\therefore$ In the expansion of $(1+x)^{-2}$, coeff of $x^4 = \binom{-2}{4}$

$= {}^{-2}C_4 = \dfrac{(-2)(-3)(-4)(-5)}{4!}$

$= 5$

coeff of x^{-4} = 0 coz x^{negative} is not possible

$(2+x)^{-2}$ coeff of x^4 = $\cancel{\binom{-2}{4} x^4 2^{-2}}$

$= 2^{-2}\left[1+\frac{x}{2}\right]^{-2}$ $\therefore$ coeff of $x^4 = 2^{-2} \cdot \binom{-2}{4} \left(\frac{1}{2}\right)^4$

$= \frac{1}{4} \cdot {}^{-2}C_4 \left(\frac{1}{2}\right)^4$

$= \frac{1}{4} \cdot \left(\frac{1}{2}\right)^4 \cdot \frac{(-2)\cancel{(-3)(-4)}(-5)}{\cancel{4!}}$

$= \frac{5}{2^6} = 5/64$

~~$^{-n}C_r = (-1)^r \binom{n+r-1}{r}$~~

$$\boxed{{}^{-n}C_r = (-1)^r \binom{n+r-1}{r}}$$

negative term

imp

Recurrence Relations

Recurrence relations occur when some term in the sequence depends upon the previous terms of the sequence.

Recurrence relation is the equation that expresses a_n in terms of one or more of the previous terms of the sequence.
Initial conditions are required to be specified to find first term of seq.

__Recursive function__ – Function that is represented by itself in terms of smaller elements.

$fib(n) = fib(n-1) + fib(n-2)$ ← R. Function

$a_n = a_{n-1} + a_{n-2}$ ← R. Relation

__Ques__ :- No. of bit strings that do not have 2 consecutive 0's

$\underbrace{\boxed{\quad\quad|1}}_{\text{(n-1) length string no consecutive 0's}} + \underbrace{\boxed{\quad\quad|1|0}}_{\text{(n-2) length string no consecutive 0's}}$

$$\boxed{f(n) = f(n-1) + f(n-2)}$$

$f(1) = 2$
$f(2) = 3$

__Ques__ :- No. of permutations of a set with n elements

$$P(n) = n\,P(n-1)$$ (factorial का रूप)

__Ques__ :- No of bit strings containing a pair of consecutive 0's

$\underbrace{\boxed{\quad\quad|1}}_{n-1} \rightarrow T(n-1)$

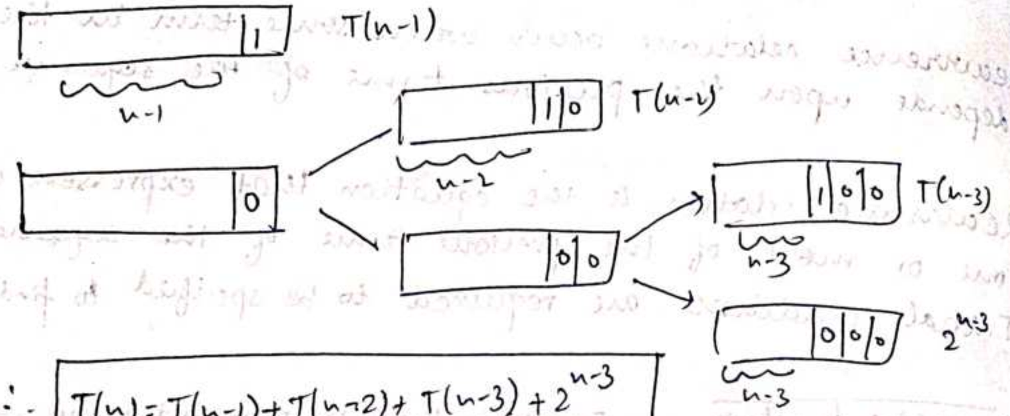
$\boxed{\quad\quad|0}$ → $\underbrace{\boxed{\quad\quad|0|0}}_{n-2}$ all strings of n-2 length are counted $= 2^{n-2}$

$\boxed{\quad\quad|0}$ → $\underbrace{\boxed{\quad\quad|1|0}}_{n-2}$ $T(n-2)$

$T(1) = 0$
$T(2) = 1$

$$\therefore \boxed{T(n) = T(n-1) + T(n-2) + 2^{n-2}}$$

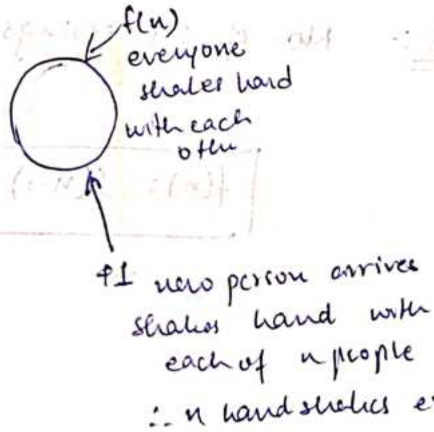
<u>Ques</u> - No. of bit strings of length n that contain three consecutive 0's.



$$\therefore \boxed{T(n) = T(n-1) + T(n-2) + T(n-3) + 2^{n-3}}$$

$f(1) = 0 \quad f(2) = 0 \quad f(3) = 1$

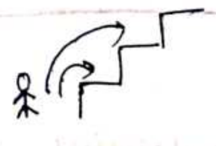
<u>Ques</u> :- Recurrence relation for number of handshakes among n people.



$$\boxed{\therefore T(n+1) = T(n) + n}$$

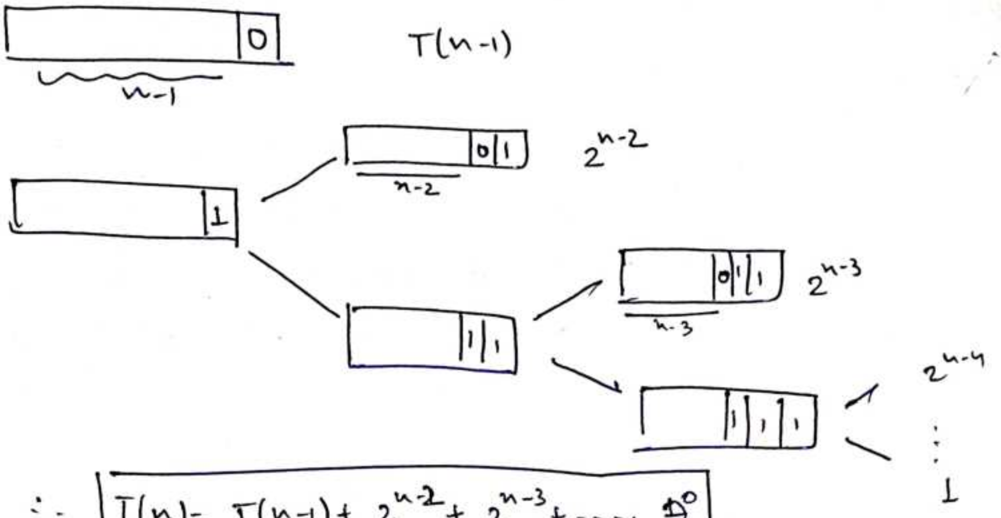
$\frac{n(n+1)}{2} \quad {}^nC_2$

<u>Ques</u> :- No. of ways of climbing n stairs if one or 2 steps can be taken



$$\therefore S(n) = S(n-1) + S(n-2)$$

Ques - Recurrence relation for the number of ~~bits~~ bit strings containing the string '01'



$$\therefore \boxed{T(n) = T(n-1) + 2^{n-2} + 2^{n-3} + \dots 2^0}$$

$$\therefore T(n) = T(n-1) + 2^{n-1} - 1$$

Graph Type	Edges	Multiple edges?	self loop?
Simple graph	undirected	✗	✗
Multigraph	undirected	✓	✗
Pseudograph	undirected	✓	✓
Simple directed graph.	directed	✗	✗
Directed multigraph.	directed	✓	✓
Mixed graph	Directed / undirected	✓	✓

GRAPH THEORY

(~~Engineer~~ Discrete Mathematics)

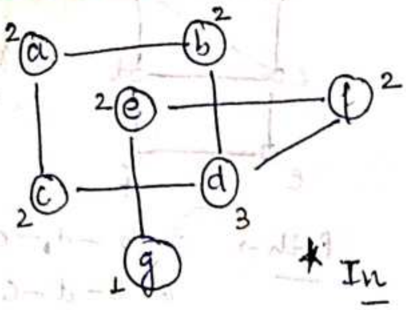
Types of graph –

- ① undirected graph ~~(...)~~
- ② directed graph
- ③ Multigraph
  - undirected graph
  - no self loop

  } multi edges / parallel ~~pa~~ edges allowed.
- ④ Pseudo graph
  - any undirected graph parallel edges, self loop allowed

Degree of a vertex – Number of edges incident on the vertex.

not defined for directed graph

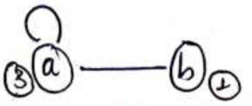


Even degree vertices = a b c e f

Odd degree vertices = d g

* In pseudograph –

Self loop gives degree of 2.



Self loop contributes to twice in the degree of a vertex.

$$\boxed{\begin{array}{l}\Sigma \text{Degree} = 2 * \text{size of graph} \\ \Sigma \text{Degree} = 2 * \text{no of edges}\end{array}}$$

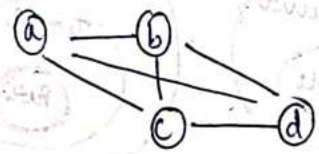
> 1. Degree 1 vertex
>   ↳ pendant vertex
>
> 2. Degree 0 vertex
>   ↳ isolated vertex

Order of graph – cardinality of vertex set.

Size of graph – cardinality of edge set.

$\delta(G) \rightarrow$ minimum degree in G.

$\Delta(G) \rightarrow$ maximum degree in G.



Order of graph = 4

Size of graph = 6

$\delta(G) \le$ Avg. degree $\le \Delta(G)$

$\Rightarrow n\delta(G) \le n *$ Avg. degree $\le n \cdot \Delta(G)$

$\Rightarrow n\delta(G) \le$ Total degree $\le n\Delta(G)$

$\rightarrow n\delta(G) \le$ No. of edges $\le n\Delta(G)$

Handshaking Theorem –

G(V, E) is any undirected graph with m edges. Then,

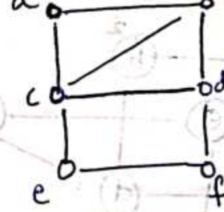
$$2m = \sum_{v \in V} \deg(v)$$

(applies even if self loop & multiple edges are present).

> For directed graph,
>
> Total indegree = Total outdegree = No. of edges

* In any graph, the no. of odd degree vertices is always even.

Walk - vertex repetition is allowed
edge repetition is allowed



length of walk = no. of edges in the walk

Trail - vertex repetition is ~~not~~ allowed
edge repetition is not allowed.

Path - vertex repetition is not allowed
edge repetition is not allowed.
$v_1 \neq v_n$

Path → a-b-d-c-b-d-f
a-b-d-c
a-c-e-f-d-c-a-b.

Trail → a-b-d-c-b

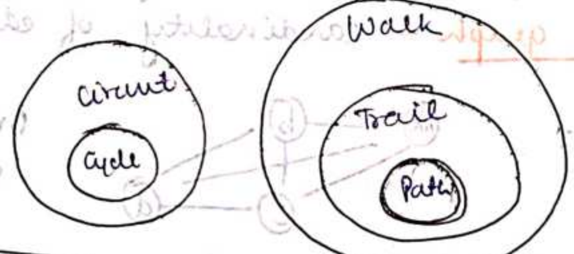
Path → a-b-d-c
a-b-d-f
a-c-e-f.

* Closed trail is called circuit.

a—b—d—c—a

* Path in which first & last vertices are same is called cycle
min. 3 vertices in a cycle.

* If u-v walk exists then, u-v path also exist.



$\boxed{n\delta \leq 2e}$ ← used often in min.

Connected Graph

A graph G is connected iff there is a path ~~from~~ b/w every pair of vertices.

Distance b/w a & b is length of shortest path from a to b.

*

$$\boxed{\text{Diameter of a connected graph} = \max_{\forall a, b} \{ dist(a,b) \}}$$

Regular graph – Every vertex has same degree.

2 reg.   2 reg.   0 regular

Analysis of regular graph –
d-regular graph.
No. of vertices = n.

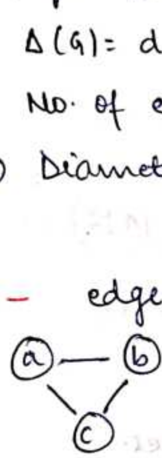
① Degree sequence = d, d, d .... d (n times)

② $\Delta(G) = d$ $\delta(G) = d$ Avg deg := d   Total deg. = nd

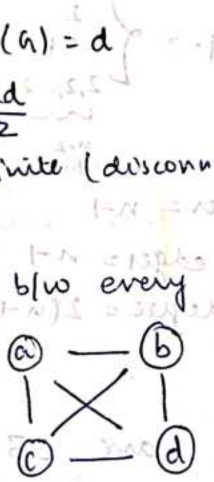
③ No. of edges = $\frac{nd}{2}$

④ Diameter = infinite (disconnected) or finite.

Complete graph – edge exists b/w every pair of vertices.



K_3



n vertices

$\frac{n(n-1)}{2}$ edges.

K_n: $|V| = n$

↳ degree of every vertex = n-1

↳ $\Delta(G) = \delta(G) = $ Avg deg = n-1

↳ (n-1) regular

↳ Deg. seq. → (n-1), (n-1), ... (n-1) [n times]

↳ Diameter = 1

↳ connected graph.

Empty graph / Null graph / Edgeless graph.

graph without edges (E_n) → n vertices.

E_n

↳ Degree of each vertex = 0

↳ 0 - Regular

↳ $\delta = \Delta =$ Avg. = 0

↳ No. of edges = 0

↳ Diameter = 0 if n=1
  ∞ if n>1

Path graph - looks like a straight line

$a_1 - a_2 - a_3 - - - - a_n$

P_1 P_2 P_3 P_4

P_n

↳ Deg. seq. = $\begin{cases} 1 & P_1 \\ 1,1 & P_2 \\ \underbrace{2,2,\ldots 2}_{n-2 \text{ times}},1,1 & P_{n\geq 3} \end{cases}$

↳ Diameter = $n-1$

↳ No of edges = $n-1$

↳ Total degree = $2(n-1)$.

Cycle Graph - must have ≥ 3 vertices.

n vertices n edges.

C_n

↳ Degree of each vertex = 2

↳ 2- regular

↳ $\delta = \Delta =$ avg = 2

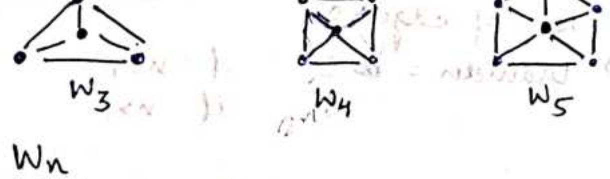
↳ No. of edges = n

↳ Diameter = $\left\lfloor \frac{n}{2} \right\rfloor$ $\begin{cases} n/2 & n \text{ is even} \\ \frac{n-1}{2} & n \text{ is odd} \end{cases}$

↳ deg. seq. → $\underbrace{2,2,2\ldots}_{n \text{ times}}$

Wheel graph

add a node to C_n and connect it with every other node



W_3 W_4 W_5

W_n → wheel graph on n+1 vertices.

only W_3 is regular

Diameter = 2

W_n

↳ (n+1) vertices

↳ Deg. seq. → n, $\underbrace{3,\ldots 3}_{n \text{ times}}$

↳ not regular

↳ edges = $\frac{nd}{2} = \frac{n^2}{2}$

$\delta = 3$ $\Delta = n$ avg. deg = $\frac{n+3n}{n+1} = 4n/n+1$

edges = 2n

Hypercube graph

Q_n.
- ↳ no. of vertices = 2^n
- ↳ no. of edges = $2^{n-1} * n$
- ↳ Diameter = n
- ↳ Degree of each vertex = n.

$Q_{(n)} \to$ n bit sequence. each seq. is a node ∴ 2^n nodes.

edge exists b/w u & v if u & v have 1 hamming distance i.e. differ by one bit position.

Boolean lattice is hypercube graph.

Ques- How many edges in Hasse diagram of $(P(A), \subseteq)$ where $|A| = n$

$|P(A)| = 2^n$ $\quad Q_n \quad$ ∴ no. of edges = $n \times 2^n$

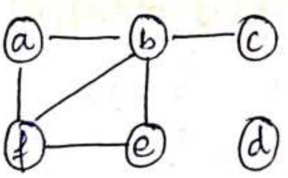
Graph H is subgraph of graph G.

iff $V(H) \subseteq V(G)$ &&
$E(H) \subseteq E(G)$

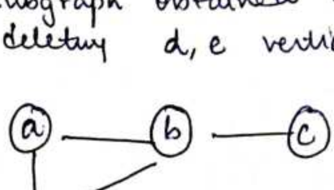
Graph G $\xrightarrow{\text{delete one or more edges / vertices or both}}$ Graph H

Subgraph
- **Spanning subgraph**: vertex deletion is not allowed
- **Induced subgraph**: edge deletion is not allowed (vertex deletion allowed — edge deleted due to vertex deletion is allowed, no extra edge should be deleted)

No. of spanning & induced subgraphs of graph G = 1



Subgraph induced by the vertices a, b, c, and f
≅ subgraph obtained after deleting d, e vertices.



Imp-
- ↦ Every induced subgraph of a complete graph is also a complete graph.
- ↦ Every graph of n vertices is subgraph of n vertices complete graph.
- ↦ There is exactly 1 subgraph of G (G itself) which is both induced and spanning subgraph.

Ques - n vertices simple graph

Max. no. of edges = $^nC_2 = \frac{n(n-1)}{2}$

$\therefore$ No. of simple graphs = $2^{\frac{n(n-1)}{2}}$

Graph isomorphism (very hard problem NP Intermediate) exp. time complexity.

graphs with same structure

⊙ 2 graphs are isomorphic if there exists bijection in the ~~no. of~~ vertices of the graph which preserves edges.

i.e. $a, b \in E_1$ iff $f(a), f(b) \in E_2$

$|A| = 3 \quad |B| = 2$

No. of bijections from $A \to B$ = 0

coz $|A| \neq |B|$

$|A| = 3 \quad |B| = 3$

No. of bijections from $A \to B$ = $3! = 6$

Graph invariates are necessary for isomorphism but not sufficient → order, degree, size, degree seq., n-length cycle etc.

- No of vertices
- $|E|$
- connectedness
- Degree sequence.

- No. of 3 length cycles
- No. of 4 " "
- No. of 5 " "

*Graph isomorphism is equivalence relation.

Complement of a graph

complement of a graph with n vertices = Complete graph with n vertices − edges in G.

$G(V, E)$

$\bar{G}(V, \bar{E}) \leftarrow$ comp.

$$\boxed{\bar{E} = E(K_n) - E}$$

$$|\bar{E}| = {}^nC_2 - |E|$$

i.e. $$|E| + |\bar{E}| = {}^nC_2$$

Self complementary graph –

If $G = \overline{G}$ i.e. G & $\overline{G}$ are isomorphic, then, G is self complementary

★ complement of disconnected graph is always connected

If G is disconnected, then, $\overline{G}$ is connected

For every simple graph G,
either G or $\overline{G}$ is definitely connected

★ If a graph has exactly 2 vertices of odd degree, then, they are connected by a path.

★ In any simple graph there is a simple path from any vertex of odd degree to some ? not every other vertex of odd degree

For a simple graph with n vertices, what is max. no of edges?

↦ if G is undirected → ${}^nC_2 = \frac{n(n-1)}{2}$

↦ if G is directed → $2 \times {}^nC_2 = n(n-1)$

Bipartite graph

$G(V, E)$ is bipartite iff
$\exists$ a ~~bijection~~ bipartition X, Y of V such that

① $X \cap Y = \phi$

② $\forall a, b \in X \quad (a, b) \notin E(G)$
$\forall a, b \in Y \quad (a, b) \notin E(G)$

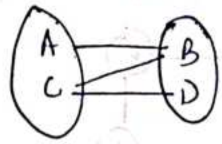
③ $X \cup Y = V$

④ X, Y can be empty.

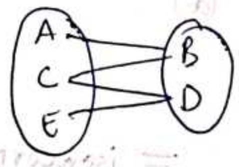
Ques – P_n is bipartite; $n = ?$

~~should be even~~

A — B — C — D

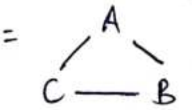
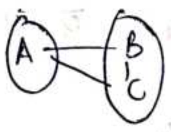


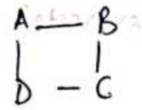
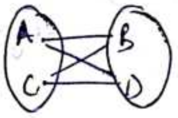
A — B — C — D — E



$\therefore$ every P_n graph is bipartite..

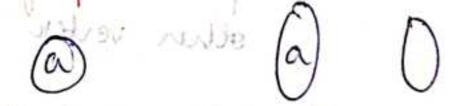
Ques – C_n is bipartite?

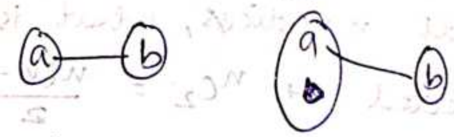
$C_3 =$   not bipartite

$C_4 =$   bipartite

$\therefore$ C_n is bipartite when n is even

Ques – K_n is bipartite?

$K_1 =$  bipartite

$K_2 =$  bipartite

$K_3 =$  not bipartite

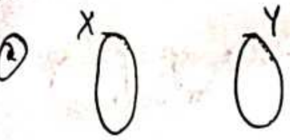
$K_4 =$  not bipartite

$\therefore$ only K_1 & K_2 are bipartite

K_n $n \geq 3$ is not bipartite

Theorem

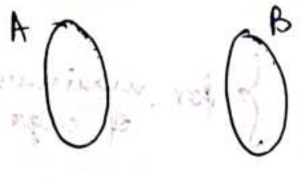
↳ A graph is bipartite if and only if it ~~has even length cycle~~ doesn't have odd length cycle.

① 

$\sum_{v \in X} deg(v) = \sum_{v \in Y} deg(v)$

Total degree of vertices in X = Total degree in Y.

For k regular bipartite graph,
all the vertices have degree k. $k \geq 1$



$\sum_{v \in A} deg(v) = k|A|$

$\sum_{v \in B} deg(v) = k|B|$

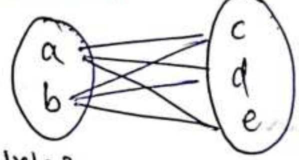
$\because \sum_{v \in A} deg(v) = \sum_{v \in B} deg(v)$

$\therefore K(|A|) = k|B| \Rightarrow |A| = |B|$

⊙ Every subgraph of a bipartite graph is bipartite.

Complete Bipartite graph

$K_{2,3}$



$|X| = 2$ $|Y| = 3$

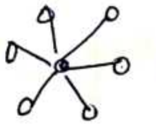
$K_{m,n} \rightarrow m \geq 1 \quad n \geq 1$

$|V| = m+n$

$|E| = m \times n$

deg. seq. = $\underbrace{n, n, n, \ldots n}_{m\ times}, \underbrace{m\ m \ldots m}_{n\ times}$

⊙ Star graph is always complete bipartite graph.



$K_{1,6}$ $K_{1,n}$

$\Delta = max(m,n)$

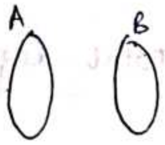
$\delta = min(m,n)$

diameter = 1 for $K_{1,1}$

2 for $K_{m,n}$ otherwise

⊙ E_n is complete bipartite graph.
↳ edgeless graph

Maximize the number of ~~vertices~~ edges in bipartite graph of n vertices



let $|A| = m$

$|B| = n - m$.

No. of edges in complete b.g. $= |A||B| = m(n-m) = mn - m^2$

$$e$$

$$\frac{de}{dm} = n - 2m = 0 \Rightarrow 2m = n \Rightarrow m = n/2$$

$\therefore \left\lfloor \frac{n}{2} \right\rfloor$ elements should be in A
$\left\lceil \frac{n}{2} \right\rceil$ elements should be in B
$\Big\}$ for maximum no. of edges.

$\underline{\text{Max}} \rightarrow \left(\frac{n}{2}\right)^2$ edges.

* If a graph on n vertices has more than $\frac{n^2}{4}$ edges, then, the graph is not <u>bipartite</u>.

Which graphs are bipartite?

(complete graph) K_n — only for $n = 1, 2$

(Path graph) P_n — ✓

(Cyclic graph) C_n — only if n is even

(wheel graph) W_n — never

(edgeless graph) Θ_n — ✓

(Hypercube graph) Q_n — ✓

(Complete bipartite graph) $K_{m,n}$ — ✓

Cyclic graph

graph containing atleast one cycle.

⊙ There are atleast 2 vertices which have more than one path between them.

Tree

- connected
- undirected
- acyclic

Forest → collection of trees

In a tree, vertex with degree 1 is called leaf.

Complete analysis of tree –

⊙ Tree on n vertices

⊙ ~~n-1~~ (n-1) edges should be there (minimally connected)

⊙ connected, acyclic

Connected graph on n vertices (n-1) edges. Is this tree? ⇒ Yes absolutely YES!

⊙ Tree is maximally acyclic

↳ adding one more edge leads to cycle

→ Every tree T with atleast 2 vertices has minimum 2 vertices of degree 1

Height of a node in rooted tree = no. of edges from that node to the farthest leaf.

~~Deapth~~ Depth of a node in rooted tree = no. of edges from the root to that node

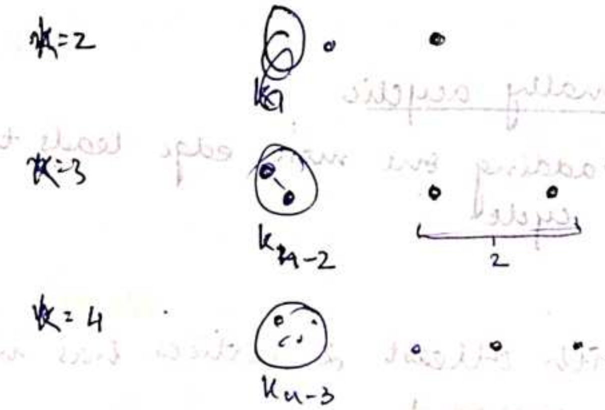
* In any cycle of cyclic graph, every vertex in it has a degree greater than or equal to 2.

* Tree + 1 edge $\rightarrow$ graph with exactly one cycle.

No. of edges in a graph with n vertices to guarantee that it is connected — $\underbrace{{}^{n-1}C_2}_{\substack{\text{complete}\\ \text{graph with}\\ n-1 \text{ vertices}}} + \underbrace{1}_{\substack{\text{to add}\\ n^{th}\\ \text{vertex}}}$

* Every simple graph with more than $(n-1)(n-2)/2$ edges is connected.

* Maximum number of possible edges in an undirected graph with n vertices and k components.



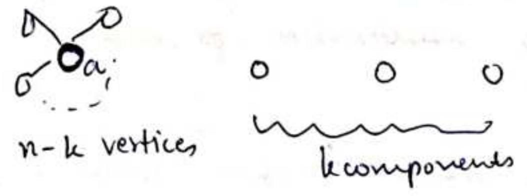
$\therefore$ for k components, $(k-1)$ components will have 0 edge
The k^{th} component will be $K_{n-(k-1)} = K_{n-k+1}$

$\therefore$ Max. no. of edges $= {}^{n-k+1}C_2 = \dfrac{(n-k+1)(n-k)}{2}$

Let G be an arbitrary graph with n vertices and k components. If a vertex is removed from G the number of components in the resultant graph must lie down b/w

A. k & n B. k-1 & k+1 C. k-1 & n-1 ✓ D. k+1 & n-k

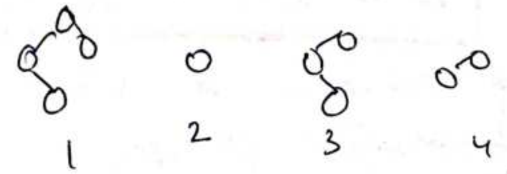
For max comp. —



On deleting a_1 no. of components generated from 1st component
$= n-k-1$

∴ Total no. of components
$= n-\cancel{k}-1+\cancel{k} = n-1$

For min comp. —



On deleting ~~com~~ 2, no. of remaining components $= k-1$

Clique

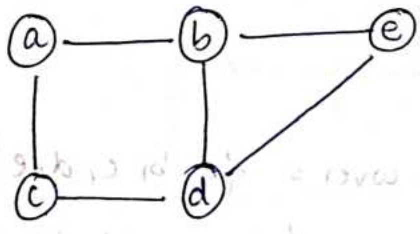
Clique of a graph is the subgraph which is complete.

Size of maximum clique = clique no. = $\omega(G)$

Independent set

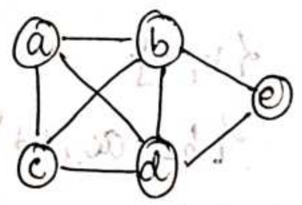
Subgraph with vertices not connected to each other.

Size of maximum independent set = independence number $\alpha(G)$ ↳ stability no.



Clique → {a, b} {b, d, e} ~~{a, d, e}~~

Independent set → {a, e} {a} ~~{a, d, e}~~ {a, d}



$\omega(G) = 4$ $\alpha(G) = 2$

{a, b, c, d} ⊇ maximum clique {a, e} ⊇ maximum independent set

C_n: $\alpha(G) = \lfloor \frac{n}{2} \rfloor$

C_n: $\omega(G) = \begin{cases} 3 & n=3 \\ 2 & n>3 \end{cases}$

K_n: $\alpha(G) = 1$

K_n: $\omega(G) = n$

P_n: $\alpha(G) = \lceil \frac{n}{2} \rceil$

P_n: $\omega(G) = \begin{cases} 1 & n=1 \\ 2 & n\geq 2 \end{cases}$

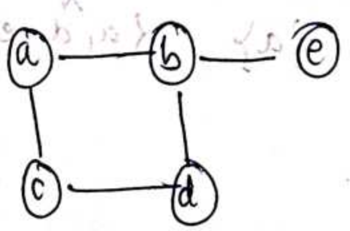
Clique in G $\rightleftarrows$ Independent set in $\bar{G}$

Vertex Cover

set of vertices that cover all the edges.

Edge Cover

edges covering all the vertices

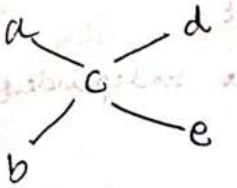


Maximum vertex cover = {a, b, c, d, e}

Maximum edge cover = {ab, ac, cd, bd, be}

Minimum ~~edge~~ vertex cover = {b, c}

Minimum edge cover = {be, ac, bd}



Vertex cover → {c}   {a, b, e, d}

Edge cover → {ac, bc, ec, dc}

★ Edge cover exists only if no isolated vertex in the graph

Size of minimum vertex cover = $\beta(G)$

Size of minimum edge cover = $\beta'(G)$

Maximum degree vertex should be in minimum vertex cover ⟶ False

Analysis of edge cover

For n vertices graph,
size of minimum edge cover, $\beta' \geq \lceil \frac{n}{2} \rceil$

K_n: β (vertex cover) $= n-1$

P_n: $\beta = \lfloor \frac{n}{2} \rfloor$, $\beta' = \lceil \frac{n}{2} \rceil$

C_n: $\beta = \lceil \frac{n}{2} \rceil$, $\beta' = \lceil \frac{n}{2} \rceil$

W_n: $\beta = \lceil \frac{n}{2} \rceil + 1$, $\beta' = \lceil \frac{n}{2} \rceil$

$$\boxed{\text{Covering number of graph} = \text{vertex cover.}}$$

Relation between vertex cover and ~~a~~ independent set.

If $S \subseteq V$ is vertex cover, then $\bar{S}$ or $V-S$ is independent set & vice versa.

$\alpha \rightarrow$ size of largest independent set.

$\beta \rightarrow$ size of minimum vertex cover.

$$\boxed{\alpha + \beta = n}$$

no. of vertices.

Matching

set of edges that are independent (non adjacent)

- A set of pairwise non adjacent edges in a graph is called a matching.
- The maximum number of edges in a matching in a graph G is called matching number of G and is denoted by ~~μ(G)~~. $\alpha'(G) = \mu(G)$

Perfect matching means that all the vertices are saturated → Matching is perfect if it covers all vertices of the graph.

Cycle graph $C_n \longrightarrow \alpha'(G) = \left\lfloor \frac{n}{2} \right\rfloor$

(perfect matching exists when n = even)

Complete graph. $K_n \longrightarrow \alpha'(G) = \left\lfloor \frac{n}{2} \right\rfloor$

Complete bipartite graph $K_{m,n} \longrightarrow \alpha'(G) = \mu(G) = \min(m,n)$

perfect matching exists only if $m = n$

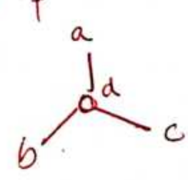
For every graph,

$\mu(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$

size of ~~perfect~~ largest matching

Prefect matching exists means no of vertices in graph is even.

Vice versa is not true



* If there exists a matching of size k,
then, every vertex cover has size $\geq k$

$\mu \leq \beta(G)$

max matching — min vertex cover

⊙ The cardinality of any matching is less than or equal to the cardinality of any vertex cover

NOTE: If S is the maximum matching then every edge of the graph is incident on some vertex covered by S.

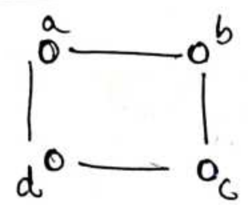
So, vertices covered by S will cover all the edges.

$\therefore \quad \mu(G) \leq \beta(G) \leq 2\mu(G)$

K_n (even): $\beta = n-1$, $\mu = \lfloor \frac{n}{2} \rfloor$

$K_{m,n}$: $\mu = \min(m,n)$, $\beta = \min(m,n)$

<u>$\mu = \beta$</u>



$\mu(G) = |\{ab, cd\}| = 2$

$\beta(G) = |\{a, c\}| = 2$

Cycle graph.

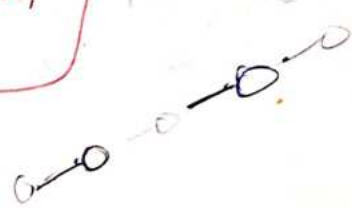
<u>$\mu = 2\beta$</u>

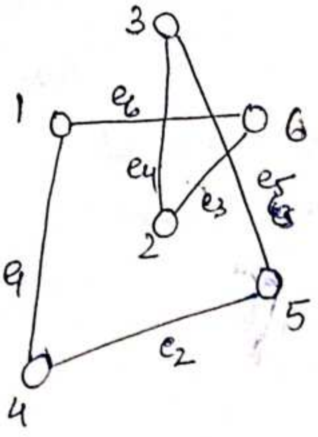
K_{odd}: $\beta = n-1$, $\mu = \lfloor \frac{n}{2} \rfloor$

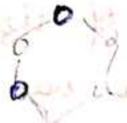
$\beta = 2\mu$

$\beta = 4$
$\mu = 2$

for bipartite graph,
$\mu = \beta$







$= C_6$

α = Independent set = 3
β = min vertex cover = 6 - 3 = 3
β' = min edge cover = $\lceil 6/2 \rceil$ = 3
μ = max matching = 3

⊙ For any graph, the size of maximum independent set is less than or equal to edge cover.

$$\boxed{\alpha \le \beta'}$$

Conclusion

α = Maximum independent set
$\alpha' = \mu$ = Maximum matching
β = Minimum vertex cover.
β' = Minimum edge cover.

① $\alpha + \beta = n = |V|$  $\alpha' + \beta' = n$

② $\mu \le \beta \le 2\mu$

③ $\beta' \ge \left\lceil \frac{n}{2} \right\rceil$

④ $\alpha' = \mu \le \left\lfloor \frac{n}{2} \right\rfloor$

⑤ $\alpha \le \beta'$

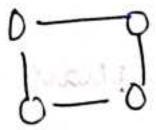
⑥ Bipartite graph $\rightarrow \mu = \beta$

Graph coloring

assignment of labels / colors to vertices of a graph such that no two adjacent vertices share the same color.

Graph is K-colorable? - graph can be colored using atmost k colors?

Minimum number of colors needed — chromatic number



1 colourable ? No
2 colorable ? Yes
3 colorable ? Yes

→ denoted by $\chi(G)$
geta

K_n complete graph $\quad \chi = n$

E_n edgeless graph $\quad \chi = 1$

P_n path graph $\quad \chi = 2$, $n \geq 2$ else 1.

~~C_n cycle graph~~ $\quad \cancel{\chi = \begin{cases} \lceil \frac{n}{2} \rceil & \text{odd length cycle} \\ \frac{n}{2} & \text{even length cycle} \end{cases}}$

$n - 2\lfloor \frac{n}{2} \rfloor + 2$ $\quad C_n$ cycle graph $\quad \chi = \begin{cases} 3 & n \text{ is odd} \\ 2 & n \text{ is even} \end{cases}$

simple connected graph which does not contain odd length cycle

Bipartite graph $\quad \chi = 2$ if edge exists
else $\chi = 1$

$K_{m,n}$ complete bipartite graph. $\quad \chi = \begin{cases} 1 & \text{if } m=0 \text{ or } n=0 \\ 2 & \text{otherwise.} \end{cases}$

$$\chi(G) \le \Delta(G)+1$$

$Q_n \rightarrow$ Hypercube graph (bipartite) $= \chi = \begin{cases} 2 & n \geq 1 \\ 1 & n = 0 \end{cases}$

$W_3 = K_4$
4 colors needed

$W_n \rightarrow$ wheel graph on $n+1$ vertices $= \chi = \begin{cases} 4 & n \text{ is odd} \\ 3 & n \text{ is even} \end{cases}$

Chromatic number is always greater than or equal to clique number

$$\chi(G) \geq \omega(G)$$

For any graph,

$$\boxed{\omega(G) \leq \chi(G) \leq \Delta(G)+1}$$

$$|E(G)| \geq \frac{\chi(G)\,(\chi(G)-1)}{2} \quad \text{for every graph}$$

χC_2

b/w every color class min 1 edge should be there.

$$|V(G)| \leq \chi(G)\,\alpha(G)$$

$\frac{n}{\chi} \rightarrow$ avg. size of color class.

$\chi = \Delta + 1$ for only 2 graphs
- $\rightarrow$ odd length cycle graph
- $\rightarrow$ complete graph (K_n)

Brooke's Theorem.

Vizing Theorem

For any simple graph,
edge chromatic number can be ~~eig~~ either $\Delta(G)$ or $\Delta(G)+1$.

$$\boxed{\Delta(G) \le \chi'(G) \le \Delta(G)+1}$$

* Bipartite graph $\rightarrow$ $\chi'(G) = \Delta(G)$

* Every graph having $\chi'(G) = \Delta(G)+1$ must have atleast three vertices of maximum degree

$$\left.\begin{matrix} C_{odd} \\ K_{odd} \\ \text{regular}_{odd} \end{matrix}\right\} \chi'(G) = \Delta(G)+1$$

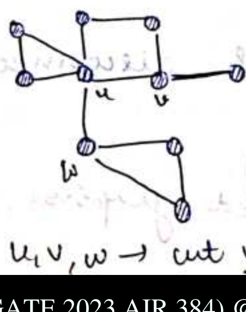
Cut vertex (Articulation point).

A vertex v of a graph G is called cut vertex of G if ~~d~~ ~~no~~ on removal of v from G, the no. of components increases.

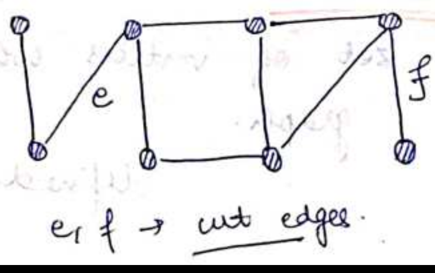
i.e. $k(G-v) > k(G)$ $\qquad$ $k(G) \rightarrow$ no. of components of G

Cut edge (Bridges)

An edge e of a graph is said to be a cut edge if $k(G-e) > k(G)$



$u, v, w \rightarrow$ cut vertices.



$e, f \rightarrow$ cut edges.

Theorem If e is the bridge incident on vertex v, then v is cut vertex of G and only if $\deg(v) \geq 2$.

$$\underbrace{k(G)-1}_{\text{when } v \text{ is isolated vertex}} \leq \underbrace{k(G-v)}_{\text{no. of components in graph after removing a vertex}} \leq \underbrace{n-1}_{\text{star graph}}$$ } Graph with isolated vertex $\delta \geq 0$

$$k(G) \leq k(G-v) \leq n-1$$ } Graph without isolated vertex $\delta \geq 1$

On deleting an edge,
no. of components = $k(G)$ or $k(G)+1$

Every path b/w 2 vertices u and v in a graph contains edge e $\Rightarrow$ e is a cut edge / bridge.

~~every graph is~~
~~every edge~~

all edges in all graphs are cut edges $\rightarrow$ graph is forest

all edges in connected graph are cut edges $\rightarrow$ graph is tree

Vertex cut

set of vertices whose removal disconnects the graph.

defined for only connected graphs.

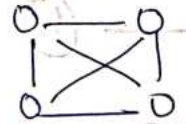
Size of smallest vertex cut = Connectivity number $K(G)$ ← kappa

If cut vertex is present in the graph,

then $\underline{K(G) = 1}$

Full definition of vertex cut -

In a connected graph, a vertex cut is a subset of vertices whose removal either disconnects the graph or a single vertex remains.

K_4 :  → $K = 3$ (single vertex remain)

$\boxed{K_n = n-1}$

$K(\text{disconnected graph}) = 0$

$K(C_n) = 2 \quad n \geq 4$

$\qquad\;\; 2 \quad n = 3$ } 2 for $\forall n$

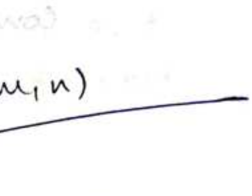
No cut vertex (articulation point)

{b} ; {a} is vertex cut

K connected graph ← $K(G) \geq k$

$K_{m,n} \rightarrow K(K_{m,n}) = \min(m, n)$

$W_n \longrightarrow K(W_n) = 3$



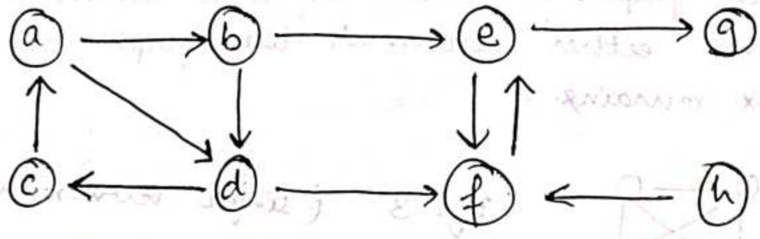
Imp -

The connectivity of a graph is atmost its minimum degree (removing all vertices around vertex with minimum degree gives only 1 vertex disconnected graph)

$K \leq \delta$

Strongly Connected Component

Strongly connected component of a directed graph G is a maximal set of vertices $C \subseteq V$ such that for every pair of vertices u & v, there is a directed path from u to v and a directed path from v to u.

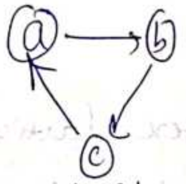


4 strongly connected components

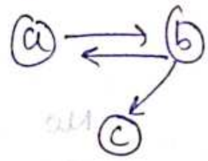
$[a]_R = \{a, b, c, d\} = [b]_R = [c]_R = [d]_R$

$[e]_R = \{e, f\} = [f]_R$

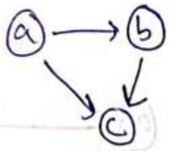
$[g]_R = \{g\}$ $\quad [h]_R = \{h\}$



No. of strongly connected components = 1



#SCC = 2



#SCC = 3

Graph is strongly connected if no of components in the graph = 1

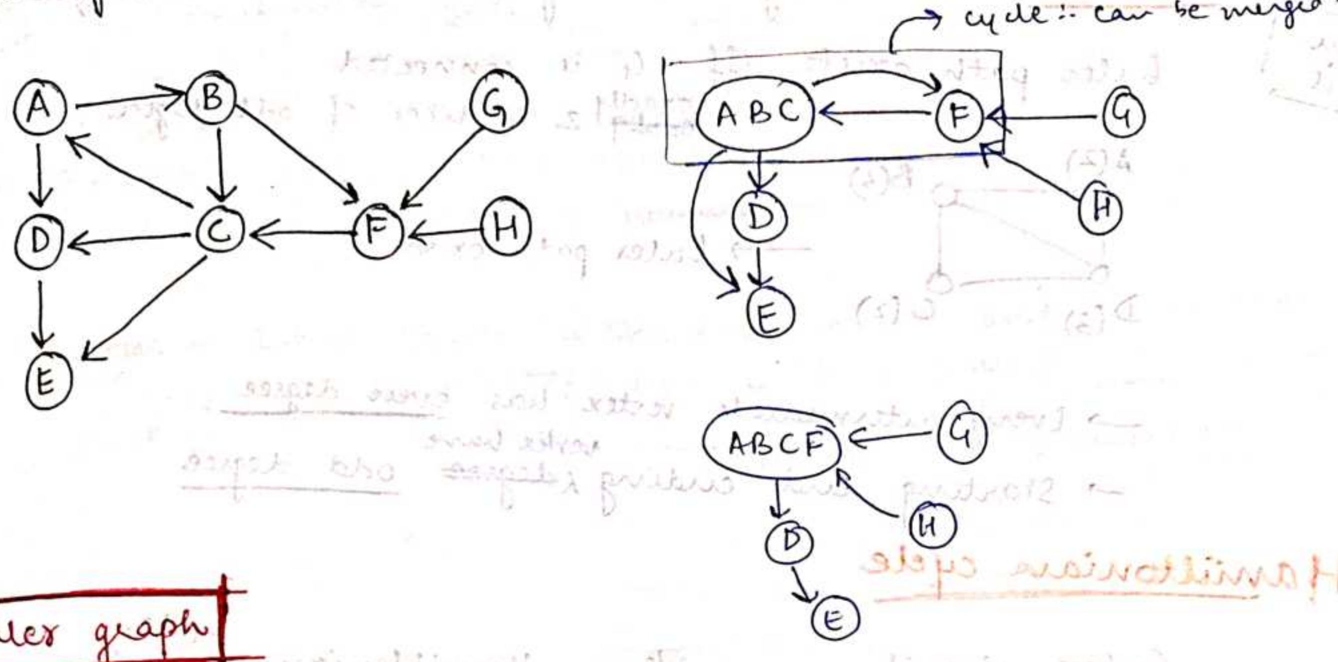
all vertices are reachable from every other vertex

- A directed graph is weakly connected if there is a path b/w every 2 vertices in the underlying undirected graph, which is the undirected graph obtained by ignoring the directions of the edges of the graph.

Every strongly connected graph is also weakly connected.

Priyanshu (GATE 2024 AIR 379, GATE 2023 AIR 384) @ https://x.com/Priyanshu__2109

* Every directed graph is DAG of strongly connected compon



Euler graph

Connected graph in which it is possible to have a walk that crosses each edge exactly once and returns to the starting point.

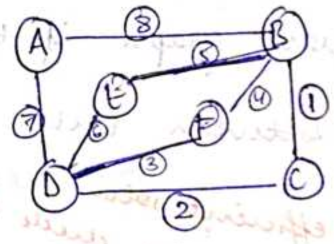
Circuit → closed trail → edge must not repeat.

starting = ending

circuit containing every edge.

Euler graph.
Connected graph with existence of Euler circuit

connected even degree graph.



B C D F B E D A B
Euler circuit

An undirected graph G is Eulerian if and only if it is connected and every node has even degree.

Graph with all vertices of even degree but not euler graph → disconnected graph

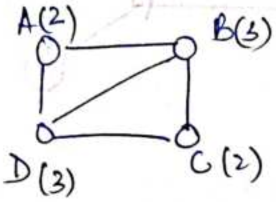
Connected graph	+	even degree vertices	=	Euler graph

isolated vertex are allowed.

Euler Path — starting & ending vertices must ~~can~~ be different.

open trail

Euler path exists iff G is connected
exactly ~~only~~ 2 vertices of odd degree



⟶ Euler path exists.

→ Every intermediate vertex has even degree
→ Starting and ending vertex have ~~degree~~ odd degree

Hamiltonian cycle

Euler circuit	v/s	Hamiltonian cycle ~~circuit~~
visit every edge exactly once and return to the starting vertex		visit every vertex exactly once and return to the starting point

G is Hamiltonian graph if there exists hamiltonian cycle

* No relationship between Euler circuit and hamiltonian cycle.

↳ no efficient soln
↳ manually check करना पड़ेगा

* Clay Institute will give $1,000,000 to the person who will give polynomial time solution

Ready, Priyanshu?

Hamiltonian cycle ~~circuit~~ exists ⟶ Hamiltonian path also exists

Imp.
- Every Hamiltonian graph must be connected.
- No tree is hamiltonian (coz acyclic)
- For each $n \geq 3$, C_n is hamiltonian
- For each $n \geq 3$, K_n is hamiltonian
- for each $n \geq 2$, $K_{n,n}$ is hamiltonian

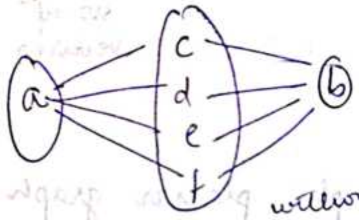
Dirac theorem – Every graph with $n \geq 3$ vertices and minimum degree at least $n/2$ has a hamiltonian cycle

$\delta(G) \geq n/2 \longrightarrow$ hamiltonian graph exists.

Planar graph

A graph is called a planar graph if there is some way to draw it in a 2D plane without any of the edges crossing.

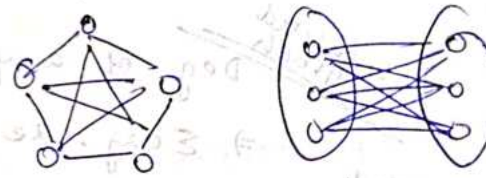
every $K_{2,n}$ is planar



without edge crossing.

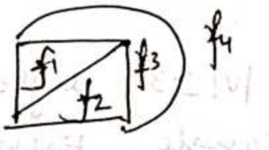
K_5 and $K_{3,3}$ are non planar

$\hookrightarrow$ most common examples.



K_5 $K_{3,3}$

Planar representation divides the plane into ~~regions~~ faces



K_4

- Any tree $\rightarrow$ always planar with 1 face.
- forest $\rightarrow$ always planar with 1 face

$\rightarrow$ All planar ~~gr~~ representations have same number of faces.

Degree of face –
Number of sides of edges the face touches is called degree of face.

For any planar graph,

$$\sum_{\forall i} Deg(f_i) = \text{Degree sum of faces} = 2|E|$$

even no. of faces have odd degree

Total degree of faces = Total degree of vertices $= 2|E|$

→ Degree of face can be zero (edgeless graph)
→ Degree of face cannot be 1.
→ ⓐ—ⓑ K_2 is only this graph has face with degree 2

Euler's formula—

No. of faces $(F) = E - V + C + 1$

very important for GATE

or $\boxed{V + F = E + C + 1}$

V: no. of vertices, F: no. of faces, E: no. of edges, C: no. of components

⊙ In a simple planar graph with ≥ 3 vertices, degree of every face is $>= 3$.

connected graphs

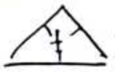
Deg of any face ≥ 3

$\Rightarrow \sum deg = 2e$ & $3 \cdot F \leq \sum deg$

connected planar graph → $\Rightarrow 3F \leq 2e \Rightarrow 3(e - v + 1 + 1) \leq 2e \Rightarrow 3e - 3v + 6 \leq 2e$

$\Rightarrow \boxed{e \leq 3v - 6}$

⊙ In any connected planar graph with $|V| \geq 3$, degree of every face ≥ 3 and if there is no triangle, then degree of every face ≥ 4.

 → only Δ leads to degree 3.

Connected No triangle in simple planar graph.

$deg(f) \geq 4$

$\Rightarrow \sum deg(f) = 2e \Rightarrow 4F \leq 2e$

$\Rightarrow 4(e - v + 2) \leq 2e \Rightarrow 4e - 4v + 8 \leq 2e \Rightarrow \boxed{e \leq 2v - 4}$

- Every planer graph is 4 colorable.
  ie. $\chi(G) \leq 4$.

  ↳ Four colorable theorem has not been proved by any human!
  It was proved by COMPUTER

Adjacency Matrix

If M is the adjacency matrix of a graph.

$M[i][j] \longrightarrow$ denotes no. of walks from vertex i to j of length 1. $i — j$

$M^2[i][j] \longrightarrow$ no. of walks of length 2 from vertex i to j

$i — \square — j$

⋮

$M^k \longrightarrow$ walks of length exactly k b/w every 2 vertices.

Applications

↳ diagonal entries in M^2 give the degree of that vertex in simple undirected graph.

$$\text{Trace}(M^2) = \Sigma \deg(v) = 2e$$

If self loops are present, then,

$$\text{Trace}(M) + \text{Trace}(\underset{}{M^2}) = 2e$$

self loop counted once

$$\Rightarrow \boxed{\text{Trace}(M + M^2) = 2e}$$

after putting all self loops

$(M + I_n)^{(n-1)}$ ← all non zero entries ⇒ graph is connected

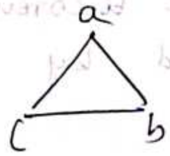
$(n-1) \longrightarrow n$ for directed graph

$(M^0 + M^1 + M^2 + \dots M^{n-1})$ ← all non zero entries ⇔ graph is connected

$+M^n$ for directed graph

Number of three length cycles in undirected simple graph

one triangle contribution in trace(M^3) = 6



a — b — c — a
a — c — b — a
b — a — c — b
b — c — a — b
c → a — b — c
c — b — a — c

$\therefore$ No. of triangles = $\frac{1}{6}$ trace(M^3)

trace (M^3) = 6 (No of triangles)

In directed simple graph,

trace(A^3) = ~~$\frac{1}{3}$ No. of tb~~ 3 (No. of triangles)

Syllabus
- magma
- semigroup
- monoid
- group
- quasi group

Algebraic structure (AS)
↳ defining operations on base set
(Base set, $*$, $\#$, - - - -)

Algebraic structure with single binary operation → GATE syllabus

1. Closure Property

Applying operation on any 2 elements of a set, the result also belongs to the set.

अगर अंदर की बात अंदर ही रहे तो closure property satisfy हो जाएगी।

$(S, \#)$ AS

S

$\forall_{a,b}\ a\#b = c$

$a\#b = c$

Set S is closed under # operation

↳ $(S, \#)$ follows closure property under # operation

$(\{0\}, +) \rightarrow$ closed

$(\{1\}, +) \rightarrow$ not closed

* Binary operation ⟺ closure property satisfied
$(G, *)$ $*$ is binary operation iff $\forall_{a,b} \in G;\ a * b \in G$

2. Associative property

$(S, \#)$ is associative iff
$$(a\#b)\#c = a\#(b\#c)$$

3. Identity Property

Identity element does not affect operation

$(S, \#)$ $a\#e = a$ & $e\#a = a$

e is fixed for all elements $e \in$ Base set

$(N, +) \longrightarrow$ No identity element

$(W, +) \longrightarrow e = 0$.

$(Z, -) \longrightarrow$ no identity element.

rational $(Q, \times) \longrightarrow e = 1$

4. Inverse property

$(S,\#)$ satisfies inverse property iff $\forall_{a\in S}$, $\exists b \in S$ such that $a\#b = e$ and $b\#a = e$

$\underline{b = a^{-1} \text{ and } a = b^{-1}}$

① an element can have multiple inverses.

inverse of element is not unique.

$(R, \times)$ does not satisfy inverse property coz 0^{-1} doesnt exist.

$(N, +)$ " " " " " coz e doesnt exist

$\boxed{e^{-1} = e}$

5. Commutative property

$(S,\#)$ satisfies commutative property iff

$\forall_{a,b\in S}$ $a\#b = b\#a$

Classification of algebraic structures based on properties —

Abelian = commutative

Magma (groupoid) → closure

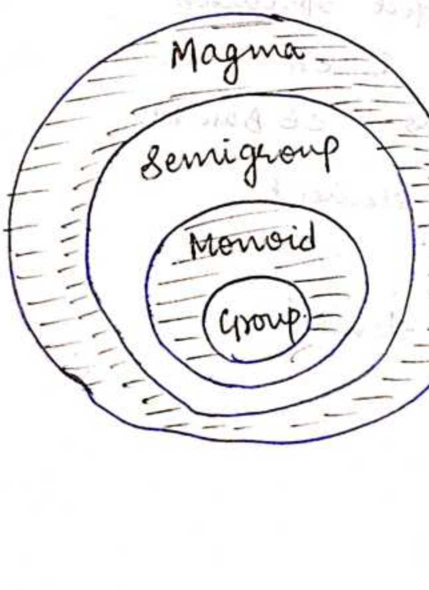
Semigroup → Closure + Associative

Monoid → Closure + Associative + Identity

Group → Closure + Associative + Identity + Inverse.

Abelian group/monoid/semigroup/magma → group/monid/semigroup/mangma + commutative property

∴ Abe



★ Order of any algebraic structure = Cardinality of Base set

★ No. of binary operations $*$ on a set of size n $= n^{n^2}$ [No of functions from $S \times S \to S$

No. of commutative binary operations $= n^n \cdot n^{\frac{n^2-n}{2}}$

$= n^{n+\frac{n^2-n}{2}} = n^{\frac{n^2+n}{2}}$

Properties of ~~mono~~ monoid.

↦ Identity element is unique (true for all algebraic structures)

↦ left and right cancellation property does not hold.

do not hold on monoid
$a \# b = a \# c \rightarrow b = c$ (left cancellation)
$b \# a = c \# a \rightarrow b = c$ (right cancellation)

$(\underbrace{\{1, \omega, \omega^2\}}_{\text{cube roots of unity}}, \times) \rightarrow$ Abelian group

$1+\omega+\omega^2=0$
$\omega^3 = 1$

closure ✓ $\omega^2 \times \omega^2 = \omega^4 = \omega \in S$
Asso ✓ coz multiplication
Identity ele $\rightarrow 1$
Inverse ✓ $1^{-1} = 1$
$\omega^{-1} = \omega^2$
$(\omega^2)^{-1} = \omega$.
Commutative ✓

∴ abelian group

Note → for all $n \geq 1$, n^{th} roots of unity are abelian groups under multiplication.

$(\{1, -1, i, -i\}, \times)$
↳ 4^{th} root of unity.

closure $(-i)\times(-i) = -1$ ✓
$i \times i = 1$

Associative ✓ coz multiplication

Identity ele $= 1$

Inverse → $1^{-1} = 1$
$-1^{-1} = -1$
$i^{-1} = -i$
$-i^{-1} = i$

Commutative ✓

∴ abelian group.

Addition modulo n is abelian group.

$\{Z_n, \oplus_n\}$ — set of 0 to n-1

$a \oplus_n b = \cancel{a \bmod b}\ (a+b) \bmod n$

Closure property ✓

Associative? ✓ → $a \oplus_n (b \oplus_n c)$ $(a \oplus_n b) \oplus_n c$

Identity ele = 0 $a \oplus_n (b+c) \bmod n$ $((a+b) \bmod n) \oplus_n c$

Inverse → n-a $[a + ((b+c) \bmod n)] \bmod n$ $(((a+b) \bmod n)+c) \bmod n$

commute → ✓ $(a+b+c) \bmod n = (a+b+c) \bmod n$

$a^{-1} = n-a \quad \forall a \neq 0$

$a^{-1} = 0 \quad \text{for } a=0.$

Ⓞ In a group, every element has unique inverse ✓

let a be 2 inverses b and c

$\therefore a^{-1} = b \;\&\; a^{-1} = c \Rightarrow a*b = e$ and $a*c = e$

$\Rightarrow a*b = a*c \Rightarrow a^{-1}*(a*b) = a^{-1}*(a*c)$

$\Rightarrow (a^{-1}*a)*b = (a^{-1}*a)*c \Rightarrow e*b = e*c \Rightarrow \underline{b=c}$

$\therefore$ unique inverse.

Proof that group has left & right cancellation property.

$a*b = b*c$ does not imply $a=c$

common sense है भाई!

In a group,

$(a*b)^{-1} = b^{-1}a^{-1}$

Proof— $(a*b)*(a*b)^{-1} = e$

$\cancel{a^{-1}*a}*b*(a*b)^{-1} = a^{-1}$

$e*b*(a*b)^{-1} = a^{-1}$

$b*(a*b)^{-1} = a^{-1}$

$b^{-1}*b*(a*b)^{-1} = b^{-1}a^{-1}$

$e*(a*b)^{-1} = b^{-1}a^{-1}$

$(a*b)^{-1} = b^{-1}\underline{a^{-1}}$

Property of numbers that allows us to remove parenthesis from expressions → **Associativity**.

Ques $a\#b\#c$

For which of the following $a\#b\#c$ makes sense (ie unambiguous)

1. Groupoid
2. Semigroup ✓
3. Monoid ✓
4. Group ✓

} bcoz associative property holds.

⓪ If operation # is ~~unambiguous~~, ~~then~~ associative, then, $a\#b\#c\#d$ is unambiguous expression.

Cayley Table | Multiplication Table | Operations Table

$(\{a, b, c\}, \#)$

	a	b	c
a	$a\#a$	$a\#b$	$a\#c$
b	$b\#a$	$b\#b$	$b\#c$
c	$c\#a$	$c\#b$	$c\#c$

→ multiplication operation नहीं है पर भी multiplication table नाम है HaHaHaHa

a, b, x are elements of a group.

Soln of $ax = b$ → $x = a^{-1}b$

Soln of $xa = b$ → $x = ba^{-1}$

	....	y	z
a		b	b

← not possible becoz soln is always unique

If it were possible,

$a*y = b \quad a*z = b$

$\Rightarrow a*y = a*z$

$\rightarrow y = z$ (left cancellation)

∴ no 2 entries in same row can be same

⓪ In a group,

if $a*b = e$ then, $b*a = e$

coz if $a = b^{-1}$ then $b = a^{-1}$ unique

⓪ If $a*a = a$, then, $a = e$

Properties of Cayley Table for groups –

- Every element $g \in G$ appears exactly once in each row and each column.
- Every row/column is permutation of all elements i.e every element appears once.
- Row and column of identity element is same as header.

(e a b c) $a^{-1}=a$ $b^{-1}=b$ $c^{-1}=c$
(e a b c) $a^{-1}=a$ $b^{-1}=c$ $c^{-1}=b$

GATE 2007

No. of isomorphic groups of order 4 = 2 (both abelian)
No. of isomorphic groups of order 3 = 1
No. of isomorphic groups of order 2 = 1
No. of isomorphic groups of order 1 = 1.

For $\forall a \in G$, $(G, *)$ is group.

$a^0 = e$ $\quad a^2 = a * a \quad a^{-n} = (a^{-1})^n$

$a^1 = a \quad a^{-1} = a^{-1} \quad a^{-2} = (a^{-1})^2 = (a^{-1}) * (a^{-1})$

'a' is an element of group.

$$\left.\begin{aligned} a^n \cdot a^{-n} &= e \\ a^m \cdot a^n &= a^{m+n} \\ (a^m)^n &= a^{mn} \end{aligned}\right\} \text{becoz of associativity.}$$

Subgroups

Subset is a group under same operation

$(H, *)$ is subgroup of $(G, *)$ iff

$(G, *)$ is a group, $H \subseteq G$ and $(H, *)$ is a group.

$G = \{1, -1, i, -i\}$ under multiplication

Subgroups of G

① $(\{1\}, \times)$

② $(\{1, -1\}, \times)$

③ $(\{1, -1, i, -i\}, \times)$

Not subgroups

$(\{-1\}, \times) \rightarrow$ not closed

$(\{i, -1\}, \times) \xrightarrow{-i}$

$(\{1, i, -1\}, \times) \xrightarrow{-i}$

* $(Z, +)$ is a group.

$\forall_{n \in Z}$, $(nZ, +)$ is a subgroup of $(Z, +)$

multiples of n.

- Identity element of G should be present in H.
- If $a_1 \in H$, then $a_1^{-1}, a_1^2, a_1^3 \ldots \in H$

$G = \{e, a, b, c \ldots\}$

Assume, $a^{-1} = d$.

Subgroup of G that contains 'a'

$H = \{e, a, a^2, a^3, a^4 \ldots, d, d^2, d^3 \ldots\}$

$(G, *)$ is a group let $a \in G$

$\langle a \rangle$ = smallest subgroup generated by a.

$\langle a \rangle = \{a^n \mid n \in Z\}$ $\quad a^0 = e \quad a^{-1} \ldots$
$a^1 \ldots \quad a^{-2} \ldots$
$a^2 \ldots$

Generator:- Group $(G, *)$ $g \in G$ is generator of G iff $\langle g \rangle = G$.

1. $(Z, +) \rightarrow$ Group G

Generator of G = 1.

2. $(\{0,1,2,3\}, \oplus_4) \rightarrow$ Group G

Generator of G = 1, 3

$\boxed{\langle a \rangle = \langle a^{-1} \rangle}$

Multiplication modulo n

$(Z_4 = \{0,1,2,3\}, \otimes_4)$

- Closure ✓
- Associative ✓
- Identity element = 1 ✓
- 0^{-1} = DNE 2^{-1} = DNE
  $1^{-1} = 1$ $3^{-1} = 3$
- commutative ✓

∴ Multiplication modulo n is abelian monoid.

NOT A GROUP.

$(U_4 = \{1,3\}, \otimes_4)$

Only those elements from Z_4 are taken which have inverse.

abelian group.

$U_n = \{m \in Z_n \mid m$ is coprime with n OR GCD(m,n)=1$\}$

$U_n \rightarrow$ contains only those elements which are coprime to n.

$(Z_8 = \{0,1,2,3,4,5,6,7\}, \otimes_8)$

$U_8 = \{1, 3, 5, 7\}, \otimes_8$

abelian group

Ques- Let G be a group which can be generated by an element 'a' which is inverse of itself. Then, what is the order of G?

Given :- $a = a^{-1} \Rightarrow a^2 = e \quad a^3 = a \cdot a^2 = a \cdot e = e$

$\therefore$ The set is $\{a, e\}$ or $\{e\}$

$\therefore$ Cardinality / Order of G. = 1 or 2

→ since order is different, 2 non isomorphic groups are possible.

each is abelian.

$$\boxed{\text{Order of an element / Period length / Period} = \text{~~Number~~ Order of the subgroup generated by that element.}}$$

→ least positive integer n such that $a^n = e$.

Cyclic group - Group that can be generated by a single element

$(\{1, 3, 5, 7\}, \otimes_8) \longrightarrow$ not cyclic group.

$(\{1, 2, 3, 4\}, \otimes_5) \longrightarrow$ cyclic group

2 is the generator

$(Z, +)$ is cyclic generated by 1

$(Z_n, \oplus_n)$ is cyclic generated by 1

when $|G| = n$, then for finite group G
$\forall a,\ a^n = e$

- Every cyclic group is abelian.

- Order of smallest group that is not cyclic = 4.

Every group of order 1, 2, 3 is cyclic.

- Order of identity element = 1.

- For any finite group,
$|a| \le |G|$ $\quad a \in G$

- $\langle g \rangle$ is finite cyclic group of order n
Then $\{1, g, g^2 \cdots g^{n-1}\}$ are distinct.

Imp Theorem
- Subgroup of cyclic group is cyclic.

Theorem —
Any cyclic group is isomorphic to either $(Z, +)$ or $(Z_n, \oplus_n)$

→ infinite cyclic group isomorphic to $(Z, +)$
→ finite cyclic group isomorphic to $(Z_n, \oplus_n)$

Lagrange's Theorem -

Order of subgroup divides order of group.

- If $|G|$ = prime, then, G is cyclic
- $|G|$ = prime $\Rightarrow$ G is abelian
  $|G| = (\text{prime})^2 \Rightarrow$ G is abelian
- $|G|$ = prime $\Rightarrow$ G is cyclic
  $|G| = (\text{prime})^2 \not\Rightarrow$ G is cyclic

$U_8 \rightarrow$ not cyclic group

$\{1, 3, 5, 7\}$

हर non cyclic grp के example में चिपका दो

Abelian group

① If every element of a group is its own inverse, then, the group is abelian. or $a^2=e$

For every group,

vice versa not true

$$(ab)^{-1} = b^{-1}a^{-1} \Rightarrow (ba)^{-1} = b^{-1}a^{-1}$$

$$\Rightarrow a^{-1}b^{-1} = b^{-1}a^{-1} \Rightarrow \underline{ab=ba}$$

② If G is a group such that $(ab)^2 = a^2b^2$ for all $a,b \in G$ then show that G is abelian

$$(ab)^2 = a^2b^2 \Rightarrow \cancel{a}ba\cancel{b} = \cancel{a}ab\cancel{b}$$

$$\Rightarrow ba = ab \qquad \therefore \text{commutative}$$

$\therefore$ abelian group.

③ Every subgroup of an abelian group have to be abelian

④ $xy = zx$ implies $y=z$

Then, G is abelian

vice versa also true

⑤ $xyz = ayz$ implies $xz = az$

Then, G is abelian

vice versa also true

Subgroup

S is subgroup of G iff S is non empty and whenever $a, b \in S$ then $ab^{-1} \in S$.

If S is subset of finite group G, then, S is a subgroup of G if and only if S is non empty and closure property holds.

GRAPH THEORY

Degree of vertex = No. of edges incident on the vertex

Pseudograph – graph in which parallel edges and self loop are allowed.

Multigraph – undirected graph in which parallel edges are allowed, no self loops.

Self loop contributes to <u>twice</u> the degree of vertex in pseudograph.

$\sum$ Degree = 2 * size of graph
= 2 * no. of edges.

Order of graph – Cardinality of vertex set

Size of graph – Cardinality of edge set.

$\delta(G) \rightarrow$ Minimum degree in G

$\Delta(G) \rightarrow$ Maximum degree in G

$$\delta(G) \leq \text{Avg. degree} \leq \Delta(G)$$

$$n\,\delta(G) \leq n * \text{Avg. degree} \leq n\,\Delta(G)$$

$$n \cdot \delta(G) \leq \text{Total degree} \leq n\,\Delta(G)$$

$$\frac{n\,\delta(G)}{2} \leq \text{No. of edges} \leq \frac{n\,\Delta(G)}{2}$$

Handshaking Theorem

$G(V, E)$ any undirected graph with m edges. Then,

$$2m = \sum_{v \in V} \deg(v)$$

No. of odd degree vertices is always even.

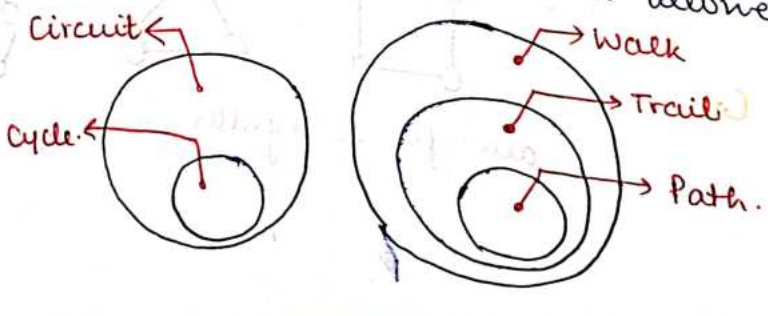
Walk – vertex repetition & edge repetition allowed.

Trail – vertex repetition is allowed & edge repetition not allowed

Path – niether vertex repetition nor edge repetition is allowed.

Closed trail is called <u>circuit</u>.

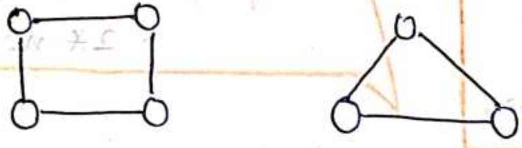
Closed path is called <u>cycle</u>



$$\boxed{\text{Diameter of a connected graph} = \max_{\forall a,b} \{ \text{dist.}(a,b) \}}$$

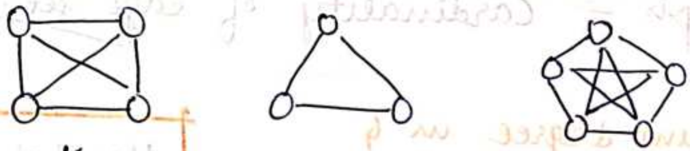
Regular graph – every vertex has ~~even~~ ~~d~~ same degree.

n-reg.



Complete graph – edge exists b/w every pair of vertices.

K_n



K_4 K_3 K_5

Empty graph – graph without edge

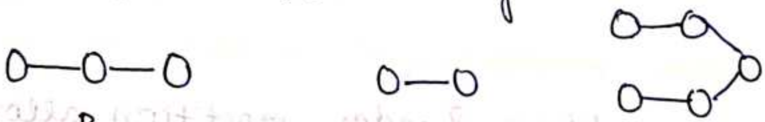
Degree of each vertex = 0.

E_n

E_1 E_2 E_3

Path graph – ~~look~~ vertices arranged in straight line.

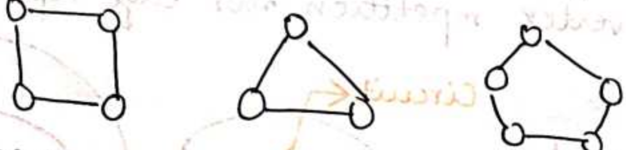
P_n



P_3 P_2 P_5

Cycle graph – ≥ 3 vertices, vertices form a cycle.

C_n

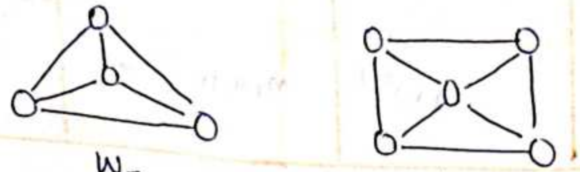


always 2 regular.

Wheel graph — add a node to C_n and connect it with every other node.

W_n contains $(n+1)$ nodes.

W_n



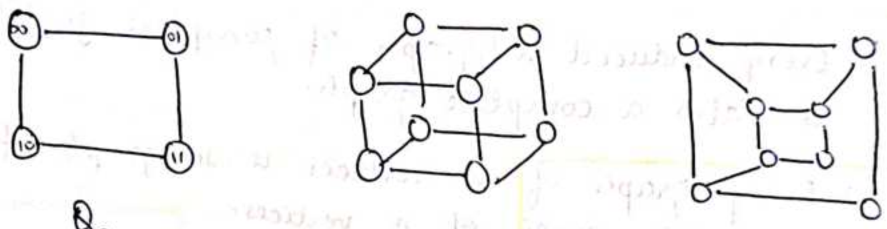
W_3 W_4.

Hypercube graph

edge exists b/w u and v iff u and v differ by only one bit

Q_n

$Q_n \rightarrow n$ bit seq. no. $\rightarrow 2^n$ nodes.



Q_2 Q_3 Q_3. (planar rep.).

Type of graph	No. of vertices	No. of edges.	Degree sequence	Diameter
d reg.	n	$\frac{nd}{2}$	$\underbrace{d, d, \ldots d}_{n \text{ times}}$	infinite (if disconnected) finite (if connected)
K_n	n	$\frac{n(n-1)}{2}$	$\underbrace{(n-1)(n-1) \ldots (n-1)}_{n \text{ times}}$. $(n-1)$ regular	1 (always connected)
P_n	n	$n-1$	$\underbrace{2, 2, \ldots 2}_{n-2 \text{ times}}, 1, 1$ $n \geq 3$	$n-1$
C_n	n	n	$\underbrace{2, 2, 2 \ldots 2}_{n \text{ times}}$	$\frac{n}{2}$ if n is even $\frac{n-1}{2}$ if n is odd

	No. of vertices	No. edges	Degree seq.	Diameter	
W_n	$n+1$	$2n$	$n, \underbrace{3, 3, \ldots 3}_{n \text{ times}}$	2	only W_3 is regular
Q_n	2^n	$n \times 2^{n-1}$	$n, n, n \ldots n$	n	

Subgraph –
- → Spanning subgraph
  vertex deletion is not allowed.
- → Induced subgraph
  edge deletion is not allowed.

↦ Every induced subgraph of complete graph is also a complete graph.

↦ Every graph of n vertices is subgraph of complete graph of n vertices.

↦ There is exactly one subgraph of G (G itself) which is both induced and spanning subgraph.

> n vertices simple graph
>
> Max. no of edges $= {}^nC_2 = \frac{n(n-1)}{2}$
>
> No. of simple graph $= 2^{\frac{n(n-1)}{2}}$

Complement of a graph $= \bar{G} =$

Complete graph with n vertices − Edges in G.

$G(V, E)$ given graph

$\bar{G}(V, \bar{E})$ complement of G.

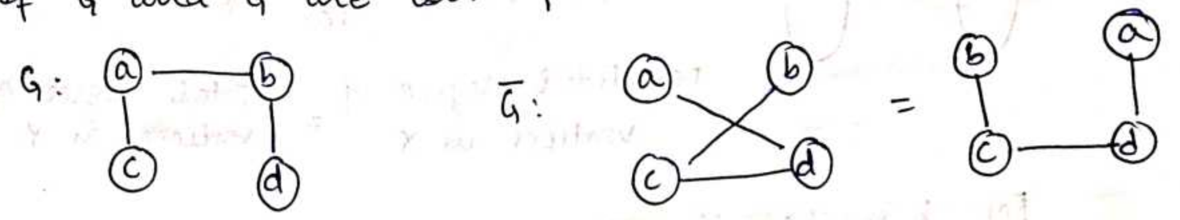
$\bar{E} = E(K_n) - E.$

$|\bar{E}| = {}^nC_2 - |E|$

i.e. $|\bar{E}| + |E| = n.$

Self Complementary graph –

If G and $\overline{G}$ are isomorphic, then, G is self complementary



⊙ Complement of disconnected graph is always connected.

⊙ For every simple graph G,
either G or $\overline{G}$ is definitely connected.

Imp.

If a graph has exactly 2 vertices of odd degree, they are connected by a path.

In any graph there is a simple path from any vertex of odd degree to some other vertex of odd degree.

Bipartite graph

$G(V,E)$ is bipartite iff $\exists$ a bijection X, Y of V such that

1. $X \cap Y = \phi$
2. $\forall a,b \in X,\ (a,b) \notin E(G)$
   $\forall a,b \in Y,\ (a,b) \notin E(G)$
3. $X \cup Y = V$
4. X, Y can be empty

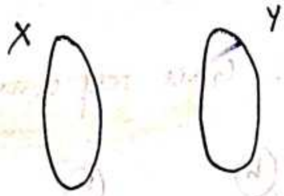
★ A graph is bipartite iff it does not contain odd length cycle.

Imp.

↦ Every P_n is bipartite graph

↦ C_n is bipartite only when n is even.

★ ↦ ~~Kn~~ only K_1 and K_2 are bipartite. $K_{n\geq 3}$ is not bipartite.



$$\sum_{v \in X} deg(v) = \sum_{v \in Y} deg(v)$$

i.e. Total degree of vertices in X = Total degree of vertices in Y —①

For k regular bipartite graph,
all vertices ~~of~~ have degree k.

$\Rightarrow k|A| = k|B|$ [From ①]

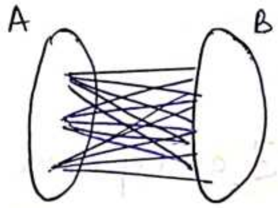
$\Rightarrow |A| = |B|$

* i.e. equal elements in A and B if bipartite graph is regular.

Imp.
Every subgraph of bipartite graph is bipartite.

Interesting result
Important
मस्त। वाह! वाह!
ये बढ़िया था गुरु !!

Complete bipartite graph.
No. of vertices in graph = n



$|A| = m$
$|B| = n-m$

No. of edges, $e = m(n-m)$
$= mn - m^2$

* **Theorem**
If a graph on n vertices has more than $n^2/4$ edges, then, graph is not complete bipartite.

— Mahatma Gandhi

For max no. of edges,

$$\frac{de}{dm} = n - 2m = 0 \Rightarrow n = 2m$$

$$\frac{d^2e}{dm^2} = -2 < 0 \therefore n = 2m \text{ is point of maxima.}$$

$$\Rightarrow m = n/2$$

$\therefore$ Maximum no. of edges $= mn - m^2$

$= \cancel{m(2m) - m^2 = 2m^2 - m^2}$

$$= \frac{n}{2} \cdot n - \left(\frac{n}{2}\right)^2 = \frac{n^2}{2} - \frac{n^2}{4} = \frac{n^2}{4}.$$

Tree

- connected, undirected, acyclic graph.
- n vertices
  - ↦ (n-1) edges
  - ↦ connected, acyclic
- Connected graph on n vertices & (n-1) edges is tree.
- Tree is <u>maximally acyclic</u>

Height of a node in rooted tree	=	No. of ~~nodes~~ edges from that node to the farthest leaf.
Depth of a node in rooted tree	=	No. of edges from the root to that node.

Tree + 1 edge → graph with exactly one cycle.

<u>Imp.</u>

Minimum number of edges in a graph with n vertices to guarantee that it is connected $= {}^{n-1}C_2 + 1$

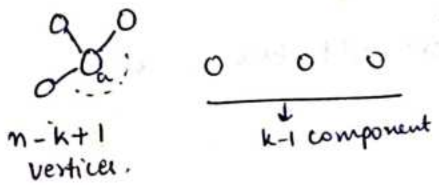
<u>Imp</u>

Every simple graph with more than $\frac{(n-1)(n-2)}{2}$ vertices is connected.

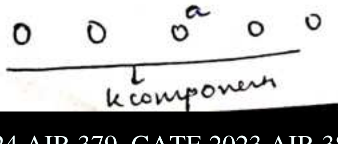
Good question.

Let G be an arbitrary graph with n vertices and k components. If a vertex is removed from G, the no. of components in the resultant graph must lie between

A) k & n B) k-1 & k+1 C) ✓ k-1 & n-1 D) k+1 & n-k



On deleting a,
no. of components $= n-\cancel{k}+\cancel{k}-1$
$= n-1$



On deleting a,
no. of components = k-1.

Clique :- clique of a ~~subgraph~~ graph is the subgraph which is complete.

Size of maximum clique = Clique no = $\omega(G)$

Independent set :-

subgraph with vertices not connected to each other.

Size of maximum independent set = Independence number = $\alpha(G)$.

Vertex cover

set of vertices that cover all the edges.

Size of minimum vertex cover = $\beta(G)$

Edge cover

~~edge~~ set of edges covering all the vertices.

Size of minimum edge cover = $\beta'(G)$

Matching

- set of edges that are independent (non adjacent)
- A set of pairwise non adjacent edges in a graph is called matching.

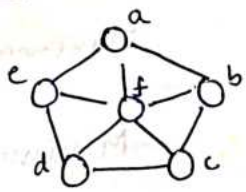
Maximum no. of edges in a matching of graph = $\alpha'(G) = \mu(G)$

→ Matching is perfect if it covers all the vertices of graph.

Imp.

- Clique in G $\rightleftarrows$ Independent set in $\overline{G}$

- Maximum degree vertex need not be included in minimum vertex cover.



Min. VC = {a, d}

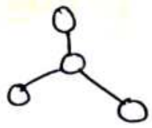
- Edge cover exists only if no isolated vertex in the graph.

- If $S \subseteq V$ is the vertex cover, then, $\overline{S}$ or $V-S$ is independent set & vice versa.

$\alpha \rightarrow$ Size of maximum independent set.
$\beta \rightarrow$ Size of minimum vertex cover.

$$\boxed{\alpha + \beta = n}$$

- If perfect matching exists, then, no. of vertices in the graph are even. (vice versa not true)



{odd no. of vertices $\Rightarrow$ no perfect matching exists}

- For every graph,

$$\boxed{\mu(G) = \underbrace{\alpha'(G)}_{\text{size of largest matching}} \leq \left\lfloor \frac{n}{2} \right\rfloor}$$

1. <u>Complete graph (K_n)</u>

Clique number $= \omega(G) = n$

Maximal independent set $= \alpha(G) = 1$

Minimum vertex cover, $\beta(G) = n-1$

Minimum edge cover

Matching number of $G = \lfloor \frac{n}{2} \rfloor$

2. <u>Cycle graph (C_n)</u>

Clique number $= \begin{cases} 3 & \text{if } n=3 \\ 2 & \text{if } n>3 \end{cases}$

Maximum independent set, $\alpha(G) = \lfloor \frac{n}{2} \rfloor$

Minimum vertex cover, $\beta(G) = \lceil \frac{n}{2} \rceil$

Minimum edge cover, $\beta'(G) = \lceil \frac{n}{2} \rceil$

Matching number $= \lfloor \frac{n}{2} \rfloor$

3. Path graph (P_n)

Clique number = $\omega(G) = \begin{cases} 1 & \text{if } n=1 \\ 2 & \text{if } n \geq 2 \end{cases}$

Maximum independent set, $\alpha(G) = \left\lceil \frac{n}{2} \right\rceil$

Minimum vertex cover, $\beta(G) = \left\lfloor \frac{n}{2} \right\rfloor$

Minimum edge cover, $\beta'(G) = \left\lceil \frac{n}{2} \right\rceil$

Matching number, $\alpha'(G) = \left\lfloor \frac{n}{2} \right\rfloor$

4. Complete bipartite graph ($K_{m,n}$)

Clique number = $\omega(G) = 2$

Maximum independent set, $\alpha(G) = \max(m, n)$

Minimum vertex cover, $\beta(G) = \min(m, n)$

Minimum edge cover, $\beta'(G) = \max(m, n)$

Matching number, $\alpha'(G) = \min(m, n)$

Imp.

- If there exists a matching of size k, then every vertex cover has size $\geq k$.

$$\boxed{\alpha'(G) = \mu \leq \beta(G)}$$

- For bipartite graph, $\alpha'(G) = \beta(G)$.

- $\alpha'(G) \leq \beta(G) \leq 2\alpha'(G)$

$C_4 \rightarrow \alpha'(G) = \beta(G)$
$K_{odd} \rightarrow 2\alpha'(G) = \beta(G)$

- For any graph, size of maximum independent set is less than or equal to edge cover.

$$\boxed{\alpha(G) \leq \beta'(G)}$$

- For any graph, size of maximum matching is less than or equal to vertex cover.

$$\boxed{\alpha'(G) \leq \beta(G)}$$

1. $\alpha(G) + \beta(G) = n$
2. $\alpha'(G) + \beta'(G) = n$
3. $\alpha'(G) \leq \beta(G) \leq 2\alpha'(G)$
4. $\beta'(G) \geq \lceil \frac{n}{2} \rceil$
5. $\alpha'(G) \leq \lfloor \frac{n}{2} \rfloor$
6. $\alpha \leq \beta'(G)$
7. $\alpha'(G) \leq \beta(G)$
8. For bipartite graph, $\mu = \beta$.

Important conclusions.
याद ही कर लो

Graph coloring -

assignment of labels or colors to vertices of a graph.

Chromatic number

Minimum number of colors required to color all the vertices of the graph such that no 2 adjacent vertices share the same color.

Denoted by $\chi(G)$

1. $\chi(K_n) = n$
2. $\chi(E_n) = 1$
3. $\chi(P_n) = \begin{cases} 1 & \text{if } n=1 \\ 2 & \text{if } n\geq 2 \end{cases}$
4. $\chi(C_n) = \begin{cases} 3 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even} \end{cases}$ $n - 2\left\lfloor \frac{n}{2} \right\rfloor + 2$
5. $\chi(K_{m,n}) = \begin{cases} 1 & \text{if } m=0 \text{ or } n=0 \\ 2 & \text{otherwise.} \end{cases}$
6. $\chi(Q_n) = \begin{cases} 2 & n \geq 1 \\ 1 & n=0 \end{cases}$
7. $\chi(W_n) = \begin{cases} 4 & \text{if } n \text{ is odd} \\ 3 & \text{if } n \text{ is even.} \end{cases}$

Imp.

- Chromatic number is always greater than or equal to clique number.

$$\boxed{\chi(G) \geq \omega(G)}$$

- For any graph,

$$\boxed{\omega(G) \leq \chi(G) \leq \Delta(G) + 1}$$

- Between every colour class, atleast one edge should be there.

$$\therefore \quad |E(G)| \geq {}^{\chi}C_2$$

i.e. $$\boxed{|E(G)| \geq \frac{\chi(G)(\chi(G)-1)}{2}}$$

$$\boxed{|V(G)| \leq \underbrace{\chi(G)}_{\text{chromatic number}}\underbrace{\alpha(G)}_{\text{Maximum independent set}}}$$

Brooke's Theorem —

$\chi(G) = \Delta(G) + 1$ for only 2 graphs —

↳ odd length cycle graph: C_{odd}

↳ complete graph K_n

Vizeing Theorem —

For edge chromatic number.

- For any simple graph, edge chromatic number can be either $\Delta(G)$ or $\Delta(G)+1$.

$$\Delta(G) \leq \chi'(G) \leq \Delta(G)+1.$$

Bipartite graph $\rightarrow$ $\chi'(G) = \Delta(G)$

- Every graph having $\chi'(G) = \Delta(G)+1$ must have atleast 3 vertices of maximum degree.

$$\left.\begin{array}{l} C_{odd} \\ K_{odd} \\ Regular_{odd} \end{array}\right\} \chi'(G) = \Delta(G)+1.$$

Cut vertex and cut edge

- If e is any ~~edge~~ bridge incident on a vertex v, then, v is cut vertex if and only if $\deg(v) \geq 2$.

$$\boxed{k(G) - 1 \leq k(G-v) \leq n-1}$$

Graph with isolated vertex. $\delta(G) \geq 0$

$$\boxed{k(G) \leq k(G-v) \leq n-1}$$

Graph without isolated vertex $\delta(G) \geq 1$.

- On deleting an edge, no. of components can be either $k(G)$ or $k(G)+1$.

- Every path b/w 2 vertices u and v contains edge e $\Rightarrow$ e is cut edge / bridge.

- All edges in graph are cut edges $\rightarrow$ G is forest.
  All edges in connected graph are cut edges $\rightarrow$ G is tree.

Vertex cut

set of vertices whose removal disconnects the graph.

$\boxed{\text{Smallest vertex cut} = \text{Connectivity number.}}$

$K(G)$

If cut vertex is present in the graph, $K(G) = 1$.

$K_n \longrightarrow K(G) = n-1$

$C_n \longrightarrow K(G) = 2$

$K_{m,n} \longrightarrow K(G) = \min(m,n)$

$W_n \longrightarrow K(G) = 3$

Imp.

Connectivity of a graph is atmost its minimum degree.

$$\boxed{K \leq \delta(G)}$$

Euler graph

connected graph with existance of euler circuit is euler graph.

closed trail

Starting = ending

edge should not repeat.

Imp.

An undirected graph is euler if and only if

- connected
- every node has even degree.

$$\boxed{\text{Connected graph} + \text{Even degree vertices} = \text{Euler graph}}$$

⇒ Euler Path –

starting and ending vertices must be different.

* Euler path exists iff

- G is connected
- exactly 2 vertices have odd degree.

* In euler path

- every intermediate vertex has even degree
- starting and ending vertex have odd degree.

Hamiltonian cycle

- visit every node exactly once and return to the starting point.

Hamiltonian cycle exists $\longrightarrow$ Hamiltonian Path also exists.

- ↦ every hamiltonian graph must be connected.
- ↦ no tree is hamiltonian (coz acyclic)
- ↦ for each

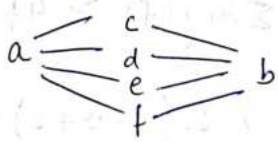
No. of hamiltonian cycles in complete graph, $K_n = \dfrac{(n-1)!}{2}$

No. of hamiltonian cycles in complete bipartite graph $k_{m,n} = \dfrac{n!\,(n-1)!}{2}$

↳ graph is hamiltonian iff $m = n \geq 2$

Planar graph

Every $K_{2,n}$ is planar



k_5 and $k_{3,3}$ are non planar

↳ most common example.

For any planar graph,

$$\boxed{\sum_{\forall i} Deg(f_i) = \text{Degree sum of faces} = 2|E|}$$

- Degree of face can be zero (edge less graph)
- Degree of face cannot be 1.
- only K_2 has degree of face 2.

Euler's formula —

$$V + F = E + K + 1$$

V → no. of vertices, F → no. of faces, E → no. of edges, K → no. of components.

⊙ In a simple graph with ≥ 3 vertices.
Degree of every face ≥ 3

<u>Connected graph</u> —

Degree of any face ≥ 3

$\Sigma deg = 2e$

$3f \leq \Sigma deg \Rightarrow 3f \leq 2e$

$\Rightarrow 3(e - v + 2) \leq 2e \Rightarrow 3e - 3v + 6 \leq 2e$

$\Rightarrow \boxed{e \leq 3v - 6}$

If no 3 length cycle is present,
Degree of every face ≥ 4

$\Sigma deg(f) \geq 4f \Rightarrow 4f \leq 2e \Rightarrow 2f \leq e$

$\Rightarrow 2(e - v + 2) \leq e \Rightarrow 2e - 2v + 4 \leq e$

$\Rightarrow \boxed{e \leq 2v - 4}$

<u>Imp.</u>

4 colorable theorem

Every planar graph is 4 colorable.

i.e. $K(G) \leq 4$.