MA 1101 : Mathematics I

Priyanshu Mahato, `21ms002` February 14, 2022

Solution 1.
Let $X, Y, Z \neq \emptyset$, let $f : X \to Y$ and let $g : Y \to Z$. We have

$$\begin{aligned} g \circ f : X &\to Z, \\ x &\mapsto g(f(x)) \end{aligned}$$

(i) If f and g are injective, for arbitrary $x_1, x_2 \in X$,

$$\begin{aligned} && (g \circ f)(x_1) &= (g \circ f)(x_2) && \\ \Rightarrow && g(f(x_1)) &= g(f(x_2)) && \\ \Rightarrow && f(x_1) &= f(x_2) && \text{(Injectivity of } g) \\ \Rightarrow && x_1 &= x_2 && \text{(Injectivity of } f) \end{aligned}$$

Hence, $g \circ f$ is injective. □

(ii) If g is surjective, it follows that for all $z_i \in Z$, there exists $y_i \in Y$ such that $g(y_i) = z_i$. If f is also surjective, it follows that for all these y_i, there exists $x_i \in X$ such that $f(x_i) = y_1$. Hence, for all $z_i \in Z$, there exists $x_i \in X$ such that $(g \circ f)(x_i) = g(f(x_i)) = z_i$. Therefore, $g \circ f$ is surjective. □

(iii) If f and g are bijective, $g \circ f$ must be injective from (i) and surjective from (ii). Therefore, $g \circ f$ is bijective. □

(iv) If $g \circ f$ is surjective, it follows that for all $z_i \in Z$, there exists $x_i \in X$ such that $g(f(x_i)) = z_i$. Since f is a function, for all these x_i, there must exist $y_i \in Y$ such that $f(x_i) = y_i$. Hence, for all $z_i \in Z$, there exists $y_i \in Y$ such that $g(y_i) = z_i$. Therefore, g is surjective. □

Consider

$$\begin{aligned} f : \{0, 1, 2\} &\to \{0, 1\}, \\ x &\mapsto 0. \\ g : \{0, 1\} &\to \{0\}, \\ x &\mapsto 0. \end{aligned}$$

Clearly, we have $g \circ f : \{0, 1, 2\} \to \{0\}, x \mapsto 0$ is surjective, yet f is not surjective since there is no $x \in \{0, 1, 2\}$ such that $f(x) = 1$.

(v) Let $x_1, x_2 \in X$ such that $x_1 \neq x_2$. We have two cases : $f(x_1) = f(x_2)$ or $f(x_1) \neq f(x_2)$. If $f(x_1) = f(x_2) = y \in Y$, we must have $(g \circ f)(x_1) = g(f(x_1)) = g(f(x_2)) = (g \circ f)(x_2)$. This contradicts the injectivity of $g \circ f$. Hence, we must have $f(x_1) \neq f(x_2)$. Therefore, f is injective. □

Consider

$$\begin{aligned} f : \{0\} &\to \{0, 1\}, \\ x &\mapsto 0. \\ g : \{0, 1\} &\to \{0\}, \\ x &\mapsto 0. \end{aligned}$$

Clearly, we have $g \circ f : \{0\} \to \{0\}, x \mapsto 0$ is injective, yet g is not injective since $g(0) = g(1) = 0$.

(vi) We have $g \circ f$ is injective and f is surjective. Let $y_1, y_2 \in Y$ such that $g(y_1) = g(y_2)$. The surjectivity of f implies that there exist $x_1, x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Hence, we have $g(f(x_1)) = g(f(x_2)) \Leftrightarrow (g \circ f)(x_1) = (g \circ f)(x_2)$. The injectivity of $g \circ f$ implies $x_1 = x_2$, from which we have $y_1 = y_2$. Therefore, g is injective. □

Solution 2.
Let $W, X, Y, Z \neq \emptyset$, and let $f : W \to X$, $g : X \to Y$ and $h : Y \to Z$. We will show that

$$(h \circ g) \circ f = h \circ (g \circ f).$$

Clearly, we have $h \circ g : X \to Z$, hence $(h \circ g) \circ f : W \to Z$. Also, $g \circ f : W \to Y$, hence $h \circ (g \circ f) : W \to Z$. Thus, the domains and codomains of both these functions are equal.

Let $w \in W$, $x = f(w) \in X$, $y = g(x) \in Y$, $z = h(y) \in Z$. Thus, $(h \circ g)(x) = h(g(x)) = h(y) = z$, so $((h \circ g) \circ f)(w) = (h \circ g)(f(w)) = (h \circ g)(x) = z$.
Again, $(g \circ f)(w) = g(f(w)) = g(x) = y$, so $(h \circ (g \circ f))(w) = h((g \circ f)(w)) = h(y) = z$.
Hence, for all $w \in W$, $((h \circ g) \circ f)(w) = (h \circ (g \circ f))(w) \in Z$. Therefore, these two functions are equal. $\square$

Solution 3.

(i) We examine

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R}, \\ x &\mapsto x^2 + x. \end{aligned}$$

Clearly, f is not injective, since $f(0) = f(-1) = 0$.
Note that for all $x \in \mathbb{R}$,

$$f(x) \;=\; x^2 + x \;=\; \left(x + \frac{1}{2}\right)^2 - \frac{1}{4} \;\geq\; -\frac{1}{4}$$

Hence, for all $y < -1/4$, e.g., $y = -1$, there is no $x \in \mathbb{R}$ such that $f(x) = y$.
Therefore, f is neither injective, nor surjective. $\square$

(ii) We examine

$$\begin{aligned} f : \mathbb{N} &\to \mathbb{N}, \\ n &\mapsto \left\lfloor \frac{n+1}{2} \right\rfloor. \end{aligned}$$

Clearly, f is not injective, since $f(1) = f(2) = 1$.
Note that for all $k \in \mathbb{N}$, $f(2k - 1) = k$. Also, $2k - 1 \in \mathbb{N}$.
Therefore, f is not injective, but is surjective. $\square$

(iii) We examine

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R}, \\ x &\mapsto x + \lfloor x \rfloor. \end{aligned}$$

Let $x_1, x_2 \in \mathbb{R}$. Thus,

$$\begin{aligned} f(x_1) &= f(x_2) \\ \Rightarrow \quad x_1 + \lfloor x_1 \rfloor &= x_2 + \lfloor x_2 \rfloor \\ \Rightarrow \quad x_1 - x_2 &= -\lfloor x_1 \rfloor + \lfloor x_2 \rfloor \end{aligned}$$

It follows that $k = x_1 - x_2 \in \mathbb{Z}$, so

$$\begin{aligned} \lfloor x_1 \rfloor &= \lfloor k + x_2 \rfloor \\ &= k + \lfloor x_2 \rfloor \\ &= x_1 - x_2 + \lfloor x_2 \rfloor \\ x_1 - x_2 &= \lfloor x_1 \rfloor - \lfloor x_2 \rfloor \end{aligned}$$

Hence, we have $x_1 = x_2$. Therefore, f is injective.
For $f(x) = 2k + 1 \in \mathbb{Z} \subset \mathbb{R}$, $k \in \mathbb{Z}$, we must have $x + \lfloor x \rfloor = 2k + 1$, so $x \in \mathbb{Z}$. Thus, $f(x) = 2x = 2k + 1 \Rightarrow x = k + \frac{1}{2} \notin \mathbb{Z}$, a contradiction. Hence, there is no $x \in \mathbb{R}$ such that $f(x) = 2k + 1$, $k \in \mathbb{Z}$.
Therefore, f is injective, but not surjective. $\square$

(iv) We examine

$$\begin{aligned} f &: \mathbb{R} \to \mathbb{R}, \\ x &\mapsto x - \lfloor x \rfloor . \end{aligned}$$

Clearly, f is not injective, since $f(0) = f(1) = 0$.

Note that $\lfloor x \rfloor$ is the *greatest* integer less than or equal to x. Let $x - \lfloor x \rfloor = \delta$, where $\delta \in \mathbb{R}$. We must have $\lfloor x \rfloor \leq x$, so $\delta \geq 0$. If $\delta \geq 1$, we would have $x - (1 + \lfloor x \rfloor) = \delta - 1 \geq 0 \Rightarrow x \geq 1 + \lfloor x \rfloor$, a contradiction. Hence, $\delta < 1$, and $f(x) < 1$ for all $x \in \mathbb{R}$, i.e., there is no $x \in \mathbb{R}$ such that $f(x) = 2$.

Therefore, f is neither injective, nor surjective. □

(v) We examine

$$\begin{aligned} f &: \mathbb{R} \setminus \{1\} \to \mathbb{R}, \\ x &\mapsto \frac{x+1}{x-1}. \end{aligned}$$

Let $x_1, x_2 \in \mathbb{R} \setminus \{1\}$. Thus,

$$\begin{aligned} & f(x_1) &=& f(x_2) \\ \Rightarrow\ & \frac{x_1+1}{x_1-1} &=& \frac{x_2+1}{x_2-1} \\ \Rightarrow\ & (x_1+1)(x_2-1) &=& (x_1-1)(x_2+1) && (x \neq 1) \\ \Rightarrow\ & x_1x_2 - x_1 + x_2 - 1 &=& x_1x_2 + x_1 - x_2 - 1 \\ \Rightarrow\ & x_1 &=& x_2 \end{aligned}$$

Hence, we have $x_1 = x_2$. Therefore, f is injective.

Note that for $f(x) = 1 \in \mathbb{R}$, we require $x+1 = x-1$, a contradiction. Hence, there is no $x \in \mathbb{R}\setminus\{1\}$ such that $f(x) = 1$.

Therefore, f is injective, but not surjective. □

(vi) We examine

$$\begin{aligned} f &: (-1,1) \to \mathbb{R}, \\ x &\mapsto \frac{x}{1-|x|}. \end{aligned}$$

Let $x_1, x_2 \in (-1, 1)$. Thus,

$$\begin{aligned} & f(x_1) &=& f(x_2) \\ \Rightarrow\ & \frac{x_1}{1-|x_1|} &=& \frac{x_2}{1-|x_2|} \\ \Rightarrow\ & x_1(1-|x_2|) &=& x_2(1-|x_1|) && (|x| \neq 1) \\ \Rightarrow\ & x_1 - x_2 &=& x_1|x_2| - x_2|x_1| \end{aligned}$$

If either x_1 or x_2 is zero, we are forced to have $x_1 = x_2 = 0$.

Note that x_1 and x_2 cannot have opposite signs, since $1 - |x| > 0$ for all $x \in (-1, +1)$.

We are left with x_1 and x_2 sharing the same sign. Thus, we have $x_1/|x_1| = x_2/|x_2| = \pm 1$, so $x_1|x_2| = x_2|x_1|$, and $x_1 = x_2$.

In all cases, we have $x_1 = x_2$. Therefore, f is injective.

We will now show that f is surjective. Let $y = f(x) \in \mathbb{R}$.

For $y = 0$, we have $x = 0$.

For $y > 0$, we have $x > 0$, so

$$y = \frac{x}{1-x} \Rightarrow x = \frac{y}{1+y} < 1 \qquad (1 + y > y > 0)$$

Clearly, for every $y > 0$, there exists $x \in (0, 1)$ such that $f(x) = y$.
For $y < 0$, we have $x < 0$, so

$$y = \frac{x}{1+x} \Rightarrow x = \frac{y}{1-y} > -1 \qquad (0 > y > y - 1)$$

Again, for every $y < 0$, there exists $x \in (-1, 0)$ such that $f(x) = y$.
Therefore, f is both injective and surjective. $\square$