PH1101 ASSIGNMENT-04

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Q1. $z = f(x,y) = x^3 y - e^{xy}$.

(a) $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^3 y - e^{xy}) = 3x^2 y - ye^{xy}$.

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^3 y - e^{xy}) = x^3 - xe^{xy}$.

(b) $\frac{\partial^2 f}{\partial x \partial y}$.

$\Rightarrow \frac{\partial f}{\partial x} = 3x^2 y - ye^{xy}$

$\Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(3x^2 y - ye^{xy})$

$\Rightarrow \frac{\partial^2 f}{\partial x \partial y} = 3x^2 - (xye^{xy} + e^{xy})$

$\frac{\partial^2 f}{\partial x \partial y} = 3x^2 - e^{xy}(1 + xy)$.

(c) $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(3x^2 y - ye^{xy})$

$\Rightarrow \frac{\partial^2 f}{\partial x^2} = 6xy - y^2 e^{xy}$.

(d) $\frac{\partial^3 f}{\partial y^3} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\right).$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(x^3 - xe^{xy}\right) = 0 - x^2e^{xy}.$$

$$\Rightarrow \frac{\partial^2 f}{\partial y^2} = -x^2e^{xy}.$$

$$\Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y^2}\right) = \frac{\partial}{\partial y}\left(-x^2e^{xy}\right) = -x^3e^{xy}.$$

$$\Rightarrow \frac{\partial^3 f}{\partial y^3} = -x^3e^{xy}.$$

(e) $\frac{\partial^3 f}{\partial x^2 \partial y}.$

$$\frac{\partial f}{\partial x} = 3x^2y - ye^{xy}. \Rightarrow \frac{\partial^2 f}{\partial x^2} = 6xy - y^2e^{xy}.$$

$$\Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x^2}\right) = 6x - \left(xy^2e^{xy} + 2ye^{xy}\right).$$

$$\Rightarrow \frac{\partial^3 f}{\partial x^2 \partial y} = 6x - ye^{xy}(xy+2).$$

Q2. (a) $y = \ln \sin 2x$.

$\frac{dy}{dx} = ?$

Let $\sin 2x = \varphi$, then,

$y = \ln \varphi$.

$$\frac{dy}{dx} = \frac{\partial y}{\partial \varphi} \cdot \frac{\partial \varphi}{\partial x} = \frac{\partial}{\partial \varphi}(\ln \varphi) \cdot \frac{\partial}{\partial x}(\sin 2x).$$

$$\frac{dy}{dx} = \frac{1}{\varphi} \cdot 2\cos 2x = \frac{2\cos 2x}{\sin 2x} = \underline{2\cot 2x}.$$

(b) $z = 2t^2 \sin t$

$\frac{dz}{dt} = ?$

Let $t^2 = f$, $\sin t = \varphi$

then $z(f, \varphi) = 2f\varphi$.

$$\cancel{\frac{dz}{dt}} \qquad \frac{dz}{dt} = \frac{\partial z}{\partial f} \cdot \frac{\partial f}{\partial t} + \frac{\partial z}{\partial \varphi} \cdot \frac{\partial \varphi}{\partial t}.$$

$$\Rightarrow \frac{dz}{dt} = \frac{\partial}{\partial f}(2f\varphi) \cdot \frac{\partial}{\partial t}(t^2) + \frac{\partial}{\partial \varphi}(2f\varphi) \cdot \frac{\partial}{\partial t}(\sin t).$$

$$\Rightarrow \frac{dz}{dt} = (2\varphi)(2t) + (2f)(\cos t).$$

$$= 4\varphi t + 2f\cos t = 4t\sin t + 2t^2\cos t.$$

$$\Rightarrow \frac{dz}{dt} = 4t\sin t + 2t^2\cos t = 2t(2\sin t + t\cos t)$$

Q3. (a) $x+e^x=t$

$\frac{dx}{dt}=?$ $\Rightarrow$ $x+e^x=t \Rightarrow \frac{dx}{dt}+\frac{d}{dt}(e^x)=\cancel{\frac{dt}{dt}}^{\,1}$

$\Rightarrow \frac{dx}{dt}+e^x\frac{dx}{dt}=1. \Rightarrow \frac{dx}{dt}(1+e^x)=1.$

$$\Rightarrow \frac{dx}{dt}=\frac{1}{1+e^x}.$$

(b) $\Rightarrow \frac{d^2x}{dt^2}=\frac{d}{dt}\left(\frac{dx}{dt}\right)=\frac{d}{dt}\left(\frac{1}{1+e^x}\right).$

$$=\frac{d}{dx}\left(\frac{1}{1+e^x}\right)\cdot\frac{dx}{dt}.$$

$$=\frac{d}{dx}(1+e^x)^{-1}\cdot\frac{dx}{dt}.$$

$$=(-1)(1+e^x)^{-2}\cdot e^x\cdot\frac{dx}{dt}.$$

$$=\frac{-e^x}{(1+e^x)^2}\times\frac{1}{(1+e^x)}$$

$$=\frac{-e^x}{(1+e^x)^3}$$

$$\Rightarrow \frac{d^2x}{dt^2}=\frac{-e^x}{(1+e^x)^3}$$

Qs. (6) $x^3 - 3y^3 + xy + 21 = 0$.

$$\Rightarrow \frac{d}{dx}(x^3) - 3\frac{d}{dx}y^3 + \frac{d}{dx}xy + \cancel{\frac{d}{dx}(21)}^{\,0} = 0$$

$$\Rightarrow 3x^2 - 9y^2 \cdot \frac{dy}{dx} + y + x\frac{dy}{dx} = 0.$$

$$\Rightarrow 3x^2 + y = (9y^2 - x)\frac{dy}{dx}.$$

$$\Rightarrow \frac{dy}{dx} = \frac{3x^2 + y}{9y^2 - x} \Rightarrow \left.\frac{dy}{dx}\right|_{(1,2)} = \frac{3+2}{36-1}.$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{(1,2)} = \frac{5}{35} = \frac{1}{7}.$$

Now, to find the equation of the tangent,

$$(y-2) = \frac{1}{7}(x-1)$$

$$\Rightarrow 7y - 14 = x - 1 \Rightarrow \boxed{7y = x + 13} \text{ ans.}$$

Q4. $z = xy \; ; \; x = \sin(s+t) \; ; \; y = (s-t)$.

$$\cancel{\frac{\partial z}{\partial x}} \quad \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}.$$

$$= \frac{\partial}{\partial x}(xy) \cdot \frac{\partial}{\partial s}(\sin(s+t))$$

$$+ \frac{\partial}{\partial y}(xy) \cdot \frac{\partial}{\partial s}(s-t).$$

$$= y \cdot \cos(s+t). + x \cdot (1-0).$$

$$\frac{\partial z}{\partial s} = x + y\cos(s+t) = \sin(s+t) + (s-t)\cos(s+t)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}.$$

$$= \frac{\partial}{\partial x}(xy) \cdot \frac{\partial}{\partial t}(\sin(s+t)) + \frac{\partial}{\partial y}(xy) \cdot \frac{\partial}{\partial t}(s-t).$$

$$= y\cos(s+t) + x \cdot (-1)$$

$$= y\cos(s+t) - x.$$

$$\Rightarrow \frac{\partial z}{\partial t} = (s-t)\cos(s+t) - \sin(s+t)$$

Sr. $x = r\cos\theta$; $y = r\sin\theta$; $r = \sqrt{x^2+y^2}$; $\theta = \tan^{-1}(y/x)$.

$x = r\cos\theta \Rightarrow \frac{\partial x}{\partial \theta} = -r\sin\theta = -y.$

$\Rightarrow \boxed{\frac{\partial x}{\partial \theta} = -y}$ — (i). Now, let's take its reciprocal,

$\Rightarrow \boxed{\frac{\partial \theta}{\partial x} = -\frac{1}{y}.}$ — (ii).

Now, let's solve $\frac{\partial \theta}{\partial x}$,

$\Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right) \Rightarrow \frac{\partial \theta}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot (-1)\, x^{-2}\,(y).$

$\Rightarrow \frac{\partial \theta}{\partial x} = \frac{-y/x^2}{1+\frac{y^2}{x^2}} = \frac{-y/\cancel{x^2}}{(x^2+y^2)/\cancel{x^2}} = \frac{-y^2}{x^2+y^2} = \frac{-y^2}{r^2}.$

$\Rightarrow \boxed{\frac{\partial \theta}{\partial x} = \frac{-y^2}{r^2}.}$ — (iii).

Now, we notice that (ii) and (iii) are not the same, ~~thus~~ and neither are (i) and (iii). Thus, we can claim that the reciprocal of $\frac{\partial x}{\partial \theta}$ is not $\frac{\partial \theta}{\partial x}$.

$\Rightarrow \boxed{\frac{1}{\frac{\partial x}{\partial \theta}} \neq \frac{\partial \theta}{\partial x}}$ also, $\boxed{\frac{\partial x}{\partial \theta} \neq \frac{\partial \theta}{\partial x}}$

QED.

Q6. $y = 1 - x^2$

String is stretched from origin to a point (x, y) on the curve.

Length of string $= \sqrt{x^2 + y^2}$

$\Rightarrow (\text{length})^2 = x^2 + y^2$

Let $f(x) = x^2 + y^2$; and $\varphi(x, y) = y + x^2$

Using Lagrange multipliers,

$F(x, y) = f(x, y) + \lambda\varphi(x, y)$.

$\Rightarrow F(x, y) = x^2 + y^2 + \lambda(x^2 + y)$

$\Rightarrow \boxed{F(x, y) = x^2(1 + \lambda) + y^2 + \lambda y.}$

$\frac{\partial F}{\partial x} = 2x(1 + \lambda) = 0 \Rightarrow \boxed{x = 0 \quad \text{OR} \quad \lambda = -1}$

$\frac{\partial F}{\partial y} = 2y + \lambda = 0 \Rightarrow \boxed{\lambda = -2y}$

$\Rightarrow$ Either $(x = 0 \text{ OR } \lambda = -1)$ or $(\lambda = -2y)$ & $(x^2 + y = 1)$

At $x = 0$, $y = 1$ and $\lambda = -2$.

At $\lambda = -1$, $y = \frac{1}{2}$ and $x = \frac{1}{\sqrt{2}}$.

ans.

Q7. $r = x + vt$; $s = x - vt$

Change of variables in $\frac{\partial^2 f}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = 0$.

$dr = dx + v\,dt$ — (i). $ds = dx - v\,dt$ — (ii).

$\frac{dr}{dx} = 1 + v\cancel{\frac{dt}{dx}}^{\,0}$; $\frac{ds}{dx} = 1$; $\frac{dr}{dt} = v$; $\frac{ds}{dt} = -v$.

$\frac{dr}{dx} = 1$.

$$\frac{\partial^2 f}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) - \frac{1}{v^2}\frac{\partial}{\partial t}\left(\frac{\partial f}{\partial t}\right).$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial f}{\partial s}\cdot\frac{\partial s}{\partial x}\right) - \frac{1}{v^2}\frac{\partial}{\partial t}\left(\frac{\partial f}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial f}{\partial s}\frac{\partial s}{\partial t}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial r}\cdot 1 + \frac{\partial f}{\partial s}\cdot 1\right) - \frac{1}{v^2}\frac{\partial}{\partial t}\left(\frac{\partial f}{\partial r}\cdot v + \frac{\partial f}{\partial s}\cdot(-v)\right).$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial r} + \frac{\partial f}{\partial s}\right) - \frac{1}{v^{\cancel{2}}}\frac{\partial}{\partial t}\left(\frac{\partial f}{\partial r}\cdot\cancel{v} - \cancel{v}\cdot\frac{\partial f}{\partial s}\right).$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial r} + \frac{\partial f}{\partial s}\right) - \frac{1}{v}\frac{\partial}{\partial t}\left(\frac{\partial f}{\partial r} - \frac{\partial f}{\partial s}\right).$$

$$= \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial r} + \frac{\partial f}{\partial s}\right)\cdot\cancel{\frac{\partial r}{\partial x}}^{\,1} + \frac{\partial}{\partial s}\left(\frac{\partial f}{\partial r} + \frac{\partial f}{\partial s}\right)\cdot\cancel{\frac{\partial s}{\partial x}}^{\,1}.$$

$$-\frac{1}{v}\left[\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial r} - \frac{\partial f}{\partial s}\right)\cdot\cancel{\frac{\partial r}{\partial t}}^{\,v} + \frac{\partial}{\partial s}\left(\frac{\partial f}{\partial r} - \frac{\partial f}{\partial s}\right)\cdot\cancel{\frac{\partial s}{\partial t}}^{\,-v}\right].$$

$$= \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial r} + \frac{\partial f}{\partial s}\right) + \frac{\partial}{\partial s}\left(\frac{\partial f}{\partial r} + \frac{\partial f}{\partial s}\right).$$

$$-\frac{1}{\cancel{v}}\left[\cancel{v}\cdot\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial r} - \frac{\partial f}{\partial s}\right) - \cancel{v}\frac{\partial}{\partial s}\left(\frac{\partial f}{\partial r} - \frac{\partial f}{\partial s}\right)\right].$$

$$= \cancel{\frac{\partial^2 F}{\partial r^2}} + \frac{\partial^2 F}{\partial s \partial r} + \frac{\partial^2 F}{\partial r \partial s} + \cancel{\frac{\partial^2 F}{\partial s^2}} - \cancel{\frac{\partial^2 F}{\partial r^2}} + \frac{\partial^2 F}{\partial s \partial r} + \frac{\partial^2 F}{\partial r \partial s}$$

$$- \cancel{\frac{\partial^2 F}{\partial s^2}}.$$

$$= 2\frac{\partial^2 F}{\partial s \partial r} + 2\frac{\partial^2 F}{\partial r \partial s}$$

If the partial derivatives (2nd) are continuous, then, we can write,

$$= 4\frac{\partial^2 F}{\partial r \partial s} = 0 \Rightarrow \frac{\partial^2 F}{\partial r \partial s} = 0 \Rightarrow \frac{\partial}{\partial r}\left(\frac{\partial F}{\partial s}\right) = 0$$

the r derivative of $\frac{\partial F}{\partial s}$ is zero. This means that $\frac{\partial F}{\partial s}$ is independent of r. If we integrate, we get, $F = f(s) + c$ as $\frac{\partial F}{\partial s}$ is a funcⁿ of 's' alone. Since, the integration was performed over a partial derivative, 'c' is a const. only as far as 's' is concerned. But it might be a funcⁿ of 'r', say $g(r)$, since $\frac{\partial}{\partial s}(g(r)) = 0$. Thus, the solⁿ looks like,

$$F = f(s) + g(r)$$

$$\Rightarrow \boxed{F = f(x - vt) + g(x + vt).}$$

d'Alembert's Equation.

Solution of the wave eqⁿ

Q8. $\frac{dI}{dx}$; $I = \int_{\pi/4}^{x} \sin t \, dt$

Applying the Newton-Leibnitz rule for differentiation of integrals,

$$\frac{dI}{dx} = \frac{d}{dx}\left[\int_{\pi/4}^{x} \sin t \, dt\right] = \sin(x)\cancel{\frac{d}{dx}(x)}^{\,1} - \sin\cancel{(\pi/4)\frac{d}{dx}\left(\frac{\pi}{4}\right)}^{\,0}$$

$$\Rightarrow \boxed{\frac{dI}{dx} = \sin x.}$$ ans.

General form of definition of differentiation of integrals,

NEWTON - LEIBNITZ RULE :-

$$\boxed{\frac{d}{dx}\int_{\varphi(x)}^{\psi(x)} f(t)\,dt = f(\psi)\frac{d\psi}{dx} - f(\varphi)\frac{d\varphi}{dx}}$$

Q9. (a) $x=0$, $y \neq 0$ -> neither max nor min

$x=1$, $y=$ max

y can be minimum at ~~$0 \leq x \leq$~~ $0<x<1$.
[Graph considered: $0 \leq x \leq 1$].

(b) $x=0$, $y=$ min

$x=1$, $y=$ max

y has local maximum and minimum at $0<x<1$.
[Graph considered: $0 \leq x \leq 1$].

(c) $x=0$, $y=$ min

$x=1$, $y=$ neither max nor min

y can be maximum at $0<x<1$.

[Graph considered: $0 \leq x \leq 1$].