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PH1101

Assignment - 5

Q1. FOURIER SERIES

A Fourier Series is an expansion or representation of a function in a series of sines and cosines, such as,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{— (i)}$$

- The coefficients a_0, a_n and b_n are related to $f(x)$ by definite integrals:

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(s) \cos ns \, ds \quad , \quad n = 0, 1, 2, \cdots \quad \text{— (ii)}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(s) \sin ns \, ds \quad , \quad n = 1, 2, \cdots \quad \text{— (iii)}$$

- a_0 is singled out for special treatment by the inclusion of the factor $\frac{1}{2}$. This is done so that (ii) will apply to all a_n, $n=0$ as well as $n>0$.

The sufficient conditions imposed on $f(x)$ to make eqn (i) valid are that $f(x)$ has only a finite no. of finite discontinuities and only a finite no. of extreme values (maxima and minima) in the interval $[0, 2\pi]$.

(DIRICHLET CONDITIONS)

Now, we try to derive the coefficients of a Fourier Series:
Let the periodic func^n $f(x)$ with period 2π be such that it may be represented as a trigo. series convergent to a given func^n in the interval $(-\pi, \pi)$; i.e., that it is the sum of this series;

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{— (IV)}.$$

Suppose that the integral of the func^n on the LHS of this eq^n is equal to the sum of the integrals of the terms of the series. This will be the case, if we assume that the numerical series made up of the coefficients of the given trigonometric series converges absolutely; i.e., that the following positive no. series converges: (V)

$$\left|\frac{a_0}{2}\right| + (|a_1| + |a_2| + \cdots + |a_n| + \cdots) + (|b_1| + |b_2| + \cdots + |b_n| + \cdots)$$

Then, series on RHS of (IV) is dominant and, consequently, it may be integrated termwise in the interval from $(-\pi)$ to π.

$$\Rightarrow \int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} \frac{a_0}{2}\,dx + \sum_{n=1}^{\infty} \left(\int_{-\pi}^{\pi} a_n \cos nx\,dx + \int_{-\pi}^{\pi} b_n \sin nx\,dx \right)$$

Evaluate each integral separately on the RHS,

$$\int_{-\pi}^{\pi} \frac{a_0}{2}\,dx = \pi a_0 \; ; \; \int_{-\pi}^{\pi} a_n \cos nx\,dx = a_n \int_{-\pi}^{\pi} \cos nx\,dx = \left. \frac{a_n \sin nx}{n} \right|_{-\pi}^{\pi} = 0$$

$$\int_{-\pi}^{\pi} b_n \sin nx\,dx = b_n = b_n \int_{-\pi}^{\pi} \sin nx\,dx = \left. -b_n \frac{\cos nx}{n} \right|_{-\pi}^{\pi} = 0$$

Consequently,

$$\int_{-\pi}^{\pi} f(x)\,dx = \pi a_0$$

where,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx.$$

To calculate other coefficients of the series, we shall need certain definite integrals, which we'll consider first.
If n and k are integers, then, we have the following eq^{n}s,

$$\left.\begin{aligned} &\int_{-\pi}^{\pi} \cos nx \cdot \cos kx \cdot dx = 0 \\ &\int_{-\pi}^{\pi} \cos nx \cdot \sin kx \, dx = 0 \\ &\int_{-\pi}^{\pi} \sin nx \cdot \sin kx \, dx = 0 \end{aligned}\right\} n \neq k. \quad \text{— (I)}$$

But if $n = k$, then,

$$\left.\begin{aligned} &\int_{-\pi}^{\pi} \cos^2 kx \, dx = \pi \\ &\int_{-\pi}^{\pi} \sin kx \cos kx \, dx = 0 \\ &\int_{-\pi}^{\pi} \sin^2 kx \, dx = \pi \end{aligned}\right\} n = k. \quad \text{— (II)}$$

By means of the formulae,

$$\left.\begin{aligned}
\cos nx \cos kx &= \frac{1}{2}\left[\cos(n+k)x + \cos(n-k)x\right] \\
\cos nx \sin kx &= \frac{1}{2}\left[\sin(n+k)x - \sin(n-k)x\right] \\
\sin nx \sin kx &= \frac{1}{2}\left[-\cos(n+k)x + \cos(n-k)x\right]
\end{aligned}\right\}.$$

Now, we can compute the coefficients a_k and b_k of (IV).
To find coefficient a_k for some definite value $k \neq 0$, multiply both sides of (IV) by $\cos kx$:

$$f(x)\cos kx = \frac{a_0}{2}\cos kx + \sum_{n=1}^{\infty}\left(a_n \cos nx \cos kx + b_n \sin nx \cos kx\right) \quad \text{— (IV')}$$

The resulting series on the right is dominated, since its terms do not exceed (absolute value $|x|$) the terms of the convergent positive series (V). We can now integrate it termwise on any interval. Integrate (IV') from $-\pi$ to π:

$$\int_{-\pi}^{\pi} f(x)\cos kx\,dx = \frac{a_0}{2}\int_{-\pi}^{\pi}\cos kx\,dx$$

$$+\sum_{n=1}^{\infty}\left(a_n\int_{-\pi}^{\pi}\cos nx\cos kx\,dx + b_n\int_{-\pi}^{\pi}\sin nx\cos kx\,dx\right)$$

Taking into account formulae (I) and (II), we see that all the integrals on the right are equal to zero, with the exception of the integral with coeff. a_k. Hence,

$$\int_{-\pi}^{\pi} f(x)\cos kx\,dx = a_k \int_{-\pi}^{\pi} \cos^2 kx\,dx = a_k \pi$$

where,

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx$$

Multiplying both sides of (IV) by $\sin kx$ and again integrating from $-\pi$ to π, we find,

$$\int_{-\pi}^{\pi} f(x)\sin kx\,dx = b_k \int_{-\pi}^{\pi} \sin^2 kx\,dx = b_k \pi$$

whence,

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx$$

The coefficients a_0, a_k and b_k are called the <u>FOURIER COEFFICIENTS</u> of the function $f(x)$.

Q2. Let us note a relationship between the coefficients $c_0, c_1, c_2, \cdots, c_n$ of the polynomial of degree n.

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n \quad \text{—} \quad (1)$$

and its derivatives of order one through n at the point $x=0$. Let us take the first n derivatives of polynomial (1) :

$$f'(x) = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots$$

$$f''(x) = 2c_2 + 2\cdot 3\, c_3 x + 3\cdot 4\, c_4 x^2 + \cdots$$

$$f'''(x) = 2\cdot 3\, c_3 + 2\cdot 3\cdot 4\, c_4 x + \cdots$$

$$\vdots$$

$$f^{(n)}(x) = n!\, c_n$$

Putting $x=0$ in the above equations and solving for the c's, we obtain,

$$c_0 = f(0);\ c_1 = f'(0);\ c_2 = \frac{f''(0)}{2};\ c_3 = \frac{f'''(0)}{3!};\ \cdots\ c_n = \frac{f^{(n)}(0)}{n!}$$

Thus, we see that polynomial (1) can be written as:

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

There is a completely analogous relation between the coefficients $c_0, c_1, c_2, \cdots, c_n$ of the power series in $(x-a)$ of degree n.

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n \quad \text{--- (11)}$$

and its derivatives of order one through n at the point $x = a$. If we take the first n derivatives of power series (11), we get:

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots$$

$$f''(x) = 2c_2 + 2\cdot 3c_3(x-a) + 3\cdot 4c_4(x-a)^2 + \cdots$$

$$f'''(x) = 2\cdot 3c_3 + 2\cdot 3\cdot 4c_4(x-a) + \cdots$$

$$\vdots$$

$$f^{(n)}(x) = n!\,c_n$$

Putting $x = a$ in the above eq^{n}s and solving for the c's we obtain:

$$c_0 = f(a);\; c_1 = f'(a);\; c_2 = \frac{f''(a)}{2!};\; c_3 = \frac{f'''(a)}{3!};\; \ldots;\; c_n = \frac{f^{(n)}(a)}{n!}$$

Thus, we see that power series (11) can be written as:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

$$\cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Since, a power series is a polynomial representation of a function, this is another polynomial representation of the func^n. Such series may be used to represent rather general func^{n}s with some interval of convergence.

Q3. Taylor's Series for $\sin x$:-

In order to use Taylor's Formula to find the power series expansion of $\sin x$, we have to find derivatives of $\sin(x)$:

$$\sin'(x) = \cos(x)$$

$$\sin''(x) = -\sin(x)$$

$$\sin'''(x) = -\cos(x)$$

$$\sin''''(x) = \sin x .$$

Since, $\sin''''(x) = \sin x$, this pattern will repeat.

Next, we will evaluate the f^n & its derivatives at 0:

$$\sin(0) = 0$$

$$\sin'(0) = 1$$

$$\sin''(0) = 0$$

$$\sin'''(0) = -1$$

$$\sin''''(0) = 0$$

and the pattern repeats.

Now, according to Taylor's Formula,

$$\sin(x) = 0 + 1x + 0x^2 + \frac{-1}{3!}x^3 + 0x^4 + \dots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

The radius of convergence R is infinity.

Taylor's Series for $f(x) = \cos x$.

$f(x) = \cos x$.

We start out by finding the derivatives of $\cos x$,

$f'(x) = -\sin x$.

$f''(x) = -\cos x$.

$f'''(x) = \sin x$.

and then the pattern repeats.

Now, the func^n can be written as,

$$\cos x = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$$= 1 + 0 \cdot x + \frac{-1}{2!}x^2 + 0 \cdot x^3 + \cdots$$

$$\Rightarrow \cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$$

Taylor Series Expansion for e^x

We consider $f(x) = e^x$.

We start by looking at the derivatives of $f(x)$,

$f'(x) = e^x$ $\qquad\qquad$ $[e^0 = 1]$

$f''(x) = e^x$

$f'''(x) = e^x$ $\rightarrow$ and the pattern continues,

$\vdots$

Now, using these values, we write the Taylor expansion,

$$e^x = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$$\Rightarrow e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$$

Q4. SHM is a periodic oscillation where the acceleration is proportional to the displacement from equilibrium, in the dirⁿ of the equilibrium position.

Eqⁿ for SHM: $y = A\sin\theta$

$$\left.\begin{aligned} &\Rightarrow y = A\sin\omega t \\ &\Rightarrow x = A\cos\omega t \end{aligned}\right\} \quad \begin{aligned} &z = A(\cos\omega t + i\sin\omega t) \\ &\approx x + iy \end{aligned}$$

$$\Rightarrow \boxed{z = Ae^{i\omega t}}$$

According to the def^n of SHM,

$$F_x = -kx$$

$$\Rightarrow m \cdot \frac{d^2x}{dt^2} = -kx$$

$$\Rightarrow \frac{d^2x}{dt^2} = -\frac{k}{m}x \quad \left[\text{Consider } \omega^2 = \frac{k}{m}\right]$$

$$\Rightarrow \frac{d^2x}{dt^2} + \omega^2 x = 0 \Rightarrow \frac{dv}{dt} + \omega^2 x = 0$$

$$\Rightarrow \frac{dv}{dx} \cdot \frac{dx}{dt} + \omega^2 x = 0 \Rightarrow v\frac{dv}{dx} + \omega^2 x = 0$$

$$\Rightarrow \int v\,dv = -\omega^2 \int x\,dx \Rightarrow \frac{v^2}{2} = \frac{-\omega^2}{2} \cdot x^2 + C$$

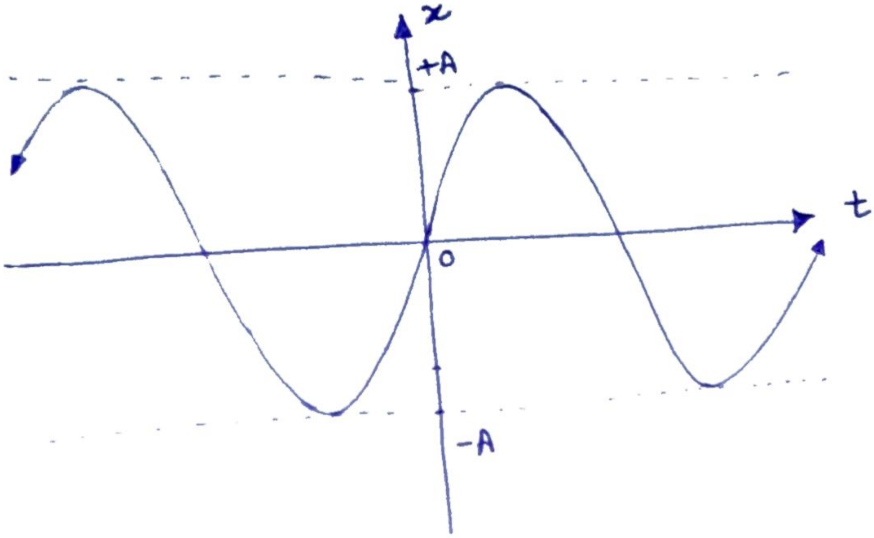
$$\Rightarrow v^2 = -\omega^2 x^2 + 2C = \omega^2\left(\frac{2C}{\omega^2} - x^2\right)$$

$$\Rightarrow \left(\frac{dx}{dt}\right)^2 = \omega^2(A^2 - x^2) \quad \left[\text{where } A = \frac{2C}{\omega^2}\right]$$

$$\Rightarrow \frac{dx}{dt} = \omega\sqrt{A^2 - x^2} \Rightarrow \frac{dx}{\sqrt{A^2 - x^2}} = \omega \cdot dt$$

$$\Rightarrow \int \frac{dx}{\sqrt{A^2 - x^2}} = \int \omega\,dt \Rightarrow \sin^{-1}\left(\frac{x}{A}\right) = \omega t + C_2$$

$$\Rightarrow \boxed{x = A\sin(\omega t + C_2)}$$



A realistic problem :—

Damped Oscillatory Motion :

$$F_x^{(1)} = -kx$$

$$F_x^{(2)} = -b\frac{dx}{dt} \Rightarrow m\cdot\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - kx.$$

$$\Rightarrow \boxed{\frac{d^2x}{dt^2} + 2\alpha\frac{dx}{dt} + \omega^2 x = 0} \text{ — (1)} \qquad \left[2\alpha = \frac{b}{m}\ ;\ \omega^2 = \frac{k}{m}\right]$$

Test funcⁿ : $e^{-at} f(t)$

$$\Rightarrow \frac{dx}{dt} = -\alpha e^{-\alpha t} f(t) + e^{-\alpha t}\frac{d}{dt} f(t)$$

$$\Rightarrow \frac{d^2x}{dt^2} = \alpha^2 e^{-\alpha t} f(t) - \alpha e^{\cdot\alpha t}\frac{d}{dt} f(t) - \alpha e^{\cdot\alpha t}\frac{d}{dt} f(t)$$

$$+ e^{\cdot\alpha t}\frac{d^2}{dt^2} f(t)$$

Substituting these values ①,

$$\alpha^2 \cancel{e^{-\alpha t}} f(t) - 2\alpha \cancel{e^{-\alpha t}} \frac{d}{dt} f(t) + \cancel{e^{-\alpha t}} \frac{d^2 f}{dt^2} - 2\alpha^2 \cancel{e^{-\alpha t}} f(t)$$

$$+ 2\alpha \cancel{e^{-\alpha t}} \frac{df}{dt} + \omega^2 \cancel{e^{-\alpha t}} f(t) = 0$$

$$\Rightarrow \alpha^2 f(t) - \cancel{2\alpha \frac{d}{dt} f(t)} + \frac{d^2}{dt^2} f(t) - 2\alpha^2 f(t) + \cancel{2\alpha \frac{d}{dt} f(t)}$$

$$+ \omega^2 f(t) = 0$$

$$\Rightarrow \frac{d^2}{dt^2} f(t) + (\omega^2 - \alpha^2) f(t) = 0$$

From the previous derivation, solution of this DE,

$$f(t) = A \sin\left(\sqrt{\omega^2 - \alpha^2}\, t + c\right)$$

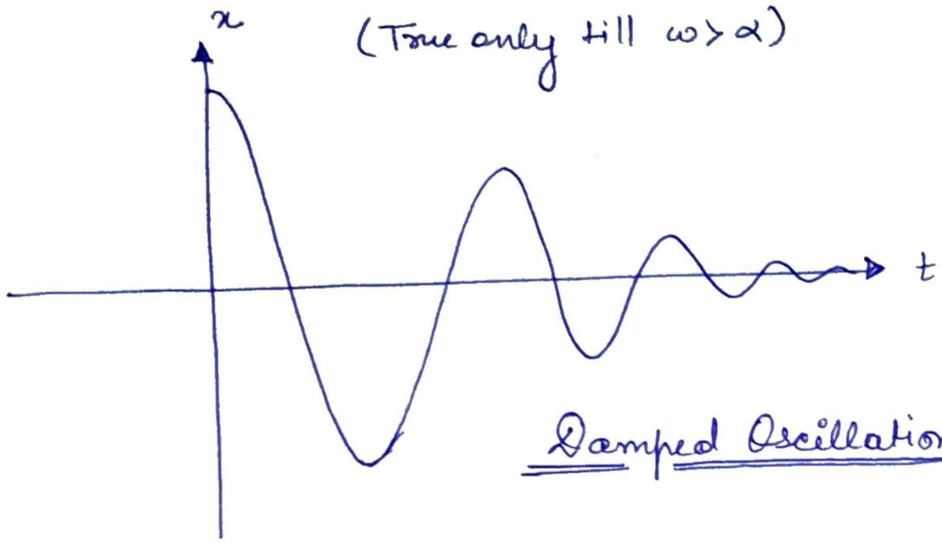
$$\Rightarrow x = e^{-\alpha t} \cdot f(t)$$

$$= e^{-\alpha t} \cdot A \cdot \sin\left(\sqrt{\omega^2 - \alpha^2}\, t + c\right)$$

$$\Rightarrow \boxed{x = A e^{-\alpha t} \sin\left(\sqrt{\omega^2 - \alpha^2}\, t + c\right)}$$

- Amplitude decreases with time.
- Damping term has reduced the frequency.

(True only till $\omega > \alpha$)

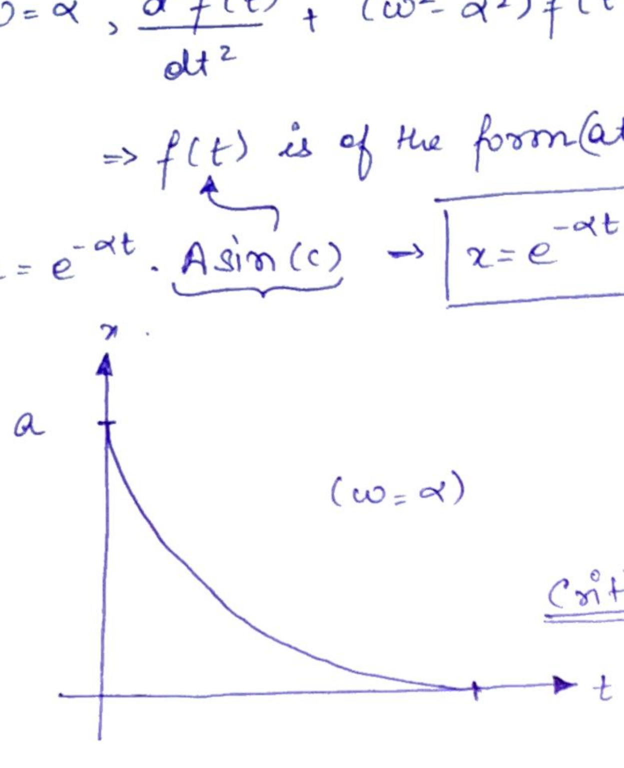


<u>Damped Oscillations</u>

If $\omega = \alpha$, $\dfrac{d^2 f(t)}{dt^2} + (\omega^2 - \alpha^2) f(t) = 0$

$\Rightarrow f(t)$ is of the form $(at + b)$.

$x = e^{-\alpha t} \cdot \underbrace{A \sin(c)}$ $\rightarrow$ $\boxed{x = e^{-\alpha t}(at + b)}$



<u>Critically Damped Oscillations.</u>

If $\alpha > \omega$, $\dfrac{d^2}{dt^2} f(t) + (\omega^2 - \alpha^2) f(t) = 0$

$\rightarrow \dfrac{d^2}{dt^2} f(t) - \underbrace{(\alpha^2 - \omega^2)}_{+ve} f(t) = 0$

$\Rightarrow \frac{d^2x}{dt^2} + \omega^2 x = 0 \rightarrow x = A\sin(\omega t + c)$

$$\left[\cos x = \frac{e^{ix} + e^{-ix}}{2} \quad ; \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}\right]$$

$$\Rightarrow x = A\left[\frac{e^{i(\omega t + c)} - e^{-i(\omega t + c)}}{2i}\right]$$

If $\frac{d^2x}{dt^2} - \omega^2 x = 0 \Rightarrow \frac{d^2x}{dt^2} + (i\omega)^2 x = 0$

$$\Rightarrow x = A\left[\frac{e^{i(i\omega t + c)} - e^{-i(i\omega t + c)}}{2i}\right]$$

$$\Rightarrow x = A\left[\frac{e^{-\omega t + ic} - e^{\omega t - ic}}{2i}\right]$$

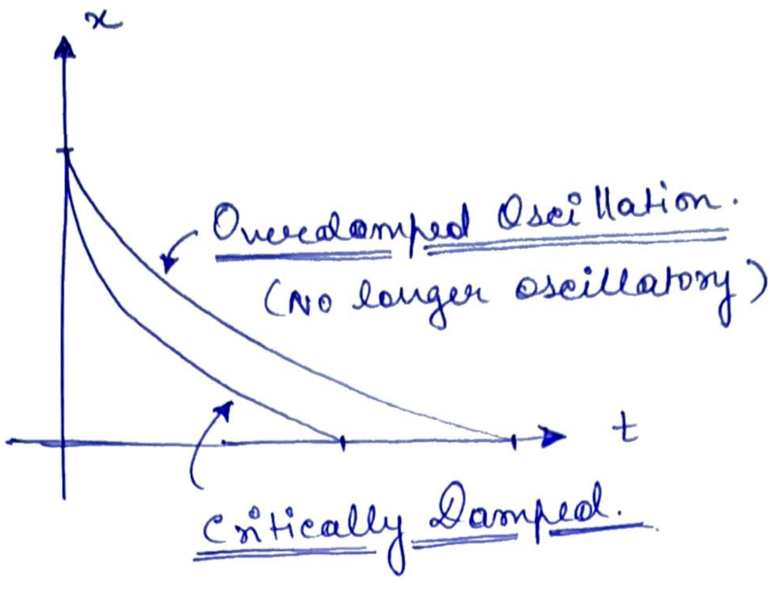
$$\Rightarrow x = A'e^{-\omega t} + B'e^{\omega t}$$

So, in case of $\frac{d^2}{dt^2} f(t) - (\alpha^2 - \omega^2) f(t) = 0$,

$$\rightarrow x = e^{-\alpha t} f(t)$$

$$\Rightarrow f(t) = A''e^{\sqrt{\alpha^2 - \omega^2}\, t} + B''e^{-\sqrt{\alpha^2 - \omega^2}\, t}$$

$$\Rightarrow \boxed{x = e^{-\alpha t}\left[A''e^{\sqrt{\alpha^2 - \omega^2}\, t} + B''e^{-\sqrt{\alpha^2 - \omega^2}\, t}\right]}$$



Q5. Let $f(x)$ be continuous over the interval $[a,b]$.
Then the average value of the function $f(x)$ (or f_{avg}) on $[a,b]$ is given by :

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

It can also be done if the function is discrete,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x$$

Avg. value of sinx :-

In the interval $[0,\pi]$, $f_{avg} = \frac{1}{\pi - 0}\int_0^\pi \sin x\,dx$

$$\Rightarrow f_{avg} = \frac{1}{\pi}\left[-\cos x\right]_0^\pi = \frac{1}{\pi}\left[-\cos\pi - (-\cos 0)\right] = \frac{2}{\pi}$$

Hence, the avg. of the func^n sinx in the interval $[0,\pi]$ is $\frac{2}{\pi}$.

But, in the interval $[0, 2\pi]$ or $[-\pi, \pi]$, the average value boils down to zero (0).

<u>Average value of cos x</u> :-

In the interval $[0, \pi/2]$,

$$f_{avg} = \frac{1}{\frac{\pi}{2} - 0} \int_0^{\pi/2} \cos(x)\, dx = \frac{1}{\pi/2} \left[\sin(x)\right]_0^{\pi/2}$$

$$= \frac{2}{\pi}\left[\sin\left(\frac{\pi}{2}\right) - \sin(0)\right]$$

$$= \frac{2}{\pi}[1-0] = \frac{2}{\pi}$$

But, in the interval $[0, 2\pi]$ or $[-\pi, \pi]$, the average value boils down to zero (0).

<u>Average value of $\sin^2 x$ and $\cos^2 x$</u> :-

- for $n \neq 0$,

$$\sin^2(nx) + \cos^2(nx) = 1$$

$$\Rightarrow \int_{-\pi}^{\pi} \sin^2 nx\, dx = \int_{-\pi}^{\pi} \cos^2 nx\, dx \Rightarrow \int_{-\pi}^{\pi} (\sin^2 nx + \cos^2 nx)\, dx = 2\pi$$

$$\Rightarrow \int_{-\pi}^{\pi} \sin^2 nx = \int_{-\pi}^{\pi} \cos^2 nx = \pi \quad \text{So, } \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin^2 nx = \frac{\pi}{2\pi} = \frac{1}{2}$$

$\therefore$ Avg. Value of $\sin^2 x = \cos^2 x = \frac{1}{2}$.