

Module 2: Number Systems

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Email: pm21ms002@iiserkol.ac.in. These are my personal notes on Number Systems. We will consider \mathbb{N} (Natural Numbers), \mathbb{Z} (Integers), and \mathbb{Q} (rational Numbers), but not \mathbb{R} (Real Numbers).

§1 Natural Numbers

The natural numbers are $1, 2, 3, 4, \dots$. The set of all natural numbers is denoted by \mathbb{N} .

Definition 1.1. We assume familiarity with the algebraic operations of addition and multiplication on the set \mathbb{N} and also with the linear order relation $<$ on \mathbb{N} defined by “ $a < b$ if $a, b \in \mathbb{N}$ and a is less than b ”.

We discuss the following fundamental properties of the set \mathbb{N} .

1. Well Ordering Property
2. Principle of Induction

§1.1 Well Ordering Property

Definition 1.2. Every non-empty subset of \mathbb{N} has a least element.

This means that if S is a non-empty subset of \mathbb{N} , then there is an element m in S such that $m \leq s$ for all $s \in S$.

In particular, \mathbb{N} itself has the least element 1.

Proof. Let S be a non-empty subset of \mathbb{N} . Let k be an element of S . Then k is a natural number.

We define a subset T by $T = \{x \in S : x \leq k\}$. The T is a non-empty subset of $\{1, 2, 3, \dots, k\}$.

Clearly, T is a finite subset of \mathbb{N} and therefore it has a least element, say m . Then $1 \leq m \leq k$.

We now show that m is the least element of S . Let s be any element of S .

If $s > k$, then the inequality $m \leq k$ implies $m < s$.

If $s \leq k$, the $s \in T$; and m being the least element of T , we have $m \leq s$.

Thus m is the least element of S . □

§1.2 Principle of Induction

Definition 1.3. Let S be a subset of \mathbb{N} such that,

- i) $1 \in S$ and,
- ii) if $k \in S$, then $k + 1 \in S$.

Then $S = \mathbb{N}$

Proof. Let $T = \mathbb{N} - S$. We prove that $T = \phi$.

Let T be non-empty. then by the *Well Ordering Property* of \mathbb{N} , the non-empty subset T has a least element, say m .

Since $1 \in S$ and 1 is the least element of \mathbb{N} , $m > 1$.

Hence, $m - 1$ is a natural number and $m - 1 \notin T$. So, $m - 1 \in S$.

But by ii) $m - 1 \in S \Rightarrow (m - 1) + 1 \in S$, i.e., $m \in S$.

This contradicts that m is the least element in T . Therefore, our assumption is wrong and $T = \phi$.

Therefore, $S = \mathbb{N}$. □

Theorem 1.4

Let $P(n)$ be a statement involving a natural number n . If,

i) $P(1)$ is true, and

ii) $P(k + 1)$ is true whenever $P(k)$ is true,

then $P(n)$ is true for all $n \in \mathbb{N}$.

Proof.

□