CHAPTER 5: DIFFERENTIABILITY

We begin by introducing the notion of derivative of a function

DEFINITION 5.1 (DERIVATIVE)

Let $-\infty < a < b < \infty$, let $f : (a,b) \to \mathbb{R}$ and let $x_0 \in (a,b)$. We say that f is differentiable at x_0 if

$$(*) \quad \boxed{\lim_{y \to x_0} \frac{f(y) - f(x_0)}{y - x_0}}$$

exists. When $(*)$ exists, we say that

$$\boxed{f'(x_0) \stackrel{\text{def}^n}{:=} \lim_{y \to x_0} \frac{f(y) - f(x_0)}{y - x_0}}$$

is the derivative of f at x_0. Furthermore, if f is differentiable at every point of (a,b), f is said to be differentiable in (a,b).

The following result gives us the relation between differentiability and continuity.

THEOREM 5.2 (DIFFERENTIABILITY & CONTINUITY)

Let $-\infty < a < b < \infty$, let $f : (a,b) \to \mathbb{R}$ be differentiable at $x_0 \in (a,b)$. Then, f is continuous at x_0.

PROOF: Define $\phi : (a,b) \to \mathbb{R}$ by

$$\phi(x) := \begin{cases} \dfrac{f(x) - f(x_0)}{x - x_0}, & \text{if } x \neq x_0 \\ f'(x_0) & \text{if } x = x_0 \end{cases}$$

~~Then, $\lim_{x \to x_0} \phi(x)$ exists and $\lim_{x \to x_0} \phi(x) = f'(x_0) = \phi(x_0)$, as f is differentiable at x_0. Therefore, using Theorem, ϕ is continuous. Hence, using Theorem 4.2.5, the function $g : (a,b) \to$ $f(x) - f(x_0$~~

Then, $\lim_{x \to x_0} \phi(x)$ exists. ~~and is equal to Therefore~~ as f is differentiable at x^0. Therefore, using thorem 4.1.8, $\lim_{x \to x_0} \phi(x)(x-x_0)$ exists and

$$\lim_{x \to x_0} \phi(x)(x-x_0) = \lim_{x \to x_0} \phi(x) \lim_{x \to x_0} (x-x_0) = 0.$$

$$\Rightarrow \lim_{x \to x_0} \left(f(x) - f(x_0)\right) = 0,$$

which shows that f is continuous at x^0. This proves the theorem. (Proved).

We now look at a few examples.

EXAMPLE 5.3

i) Define $f : \mathbb{R} \to \mathbb{R}$ by

$$\boxed{f(x) := x^2, \text{ for all } x \in \mathbb{R}.}$$

Then, f is differentiable in $\mathbb{R}$ and $f'(x) = 2x$ for all $x \in \mathbb{R}$.

To see this, let $a \in \mathbb{R}$. Then

$$\lim_{y \to a} \frac{f(y) - f(a)}{y-a} = \lim_{y \to a} \frac{y^2 - a^2}{y-a} = \lim_{y \to a} (y+a) = 2a.$$

Hence, $f'(x) = 2x$ for all $x \in \mathbb{R}$.

ii) Define $f : \mathbb{R} \to \mathbb{R}$ by

$$\boxed{f(x) := \begin{cases} x \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}}$$

Then, f is not differentiable at 0, as

$$\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{x\sin\frac{1}{x}}{x} = \lim_{x\to 0} \sin\frac{1}{x} \text{ does not exist.}$$

iii) Define $f:\mathbb{R}\to\mathbb{R}$ by

$$\boxed{f(x) := \begin{cases} x^2\sin\frac{1}{x}, & \text{if } x\neq 0 \\ 0, & \text{if } x=0. \end{cases}}$$

Then, f is differentiable at 0 as

$$\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} x\sin\frac{1}{x} = 0.$$

Note that, $f'(0)=0$.

iv) Define $f:\mathbb{R}\to\mathbb{R}$ by

$$\boxed{f(x) := |x|, \text{ for all } x\in\mathbb{R}}$$

Then, f is not differentiable at 0. (Exercise).

We now state a few properties of the derivative.

THEOREM 5.4 (PROPERTIES OF DERIVATIVE)
Let $-\infty<a<b<\infty$, and let $f,g:(a,b)\to\mathbb{R}$ be differentiable at $x_0\in(a,b)$. Then,

i) $f+g$ is differentiable at x_0, and $\boxed{(f+g)'(x_0) = f'(x_0)+g'(x_0)}$

ii) for all $\alpha\in\mathbb{R}$, αf is differentiable at x_0, and
$$\boxed{(\alpha f)'(x_0) = \alpha f'(x_0).}$$

iii) fg is differentiable at x_0, and
$$\boxed{(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)}$$

iv) if $g(x_0) \neq 0$, f/g is differentiable at x_0, and

$$\boxed{\left(f/g\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}}$$

PROOF: We shall prove iii). Others are left as exercise. Note that, for all $x \in (a,b)$ with $x \neq x_0$,

$$\cancel{\frac{f(x)g(x) - (f'(x_0)g(x_0) + f(x_0)g'(x_0))}{x - x_0} = f(x)g(x) - f(x)g(x_0) + f(x)g(x_0)}$$

$$\frac{f(x)g(x) - f(x^0)g(x^0)}{x - x^0} = f(x)\frac{g(x) - g(x_0)}{x - x_0} + g(x_0)\frac{f(x) - f(x_0)}{x - x_0}$$

Therefore, as f is differentiable (and hence continuous), and g is differentiable, we have that

$$\lim_{x \to x_0} \frac{f(x)g(x) - f(x^0)g(x^0)}{x - x^0} \text{ exists, and}$$

$$(fg)'(x_0) = f(x_0)g'(x_0) + f'(x_0)g(x_0),$$

which proves iii). (Proved)

THEOREM 5.5 (DIFFERENTIATION & COMPOSITION) Let $-\infty < a < b < \infty$, $-\infty < c < d < \infty$, let $f : (a,b) \to \mathbb{R}$, $g : (c,d) \to \mathbb{R}$ with $f((a,b)) \subseteq (c,d)$. Let us suppose that f is differentiable at $x^0 \in (a,b)$ and g is differentiable at $f(x^0)$. Then, $g \circ f : (a,b) \to \mathbb{R}$ is differentiable at x^0 and

$$\boxed{(g \circ f)'(x_0) = g'(f(x^0))\, f'(x^0).}$$

PROOF: Exercise.

Let $-\infty < a < b < \infty$ and let $f : (a, b) \longrightarrow \mathbb{R}$ be differentiable. Then, $f' : (a, b) \longrightarrow \mathbb{R}$ is a well-defined function. If f' is differentiable at $x_0 \in (a, b)$, the derivative of f' at x_0 is called the second derivative of f at x_0, and is denoted by $f''(x_0)$.

Note that, for $f''(x_0)$ to be defined f' needs to be defined on an interval around x_0 i.e. f needs to be differentiable in an open interval around x_0.