__/__/____

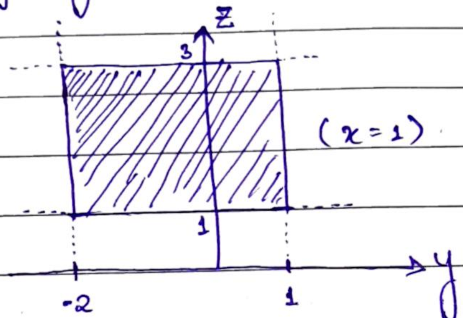
PH1201 MID-SEMESTER EXAMINATION

Name : Priyanshu Mahato

Roll No. : pm21ms002.

Q1. $\vec{v} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$

$S \Rightarrow (x=1)$



$(x=1)$ $\qquad ds = dy\,dz\,(-\hat{i})$

$$-\int_1^3\int_{-2}^1 4xz\,dy\,dz = -\int_1^3\int_{-2}^1 4z\,dy\,dz \quad (\text{as } x=1)$$

$$= -\int_1^3 4z\,[y]_{-2}^1\,dz$$

$$= -\int_1^3 4z\cdot 3\cdot dz = -\left[6z^2\right]_1^3$$

$= 6 - 54 = \underline{-48}$ ans.

(pointing outside the cube or, in $-\hat{i}$ dir^n)

[and 48 pointing into the cube, or in $\hat{i}$ dir^n)

_/_/_

Q2. For the surface on $x=5$.

$$d\vec{S} = dy\,dz\,(\hat{i}).$$

$$\Rightarrow \int_S \vec{v}\cdot d\vec{S} = \int_S (4xz\hat{i} - y^2\hat{j} + yz\hat{k})\cdot(dy\,dz)\hat{i}$$

$$= \int_S 4xz\,dy\,dz = \int_{-2}^{1} 4\times 5\left[\int_1^3 z\,dz\right]dy.$$

$$= 20\int_{-2}^{1}\left[\frac{z^2}{3}\Big|_1^3\right]dy = 20\int_{-2}^{1}\left[\frac{1}{2}(9-1)\right]dy$$

$$= 20\int_{-2}^{1} 4\,dy = 80\,[y]_{-2}^{1} = \underline{240} \text{ ans.}$$

Q3. $S: y=-2$ $\quad d\vec{S} = dx\,dz\,(-\hat{j})$

$$\Rightarrow \int_S \vec{v}\cdot d\vec{S} = -\int_1^3\int_1^5 y^2\,dx\,dz$$

$$= \int_1^3 y^2\,[x]_1^5\,dz = \int_1^3 4y^2\,dz$$

$$= 4y^2\,[z]_1^3 = 8y^2$$

_/_/_

at $y = -2$,

$$\int_S \vec{v} \cdot d\vec{s} = 8(-2)^2 = \underline{\underline{32}}. \text{ ans.}$$

Q4. S: $y = 1$ $\qquad d\vec{s} = dx\,dz\,(\hat{j})$

$$\int_S \vec{v} \cdot d\vec{s} = -\int_1^3 \int_1^5 y^2 \, dx \, dz$$

$$= -\int_1^3 y^2 [x]_1^5 \, dz = -\int_1^3 4y^2 \, dz$$

$$= -4y^2 [z]_1^3 = -4y^2 (2) = -8y^2$$

at $y = 1$,

$$\int_S \vec{v} \cdot d\vec{s} = \underline{\underline{-8}}. \text{ ans.}$$

Q5. $S: z = 1$

$$d\vec{S} = dx\,dy\,(-\hat{k})$$

$$\int_S \vec{v}\cdot d\vec{S} = -\iint yz\,dx\,dy = -\int_{-2}^{1}\int_{1}^{5} yz\,dx\,dy.$$

$$= \int_{-2}^{1} yz\,[x]_1^5\,dy.$$

$$= -\int_{-2}^{1} 4yz\,dy = -4z\left[\frac{y^2}{2}\right]_{-2}^{1}$$

$$= -4z\left(\frac{1}{2} - 2\right] = 4z \times \frac{3}{2}$$

$$= 6z$$

at $z = 1$,

$$\int_S \vec{v}\cdot d\vec{S} = \underline{6}.\ \text{ans}.$$

Q6. $S: z=3$

$d\vec{s} = dx\,dy\,(\hat{k})$

$$\int_S \vec{v}\cdot d\vec{s} = \int_{-2}^{1}\int_{1}^{5} yz\,dx\,dy = \int_{-2}^{1} yz\,[x]_1^5\,dy = \int_{-2}^{1} 4yz\,dy$$

$$= 4z\left[\frac{y^2}{2}\right]_{-2}^{1} = 4z \times \left(\frac{-3}{2}\right) = -6z$$

at $z=3$,

$$\int_S \vec{v}\cdot d\vec{s} = \underline{-18} \text{ ans.}$$

Q7. $\int_V (\vec{\nabla}\cdot\vec{v})\,dv \qquad dv = dx\,dy\,dz$

$$\vec{\nabla}\cdot\vec{v} = 4z - 2y + y = 4z - y$$

$$\Rightarrow \int_V (\vec{\nabla}\cdot\vec{v})\,dv = \int_V (4z-y)\,dx\,dy\,dz$$

$$= \int_{1}^{3}\int_{-2}^{1}\int_{1}^{5} (4z-y)\,dx\,dy\,dz$$

$$= \int_{1}^{3}\int_{-2}^{1} \{ 4z[x]_1^5 - y[x]_1^5 \} \, dy\, dz$$

$$= \int_{1}^{3}\int_{-2}^{1} (16z - 4y)\, dy\, dz = \int_{1}^{3} \{ 16z[y]_{-2}^{1} - 4\left[\frac{y^2}{2}\right]_{-2}^{1} \} dz$$

$$= \int_{1}^{3} (48z + 6)\, dz = 48\left[\frac{z^2}{2}\right]_1^3 + 6[z]_1^3$$

$$= \left[48 \cdot \left(\frac{9}{2} - \frac{1}{2}\right)\right] + 12$$

$$= \left(48 \times \frac{8}{2}\right) + 12.$$

$$= 192 + 12 = \underline{\underline{204}} \text{ ans.}$$

Q8. $||\vec{r} - \vec{r}'|| = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$

Substituting $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$, we get

$$||\vec{r} - \vec{r}'|| = \sqrt{r^2 + r'^2 - 2rr'(\sin\theta\sin\theta'\cos\varphi\cos\varphi' + \sin\theta\sin\theta'\sin\varphi\sin\varphi' + \cos\theta\cos\theta')}$$

$$= \sqrt{r^2 + r'^2 - 2rr'(\sin\theta\sin\theta'\cos(\varphi - \varphi') + \cos\theta\cos\theta')}$$

$$= \sqrt{1^2 + 1^2 - 2\left(\sin\frac{\pi}{4}\sin\frac{3\pi}{4}\cos(0-\pi) + \cos\frac{\pi}{4}\cos\frac{3\pi}{4}\right)}$$

$$= \sqrt{2 - 2\left(\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} \times (-1) + \frac{1}{\sqrt{2}} \times \left(-\frac{1}{\sqrt{2}}\right)\right)}$$

$$= \sqrt{2 - 2\left(-\frac{1}{2} - \frac{1}{2}\right)} = \sqrt{2 - 2(-1)} = \sqrt{4} = \underline{\underline{2}}$$

$=$ __2 units__ ans.

__/__/__

Q4. Applying Stoke's Theorem, $\iint_S \vec{\nabla} \times \vec{v} \cdot d\vec{S} = \oint_C \vec{v} \cdot d\vec{r}$

$$\Rightarrow \int (0\hat{i} + xy\hat{j} + 0\hat{k}) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \boxed{\int xy \cdot dy}$$

$$\Rightarrow \int_0^4 xy \cdot dy$$ in XY plane,

$r \Rightarrow$ path $\Rightarrow y = -2x + 4$

$\Rightarrow x = 2 - \frac{y}{2}$.

$$= \int_0^4 \left(2 - \frac{y}{2}\right) y \cdot dy$$

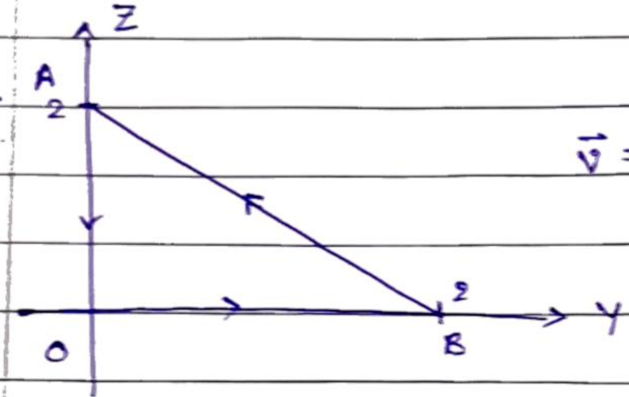
$$= \int_0^4 \left(2y - \frac{y^2}{2}\right) dy = \left[y^2 - \frac{y^3}{6}\right]_0^4 = 16 - \frac{64}{6}$$

$$= 16 - \frac{32}{3} = \boxed{\frac{16}{3}}$$

$\int xy \cdot dy$ is zero (0) in the YZ plane and ZX plane as $xy = 0$ for YZ and ZX planes.

So, $\boxed{\iint_S \vec{\nabla} \times \vec{v} \cdot d\vec{S} = \oint_C \vec{v} \cdot d\vec{r} = \frac{16}{3}}$ ans.

Q10.



$\vec{v} = xy\hat{i} + 2yz\hat{j} + 3zx\hat{k}$

(i) $A \to O$

$d\vec{r} = dz\,\hat{k} \Rightarrow \vec{v}\cdot d\vec{r} = 3xz\,dz = 0 \quad (\because x = 0)$

NOTE : x may be taken to be any other constant as it will not affect the value of $\oint \vec{v}\cdot d\vec{r}$

(ii) $O \to B$

$d\vec{r} = dy\,\hat{j}$

$\int \vec{v}\cdot d\vec{r} = \int 2yz\,dy = 0 \quad (\because z = 0)$

(iii) $B \to A$

$dz = -dy \Rightarrow d\vec{r} = dy\,\hat{j} + dz\,\hat{k}$

$\therefore \vec{v}\cdot d\vec{r} = 2yz\,dy + 3xz\,dz = 2yz\,dy \quad (\because x = 0)$

$$\vec{v} \cdot d\vec{r} = 2y(2-y)\, dy.$$

$$= (4y - 2y^2)\, dy.$$

$$\int \vec{v} \cdot d\vec{r} = \int_2^0 (4y - 2y^2)\, dy.$$

$$= 2\left[y^2\right]_2^0 - \frac{2}{3}\left[y^3\right]_2^0$$

$$= -8 + \frac{16}{3} = \frac{-8}{3}$$

$$\Rightarrow \oint \vec{v} \cdot d\vec{r} = 0 + 0 + \left(-\frac{8}{3}\right) = -\frac{8}{3}$$

$$\therefore \int_S (\vec{\nabla} \times \vec{v}) \cdot d\vec{S} = \oint_C \vec{v} \cdot d\vec{r} = \frac{-8}{3} \quad \text{ans.}$$

$$= -2.67 \quad \text{ans.}$$

Q11. $\rho = 2r\sin\theta\,(\cos\varphi + \sin\varphi)$

$x = r\sin\theta\cos\varphi$

$y = r\sin\theta\sin\varphi$

we get,

$\rho = 2(x+y)$

$\therefore$, at $(1,1,1)$, $\rho = 2(1+1) = \underline{\underline{4}}$ ans.

Q12. Total Charge inside a volume = $\int \rho \, dV$

$$= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} 2(x+y)\, dx\, dy\, dz$$

$$= \int_{-1}^{1}\int_{-1}^{1} \left(2\left[\frac{x^2}{2}\right]_{-1}^{1} + 2y\left[x\right]_{-1}^{1} \right) dy\, dz$$

$$= \int_{-1}^{1}\int_{-1}^{1} 4y\, dy\, dz = \int_{-1}^{1} 4\left[\frac{y^2}{2}\right]_{-1}^{1} dz$$

$$= \int_{-1}^{1} (0)\, dz = \underline{0 \text{ units}} \text{ ans}$$