Trace

Sum of diagonal elements of a (square) matrix. $tr(A) = \sum_{i=1}^{n} a_{ii}$ $A \in M_n(F)$

- ★ $tr(A + B) = tr(A) + tr(B)$
- ★ $tr(A^T) = tr(A)$
- ★ $tr(kA) = k\ tr(A)$
- ★ $tr(AB) = tr(BA)$

Proof : $tr(A + B) = \sum_i (a_{ii} + b_{ii}) = \sum_i a_{ii} + \sum_i b_{ii} = tr(A) + tr(B)$ $\square$

Proof :
$$
\begin{aligned}
tr(AB) &= \textstyle\sum_i [AB]_{ii} \\
&= \textstyle\sum_i \left[\sum_k A_{ik} B_{ki} \right] \\
&= \textstyle\sum_k \left[\sum_i B_{ki} A_{ik} \right] \\
&= \textstyle\sum_k [BA]_{kk} \\
&= tr(BA). \quad \square
\end{aligned}
$$

Determinant

$$n = 1 \quad det(A) = \quad |a_{11}| \quad = a_{11}$$

$$n = 2 \quad det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$- \qquad\qquad +$$

Linear equations:

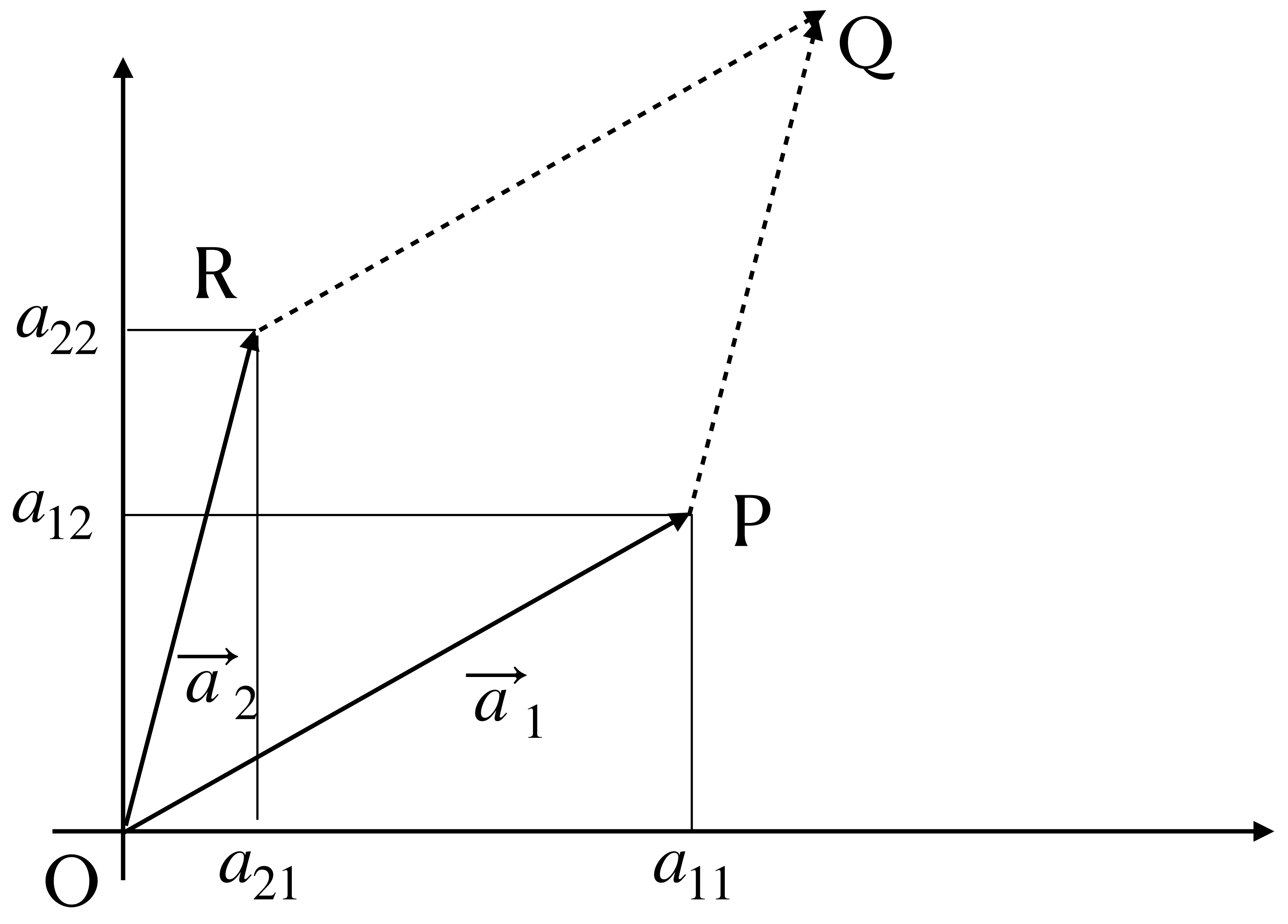
$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

$$x_1 = \frac{b_1a_{22} - b_2a_{12}}{a_{11}a_{22} - a_{12}a_{21}} = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \qquad x_2 = \frac{b_2a_{11} - b_1a_{12}}{a_{11}a_{22} - a_{12}a_{21}} = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{12} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

Solution exists for $det(A) \neq 0$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \implies \begin{array}{l} \vec{a}_1 = a_{11}\hat{i} + a_{12}\hat{j} \\ \vec{a}_2 = a_{21}\hat{i} + a_{22}\hat{j} \end{array}$$

$det(A)$ = (signed) area of the parallelogram OPQR



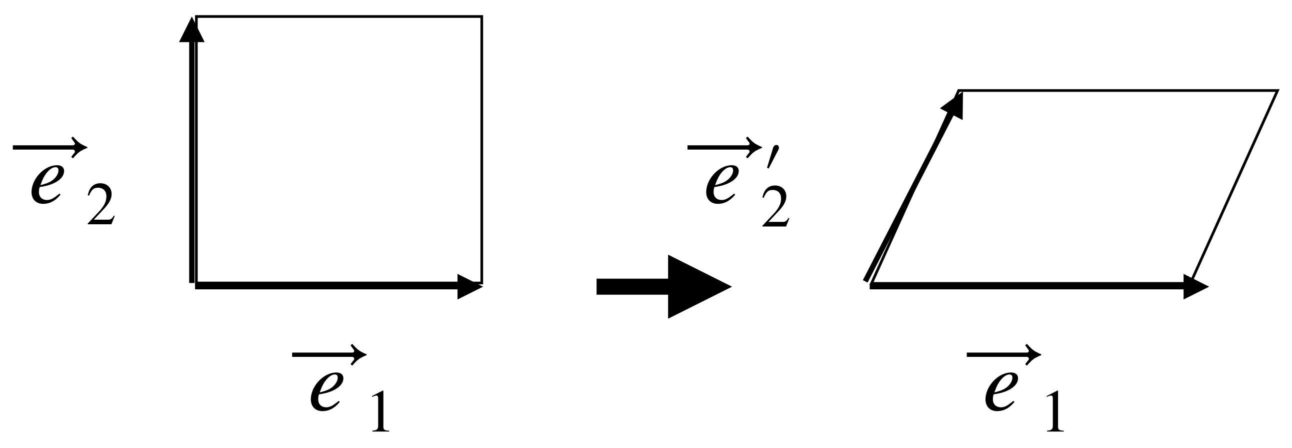
Shear matrix:

$$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \vec{e} = \vec{e}\,'$$

$$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha \\ 1 \end{pmatrix}$$

Area=1 $\quad det(EA) = 1$



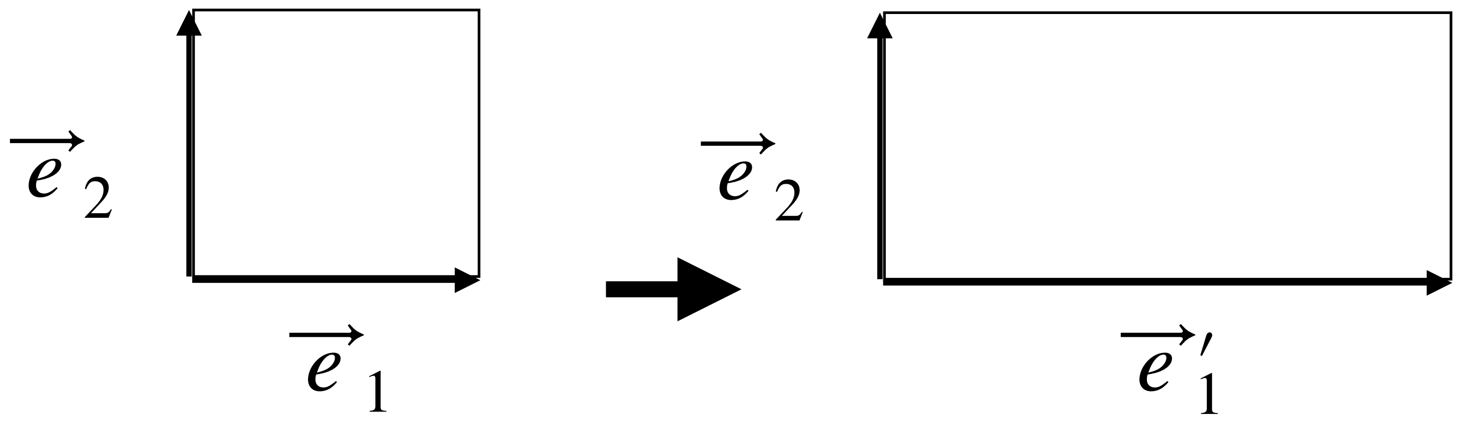
Scaling/dilation matrix:

$$\begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix} \vec{e} = \vec{e}\,'$$

$$\begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \beta \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Area=1 $\quad det(EA) = \beta$



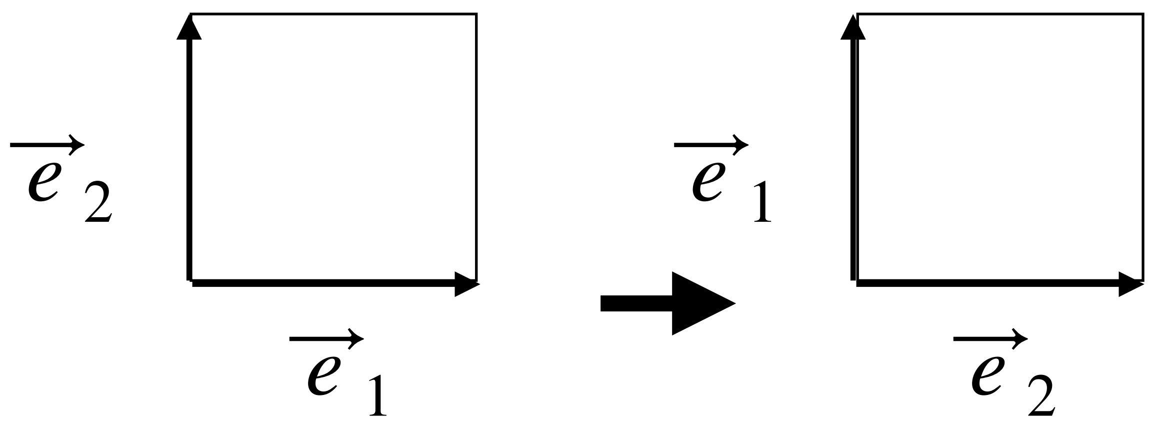
Exchange/reflection matrix:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \vec{e} = \vec{e}\,'$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Area=1 $\quad det(EA) = -1$



$|E\ A| = |E|\ |A|$

Determinants of matrix order 3

$$det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}B_{11} \ + \ a_{12}B_{12} \ + \ a_{13}B_{13}$$

Minors Determinant of a (sub)matrix obtained by deleting i-th row and j-th column of nXn matrix: $[M_{ij}]_{(n-1)\times(n-1)}$

Cofactors $B_{ij} = (-1)^{i+j}|M_{ij}|$

Example:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \implies |M_{23}| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} = -6 \qquad \therefore B_{23} = (-1)^{2+3}|M_{23}| = -1(-6) = 6$$

Laplace expansion: Generalization of determinants

$$|A| = a_{i1}B_{i1} + a_{i2}B_{i2} + \cdots = \sum_{j=1}^{n} a_{ij}B_{ij} \qquad \text{For any row } i$$

$$= a_{1j}B_{1j} + a_{2j}B_{2j} + \cdots = \sum_{i=1}^{n} a_{ij}B_{ij} \qquad \text{For any column } j$$

$\star$ Theorem : $|AB| = |A||B|$

Proof:

Lemma: Any invertible matrix is a product of elementary matrices.

$$E_q E_{q-1} \cdots E_2 E_1 P = \mathbb{I} \implies P^{-1} = E_q E_{q-1} \cdots E_2 E_1 \mathbb{I} \quad \square$$

$$\begin{aligned} \text{Let } \ A &= E_1 E_2 \cdots E_k \\ |AB| &= |E_1 E_2 \cdots E_k B| \\ &= |E_1||E_2 \cdots E_k B| \qquad \because |EX| = |E||X| \\ &= |E_1||E_2| \cdots |E_k||B| \\ &= |E_1 E_2 \cdots E_k||B| \\ &= |A||B| \quad \square \end{aligned}$$

$\star$Theorem: If A and B are row equivalent then $|B|=0$ iff $|A|=0$.

$$E_k \cdots E_2 E_1 A = B \implies |B| = |E_k \cdots E_2 E_1 A| = |E_k \cdots E_2 E_1||A|$$

$$|E_i| \neq 0 \forall i \implies |B| = 0 \quad \text{iff} \quad |A| = 0 \quad \square$$

$\star$Theorem: If two matrices are row(or column) exchanged $|B| = -|A|$.

$$B = E_{ex} A \implies |B| = |E_{ex}||A| = -|A| \qquad \because |E_{ex}| = -1 \qquad \square$$

★Theorem: If A is invertible $\Leftrightarrow |A| \neq 0$.

Proof: A is invertible $\implies E_k \cdots E_2 E_1 A = \mathbb{I}$

$$|E_k \cdots E_2 E_1 A| = |\mathbb{I}| = 1 \neq 0$$
$$|E_k \cdots E_2 E_1||A| \neq 0$$
$$|E_k| \cdots |E_2||E_1||A| \neq 0$$
$$|E_i| \neq 0 \forall i \implies |A| \neq 0 \quad \square$$

★Theorem: If A has a row (or column) of all zeros then $|A|=0$.

Proof:
$$|A| = a_{i1}B_{i1} + a_{i2}B_{i2} + \cdots = \sum_{j=1}^{n} a_{ij}B_{ij} \qquad \text{For any row } i$$
$$= a_{1j}B_{1j} + a_{2j}B_{2j} + \cdots = \sum_{i=1}^{n} a_{ij}B_{ij} \qquad \text{For any column } j \quad \square$$

★If A has two identical rows the $|A|=0$.

$$EA \to \begin{pmatrix} \vdots & & & \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \end{pmatrix} \implies |A| = 0$$

$\star$Determinant of a diagonal matrix = $\prod_{i=1}^{n} d_i$.

$\star$Determinant of an upper triangular matrix = $\prod_{i=1}^{n} d_i$.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix} \implies |A| = a_{11} \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & a_{33} & \cdots & a_{3n} \\ \vdots & & & \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix} \cdots = a_{11}a_{22}\cdots a_{nn} = \prod_{i=1}^{n} d_i$$

$\star$For any elementary matrix = $|E| = |E^T|$.

$\star$Theorem: $|A| = |A^T|$.

$$A \equiv \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \quad \& \quad A^T \equiv \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & & & \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix}$$

Using cofactor expansion along first column of A

$$|A| = a_{11}|A_{11}| + a_{21}|A_{21}| + \cdots + a_{n1}|A_{n1}|$$

Using cofactor expansion along first row of A^T

$$|A^T| = a_{11}|(A^T)_{11}| + a_{21}|(A^T)_{12}| + \cdots + a_{n1}|(A^T)_{1n}|$$

It is easy to see $\quad (A^T)_{ij} = (A_{ji})^T \implies |(A^T)_{ij}| = |(A_{ji})^T| = |A_{ji}| \quad \square$