Vector Space Recap

A vector space consists of vectors and a set of scalars such that the set of vectors are **closed** under vector addition and scalar multiplication.

Let us consider a set of vectors consisting of column vectors:

$$U = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \qquad V = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \qquad U, V \in \mathbb{R}^3$$

$$\implies W = \alpha U + \beta V = \begin{pmatrix} \alpha u_1 + \beta v_1 \\ \alpha u_2 + \beta v_2 \\ \alpha u_3 + \beta v_3 \end{pmatrix} \qquad W \in \mathbb{R}^3$$

The set of vectors $\{u_1, u_2, \cdots, u_n\}$are **linearly independent** if for any scalars $\{c_1, c_2, \cdots, c_n\}$ the equation

$$c_1u_1 + c_2u_2 + \cdots + c_nu_n = 0 \implies c_1 = c_2 = \cdots = c_n = 0$$

Example:

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad u_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$c_1u_1 + c_2u_2 + c_3u_3 = 0 \implies c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies c_1 = c_2 = c_3 = 0$$

Given a set of vectors one can generate a vector space by forming linear combinations of that set of vectors.

$\Longrightarrow$ The set of vectors **span** the vector space.

$$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}2\\3\\0\end{pmatrix} \rightarrow$$ Spans the vector space of all 3x1 matrices with zero in the third row.

The smallest set of vectors needed to span the vector space form the **basis**.

Three possible basis: $\left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right); \ \left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}2\\3\\0\end{pmatrix}\right); \ \left(\begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}2\\3\\0\end{pmatrix}\right)$

Preferred basis since this forms an orthonormal set.

The number of vectors in a basis gives the **dimension** of the vector space.

The set of vectors $\{u_1, u_2, \cdots, u_n\} \in \mathbb{V}$ are said to span $\mathbb{V}$ or form a spanning set of $\mathbb{V}$ if every $v \in \mathbb{V}$ can be expressed as a linear combination

$$v = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n$$

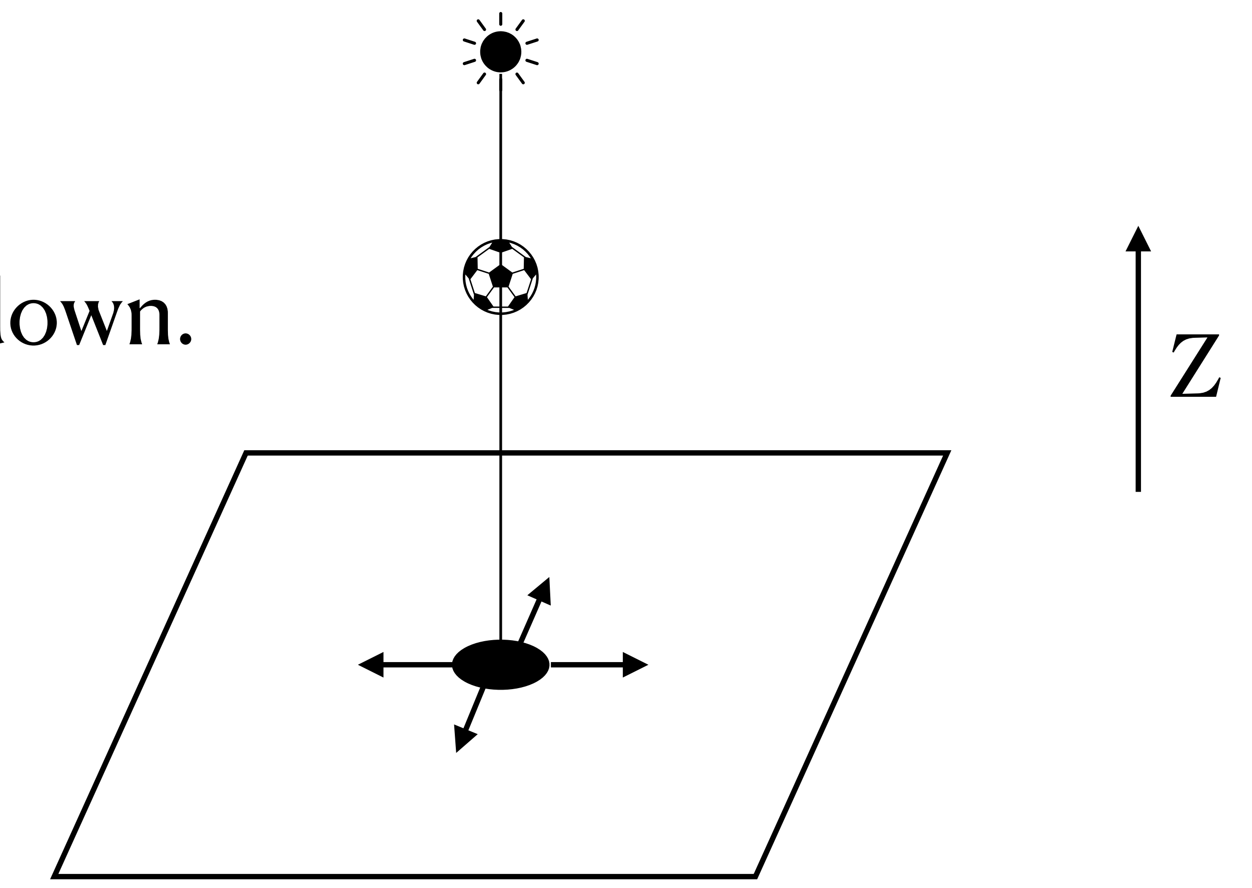
The **null space** of a matrix A is the vector space of all column vectors x satisfying A x = 0 and is denoted as **Null(A)**.

$$x, y \in Null(A) \implies A(x+y) = Ax + Ay = 0 + 0 = 0$$

$$\implies A(\alpha x) = \alpha A x = 0$$

Null(A) is closed under vector addition and scalar multiplication.

Physical interpretation of Null Space:
Z is the null space as the projection does not move though the ball moves up or down.



Let the position of the ball be $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ and the projection $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

The position of the shadow: $Av = \begin{pmatrix} v_1 \\ v_2 \\ 0 \end{pmatrix}$

Finding Null(A): $Ax = 0 \implies \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Null space is spanned by Z axis.