Module 2: Number Systems

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Email: pm21ms002@iiserkol.ac.in.These are my personal notes on Number Systems. We will consider $\mathbb{N}$ (Natural Numbers), $\mathbb{Z}$ (Integers), and $\mathbb{Q}$ (rational Numbers), but not $\mathbb{R}$ (Real Numbers).

§1 Natural Numbers

The natural numbers are $1, 2, 3, 4, \ldots$. The set of all natural numbers is denoted by $\mathbb{N}$.

Definition 1.1. We assume familiarity with the algebraic operations of addition and multiplication on the set $\mathbb{N}$ and also with the linear order relation $<$ on $\mathbb{N}$ defined by "$a < b$ if $a, b \in \mathbb{N}$ and a is less than b".

We discuss the following fundamental properties of the set $\mathbb{N}$.

1. Well Ordering Property
2. Principle of Induction

§1.1 Well Ordering Property

Definition 1.2. Every non-empty subset of $\mathbb{N}$ has a least element.

This means that if S is a non-empty subset of $\mathbb{N}$, then there is an element m in S such that $m \leq s$ for all $s \in S$.
In particular, $\mathbb{N}$ itself has the least element 1.

Proof. Let S be a non-empty subset of $\mathbb{N}$. Let k be an element of S. Then k is a natural number.
We define a subset T by $T = \{x \in S : x \leq k\}$. The T is a non-empty subset of $\{1, 2, 3, \ldots, k\}$.
Clearly, T is a finite subset of $\mathbb{N}$ and therefore it has a least element, say m. Then $1 \leq m \leq k$.
We now show that m is the least element of S. Let s be any element of S.
If $s > k$, then the inequality $m \leq k$ implies $m < s$.
If $s \leq k$, the $s \in T$; and m being the least element of T, we have $m \leq s$.
Thus m is the least element of S. □

§1.2 Principle of Induction

Definition 1.3. Let S be a subset of $\mathbb{N}$ such that,

1. $1 \in S$ and,
2. if $k \in S$, then $k + 1 \in S$.

Then $S = \mathbb{N}$

Proof. Let $T = \mathbb{N} - S$. We prove that $T = \phi$.
Let T be non-empty. then by the *Well Ordering Property* of $\mathbb{N}$, the non-empty subset T has a least element, say m.

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