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MA1202 END-SEMESTER EXAMINATION

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Q3. $y'' - y' - 6y = 12x - e^{x}$

Consider a linear eq^n with constant coefficients,

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = f(x)$$

Composing the characteristic equation,

$$a_0 \lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1} \lambda' + a_n = 0 :$$

$$\lambda^2 - \lambda - 6 = 0 \Rightarrow (\lambda - 3)(\lambda + 2) = 0$$

$$\lambda - 3 \rightarrow \lambda_1 = 3 \rightarrow k = 1 \rightarrow \tau : Ce^{3x}$$

$$\lambda + 2 \rightarrow \lambda_2 = -2 \rightarrow k = 1 \rightarrow \tau : \frac{C_1}{e^{2x}}$$

where, $\lambda_1, \lambda_2, \cdots, \lambda_n$ are roots, k is the multiplicity of the root and τ is the summand of the root.

$\therefore$ General Solution : $\bar{y} = Ce^{3x} + \dfrac{C_1}{e^{2x}}$ — ①

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Now, searching for particular solutions,

Particular solⁿ for the right side,

$f_1 + f_2 + f_3 + \cdots + f_p = 12x - e^x$

equal to the sum of particular solutions for the RHS,

$f_1, f_2, f_3, \cdots, f_p = 12x, -e^x$

For the right side:

$$e^{\alpha x}(P_m(x)\cos\beta x + Q_m(x)\sin\beta x)$$

A particular solⁿ is sought in the form

$$y_i = x^s e^{\alpha x}(R_m(x)\cos\beta x + T_m(x)\sin\beta x)$$

where $s=0$, if $\alpha+\beta i$ is not a root of the characteristic equation and $s=k$ if roots are $(\lambda_1, \lambda_2, \cdots, \lambda_n)$.

Particular solution for $12x$;

$\alpha + \beta \hat{i} = 0 \rightarrow s = 0$

$y_0 = Ax + B$

$\Rightarrow y_0' = A$

$y_0'' = 0$

Substituting in the original eqⁿ:

$-6Ax - 6B - A = 12x$

Finding coefficients.

$$\left.\begin{aligned} -6A &= 12 \\ -6B - A &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} A &= -2 \\ B &= 1/3 \end{aligned}$$

$\Rightarrow y_0 = \frac{1}{3} - 2x$ — (ii)

Particular solⁿ for $-e^x$:

$\alpha + \beta i = 1 \rightarrow s = 0$

$y_1 = Ae^x$

$\Rightarrow y_1' = Ae^x$
$y_1'' = Ae^x$

Substituting in original eqⁿ:

$-6Ae^x = -e^x$

Finding coefficients,

$-6A = -1 \Rightarrow A = \frac{1}{6}$

Substituting,

$y_1 = \frac{e^x}{6}$ — (iii)

$\therefore$ Solution: y = General Solⁿ + Particular Solⁿ
$= \bar{y} + y_0 + y_1$

$\therefore$ $\boxed{y = Ce^{3x} + \frac{e^x}{6} + \frac{C_1}{e^{2x}} - 2x + \frac{1}{3}}$

$$y' = 3Ce^{3x} + \frac{e^x}{6} - \frac{2C_1}{e^{2x}} - 2$$

at $x=0, y=1, y'=-2$,

$\Rightarrow 1 = C_1 + C + \frac{1}{2}$; $-2 = -2C_1 + 3C - \frac{11}{6}$

$\Rightarrow C = \frac{1}{6}, \; C_1 = \frac{1}{3}$

$$\therefore y(x) = \frac{e^{3x}}{6} + \frac{e^x}{6} + \frac{1}{3e^{2x}} - 2x + \frac{1}{3}$$

$\therefore$ $\boxed{y(x) = \frac{((-12x + e^{3x} + e^x + 2)e^{2x} + 2)e^{-2x}}{6}}$

ans.

Substituting $x = 0.87$ in the above solution,

$\boxed{y(x=0.87) = 1.32}$ ans.

Q4. $\det(A) = \frac{1}{3}$ $[\det(KA) = K^n \det(A)]$

$A_{n \times n}$

a) $\det(3A) = 3^4 \det(A)$

$= 3^4 \times \frac{1}{3} = 27$.

b) $\det(A^{-1}) = \frac{1}{\det(A)} = 3$

c) $\det(AA^{-1}) = \det(A)\det(A^{-1})$

$= \frac{1}{3} \times 3 = 1$

d) $\det(-A) = (-1)^4 \det(A)$

$= 1 \times \frac{1}{3} = \frac{1}{3}$

e) $\det(A^2) = \det(AA) = \det(A)\det(A) = \frac{1}{3} \times \frac{1}{3}$

$= \frac{1}{9}$

Q5.

$$\begin{bmatrix} 2 & 2 & 2 & 2 & \cdots & 2 \\ 2 & 2 & 2 & 2 & \cdots & 2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & 2 & 2 & & 2 \end{bmatrix}_{26 \times 26}$$

Rank of the matrix = 1 , $\therefore$ It will have 0 as an eigenvalue.

The matrix will have a non-zero eigenvalue.

Trace of the matrix $= 2+2+2+\cdots+2$ (26 times)

$= 2 \times 26$

$= \underline{\underline{52}}$.

Trace of matrix $A = \sum_{i=1}^{26} \lambda_i$ where λ_i is an eigenvalue.

$2 \times 26 = 0 \times 25 + \lambda$

$\therefore \boxed{\lambda = 52}$ ans.

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Q2. $f(z) = \oint_{|r|=3} \frac{3r^2+7r+1}{r-z} dr$

Using Cauchy Integral,

$\frac{1}{2\pi i} \oint_C \frac{f(z)dz}{z-z_0} = f(z_0)$

$\therefore f(z) = \oint_{|r|=3} (3r^2+7r+1) \cdot \frac{dr}{r-z}$

where, $r_0 = z$

$f(r) = 3r^2 + 7r + 1$

$\Rightarrow f(z) = 3z^2 + 7z + 1$

$\Rightarrow f(z) = f(r_0) \cdot 2\pi i$

$= f(z) \cdot 2\pi i$

$= (3z^2 + 7z + 1) \cdot 2\pi i$

$\Rightarrow f'(z) = 2\pi i \, (6z+7)$

$\Rightarrow f'(1+i) = 2\pi i \, [6(1+i)+7]$

$= 2\pi i \, [13+6i]$

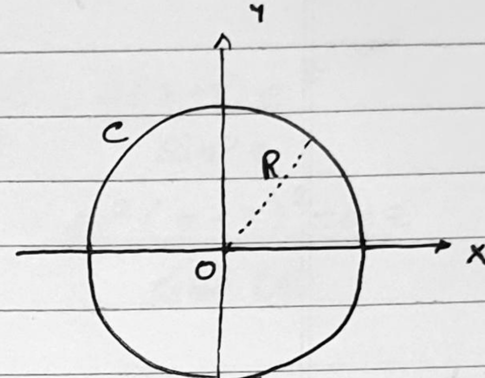
$= 26\pi i + 12\pi (i)^2$

$= -12\pi + 26\pi i$

$$\Rightarrow |f'(1+i)| = \sqrt{(-12\pi)^2 + (26\pi)^2} = \underline{89.96} \text{ ans.}$$

Q1. <u>We have 2nd order poles at:</u>

$z = 0, \pi, -\pi$



Now, we know that,

Residue $(f(z))$ at $z\,(=a)$

$$= \frac{1}{(m-1)!} \lim_{z \to a} \left[\frac{d^{m-1}}{dz^{m-1}} (f(z))(z-a)^m \right]$$

where, 'm' is the order of the pole.

<u>Evaluating the expression,</u> $(m=2)$

$$\frac{d^{m-1}}{dz^{m-1}} \left[f(z)(z-a)^m \right] = \frac{d}{dz} \left[f(z)(z-a)^2 \right]$$

$$= \frac{d}{dz} \left[\frac{e^z (z-a)^2}{\sin^2 z} \right] = \frac{e^z (z-a)^2}{\sin^2 z} + \frac{2(z-a)e^z}{\sin^2 z} - \frac{2e^z (z-a)^2 \cos z}{\sin^3 z}$$

- For pole at $z=0$:- $\{a=0\}$

$$\text{Residue}(f(z)) = \lim_{z\to 0}\left[\frac{e^z (z)^2}{\sin^2 z} + \frac{2ze^z}{\sin^2 z} - \frac{2e^z z^2 \cos z}{\sin^3 z}\right]$$

$$= 1 + \lim_{z\to 0} \frac{2ze^z}{\sin z}\left(\frac{\sin z - z\cos z}{\sin^2 z}\right)$$

$$= 1 + \lim_{z\to 0} \frac{2ze^z}{\sin z}\left(\frac{\sin z - z\cos z}{\sin^2 z}\right)$$

$$= 1 + 0 = \underline{\underline{1}}.$$

- For pole at $z=\pi$:- $\{a=\pi\}$

$$\text{Residue}(f(z)) = \lim_{z=\pi} \frac{e^z (z-\pi)^2}{\sin^2 z} + \frac{2(z-\pi)e^z}{\sin^2 z} - \frac{2e^z (z-\pi)^2 \cos z}{\sin^3 z}$$

Let $z-\pi = p$, then,

$$= \lim_{p\to 0} \frac{e^{\pi+p}\, p^2}{\sin^2(\pi+p)} + \frac{2pe^{(\pi+p)}}{\sin^2(\pi+p)} - \frac{2e^{(\pi+p)} p^2 \cos(\pi+p)}{\sin^3(\pi+p)}$$

$$= \lim_{p\to 0} e^{\pi}\left(\frac{p^2 e^p}{\sin^2 p} + \frac{2pe^p}{\sin^2 p} - \frac{2p^2 e^p \cos p}{\sin^3 p}\right)$$

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$$= e^{\pi} \cdot (1+0+0) = \underline{e^{\pi}}.$$

- For pole at $z = -\pi$:- $\{a = -\pi\}$

$$\text{Residue}\,(f(z)) = \lim_{z \to -\pi} \frac{e^{z}(z+\pi)^2}{\sin^2 z} + \frac{2(z+\pi)e^{z}}{\sin^2 z} - \frac{2e^{z}(z+\pi)^2 \cos z}{\sin^3(z)}$$

Using similar calculations as above, we get :

$$\text{Residue}\,(f(z)) = e^{-\pi} \cdot (1+0-0) = \underline{e^{-\pi}}.$$

Now, using Cauchy's Integral Theorem, we get :

$$\oint_C \frac{e^{z}}{\sin^2 z}\, dz = 2\pi i \sum \text{Residue}\,(f(z))$$

$$= 2\pi i \left[1 + e^{\pi} + e^{-\pi}\right]$$

$$\Rightarrow \boxed{\frac{1}{2\pi i} \oint_C \frac{e^{z}}{\sin^2 z}\, dz = (1 + e^{\pi} + e^{-\pi})}$$ ans.