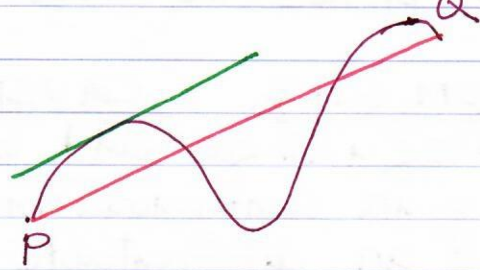
CHAPTER 6: MEAN VALUE THEOREMS

In this chapter, we shall study a family of theorems known as mean value theorems, which roughly state that, for a given planar arc between two points, there is atleast one point on the arc at which tangent to the arc is parallel to the secant through its end points.



While these theorems, in some form, were known for many years, it was Cauchy who proved the mean value in its modern form.

SECTION 6.1 FIRST-ORDER MEAN VALUE THEOREMS

We begin with the following theorem, the proof of which is beyond the scope of this course.

THEOREM 6.1.1 (ROLLE'S THEOREM)
Let $-\infty < a < b < \infty$ and let $f: [a,b] \longrightarrow \mathbb{R}$ be such that

i) f is continuous in $[a,b]$,
ii) f is differentiable in (a,b), and
iii) $f(a) = f(b)$.

Then, there exists $c \in (a,b)$ such that $f'(c) = 0$.

THEOREM 6.1.2 (LAGRANGE'S MEAN VALUE THEOREM)
Let $-\infty < a < b < \infty$ and let $f: [a,b] \rightarrow \mathbb{R}$ be continuous, and let f be differentiable in (a,b). Then, there exists $c \in (a,b)$ such that

$$\boxed{f(b) - f(a) = f'(c)(b-a)}$$

PROOF: Define $\phi: [a,b] \longrightarrow \mathbb{R}$ by

$$\phi(x) := x\big(f(b) - f(a)\big) - f(x)(b-a), \text{ for all } x \in [a,b]$$

Then, ϕ is continuous in $[a,b]$, ϕ is differentiable in (a,b) and $\phi(a) = af(b) - f(a)b = \phi(b)$. Hence, using Rolle's theorem, 6.1.1 we find $c \in (a,b)$ such that

$$\phi'(c) = 0$$
$$\Rightarrow f(b) - f(a) = f'(c)(b-a),$$

which proves the theorem. <u>(proved)</u>

THEOREM 6.1.3 (CAUCHY MEAN VALUE THEOREM)
Let $-\infty < a < b < \infty$ and let $f, g : [a,b] \to \mathbb{R}$ be such that
i) f, g are continuous in $[a,b]$, and
ii) f, g are differentiable in (a,b).

Then, for some $c \in (a, b)$

$$\boxed{(g(b) - g(a))\, f'(c) = (f(b) - f(a))\, g'(c).}$$

PROOF: Define $\phi : [a,b] \to \mathbb{R}$ by

$$\phi(x) := (g(b) - g(a)) f(x) - (f(b) - f(a)) g(x), \text{ for all } x \in [a,b].$$

Then, ϕ is continuous in $[a,b]$, differentiable in (a,b) and

$$\phi(a) = g(b) f(a) - f(b) g(a) = \phi(b).$$

Hence, using Theorem 6.1.1, we find $c \in (a,b)$ such that

$$0 = \phi'(c) = (g(b) - g(a)) f'(c) - (f(b) - f(a)) g'(c),$$

which proves the theorem. <u>(proved)</u>

We now look at a few applications of the mean value theorems.

THEOREM 6.1.4
Let $-\infty < a < b < \infty$ and let $f : (a,b) \to \mathbb{R}$ be differentiable. Then, f is constant if and only if $f'(x) = 0$, for all $x \in (a,b)$.

PROOF: If f is constant, clearly $f'(x) = 0$ for all $x \in (a,b)$. Let us prove the converse. Let $f : (a,b) \to \mathbb{R}$ be differentiable with $f'(x)=0$, for all $x\in(a,b)$. We shall prove that f is constant. Let $t,s\in(a,b)$ with $t<s$. Then, f is continuous in $[t,s]$ and differentiable in (t,s). Hence, using Theorem 6.1.2, we find $\theta\in(t,s)$ such that

$$f(s)-f(t)=(s-t)f'(\theta)=0$$
$$\Rightarrow f(s)=f(t).$$

Hence, f is constant. This proves the theorem. (Proved)

THEOREM 6.1.5

Let $-\infty\leq a<b<\infty$ and let $f:[a,b]\to\mathbb{R}$ be differentiable satisfying $f'(x)>0$, for all $x\in(a,b)$. Then, f is strictly increasing i.e. $f(x)>f(y)$, for all $x,y\in[a,b]$ with $x>y$.

PROOF: Exercise

We now use mean value theorems to prove a few inequalities.

EXAMPLE 6.1.6 (APPLICATIONS TO INEQUALITIES)

i) For all $x\in(0,\frac{\pi}{2})$,

$$\boxed{\frac{2}{\pi}x<\sin x<x<\tan x}$$

Define $f,g,h:(0,\frac{\pi}{2})\to\mathbb{R}$ by

$$f(x):=\tan x-x,\quad g(x):=x-\sin x,\quad h(x):=\frac{\sin x}{x},\ \text{for all } x\in(0,\tfrac{\pi}{2}).$$

Note that, $f'(x)=\sec^2 x-1>0$, $g'(x)=1-\cos x>0$, for all $x\in(0,\frac{\pi}{2})$.

Therefore, f,g are ~~i~~strictly increasing functions on $(0,\frac{\pi}{2})$.

Hence,
$$f(x)>f(y),\quad\text{and}\quad g(x)>g(y),\ \text{for all } x,y\in(0,\tfrac{\pi}{2}),\ x>y.$$

Therefore, $f(x)>\lim_{y\to0+}f(y)$, $\quad g(x)>\lim_{y\to0}g(y)$, for all $x\in(0,\frac{\pi}{2})$.

$$\Rightarrow x-\tan x>0,\quad x-\sin x>0,\ \text{for all } x\in(0,\tfrac{\pi}{2}).$$

ie. $\cancel{x}$ $\tan x > x > \sin x$, for all $x \in (0, \frac{\pi}{2})$.

Now, $h'(x) = \dfrac{x \cos x - \sin x}{x^2} = \cos x \cdot \left(\dfrac{x - \tan x}{x^2} \right) < 0$,
for all $x \in (0, \frac{\pi}{2})$.

Hence, $h(x)$ is a strictly decreasing function. Therefore,

$$h(x) > h(y), \text{ for all } x, y \in (0, \tfrac{\pi}{2}),\ x < y.$$

Hence, $h(x) > \lim_{y \to \pi/2} h(y) = \frac{2}{\pi}$, for all $x \in (0, \frac{\pi}{2})$.

$\Rightarrow \dfrac{\sin x}{x} > \dfrac{2}{\pi} \Rightarrow \sin x > \dfrac{2}{\pi} x$, for all $x \in (0, \frac{\pi}{2})$.

ii) ~~For~~ all $x \in (0, \infty)$,

$$\boxed{x > \ln(1+x) > \frac{x}{1+x}}$$

Define $f, g : [0, \infty) \to \mathbb{R}$ by

$f(x) := x - \ln(1+x)$, $g(x) := \ln(1+x) - \dfrac{x}{1+x}$,
for all $x \in (0, \infty)$.

~~Define~~ Then,

$f'(x) = 1 - \dfrac{1}{1+x} = \dfrac{x}{1+x} > 0$, for all $x \in (0, \infty)$

$g'(x) = \dfrac{1}{1+x} - \dfrac{1}{1+x} + \dfrac{x}{(1+x)^2} = \dfrac{x}{1+x} > 0$, for all $x \in (0, \infty)$

Therefore, using Theorem 6.1.5,

$f(x) > f(0)$, and $g(x) > g(0)$, for all $x \in (0, \infty)$

This implies that

$x > \ln(1+x)$ and $\ln(1+x) > \dfrac{x}{1+x}$, for all $x \in (0, \infty)$.

THEOREM 6.1.7 (L'HÔPITAL RULE)

Let $a<c<b$ and let $f,g: (a,b) \to \mathbb{R}$ be continuous, and let f,g be differentiable in $(a,b) \setminus \{c\}$. Let us suppose that

i) $f(c) = g(c) = 0$.
ii) $g(x) \neq 0$, for all $x \in (a,b)$, $x \neq c$
iii) $g'(x) \neq 0$, for all $x \in (a,b)$, $x \neq c$

While the theorem carries L'Hôpital's name, it is believed to have been proved by Bernoulli

If $\lim_{x \to c} \frac{f'(x)}{g'(x)}$ exists, then,

$\lim_{x \to c} \frac{f(x)}{g(x)}$ exists and $\boxed{\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}}$.

PROOF: Let us define $L \in \mathbb{R}$ by

$$L := \lim_{x \to c} \frac{f'(x)}{g'(x)}.$$

We shall prove that $\lim_{x \to c} \frac{f(x)}{g(x)} = L$. To prove this, let $\epsilon > 0$ be given.

We find a $\delta > 0$ such that, for all $x \in (a,b)$,

$$0 < |x-c| < \delta \Rightarrow \left| \frac{f'(x)}{g'(x)} - L \right| < \epsilon. \quad (*)$$

We claim that, for all $x \in (a,b)$,

$$0 < |x-c| < \delta \Rightarrow \left| \frac{f(x)}{g(x)} - L \right| < \epsilon. \quad (*)(*)$$

We consider two cases.

CASE 1. $c < x < c+\delta$

Using Theorem 6.1.3, we find $c_1 \in (c,x) \subset (c, c+\delta)$ such that, using $(*)$.

$$\left|\frac{f(x)}{g(x)}-L\right| = \left|\frac{f(x)-f(c)}{g(x)-g(c)}-L\right| = \left|\frac{f'(c_1)}{g'(c_1)}-L\right| < \epsilon$$

Again, using Theorem 6.1.3 and (*), we find $c_2 \in (c-\delta, c)$ such that,

$$\left|\frac{f(x)}{g(x)}-L\right| = \left|\frac{f(x)-f(c)}{g(x)-g(c)}-L\right| = \left|\frac{f'(c_2)}{g'(c_2)}-L\right| < \epsilon.$$

Therefore, for all $x \in (a,b)$ with

$$0 < |x-c| < \delta \Rightarrow \left|\frac{f(x)}{g(x)}-L\right| < \epsilon.$$

Hence, $\displaystyle\lim_{x\to c}\frac{f(x)}{g(x)} = L.$ (Proved)

EXAMPLE 6.1.8

i) $\displaystyle\lim_{x\to 0}\frac{e^x-1}{x^2+x} = \lim_{x\to 0}\frac{e^x}{2x+1} = 1.$

ii) $\displaystyle\lim_{x\to 0}\left(\frac{\sin x - x}{x\sin x}\right) = \lim_{x\to 0}\left(\frac{\cos x - 1}{\sin x + x\cos x}\right) = \lim_{x\to 0}\frac{-\sin x}{\cos x - x\sin x + \cos x}$

$= 0$

SECTION 6.2 SECOND-ORDER MEAN-VALUE THEOREM

We conclude the chapter with a ~~theorem~~ mean value theorem involving the second order derivatives. This theorem will have applications in finding maxima or minima of a function.

THEOREM 6.2.1 (TAYLOR'S THEOREM OF SECOND ORDER).

Let $a < b < \infty$ and let $f : [a,b] \to \mathbb{R}$ be such that

i) f, f' are continuous in $[a,b]$.
ii) f' is differentiable in (a,b).

Then, there exists a $c \in (a,b)$ such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{1}{2} f''(c)(b-a)^2$$

PROOF: Let us define $\varphi : [a,b] \longrightarrow \mathbb{R}$ by

$$\varphi(x) := f(b) - f(x) - f'(x)(b-x) - A(b-x)^2, \text{ for all } x \in [a,b],$$

where $A \in \mathbb{R}$ will be chosen later. Clearly φ is continuous in $[a,b]$, and differentiable in (a,b).

We now choose $\boxed{A \in \mathbb{R} \text{ such that } \varphi(a) = \varphi(b).}$
Therefore,

$$f(b) - f(a) - f'(a)(b-a) - A(b-a)^2 = 0.$$

$$\Rightarrow \quad A = \frac{1}{(b-a)^2}\left[f(b) - f(a) - f'(a)(b-a)\right].$$

Using Theorem 6.1.1, we find $c \in (a,b)$ such that

$$0 = \varphi'(c) = -\cancel{f'(c)} - f''(c)(b-c) + \cancel{f'(c)} + 2A(b-c)$$

$$\Rightarrow \quad A = \frac{1}{2} f''(c).$$

Hence, we have

$$f(b) - f(a) - f'(a)(b-a) = \frac{1}{2} f''(c)(b-a)^2$$

$$\Rightarrow f(b) = f(a) + f'(a)(b-a) + \frac{1}{2} f''(c)(b-a)^2,$$

which proves the theorem. (Proved)