PH1101 ASSIGNMENT - 6

NAME : Priyanshu Mahato

ROLL NO. : pm21ms002

Q1. Determining the coefficients of a series from Fourier's formulae.

Let the periodic function $f(x)$ with period 2π be such that it may be represented as a trigonometric series convergent to a given function in the interval $(-\pi, \pi)$; i.e., that it is the sum of this series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{— (I)}$$

Suppose that the integral of the function on the LHS of the equation is equal to the sum of the integrals of the terms of the series (I). This will be the case, for example, if we assume that the numerical series made up of the coefficients of the given trigonometric series converges absolutely; i.e., that the following positive number series converges:

$$\left|\frac{a_0}{2}\right| + |a_1| + |b_1| + |a_2| + |b_2| + \cdots + |a_n| + |b_n| + \cdots \quad \text{— (II)}$$

Then series (I) is dominated and, consequently, it may be integrated termwise in the interval from $-\pi$ to π. Let us take advantage of this for computing the coefficient 'a_0'.

Integrate both sides of ① from $-\pi$ to π :

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} \frac{a_0}{2}\,dx + \sum_{n=1}^{\infty}\left(\int_{-\pi}^{\pi} a_n \cos nx\,dx + \int_{-\pi}^{\pi} b_n \sin nx\,dx\right)$$

Evaluate separately each integral on the RHS :

$$\int_{-\pi}^{\pi} \frac{a_0}{2}\,dx = \pi a_0$$

$$\Rightarrow \int_{-\pi}^{\pi} a_n \cos nx\,dx = a_n \int_{-\pi}^{\pi} \cos nx\,dx = \left.\frac{a_n \sin nx}{n}\right|_{-\pi}^{\pi} = 0$$

$$\Rightarrow \int_{-\pi}^{\pi} b_n \sin nx\,dx = b_n \int_{-\pi}^{\pi} \sin nx\,dx = \left.-b_n \frac{\cos nx}{n}\right|_{-\pi}^{\pi} = 0$$

Consequently,

$$\int_{-\pi}^{\pi} f(x)\,dx = \pi a_0$$

whence,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

To calculate the other coefficients of the series, we shall need certain definite integrals, which we will consider first.

If n and k are integers, then we have the following equations:

if $n \neq k$ then,

$$\left.\begin{aligned}
&\int_{-\pi}^{\pi} \cos nx \cos kx \, dx = 0 \\
&\int_{-\pi}^{\pi} \cos nx \sin kx \, dx = 0 \\
&\int_{-\pi}^{\pi} \sin nx \sin kx \, dx = 0
\end{aligned}\right\} \text{ — } (\mathrm{I})$$

but if $n = k$, then,

$$\left.\begin{aligned}
&\int_{-\pi}^{\pi} \cos^2 kx \, dx = \pi \\
&\int_{-\pi}^{\pi} \sin kx \cos kx \, dx = 0 \\
&\int_{-\pi}^{\pi} \sin^2 kx \, dx = \pi
\end{aligned}\right\} \text{ — } (\mathrm{II})$$

To take an example, evaluate the first integral of group (I). Since,

$$\cos nx \cos kx = \frac{1}{2}\left[\cos(n+k)x + \cos(n-k)x\right]$$

it follows that,

$$\int_{-\pi}^{\pi} \cos nx \cos kx \, dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos(n+k)x \, dx + \frac{1}{2}\int_{-\pi}^{\pi} \cos(n-k)x \, dx = 0$$

The other formulae of (I) are obtained in similar fashion.
The integrals of group (II) are computed directly.

Now, we can compute the coefficients a_k and b_k of series (1). To find the coefficient a_k for some definite value $k \neq 0$, multiply both sides of (1) by $\cos kx$:

$$f(x)\cos kx = \frac{a_0}{2}\cos kx + \sum_{n=1}^{\infty}\left(a_n \cos nx \cos kx + b_n \sin nx \cos kx\right) - (1')$$

The resulting series on the right is dominated, since its terms do not exceed (in absolute value) the terms of the convergent positive series (II). We can therefore integrate it termwise on any interval. Integrate (1') from $-\pi$ to π:

$$\int_{-\pi}^{\pi} f(x)\cos kx \, dx = \frac{a_0}{2}\int_{-\pi}^{\pi} \cos kx \, dx$$

$$+\sum_{n=1}^{\infty}\left(a_n \int_{-\pi}^{\pi} \cos nx \cos kx \, dx + b_n \int_{-\pi}^{\pi} \sin nx \cos kx \, dx\right)$$

Taking into account formulae (II') and (I), we see that all the integrals on the right are equal to zero (0), with the exception of the integral with coefficient a_k. Hence,

$$\int_{-\pi}^{\pi} f(x)\cos kx\, dx = a_k \int_{-\pi}^{\pi} \cos^2 kx\, dx = a_k \pi .$$

whence,

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\, dx \quad \text{— (III)}$$

Multiplying both sides of (1) by $\sin kx$ and again integrating from $-\pi$ to π, we find,

$$\int_{-\pi}^{\pi} f(x)\sin kx\, dx = b_k \int_{-\pi}^{\pi} \sin^2 kx\, dx = b_k \pi$$

whence,

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\, dx \quad \text{— (IV)}$$

and, a_0 was found to be,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx \quad \text{— (V)}$$

The coefficients determined from formulae (III), (IV) and (V), are called Fourier Coefficients of the function $f(x)$, and the trigonometric series of the form,

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

with such coefficients is called a Fourier series of the function $f(x)$.

Q2. $r = x + vt$; $s = x - vt$

Change of variables in $\dfrac{\partial^2 F}{\partial x^2} - \dfrac{1}{v^2}\dfrac{\partial^2 F}{\partial t^2} = 0$

$dr = dx + v\,dt$ — (i) $\qquad ds = dx - v\,dt$ — (ii).

$$\frac{\partial r}{\partial x} = 1 + \cancelto{0}{v\frac{\partial t}{\partial x}} = 1 \; ; \; \frac{\partial s}{\partial x} = 1 \; ; \; \frac{\partial r}{\partial t} = v \; ; \; \frac{\partial s}{\partial t} = -v$$

$$\Rightarrow \frac{\partial^2 F}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 F}{\partial t^2} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial x}\right) - \frac{1}{v^2}\frac{\partial}{\partial t}\left(\frac{\partial F}{\partial t}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial F}{\partial s}\cdot\frac{\partial s}{\partial x}\right) - \frac{1}{v^2}\frac{\partial}{\partial t}\left(\frac{\partial F}{\partial r}\cdot\frac{\partial r}{\partial t} + \frac{\partial F}{\partial s}\cdot\frac{\partial s}{\partial t}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial r} + \frac{\partial F}{\partial s}\right) - \frac{1}{v^{\cancel{2}}}\frac{\partial}{\partial t}\left(\cancel{v}\cdot\frac{\partial F}{\partial r} - \cancel{v}\,\frac{\partial F}{\partial s}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial r} + \frac{\partial F}{\partial s}\right) - \frac{1}{v}\frac{\partial}{\partial t}\left(\frac{\partial F}{\partial r} - \frac{\partial F}{\partial s}\right)$$

$$= \frac{\partial}{\partial r}\left(\frac{\partial F}{\partial r} + \frac{\partial F}{\partial s}\right)\cdot\cancelto{1}{\frac{\partial r}{\partial x}} + \frac{\partial}{\partial s}\left(\frac{\partial F}{\partial r} + \frac{\partial F}{\partial s}\right)\cdot\cancelto{1}{\frac{\partial s}{\partial x}}$$

$$- \frac{1}{v}\left[\frac{\partial}{\partial r}\left(\frac{\partial F}{\partial r} - \frac{\partial F}{\partial s}\right)\cdot\cancelto{v}{\frac{\partial r}{\partial t}} + \frac{\partial}{\partial s}\left(\frac{\partial F}{\partial r} - \frac{\partial F}{\partial s}\right)\cdot\cancelto{-v}{\frac{\partial s}{\partial t}}\right]$$

$$= \frac{\partial}{\partial r}\left(\frac{\partial F}{\partial r} + \frac{\partial F}{\partial s}\right) + \frac{\partial}{\partial s}\left(\frac{\partial F}{\partial r} + \frac{\partial F}{\partial s}\right)$$

$$- \frac{1}{\cancel{v}}\left[\cancel{v}\cdot\frac{\partial}{\partial r}\left(\frac{\partial F}{\partial r} - \frac{\partial F}{\partial s}\right) - \cancel{v}\cdot\frac{\partial}{\partial s}\left(\frac{\partial F}{\partial r} - \frac{\partial F}{\partial s}\right)\right]$$

$$= \cancel{\frac{\partial^2 F}{\partial r^2}} + \frac{\partial^2 F}{\partial s\,\partial r} + \frac{\partial^2 F}{\partial r\,\partial s} + \cancel{\frac{\partial^2 F}{\partial s^2}} - \cancel{\frac{\partial^2 F}{\partial r^2}} + \frac{\partial^2 F}{\partial s\,\partial r} + \frac{\partial^2 F}{\partial r\,\partial s} - \cancel{\frac{\partial^2 F}{\partial s^2}}$$

$$= 2\frac{\partial^2 F}{\partial s\,\partial r} + 2\frac{\partial^2 F}{\partial r\,\partial s}$$

If the partial derivatives (2nd) are continuous, then, we can write,

$$= 4 \cdot \frac{\partial^2 F}{\partial r \partial s} = 0 \Rightarrow \frac{\partial^2 F}{\partial r \partial s} = 0 \Rightarrow \frac{\partial}{\partial r}\left(\frac{\partial F}{\partial s}\right) = 0$$

The 'r' derivative of $\frac{\partial F}{\partial s}$ is zero. This means that $\frac{\partial F}{\partial s}$ is independent of r. If we integrate, we get, $F = f(s) + c$ as $\frac{\partial F}{\partial s}$ is a funcⁿ of 's' alone. Since the integration was performed over a partial derivative, 'c' is a const. only as far as 's' is concerned. But it might be a funcⁿ of 'r', say $g(r)$, since, $\frac{\partial}{\partial s}(g(r)) = 0$. Thus, the solution looks like,

$$F = f(s) + g(r)$$

$$\Rightarrow \boxed{F = f(x - vt) + g(x + vt)}$$

This is the d'Alembert's Equation which is the solution of the wave eqⁿ

Q3. $\vec{r} = r.\hat{r}$

Velocity is the rate of change of position, so, differentiating both sides,

$$\frac{d\vec{r}}{dt} = \vec{v} = \dot{r}\hat{r} + r\dot{\hat{r}}$$

Now, $\hat{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} \Rightarrow \frac{d\hat{r}}{dt} = \frac{d\hat{r}}{d\theta}\cdot\frac{d\theta}{dt} = \hat{\theta}\cdot\dot{\theta}$

as, $\Rightarrow \frac{d\hat{r}}{d\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} = \hat{\theta}$

$\hat{\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} \Rightarrow \frac{d\hat{\theta}}{dt} = \frac{d\hat{\theta}}{d\theta}\cdot\frac{d\theta}{dt} = -\hat{r}\cdot\dot{\theta}$

as, $\frac{d\hat{\theta}}{d\theta} = -\cos\theta\,\hat{i} - \sin\theta\,\hat{j} = -\hat{r}$

$$\Rightarrow \frac{d\vec{r}}{dt} = \vec{v} = \dot{r}\hat{r} + r\dot{\hat{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \Rightarrow \boxed{\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}}$$

For translatory motion, the angle would remain constant,

$\Rightarrow \dot{\theta} = 0 \Rightarrow \boxed{\vec{v} = \dot{r}\hat{r}}$ ← Translational Velocity

For uniform circular Motion, only the angle would change at const. rate and radius would remain const.

$\Rightarrow \dot{r} = 0 \Rightarrow \boxed{\vec{v} = r\dot{\theta}\hat{\theta}}$ → Tangential Velocity (UCM)

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}) = \ddot{r}\hat{r} + \dot{r}\frac{d}{dt}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{d}{dt}\hat{\theta}$$

$$\rightarrow \vec{a} = \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} - r\dot{\theta}^2\hat{r} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$$

$$\rightarrow \boxed{\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}}$$

The term $\ddot{r}\hat{r}$ is a linear accln in the radial dirn due to change in radial speed. Similarly, $r\ddot{\theta}\hat{\theta}$ is a linear accln in the tangential dirn due to change in the magnitude of the angular velocity.

- The term $r\dot{\theta}^2\hat{r}$ is the centripetal accln.

$2\dot{r}\dot{\theta}\hat{\theta}$ is the Coriolis Accln. It appears as a fictitious force in a rotating coordinate system. However, Coriolis Accln we are discussing here is a real accln and which is present when r and θ both change with time.

-

84. $\vec{u} = x\hat{i} + y\hat{j} + z\hat{k}$

where, $x = |\vec{u}|\cos\alpha$, α = angle by $\vec{u}$ with the x-axis

$y = |\vec{u}|\cos\beta$, β = angle by $\vec{u}$ with the y-axis

$z = |\vec{u}|\cos\gamma$, γ = angle by $\vec{u}$ with the z-axis

$$\rightarrow \vec{v} = \frac{d\vec{u}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

$$\Rightarrow \vec{a} = \frac{d^2\vec{u}}{dt^2} = \frac{d\vec{v}}{dt} = \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j} + \frac{d^2z}{dt^2}\hat{k}$$

$$\Rightarrow \int_{\vec{u}_0}^{\vec{u}} d\vec{u} = \int_{t_0}^{t} \vec{v}\, dt$$

$$\Rightarrow \boxed{\vec{u} - \vec{u}_0 = \int_{t_0}^{t} \vec{v}\, dt} \quad \text{— (I)}$$

$$\Rightarrow \int_{\vec{v}_0}^{\vec{v}} d\vec{v} = \int_{t_0}^{t} \vec{a}\, dt$$

$$\Rightarrow \boxed{\vec{v} - \vec{v}_0 = \int_{t_0}^{t} \vec{a}\, dt} \quad \text{— (II)}$$

<u>Case</u>: $\vec{v} = \text{const.}$

$$\Rightarrow \boxed{\vec{u} - \vec{u}_0 = \vec{v}\,(t - t_0)} \quad \text{— (III)}$$

<u>Case</u>: $\vec{a} = \text{const}$

$$\boxed{\vec{v} - \vec{v}_0 = \vec{a}\,(t - t_0)} \quad \text{— (IV)}$$

putting (IV) in (I), we get,

$$\Rightarrow \vec{u} - \vec{u}_0 = \int_{t_0}^{t} (\vec{v}_0 + \vec{a}\,(t - t_0))\, dt$$

$$\Rightarrow \vec{u} - \vec{u}_0 = \int_{t_0}^{t} \vec{v}_0\, dt + \int_{t_0}^{t} \vec{a}\,(t - t_0)\, dt$$

$$\Rightarrow \boxed{\vec{u} - \vec{u}_0 = \vec{v}_0 (t - t_0) + \frac{\vec{a}}{2} (t - t_0)^2} \quad \text{for constant } \vec{a}$$

Q.⑤ $\vec{A} \times \vec{B} = (8\hat{i} + 4\hat{j} - 5\hat{k}) \times (-\hat{i} + 2\hat{j} + 6\hat{k})$

$= 6(\hat{i} \times \hat{j}) + 18(\hat{i} \times \hat{k}) - 4(\hat{j} \times \hat{i}) + 24(\hat{j} \times \hat{k}) + 5(\hat{k} \times \hat{i}) -$
$10(\hat{k} \times \hat{i})$

$= 6\hat{k} - 18\hat{j} + 4\hat{k} + 24\hat{i} + 5\hat{i} + 10\hat{i}$

$\vec{A} \cdot (\vec{A} \times \vec{B}) = (3\hat{i} + 4\hat{j} - 5\hat{k}) \cdot (34\hat{i} - 13\hat{j} + 10\hat{k})$

$= 102 - 52 - 50 = \underline{0}$.