Row space of a matrix

$$A = [a_{ij}]_{m\times n} \equiv \begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{pmatrix} \qquad \text{Row space(A)} = \text{span}\{R_1, R_2, \cdots, R_m\}$$

Row equivalent matrices have the same row space.

$$A = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{pmatrix} \underset{\substack{R2 \leftarrow R2 - 2R1 \\ R3 \leftarrow R3 - 3R1}}{\Longrightarrow} \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 6 & -16 \end{pmatrix} \underset{R3 \leftarrow R3 - 2R2}{\Longrightarrow} \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & -3 & 8 \\ 0 & 0 & 0 & 0 \end{pmatrix} \underset{\substack{R2 \leftarrow R2/3 \\ R1 \leftarrow R1 + R2}}{\Longrightarrow} \begin{pmatrix} 1 & 2 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{8}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 2 & -4 & 11 \\ 2 & 4 & -5 & 14 \end{pmatrix} \underset{R2 \leftarrow R2 - 2R1}{\Longrightarrow} \begin{pmatrix} 1 & 2 & -4 & 11 \\ 0 & 0 & 3 & -8 \end{pmatrix} \underset{\substack{R2 \leftarrow R2/3 \\ R1 \leftarrow R1 + 4R2}}{\Longrightarrow} \begin{pmatrix} 1 & 2 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{8}{3} \end{pmatrix}$$

$$RREF(A) = \left(\begin{array}{c|c} \mathbb{I}_k & \overbrace{}^{\text{dependent columns}} \\ \hline 0 & 0 \end{array}\right)$$

A and B are row equivalent with row space spanned by $\{1,2,0,\frac{1}{3}\}$ & $\{0,0,1,-\frac{8}{3}\}$

Number of independent rows = Number of independent columns

dim[row(A)] = dim[column(A)]

Column space of a matrix

Column space of a matrix is the vector space spanned by the columns of the matrix.

$$A = [a_{ij}]_{m\times n} \equiv \begin{pmatrix} C_1 & C_2 & \cdots & C_n \end{pmatrix} \qquad \text{Column space(A)} = \text{span}\{C_1, C_2, \cdots, C_n\}$$

When a matrix multiplied by a column vector the resultant vector is in column space of A.

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix} = x\begin{pmatrix} a \\ c \end{pmatrix} + y\begin{pmatrix} b \\ d \end{pmatrix}$$

In general Ax is a linear combination of columns of A.

From the previous example:

$$A = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & 7 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 2 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{8}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix} = RREF(A)$$

$$A \equiv \begin{pmatrix} C_1 & C_2 & C_3 & C_4 \end{pmatrix}$$

$$RREF(A) \equiv \begin{pmatrix} C_1' & C_2' & C_3' & C_4' \end{pmatrix}$$

$$C_2' = \alpha_1 C_1' \leftrightarrow C_2 = \beta_1 C_1$$

$$C_4' = \alpha_1 C_1' + \alpha_3 C_3' \leftrightarrow C_4 = \beta_1 C_1 + \beta_3 C_3$$

Columns of A corresponding to the pivot columns of RREF(A) form the basis of column space.

From an earlier example:

$$A = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Pivot columns $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ are linearly independent.

A basis for column space $\left(\begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix} \right)$

Dimension of column space = no. of pivot columns = 2

If $A \in M_{m,n}(F)$ then $dim(col(A)) + dim(null(A)) = n$

A necessary and sufficient condition that Ax=b has a solution is that b is spanned by the columns vectors of A.

$$Ax = b \implies \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{pmatrix} \implies x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \end{pmatrix} \implies x_1\bar{c}_1 + x_2\bar{c}_2 + \cdots + x_n\bar{c}_n = \bar{b}$$

Example

$$\begin{pmatrix} x_1 + 2x_2 + 4x_3 = 0 \\ 2x_1 + 3x_2 + 7x_3 = 0 \\ 3x_1 + 3x_2 + 9x_3 = 0 \end{pmatrix} \implies \left(\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 2 & 3 & 7 & 0 \\ 3 & 3 & 9 & 0 \end{array}\right) \implies \left(\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & -3 & -3 & 0 \end{array}\right) \implies \left(\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -3 & -3 & 0 \end{array}\right) \implies \left(\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \implies \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$R2 \leftarrow R2 - 2R1 \quad R3 \leftarrow R3 - 3R1 \qquad R2 \leftarrow -R2 \qquad R3 \leftarrow R3 + 3R2 \qquad R1 \leftarrow R1 - 2R2$$

$$\begin{aligned} x_1 + 2x_3 &= 0 \\ x_2 + x_3 &= 0 \end{aligned}$$

Let $x_3 = t \implies x_1 = -2t$ & $x_2 = -t$ hence $\bar{x} = \{-2t, -t, t\}$ The solution is not unique.

$$x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{has no unique solution} \{x_1, x_2, x_3\}.$$

Number of linearly independent rows = number of linearly independent columns

$$\begin{pmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow RREF \sim \left(\begin{array}{c|c} \mathbb{I}_k & X \\ \hline 0 & 0 \end{array}\right)$$

$$\begin{pmatrix} 1 & 2 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{8}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow RREF \sim \left(\begin{array}{c|c} \mathbb{I}_k & X \\ \hline 0 & 0 \end{array}\right)$$

The **rank** of a matrix A is the dimension of the column space = $r(A)$=the number of leading ones in RREF(A).

Let $A \in M_{m,n}(F)$ & $b \in F$ then Ax=b has a solution iff $r(A)$=$r(A|b)$ i.e. the rank of matrix A and augmented matrix $A|b$ are equal.

Example $\begin{pmatrix} x_1 + x_2 = 2 \\ x_1 + x_2 = 3 \end{pmatrix} \implies A\,.\,x = b \implies A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \implies \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = RREF(A) \implies r(A) = 1$

$$A\,|\,b = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 3 \end{pmatrix} \implies \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \implies \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = RREF(A\,|\,b) \implies r(A\,|\,b) = 2$$

Invertible matrices

$A \in M_n$ is nonsingular if $Ax = 0 \implies x = 0$

$\star\ r(A) = n = dim(col(A)) = dim(row(A))$

$\star\ A$ is invertible $AA^{-1} = A^{-1}A = \mathbb{I}$

$\star\ dim(null(A)) = 0$

$\star$ rows and columns of A are linearly independent

$\star\ Ax = 0 \implies x = 0$

$\star\ Ax = b$ has unique solution

$\star$ 0 is not an eigenvalue of A

$\star\ detA \neq 0$

Theorem: ***Inverse of a matrix is unique.***

Let A has two inverses B and C such that AB=BA=I & AC=CA=I.

B=BI
 =B(AC)
 =(BA)C
 =IC
 =C

Contradiction hence proved. $\square$

Elementary Matrices

Let matrix E denote an **elementary row operation** giving A'=E A.

Examples

$$E_k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \to A' = E_k A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{pmatrix} \implies R2 \leftrightarrow R3$$

Row exchange

$$E_m = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \to A' = E_m A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ \alpha b_1 & \alpha b_2 & \alpha b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \implies R2 \leftarrow \alpha R2$$

Row scaling

$$E_n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \beta & 0 & 1 \end{pmatrix} \to A' = E_n A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \beta & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ \beta a_1 + c_1 & \beta a_2 + c_2 & \beta a_3 + c_3 \end{pmatrix} \implies R3 \leftarrow R3 + \beta R1$$

Examples

$$A = \begin{pmatrix} -3 & 2 & -1 \\ 6 & -6 & 7 \\ 3 & -4 & 4 \end{pmatrix}$$

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad M_1A = \begin{pmatrix} -3 & 2 & -1 \\ 0 & -2 & 5 \\ 3 & -4 & 4 \end{pmatrix} \qquad R2 \leftarrow R2 + 2R1$$

$$M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad M_2M_1A = \begin{pmatrix} -3 & 2 & -1 \\ 0 & -2 & 5 \\ 0 & -2 & 3 \end{pmatrix} \qquad R3 \leftarrow R3 + R1$$

$$M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \qquad M_3M_2M_1A = \begin{pmatrix} -3 & 2 & -1 \\ 0 & -2 & 5 \\ 0 & 0 & -2 \end{pmatrix} \qquad R3 \leftarrow R3 - R2$$

$$M_3M_2M_1A = U$$ Upper Triangular Matrix

Finding inverse of a matrix

Let A be an invertible matrix and let there be a sequence of row operations performed by elementary matrices such that

$$E_q E_{q-1} \cdots E_2 E_1 A = \mathbb{I}$$

$$\implies (E_q E_{q-1} \cdots E_2 E_1 \mathbb{I}) A = \mathbb{I}$$

$$\implies (E_q E_{q-1} \cdots E_2 E_1 \mathbb{I}) A A^{-1} = \mathbb{I} A^{-1}$$

$$\implies A^{-1} = E_q E_{q-1} \cdots E_2 E_1 \mathbb{I}$$

$$[A \,|\, \mathbb{I}] \xrightarrow{\text{Eq ... E2 E1}} [\mathbb{I} \,|\, A^{-1}] \qquad A, \mathbb{I} \in M_n(F)$$

Example: Find inverse of the matrix $A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix}$

$$[A \,|\, \mathbb{I}] = \begin{pmatrix} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 4 & 1 & 8 & 0 & 0 & 1 \end{pmatrix} \implies \begin{pmatrix} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \end{pmatrix} \implies \begin{pmatrix} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -6 & 1 & 1 \end{pmatrix} \implies$$

$R2 \leftarrow 2R1$

$R3 \leftarrow R3 - 4R1$

$R3 \leftarrow R3 + R2$

$$\implies \begin{pmatrix} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{pmatrix} \implies \begin{pmatrix} 1 & 0 & 0 & -11 & 2 & 2 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{pmatrix} = [\mathbb{I} \,|\, A^{-1}]$$

$R2 \leftarrow -R2$

$R3 \leftarrow -R3$

$R1 \leftarrow R1 - 2R3$

$R2 \leftarrow R2 - R3$