

Module 2: Number Systems

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Email: pm21ms002@iiserkol.ac.in. These are my personal notes on Number Systems. We will consider \mathbb{N} (Natural Numbers), \mathbb{Z} (Integers), and \mathbb{Q} (rational Numbers), but not \mathbb{R} (Real Numbers).

§1 Natural Numbers

The natural numbers are $1, 2, 3, 4, \dots$. The set of all natural numbers is denoted by \mathbb{N} .

Definition 1.1. We assume familiarity with the algebraic operations of addition and multiplication on the set \mathbb{N} and also with the linear order relation $<$ on \mathbb{N} defined by “ $a < b$ if $a, b \in \mathbb{N}$ and a is less than b ”.

We discuss the following fundamental properties of the set \mathbb{N} .

1. Well Ordering Property
2. Principle of Induction

§1.1 Well Ordering Property

Definition 1.2. Every non-empty subset of \mathbb{N} has a least element.

This means that if S is a non-empty subset of \mathbb{N} , then there is an element m in S such that $m \leq s$ for all $s \in S$.

In particular, \mathbb{N} itself has the least element 1.

Proof. Let S be a non-empty subset of \mathbb{N} . Let k be an element of S . Then k is a natural number.

We define a subset T by $T = \{x \in S : x \leq k\}$. The T is a non-empty subset of $\{1, 2, 3, \dots, k\}$.

Clearly, T is a finite subset of \mathbb{N} and therefore it has a least element, say m . Then $1 \leq m \leq k$.

We now show that m is the least element of S . Let s be any element of S .

If $s > k$, then the inequality $m \leq k$ implies $m < s$.

If $s \leq k$, the $s \in T$; and m being the least element of T , we have $m \leq s$.

Thus m is the least element of S . □

§1.2 Principle of Induction

Definition 1.3. Let S be a subset of \mathbb{N} such that,

- i) $1 \in S$ and,
- ii) if $k \in S$, then $k + 1 \in S$.

Then $S = \mathbb{N}$

Proof. Let $T = \mathbb{N} - S$. We prove that $T = \phi$.

Let T be non-empty. then by the *Well Ordering Property* of \mathbb{N} , the non-empty subset T has a least element, say m .

Since $1 \in S$ and 1 is the least element of \mathbb{N} , $m > 1$.

Hence, $m - 1$ is a natural number and $m - 1 \notin T$. So, $m - 1 \in S$.

But by ii) $m - 1 \in S \Rightarrow (m - 1) + 1 \in S$, i.e., $m \in S$.

This contradicts that m is the least element in T . Therefore, our assumption is wrong and $T = \phi$.

Therefore, $S = \mathbb{N}$. □

Theorem 1.4

Let $P(n)$ be a statement involving a natural number n . If,

- i) $P(1)$ is true, and
 - ii) $P(k + 1)$ is true whenever $P(k)$ is true,
- then $P(n)$ is true for all $n \in \mathbb{N}$.

Proof. Let S be the set of those natural numbers for which the statement $P(n)$ is true.

Then S has the properties,

- (a) $1 \in S$, by (i)
- (b) $k \in S \Rightarrow k + 1 \in S$ by (ii).

By the *Principle of Induction*, $S = \mathbb{N}$.

Therefore, $P(n)$ is true for all $n \in \mathbb{N}$. □

Remark 1.5. Let a statement $P(n)$ satisfies the conditions,

- (i) for some $m \in \mathbb{N}$, $P(m)$ is true (m being the least possible); and
- (ii) $P(k)$ is true $\Rightarrow P(k + 1)$ is true for all $k \geq m$.

Then $P(n)$ is true for all natural numbers $\geq m$.

Worked Examples**Example 1.6**

Prove that for each $n \in \mathbb{N}$, $1 + 2 + 3 + 4 + \dots + n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

The statement is true for $n = 1$, because $1 = \frac{1(1+1)}{2}$.
Let the statement be true for some natural number k .

Then $1 + 2 + 3 + 4 + \dots + k = \frac{(k+1)}{2}$ and therefore,

$$(1 + 2 + \dots + k) + (k + 1) = \frac{k(k+1)}{2} + (k + 1)$$

$$\text{or, } 1 + 2 + 3 + \dots + (k + 1) = \frac{(k+1)(k+2)}{2}.$$

This shows that the statement is true for the natural number $k + 1$ if it is true for k .
By the principle of induction, the statement is true for all natural numbers.

Example 1.7

Prove that for each $n \geq 2$, $(n + 1)! > 2^n$.

The equality holds for $n = 2$ since $(2 + 1)! > 2^2$.

Let the inequality hold for some natural number $k \geq 2$.

Then, $(k + 1)! > 2^k$

$$\begin{aligned} \text{and } (k + 2)! &= (k + 2)(k + 1)! \\ &> 2 \cdot 2^k, \text{ since } k + 2 > 2 \\ \text{or, } (k + 2)! &> 2^{k+1} \end{aligned}$$

This shows that if the inequality holds for $k(\geq 2)$ then it also holds for $k + 1$.

By the principle of induction, the inequality holds for all natural numbers ≥ 2 .

[Note that the inequality does not hold for $n = 1$.]

§1.3 Second Principle of Induction (or, Principle of Strong Induction)

Definition 1.8. Let S be a subset of \mathbb{N} such that

- (i) $1 \in S$, and
- (ii) if $\{1, 2, 3, 4, \dots\} \subset S$, then $k + 1 \in S$.

Then $S = \mathbb{N}$

Proof.

□