ASSIGNMENT - 07

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Q1. Solutions for the Harmonic Oscillator :

The simplest Harmonic Oscillator we can think of is an oscillating block attached to a spring.

$$F_{spring} = -kx$$

$$\Rightarrow m\ddot{x} = -kx \Rightarrow m \cdot \frac{d^2x}{dt^2} = -kx$$

$$\Rightarrow \frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \quad \text{(2nd Order ODE)}$$

Consider $\omega_0^2 = \frac{k}{m}$,

$$\Rightarrow D^2x + \omega_0^2 x = 0 \qquad \left[D = \frac{d}{dt} \; ; \; D^2 = \frac{d^2}{dt^2}\right]$$

$$\Rightarrow (D^2 + \omega_0^2)x = 0$$

$$\Rightarrow (D + i\omega_0)(D - i\omega_0)x = 0$$

$$(D + i\omega_0)x = 0$$

$$\Rightarrow Dx = -i\omega_0 x$$

$$\Rightarrow \frac{dx}{dt} = -i\omega_0 x$$

$$\Rightarrow \int \frac{dx}{x} = \int -i\omega_0 dt$$

$$\Rightarrow \ln x = -i\omega_0 t + c$$

$$\Rightarrow x = Ae^{-i\omega_0 t}$$

$$(D - i\omega_0)x = 0$$

$$\Rightarrow x = Be^{i\omega_0 t}$$

$$\Rightarrow x = Ae^{-i\omega_0 t} + Be^{i\omega_0 t} \longrightarrow \begin{cases} e^{i\omega_0 t} = \cos\omega_0 t + i\sin\omega_0 t \\ e^{-i\omega_0 t} = \cos\omega_0 t - i\sin\omega_0 t \end{cases}$$

$x = (A+B)\cos\omega_0 t - i(A-B)\sin\omega_0 t$

Real Part :

$x = x_0 \cos\omega t$

Generally,

$\boxed{x = x_0 \cos(\omega t + \varphi)}$ → Sol^n^ of the Harmonic Oscillator.

x_0 = Amplitude

ω_0 = Frequency

- Now, to prove that the total energy of a Harmonic Oscillator is constant,

$x = x_0 \cos\omega_0 t$

$v = \dot{x} = -x_0\omega_0 \sin\omega_0 t$

$v^2 = x_0^2 \omega_0^2 \sin^2(\omega_0 t)$

$KE = \frac{1}{2}mv^2 = \frac{1}{2}m\omega_0^2 x_0^2 \sin^2\omega_0 t$

$PE = \frac{1}{2}kx^2 = \frac{1}{2}k x_0^2 \cos^2\omega_0 t$

$\because m\omega_0^2 = k \,;\, \frac{k}{m} = \omega_0^2$

$\Rightarrow E = K + V = \frac{1}{2}kx_0^2(\sin^2\omega_0 t + \cos^2\omega_0 t) \Rightarrow \boxed{E = \frac{1}{2}kx_0^2}$

Total Energy is const.

QED.

Q2. Angular momentum : $\vec{L} = \vec{r} \times \vec{p}$

Angular Momentum about A,

$$\vec{L}_{sys}(A) = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2$$
$$= \vec{r} \times \vec{p} + (-\vec{r}) \times (-\vec{p})$$
$$= 2(\vec{r} \times \vec{p})$$

Angular Momentum about B,

$$\vec{L}_{sys}(B) = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2$$
$$= (-\vec{d}) \times \vec{p} + (-(\vec{d} + 2\vec{r})) \times (-\vec{p})$$
$$= -\vec{d} \times \vec{p} + (\vec{d} + 2\vec{r}) \times \vec{p}$$
$$= \cancel{-\vec{d} \times \vec{p}} + \cancel{\vec{d} \times \vec{p}} + 2\vec{r} \times \vec{p}$$
$$= 2(\vec{r} \times \vec{p})$$

Here, we see that angular momentum calculated from points A and B, both turn out to be same, and hence we can conclude that when the linear momentum of a system adds up to zero (0), the angular momentum doesn't depend on the position from where it is calculated.

Q3. The Parallel Axis Theorem :-

Consider the moment of inertia of the body around an axis that we choose to lie in the z-directions. The $\perp^r$ vector from the z-axis to particle 'j' is,

$$\rho_j = x_j \hat{i} + y_j \hat{j}$$

and,
$$I = \sum_j m_j \rho_j^2$$

If the Centre of Mass is at $\vec{R} = X\hat{i} + Y\hat{j} + Z\hat{k}$, the vector $\perp^r$ from the z-axis to the Centre of Mass is,

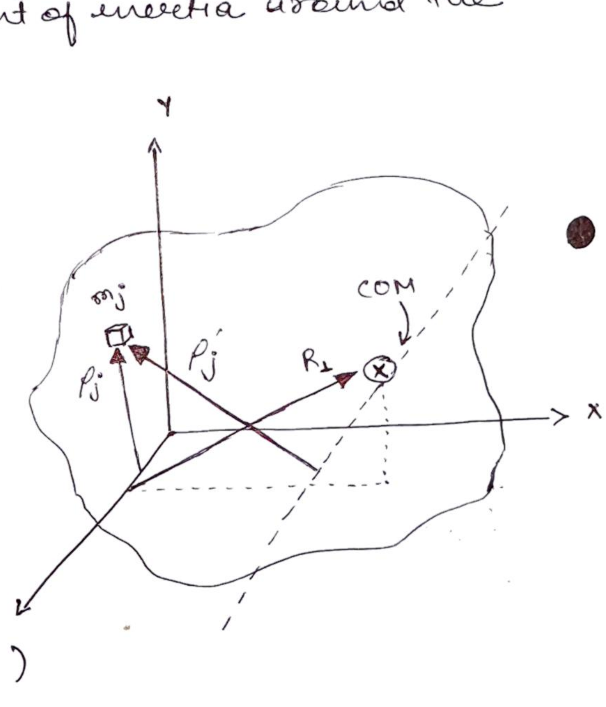
$$\vec{R}_\perp = X\hat{i} + Y\hat{j}$$

If the vector from the axis through the center of mass to particle j is ρ_j', then the moment of inertia around the center of mass is

$$I_0 = \sum m_j \rho_j'^2$$

From the diagram,

$$\vec{\rho_j} = \vec{\rho_j}' + \vec{R}_\perp$$



So that,

$$\begin{aligned} I &= \sum m_j \rho_j^2 \\ &= \sum m_j (\vec{\rho_j}' + \vec{R}_\perp)^2 \\ &= \sum m_j (\rho_j'^2 + 2\vec{\rho_j}' \cdot \vec{R}_\perp + R_\perp^2) \end{aligned}$$

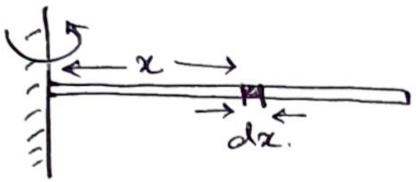
The middle term vanishes by defn of the center of mass:

$$\sum m_j \vec{\rho_j}' = \sum m_j (\vec{\rho_j} - R_\perp) = M(\vec{R}_\perp - \vec{R}_\perp) = \underline{\underline{0}}$$

If we designate the magnitude of $\vec{R}_\perp$ by l, then,

$$\boxed{I = I_0 + Ml^2}$$

Verification of Parallel Axis Theorem :-



Let the mass of the rod be 'm' and its length be 'l'.

$$\frac{dm}{dx} = \lambda = \frac{M}{l}.$$

$\Rightarrow$ Moment of Inertia $(I) = \int dm \cdot x^2.$

$$= \int \lambda \, dx \cdot x^2.$$

$$= \lambda \int x^2 \, dx.$$

$$= \frac{M}{l} \int_0^l x^2 \, dx.$$

$$= \frac{M}{l} \left[\frac{x^3}{3}\right]_0^l$$

$$= \frac{M}{\cancel{l}} \cdot \frac{l^{\cancel{3}\,2}}{3}$$

$$= \frac{Ml^2}{3}.$$

$\Rightarrow \boxed{I = \frac{Ml^2}{3}}$ → from the attached end.

Verifying this using Parallel Axis Theorem:

$$I = I_{COM} + Mx^2$$

$$= \frac{Ml^2}{12} + M\left(\frac{l}{2}\right)^2.$$

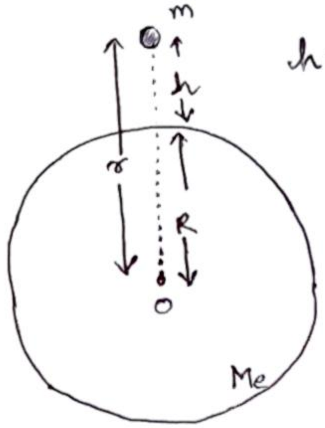
$$= \frac{Ml^2}{12} + \frac{Ml^2}{4}$$

$$= \frac{Ml^2 + 3Ml^2}{12}$$

$$= \frac{\cancel{4}Ml^2}{\cancel{12}\,3}$$

$$= \frac{Ml^2}{3}.$$ (which is equal to what we derived analytically).

4. Gravitational Potential Energy near the surface of the earth :-



$h << R$

Consider the Earth and mass system, with r, the distance between the mass 'm' and the Earth's centre. Then the gravitational potential energy,

$$U = -\frac{GM_e m}{r}$$

Here, $r = R_e + h$, where R_e is the radius of the Earth. h is the height above the Earth's surface,

$$U = -G\frac{M_e m}{(R_e + h)}$$

If $h << R_e$, the equation can be modified as,

$$U = -G\frac{M_e m}{R_e(1 + h/R_e)}$$

$$\Rightarrow U = -G\frac{M_e m}{R_e}\left(1 + \frac{h}{R_e}\right)^{-1}$$

By using Binomial Expansion and neglecting the higher order,

$$U = -G\frac{M_e m}{R_e}\left(1 - \frac{h}{R_e}\right) \quad — *_1$$

We know that, for a mass m on the Earth's surface,

$$G\frac{M_e m}{R_e} = mgR_e \quad — *_2$$

Substituting $*_2$ in $*_1$, we get,

$$\boxed{U = -mgR_e + mgh}$$

It is clear that the first term in the above expression is independent of the height 'h'. for example, if the object is taken from height h_1 to h_2 then the potential energy at h_1 is:

$$U(h_1) = -mgR_e + mgh_1$$

$$U(h_2) = -mgR_e + mgh_2$$

∴ The potential energy difference between h_1 & h_2 is,

$$\boxed{U(h_2) - U(h_1) = \Delta U = mg(h_1 - h_2)}$$

S5. The work done by a conservative force along any path from a to b is,

$$\oint_{\vec{r}_a}^{\vec{r}_b} \vec{F}\cdot d\vec{r} = \text{function of } (\vec{r}_b) - \text{function of } (\vec{r}_a)$$

or,
$$\oint_{\vec{r}_a}^{\vec{r}_b} \vec{F}\cdot d\vec{r} = -U(\vec{r}_b) + U(\vec{r}_a)$$

where, $U(\vec{r})$ is a function, defined by the above expression, known as the potential energy function. For a conservative force, the work-energy theorem $W_{ba} = K_b - K_a$ becomes,

$$W_{ba} = -U_b + U_a$$
$$= K_b - K_a$$

or, rearranging,

$$K_a + U_a = K_b + U_b$$

The left-hand side of this equation, $K_a + U_a$, depends on the speed of the particle and its potential energy at $\vec{r}_a$, without reference to $\vec{r}_b$. Similarly, the right-hand side depends on the speed and potential energy at $\vec{r}_b$, without reference to $\vec{r}_a$. Because $\vec{r}_a$ and $\vec{r}_b$ are arbitrary and not specially chosen points, this can be true only if each side of the equation equals a constant. Denoting this constant by E, we have,

$$K_a + U_a = K_b + U_b = E$$

E is called the <u>total mechanical energy</u> of the particle or less precisely, the total energy. In this case, the total energy is conserved.

A peculiar property of energy is that the value of E is arbitrary; only changes in E have physical significance. This comes about because the equation

$$U_b - U_a = -\oint_a^b \vec{F} \cdot d\vec{r}$$

defines only the difference in potential energy between a and b and not the potential energy itself. However, since $E = K + U$, adding an arbitrary constant to U increases E by the same amount. As a corollary, the above equation implies that the work by a conservative force $\vec{F}$ around a closed path is zero:

$$\oint \vec{F} \cdot d\vec{r} = 0$$

For non-conservative forces :-

$$\vec{F} = \vec{F}^c + \vec{F}^{nc}$$

where $\vec{F}^c$ and $\vec{F}^{nc}$ are the conservative and the non-conservative forces, respectively. Since the work-energy theorem is true whether or not the forces are conservative, the total work by $\vec{F}$ as the particle moves from a to b is

$$W_{ba}^{total} = \oint_a^b \vec{F} \cdot d\vec{r}$$

$$= \oint_a^b \vec{F}^c \cdot d\vec{r} + \oint_a^b \vec{F}^{nc} \cdot d\vec{r}$$

$$= -U_b + U_a + W_{ba}^{nc}$$

Here, U is the potential energy associated with the conservative force and W_{ba}^{nc} is the work by the non-conservative force. The work-energy theorem, ~~W^{nc}~~ $W_{ba}^{total} = K_b - K_a$, now has the form,

$$-U_b + U_a + W_{ba}^{nc} = K_b - K_a$$

or,

$$K_b + U_b - (K_a + U_a) = W_{ba}^{nc}$$

If we define the total mechanical energy by $E = K + U$, as before, then, E is no longer constant, but depends on the state of the system. We have,

$$E_b - E_a = W_{ba}^{nc}$$

this result is a generalisation of the statement of conservation of mechanical energy.

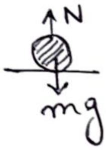
Q6. Time average of a force : $\vec{F}_{avg} = \dfrac{\int_{t_i}^{t_f} F(t) \cdot dt}{t_f - t_i}$

Initial State : $v_b \downarrow$ (Before bouncing) at t_b

Final State : $v_a \uparrow$ (After bouncing) at t_a

From Momentum Principle : $\int \vec{F} \cdot dt = \Delta \vec{p}$

Free Body:



$$\Rightarrow \int_{t_b}^{t_a} (N - mg)\, dt = P_y(t_a) - P_y(t_b)$$

For N_{avg} at a given time interval ($\because$, assume the time interval is very small)

$$(N_{avg} - mg)\Delta t = P_y(t_a) - P_y(t_b)$$
$$= mv_a - m(-v_b)$$

$$\Rightarrow (N_{avg} - mg)\, \Delta t = m(v_a + v_b)$$

$$\Rightarrow N_{avg} = \frac{m(v_a + v_b)}{\Delta t} + mg.$$

In the question, Δt is given as T, so the answer becomes,

$$\boxed{N_{avg} = \frac{m(v_a + v_b)}{T} + mg.}$$

For very small values of $\Delta t = T$, the N_{avg} is larger.

Q7. $mr\omega^2 = G\frac{mM}{r^2}$

$$\Rightarrow mr\left(\frac{2\pi}{T}\right)^2 = G\frac{mM}{r^2}$$

$$\Rightarrow T^2 = \left(\frac{4\pi^2}{GM}\right) r^3 \qquad [T^2 \propto r^3]$$

The above formula works only for ideal circular orbits. In the above question, putting in all the values, we get,

$$\Rightarrow T^2 = \left(\frac{4\pi^2}{Gm_1}\right) r^3 \Rightarrow r^3 = T^2 \cdot \frac{Gm_1}{4\pi^2}.$$

$$\Rightarrow r_{sat}^3 = \frac{T^2 G m_1}{4\pi^2} \Rightarrow r_{sat} = \left(\frac{T^2}{(2\pi)^2} \cdot Gm_1\right)^{1/3}$$

$$\Rightarrow \boxed{r_{sat} = \left(\frac{T}{2\pi}\right)^{2/3} (Gm_1)^{1/3}}$$

Q8. Let the neutral point between the planets be situated at a distance $x = r$ from the origin or, $x=0$. For it to remain neutral,

$$\frac{\cancel{G}m_1}{r^2} = \frac{\cancel{G}m_2}{(d-r)^2}$$

$$\Rightarrow (d-r)^2 = \left(\frac{m_2}{m_1}\right) r^2.$$

$$\Rightarrow d^2 + r^2 - 2rd = cr^2, \quad \text{where } c = \frac{m_2}{m_1}$$

$$\Rightarrow r^2 - cr^2 - 2rd + d^2 = 0$$

$$\Rightarrow (1-c)r^2 - 2rd + d^2 = 0$$

$$\Rightarrow r = \frac{2d \pm \sqrt{4d^2 - 4\cdot(1-c)\cdot d^2}}{2(1-c)}$$

$$\Rightarrow \quad r = \frac{2d \pm \sqrt{4d^2 - 4d^2(1-c)}}{2(1-c)}$$

$$= \frac{\cancel{2}d \pm \cancel{2}d\sqrt{1-(1-c)}}{\cancel{2}(1-c)}$$

$$= \frac{d \pm d\sqrt{\cancel{1}-\cancel{1}+c}}{(1-c)}$$

$$= d \quad \frac{1 \pm \sqrt{c}}{(1-c)}$$

$$= d\left(\frac{1+\sqrt{c}}{1-c}\right) \quad \text{or} \quad d\left(\frac{1-\sqrt{c}}{1-c}\right).$$

$$= d\left(\frac{\cancel{1-\sqrt{c}}}{(1+\sqrt{c})\cancel{(1-\sqrt{c})}}\right)$$

$$\therefore r = d\left(\frac{1+\sqrt{c}}{1-c}\right) \quad \text{or} \quad r = d\left(\frac{1}{1+\sqrt{c}}\right)$$

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$$\therefore \boxed{r = d\left(\frac{1}{1+\sqrt{c}}\right)} \quad \text{where, } c = \frac{m_2}{m_1} = \text{constant}.$$

Q9. Derive velocity & accel^n in polar coordinates

$$\left.\begin{aligned}\hat{r}(\theta) &= \cos\theta\,\hat{i} + \sin\theta\,\hat{j} \\ \hat{\theta}(\theta) &= -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}\end{aligned}\right\} \text{Polar Coordinates}$$

$\vec{r} = x\hat{i} + y\hat{j} \rightarrow$ Cartesian Coordinates

Combining the two, we get,

$$x\hat{i} + y\hat{j} = r(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$$

By orthogonality we have,

$$x = r\cos\theta$$
$$y = r\sin\theta$$

as we expect.

Using Newton's notation for time derivatives can help make equations easier to read.

$$\frac{d\theta}{dt} = \dot{\theta}$$

$$\frac{d^2\theta}{dt^2} = \ddot{\theta}$$

$$\Rightarrow \frac{d\hat{r}}{dt} = \frac{d}{dt}(\cos\theta)\hat{i} + \frac{d}{dt}(\sin\theta)\hat{j}$$

$$= -\sin\theta\cdot\dot{\theta}\hat{i} + \cos\theta\cdot\dot{\theta}\hat{j}$$

$$= (-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})\dot{\theta}$$

$$\therefore \frac{d\hat{r}}{dt} = \dot{\theta}\,\hat{\theta}$$

Similarly, we have,

$$\frac{d\hat{\theta}}{dt} = (-\cos\theta\,\hat{i} - \sin\theta\,\hat{j})\,\dot{\theta}$$

$$= -\dot{\theta}\,\hat{r}$$

$\therefore$ Summarising the above results,

$$\frac{d\hat{r}}{dt} = \dot{\theta}\,\hat{\theta}$$

$$\frac{d\hat{\theta}}{dt} = -\dot{\theta}\,\hat{r}$$

- $\vec{v} = \frac{d}{dt}(r\hat{r}) = \dot{r}\hat{r} + r\frac{d\hat{r}}{dt}$

$$\Rightarrow \boxed{\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}}$$

Cross-Radial Component of Velocity

Radial Velocity Comp.

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})$$

$$= \ddot{r}\hat{r} + \dot{r}\frac{d}{dt}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{d}{dt}\hat{\theta}$$

- $\Rightarrow \vec{a} = \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} - r\dot{\theta}^2\hat{r}$

$$\Rightarrow \boxed{\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}}$$

Radial / Centripetal Acceleration.

Tangential Acceleration.
OR
CORIOLIS Acceleration.

Case 1: <u>Radial Acceleration</u>. The term $\ddot{r}\hat{r}$ is the acceleration due to change in radial speed. The second term $-r\dot{\theta}^2\hat{r}$ is the centripetal acceleration.

Case 2: <u>Tangential Acceleration</u>: The term $r\ddot{\theta}\hat{\theta}$ is the accelⁿ that arises from the changing tangential speed. The next term $2\dot{r}\dot{\theta}\hat{\theta}$ is the Coriolis accelⁿ. It occurs due to Coriolis force which is a fictitious force that appears to act in a rotating coordinate system.

Q10. Polar curve : $r = 2(\cos\theta - \sin\theta)$ — (1)

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\} \text{ Converting to Cartesian Coordinates}$$

Substituting these in (1), we get,

$$r = \frac{2x}{r} - \frac{2y}{r}$$

$$\Rightarrow r^2 = 2x - 2y.$$

We know, $r^2 = x^2 + y^2$

$\therefore x^2 + y^2 = 2x - 2y$ since, $r^2 = 2x - 2y$.

So, eqⁿ of circle becomes,

$$\left(x - (2/2)\right)^2 + \left(y + (2/2)^2\right)^2 = \sqrt{\frac{2^2 + 2^2}{2}}$$

$\Rightarrow \boxed{(x-1)^2 + (y+1)^2 = 2}$ which is the eqⁿ of a circle with centre at $(1,-1)$ and radius $= \sqrt{2}$ units.