PH1202

Physics Laboratory II

Experiment Number - 3

Coupled Pendula

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June 3, 2022

Contents

§1 Aim

Measurement of effective spring constant of a string-mass assembly using two coupled pendula.

§2 Apparatus Required

a. Two fairly identical masses

b. A string-mass assembly

c. A fixed supporting metal frame

d. Meter scale / Ruler

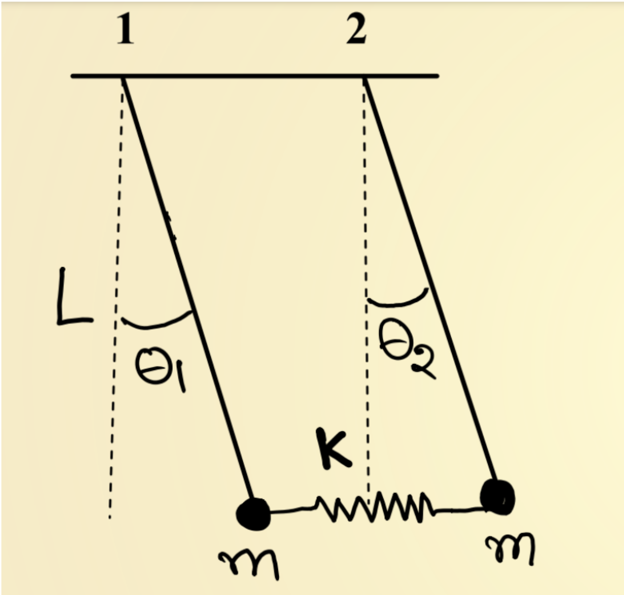
e. A stopwatch

f. A digital balance

§3 Experimental Setup

The experimental setup mainly consists of two pendula having bobs of almost same masses supported at the top by a fixed metal frame.The pendula are free to oscillate to-and-fro about the common metal axis at the top. A ruler is kept just behind the pendula at the same level as the two bobs to measure the displacements of the bobs from their respective mean positions. A string-mass assembly is also set up to couple the two pendula as shown below :

§4 Theory



The coupled pendulum consists of two pendula (say 1 and 2).

For Pendulum 1:

$$mL\frac{d^2\theta_1}{dt^2} = -mg\sin\theta_1 - k(L\sin\theta_1 - L\sin\theta_2) \tag{1}$$

For Pendulum 2:

$$mL\frac{d^2\theta_2}{dt^2} = -mg\sin\theta_2 - k(L\sin\theta_2 - L\sin\theta_1) \tag{2}$$

Adding (1) and (2) we get:

$$mL\left(\frac{d^2\theta_1}{dt^2} + \frac{d^2\theta_2}{dt^2}\right) = -mg(\theta_1 + \theta_2) \tag{3}$$

Similarly, on subtracting (2) from (1), we get:

$$mL\left(\frac{d^2\theta_1}{dt^2} - \frac{d^2\theta_2}{dt^2}\right) = -(mg + 2kL)(\theta_1 - \theta_2) \tag{4}$$

In order to solve the above coupled differential equations, we implement the following substitution :

$$q_1 = (\theta_1 + \theta_2); q_2 = (\theta_1 - \theta_2)$$

So the equations (3) and (4) become:

$$\frac{d^2q_1}{dt^2} = -\left(\frac{g}{l}\right)q_1 = -\omega_1^2 q_1 \tag{5}$$

$$\frac{d^2q_2}{dt^2} = -\left(\frac{g}{l} + \frac{2k}{m}\right)q_2 = -\omega_2^2 q_2 \tag{6}$$

where, $\omega_1 = \sqrt{\frac{g}{l}}$ and, $\omega_2 = \sqrt{\frac{g}{l} + \frac{2k}{m}}$

Solving the differential equations, we get:

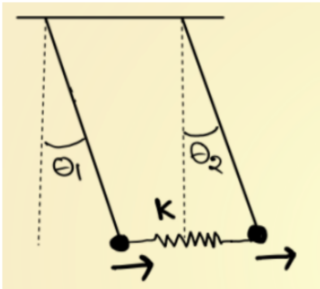
$$q_1 = A_1 cos(\omega_1 t + \varphi_1)$$
$$q_2 = A_2 cos(\omega_2 t + \varphi_2)$$

The expressions for the original variables i.e. θ_1 and θ_2 become:

$$\theta_1 = \frac{1}{2}\left(A_1 cos(\omega_1 t + \varphi_1) + A_2 cos(\omega_2 t + \varphi_2)\right)$$
$$\theta_2 = \frac{1}{2}\left(A_1 cos(\omega_1 t + \varphi_1) - A_2 cos(\omega_2 t + \varphi_2)\right)$$

Now for the given setup, both A_1 and A_2 are equal (say, $= A_0$)
Two special cases can be formed here depending on the initial conditions of θ_1 and θ_2:

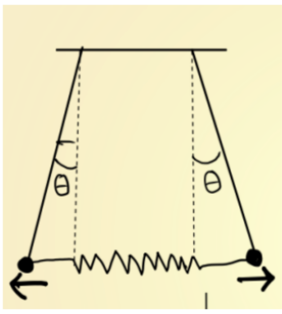
§4.1 In-Phase Motion:



At t = 0 sec,
$\theta_1 = \theta_2$
$\theta_1 = A_0 cos\left(\omega_1 t + \varphi_1\right)$ and $\theta_2 = A_0 cos\left(\omega_1 t + \varphi_1\right)$
Thus the In-phase frequency, ω_+ is given by :

$$\omega_+ = \sqrt{\frac{g}{L}}$$

§4.2 Out of Phase Motion:



At $t = 0$ sec
$\theta_1 = -\theta_2$
$\theta_1 = A_0 \cos(\omega_1 t + \phi_1)$ and $\theta_2 = -A_0 \cos(\omega_1 t + \phi_1)$

Thus the In-phase frequency, ω_+ is given by:

$$\omega_- = \sqrt{\frac{g}{L} + \frac{2k}{m}}$$

Thus the effective spring constant can be calculated by the formula:

$$\omega_-^{\,2} = \omega_+^{\,2} + \frac{2k}{m}$$

§4.3 Beat Frequency:

Keeping the initial conditions at t = 0 sec as : $\theta_1 = A_0$; $\theta_2 = 0$; $\varphi_1 = \varphi_2 = 0$

$$\theta_1 = \frac{1}{2}A_0\left[\cos\left(\omega_+ t\right) + \cos\left(\omega_- t\right)\right] = A_0\cos\left(\frac{\omega_- + \omega_+}{2}t\right)\cos\left(\frac{\omega_- - \omega_+}{2}t\right)$$

$$\theta_2 = \frac{1}{2}A_0\left[\cos\left(\omega_+ t\right) - \cos\left(\omega_- t\right)\right] = A_0\sin\left(\frac{\omega_- + \omega_+}{2}t\right)\sin\left(\frac{\omega_- - \omega_+}{2}t\right)$$

Here, $\omega_{beat} = \omega_- - \omega_+$ and $T = \frac{2\pi}{\omega_{beat}}$

§4.4 Working Formulae:

- The beat frequency, ω_{beat} of the coupled pendulum is given by :

$$\omega_{beat} = \omega_- - \omega_+ \qquad \text{… (Equation 1)}$$

- The moment of inertia, $I_{pendulum}$ for the given pendulum is given by :

$$I_{pendulum} = I_{rod} + I_{bob} \qquad \text{… (Equation 2)}$$

Also,

$$I_{rod} = \frac{1}{3}M_{rod}L^2\left(1 - \frac{3h}{L}\right) \qquad \text{… (Equation 3)}$$

and

$$I_{bob} = M_{bob}L^2\left(1 - \frac{h}{L}\right) \qquad \text{… (Equation 4)}$$

- The experimental value of spring constant, k_{expt} is given by :

$$k_{expt} = \frac{I_{pendulum}}{2l^2}\left(\omega_-^{\,2} - \omega_+^{\,2}\right) \qquad \text{… (Equation 5)}$$

- The estimated value of spring constant, k_{estim} is given by :

$$k_{estim} = \frac{mg}{2}\left(\frac{L_s^{\,2}}{\left(L_s^{\,2} - d^2\right)^{3/2}}\right) \qquad \text{… (Equation 6)}$$

where,

ω_{beat} = beat frequency of the coupled pendulum

ω_- = out of phase frequency

ω_+ = in phase frequency

M_{rod} = mass of the pendulum rod

L = length of the rod of the pendulum

h = height of the bob

M_{bob} = mass of the bob

m = coupling mass

g = acceleration due to gravity (= 9.8 m/s)

L_s = length of the string of the coupling assembly

d = distance between the two coupled pendula

§5 Measurements and Data Tabulation

1. Least Count of the stopwatch = 0.01 s
2. Least Count of the meter scale = 0.1 cm
3. Least Count of the digital balance = 0.01 gm

§5.1 Measurement of frequency of oscillation of a single pendulum

T = Time Period of oscillation

Sr. No.	20 × T (in s)	T (in s)	$\omega = \frac{2\pi}{T}$ (in Hz)	Average ω (in Hz)
1	35.13	1.7565	3.577	3.5814
2	35.00	1.7500	3.590	
3	35.03	1.7515	3.587	
4	35.12	1.7560	3.578	
5	35.15	1.7575	3.575	

But the frequency of the oscillation of a single pendulum taken alone corresponds to its in-phase frequency, ω_+. Thus,

$$\omega_+ = 3.5814 \text{ Hz}$$

§5.2 Measurement of beat frequency for l = 35 cm:

Sr. No.	T_{beat} (in s)	$\omega = \frac{2\pi}{T_{beat}}$ (in Hz)	Average ω (in Hz)
1	53.97	0.1164	0.1160
2	53.84	0.1167	
3	54.23	0.1159	
4	54.62	0.1150	

So the beat frequency of the coupled pendulum for l = 35 cm is given by:

$$\omega_{beat\,,\,35\,cm} = 0.1160 \text{ Hz}$$

§5.3 Measurement of beat frequency for l = 45 cm:

Sr. No.	T_{beat} (in s)	$\omega = \frac{2\pi}{T_{beat}}$ (in Hz)	Average ω (in Hz)
1	36.47	0.1723	0.1724
2	36.15	0.1738	
3	37.41	0.1680	
4	35.84	0.1753	

So the beat frequency of the coupled pendulum for l = 45 cm is given by:

$$\omega_{beat\,,\,45\,cm} = 0.1724 \text{ Hz}$$

§5.4 Determination of the moment of inertia of the pendulum:

Given that,
Length of the pendulum rod, L = 85 cm = 0.85m
Mass of the rod of the pendulum, $M_{rod} = 141g = 0.141kg$
Height of the bob, $h = 3cm = 0.03m$
Mass of the bob, $M_{bob} = 176g = 0.176kg$

Now from equations (2) and (3) we get:

$$I_{rod} = \frac{1}{3} M_{rod} L^2 \left(1 - \frac{3h}{L}\right) = \frac{1}{3} \times 0.141 \times 0.85^2 \times \left(1 - \frac{3\times0.03}{0.85}\right) = 0.0304\ kg\ m^2$$

and $$I_{bob} = M_{bob} L^2 \left(1 - \frac{h}{L}\right) = 0.176 \times 0.85^2 \times \left(1 - \frac{0.03}{0.85}\right) = 0.1227\ kg\ m^2$$

Also, $$I_{pendulum} = I_{rod} + I_{bob} = 0.1227 + 0.0304 = 0.1531\ kg\ m^2$$

Thus the moment of inertia of the pendulum about the point of oscillation is given by:

$$I_{pendulum} = 0.1531\ kg\ m^2$$

§5.5 Estimation of the effective spring constant of the string-mass system:

Given that,
Length of the string, L_s of the coupling assembly = 19.0 cm = 0.19 m
Coupling mass, m = 13.6 g = 0.0136 kg
Distance between the coupled pendula, d = 8.5 cm = 0.085 m

Now, using equation (6), we get:

$$k_{estim} = \frac{mg}{2}\left(\frac{L_s^2}{(L_s^2 - d^2)^{\frac{3}{2}}}\right) = \frac{0.0136 \times 9.8}{2}\left(\frac{0.19^2}{(0.19^2 - 0.085^2)^{\frac{3}{2}}}\right) = 0.4903 Nm^{-1}$$

Thus the estimated spring constant of the string-mass assembly is given by:

$$k_{estim} = 0.4903\ Nm^{-1}$$

§6 Calculations

Now we are required to calculate the experimental value of the effective spring constant of the string-mass assembly. For that we have two sets of data : one when l = 35 cm and the other when l = 45 cm. We will calculate the spring constants for the two data sets and then take their arithmetic mean to get the final experimental value of the spring constant, k_{expt} . Firstly from equation (1), we get :

$$\omega_{beat} = \omega_- - \omega_+ \Rightarrow \omega_- = \omega_{beat} + \omega_+$$

a. For l = 35 cm:

$$\omega_{-,\,35\,cm} = \omega_{beat,\,35\,cm} + \omega_{+,\,35\,cm} = 0.1160 + 3.5814 = 3.6974\ Hz$$

Now, from equation (5)

$$k_{expt,\,35\,cm} = \frac{I_{pendulum}}{2l_1^{\ 2}}(\omega_{-,\,35\,cm}^{\ \ 2} - \omega_{+,\,35\,cm}^{\ \ 2}) = \frac{0.1531}{2 \times 0.35^2}(3.6974^2 - 3.5841^{\ 2})$$
$$= 0.5155\ Nm^{-1}$$

Thus the value of k_{expt} for l = 35 cm is 0.5155 Nm^{-1}.

For l = 45 cm:

$$\omega_{-,\,45\,cm} = \omega_{beat,\,45\,cm} + \omega_{+,\,45\,cm} = 0.1724 + 3.5814 = 3.7538\,Hz$$

From equation (5)

$$k_{expt,\,45\,cm} = \frac{I_{pendulum}}{2l_2^{\,2}}(\omega_{-,\,45\,cm}^{\,2} - \omega_{+,\,45\,cm}^{\,2}) = \frac{0.1531}{2\times 0.45^2}(3.7538^2 - 3.5841^{\,2})$$
$$= 0.4707\,Nm^{-1}$$

Thus the value of k_{expt} for l = 45 cm is 0.4707 Nm^{-1}.

$$\text{Now average } k_{expt} = \frac{k_{expt,\,35\,cm} + k_{expt,\,45\,cm}}{2} = \frac{0.5155 + 0.4707}{2} = 0.4931\,Nm^{-1}$$

Thus, the average experimental value of spring constant of the string-mass system is 0.4931 Nm^{-1}

§7 Error Analysis

We now know both the experimental as well as the estimated value of the required spring constant.

§7.1 Absolute Error

$$\begin{aligned} Absolute\ Error &= |Experimental\ Value(k_{expt}) - Estimated\ Value(k_{estim})| \\ &= |0.4931 - 0.4903| \\ &= 0.0028\ Nm^{-1} \end{aligned}$$

§7.2 Percentage Error

$$Percentage\ Error = \frac{Absolute\ Error}{Estimated\ Value} \times 100\% = \frac{0.0028}{0.4903} \times 100\% = 0.57\%$$

§7.3 Estimated Error

Given that,

$$\delta l = \delta L = \delta h = 0.001\,m$$
$$\delta M_{bob} = \delta M_{rod} = 10^{-5}\,kg$$
$$\delta T_{beat} = \delta T_{+} = 0.01\,s$$

$$k_{expt} = \frac{I_{pendulum}}{2l^2}(\omega_{-}^{\,2} - \omega_{+}^{\,2}) = \frac{I_{pendulum}}{2l^2} \times \left(\omega_{beat} + 2\omega_{+}\right) \times \omega_{beat}$$
$$= \frac{I_{pendulum}}{2l^2} \times \left(\frac{2\pi}{T_{beat}} + \frac{4\pi}{T_{+}}\right) \times \frac{2\pi}{T_{beat}}$$

We first need to find $\delta l_{pendulum}$. For that we have,

$$I_{pendulum} = I_{rod} + I_{bob} = \frac{1}{3}M_{rod}\left(L^2 - 3hL\right) + M_{bob}\left(L^2 - hL\right)$$

$$\delta I_{pendulum} = \left|\frac{\partial I_{pendulum}}{\partial M_{rod}}\right|\delta M_{rod} + \left|\frac{\partial I_{pendulum}}{\partial M_{bob}}\right|\delta M_{bob} + \left|\frac{\partial I_{pendulum}}{\partial L}\right|\delta L + \left|\frac{\partial I_{pendulum}}{\partial h}\right|\delta h$$

$$= \left|\frac{1}{3}(L^2 - 3hL)\right|10^{-5} + \left|(L^2 - hL)\right|10^{-5} + \left|\frac{1}{3}M_{rod}(2L - 3h) + M_{bob}(2L - h)\right|10^{-3} + \left|M_{rod}L + M_{bob}L\right|10^{-3}$$

$$= 1.52 \times 10^{-3}\, Nm^{-1}$$

Now using the knowledge of partial derivatives, we can write:

$$\delta k_{expt} = \left|\frac{\partial k_{expt}}{\partial l}\right|\delta l + \left|\frac{\partial k_{expt}}{\partial I_{pendulum}}\right|\delta I_{pendulum} + \left|\frac{\partial k_{expt}}{\partial T_{beat}}\right|\delta T_{beat} + \left|\frac{\partial k_{expt}}{\partial T_{+}}\right|\delta T_{+}$$

$$= \left|\frac{2k_{expt}}{l}\right|10^{-3} + \left|\frac{k_{expt}}{I_{pendulum}}\right|1.52 \times 10^{-3} + \left|\frac{I_{pendulum}}{2l^2}\left(\frac{32\pi^2}{T_{beat}^{\ 3}} + \frac{8\pi^2}{T_+T_{beat}}\right)\right|0.01 + \left|\frac{I_{pendulum}}{2l^2}\left(\frac{8\pi^2}{T_+^{\ 2}T_{beat}}\right)\right|0.01$$

$$= |2.466| \times 10^{-3} + |3.221| \times 1.52 \times 10^{-3} + |0.020| \times 0.01 + |0.001| \times 0.01$$

$$= 7.57 \times 10^{-3} Nm^{-1}$$

$$\text{Thus, } \frac{\delta k_{expt}}{k_{expt}} \times 100 = \frac{7.57 \times 10^{-3}}{0.4707} \times 100 = 1.61\,\%$$

Thus the estimated propagation error in the calculation of effective spring constant of the string-mass assembly is 1.61 % .

§8 Sources of Error

- The oscillation of the two coupled pendula doesn't remain constrained to the theoretical plane of oscillation. Instead the two bobs wobble and deviate out of their original plane of oscillation thus leading to error in experiment.
- The friction at the contact points (eg. at the top end of each pendulum) also resists the motion of the pendulum which hasn't been taken into account.
- There is dragging force due to air which also contributes to the error in the final value of spring constant.
- Inaccuracy in working with the stopwatch. Exact readings of time elapsed cannot be taken due to a finite reaction time of the person performing the experiment.
- Errors arising due to limited precision (least count) of the instruments used i.e. the meter scale and the stopwatch.

§9 Measures taken to increase precision and accuracy, and reduce errors

- In order to increase accuracy in the measurement of time period, time elapsed is measured for 20 oscillations instead of a single oscillation. This reduced the error associated with the time period significantly.
- Small oscillations are taken into consideration in order to minimize the wobbly motion of the pendula bobs and so that the motion of the pendula remains confined within the required plane of oscillation.
- Further, the experiment is performed with two set of values of l (35 cm and 45 cm) instead of one. This gives more data values of the spring constant and thus further reduces the error.

§10 Conclusion

The value of effective spring constant of the given string-mass assembly is calculated to be 0.4931 Nm^{-1} with a deviation of 0.57 % from the estimated value (= 0.4903 Nm^{-1}).

The propagation error due to limited accuracy of the instruments is 1.61 %.