MA 1101 : Mathematics I

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Solution 1.
Let A, B, C be sets.

(i) We wish to prove $A \cup B = B \cup A$. We do so by showing that $A \cup B \subseteq B \cup A$ and $B \cup A \subseteq A \cup B$.

Let $x \in A \cup B$. This implies $x \in A$ or $x \in B$, which is the same as $x \in B$ or $x \in A$. Thus, $x \in B \cup A$. This proves $A \cup B \subseteq B \cup A$.

Similarly, let $x \in B \cup A$. This implies $x \in B$ or $x \in A$, which is the same as $x \in A$ or $x \in B$. Thus, $x \in A \cup B$. This proves $B \cup A \subseteq A \cup B$, and we are done. $\square$

Next, we wish to prove $A \cap B = B \cap A$. We do so by showing that $A \cap B \subseteq B \cap A$ and $B \cap A \subseteq A \cap B$.

Let $x \in A \cap B$. This implies $x \in A$ and $x \in B$, which is the same as $x \in B$ and $x \in A$. Thus, $x \in B \cap A$. This proves $A \cap B \subseteq B \cap A$.

Similarly, let $x \in B \cap A$. This implies $x \in B$ and $x \in A$, which is the same as $x \in A$ and $x \in B$. Thus, $x \in A \cap B$. This proves $B \cap A \subseteq A \cap B$, and we are done. $\square$

(ii) We wish to prove $(A \cup B) \cup C = A \cup (B \cup C)$. We do so by showing that $(A \cup B) \cup C \subseteq A \cup (B \cup C)$ and $A \cup (B \cup C) \subseteq (A \cup B) \cup C$.

Let $\wedge$ denote 'and' and $\vee$ denote 'or'. Let

$$\begin{aligned} x \in (A \cup B) \cup C &\Rightarrow x \in (A \cup B) \vee x \in C \\ &\Rightarrow (x \in A \vee x \in B) \vee x \in C \\ &\Rightarrow x \in A \vee x \in B \vee x \in C \\ &\Rightarrow x \in A \vee (x \in B \vee x \in C) \\ &\Rightarrow x \in A \vee x \in (B \cup C) \\ &\Rightarrow x \in A \cup (B \cup C) \end{aligned}$$

This proves, $(A \cup B) \cup C \subseteq A \cup (B \cup C)$. Similarly, let

$$\begin{aligned} x \in A \cup (B \cup C) &\Rightarrow x \in A \vee x \in (B \cup C) \\ &\Rightarrow x \in A \vee (x \in B \vee x \in C) \\ &\Rightarrow x \in A \vee x \in B \vee x \in C \\ &\Rightarrow (x \in A \vee x \in B) \vee x \in C \\ &\Rightarrow x \in (A \cup B) \vee x \in C \\ &\Rightarrow x \in (A \cup B) \cup C \end{aligned}$$

This proves, $A \cup (B \cup C) \subseteq (A \cup B) \cup C$, and we are done. $\square$

Next, we wish to prove $(A \cap B) \cap C = A \cap (B \cap C)$. We do so by showing that $(A \cap B) \cap C \subseteq A \cap (B \cap C)$ and $A \cap (B \cap C) \subseteq (A \cap B) \cap C$. Let

$$\begin{aligned} x \in (A \cap B) \cap C &\Rightarrow x \in (A \cap B) \wedge x \in C \\ &\Rightarrow (x \in A \wedge x \in B) \wedge x \in C \\ &\Rightarrow x \in A \wedge x \in B \wedge x \in C \\ &\Rightarrow x \in A \wedge (x \in B \wedge x \in C) \\ &\Rightarrow x \in A \wedge x \in (B \cap C) \\ &\Rightarrow x \in A \cap (B \cap C) \end{aligned}$$

This proves, $(A \cap B) \cap C \subseteq A \cap (B \cap C)$. Similarly, let

$$\begin{aligned}
x \in A \cap (B \cap C) &\Rightarrow x \in A \wedge x \in (B \cap C) \\
&\Rightarrow x \in A \wedge (x \in B \wedge x \in C) \\
&\Rightarrow x \in A \wedge x \in B \wedge x \in C \\
&\Rightarrow (x \in A \wedge x \in B) \wedge x \in C \\
&\Rightarrow x \in (A \cap B) \wedge x \in C \\
&\Rightarrow x \in (A \cap B) \cap C
\end{aligned}$$

This proves, $A \cap (B \cap C) \subseteq (A \cap B) \cap C$, and we are done. $\square$

(iii) We wish to prove $A \subseteq B$ if and only if $A \cup B = B$. We first show that $A \subseteq B$ if $A \cup B = B$.

$$\begin{aligned}
x \in A &\Rightarrow x \in A \vee x \in B \\
&\Rightarrow x \in A \cup B \\
&\Rightarrow x \in B && (A \cup B = B)
\end{aligned}$$

Thus, $A \cup B = B \Rightarrow A \subseteq B$. Next, we show that if $A \cup B = B$ if $A \subseteq B$.

$$\begin{aligned}
x \in A \cup B &\Rightarrow x \in A \vee x \in B \\
&\Rightarrow x \in B \vee x \in B && (A \subseteq B) \\
&\Rightarrow x \in B
\end{aligned}$$

$$\begin{aligned}
x \in B &\Rightarrow x \in B \vee x \in A \\
&\Rightarrow x \in A \vee x \in B \\
&\Rightarrow x \in A \cup B
\end{aligned}$$

Thus, $A \subseteq B \Rightarrow A \cup B = B$.
This proves $A \subseteq B \Leftrightarrow A \cup B = B$. $\square$

(iv) We wish to prove $A \subseteq B$ if and only if $A \cap B = A$. We first show that $A \subseteq B$ if $A \cap B = A$.

$$\begin{aligned}
x \in A &\Rightarrow x \in A \cap B && (A \cap B = A) \\
&\Rightarrow x \in A \wedge x \in B \\
&\Rightarrow x \in B
\end{aligned}$$

Thus, $A \cap B = A \Rightarrow A \subseteq B$. Next, we show that $A \cap B = A$ if $A \subseteq B$.

$$\begin{aligned}
x \in A \cap B &\Rightarrow x \in A \wedge x \in B \\
&\Rightarrow x \in A
\end{aligned}$$

$$\begin{aligned}
x \in A &\Rightarrow x \in A \wedge x \in A \\
&\Rightarrow x \in A \wedge x \in B && (A \subseteq B) \\
&\Rightarrow x \in A \cap B
\end{aligned}$$

Thus, $A \subseteq B \Rightarrow A \cap B = A$.
This proves $A \subseteq B \Leftrightarrow A \cap B = A$. $\square$

(v) We wish to prove $A \subseteq B$ if and only if $A \setminus B = \emptyset$. We first show that $A \subseteq B$ if $A \setminus B = \emptyset$.

$$\begin{aligned}
x \in A &\Rightarrow x \in A \wedge (x \in B \vee x \notin B) \\
&\Rightarrow (x \in A \wedge x \in B) \vee (x \in A \wedge x \notin B) \\
&\Rightarrow (x \in A \wedge x \in B) \vee x \in A \setminus B \\
&\Rightarrow (x \in A \wedge x \in B) \vee x \in \emptyset && (A \setminus B = \emptyset) \\
&\Rightarrow x \in A \wedge x \in B && (x \in A) \\
&\Rightarrow x \in B
\end{aligned}$$

Thus, $A \setminus B = \emptyset \Rightarrow A \subseteq B$. Next, we show that $A \setminus B = \emptyset$ if $A \subseteq B$.

$$\begin{aligned} x \in A \setminus B &\Rightarrow x \in A \wedge x \notin B \\ &\Rightarrow x \in B \wedge x \notin B && (A \subseteq B) \end{aligned}$$

However, there is no such x which is simultaneously in and not in B. Hence, the set $A \setminus B$ is empty, that is, $A \subseteq B \Rightarrow A \setminus B = \emptyset$.

This proves $A \subseteq B \Leftrightarrow A \setminus B = \emptyset$. □

(vi) We wish to prove $A \setminus (A \setminus B) = A \cap B$.

Note that for sets X and Y,

$$\begin{aligned} X \setminus Y &= \{x : x \in X \wedge x \notin Y\} \\ &= \{x : x \in X \wedge x \in Y^C\} \\ &= X \cap Y^C \end{aligned}$$

Thus, $X \cap X^C = \{x : x \in X \wedge x \notin X\} = \emptyset$. Also note that $(X^C)^C = X$, since

$$\begin{aligned} x \in X &\Leftrightarrow x \notin X^C \\ &\Leftrightarrow x \in (X^C)^C \end{aligned}$$

Thus, we have

$$\begin{aligned} A \setminus (A \setminus B) &= A \setminus (A \cap B^C) \\ &= A \cap (A \cap B^C)^C \\ &= A \cap (A^C \cup (B^C)^C) && \text{(De Morgan's Law)} \\ &= A \cap (A^C \cup B) \\ &= (A \cap A^C) \cup (A \cap B) && \text{(Distributive Law)} \\ &= \emptyset \cup (A \cap B) \\ &= A \cap B \end{aligned}$$
□

(vii) We wish to prove $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

$$\begin{aligned} A \setminus (B \cup C) &= A \cap (B \cup C)^C \\ &= A \cap (B^C \cap C^C) && \text{(De Morgan's Law)} \\ &= (A \cap B^C) \cap (A \cap C^C) && \text{(Distributive Law)} \\ &= (A \setminus B) \cap (A \setminus C) \end{aligned}$$
□

(viii) We wish to prove $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

$$\begin{aligned} A \setminus (B \cap C) &= A \cap (B \cap C)^C \\ &= A \cap (B^C \cup C^C) && \text{(De Morgan's Law)} \\ &= (A \cap B^C) \cup (A \cap C^C) && \text{(Distributive Law)} \\ &= (A \setminus B) \cup (A \setminus C) \end{aligned}$$
□

(ix) We wish to prove $A \Delta B = (A \cup B) \setminus (A \cap B)$.

Let U be a universal set. Note that for a set X, $X \cup X^C = \{x : x \in X \vee x \notin X\} = U$. Also,

$X \cap U = \{x : x \in X \wedge x \in U\} = X$.

$$
\begin{aligned}
A \Delta B &= (A \setminus B) \cup (B \setminus A) \\
&= (A \cap B^C) \cup (B \cap A^C) \\
&= ((A \cap B^C) \cup B) \cap ((A \cap B^C) \cup A^C) && \text{(Distributive Law)} \\
&= (B \cup (A \cap B^C)) \cap (A^C \cup (A \cap B^C)) \\
&= ((B \cup A) \cap (B \cup B^C)) \cap ((A^C \cup A) \cap (A^C \cup B^C)) && \text{(Distributive Law)} \\
&= ((B \cup A) \cap U) \cap (U \cap (A^C \cup B^C)) \\
&= (B \cup A) \cap (A^C \cup B^C) \\
&= (A \cup B) \cap (A \cap B)^C && \text{(De Morgan's Law)} \\
&= (A \cup B) \setminus (A \cap B) && \square
\end{aligned}
$$

(x) We wish to prove $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$.

$$
\begin{aligned}
(A \cap B) \Delta (A \cap C) &= ((A \cap B) \cup (A \cap C)) \setminus ((A \cap B) \cap (A \cap C)) && \text{(From (ix))} \\
&= (A \cap (B \cup C)) \setminus (A \cap B \cap A \cap C) && \text{(Distributive Law)} \\
&= (A \cap (B \cup C)) \setminus (A \cap B \cap C) \\
&= (A \cap (B \cup C)) \cap (A \cap (B \cap C))^C \\
&= (A \cap (B \cup C)) \cap (A^C \cup (B \cap C)^C) && \text{(De Morgan's Law)} \\
&= (A \cap (B \cup C) \cap A^C) \cup (A \cap (B \cup C) \cap (B \cap C)^C) && \text{(Distributive Law)} \\
&= (A \cap A^C \cap (B \cup C)) \cup (A \cap (B \cup C) \cap (B \cap C)^C) \\
&= (\emptyset \cap (B \cup C)) \cup (A \cap (B \cup C) \setminus (B \cap C)) \\
&= \emptyset \cup (A \cap (B \Delta C)) && \text{(From (ix))} \\
&= A \cap (B \Delta C) && \square
\end{aligned}
$$

(xi) We wish to prove $A \Delta (B \Delta C) = (A \Delta B) \Delta C$.
Note that $A \Delta B = B \Delta A$, since

$$
\begin{aligned}
A \Delta B &= (A \cup B) \setminus (A \cap B) \\
&= (B \cup A) \setminus (B \cap A) \\
&= B \Delta A
\end{aligned}
$$

First, we expand

$$
\begin{aligned}
A \Delta (B \Delta C) &= (A \setminus (B \Delta C)) \cup ((B \Delta C) \setminus A) \\
&= (A \setminus ((B \setminus C) \cup (C \setminus B))) \cup (((B \setminus C) \cup (C \setminus B)) \setminus A) \\
&= (A \cap ((B \cap C^C) \cup (C \cap B^C))^C) \cup (((B \cap C^C) \cup (C \cap B^C)) \cap A^C) \\
&= (A \cap ((B \cap C^C)^C \cap (C \cap B^C)^C)) \cup (((B \cap C^C) \cup (C \cap B^C)) \cap A^C) \\
&= (A \cap ((B^C \cup C) \cap (C^C \cup B))) \cup (((B \cap C^C) \cup (C \cap B^C)) \cap A^C) \\
&= (A \cap ((B^C \cap (C^C \cup B)) \cup (C \cap (C^C \cup B)))) \cup (((B \cap C^C) \cup (C \cap B^C)) \cap A^C) \\
&= (A \cap ((B^C \cap C^C) \cup (B^C \cap B) \cup (C \cap C^C) \cup (C \cap B))) \cup (((B \cap C^C) \cup (C \cap B^C)) \cap A^C) \\
&= (A \cap ((B^C \cap C^C) \cup \emptyset \cup \emptyset \cup (C \cap B))) \cup (((B \cap C^C) \cap A^C) \cup ((C \cap B^C) \cap A^C)) \\
&= (A \cap ((B^C \cap C^C) \cup (C \cap B))) \cup ((B \cap C^C \cap A^C) \cup (C \cap B^C \cap A^C)) \\
&= ((A \cap (B^C \cap C^C)) \cup (A \cap (C \cap B))) \cup ((B \cap C^C \cap A^C) \cup (C \cap B^C \cap A^C)) \\
&= ((A \cap B^C \cap C^C) \cup (A \cap B \cap C)) \cup ((A^C \cap B \cap C^C) \cup (A^C \cap B^C \cap C)) \\
&= (A \cap B \cap C) \cup (A \cap B^C \cap C^C) \cup (A^C \cap B \cap C^C) \cup (A^C \cap B^C \cap C)
\end{aligned}
$$

Similarly,

$$\begin{aligned}
(A\Delta B)\Delta C &= ((A\Delta B)\setminus C)\cup(C\setminus(A\Delta B)) \\
&= (((A\setminus B)\cup(B\setminus A))\setminus C)\cup(C\setminus((A\setminus B)\cup(B\setminus A))) \\
&= (((A\cap B^C)\cup(B\cap A^C))\cap C^C)\cup(C\cap((A\cap B^C)\cup(B\cap A^C))^C) \\
&= (((A\cap B^C)\cup(B\cap A^C))\cap C^C)\cup(C\cap((A\cap B^C)^C\cap(B\cap A^C)^C)) \\
&= (((A\cap B^C)\cup(B\cap A^C))\cap C^C)\cup(C\cap((A^C\cup B)\cap(B^C\cup A))) \\
&= (((A\cap B^C)\cup(B\cap A^C))\cap C^C)\cup(C\cap((A^C\cap(B^C\cup A))\cup(B\cap(B^C\cup A)))) \\
&= (((A\cap B^C)\cup(B\cap A^C))\cap C^C)\cup(C\cap((A^C\cap B^C)\cup(A^C\cap A)\cup(B\cap B^C)\cup(B\cap A))) \\
&= (((A\cap B^C)\cap C^C)\cup((B\cap A^C)\cap C^C))\cup(C\cap((A^C\cap B^C)\cup\emptyset\cup\emptyset\cup(B\cap A))) \\
&= ((A\cap B^C\cap C^C)\cup(B\cap A^C\cap C^C))\cup(C\cap((A^C\cap B^C)\cup(B\cap A))) \\
&= ((A\cap B^C\cap C^C)\cup(B\cap A^C\cap C^C))\cup((C\cap(A^C\cap B^C))\cup(C\cap(B\cap A))) \\
&= ((A\cap B^C\cap C^C)\cup(A^C\cap B\cap C^C))\cup((A^C\cap B^C\cap C)\cup(A\cap B\cap C)) \\
&= (A\cap B\cap C)\cup(A\cap B^C\cap C^C)\cup(A^C\cap B\cap C^C)\cup(A^C\cap B^C\cap C)
\end{aligned}$$

Thus, $A\Delta(B\Delta C)$ and $(A\Delta B)\Delta C$ expand to the same expression, proving them to be equal. □

(xii) We wish to prove $A\Delta B = A\Delta C$ if and only if $B = C$.

Note that for a set X, $X\Delta X = (X\setminus X)\cup(X\setminus X) = \emptyset$, and $X\Delta\emptyset = \emptyset\Delta X = (X\setminus\emptyset)\cup(\emptyset\setminus X) = X$. Using the result from (xi)

$$\begin{aligned}
(A\Delta A)\Delta B &= A\Delta(A\Delta B) \\
&= A\Delta(A\Delta C) \\
&= (A\Delta A)\Delta C \\
\emptyset\Delta B &= \emptyset\Delta C \\
B &= C
\end{aligned}$$

□

Solution 2. Let A, B, C, D be sets.

(i) We wish to prove $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

$$\begin{aligned}
(x, y) \in A \times (B \cup C) &\Leftrightarrow x \in A \wedge y \in (B \cup C) \\
&\Leftrightarrow (x \in A) \wedge (y \in B \vee y \in C) \\
&\Leftrightarrow (x \in A \wedge y \in B) \vee (x \in A \vee y \in C) \\
&\Leftrightarrow ((x, y) \in A \times B) \vee ((x, y) \in A \times C) \\
&\Leftrightarrow (x, y) \in (A \times B) \cup (A \times C) \qquad \square
\end{aligned}$$

(ii) We wish to prove $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

$$\begin{aligned}
(x, y) \in A \times (B \cap C) &\Leftrightarrow x \in A \wedge y \in (B \cap C) \\
&\Leftrightarrow (x \in A) \wedge (y \in B \wedge y \in C) \\
&\Leftrightarrow (x \in A \wedge y \in B) \wedge (x \in A \wedge y \in C) \\
&\Leftrightarrow ((x, y) \in A \times B) \wedge ((x, y) \in A \times C) \\
&\Leftrightarrow (x, y) \in (A \times B) \cap (A \times C) \qquad \square
\end{aligned}$$

(iii) We wish to prove $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$.

$$\begin{aligned}
(x, y) \in A \times (B \setminus C) &\Rightarrow x \in A \wedge y \in (B \setminus C) \\
&\Rightarrow (x \in A) \wedge (y \in B \wedge y \notin C) \\
&\Rightarrow (x \in A \wedge y \in B) \wedge (y \notin C) \\
&\Rightarrow (x, y) \in A \times B) \wedge ((x, y) \notin A \times C) \\
&\Rightarrow (x, y) \in (A \times B) \setminus (A \times C)
\end{aligned}$$

$$\begin{aligned}
(x, y) \in (A \times B) \setminus (A \times C) &\Rightarrow ((x, y) \in A \times B) \wedge ((x, y) \notin A \times C) \\
&\Rightarrow (x \in A \wedge y \in B) \wedge (x \notin A \vee y \notin C) \\
&\Rightarrow (x \in A \wedge y \in B \wedge x \notin A) \vee (x \in A \wedge y \in B \wedge y \notin C) \\
&\Rightarrow (x \in \emptyset) \vee (x \in A \wedge y \in (B \setminus C)) \\
&\Rightarrow x \in A \times (B \setminus C)
\end{aligned}$$

Since each side is a subset of the other, they are equal. $\square$

(iv) We wish to determine whether $\mathcal{P}(A \times B) = \mathcal{P}(A) \times \mathcal{P}(B)$. This can be shown to be false in general. As a counterexample, consider $A = \{a\}$, $B = \{b\}$.

$$\begin{aligned}
A \times B &= \{(a, b)\} \\
\mathcal{P}(A \times B) &= \{\emptyset, \{(a, b)\}\} \\
\mathcal{P}(A) &= \{\emptyset, \{a\}\} \\
\mathcal{P}(B) &= \{\emptyset, \{b\}\} \\
\mathcal{P}(A) \times \mathcal{P}(B) &= \{(\emptyset, \emptyset), (\emptyset, \{b\}), (\{a\}, \emptyset), (\{a\}, \{b\})\} \qquad \square
\end{aligned}$$

(v) We wish to determine whether $(A \cap C) \times (B \cap D) = (A \times B) \cap (C \times D)$. We prove this by selecting

$$\begin{aligned}
(x, y) \in (A \cap C) \times (B \cap D) &\Leftrightarrow x \in (A \cap C) \wedge y \in (B \cap D) \\
&\Leftrightarrow x \in A \wedge x \in C \wedge y \in B \wedge y \in D \\
&\Leftrightarrow x \in A \wedge y \in B \wedge x \in C \wedge y \in D \\
&\Leftrightarrow ((x, y) \in A \times B) \wedge ((x, y) \in C \times D) \\
&\Leftrightarrow (x, y) \in (A \times B) \cap (B \times C) \qquad \square
\end{aligned}$$

(vi) We wish to determine whether $(A \cup C) \times (B \cup D) = (A \times B) \cup (C \times D)$. This can be shown to be false in general. As a counterexample, consider

$$\begin{aligned} A &= \{a\} \\ B &= \{b\} \\ C &= \{c\} \\ D &= \{d\} \\ A \cup C &= \{a, c\} \\ B \cup D &= \{b, d\} \\ (A \cup C) \times (B \cup D) &= \{(a, b), (a, d), (c, b), (c, d)\} \\ (A \times B) &= \{(a, b)\} \\ (C \times D) &= \{(c, d)\} \\ (A \times B) \cup (C \times D) &= \{(a, b), (c, d)\} \end{aligned}$$

$\square$

Solution 3. Let $n \in \mathbb{N}$ and let X be a set of n elements.

(i) The number of subsets of X is 2^n.

A subset of X must have $k \in \{0, 1, 2, \dots, n\}$ elements. For a given k, there are exactly $\binom{n}{k}$ ways of selecting k elements from X, hence there are as many subsets of X with k elements. Thus, the total number of subsets of X is

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \qquad \square$$

(ii) The number of non-empty subsets of X is $2^n - 1$.

Of the 2^n subsets of X, the number of empty subsets, that is, sets with zero elements, is exactly $\binom{n}{0} = 1$. Removing the empty set from our count gives $2^n - 1$. $\square$

(iii) The number of ways one can choose two disjoint subsets of X is $(3^n + 1)/2$.

Let us choose two disjoint subsets A and B of X. Each $x \in X$ has 3 choices: it can be placed either in A, or in B, or in neither. This gives us 3^n ways of constructing A and B. Note that we are not concerned about the order in which we choose A and B, so we have precisely double counted the cases when $A \neq B$, i.e., all but one, giving us $(3^n - 1)/2$. The only remaining case is $A = B = \emptyset$, which we add back on, giving a total of $(3^n + 1)/2$. $\square$

(iv) The number of ways one can choose two non-empty disjoint subsets of X is $(3^n - 2^{n+1} + 1)/2$.

Again, let us choose two disjoint subsets A and B of X. Of the 3^n ways of placing some $x \in X$ in A, B, or neither, note that A remains empty in exactly 2^n cases. This is because each $x \in X$ has 2 choices: it can be placed either in B, or in neither A nor B. Similarly, B remains empty in exactly 2^n cases, since each $x \in X$ can be placed either in A or in neither A nor B. We have excluded the case where $A = B = \emptyset$ twice, so we have $3^n - 2^n - 2^n + 1$. Again, symmetry gives us a total of $(3^n - 2^{n+1} + 1)/2$ unordered pairs of disjoint non-empty subsets of X. $\square$