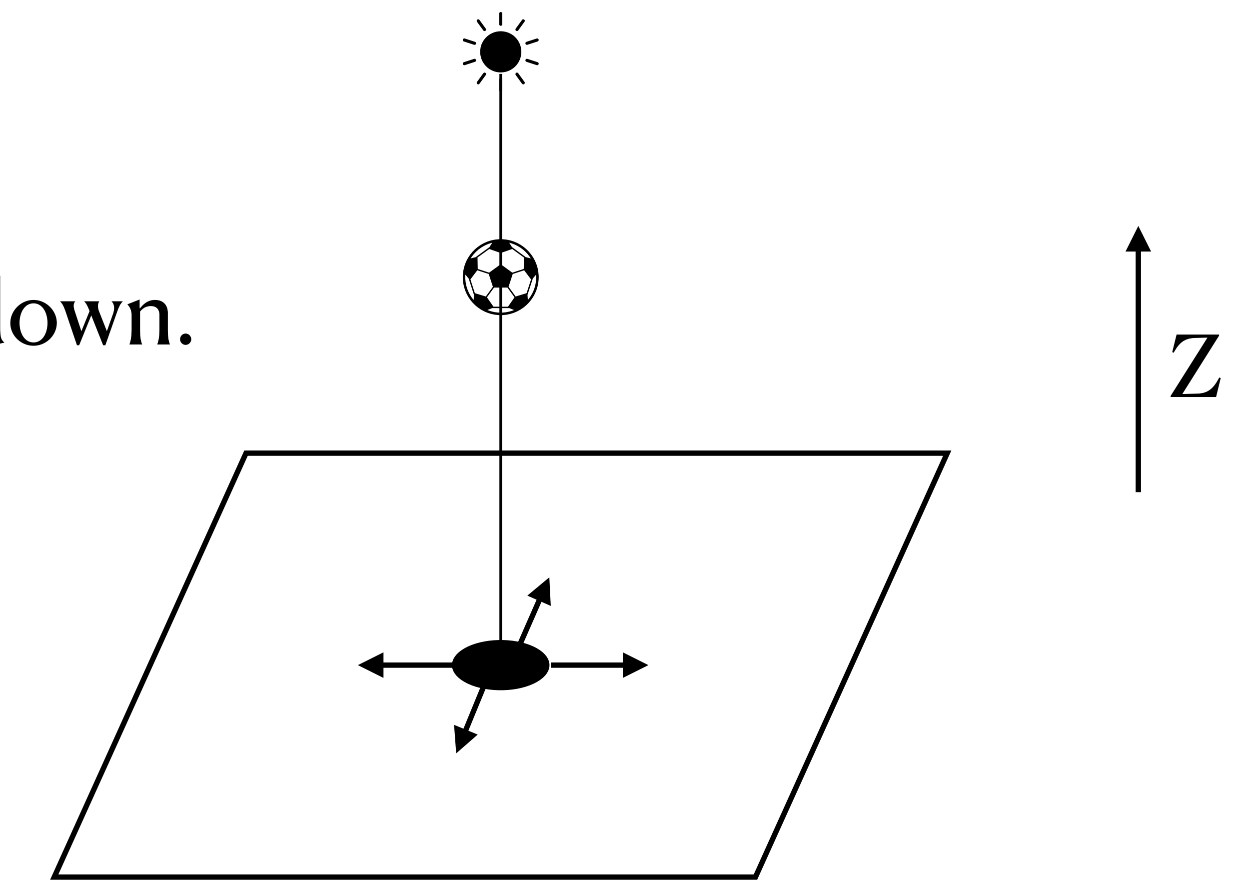
The **null space** of a matrix A is the vector space of all column vectors x satisfying A x = 0 and is denoted as **Null(A)**.

$$x, y \in Null(A) \implies A(x+y) = Ax + Ay = 0 + 0 = 0$$

$$\implies A(\alpha x) = \alpha A x = 0$$

Null(A) is closed under vector addition and scalar multiplication.

Physical interpretation of Null Space:
Z is the null space as the projection does not move though the ball moves up or down.



Let the position of the ball be $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ and the projection $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

The position of the shadow: $Av = \begin{pmatrix} v_1 \\ v_2 \\ 0 \end{pmatrix}$

Finding Null(A): $Ax = 0 \implies \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Null space is spanned by Z axis.

Example: Find Null(A)

$$A = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 2 & 3 & 1 \\ -3 & 6 & -1 & 1 & -7 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 5 & 10 & -10 \\ 0 & 0 & 1 & 2 & -2 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 5 & 10 & -10 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$R2 \leftarrow R1$

$R1 \leftarrow R2$

$R1 \equiv R1$

$R2 \leftarrow R2 + 3R1$

$R3 \leftarrow R3 - 2R1$

$R1 \equiv R1$

$R2 \leftarrow R3$

$R3 \leftarrow R2$

$R1 \equiv R1$

$R2 \equiv R2$

$R3 \leftarrow R3 - 5R2$

$R1 \leftarrow R1 - 2R2$

$$\begin{pmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

RREF(A)

$$Ax = 0 \implies \begin{pmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x_1 - 2x_2 - x_4 + 3x_5 = 0$$
$$x_3 + 2x_4 - 2x_5 = 0$$

$$x_3 = -2x_4 + 2x_5$$
$$x_1 = 2x_2 + x_4 - 3x_5$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \equiv \begin{pmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{pmatrix} = x_2 \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}$$

x_2, x_4, x_5 can take any values

$$Null(A) = \left(\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right)$$

dim(Null(A)) = number of non pivot columns of RREF(A)

Application of Null Space in expressing solutions of underdetermined system of linear equations.

Let $Av = b$ and $u \in Null(A) \implies Au = 0$

It is easy to see that $x = u + v$ is also a solution of $Ax = b$

Example $\begin{aligned} 2x_1 + 2x_2 + x_3 &= 0 \\ 2x_1 - 2x_2 - x_3 &= 1 \end{aligned} \quad \Rightarrow \begin{pmatrix} 2 & 2 & 1 \\ 2 & -2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$Au = 0 \implies \left(\begin{array}{ccc|c} 2 & 2 & 1 & 0 \\ 2 & -2 & -1 & 0 \end{array}\right) \underset{\substack{R1 \leftarrow R1 + R2 \\ R2 \leftarrow R2 - R1}}{\implies} \left(\begin{array}{ccc|c} 4 & 0 & 0 & 0 \\ 0 & -4 & -2 & 0 \end{array}\right) \underset{\substack{R1 \leftarrow R1/4 \\ R2 \leftarrow -R2/4}}{\implies} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \end{array}\right) \implies \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{aligned} u_1 &= 0 \\ u_2 + \frac{u_3}{2} &= 0 \end{aligned}$$

Null space is spanned by: $\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{u_3}{2} \\ u_3 \end{pmatrix} = \frac{u_3}{2} \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} = \alpha \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$

$$Av = b \implies \left(\begin{array}{ccc|c} 2 & 2 & 1 & 0 \\ 2 & -2 & -1 & 1 \end{array}\right) \implies \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{4} \\ 0 & 1 & \frac{1}{2} & -\frac{1}{4} \end{array}\right) \implies \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ -\frac{1}{4} \end{pmatrix} \implies \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ -\frac{1}{4} \\ 0 \end{pmatrix}$$

General solution: $x = u + v = \alpha \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} + \frac{1}{4} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

Row space of a matrix

$$A = [a_{ij}]_{m\times n} \equiv \begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{pmatrix} \qquad \text{Row space(A)} = \text{span}\{R_1, R_2, \cdots, R_m\}$$

Row equivalent matrices have the same row space.

$$A = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{pmatrix} \underset{\substack{R2 \leftarrow R2 - 2R1 \\ R3 \leftarrow R3 - 3R1}}{\Longrightarrow} \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 6 & -16 \end{pmatrix} \underset{R3 \leftarrow R3 - 2R2}{\Longrightarrow} \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & -3 & 8 \\ 0 & 0 & 0 & 0 \end{pmatrix} \underset{\substack{R2 \leftarrow R2/3 \\ R1 \leftarrow R1 + R2}}{\Longrightarrow} \begin{pmatrix} 1 & 2 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{8}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 2 & -4 & 11 \\ 2 & 4 & -5 & 14 \end{pmatrix} \underset{R2 \leftarrow R2 - 2R1}{\Longrightarrow} \begin{pmatrix} 1 & 2 & -4 & 11 \\ 0 & 0 & 3 & -8 \end{pmatrix} \underset{\substack{R2 \leftarrow R2/3 \\ R1 \leftarrow R1 + 4R2}}{\Longrightarrow} \begin{pmatrix} 1 & 2 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{8}{3} \end{pmatrix}$$

dependent columns

$$RREF(A) = \left(\begin{array}{c|c} \mathbb{I}_k & \\ \hline 0 & 0 \end{array}\right)$$

A and B are row equivalent with row space spanned by $\{1,2,0,\frac{1}{3}\}$ & $\{0,0,1,-\frac{8}{3}\}$

Number of independent rows = Number of independent columns

dim[row(A)] = dim[column(A)]

Linear Algebra: Hoffman&Kunze

Schaum's outline of linear algebra: Lipschutz