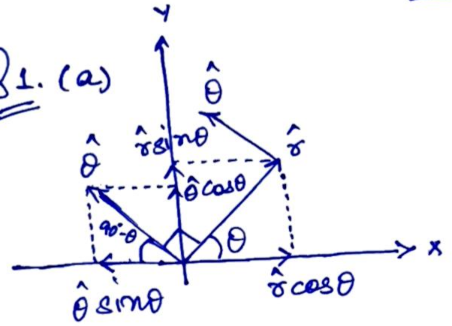
PH1101 Assignment-3

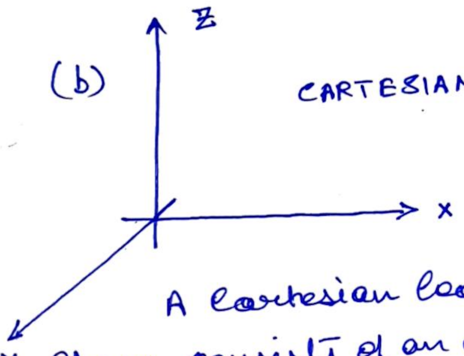
Q1. (a)



$$\boxed{\begin{aligned}\hat{r} &= \cos\theta\,\hat{i} + \sin\theta\,\hat{j}\\ \hat{\theta} &= -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}\end{aligned}}$$

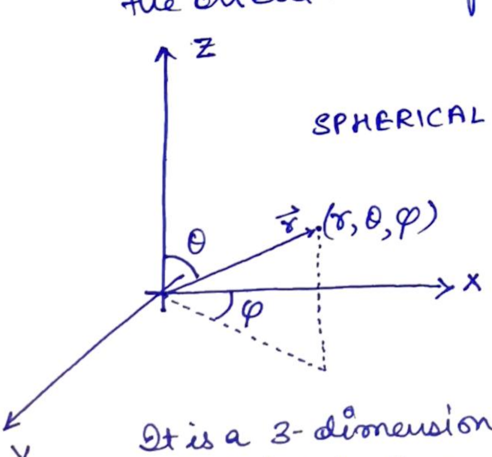
$$\Rightarrow \boxed{\begin{aligned}\hat{i} &= \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\\ \hat{j} &= \sin\theta\,\hat{r} + \cos\theta\,\hat{\theta}\end{aligned}} \quad \text{ans.}$$

(b)



CARTESIAN COORDINATE SYSTEM

A Cartesian Coordinate System for a three-dimensional space consists of an ordered triplet of lines (the axes) that go through a common point (the origin), and are pair-wise perpendicular. Each pair of axes define a coordinate hyperplane. These hyperplanes divide space into eight trihedra, called octants. Each coordinate of a point P can be taken as the distance from P to the hyperplane defined by the other two axes, with the sign determined by the orientation of the corresponding axis.



SPHERICAL POLAR COORDINATE SYSTEM

It is a 3-dimensional space where the position of a point is specified by three nos. the radial distance of that point from a fixed origin, its polar angle measured from a fixed zenith direction and the azimuthal angle of its orthogonal projection on a reference plane that passes through the origin and is orthogonal to the zenith.

(c) $P \cdot (x, y, z)$
(r, θ, φ)

$$\left.\begin{aligned} x &= r\sin\theta\cos\varphi \\ y &= r\sin\varphi\sin\theta \\ z &= r\cos\theta \end{aligned}\right\}$$ This is how the two coordinate systems are related.

(d) $\frac{d\hat{r}}{dt} = \frac{d\hat{r}}{d\theta} \cdot \frac{d\theta}{dt}$

$= \hat{\theta}\dot{\theta}$

$\hat{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$

$\frac{d\hat{r}}{d\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} = \hat{\theta}$

$\frac{d\hat{\theta}}{dt} = \frac{d\hat{\theta}}{d\theta} \cdot \frac{d\theta}{dt}$

$= -\hat{r}\dot{\theta}$

$\hat{\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$

$\frac{d\hat{\theta}}{d\theta} = -\cos\theta\,\hat{i} - \sin\theta\,\hat{j} = -\hat{r}$

Q2. (a) $\vec{r} = r.\hat{r} \Rightarrow$ Velocity is rate of change of position, so, differentiating both sides,

$$\frac{d\vec{r}}{dt} = \vec{v} = \dot{r}\hat{r} + r\dot{\hat{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

[$\dot{\hat{r}} = \dot{\theta}\hat{\theta}$ from previous problem]

$$\therefore \boxed{\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}}$$

For translatory motion, the angle would remain constant,

$\Rightarrow \dot{\theta} = 0 \Rightarrow \boxed{\vec{v} = \dot{r}\hat{r}} \leftarrow$ Translational Velocity

For uniform circular motion, only the angle would change at const. rate and radius would remain constant.

$\Rightarrow \dot{r} = 0 \Rightarrow \boxed{\vec{v} = r\dot{\theta}\hat{\theta}} \rightarrow$ Tangential Velocity (UCM)

(b) $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}) = \ddot{r}\hat{r} + \dot{r}\frac{d}{dt}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{d}{dt}\hat{\theta}$

$$\vec{a} = \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} - r\dot{\theta}^2\hat{r} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$$

$$\Rightarrow \boxed{\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}}$$

The term $\ddot{r}\hat{r}$ is a linear accl^n in the radial dir^n due to change in radial speed. Similarly, $r\ddot{\theta}\hat{\theta}$ is a linear accl^n in the tangential dir^n due to change in the magnitude of the angular velocity.

The term $r\dot{\theta}^2\hat{r}$ is the centripetal accl^n.

$2\dot{r}\dot{\theta}\hat{\theta}$ is the Coriolis Accl^n. It appears as a fictitious force in a rotating coordinate System. However, Coriolis accl^n we are discussing here is a real accl^n and which is present when r and θ both change with time.

Q3. (a) In polar coordinates : $r = ut$, $\dot{r} = u$, $\dot{\theta} = \omega$

$\Rightarrow \vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} = u\hat{r} + u\omega t\,\hat{\theta} \Rightarrow \boxed{\vec{v} = u\hat{r} + u\omega t\,\hat{\theta}}$ ans.

$\vec{r} = (r, \theta) = (ut, \omega t)$

(b) In Cartesian coordinates : $v_x = v_r\cos\theta - v_\theta\sin\theta$

$v_y = v_r\sin\theta + v_\theta\cos\theta$

Since, $v_r = u$, $v_\theta = r\omega = \omega ut$, $\theta = \omega t$,

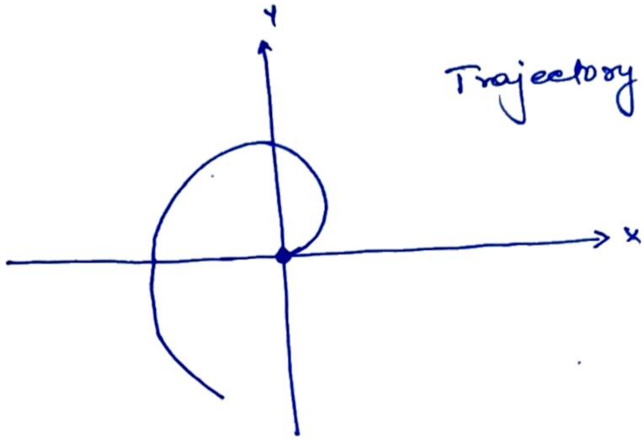
$$\boxed{\vec{v} = (u\cos\omega t - ut\omega\sin\omega t)\hat{i} + (u\sin\omega t + ut\omega\cos\omega t)\hat{j}}$$ ans.

(c) $\vec{a} = \frac{dv}{dt} = \frac{d}{dt}(u\hat{r} + u\omega t\hat{\theta})$

$= (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$

$\Rightarrow \boxed{\vec{a} = -ut\omega^2\hat{r} + 2u\omega\hat{\theta}}$ ans.

(d)



Trajectory of the bead.

Q4. (a) $\theta = \omega t$, $\vec{r} = re^{\alpha t}$

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} = \alpha r e^{\alpha t}\hat{r} + re^{\alpha t}\omega\hat{\theta}$$

$$\Rightarrow \boxed{\vec{v}(t) = \alpha r e^{\alpha t}\hat{r} + r\omega e^{\alpha t}\hat{\theta}} \text{ ans.}$$

(b) $r = \sqrt{x^2+y^2} = \sqrt{2^2+6^2} = \sqrt{4+36} = \sqrt{40}$ — (i)

$\tan\theta = -\cancel{\frac{6}{2}}3 = -3 \Rightarrow \theta = \tan^{-1}(-3)$ — (ii).

$\therefore$ $(2,-6)$ in polar coordinates would be $(\sqrt{40}, \tan^{-1}(-3))$.

(c) $\tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{4} \Rightarrow \frac{y}{x} = \tan\left(\frac{\pi}{4}\right) \Rightarrow y = x$

$\Rightarrow 2 = \sqrt{x^2+y^2} = \sqrt{2x^2} = \sqrt{2}\cdot x$.

$\Rightarrow x = \frac{2}{\sqrt{2}} = \sqrt{2}$ $\therefore$ $\left(2, \frac{\pi}{4}\right)$ in Cartesian coordinates would be $(\sqrt{2}, \sqrt{2})$.

(d) $r = 4\tan\theta\sec\theta \Rightarrow \theta = \tan^{-1}(y/x) \Rightarrow \tan\theta = (y/x)$ — (i)

$r = \sqrt{x^2+y^2}$ $\Rightarrow \cos\theta = \frac{x}{\sqrt{x^2+y^2}} \Rightarrow \sec\theta = \frac{\sqrt{x^2+y^2}}{x}$ — (ii).

$$\Rightarrow \cancel{\sqrt{x^2+y^2}} = 4\cdot\frac{y}{x}\cdot\frac{\cancel{\sqrt{x^2+y^2}}}{x} \Rightarrow 1 = 4\frac{y}{x^2}$$

$$\Rightarrow \boxed{x^2 = 4y} \text{ ans.}$$

Q5 (a) $\dfrac{4x}{3x^2+3y^2} = 6-xy$

$\Rightarrow \dfrac{4r\cos\theta}{3r^2\sin^2\theta + 3r^2\cos^2\theta} = 6-r^2\sin\theta\cos\theta$

$\Rightarrow \dfrac{4\cancel{r}\cos\theta}{3r\cancel{^2}(\sin^2\theta+\cos^2\theta)} = 6-r^2\sin\theta\cos\theta$

$\Rightarrow \dfrac{4\cos\theta}{3r} = 6-r^2\sin\theta\cos\theta$

$\Rightarrow 4\cos\theta = 18r - 3r^3\sin\theta\cos\theta$

$\Rightarrow \boxed{18r = 3r^3\sin\theta\cos\theta + 4\cos\theta}$ ans.

(b) $x^2 = \dfrac{4x}{y} - 3y^2 + 2$

$\Rightarrow r^2\cos^2\theta = \dfrac{4r\cos\theta}{r\sin\theta} - 3r^2\sin^2\theta + 2$

$\Rightarrow r^2\cos^2\theta + 3r^2\sin^2\theta = 4\cot\theta + 2$

$\Rightarrow 3r^2\cos^2\theta + 3r^2\sin^2\theta - 2r^2\cos^2\theta = 4\cot\theta + 2$

$\Rightarrow 3r^2(\sin^2\theta+\cos^2\theta) - 2r^2\cos^2\theta = 4\cot\theta + 2$

$\Rightarrow 3r^2 - 2r^2\cos^2\theta = 4\cot\theta + 2$

$\Rightarrow r^2(3-2\cos^2\theta) = 4\cot\theta + 2$

$\Rightarrow r^2 = \dfrac{4\cot\theta+2}{3-2\cos^2\theta} \Rightarrow \boxed{r = \sqrt{\dfrac{4\cot\theta+2}{3-2\cos^2\theta}}}$ ans.

(c) $r = 2(\cos\theta - \sin\theta) \Rightarrow r^2 = 2(r\cos\theta - r\sin\theta)$

$\Rightarrow x^2+y^2 = 2(x-y) \Rightarrow x^2+y^2-2x+2y=0$

Centre $\equiv (-g, -f) \equiv \underline{(1,-1)}$ ans.

Radius $= \sqrt{g^2+f^2-c} = \sqrt{1+1-0} = \underline{\sqrt{2}}$ ans.

Q6. (a) The polar system of coordinates would be the most suitable way to describe this motion. We see that there is circular motion taking place in the annular layers, suggesting the use of polar coordinates. Converting the given polar system of velocity would also give us Cartesian coordinates dependent forms of eqn, but they'd be of higher order and describing trajectory using those equations wouldn't be that easy. So, polar coordinates system makes the system easier to analyse and less complicated.

Q6 (c) $v_r = \frac{1}{r^2}(2\cos^2\theta - \sin^2\theta)$

$\Rightarrow \dot{r} = \frac{1}{r^2}(2\cos^2\theta - \sin^2\theta) \Rightarrow \frac{dr}{dt} = \frac{1}{r^2}(2\cos^2\theta - \sin^2\theta)$ — (i)

$\Rightarrow \dot{\theta} = \frac{1}{r^3}(\sin\theta\cos\theta)$ [Angular Velocity)

$\Rightarrow \frac{d\theta}{dt} = \frac{1}{r^3}(\sin\theta\cos\theta) \Rightarrow dt = \frac{r^3 . d\theta}{\sin\theta . \cos\theta}$ — (ii)

from (i) & (ii),

$$\frac{dr}{d\theta} = r\frac{(2\cos^2\theta - \sin^2\theta)}{\sin\theta\cos\theta} \Rightarrow \frac{dr}{d\theta} = r(2\cot\theta - \tan\theta)$$

$$\int_6^r \frac{dr}{r} = \int_{\pi/3}^{\theta} (2\cot\theta - \tan\theta)\, d\theta$$

$$\Rightarrow [\ln r]_6^r = [\ln(\sin^2\theta\cos\theta)]_{\pi/3}^{\theta}$$

$$\Rightarrow \ln r - \ln 6 = \ln(\sin^2\theta\cos\theta) - \ln\left(\frac{3}{4}\cdot\frac{1}{2}\right)$$

$$\Rightarrow \ln r = \ln(\sin^2\theta) + \ln\left(\frac{\cancel{6}^{2}\times 8}{\cancel{3}}\right)$$

$$\Rightarrow \boxed{r = 16\sin^2\theta\cos\theta} \Rightarrow c = 16 \text{ (say)}$$

$$\Rightarrow \dot{\theta} = \frac{1}{(c\sin^2\theta\cos\theta)^3}\sin\theta\cos\theta = \frac{1}{c^3\sin^5\theta\cos^2\theta}$$

$$\Rightarrow \frac{d\theta}{dt} = \left(\frac{1}{c^3}\right)\frac{1}{\sin^5\theta\cos^2\theta} \Rightarrow \sin^5\theta\cos^2\theta\, d\theta = \frac{dt}{c^3}$$

Let $\cos\theta = u \Rightarrow -\sin\theta\, d\theta = du$

$$\Rightarrow u^2(1-u^2)^2\, du = -\frac{dt}{c^3}$$

$$\Rightarrow \int (u^6 - 2u^4 + u^2)\, du = -\int \frac{dt}{c^3}$$

$$\Rightarrow \frac{u^7}{7} - \frac{2u^5}{5} + \frac{u^3}{3} = -\frac{t}{c^3} + B$$

$$\Rightarrow \boxed{\frac{\cos^3\theta}{3} - \frac{2\cos^5\theta}{5} + \frac{\cos^7\theta}{7} = At + B} \quad \boxed{A = -\frac{1}{16^3}}$$

At $t=0$, $\theta = \frac{\pi}{3} \Rightarrow B = \frac{1}{24} - \frac{1}{80} + \frac{1}{896} = 0.03$

$$\Rightarrow \boxed{\frac{\cos^3\theta}{3} - \frac{2\cos^5\theta}{5} + \frac{\cos^7\theta}{7} = 0.03 - \frac{t}{16^3}}$$

Putting $t=1$,

We get $\boxed{\theta = 5.23}$

$r = 16 \sin^2\theta \cos\theta = 5.97$

$\therefore$ final coordinates $\equiv \boxed{(5.97, 5.23)}$ ans.

Q6 (b). Given, $v_r = \frac{1}{r^2}(2\cos^2\theta - \sin^2\theta)$ $[v_r = \dot{r}]$

$v_\theta = \dot{\theta}$ [Angular Velocity]

$= \frac{1}{r^3}(\sin\theta\cos\theta)$

Thus radial velocity $= v_r \hat{r} = \frac{1}{r^2}(2\cos^2\theta - \sin^2\theta)\hat{r}$

Tangential Velocity $= v_\theta \hat{\theta} = r\dot{\theta}\hat{\theta} = \frac{1}{r^2}(\sin\theta\cos\theta)\hat{\theta}$