Determinants of matrix order 3

$$det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}B_{11} \ + \ a_{12}B_{12} \ + \ a_{13}B_{13}$$

Minors Determinant of a (sub)matrix obtained by deleting i-th row and j-th column of nXn matrix: $[M_{ij}]_{(n-1)\times(n-1)}$

Cofactors $B_{ij} = (-1)^{i+j}|M_{ij}|$

Example:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \implies |M_{23}| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} = -6 \qquad \therefore B_{23} = (-1)^{2+3}|M_{23}| = -1(-6) = 6$$

Laplace expansion: Generalization of determinants

$$|A| \quad = a_{i1}B_{i1} + a_{i2}B_{i2} + \cdots = \textstyle\sum_{j=1}^{n} a_{ij}B_{ij} \qquad \text{For any row } i$$

$$= a_{1j}B_{1j} + a_{2j}B_{2j} + \cdots = \textstyle\sum_{i=1}^{n} a_{ij}B_{ij} \qquad \text{For any column } j$$

Classical adjoint=**adjugate**=**adjunct**=*adj*(*A*)

$$A = [a_{ij}]_{n\times n} \implies \text{cofactor}: \quad B_{ij} = (-1)^{i+j}|M_{ij}| \implies adj(A) = [B_{ij}]^T$$

Example

$$A = \begin{pmatrix} 2 & 3 & -4 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{pmatrix}$$

$$B_{11} = + \begin{vmatrix} -4 & 2 \\ -1 & 5 \end{vmatrix} = -18 \qquad B_{12} = - \begin{vmatrix} 0 & 2 \\ 1 & 5 \end{vmatrix} = 2 \qquad B_{13} = + \begin{vmatrix} 0 & -4 \\ 1 & -1 \end{vmatrix} = 4$$

$$B_{21} = - \begin{vmatrix} 3 & -4 \\ -1 & 5 \end{vmatrix} = -11 \qquad B_{22} = + \begin{vmatrix} 2 & -4 \\ 1 & 5 \end{vmatrix} = 14 \qquad B_{23} = - \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} = 5$$

$$B_{31} = + \begin{vmatrix} 3 & -4 \\ -4 & 2 \end{vmatrix} = -10 \qquad B_{32} = - \begin{vmatrix} 2 & -4 \\ 0 & 2 \end{vmatrix} = -4 \qquad B_{33} = + \begin{vmatrix} 2 & 3 \\ 0 & -4 \end{vmatrix} = -8$$

$$adj(A) = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix}^T = \begin{pmatrix} -18 & 2 & 4 \\ -11 & 14 & 5 \\ -10 & -4 & -8 \end{pmatrix}^T = \begin{pmatrix} -18 & -11 & -10 \\ 2 & 14 & -4 \\ 4 & 5 & -8 \end{pmatrix}$$

Theorem $A\ adj(A) = adj(A)\ A = |A|\,\mathbb{I}$

$$A^{-1} = \frac{adj(A)}{|A|}$$

Proof: Let $A = [a_{ij}]$ & $A\ adj(A) = [b_{ij}] = B$

i^{th} row of $A : [a_{i1}\ a_{i2}\ \cdots\ a_{in}]$ (1)

j^{th} $column$ of $adj(A) : [A_{j1}\ A_{j2}\ \cdots\ A_{jn}]^T$ (2)

ij^{th} *entry of* $Aadj(A)$ *is obtained by multiplying* (1) & (2) : $b_{ij} = a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn} = |A|$ if $i = j$

From Laplace expansion:

$|A| = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots = \sum_{j=1}^{n} a_{ij}A_{ij}$ For any row i

To show : $b_{ij} = 0$ for $i \neq j$

$$\text{Let } A' = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & & & \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$\longrightarrow$ i-th row is replaced by j-th row $\implies |A'| = a_{j1}A_{i1} + a_{j2}A_{i2} + \cdots + A_{jn}A_{in} = 0$

Similarly for column replacement.

$A\ adj(A) = |A|\,\mathbb{I}$ $similarly$ $= adj(A)\ A = |A|\,\mathbb{I}$ $\square$

Polynomials of Matrices

$$f(t) = a_n t^n + \cdots + a_2 t^2 + a_1 t + a_0 \implies f(A) = a_n A^n + \cdots + a_2 A^2 + a_1 A + a_0 \mathbb{I} \qquad [A^n \equiv \underbrace{A \,.\, A \cdots\cdots A}_{n-times}]$$

$$If \;\; f(t^*) = 0 \implies t^* \to root\ of\ f \;\; similarly \;\; f(A^*) = 0 : A^* \to root\ of\ polynomial\ f$$

Example:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad A^2 = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix} \qquad g(t) = t^2 - 5t - 2 \implies g(A) = A^2 - 5A - 2\mathbb{I} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

A=zero of g(A)

Matrix with polynomial elements = polynomial with matrix coefficients

$$A = \begin{pmatrix} t^2 + 2t & t^3 - 1 \\ 5t & 3t^2 + 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} t^3 + \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} t^2 + \begin{pmatrix} 2 & 0 \\ 5 & 0 \end{pmatrix} t + \begin{pmatrix} 0 & -1 \\ 0 & 2 \end{pmatrix}$$

Characteristic polynomial

$$A \in M_n(F) \implies f(t) = det(t\mathbb{I} - A) = (-1)^n det(A - t\mathbb{I})$$

$$-A = -\mathbb{I}A \implies det(-A) = det(-\mathbb{I})det(A) = (-1)^n det(A)$$

A simple case: if A is a triangular matrix $\implies$ $t\mathbb{I} - A$ is triangular

$$det(t\mathbb{I} - A) = (t - a_{11})(t - a_{22})\cdots = \prod_{i=1}^{n}(t - a_{ii}) = f(t)$$

Cayley-Hamilton Theorem: Every matrix is a root of its characteristic polynomial.

Example of characteristic polynomial n=2

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \implies f(t) = det(t\mathbb{I} - A) = \left| \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} - \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \right| = \left| \begin{pmatrix} t - a_{11} & -a_{12} \\ -a_{21} & t - a_{22} \end{pmatrix} \right| = t^2 - (a_{11} + a_{22})t + a_{11}a_{22} - a_{12}a_{21}$$

$$= t^2 - tr(A)\ t + det(A)$$

Verify $\quad f(A) = A^2 - (a_{11} + a_{22})\ A + (a_{11}a_{22} - a_{12}a_{21})\mathbb{I} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Cayley-Hamilton Theorem: Every matrix is a root of its characteristic polynomial.

This is not done in class

Proof: Let $A \in M_n(F) \implies f(t) = det(t\mathbb{I} - A) = t^n + q_{n-1}t^{n-1} + \cdots + q_2t^2 + q_1t + q_0$

Let $B(t)$ be the classical adjoint of $Q \equiv (t\mathbb{I} - A) \leftrightarrow$ elements of $B(t)$ are cofactors of $Q \equiv (t\mathbb{I} - A)$

$B(t) \equiv P_{n-1}t^{n-1} + P_{n-2}t^{n-2} + \cdots \ P_1t + P_0$ a polynomial of maximum degree (n-1) with matrix coefficients

$$Qadj(Q) = |Q|\mathbb{I} \implies (t\mathbb{I} - A)B(t) = |t\mathbb{I} - A|\mathbb{I}$$

$$(t\mathbb{I} - A)[P_{n-1}t^{n-1} + P_{n-2}t^{n-2} + \cdots \ P_1t + P_0] = t^n + q_{n-1}t^{n-1} + \cdots + q_2t^2 + q_1t + q_0$$

Equate powers of t:

$$\begin{array}{rcccc}
t^n : & P_{n-1} = \mathbb{I} & \times A^n & \implies & \cancel{A^n}P_{n-1} = A^n\mathbb{I} \\
t^{n-1} : & P_{n-2} - AP_{n-1} = q_{n-1}\mathbb{I} & \times A^{n-1} & \implies & A^{n-1}P_{n-2} - \cancel{A^n}P_{n-1} = q_{n-1}A^{n-1}\mathbb{I} \\
 & \vdots & & & \vdots \\
t : & P_0 - AP_1 = q_1\mathbb{I} & \times A & \implies & \cancel{A}P_0 - A^2P_1 = q_1A\mathbb{I} \\
1 : & -AP_0 = q_0\mathbb{I} & \times \mathbb{I} & \implies & -\cancel{A}P_0 = q_0\mathbb{I}
\end{array}$$

$$A^n + q_{n-1}A^{n-1} + \cdots + q_2A^2 + q_1A + q_0 = f(A) = 0 \qquad \square$$

Similarity Transformation

A and B are **similar** matrices if there exists a matrix, P, such that $B = P^{-1}\ A\ P$

Theorem: For similar matrices $det(A) = det(B)$

$tr(A) = tr(B)$

$r(A) = r(B)$

$$
\begin{aligned}
det(B) &= det(P^{-1}AP) \\
&= det(P^{-1})det(A)det(P) \\
&= det(P^{-1})det(P)det(A) \\
&= det(P^{-1}P)det(A) \\
&= det(\mathbb{I})det(A) \\
&= det(A) \quad \square
\end{aligned}
$$

Theorem: Similar matrices have same characteristic polynomial.

$$
\begin{aligned}
f_B(t) &= det(t\mathbb{I} - B) \\
&= det(P^{-1}t\mathbb{I}P - B) \\
&= det(P^{-1}t\mathbb{I}P - P^{-1}AP) \\
&= det(P^{-1}(t\mathbb{I} - A)P) \\
&= det(P^{-1})det(t\mathbb{I} - A)det(P) \\
&= det(P^{-1})det(P)det(t - \mathbb{I}A) \\
&= det(t\mathbb{I} - A) \\
&= f_A(t) \quad \square
\end{aligned}
$$

Diagonalization

Find invertible and non-singular matrix P such that $D = P^{-1}AP$ is a **diagonal** matrix obtained from matrix A.

Similarly for matrix representation of operators.

Let $A \in M_n(F)$ and there exists a basis $S = \{u_1\ u_2 \cdots u_n\}$ such that

$$\begin{array}{lcl} Au_1 & = & k_1u_1 \\ Au_2 & = & k_2u_2 \\ \vdots & & \\ Au_n & = & k_nu_n \end{array} \implies [A]_{n\times n}\underbrace{[u_1\ u_2\ \cdots\ u_n]}_{\text{P}} \equiv \begin{pmatrix} k_1 & 0 & \cdots & \\ 0 & k_2 & 0 & \cdots \\ & & \ddots & \\ \cdots & & 0 & k_n \end{pmatrix} \underbrace{[u_1\ u_2\ \cdots\ u_n]}_{\text{P}}$$

$$AP = PD \implies D = P^{-1}AP$$

then we say that the matrix A is **diagonalizable** and matrix A is similar to the diagonal matrix D.

$$\begin{array}{l} S^{-1}AS = \Lambda \\ P^{-1}BP = \Lambda \end{array} \implies S^{-1}AS = P^{-1}BP \implies PS^{-1}ASP^{-1} = B \implies U^{-1}AU = B$$

(A,B) are similar

where $U = SP^{-1}$ & $U^{-1} = (SP^{-1})^{-1} = PS^{-1}$