CHAPTER 4 : LIMIT & CONTINUITY

Let us start with the notion of the limit of a function.

SECTION 4.1 LIMIT OF A FUNCTION

Let us start with the definition.

DEFINITION 4.1.1 (Limit of a Function)

Let $-\infty < a < b < \infty$ and, let $f : (a, b) \to \mathbb{R}$, let $x_0 \in [a, b]$ and let $L \in \mathbb{R}$. We say that L is the limit of f at x_0, denoted by $L = \lim_{x \to x_0} f(x)$ if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x \in (a, b)$,

$$\boxed{0 < |x - x_0| < \delta \implies |f(x) - L| < \varepsilon. \quad (*)}$$

REMARK 4.1.2

Note that, if $\delta > 0$ satisfies $(*)$, any δ' satisfying $0 < \delta' < \delta$ satisfies $(*)$ as well. Therefore, for a given $\varepsilon > 0$, $\delta > 0$ satisfying $(*)$ is not unique.

~~The limit of a function, whenever exists, is unique. In other words, if $L_1, L_2 \in \mathbb{R}$ be such that~~

$$\cancel{L_1 = \lim_{x \to x_0} f(x) \quad \text{and} \quad L_2 = \lim_{x \to x_0} f(x),}$$

~~then $L_1 = L_2$. This can be seen as follows. Let us suppose, to the contrary, that $L_1 \neq L_2$.~~

We look at a few examples.

EXAMPLE 4.1.3

Define $f : (-1, 1) \to \mathbb{R}$ by

$$\boxed{f(x) = x^2, \quad \text{for all } x \in (-1, 1).}$$

Then, $\lim_{x \to 0} f(x) = 0$.

To see this, let $\varepsilon > 0$ be given. Let us set $\delta = \sqrt{\varepsilon}$.

Then, for all $x \in (-1, 1)$,

$$0 < |x-0| = |x| < \delta = \sqrt{\epsilon} \Rightarrow |f(x) - 0| = |x^2| = |x|^2 < \delta^2 = \epsilon.$$

Hence, $\lim_{x \to 0} f(x) = 0$.

EXAMPLE 4.1.4

Let $a \in \mathbb{R}$ and let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be defined as

$$\boxed{g(x) := \begin{cases} x \sin \frac{1}{x}, & \text{if } x \in \mathbb{R}, x \neq 0, \\ a, & \text{if } x = 0. \end{cases}}$$

Then, $\lim_{x \to 0} g(x) = 0$.

To see this, let $\epsilon > 0$ be given. Let us set $\delta = \epsilon$. Then, for all $x \in \mathbb{R}$,

$$0 < |x - 0| = |x| < \delta \Rightarrow |g(x) - 0| = \left|x \sin \tfrac{1}{x}\right| \leq |x| < \delta = \epsilon.$$

Hence, $\lim_{x \to 0} g(x) = 0$. Note that, the value of g at 0 is of no relevance.

EXAMPLE 4.1.5

Let us define $h : \mathbb{R} \longrightarrow \mathbb{R}$ as

$$\boxed{h(x) := \begin{cases} \dfrac{x^2 - 4}{x - 2}, & \text{if } x \in \mathbb{R} \text{ with } x \neq 2, \\ 2021, & \text{if } x = 2. \end{cases}}$$

Then, $\lim_{x \to 2} h(x) = 4$.

Indeed, to see this, let $\epsilon > 0$ be given. Let us choose $\delta = \epsilon$. Then, for all $x \in \mathbb{R}$,

$$0 < |x - 2| < \delta \Rightarrow |h(x) - 4| \underset{(\text{as } x \neq 2)}{=} \left|\frac{x^2 - 4}{x - 2} - 4\right| = |x + 2 - 4| = |x-2| < \delta = \epsilon.$$

Hence, $\lim_{x \to 2} h(x) = 4$.

EXAMPLE 4.1.6

Let us define $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$\boxed{\phi(x) := \begin{cases} \sin \frac{1}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}}$$

Then, $\lim_{x \to 0} \phi(x)$ does not exist.

We prove this through contradiction. Let us suppose, to the contrary, that, $\lim_{x\to 0} \phi(x)$ exists. Let us write

$$L := \lim_{x \to 0} \phi(x).$$

We consider two cases.

Case 1. $L \neq 0$.

In this case, we find $\delta > 0$ such that, for all $x \in \mathbb{R}$,

$$0 < |x| < \delta \Rightarrow |\phi(x) - L| = \left|\sin \tfrac{1}{x} - L\right| < \tfrac{1}{2}|L|. \quad (*_1)$$

Let us choose $N \in \mathbb{N}$ such that $0 < \frac{1}{N\pi} < \delta$. Then, it follows from $(*_1)$ that

$$\left|\phi\left(\tfrac{1}{N\pi}\right) - L\right| = |\sin(N\pi) - L| = |L| < \tfrac{1}{2}|L|,$$

which is a contradiction as $L \neq 0$. It remains to consider the second case when

Case 2. $L = 0$.

In this case, we find $\delta' > 0$ such that, for all $x \in \mathbb{R}$,

$$0 < |x| < \delta' \Rightarrow |\phi(x) - 0| = \left|\sin \tfrac{1}{x}\right| < 1/2. \quad (*_2)$$

Let us choose $M \in \mathbb{N}$ such that $0 < \frac{1}{2M\pi + \pi/2} < \delta'$. Using $(*_2)$, it follows that

$1 = \left|\sin\left(2M\pi + \frac{\pi}{2}\right)\right| < 1/2$, which is absurd. ~~Hence~~,

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Therefore, $\lim_{x \to 0} \sin \frac{1}{x}$ does not exist.

Exercise 4.1.7

Draw the graphs of functions in Example 4.1.4 and Example 4.1.6.

The following theorem states the elementary properties of the limit of a function.

Theorem 4.1.8 (Properties of Limit)

Let $-\infty < a < b < \infty$, let $x_0 \in [a, b]$ and let $f, g : (a, b) \to \mathbb{R}$ be such that $\lim_{x \to x_0} f(x)$ and $\lim_{x \to x_0} g(x)$ exist. Then

i) $\lim_{x \to x_0} (f(x) + g(x))$ exist and $\lim_{x \to x_0} (f(x) + g(x)) = \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x)$

ii) $\lim_{x \to x_0} (f(x) g(x))$ exist, and $\lim_{x \to x_0} f(x) g(x) = \left(\lim_{x \to x_0} f(x)\right)\left(\lim_{x \to x_0} g(x)\right)$.

iii) for all $\alpha \in \mathbb{R}$,

$$\lim_{x \to x_0} \alpha f(x) = \alpha \lim_{x \to x_0} f(x)$$

iv) If $\lim_{x \to x_0} g(x) \neq 0$, then $\lim_{x \to x_0} \frac{f(x)}{g(x)}$ exists and

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x \to x_0} f(x)}{\lim_{x \to x_0} g(x)}.$$

Proof: Let us write

$$L := \lim_{x \to x_0} f(x), \text{ and } M := \lim_{x \to x_0} g(x).$$

We shall prove ii). Others are left as exercise.

To prove (ii), let $\varepsilon > 0$ be given. We find $\delta_1, \delta_2, \delta_3 > 0$ such that, for all $x \in (a, b)$,

$$0 < |x - x_0| < \delta_1 \Rightarrow |g(x) - M| < 1. \quad (*_1)$$

$$0 < |x - x_0| < \delta_2 \Rightarrow |g(x) - M| < \frac{\varepsilon}{1 + |L| + |M|} \quad (*_2)$$

$$0 < |x - x_0| < \delta_3 \Rightarrow |f(x) - L| < \frac{\varepsilon}{1 + |L| + |M|} \quad (*_3).$$

Let us take $\delta := \min\{\delta_1, \delta_2, \delta_3\}$.

Then, for all $x \in (a, b)$, with $0 < |x - x_0| < \delta$, we have

$$|f(x)g(x) - LM| \le \cancel{|f(x)|} g \cancel{|g(x)||f(x) - L|} +$$

$$= |(f(x) - L)g(x) + L(g(x) - M)|$$

$$\le |f(x) - L||g(x)| + |L||g(x) - M|$$

$$\le (|g(x) - M| + |M|)|f(x) - L| + |L||g(x) - M|$$

$$< (1 + |M|)\frac{\varepsilon}{1 + |L| + |M|} + |L|\frac{\varepsilon}{1 + |L| + |M|} = \varepsilon,$$

which shows that

$$\lim_{x \to x_0} (f(x)g(x)) = LM. \quad \text{(Proved)}$$

SECTION 4.2. CONTINUITY

Let us begin with the definition.

DEFINITION 4.2.1 (CONTINUITY AT A POINT)

Let $-\infty < a < b < \infty$, let $f: [a, b] \to \mathbb{R}$ and let $x_0 \in [a, b]$.

We say that f is continuous at x_0, if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that, for all $x \in [a, b]$,

$$\boxed{(\ast_1) \quad |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon.}$$

If f is continuous at every point of $[a, b]$, we say that f is continuous on $[a, b]$.

While Definitions 4.1.1 and 4.2.1 look similar, note the differences.

THEOREM 4.2.2 (RELATION BETWEEN LIMIT & CONTINUITY)

Let $-\infty < a < b < \infty$, let $f: [a, b] \to \mathbb{R}$ and let $x_0 \in [a, b]$. Then, f is continuous at x_0 if and only if

i) $\lim_{x \to x_0} f(x)$ exists, and

ii) $\lim_{x \to x_0} f(x) = f(x_0)$.

PROOF: Easy.

EXAMPLE 4.2.3

Let us define $f: \mathbb{R} \to \mathbb{R}$ by

$$\boxed{f(x) := x^2, \text{ for all } x \in \mathbb{R}.}$$

Then, f is continuous on $\mathbb{R}$. To see this let $a \in \mathbb{R}$. We show that f is continuous at a. Let us first consider the case when $a \neq 0$. Let $\varepsilon > 0$ be given. We set

$$\delta := \min\left\{ \frac{\varepsilon}{3|a|}, |a| \right\}.$$

Then, for all $x \in \mathbb{R}$ with $|x - a| < \delta$, we have,

$$|x^2 - a^2| = |x - a||x + a| \leq |x - a|(|x - a| + 2|a|) < \delta(\delta + 2|a|)$$

$$\leq \frac{\varepsilon}{3|a|}(|a| + 2|a|) = \varepsilon.$$

Therefore, f is continuous at a, when $a \neq 0$. The case when $a = 0$ is left as an exercise.

EXAMPLE 4.2.4

Let us define $g: \mathbb{R} \longrightarrow \mathbb{R}$ by

$$\boxed{g(x) := \begin{cases} 1, & \text{if } x \geq 0 \\ -1, & \text{if } x < 0. \end{cases}}$$

Then, g is not continuous at 0. For if, g is continuous at 0, there exists a $\delta > 0$ such that, for all $x \in \mathbb{R}$ satisfying $|x| < \delta$, we have $|g(x) - 1| < 1$.

In particular, we have $|g(-\delta/2) - 1| < 1$ i.e. $2 < 1$, a contradiction. Hence, g is not continuous at 0.

THEOREM 4.2.5

Let $-\infty < a < b < \infty$, let $f, g : [a, b] \longrightarrow \mathbb{R}$ be continuous at $x_0 \in [a, b]$. Then,

i) $f + g$ is continuous at x_0.
ii) αf is continuous at x_0, for all $\alpha \in \mathbb{R}$.
iii) fg is continuous at x_0
iv) if $g(x_0) \neq 0$, f/g is continuous at x_0.

PROOF: Exercise.

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