Expt. #04: Young's modulus

Aim: To determine the Young's modulus of a metal bar by flexure method.

Prerequisite: You must thoroughly read and understand the content of appendix-I: supplementary reading.

Apparatus: Metal bar, knife-edge support, weight hanger, weights of different size, meter scale, screw gauge, Vernier Caliper, travelling microscope etc.

Principle and working formula: A metal bar is kept horizontal with two knife-edge support at the two ends. When different suitable weight is hanged at the middle point of the bar it gets depressed (Fig. 1). The amount of depression d is related to the Young's modulus Y of the bar material. By experimentally measuring length, breadth and thickness of the bar and depression d for different weights W Young's modulus Y can be determined.

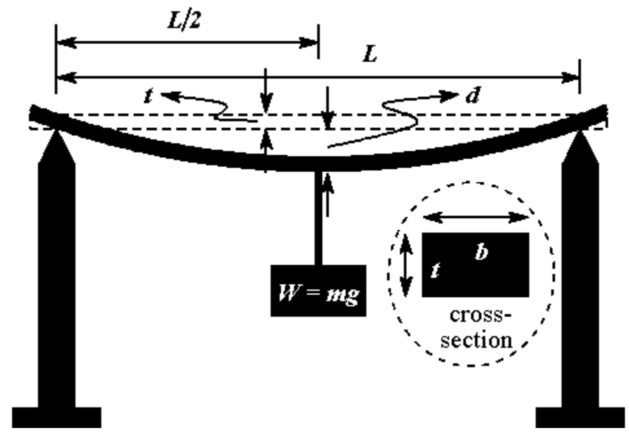

Figure 1: Schematic of Young's modulus experiment.

The bar in Fig. 1 can be regarded as two cantilevers, each of length $L/2$, fixed at the center (where weight is hanged) and loaded by $W/2$ at the two knife-edges. Each cantilever undergoes depression d given by

$$d = \frac{WL^3}{4Ybt^3} = \frac{mgL^3}{4Ybt^3} \qquad (1)$$

**** See appendix-I for derivation. ****

Rearranging Eq. (1) we get

$$Y = \frac{mgL^3}{4bdt^3} \qquad (2) \qquad [\textbf{Working Formula}]$$

Actually, it is difficult to know the absolute value of depression, because, even without any additional weight, the bar will be depressed slightly due to its own weight. So, one measures relative depression as a function of added weights and plots a graph: d vs. m, which is a straight line with a slope

$$s = \frac{gL^3}{4Ybt^3} \qquad (3)$$

From the slope s of the d vs. m graph one can calculate Young's modulus as

$$Y = \frac{gL^3}{4sbt^3} \qquad (4)$$

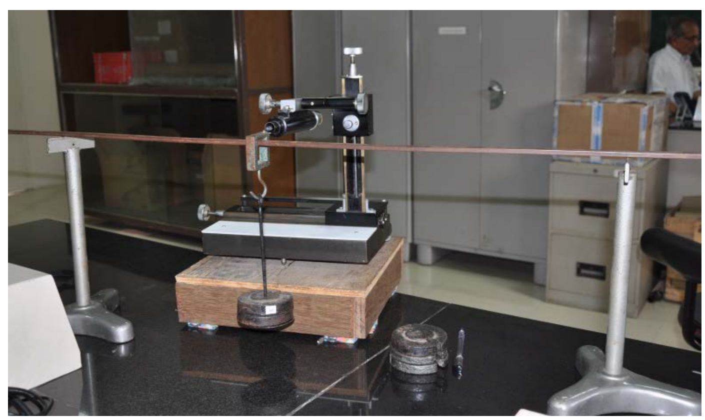

Figure 2: Photograph of experimental setup.

Procedure:

1. Measure the breadth (b) of the bar using a Vernier caliper and tabulate the data.
2. Measure the thickness (t) of the bar using a screw gauge and tabulate the data.
3. Measure the length (L) of bar between two knife-edges i.e. separation between two knife-edges (Fig. 1) using a meter scale and tabulate the data.
4. Mark the point on the bar exactly at the center of the two knife-edges. Put the hanger of the weights exactly at that point. The hanger has pointer on top.
5. Focus the travelling microscope such that the pointer is clearly seen in it. Position the travelling microscope such that the pointer is touching the horizontal cross wire. Note down the reading of the vertical scale of the travelling microscope.
6. Add one weight block on the hanger. [Mass of all the weight blocks should be already written on them. If it is missing, weigh the block and write it down.] The bar will depress and the pointer will move in the microscope field of view. Adjust

microscope vertical position to align the pointer back to the horizontal cross wire. Note the reading of the vertical scale of the travelling microscope.

7. Similarly, note microscope reading for addition of 4-5 more weight blocks one after another.
8. Then start reducing the weight one by one and each time adjust microscope to align the pointer at the horizontal cross wire and take microscope reading till the hanger is empty again.
9. Record data for steps 5-8 in a table. Calculate average depression d for a given amount of mass m on the hanger.

Calculation and error analysis: Plot d vs. m graph (plot data as points only, do not join them by line), fit a straight line, note its slope. Use Eq. (4) to calculate Young's modulus Y.

Do the error analysis yourself. See the notes on error analysis available in course webpage at WeLearn. Compare your result with standard value. You may take standard value of Young's modulus for the metal bar given to you as 2×10^{12} dynes/cm^2.

Appendix – I: supplementary reading

Elasticity: When an external force acts on a material body it tries to resist the change in its shape and size and it tries to recover its original shape and size when external deforming force is removed. This property of materials is known as elasticity. If a material body completely recovers its original shape and size when external deforming force is removed, it is known as perfectly elastic material. The materials which does not at all recover their original shape and size when external deforming force is removed, are known as perfectly plastic materials. There are no perfectly elastic or perfectly plastic materials. Quartz fiber and phosphor bronze are nearly perfect elastic materials and putty and cork are nearly perfect plastic materials.

Stress: When an external force is applied to a body to deform its shape and size, a reaction force comes into play internally inside the body. This reaction force tends to balance the external force, tries to prevent any change in shape and size of the body, and tries to restore its original shape and size when external deforming force is removed. This restoring force (equal to the applied force F within elastic limit) per unit area A of the body is known as stress, $S = F/A$. Stress has the same dimension and unit as pressure. Actually, stress is a tensor quantity having several components in general. But for isotropic material we can take it as a scalar constant.

Strain: A body is deformed (change in shape and/or size) under application of external force. The ratio of change in a dimension of the body to its original dimension is known as strain. It can be any one of the following: change per unit length (linear or tensile strain), change per unit volume (volume strain) and change per unit shape (shear strain). Further clarification will be given subsequently.

Hooke's law: In 1678, British scientist Robert Hooke gave the following law of elasticity: stress is proportional to strain or the ratio of stress to strain is a constant. The constant is known as coefficient of elasticity or modulus of elasticity. Depending on the type of stress and strain (linear, volume or shear) on can define various different elastic moduli. Further details will follow.

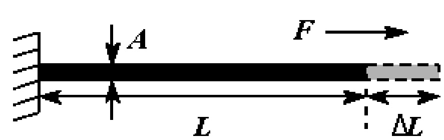

Figure 3: Linear elongation - Young's modulus.

Elastic limit: If the stress on a body is increased beyond a certain maximum value, the body does not recover its original shape and size even after deforming force is removed. This is known as elastic limit. Hooke's law is not valid beyond elastic limit.

Different types of elastic moduli: Based on the three types of strain, three different elastic moduli can be defined: Young's Modulus (corresponding to linear strain), Bulk Modulus (corresponding to volume strain) and Rigidity (or shear) modulus (corresponding to shear strain).

Young's Modulus: Consider Fig. 1. A rod of length L and cross-section A is elongated by an amount ΔL under the action of a force F. Then the linear stress is F/A, the linear strain is $\Delta L/L$ and Young's Modulus = linear stress / linear strain is given by $Y = (F/A)/(\Delta L/L) = FL/A\Delta L$.

Bulk Modulus: Consider Fig. 2. A cube of volume V is compressed hydrostatically (equal pressure is applied from all directions). The force F is acting on the cross-section area A. If the volume reduction is ΔV, then the volume strain $\Delta V/V$. The hydrostatic or bulk stress is F/A. The bulk modulus = volume stress / volume strain is given by $K = (F/A)/(\Delta V/V) = FV/A\Delta V$. The inverse $(1/K)$ of bulk modulus is known as ***compressibility***.

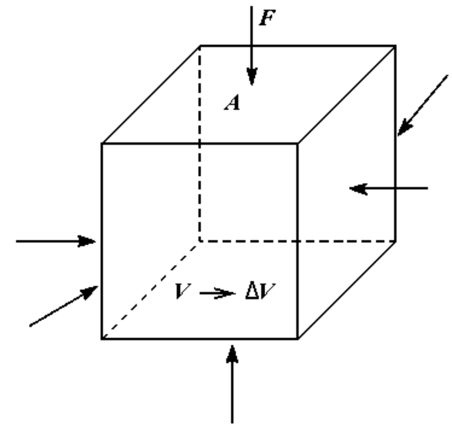

Figure 6: Volume compression - bulk modulus.

Rigidity (Shear) Modulus: Consider Fig. 3. A tangential force F is applied to a cube of cross-section A. It actually gives a torque. As a result the cube is deformed as shown in Fig. 3. Here, the stress is F/A. The shear strain is $\theta \approx \Delta L/L$. The rigidity (shear) modulus = shear stress / shear strain is given by $\eta = (F/A)/(\Delta L/L) = FL/A\Delta L$.

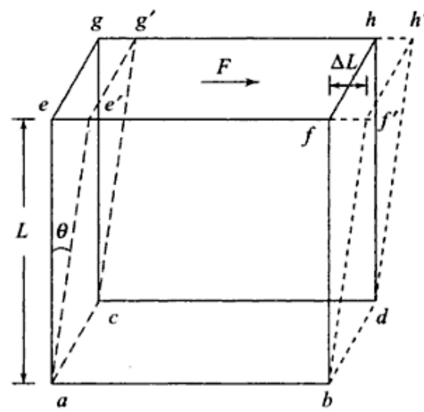

Figure 6: Shearing strain - shear modulus.

Poisson's ratio: Consider the case of linear elongation of a rod. When it is elongated in its length, its cross-section area decreases and when it is compressed in length, its cross-section area increases. So, we can say that a longitudinal strain (in the length) in the rod is accompanied by a transverse strain (in the cross-section). The transverse strain is given by the ratio of the change in diameter of the rod to its original diameter. The ratio of transverse strain to the longitudinal strain is known as Poisson's ratio, named after the French scientist Siméon Denis Poisson. So, Poisson's ratio is given by σ = transvers strain / longitudinal strain.

Interrelations among various elastic constants: The three elastic moduli and Poisson's ratio are interrelated in many ways. One of them can always be expressed in terms of any other two of them. Some of these interrelations are given below. We use Y= Young's modulus, K = Bulk modulus, η = Rigidity modulus, and σ = Poisson's ratio.

$$Y = 3K(1-2\sigma) = 2\eta(1+\sigma) = \frac{9K\eta}{3K+\eta} \qquad (1)$$

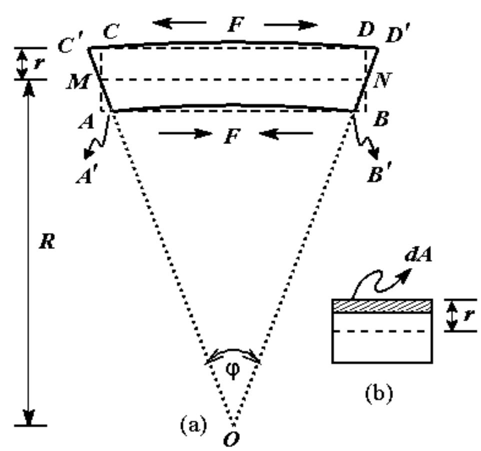

Figure 6: Bending of beam.

Bending moment of a beam: Consider small bending (we mean that strain due to bending is small) of a beam; see Fig. 5(a). The beam $ABCD$ is bent to the take the shape $A'B'C'D'$. Due to bending, planes in the upper half (convex part) of the beam are stretched and planes in the lower half (concave part) of the beam are compressed. There is a plane [MN in Fig. 5(a)] passing through the center of mass remains unstrained. This is known as neutral plane. The planes above neutral plane feel a tensile force and get elongated, whereas the planes below neutral plane feel a compressive force and get compressed. Therefore, equal and opposite forces act on planes above and below the neutral planes. They give rise to a "bending moment" (a torque about the neutral plane). For a symmetrically situated pair of planes, **bending moment = force experienced by the planes x distance from the neutral plane**. Consider two planes ($C'D'$ and $A'B'$) symmetrically placed at a distance r above and below the neutral plane MN. The plane $C'D'$ is elongated compared to plane MN and feels tensile strain, while the plane $A'B'$ is compressed compared to plane MN and feels compressive strain. If the radius of curvature of the beam after bending is R, then we have for plane $C'D'$

$$\text{strain} = \frac{\text{change in length}}{\text{original length}} = \frac{(R+r)\phi - R\phi}{R\phi} = \frac{r}{R} \qquad (8)$$

Equation (8) shows that the strain at plane is proportional to its distance from the neutral plane. We assume that because of this strain, a restoring force dF acts on the elementary plane of cross-section dA [see the transverse cross-section of the beam in Fig. 5(b)]. We know, stress = Young's modulus x strain. So, we can write,

$$\frac{dF}{dA} = Y\frac{r}{R} \;\Rightarrow\; dF = \frac{Y}{R} r dA \qquad (9)$$

Here, dF is the tensile force on elementary plane at $C'D'$. Same amount of compressive force acts on the elementary plane at $A'B'$. The bending moment due to these two planes is then given by rdF. Total bending moment acting on the bar can be obtained by integrating over all planes as

$$\mathcal{M} = \int r dF = \frac{Y}{R}\int r^2 dA = \frac{Y I_g}{R} \qquad (10)$$

The quantity $I_g = \int r^2 dA$ is known as geometric moment of inertia – the moment of inertia about axis through the center of mass of a hypothetical transverse cross-section with unit mass per unit area. The calculation of bending moment is very important for example in construction of bridges.

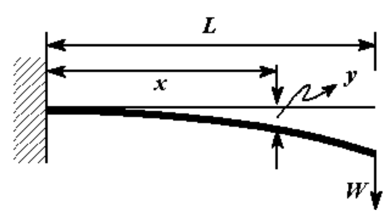

Figure 7: Depression of a cantilever.

Depression of a cantilever: A cantilever usually is a horizontal beam which is fixed at one end and loaded at the free end (see Fig. 6). Due the load at one end, the beam bends. Amount of depression from the horizontal (unbent) position is maximum at the free end and it decreases as we move towards fixed end, approaching zero at the fixed end. Let us assume that the amount of depression is y at distance x from the fixed end. We want to calculate $y(x)$, which is the equation of a curve which gives the bent shape of the cantilever. We shall do it under following assumptions: (a) Bending is small, i.e. depression at the free end is small compared to the length L of the cantilever beam (small strain), (b) Length of the beam is much larger that its width and breadth, (c) The cross-section of the beam does not change due to bending [this holds good under assumption of (a) and (b)], and (d) mass of the beam is negligible compared to the load at free end, i.e. bending due to mass of the beam is neglected compared to that caused by the load applied at the free end.

Radius of curvature R of any curve $y(x)$ is given by

$$\frac{1}{R} = \frac{\frac{d^2y}{dx^2}}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}} \qquad (11)$$

Since, we are interested in small bending, we can neglect $\frac{dy}{dx}$ compared to 1 in Eq. (11). So, we get

$$\frac{1}{R} = \frac{d^2y}{dx^2} \qquad (12)$$

The load W causes bending of the beam. It gives a bending moment which is equal to the torque about the neutral axis of the transverse cross-section at x exerted by the load W. It is given by

$$\mathcal{M}(x) = W(L - x) \qquad (13)$$

Using Eqs. (10), (12) and (13) we get

$$W(L - x) = \frac{YI_g}{R} = YI_g \frac{d^2y}{dx^2} \quad \Rightarrow \quad \frac{d^2y}{dx^2} = \frac{W}{YI_g}(L - x) \qquad (14)$$

We can integrate Eq. (14) twice and use the boundary conditions that at $x = 0$, both $y = 0$ and $dy/dx = 0$ holds. Then we get,

$$y(x) = \frac{W}{YI_g}\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right) \qquad (15)$$

The maximum depression is at the free end, $x = L$. It is given by

$$y(L) = \frac{WL^3}{3YI_g} \qquad (16)$$

The depression at the end increases as the cube of the beam length. The geometrical moment of inertia I_g depends on the cross-section of the beam. For example, for the rectangle and the circle in Fig. 7, I_g about the axis AB is given respectively by $ab^3/12$ and $\pi r^4/4$.

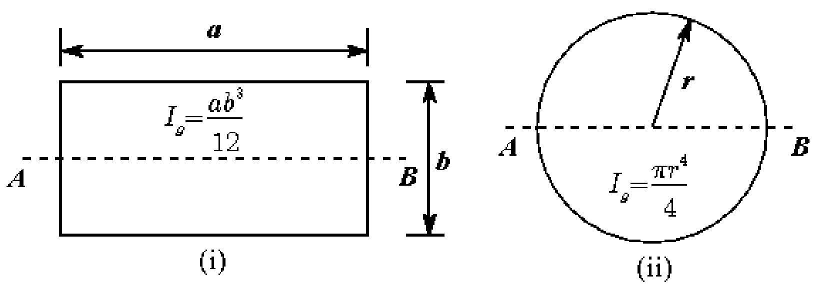

Figure 8: Geometrical moment of inertia for rectangular and circular cross-section.