

## Matrix Multiplication

$$C = A * B \equiv c_{ij} = \sum_{k=1}^K a_{ik} b_{kj} \quad \forall \quad 1 \leq i \leq M \quad \& \quad 1 \leq j \leq N$$

$$C_{M,N} = A_{M,K} * B_{K,N}$$

Associative:  $A(BC) = (AB)C$

Distributive:  $A(B \pm C) = AB \pm AC$   
 $(A \pm B)C = AC \pm BC$

Identity:  $A \mathbb{I} = \mathbb{I} A = A$

$$c(A B) = (c A) B$$

$$A \mathbf{O} = \mathbf{O} A = \mathbf{O}$$

$$A^1 = A \quad A^2 = A * A \quad A^n = A * \dots * A$$

$$AB \neq BA$$

matrix multiplication is not commutative

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad \& \quad B = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$AB = \begin{pmatrix} -1 & 3 \\ -3 & 7 \end{pmatrix}$$

$$\neq BA = \begin{pmatrix} 2 & 2 \\ 3 & 4 \end{pmatrix}$$

$$AC = BC \not\Rightarrow A = B$$

$$AC = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix}$$

$$BC = \begin{pmatrix} 1 & 3 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix}$$

# Matrix Transpose

Switches rows and columns of a matrix

$$a_{ij} \rightarrow a_{ji}^T$$

$$A \in M_{m,n}(F) \longrightarrow A^T \in M_{n,m}(F)$$

$$A \equiv \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \vdots & & & \ddots \end{pmatrix}_{m \times n} \qquad A^T \equiv \begin{pmatrix} a_{11} & a_{21} & a_{31} & \cdots \\ a_{12} & a_{22} & a_{32} & \cdots \\ a_{13} & a_{23} & a_{33} & \cdots \\ \vdots & & & \ddots \end{pmatrix}_{n \times m}$$

$$a_{ij} \rightarrow \bar{a}_{ji}^T$$

$$A \in M_{m,n}(\mathbb{C}) \longrightarrow A^\star \in M_{n,m}(\mathbb{C})$$

$A^\star \longrightarrow$  Hermitian Adjoint  
Matrix Adjoint  
**Conjugate transpose**  
Adjoint operator

- If a square matrix is such that
- $A^T = A \rightarrow$  **Symmetric** matrix
  - $A^T = -A \rightarrow$  Skew-symmetric matrix
  - $A^\star = A \rightarrow$  **Hermitian** matrix
  - $A^\star = -A \rightarrow$  Skew-hermitian matrix

## Laws of Transpose

$$\star \quad (A^T)^T = A$$

$$\star \quad (A^\star)^\star = A$$

$$\star \quad (A \pm B)^T = A^T \pm B^T$$

$$\star \quad (c A)^T = c A^T$$

$$\star \quad (AB)^T = B^T A^T$$

**Theorem:** For any matrix  $A \in M_N(F)$   $A - A^T \rightarrow$  skew-symmetric and  $A + A^T \rightarrow$  symmetric

*Proof:*  $(A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T) \quad \square$

## Inner product

$$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \quad v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \quad \langle u, v \rangle = u^T v = (u_1 \ u_2 \ u_3) \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

$$\langle u, v \rangle = 0 \implies u, v \text{ are } \mathbf{orthogonal}$$

$$\mathbf{Norm} \text{ of a vector: } ||u|| = [u^T u]^{1/2} = [u_1^2 + u_2^2 + u_3^2]^{1/2}$$

$$||u|| = 1 \implies \mathbf{normalised}$$

Set of normalised vectors mutually orthogonal — **orthonormal**

## Outer product

$$u \ v^T = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} (v_1 \ v_2 \ v_3) = \begin{pmatrix} u_1 v_1 & u_1 v_2 & u_1 v_3 \\ u_2 v_1 & u_2 v_2 & u_2 v_3 \\ u_3 v_1 & u_3 v_2 & u_3 v_3 \end{pmatrix}$$

## Matrix Inverse

If for a square matrix  $A$  there exists  $A^{-1}$  such that  $AA^{-1} = A^{-1}A = \mathbb{I}$

$A$  is called **invertible**

Matrix  $A^{-1}$  is the **inverse** of matrix  $A$

## Matrix Inverse Calculation for $N = 2$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} ax_1 + by_1 & ax_2 + by_2 \\ cx_1 + dy_1 & cx_2 + dy_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

If  $ad - bc \neq 0 \Rightarrow A^{-1}$  exists

**Theorem:** Inverse of a matrix is unique.

*Proof:* Let B and C both inverses of A such that  $AB=BA=I$  and  $AC=CA=I$ .

$$B=B.I = B(AC)=(BA)C=I.C=C \quad \square$$

$$\star \quad (AB)^{-1} = B^{-1}A^{-1}$$

$$\star \quad (A^T)^{-1} = (A^{-1})^T$$

$$\star \quad A \text{ is invertible} \implies A^T \text{ is invertible}$$

$$\star \quad AC = BC = Z \implies A = B = ZC^{-1} \quad \text{if } C^{-1} \text{ exists}$$

## Orthogonal Matrices

A square matrix  $Q \in M_N(\mathbb{R})$  such that  $Q^{-1} = Q^T \rightarrow QQ^T = Q^TQ = \mathbb{I}$

Row vectors and column vectors are orthonormal.

**Orthogonal matrices preserve norm:**

$$x \rightarrow Qx \implies ||Qx||^2 = (Qx)^T(Qx) = x^T Q^T Q x = x^T x = ||x||^2$$

**Unitary Matrices:**  $U \in M_N(\mathbb{C})$  such that  $U^{-1} = U^\star$

Examples:

$$\begin{pmatrix} \cos \theta & i \sin \theta \\ -i \sin \theta & -\cos \theta \end{pmatrix}$$

$$\frac{1}{2} \begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1+i \\ 1-i & 2 \end{pmatrix}$$



# Linear System of Equations

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \\ & \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m \end{array}$$



$$\begin{array}{rcl} u_{11}x_1 + u_{12}x_2 + \cdots + u_{1n}x_n & = & b'_1 \\ & & u_{22}x_2 + \cdots + u_{2n}x_n = b'_2 \\ & & \vdots \\ & & u_{nn}x_n = b'_n \end{array}$$

*m=n*

$$A . \overrightarrow{x} = \overrightarrow{b}$$

$$U . \overrightarrow{x} = \overrightarrow{b'}$$

Solve by back substitution

- Interchange two equations
- Multiply/scale any equation with a nonzero constant
- Add a multiple of one equation to another

## Example

$$\begin{array}{rcl} -3x_1 + 2x_2 - x_3 & = & -1 \\ 6x_1 - 6x_2 + 7x_3 & = & -7 \\ 3x_1 - 4x_2 + 4x_3 & = & -6 \end{array} \implies \begin{pmatrix} -3 & 2 & -1 \\ 6 & -6 & 7 \\ 3 & -4 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ -7 \\ -6 \end{pmatrix}$$

Let us construct an **augmented** matrix

$$\left( \begin{array}{ccc|c} -3 & 2 & -1 & -1 \\ 6 & -6 & 7 & -7 \\ 3 & -4 & 4 & -6 \end{array} \right) \xrightarrow[\begin{array}{l} R_1=R_1 \\ R_2=2R_1+R_2 \\ R_3=R_1+R_3 \end{array}]{\hspace{1cm}} \left( \begin{array}{ccc|c} -3 & 2 & -1 & -1 \\ 0 & -2 & 5 & -9 \\ 0 & -2 & 3 & -7 \end{array} \right) \xrightarrow[R_3=(-1)R_2+R_3]{\hspace{1cm}} \left( \begin{array}{ccc|c} -3 & 2 & -1 & -1 \\ 0 & -2 & 5 & -9 \\ 0 & 0 & -2 & 2 \end{array} \right)$$

Solved by back substitution

$$\begin{array}{rcl} -3x_1 + 2x_2 - x_3 & = & -1 \\ -2x_2 + 5x_3 & = & -9 \\ -2x_3 & = & 2 \end{array} \implies \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$$

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \\ \vdots & & \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n & = & b_n \end{array} \quad \xrightarrow{R2 \rightarrow R2 - R1 \frac{a_{21}}{a_{11}}}$$

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ 0 + a_{22}^{(2)}x_2 + \cdots + a_{2n}^{(2)}x_n & = & b_2^{(2)} \\ \vdots & & \end{array}$$

2nd generation

$$\forall i,j = 2,3,\cdots,n \quad m_{i1} = \frac{a_{i1}^{(1)}}{a_{11}^{(1)}} \qquad \begin{array}{l} a_{ij}^{(2)} = a_{ij}^{(1)} - m_{i1}a_{ij}^{(1)} \\ b_i^{(2)} = b_i^{(1)} - m_{i1}b_i^{(1)} \end{array}$$

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ 0 + a_{22}^{(2)}x_2 + \cdots + a_{2n}^{(2)}x_n & = & b_2^{(2)} \\ \vdots & & \\ 0 + a_{n2}^{(2)}x_2 + \cdots + a_{nn}^{(2)}x_n & = & b_n^{(2)} \end{array}$$

(k+1) generation

$$\forall i,j = k + 1,k + 2,\cdots,n \quad m_{ik} = \frac{a_{i1}^{(k)}}{a_{kk}^{(k)}} \qquad \begin{array}{l} a_{ij}^{(k+1)} = a_{ij}^{(k)} - m_{ik}a_{kj}^{(k)} \\ b_i^{(k+1)} = b_i^{(k)} - m_{ik}b_k^{(k)} \end{array}$$

$$\begin{array}{rcl} u_{11}x_1 + u_{12}x_2 + \cdots + u_{1n}x_n & = & g_1 \\ u_{22}x_2 + \cdots + u_{2n}x_n & = & g_2 \\ \vdots & & \\ u_{n-1,n-1}x_{n-1} + u_{n-1,n}x_n & = & g_{n-1} \\ u_{nn}x_n & = & g_n \end{array} \qquad \begin{array}{l} u_{ij} = a_{ij}^{(i)} \\ g_i = b_i^{(i)} \end{array}$$

Solution by back substitution:

$$x_i = \frac{g_i - \sum_{j=i+1}^n u_{ij}x_j}{u_{ii}}$$

Pivot  $u_{kk} \neq 0$

**Example contd.**

$$\begin{array}{rcl} -3x_1 + 2x_2 - x_3 & = & -1 \\ 6x_1 - 6x_2 + 7x_3 & = & -7 \\ 3x_1 - 4x_2 + 4x_3 & = & -6 \end{array} \Rightarrow \begin{pmatrix} -3 & 2 & -1 \\ 6 & -6 & 7 \\ 3 & -4 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ -7 \\ -6 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} -3 & 2 & -1 & \bigg| & -1 \\ 6 & -6 & 7 & \bigg| & -7 \\ 3 & -4 & 4 & \bigg| & -6 \end{pmatrix} \xrightarrow{\substack{R_1=R_1 \\ R_2=2R_1+R_2 \\ R_3=R_1+R_3}} \begin{pmatrix} -3 & 2 & -1 & \bigg| & -1 \\ 0 & -2 & 5 & \bigg| & -9 \\ 0 & -2 & 3 & \bigg| & -7 \end{pmatrix} \xrightarrow{R_3=(-1)R_2+R_3} \begin{pmatrix} -3 & 2 & -1 & \bigg| & -1 \\ 0 & -2 & 5 & \bigg| & -9 \\ 0 & 0 & -2 & \bigg| & 2 \end{pmatrix}$$

$$\begin{pmatrix} -3 & 2 & -1 & \bigg| & -1 \\ 0 & -2 & 5 & \bigg| & -9 \\ 0 & 0 & -2 & \bigg| & 2 \end{pmatrix} \xrightarrow{R_3=R_3/-2} \begin{pmatrix} -3 & 2 & -1 & \bigg| & -1 \\ 0 & -2 & 5 & \bigg| & -9 \\ 0 & 0 & 1 & \bigg| & -1 \end{pmatrix} \xrightarrow{\substack{R_1=R_1+R_3 \\ R_2=R_2-5R_3}} \begin{pmatrix} -3 & 2 & 0 & \bigg| & -2 \\ 0 & -2 & 0 & \bigg| & -4 \\ 0 & 0 & 1 & \bigg| & -1 \end{pmatrix} \xrightarrow{R_2=R_2/-2} \begin{pmatrix} -3 & 2 & 0 & \bigg| & -2 \\ 0 & 1 & 0 & \bigg| & 2 \\ 0 & 0 & 1 & \bigg| & -1 \end{pmatrix}$$

$$\begin{pmatrix} -3 & 2 & 0 & \bigg| & -2 \\ 0 & 1 & 0 & \bigg| & 2 \\ 0 & 0 & 1 & \bigg| & -1 \end{pmatrix} \xrightarrow{R_1=R_1-2R_2} \begin{pmatrix} -3 & 0 & 0 & \bigg| & -6 \\ 0 & 1 & 0 & \bigg| & 2 \\ 0 & 0 & 1 & \bigg| & -1 \end{pmatrix} \xrightarrow{R_1=R_1/-3} \begin{pmatrix} 1 & 0 & 0 & \bigg| & 2 \\ 0 & 1 & 0 & \bigg| & 2 \\ 0 & 0 & 1 & \bigg| & -1 \end{pmatrix}$$

## Row Reduced Echelon Form (RREF)

- Each non-zero row has 1 as the first entry = leading entry=**pivot**
- All column entries above and below the pivot is zero
- Pivot(=1) is the only entry in its column
- The leading element of a row is the leftmost of all leading elements in rows below
- All zero rows are at the bottom

$$\begin{pmatrix} 1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 5 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

## Row Reduced Echelon Form (RREF) - Gauss Jordan Algorithm

- **Start**  $j=1, k=1$  equivalently  $j=N, k=N$
- If  $a_{jk} = 0$  swap  $j$  row with some other row such that  $a_{jk} \neq 0$
- Divide  $j$  row by  $a_{jk}$  such that  $a_{jk} = 1$  (pivot)
- If all entries in the column are zero  $k=k+1$
- Else eliminate all entries in the  $k$  column by suitable multiples of  $j$  row from other rows
- $j=j+1, k=k+1$ . **Repeat**

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{pmatrix} \xrightarrow{\substack{R_2=R_2-4R_1 \\ R_3=R_3-6R_1}} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 0 & -5 & -10 & -15 \end{pmatrix} \xrightarrow{\substack{R_2=R_2/-3 \\ R_3=R_3/-5}} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{pmatrix} \xrightarrow{\substack{R_1=R_1-2R_2 \\ R_3=R_3-R_2}} \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

**Every matrix has an unique Row Reduced Echelon Form (RREF)**