## MARCH 1st, 2022 : Linear differential equation :

Any linear differential equation of order $n$ has a solution containing $n$ independent arbitrary constants.

$$\boxed{\begin{array}{c} y'' + y' + y = 0 \\ c_1 y_1 + c_2 y_2 \end{array}}$$

$\leftarrow$ 2nd order

Ex: $\dfrac{d^2x}{dt^2} = g \overset{\text{Integration}}{\Longrightarrow} \dfrac{dx}{dt} = gt + \text{Constant}_1$

further integration

$$x = +\frac{1}{2}gt^2 + \text{Constant}_1 t + \text{Constant}_2$$

$$\boxed{x = \frac{1}{2}gt^2 + c_1 t + c_2}$$

Suppose: $a_2 \dfrac{d^2y}{dx^2} + a_1 \dfrac{dy}{dx} + a_0 y = 0$ take $\dfrac{d}{dx} = D$

$$\boxed{a_2 D^2 y + a_1 Dy + a_0 y = 0}$$

Ex:- $y'' + 5y' + 4y = 0$ —— (1)

$D = \dfrac{d}{dx} = y'$ $D^2 = \dfrac{d^2}{dx^2} = y''$

Equ (1) $\Rightarrow$ $D^2y + 5Dy + 4y = 0$

$$(D^2 + 5D + 4)y = 0 \quad \text{--- (2)}$$

From equ (2) $(D^2 + 5D + 4) = 0$

$$(D+1)(D+4)y = 0$$

$$(D+1)y = 0 \quad \text{--- (A)}$$

or $(D+4)y = 0$ ——— (B)

From (A) $(D+1)y = 0$

$$Dy + y = 0$$

$$Dy = -y$$

$$e^{(A+B)} = e^A \times e^B$$

$$\frac{dy}{dx} = -y$$

Variable seperation $\Rightarrow$ $\int \frac{dy}{y} = \int -dx$

$$\boxed{\ln y = -x + C_1}$$

$$y = C_1 e^{-x}$$

Similary from (B) $y = C_2 e^{-4x}$

$(D+4)y=0$

$$(D+1)(D+4)y = 0$$

$$\boxed{y = C_1 e^{-x} + C_2 e^{-4x}}$$

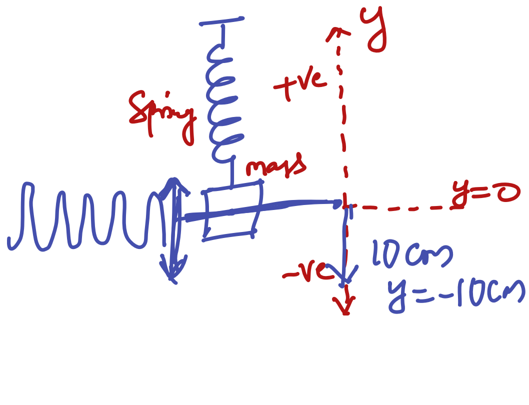
☆

$y = c_1 e^{ax} + c_2 e^{bx}$ is the general solution of $(D-a)(D-b)y=0$ where $a \neq b$

$$\Rightarrow (D-a)(D-a)y = 0$$

$$\boxed{y = (Ax+B)e^{ax}}$$

Ex: Suppose the mass is held at rest at a distance 10cm below equilibrium and then suddenly let go. If we agree to call $y$ positive when $m$ is above the equilibrium position, then at $t=0$, we have $y=-10$ and $\frac{dy}{dt} = 0$.



$$F = ma = m\frac{d^2y}{dt^2} \quad \text{— (A)}$$

$$F = -Ky \quad \text{—— (B)}$$

$$(A) = (B) \Rightarrow \frac{d^2y}{dt^2} = \frac{-K}{m}y \qquad \frac{k}{m} = \omega^2$$

$$\frac{d^2y}{dt^2} + \omega^2 y = 0$$

$$D^2y + \omega^2 y = 0$$

$$(D^2 + \omega^2)y = 0$$

$$D = \pm i\omega$$

$$\boxed{y = Ae^{i\omega t} + Be^{-i\omega t}}$$

Simple harmonic motion

$$y = c_1 \sin \omega t + c_2 \cos \omega t \quad \text{— (C)}$$

We have the situation | $\frac{dy}{dt} = 0$ $\quad \because y = c_1 \sin \omega t + c_2 \cos \omega t$

$$\frac{dy}{dt} = c_1 \omega \cos \omega t - c_2 \omega \sin \omega t \quad \text{— (D)}$$

Substituting Initial Conditions in eqn (C)

$t=0 \quad y=-10$ cm

$$-10 = c_1 \sin \omega(0) + c_2 \cos \omega(0)$$

$$-10 = c_1 \times 0 + c_2 \times 1$$

Substituting $c_2$ value in (D)  $\boxed{c_2 = -10}$ — (E)

From (D)

$$\frac{dy}{dt} = c_1 \omega \cos \omega t - (-10)\, \omega \sin \omega t$$

$$\because \left.\begin{array}{l} \frac{dy}{dt} = 0 \\ t = 0 \end{array}\right\} \Rightarrow \quad 0 = c_1 \omega \cos \omega(0) + 10 \sin \omega(0)$$

$$0 = c_1 \omega \times 1 + 0$$

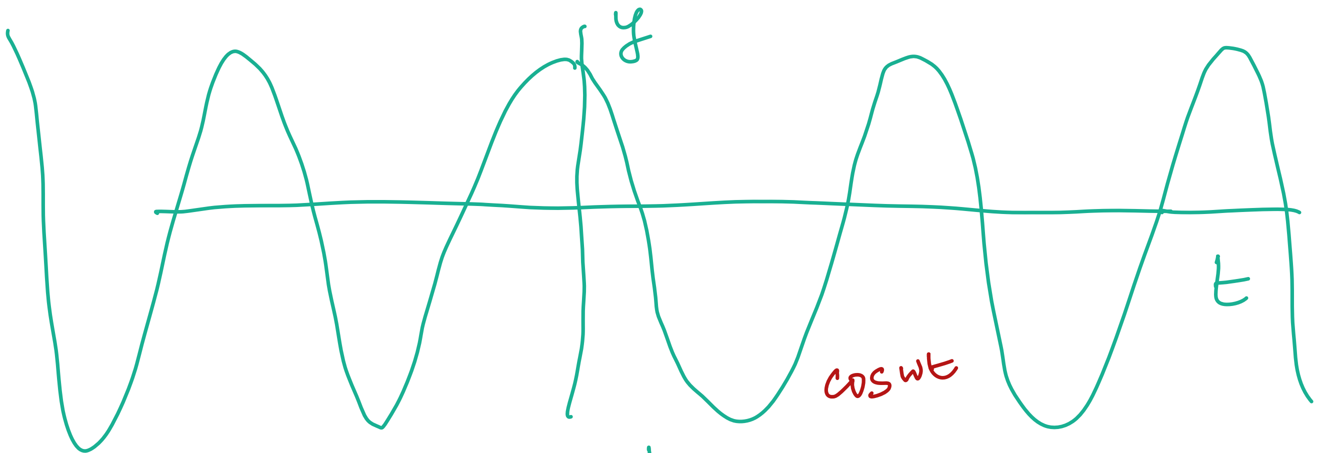
$$0 = c_1 \omega$$

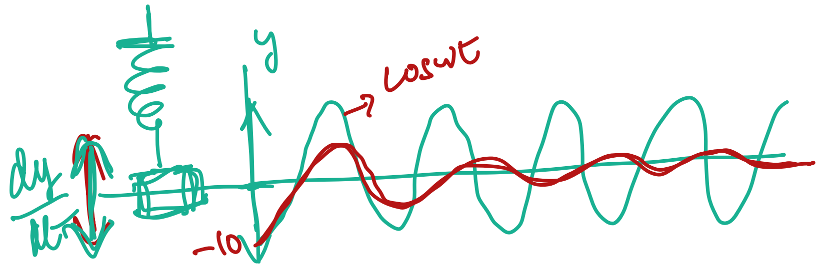
$$\boxed{c_1 = 0} \text{ — (F)}$$

Substituting (E) & (F) in the solution (C)

$$y = 0 \times \sin \omega t + (-10) \cos \omega t$$

$$\boxed{y = -10 \cos \omega t}$$



$$m\frac{d^2y}{dt^2} = -Ky \Rightarrow \text{Initial equation}$$

$$\boxed{\text{Retarding force } -l\frac{dy}{dt} \quad (l>0)}$$

$$m\frac{d^2y}{dt^2} = -Ky - l\frac{dy}{dt}$$

$$\frac{d^2y}{dt^2} + \frac{K}{m}y + \frac{l}{m}\frac{dy}{dt} = 0$$

$$w^2 = \frac{K}{m} \qquad \frac{l}{m} = 2b$$

$$\frac{d^2y}{dt^2} + 2b\frac{dy}{dt} + w^2y = 0$$

$$\underbrace{(D^2 + 2bD + w^2)}\,y = 0$$

$$D = \frac{-2b \pm \sqrt{4b^2 - 4w^2}}{2}$$

$$\boxed{D = -b \pm \sqrt{b^2 - w^2}}$$

$b^2 > w^2 \longrightarrow$ overdamp

$b^2 = w^2 \longrightarrow$ Critically damp

$b^2 < w^2 \longrightarrow$ Underdamp or oscillatory