# CHAPTER 7: MAXIMA & MINIMA

In this chapter, we ~~say~~ shall employ ~~the~~ techniques of calculus to find points of minimum/ maximum of a function. Let us start with a definition.

**DEFINITION 7.1 (LOCAL MAXIMUM/ MINIMUM)**
Let $I \subseteq \mathbb{R}$ be an interval ~~and~~, let $f: I \to \mathbb{R}$ and let $c \in I$. Then, $c$ is said to be a point of local maximum (minimum) of $f$ if there exists a $\delta > 0$ such that

$$\boxed{\begin{array}{l} f(c) \geq f(x) \text{ for all } x \in (c-\delta, c+\delta) \cap I. \\ (f(c) \leq f(x) \text{ for all } x \in (c-\delta, c+\delta) \cap I). \end{array}}$$

**REMARK 7.2**
If $I \subseteq \mathbb{R}$ is an OPEN interval, $f: I \to \mathbb{R}$ and if $c \in I$ is a point of local maximum (minimum) of $f$, then there exists $\delta > 0$ such that

i) $(c-\delta, c+\delta) \subseteq I$.
ii) $f(c) \geq f(x)$ for all $x \in (c-\delta, c+\delta)$
$(f(c) \leq f(x)$ for all $x \in (c-\delta, c+\delta))$

**THEOREM 7.3 (NECESSARY CONDITION)**
Let $-\infty \leq a < b \leq \infty$, let $f: (a,b) \to \mathbb{R}$ be differentiable at $c \in (a,b)$. If $c$ is a point of local maximum/minimum of $f$, then $f'(c) = 0$.

**PROOF:** Let us suppose that $c$ is a point of local maximum. Then, using Remark 7.2, we find a $\delta > 0$ such that

$$(c-\delta, c+\delta) \subseteq (a,b), \text{ and } f(c) \geq f(x) \text{ for all } x \in (a,b).$$

Let $\varepsilon > 0$ be given. As $f$ is differentiable, there exists $\delta_1 > 0$, $0 < \delta_1 < \delta$ such that, for all $x \in (a,b)$,

$$0 < |x - c| < \delta_1 \Rightarrow \left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < \epsilon \quad (*).$$

In other words,

$$f'(c)-\varepsilon < \frac{f(x)-f(c)}{x-c} < f'(c)+\varepsilon, \text{ for all } x \in (c-\delta_1, c) \cup (c, c+\delta_1).$$

In particular,

$$(*1).\quad f'(c)-\varepsilon < \frac{f(x)-f(c)}{x-c} \leq 0, \text{ for all } x \in (c, c+\delta_1).$$

$$(*2)\qquad 0 \leq \frac{f(x)-f(c)}{x-c} < f'(c)+\varepsilon, \text{ for all } x \in (c-\delta_1, c)$$

In particular,

$$-\varepsilon < f'(c) < \varepsilon, \text{ for all } \varepsilon > 0.$$

Therefore, $f'(c)=0$. This proves the theorem. (Proved)

**THEOREM 7.4 (SUFFICIENT CONDITION)**

Let $-\infty \leq a < b \leq \infty$ and let $f : (a,b) \to \mathbb{R}$ be such that $f, f', f''$ ~~are~~ exist and are continuous on $(a,b)$, let $c \in (a,b)$ be such that

$$\boxed{f'(c)=0, \text{ and } f''(c) \neq 0.}$$

Then,

i) $c$ is a point of strict local minimum if $f''(c) > 0$.

ii) $c$ is a point of strict local maximum if $f''(c) < 0$.

**REMARK 7.5**

If $f''(c)=0$, the test in inconclusive. To see this, we refer to the following three functions $f, g, h : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) := x^3, \quad g(x) := x^4, \quad h(x) = -x^4, \text{ for all } x \in \mathbb{R}.$$

Note that, 0 is a point of minimum for $g$, a point of maximum for $h$, and 0 is ~~neither~~ a point of neither maximum nor minimum for $f$.

We shall now prove Theorem 7.4.

PROOF: Since $f''(c) \neq 0$ and $f''$ is continuous at $c$, there exists a $\delta > 0$ such that

(∗1) $[c-\delta, c+\delta] \subset (a, b)$, & $f''(x) f''(c) > 0$, for all $x \in (c-\delta, c+\delta)$.

Using Theorem 6.2.1, ~~we find~~ for each $x \in [c-\delta, c+\delta]$, $x \neq c$, we find $\theta_x$ lying ~~between~~ $x$ and $c$, such that

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(\theta_x)}{2}(x-c)^2$$

$$= f(c) + \frac{1}{2} f''(\theta_x)(x-c)^2.$$

i) When $f''(c) > 0$, we have, using (∗1), $f''(\theta_x) > 0$. Hence, $f(x) > f(c)$, for all $x \in [c-\delta, c+\delta]$ with $x \neq c$. Therefore, $c$ is a point of strict local minimum.

ii) When $f''(c) < 0$, we have, using (∗1), $f''(\theta_x) < 0$ for all $x \in (c-\delta, c+\delta)$ with $x \neq c$. Hence, $f(x) < f(c)$, for all $x \in (c-\delta, c+\delta)$ with $x \neq c$. Therefore, $c$ is a point of strict local maximum.

## EXAMPLE 7.6

Let us consider the map $f: \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) := x^5 - 5x^4 + 5x^3 + 10, \text{ for all } x \in \mathbb{R}.$$

Then, $f'(x) = 5x^4 - 20x^3 + 15x^2 = 5x^2(x^2 - 4x + 3)$
$= 5x^2(x-3)(x-1)$, for all $x \in \mathbb{R}$.

Hence, $\{x \in \mathbb{R} \mid f'(x) = 0\} = \{0, 1, 3\}$.

Now,

$$f''(x) = 20x^3 - 60x^2 + 30x = 10x(2x^2 - 6x + 3)$$

Therefore,

$$f''(0) = 0, \quad f''(1) = -10 < 0, \quad f''(3) = 90 > 0.$$

Hence, 1 is a point of local maximum, 3 is a point of local minimum. Theorem 7.4 does not say anything about 0.