~~Relationship Problems~~ Problems on Relations

Debmalya Bandyopadhyay

January 2022

1. $(x_1, x_2) \sim (y_1, y_2)$ if $x_1 = y_1$

Soln: Reflexive: $(x_1, x_2) = (x_1, x_2)$
Symmetric: if $(x, y) \sim (x, z)$ then $(x, z) \sim (x, y)$
Transitive: if $(x_1, x_2) = x \sim y = (y_1, y_2)$, clearly $x_1 = y_1$ and if $y = (y_1, y_2) \sim (z_1, z_2) = z$, then $y_1 = z_1$. Combining the two, $x_1 = y_1 = z_1$. So $x \sim z$

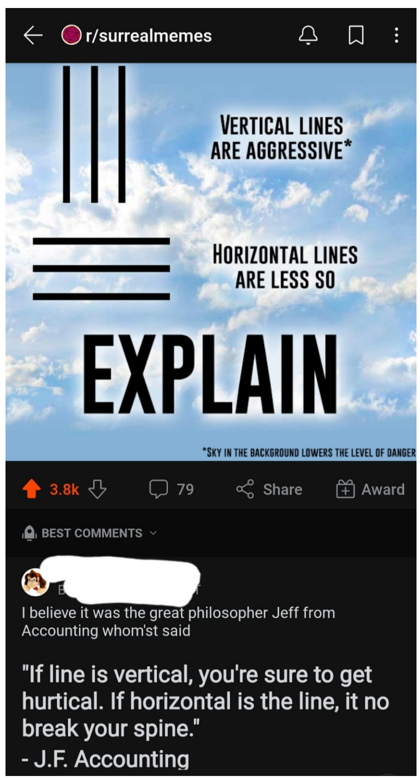

Figure 1: Answer for part (ii) is vertical lines

2. $(x_1, x_2) \sim (y_1, y_2)$ if $x_1^2 + x_2^2 = y_1^2 + y_2^2$

Soln: try yourself lol[1]

Figure 2: Answer for part (ii) is concentric circles

3. $(m, n) \sim (p, q)$ if $m + q = p + n$

Soln: Reflexive: $(x, y) \sim (x, y)$ as $x + y = y + x$ (slow claps)
Symmetric: $(x, y) \sim (a, b)$ then $x + b = a + y$ and that also means $(a, b) \sim (x, y)$
Transitive: Let $(x, y) \sim (p, q)$ and $(p, q) \sim (a, b)$. Then we have $x + q = y + p$ and $p + b = q + a$. Subtracting these two equations, we get $x - a = y - b$. Rearranging gives us $x + b = y + a$, so $(x, y) \sim (a, b)$.

For part (ii) note that if $m + q = p + n$, then $m - n = p - q$. Since $m, n, p, q \in \mathbb{N}$, let us fix any integer $k \in \mathbb{Z}$. Then the equivalence classes are integer points on the straight lines $\{(x, y) : x - y = k\}$ varying over all $k \in \mathbb{Z}$

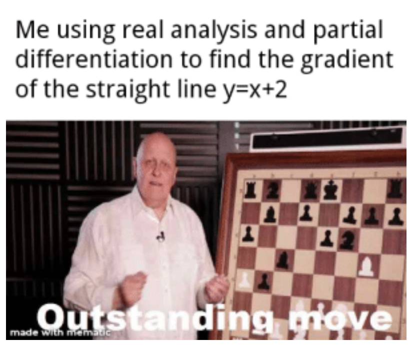

Figure 3: Answer for part (ii) is integer points on straight lines with unit slope

[1] Literally every math textbook ever

4. $(x_1, x_2) \sim (y_1, y_2)$ if $(y_1, y_2) = \alpha(x_1, x_2)$ for some $\alpha \neq 0$

Soln:(ii) Notice that if $(x_1, x_2) \sim (y_1, y_2)$, then the two points lie on a straight line that passes through origin. (That's profound, but how?) Instead of me putting spoilers here, try proving this yourself!

Figure 4: The equivalence classes are straight lines passing through origin (which is excluded). Looks somewhat like this $\uparrow$

5.(i) Count the number of relations on a set X, $|X| = n$

Soln:In how many ways can we write $x \sim y$, where both x and y are from the set X? That's right, n^2 ways. Now let us try to construct a relation. Some of these n^2 couples will be in our relation. To decide which ones, let us go element by element like before. For every pair we have two choices: whether to put it in the relation or to exclude it. Thus the total number of relations possible is $2 \times 2 \times ...$ (n^2 times) $= 2^{n^2}$.

(ii) Number of reflexive relations on X.

Soln: This means we have to include all the pairs of the form (x, x). There are n such pairs (one for each element). For the rest of the pairs, we don't really care whether they are included in our relation or not, the relation would still remain reflexive. So we have got 2 choices for each of the remaining pairs. Thus the number of such relations is 2^{n^2-n}.

(iii) Number of symmetric relations on X.

Soln: This means that if (a, b) is in the relation, then (b, a) will also be there. We can choose two distinct numbers a and b in $\binom{n}{2}$ ways. For each of these couples, we can either put (a, b) and (b, a) in our relation, or choose not to. We also choose to include the (x, x) cases or not to, our relation remains symmetric regardless. So the total number of symmetric relations is nothing but $2^{\binom{n}{2}+n} = 2^{\frac{n(n+1)}{2}}$.

(iv) Symmetric and Reflexive relations.

Soln: In this case we know we have to include all the (x, x) couples. Then the choice of including or excluding a pair is only permitted for couples (a, b) where $a \neq b$. There are $\binom{n}{2}$ such couples, so total number of symmetric and reflexive relations is $2^{\binom{n}{2}}$

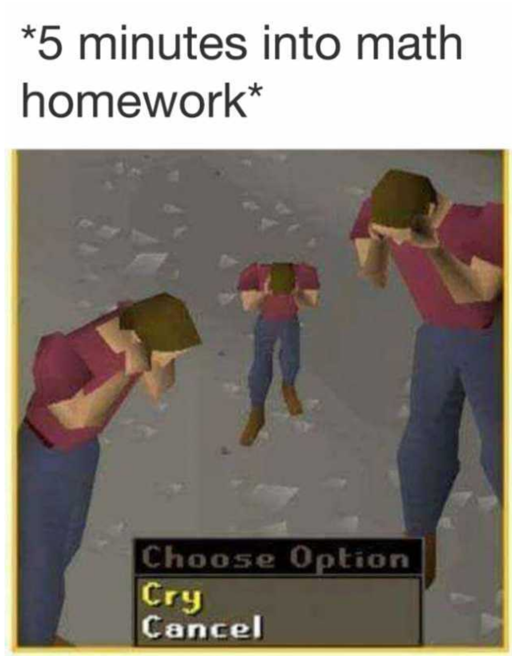

Figure 5: No caps, we all feel the pain at times