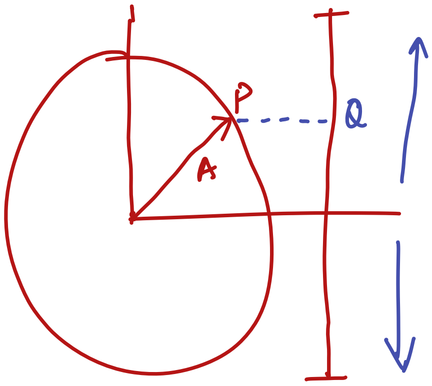
FEB 11TH, 2022 : Average value of a function :



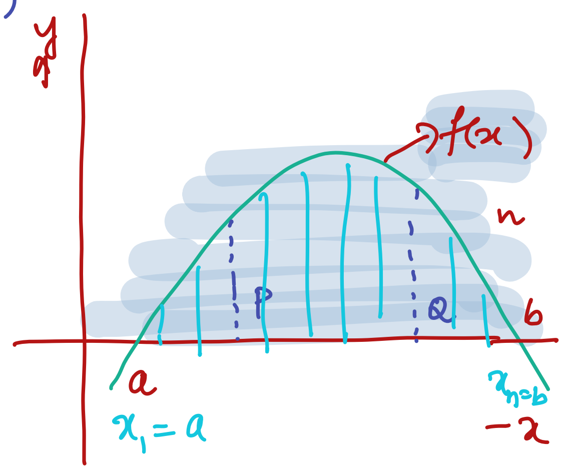
$$\boxed{y = A\sin\omega t}$$

$$\frac{dy}{dt} = A\omega\cos\omega t$$

$$\boxed{y = A\sin(\omega t + \phi)}$$

Fourier Series :- (Cos & Sin)

Average of a function :

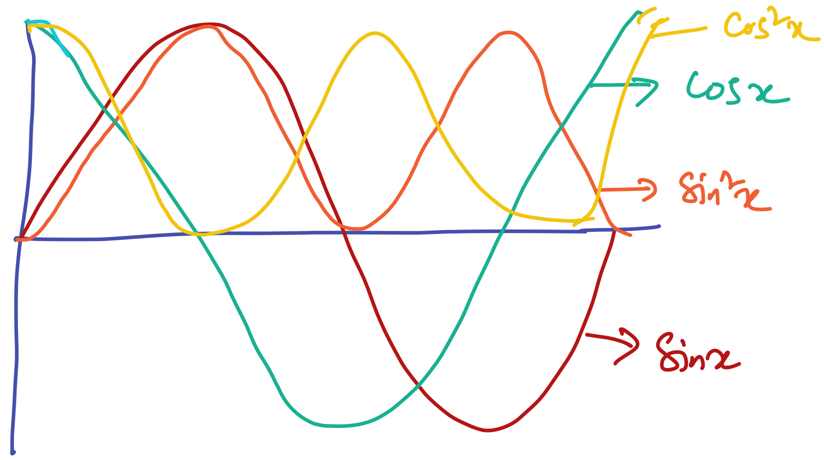


$$\boxed{\frac{\int_P^Q f(x)\,dx}{Q-P}}$$

$$\frac{f(x_1) + f(x_2) + f(x_3) \,----\, f(x_n)}{n}$$

It may happen that the average value of a given function is zero

Example $\sin x$

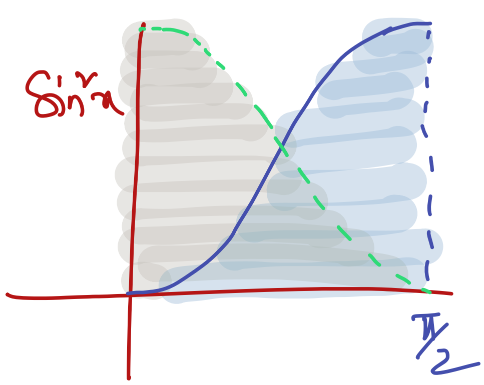


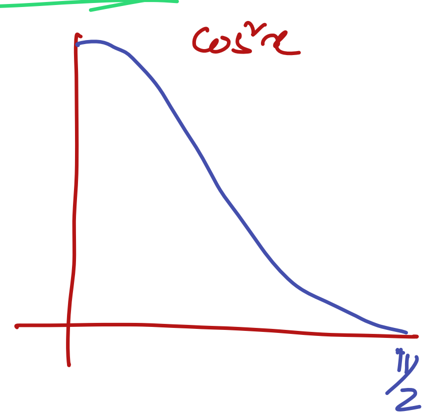
Average of $\sin x = 0$

Similarly Average of $\cos x = 0$

However if we take the average of $\sin^2 x$ or $\cos^2 x \neq 0$

$$\boxed{\int_{-\pi}^{\pi} \sin^2 x \, dx = \int_{-\pi}^{\pi} \cos^2 x \, dx}$$





for $\underline{n \neq 0}$

$$\boxed{\int_{-\pi}^{\pi} \sin^2 \underline{nx} \, dx = \int_{-\pi}^{\pi} \cos^2 \underline{nx} \, dx}$$

We have $\boxed{\sin^2 nx + \cos^2 nx = 1}$

$$\int_{-\pi}^{\pi} (\sin^2 nx + \cos^2 nx)\, dx = \int_{-\pi}^{\pi} dx = 2\pi$$

$$\int_{-\pi}^{\pi} \sin^2 nx\, dx = \int_{-\pi}^{\pi} \cos^2 nx\, dx = \pi$$

$$\boxed{\frac{1}{2\pi}\int_{-\pi}^{\pi} \sin^2 nx\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos^2 nx\, dx = \frac{\pi}{2\pi} = \frac{1}{2}}$$

Fourier Series: Expansion of a periodic function in series of sines and cosines.

2π is the periodicity for $\sin x$ & $\cos x$

$$\boxed{\sin n(x+2\pi) = \sin(nx+2n\pi) = \sin nx}$$

$$\boxed{\begin{aligned} f(x) = \frac{1}{2}a_0 &+ a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \cdots \\ &+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots \end{aligned}} \Rightarrow (A)$$

① The average value of $\sin mx \cos nx$ (over a period)

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0$$

② The average value of $\sin mx \sin nx$ (over a period)

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} 0 & m \neq n \\ \frac{1}{2} & m = n \neq 0 \\ 0 & m = n = 0 \end{cases}$$

③ The average value of $\cos mx \cos nx$ (over a period)

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \begin{cases} 0 & m \neq n \\ \frac{1}{2} & m = n \neq 0 \\ 1 & m = n = 0 \end{cases}$$

Now we try to find the fourier Coefficients : $[a_0, a_1, a_2 \ldots b_1, b_2 \ldots]$

Integrating eq (A) over a period

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \underbrace{\frac{1}{2}a_0 \frac{1}{2\pi}\int_{-\pi}^{\pi} dx}_{} + a_1 \frac{1}{2\pi}\int_{-\pi}^{\pi} \underline{\cos x \, dx} + \cdots$$

$$+ b_1 \frac{1}{2\pi}\int_{-\pi}^{\pi} \underline{\sin x \, dx} + \cdots$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2}a_0 \frac{1}{2\pi}\int_{-\pi}^{\pi} dx + a_1(0) + \cdots b_1(0)\ldots$$

$$= \frac{a_0}{2}\frac{1}{\cancel{2\pi}}(\cancel{2\pi})$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2}a_0 \Rightarrow \boxed{a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx}$$

Evaluating a_1 $\Rightarrow$ multiply $f(x)\times\cos x$, and integrate over a period

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos x\,dx = \frac{a_0}{2}\frac{1}{2\pi}\int_{-\pi}^{\pi}\underbrace{\cos x\,dx}_{=0} + a_1\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos^2 x\,dx$$

$$+\, a_2\frac{1}{2\pi}\int_{-\pi}^{\pi}\underbrace{\cos 2x\cos x}_{=0}\,dx + \cdots \quad (2)$$

$$+\, b_1\frac{1}{2\pi}\int_{-\pi}^{\pi}\underbrace{\sin x\cos x}_{=0}\,dx + \cdots \quad (1)$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos x\,dx = a_1\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos^2 x\,dx = \frac{1}{2}a_1$$

$$\boxed{a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos x\,dx}$$

Similarly, one can evaluate all n terms

$$\boxed{a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx}$$

$\Rightarrow$ Similarly $\boxed{b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx}$