Reading Project Report on Einstein's Special Theory of Relativity

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Reading Project Report on Einstein's Special Theory of Relativity

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Abstract: This is a reading project report that I prepared and is essentially a summary of what I have learnt now about the Special Theory of Relativity. It covers partly the insuffiencies faced while using non relativistic mechanics, then moves over to the famous Michelson-Morley experiment. After that, is discussed the postulates of Einstein's theory and an eventual build-up of the Lorentz Transformation. There is a mathematically evaluated consequence of Lorentz Transform; namely Length Contraction and Time Dilation. I have also covered Velocity Transformation, problems on causality, Minkowski Space and Four Vectors. The report ends with two consequences of the energy-momentum relations derived from Special Relativity namely, Compton Effect and Doppler Effect

Contents

1 Difficulties with Newtonian Mechanics and Galilean Relativity

1.1 Independence of Newton's Laws in all inertial frames of reference

To begin with, I shall be stating and then proving that Newton's Laws of motion are essentially the same irrespective of which inertial frame of reference a person or an observer is standing and viewing the motion of another object.

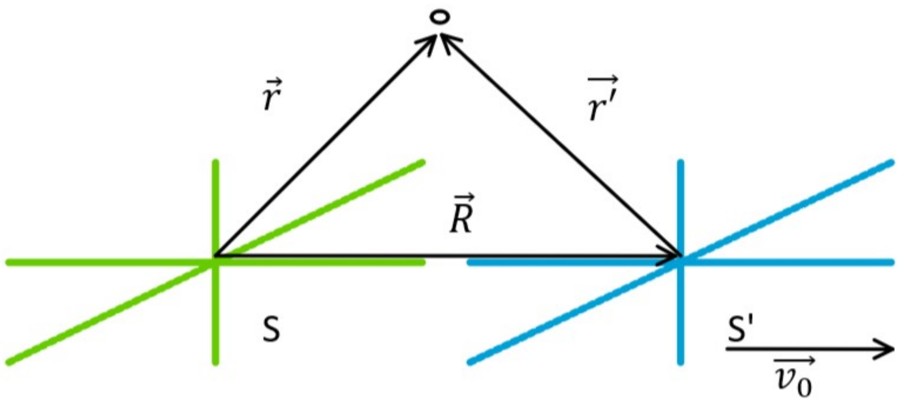

Figure 1: An Object w.r.t Different Inertial Frames

As we can see in figure 1 S and S' are two different frames of references where S is stationary and S' is moving with a veloctiy $\vec{v_0}$.
From the general triangle law of euclidean vector addition, we have

$$\vec{r} = \vec{R} + \vec{r'}$$

$$\Rightarrow \frac{d\vec{r}}{dt} = \frac{d\vec{R}}{dt} + \frac{d\vec{r'}}{dt}$$

$$\Rightarrow \vec{v} = \vec{v_0} + \vec{v'}$$

(Note that $\vec{v_0}$ is constant as the frames are inertial)

$$\Rightarrow \frac{d\vec{v}}{dt} = \frac{d\vec{v_0}}{dt} + \frac{d\vec{v'}}{dt}$$

$$\Rightarrow \vec{a} = \vec{a'}$$

$$\Rightarrow m\vec{a} = m\vec{a'}$$

$$\Rightarrow \vec{F} = \vec{F'} \tag{1}$$

Hence we can immediately see that acceleration is independent of inertial frames and hence Newton's Law is valid and same in all inertial frames of references.

1.2 A Pressing problem

We know, the speed of light in vacuum is given by $c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$. By Galilean Relativity, this speed should change for different inertial frames. But does that mean that ϵ_0 and μ_0 are dependent on inertial frames? This seems somewhat absurd because those are fundamental constants. The only logical expalnation would be that the expression for speed of light must be valid in one universal frame that is at **absolute rest** and that universal frame must be filled with a unique medium that we shall call **ether**. This in turn demands for the concept of **absolute velocity** which shall be the velocity of an object relative to ether. The speed of light in all other inertial frames can then be measured using Galilean Transformations that we are familiar with. But does this ether actually exist? These are some problems that I hope to tackle by the end of this report.

2 Michelson-Morley Experiment

2.1 Experimental setup

What we shall see now is a simplified form of the famous Michelson-Morley experiment. This experiment was extremely crucial as it had negated all the theories regarding the existence of the ether.
Let us look at the simplified diagram of the experimental setup.

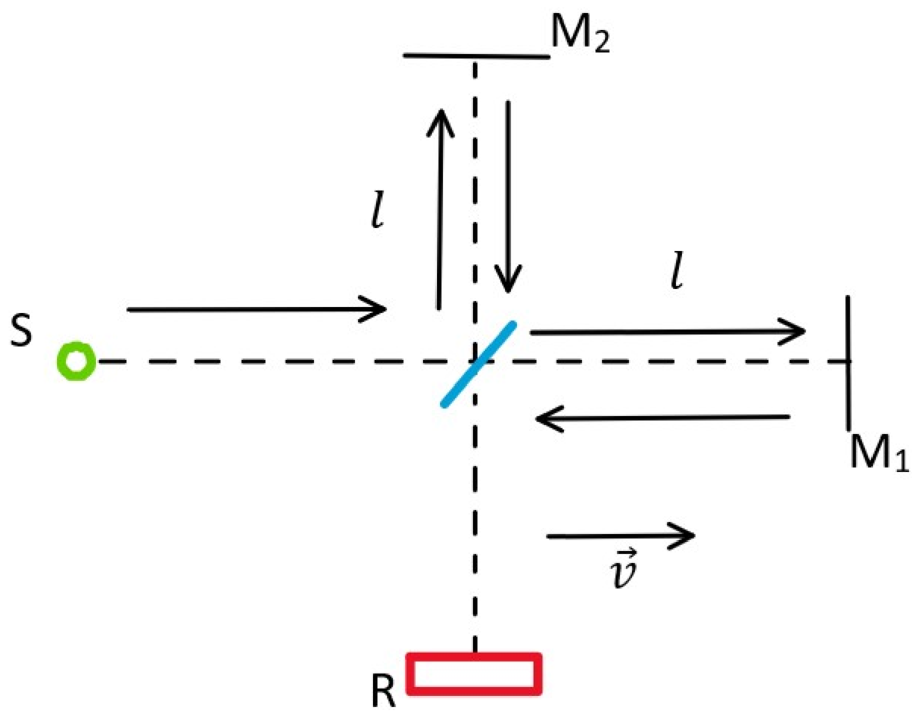

Figure 2: Experimental Diagram of Michelson-Morley

Here, S is the source of light, R is the receiver, M_1 and M_2 are the mirrors. The blue object at the centre is a partially refracting and reflecting thin glass. We consider the distance between both the mirrors and the thin glass to be equal (l). The entire setup is moving towards the right with a speed $\vec{v}$.The absolute speed of light is c (in ether) and hence an observer in the experimental frame shall observe a different velocity of light, i.e. $\vec{c} - \vec{v}$. Now, we calculate the time difference between the refracted ($\|$) and reflected ($\perp$) rays when they come back at the glass in the middle after being reflected by the mirrors M_1 and M_2 respectively.

$$t_{\parallel} = \frac{l}{c-v} + \frac{l}{c+v} = \frac{2lc}{c^2 - v^2} = \frac{2l}{c(1-\frac{v^2}{c^2})} = \frac{2l}{c}\left(1 + \frac{v^2}{c^2}\right) \qquad (\because v \ll c)$$

‘ Similarly, we also have:

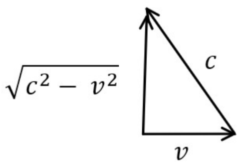

Figure 3: Component split up of the refracted ray

$$t_{\perp} = \frac{2l}{\sqrt{c^2 - v^2}} = \frac{2l}{c}\left(1 + \frac{v^2}{2c^2}\right) \qquad (\because v \ll c)$$

Now we take the time difference:

$$\Delta t = t_{\parallel} - t_{\perp} = \frac{l}{c}\left(\frac{v^2}{c^2}\right)$$

Now, the experimental setup is slowly rotated counter-clockwise:

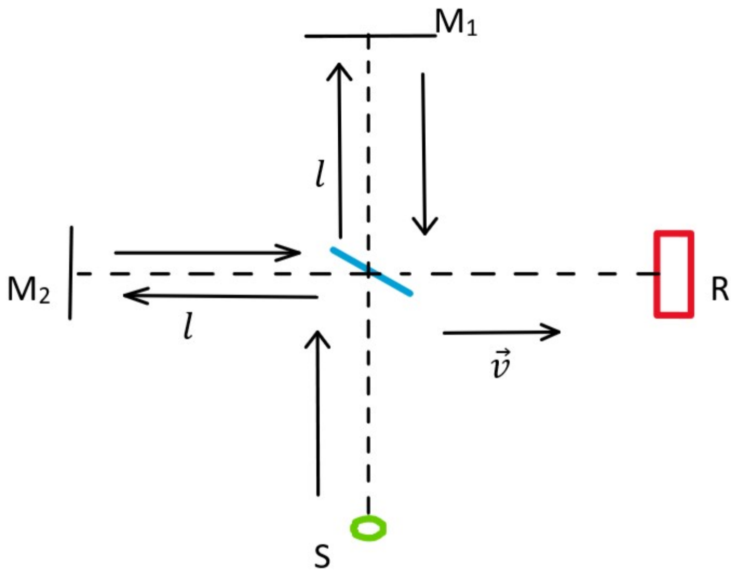

Figure 4: Setup Rotated by 90 degrees

We again compute the time difference between refracted and reflected ray:

$$\Delta t' = -\frac{l}{c}\left(\frac{v^2}{c^2}\right)$$

While the experimental setup is rotated, the path difference (δ) between the refracted and reflected ray also changes as the time difference changes:

$$\delta - \delta' = (n - n')\lambda$$

Dividing throughout by c, we have:

$$\Delta t - \Delta t' = \Delta n \frac{\lambda}{c}$$

$$\Rightarrow \Delta n = \frac{\Delta t - \Delta t'}{\frac{\lambda}{c}} = \frac{2l}{c}\left(\frac{v^2}{c^2}\right)\frac{c}{\lambda} = \frac{2l}{\lambda}\left(\frac{v^2}{c^2}\right)$$

Hence we can calculate the change in the order of interference fringes.

2.2 Results

As disappointing as it may seem, the experimental results were absolutely null. All attempts to calculate the change in the order of the fringe were negative as there was none. This further implied that there was no path difference. The experiment was repeated several times with even more sensitive apparatus, but in vain. This therefore meant that the so called 'ether' was completely non-existent. What then is the solution? **We are now in a position to discuss the postulates of Einstein's Special Theory of Relativity**.

3 Postulates of Einstein's Special Theory of Relativity

3.1 The Postulates

1. The Laws of Physics are same in all inertial frames of reference. No preferred inertial frame exists.

2. The speed of light 'c' is constant in all inertial frames of references.

3.2 Consequences

The ground beaking postulates given by Einstein is something that completely shattered the view of traditonal Classical Physicists. It shall be very easy to see that if we abide by the above postulates, the transformations according to Galilean Relativity does not hold. For example, let us look at the following problem.

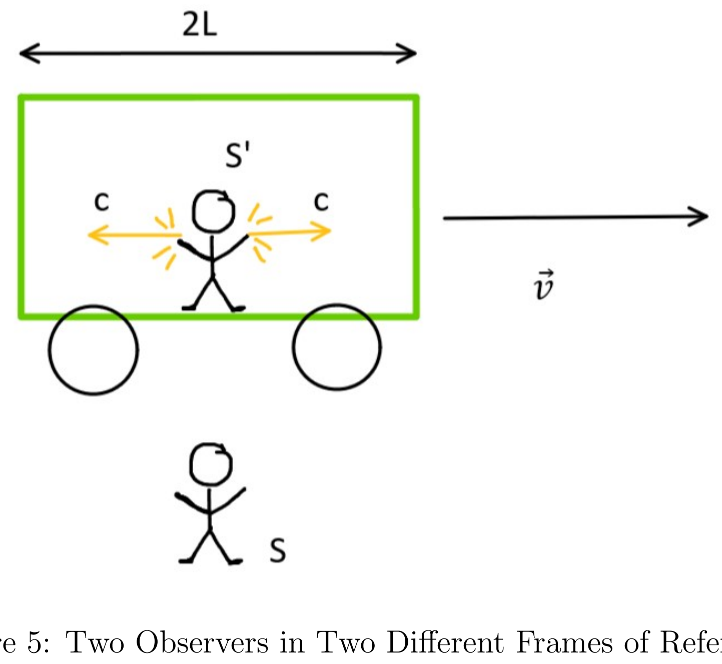

Figure 5: Two Observers in Two Different Frames of Reference

Let us have two observers in two different frames of reference. S is the stationary ground frame and S' is moving train frame. Let the observer in the S' frame of reference stand exactly in the middle of the train which shall be

his origin. Suppose the observer in S' shines light in both directions front and back, at the moment when the origin of S and S' coincide. According to the observer in S' frame of reference, time taken to hit the front wall = time taken to hit the back wall = $t' = \frac{L}{c}$. By Galilean Relativity, for the observer in S,
For front wall,
distance travelled by light = L + distance travelled by train

$$\Rightarrow (c+v)t_f = L + vt_f \Rightarrow t_f = \frac{L}{c}$$

For back wall,
distance travelled by light = L - distance travelled by train

$$\Rightarrow (c-v)t_b = L - vt_b \Rightarrow t_b = \frac{L}{c}$$

And so it is simultaneous in both the frames. Such a result is not surprising. However, if we invoke the Special Theory of Relativity, speed of light must be c in S frame too! Accordingly changing the equation,

$$ct_f = L + vt_f \Rightarrow t_f = \frac{L}{c-v}, \qquad \text{and}$$
$$ct_b = L - vt_b \Rightarrow t_b = \frac{L}{c+v}$$

And hence what we see is that the two times are not simultaneous! Thus according to the observer in S light hits the back wall before it hits the front wall.
What we thus seek is a new type of transformation which eventually we shall know as the Lorentz Transformation, named after its discoverer.

4 Lorentz Transformation

4.1 A Logical Approach

Let us consider a linear transformation of the form:

$$x' = C_{xx}x + C_{xy}y + C_{xz}z + C_{xt}t + C_1$$

$$y' = C_{yx}x + C_{yy}y + C_{yz}z + C_{yt}t + C_2$$

$$z' = C_{zx}x + C_{zy}y + C_{zz}z + C_{zt}t + C_3$$

$$t' = C_{tx}x + C_{ty}y + C_{tz}z + C_{tt}t + C_4$$

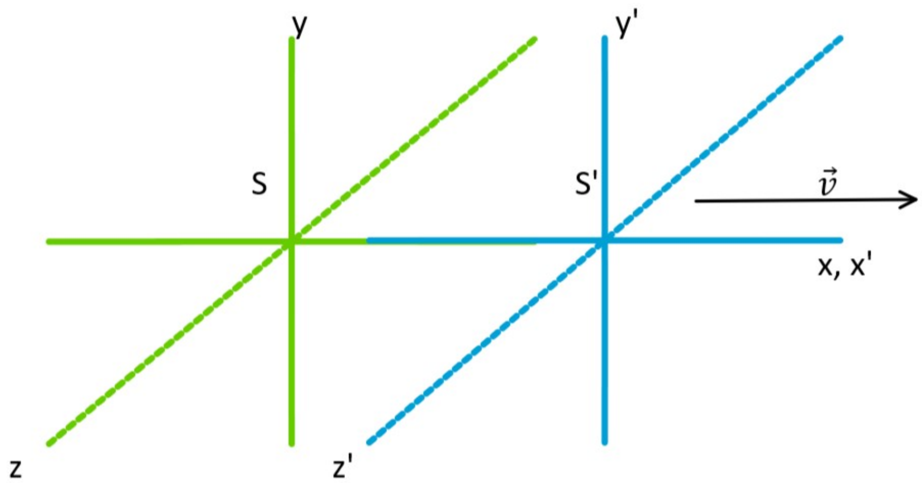

Figure 6: Two Different Frames of References

Now, we have good reason to believe that the transformation should be linear, because had it not been so, the transformation would give different results on the choice of units. Suppose there is a thin rod lying on the x axis from $x = 0$ to $x = 1$m at $t = 0$. Then $x' = C_{xx}x^n + 0 + 0 + 0 + C_1$. Naturally, we shall have $x' = 0$ to $x' = (C_{xx} + C_1)$m. In cm, that would be $100(C_{xx} + C_1)$cm. If we change the units of x to cm, the enlargement would be $(100^n C_{xx} + C_1)$cm. This is somewhat absurd and hence cannot be considered a possibility.

We shall note that **we have considered the clocks to be synchronised when the origin of the S frame (stationary) coincides with that of S' frame (moving)**. Then, if an **event** occurs at $x = y = z = t = 0$, then

it should also occur at $x' = y' = z' = t' = 0$. Substituting in the above equations, we get $C_1 = C_2 = C_3 = C_4 = 0$.
Hence, the modified set of equations is:

$$x' = C_{xx}x + C_{xy}y + C_{xz}z + C_{xt}t$$

$$y' = C_{yx}x + C_{yy}y + C_{yz}z + C_{yt}t$$
$$z' = C_{zx}x + C_{zy}y + C_{zz}z + C_{zt}t$$
$$t' = C_{tx}x + C_{ty}y + C_{tz}z + C_{tt}t$$

Again, we shall suppose that an event occurs along the x-y plane in S frame. Then, intuitively, it shall also occur in the same plane, i.e. x'-y' plane for the observer in S' frame also. This implies that if $z = 0$ then $z' = 0$. Thus,

$$0 = C_{zx}x + C_{zy}y + C_{zz} \times 0 + C_{zt}t$$

$$\Rightarrow C_{zx}x + C_{zy}y + C_{zt}t = 0$$

$\because$ x, y and t are linearly independent quantities (dimensions), the above equation implies: $C_{zx} = C_{zy} = C_{zt} = 0$.
In exactly the same way, had the event occured in the x-z plane,we could have deduced $C_{yx} = C_{yz} = C_{yt} = 0$.
Hence, we have modified our equations further:

$$x' = C_{xx}x + C_{xy}y + C_{xz}z + C_{xt}t$$

$$y' = C_{yy}y$$
$$z' = C_{zz}z$$
$$t' = C_{tx}x + C_{ty}y + C_{tz}z + C_{tt}t$$

If, however, an event occurs in the y-z plane for an observer in S, it should also occur at the same plane for S' but shifted by a distance of $vt \because$ the S' frame is moving. Thus we can write $x' = C_{xx}(x - vt)$.
Now, we shall speculate the situation a little bit. When we say that our choice of x-axis is specified, what we mean is that we choose our x-axis along the direction in which the moving frame moves away from the stationary frame. However, if we think carefully, we have freedom to choose our y and z axes. Of course, these axes must be perpendicular to the x-axis and also to each other, but there are infinite ways of choosing such a pair of y and z axes. As a matter of fact, whatever y and z axes we may have, we can simply rotate them in a plane perpendicular to the x-axis, to obtain a new set of y and z axes. However, this should not affect our transformation in any way,

otherwise, just by changing or rotating the y-z axes, we shall have different transformations, i.e. the transformation would depend on our choice of y and z axes. This, we do not want. Hence we must have $C_{ty} = C_{tz} = 0$.
Finally, we have our new transformaton:

$$x' = C_{xx}(x - vt)$$

$$y' = C_{yy}y$$

$$z' = C_{zz}z$$

$$t' = C_{tx}x + C_{tt}t$$

What is interesting to note is that, the Galilean transformation is a special case of the above general form:

$$x' = (x - vt)$$

$$y' = y$$

$$z' = z$$

$$t' = t$$

4.2 The Final Form

Now it is time for us to determine the rest of the constants.

$$\begin{aligned} x' &= C_{xx}(x - vt) && (2) \\ y' &= C_{yy}y && (3) \\ z' &= C_{zz}z && (4) \\ t' &= C_{tx}x + C_{tt}t && (5) \end{aligned}$$

Let us supose an object (rod) of length 1 unit is kept in the S frame of reference on the y-axis. When viewed from S' frame of reference, it's length shall appear to be C_{yy} units. If the same object was kept on the S' frame of reference on its y-axis (i.e. along y') and was measured 1 unit, in S frame it would appear to be $\frac{1}{C_{yy}}$ units. This means, that it enlarges in one frame of reference but shrinks in the other. One can then distinguish between the two frames and state which one is stationary and which one is the moving frame. **This is in strict violation of the first postulate of the Special Theory of Relativity, where all inertial frames are equivalent**. Refer 3.1 on page 6.

Hence, we must have $C_{yy} = 1$.
Extending the same argument to an object (rod) placed on the z-axis, we shall see that $C_{zz} = 1$.

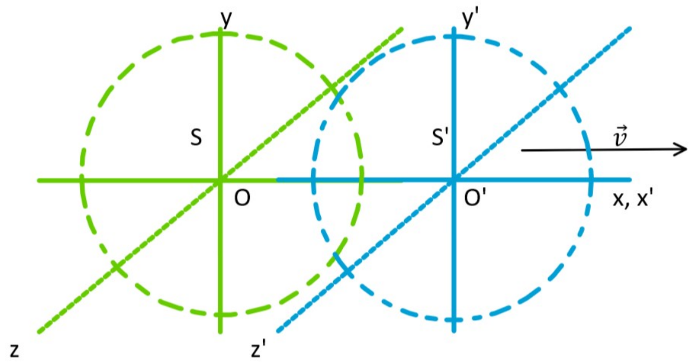

Figure 7: Spreading of light waves as seen by Two Observers in Different Frames

To determine the other constants, we now consider a different situation. Let us consider an event; a light wave is emitted at a moment when the origin of the two frames S and S' are coincident (i.e. $x = y = z = t = 0$ and $x' = y' = z' = t' = 0$). Obviously, the observer in the S frame of refernce standing at its origin will see a spherical wavefront centred at O. **Invoking the second postulate, an observer in S' frame will also notice a spherical wavefront which is centred at O', $\because$ speed of light is constant in all inertial frames of references**. Hence, we have:

$$x^2 + y^2 + z^2 = c^2t^2 \qquad and$$
$$x'^2 + y'^2 + z'^2 = c^2t'^2$$

And both these equations must be the same, i.e. of a sphere with the same radius.
Substituting the values of x', y', z' and t' from (2), (3), (4) and (5), and then equating the coefficients of x^2, y^2, z^2, t^2 and xt, we get the following:

$$\begin{aligned} C_{xx}^2 - c^2 C_{tx}^2 &= 1, & (6)\\ C_{tt}C_{tx}c^2 + C_{xx}^2 v &= 0 \qquad \text{and} & (7)\\ C_{tt}^2 c^2 - C_{xx}^2 v^2 &= c^2 & (8)\end{aligned}$$

If an event occurs at $t = 0$ and $x > 0$, then we must also have $x' > 0$ since at $t = 0$, the origins are coincident. Hence, from (2), we must have $C_{xx} > 0$.

Similarly, if an event occurs at origin of S ($x = 0$) at a time $t > 0$ we must also have $t' > 0$ since, at $t = 0$ the origins were coincident and also we had $t' = 0$. This implies, $C_{tt} > 0$, from (5).
Since, $C_{tt} > 0$ and $C_{xx} > 0$, we must have $C_{tx} < 0$, from (7).
Keeping in mind our conclusions and solving (6), (7) and (8), we get:

$$C_{xx} = +\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$C_{tt} = +\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$C_{tx} = -\frac{v}{c^2}\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right)$$

For our convenience, we shall write $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$. Hence we have our Lorentz Transformaton Equations as follows:

$$\begin{aligned} x' &= \gamma(x - vt) & (9)\\ y' &= y & (10)\\ z' &= z & (11)\\ t' &= \gamma\left(t - \frac{vx}{c^2}\right) & (12) \end{aligned}$$

Similarly, we also have the inverse transformation as:

$$\begin{aligned} x &= \gamma(x' + vt') & (13)\\ y &= y' & (14)\\ z &= z' & (15)\\ t &= \gamma\left(t' + \frac{vx'}{c^2}\right) & (16) \end{aligned}$$

5 Length Contraction and Time Dilation

5.1 Concept of Proper Length and Proper Time

Before we move on to the concept of Length Contraction and Time Dilation I feel it is important to define the terms **proper length** and **proper time**.

Proper Length: The length of an object measured in a frame in which the object is stationary.
We must note an important thing here. Let us define the components of the length of an object as the simple difference between the coordinates of its two ends.now, this definition holds true only if the object is stationary w.r.t a frame. In all other frames in which the object is moving, we have to take into account the distance moved by the object in a given time difference $-$ the simple difference of the coordinates of the two ends does not account for the length of the object. To get the length of the object in a frame in which the object is moving, we must measure the coordinates of the two ends **at the same time** for the difference to give the length. However as told earlier, this is not the case when the object is stationary in a frame of reference (we can measure the coordinates of two ends at different times and still obtain the length)
Proper Time Interval: The time interval between two events in a frame in which the events occur at exactly the same position.

5.2 Contraction of Length

It is simple to understand that $\because$ Lorentz Transformation and its inverse is a linear transformation (by which I mean that the transformation operator is a linear operator), we can easily show by simple subtraction, that:

$$\Delta x = \gamma(\Delta x' + v\Delta t')$$

$$\Delta y = \Delta y'$$

$$\Delta z = \Delta z'$$

$$\Delta t = \gamma\left(\Delta t' + \frac{v\Delta x'}{c^2}\right)$$

Let an object be stationary in the S' frame. Then the x, y and z components of the **proper length** is given by $\Delta x'$, $\Delta y'$, $\Delta z'$. Also we notice that for Δx, Δy and Δz to give the components of the length as seen in S, we must have

$\Delta t = 0$, i.e. time difference between measuring the coordinates of the ends of the rod in S frame must be 0:

$$0 = \gamma\left(\Delta t' + \frac{v\Delta x'}{c^2}\right) \Rightarrow \Delta t' = -\frac{v\Delta x'}{c^2}$$

Thus we have:

$$\Delta x = \gamma(\Delta x' + v\Delta t') = \gamma(\Delta x' - v\frac{v\Delta x'}{c^2}) = \gamma\Delta x'(1-\frac{v^2}{c^2}) = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\Delta x'(1-\frac{v^2}{c^2}) = \frac{\Delta x'}{\gamma}$$

$$or, \qquad \Delta x = \frac{\Delta x'}{\gamma}$$

Thus we have mathematically shown that the length as perceived by an observer in the S frame of reference is shortened by a factor of γ ($\Delta x < \Delta x'$) **only along the x direction**. In fact this is true whenever we observe this from any other frame of reference in which this rod is not stationary. However, we notice that $\Delta y = \Delta y'$ and $\Delta z = \Delta z'$, i.e. the components of the length remain constant in the y and z dimensions.

5.3 Dilation of Time

Suppose we have two events. Let both the events occur at exactly the same position in the S frame of reference, i.e. $\Delta x = 0$. Then according to our discussion earlier, Δt must be the **proper time interval**. Then we have:

$$\Delta x = \gamma(\Delta x' + v\Delta t') \Rightarrow 0 = \gamma(\Delta x' + v\Delta t') \Rightarrow \Delta x' = -v\Delta t'$$

Again,

$$\Delta t = \gamma\left(\Delta t' + \frac{v\Delta x'}{c^2}\right) = \gamma\left(\Delta t' - \frac{v^2\Delta t'}{c^2}\right) = \gamma\Delta t'\left(1 - \frac{v^2}{c^2}\right) = \frac{\Delta t'}{\gamma}$$

$$\Delta t' = \gamma\Delta t$$

Thus we have mathematically shown that the time between the two events is dilated in the S' frame of reference ($\Delta t' > \Delta t$). As a matter of fact time is dilated in all those frames of references where the two events do not seem occur at the same position as in S frame of reference.

5.4 An Interesting Example

For quite a few time, Classical Physicists had a problem with describing a phenomenon known as **muon-shower**. Muons are elementary particles that are formed in the upper atmosphere due to incoming cosmic rays and have a very short average life time. Classical Physicists tried to calculate the fraction of muons that reached the surface of the Earth using the information of the distance of upper atmosphere, speed of muon as seen from Earth and its average life.

Surprisingly, when experiments were carried out to measure the fraction of muons that actually reached the Earth, the number came out to be far far greater than what was calculated.

Here comes the Special Theory of Relativity to the rescue. Let us consider two frames of references − an observer standing on the surface of the Earth (E) and a hypothetical observer sitting on the muon (M). Now for M frame, the average life after which the muon decays is the **proper time**, since the muon is at rest in M frame. As soon as we shift to the obsever on Earth, i.e. E frame, this time is dilated! so, its average lifetime increases by a factor of γ in E frame. Precise calculations match exactly with the experimental results.

This can also be interpreted in another way. In E frame, the distance between the surface of the Earth and the upper atmosphere can be imagined as a vertical rod of that length that is at rest in the E frame. Hence the **length is proper** in E frame. However, according to Special Relativity, this same length must be contracted in M frame! Hence, the hypothetical observer sees that this distance is shorter by a factor of γ and so the muon has to travel a shoter distance in M frame. Thus we are able to describe this phenomenon perfectly using Special Relativity.

6 Velocity Transformation

We have now come to a position where we can discuss the velocity transformations.
From the inverse transformation we have:

$$\Delta t = \gamma\left(\Delta t' + \frac{v\Delta x'}{c^2}\right) \Rightarrow \lim_{\Delta t' \to 0} \Delta t = \gamma\left(\lim_{\Delta t' \to 0} \Delta t' + \frac{v \lim_{\Delta t' \to 0} \Delta x'}{c^2}\right) \Rightarrow \lim_{\Delta t' \to 0} \Delta t = 0$$

$$\therefore \text{as } \Delta t' \to 0, \text{ we have} \Delta t \to 0$$

We can thus show the transformation in the x component of velocity:

$$\frac{\Delta x'}{\Delta t'} = \frac{\gamma(\Delta x - v\Delta t)}{\gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right)} \Rightarrow \lim_{\Delta t' \to 0} \frac{\Delta x'}{\Delta t'} = \lim_{\Delta t \to 0} \frac{(\frac{\Delta x}{\Delta t} - v)}{\left(1 - \frac{v}{c^2}\frac{\Delta x}{\Delta t}\right)} \Rightarrow \lim_{\Delta t' \to 0} \frac{\Delta x'}{\Delta t'} = \frac{(\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} - v)}{\left(1 - \frac{v}{c^2} \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}\right)}$$

$$\Rightarrow u'_x = \frac{(u_x - v)}{\left(1 - \frac{vu_x}{c^2}\right)}$$

Similarly, we also have the transformation for the y and z components of velocity:

$$\frac{\Delta y'}{\Delta t'} = \frac{\Delta y}{\gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right)} \Rightarrow \lim_{\Delta t' \to 0} \frac{\Delta y'}{\Delta t'} = \frac{\lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t}}{\gamma\left(1 - \frac{v}{c^2} \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}\right)} \Rightarrow u'_y = \frac{u_y}{\gamma\left(1 - \frac{vu_x}{c^2}\right)}$$

$$\frac{\Delta z'}{\Delta t'} = \frac{\Delta z}{\gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right)} \Rightarrow \lim_{\Delta t' \to 0} \frac{\Delta z'}{\Delta t'} = \frac{\lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t}}{\gamma\left(1 - \frac{v}{c^2} \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}\right)} \Rightarrow u'_z = \frac{u_z}{\gamma\left(1 - \frac{vu_x}{c^2}\right)}$$

Rearranging the final forms, we have the set of velocity transformations as follows:

$$u'_x = \frac{(u_x - v)}{\left(1 - \frac{vu_x}{c^2}\right)} \tag{17}$$

$$u'_y = \frac{u_y}{\gamma\left(1 - \frac{vu_x}{c^2}\right)} \tag{18}$$

$$u'_z = \frac{u_z}{\gamma\left(1 - \frac{vu_x}{c^2}\right)} \tag{19}$$

The inverse transformation will naturally be given by:

$$u_x = \frac{(u'_x + v)}{\left(1 + \frac{vu'_x}{c^2}\right)} \tag{20}$$

$$u_y = \frac{u'_y}{\gamma\left(1 + \frac{vu'_x}{c^2}\right)} \tag{21}$$

$$u_z = \frac{u'_z}{\gamma\left(1 + \frac{vu'_x}{c^2}\right)} \tag{22}$$

A point to note is that if the speed of a particle in one frame of reference is c, then in a diferent frame also, it comes out to be c. This can be easily checked by taking simple examples. What is important is that, the individual x, y, and z components of velocity might not be conserved for such a particle but the magnitude (speed) remains conserved in both the frames, i.e. $\sqrt{u_x^2 + u_y^2 + u_z^2} = \sqrt{u_x'^2 + u_y'^2 + u_z'^2} = c$.

7 The Concept of Time-Like and Space-Like Separated Events

One of the most famous consequences of Einstein's Special Theory of Relativity is that it is not possible to have speeds greater than the speed of light. A simple explanation from mathematical point of view would be that $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ would turn out to be imaginary if $v > c$. However, there is also an intuitive explanation that deals with the concept of cause and effect. Before moving on to that, it is important that we discuss what is known as time-like and space-like separated events.

7.1 Time-Like Separated Events

Let us have two events E1 and E2 occuring on the x-axis, viewed in two different frames of references S and S'.Let us have in S frame the postion difference between E1 and E2 as $\Delta x = x_2 - x_1$ and the time difference between E1 and E2 as $\Delta t = t_2 - t_1$. In S', the differences are $\Delta x' = x'_2 - x'_1$ and $\Delta t' = t'_2 - t'_1$.

When $\Delta x < c\Delta t$, we call the events time-like separated. From Lorentz Transformation:

$$\Delta x' = \gamma(\Delta x - v\Delta t) \qquad \text{and}$$
$$\Delta t' = \frac{\gamma}{c}\left(c\Delta t - \frac{v\Delta x}{c}\right)$$

Giving it a little thought, we can observe from the above two equations that $\Delta x'$ may be positive or negative or zero but $\Delta t'$ must always be positive (unless $v > c$). $\because \Delta t' > 0$, there does not exist any frame S' in which these two events appear to occur simultaneously, hence the name−time-like separated. However, there may exist a frame S' in which $\Delta x' = 0$, by adjusting the value of v while it is still less than c.

7.2 Space-Like Separated Events

For the same two events E1 and E2, when $\Delta x > c\Delta t$, we call the events space-like separated. From Lorentz Transformation:

$$\Delta x' = \gamma(\Delta x - v\Delta t) \qquad \text{and}$$

$$\Delta t' = \frac{\gamma}{c}\left(c\Delta t - \frac{v\Delta x}{c}\right)$$

We again observe from the same two equations that this time $\Delta t'$ may be positive or negative or zero but $\Delta x'$ must always be positive (unless $v > c$). $\because$ $\Delta x' > 0$, there does not exist any frame S' in which these two events appear to occur at the same position, hence the name$-$space-like separated. However, there may exist a frame S' in which $\Delta t' = 0$, by adjusting the value of v while it is still less than c.

8 Violation of Causality for Speeds Greater than that of Light

8.1 For Time-Like Separated Events

We shall understand this with a simple example.

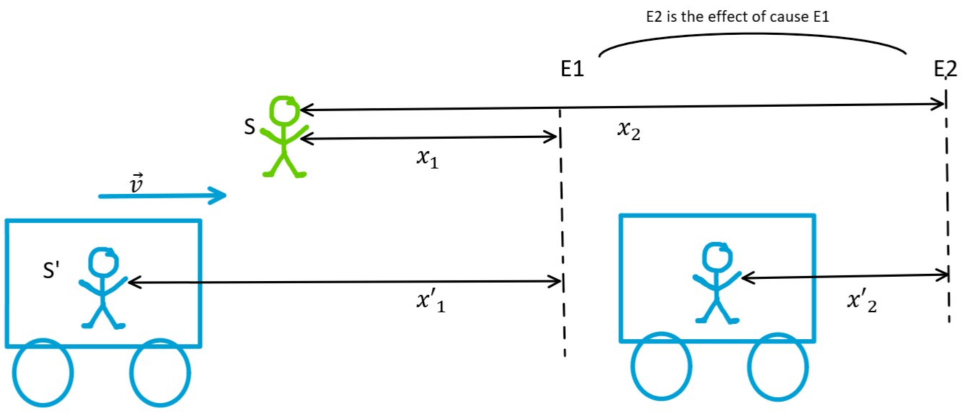

Figure 8: Time-Like Separated Events

Supposing that, E1 and E2 are cause-effect related, then we see an example in the above figure, where $\Delta x = x_2 - x_1 > 0$ but $\Delta x' = x_2' - x_1' < 0$. However, since E2 is the effect of cause E1, there exists no frame in which the sign of $\Delta t'$ is reversed, i.e. there is no frame in which the effect E2 seems to happen before the cause E1. As we have seen earlier, this is only possible if $v > c$ that violates causality and hence $v > c$ is not allowed.

8.2 For Space-Like Separated Events

In order to explain this part, let us take an example:

Suppose an observer is standing in a frame of reference S that is stationary w.r.t. himself/herself. Let there occur two events E1 and E2 at coordinates $(0, 0, 0, 0)-(\mathrm{E1})$ and $(5, 0, 0, 10^{-8})-(\mathrm{E2})$. (Note that the coordinates are in the format (x, y, z, t))
We can immediatey observe that $5 = \Delta x > c\Delta t = 3$. hence the events are space like separated. We can thus find a frame S' where the order of the two events can be reversed.

$$\Delta t' < 0 \Rightarrow \frac{\gamma}{c}\left(c\Delta t - \frac{v\Delta x}{c}\right) < 0 \Rightarrow c\Delta t - \frac{v\Delta x}{c} < 0 \Rightarrow v > \frac{c\Delta t}{\Delta x}c \Rightarrow v > \frac{3c}{5}$$

Now, what if the two events were cause and effect related? Suppose E1 is shooting of a bullet and E2 is the bullet hitting the target. In such a case the speed of bullet in S frame would be: $u = \frac{5}{10^{-8}} = 5 \times 10^8 > c$. Hence the bullet would have to travel faster than light.

Thus we see that $\Delta x > c\Delta t$ can never occur for events related by cause and effect because then the information (here bullet) would have to travel faster that light. Otherwise, we would have been able to find a frame where the time order of the two events get reversed and causality violated.

9 Minkowski Space and Four Vectors

9.1 Concept of Four Vectors

We are familiar with the concept of classical Euclidean vectors, wherein we have three dimensions, namely− x, y and z. However, as we have seen till now, time is also not absolute and it needs to be considered as a dimension. But, in order to form a vector space with the three spacial axes and time, we need to define the vectors in such a way that the general properties of a linear vector space are maintained. For example, the inner product (dot product in Euclidean vectors) of two vector quantities must be a scalar; and as we know, a scalar must be independent of any frame. For example the magnitude of length of a vector in Euclidean space $\vec{v}$, i.e $\sqrt{\vec{v}.\vec{v}}$ must be same in all frames in Classical Physics, i.e. $\sqrt{\vec{v}.\vec{v}} = \sqrt{\vec{v'}.\vec{v'}}$.

Now , we shall define the four components of a **position four vector** as following: (x, y, z, ict)
Taking $\beta = \frac{v}{c}$, $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$, and from our knowledge of Lorentz transformation:

$$\begin{pmatrix} x' \\ y' \\ z' \\ ict' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ ict \end{pmatrix}$$

where, x and x' axes are coincident and chosen along the direction of relative velocity v. In exactly the same way, we can also infer the inverse transformation as follows:

$$\begin{pmatrix} x \\ y \\ z \\ ict \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ ict \end{pmatrix}$$

Let us denote the operator for direct transformation as A. Since $|A| \neq 0$, the inverse transformaton operator exists as above and is represented by A^{-1}. We can directly observe that A^{-1} is nothing but the transpose conjugate of A; hence $A^{-1} = A^\dagger$. Thus, A is a unitary operator and preserves the inner (scalar/dot) product:

$$\langle r'|r'\rangle = \langle r| A^\dagger A |r\rangle = \langle r| A^{-1} A |r\rangle = \langle r|r\rangle$$

where,

$$|r\rangle = \begin{pmatrix} x \\ y \\ z \\ ict \end{pmatrix}, \langle r| = \begin{pmatrix} x & y & z & ict \end{pmatrix}, A = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \text{ and,}$$

$$A^{\dagger} = A^{-1} = \begin{pmatrix} \gamma & 0 & 0 & -i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ i\beta\gamma & 0 & 0 & \gamma \end{pmatrix}$$

9.2 Proper Time as a Four Scalar

Let us now come up with a new definition of proper time. We have our displacement four vector as $\Delta\underset{\sim}{s} = \Delta x\,\hat{i} + \Delta y\,\hat{j} + \Delta z\,\hat{k} + ic\Delta t\,\hat{t}$. As we have just now seen that the magnitude $\Delta s = |\Delta\underset{\sim}{s}| = \sqrt{\Delta\underset{\sim}{s}.\Delta\underset{\sim}{s}} = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2 - (c\Delta t)^2}$ is a scalar, which may or may not be imaginary. Now we write proper time as:

$$\Delta\tau = \sqrt{-\frac{\Delta s^2}{c^2}}$$

All that is left for us to check is that whether our previous definition of proper time interval fits into the new definition.

Let us consider the linear one dimensional motion between of two frames ($\Delta y = \Delta y' = 0 = \Delta z = \Delta z'$). Let S be the stationary frame. Let two events be separated by Δx and Δt in S frame, then it is possible to find a frame of reference S' where $\Delta x' = 0$, such that $\Delta t'$ represents the proper time interval (i.e. the two events occur at the same position w.r.t. S' frame). Then:

$$\Delta x' = \gamma(\Delta x - v\Delta t) \Rightarrow 0 = \gamma(\Delta x - v\Delta t) \Rightarrow \Delta x = v\Delta t \Rightarrow v = \frac{\Delta x}{\Delta t}$$

We substitute the value of v in the time transformation equation:

$$\Delta t' = \gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right) = \left(\frac{1}{\sqrt{1 - \frac{\Delta x^2}{\Delta t^2 c^2}}}\right) \times \Delta t\left(1 - \frac{\Delta x^2}{\Delta t^2 c^2}\right)$$

$$= \sqrt{-\frac{\Delta x^2 - c^2\Delta t^2}{c^2}}$$

$$or, \qquad \Delta t' = \sqrt{-\frac{\Delta s^2}{c^2}}$$

Thus we see that our previous definition of proper time beautifully fits into the new definition! Not only that but we have a very interesting observation: for space-like separated events ($\Delta x > c\Delta t$), this proper time comes out to be imaginary, which is indeed expected as it is impossible to find a frame where the two events can occur simultaneously (i.e. $\Delta x' = 0$ is not possible for any frame).

9.3 Velocity Four Vector

We must first understand that since time is not frame independent in the Minkowski space, velocity of a particle $u = \lim_{\Delta t \to 0} \frac{\Delta \underset{\sim}{s}}{\Delta t}$ no longer remains a four vector (i.e. does not follow the same transformation as described earlier). However, if we give a new definition where $\underset{\sim}{u} = \lim_{\Delta \tau \to 0} \frac{\Delta \underset{\sim}{s}}{\Delta \tau}$ we shall obtain a four vector because proper time τ is a four scalar. This is what we call a **velocity four vector**.

All we are left to show is the relation between Δt and $\Delta \tau$:

$$\Delta \tau = \sqrt{-\frac{\Delta s^2}{c^2}} = \sqrt{-\frac{\Delta x^2 + \Delta y^2 + \Delta z^2 - (c\Delta t)^2}{c^2}} = \sqrt{\Delta t^2 - \frac{\Delta x^2 + \Delta y^2 + \Delta z^2}{c^2}}$$

$$\Delta \tau = \sqrt{\Delta t^2 - \frac{(u\Delta t)^2}{c^2}}$$

$$\Delta \tau = \frac{\Delta t}{\gamma_u}$$

Writing in terms of differentials,

$$d\tau = \frac{dt}{\gamma_u}$$

Hence we have: $\underset{\sim}{u} = \frac{d\underset{\sim}{s}}{d\tau} = \gamma_u \frac{d\underset{\sim}{s}}{dt} = \gamma_u u$. This expression that we have obtained is very important as it relates the velocity four vector ($\underset{\sim}{u}$) with the velocity (u) of an object in a particular frame. Now this velocity four vector obeys the same transformation equations as any four vector, which we shall write as follows:

$$\begin{pmatrix} \underset{\sim}{u'_x} \\ \underset{\sim}{u'_y} \\ \underset{\sim}{u'_z} \\ ic\gamma_{u'} \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} \underset{\sim}{u_x} \\ \underset{\sim}{u_y} \\ \underset{\sim}{u_z} \\ ic\gamma_u \end{pmatrix}$$

$$or, \quad \begin{pmatrix} \gamma_{u'} u'_x \\ \gamma_{u'} u'_y \\ \gamma_{u'} u'_z \\ \gamma_{u'} ic \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} \gamma_u u_x \\ \gamma_u u_y \\ \gamma_u u_z \\ \gamma_u ic \end{pmatrix}$$

9.4 Why Do We Need To Redefine Momentum?

We shall now see in an example that from the classical definiton, the law of conservation of momentum does not hold when we shift from one relativistic frame to another.

Let us take an example:

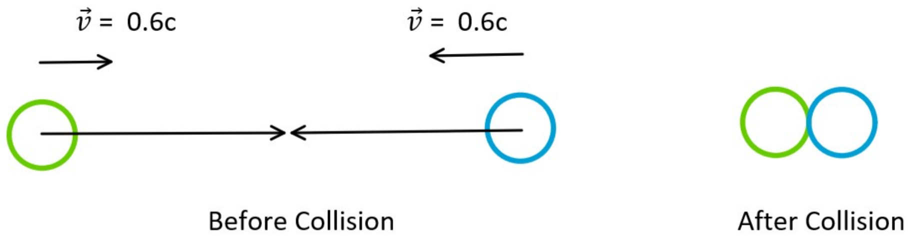

Figure 9: An Inelastic Collision

Suppose we have two identical bodies (same mass and dimension) that are moving towards each other with a speed 0.6c each in a rame S. Let us have and inelastic collision wherein the two bodies stick to each other. Now clearly, we can see that $initial\ momentum = o.6c \times m - 0.6c \times m = 0$. From Law of Conservation of Momentum, we must have $final\ momentum = initial\ momentum = 0$. Hence, after collision both the masses appear to be at rest (while stuck to each other).
Now let us observe this again but from a different frame S' that is moving with a speed 0.6c with respect to the previous frame S. We obtain the initial speeds of the two objects from our relativistic velocity transformaton.

$$u'_{1x} = \frac{(u_x - v)}{\left(1 - \frac{vu_x}{c^2}\right)} = \frac{(0.6c - 0.6c)}{\left(1 - \frac{(0.6c)\times(0.6c)}{c^2}\right)} = 0$$

$$u'_{2x} = \frac{(u_x - v)}{\left(1 - \frac{vu_x}{c^2}\right)} = \frac{(-0.6c - 0.6c)}{\left(1 - \frac{(0.6c)\times(-0.6c)}{c^2}\right)} = \frac{-1.2c}{1 + 0.36} = -\frac{1.2}{1.36}c$$

The transformation of the final speed after collision from S to S' frame is:

$$u'_f = \frac{(u_x - v)}{\left(1 - \frac{vu_x}{c^2}\right)} = \frac{(0 - 0.6c)}{1 - 0} = -0.6c$$

Hence we have:

$$initial\ momentum = m \times u'_{1x} + m \times u'_{2x} = m \times 0 - m \times \frac{1.2}{1.36}c = -\frac{1.2}{1.36}mc$$

$$final\ momentum = 2m \times u'_f = -1.2mc$$

Thus we see that, $initial\ momentum \neq final\ momentum$ when seen from S' frame of reference! Hence, we need to have a new definition in order to conserve momentum in all inertial frames of reference that is consistent even in relativistic speeds.

9.5 Momentum-Energy Four Vector

Let us define a momentum four vector as $\underset{\sim}{p} = m_0\underset{\sim}{u} = m_0\gamma_u(u_x, u_y, u_z)$ where we define:

$$p_x = m_0\gamma_u u_x$$

$$p_y = m_0\gamma_u u_y$$

$$p_z = m_0\gamma_u u_z$$

What is important to note here is that m_0 is defined as four scalar such that it is the mass of an object in a frame in which the object is at rest. In other words we also call it **rest mass**. Here, we can thus interpret $m_0\gamma_u$ as the effective mass of the object as observed in the frame in which the speed of the object is u. We must also see that there is a fourth component of velocity four vector that we have to take into consideration. Denoting that component as 'A_4':

$$A_4 = m_0\gamma_u ic$$

Our new definition has the same transformation matrix as used earlier:

$$\begin{pmatrix} m_0\gamma_{u'}u'_x \\ m_0\gamma_{u'}u'_y \\ m_0\gamma_{u'}u'_z \\ m_0\gamma_{u'}ic \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} m_0\gamma_u u_x \\ m_0\gamma_u u_y \\ m_0\gamma_u u_z \\ m_0\gamma_u ic \end{pmatrix}$$

$$or, \quad \begin{pmatrix} p'_x \\ p'_y \\ p'_z \\ A'_4 \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ p_z \\ A_4 \end{pmatrix}$$

Let us denote the sum of inital and final momenta of n particles as:

$$\sum_{k=1}^{N} p_{kI} \quad \text{and} \quad \sum_{k=1}^{N} p_{kF}$$

Since we are dealing with a linear vector space, a linear transformation like summation will not change our transformation matrix.

$$\begin{pmatrix} \sum_{k=1}^{N} p'_{xkI} \\ \sum_{k=1}^{N} p'_{ykI} \\ \sum_{k=1}^{N} p'_{zkI} \\ \sum_{k=1}^{N} A'_{4kI} \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} \sum_{k=1}^{N} p_{xkI} \\ \sum_{k=1}^{N} p_{ykI} \\ \sum_{k=1}^{N} p_{zkI} \\ \sum_{k=1}^{N} A_{4kI} \end{pmatrix}$$

Since in y and z components, the summation of momenta remains same in both frames, it is the x component in which we are interested.

$$\sum_{k=1}^{N} p'_{xkI} = \gamma \sum_{k=1}^{N} p_{xkI} + i\beta\gamma \sum_{k=1}^{N} A_{4kI}$$

Now, let $\sum_{k=1}^{N} p_{xkI} = \sum_{k=1}^{N} p_{xkF}$, i.e. sum of x components of momenta remains conserved in S frame after an interaction. We want $\sum_{k=1}^{N} p'_{xkI} = \sum_{k=1}^{N} p'_{xkF}$ i.e. we want this conservation in S' frame also. Hence, we must have $\sum_{k=1}^{N} A_{4kI}$ conserved before and after an interaction. A necessary result of conservation of $\sum_{k=1}^{N} A_{4kI}$ and $\sum_{k=1}^{N} p_{xkI}$ is the conservation of $\sum_{k=1}^{N} A'_{4kI}$, as we see:

$$\sum_{k=1}^{N} A'_{4kI} = -i\beta\gamma \sum_{k=1}^{N} p_{xkI} + \gamma \sum_{k=1}^{N} A_{4kI}$$

This brings us to the question: **What is A_4?**

$$A_4 = m_0 \gamma_u ic = \frac{i}{c} \times mc^2$$

Now here comes Einstein's genius: he stated that this term that appears above is a new form of energy and is famously known as the **mass energy equivalence**. He stated $E = mc^2$! Therefore, the fourth component can now be written as:

$$A_4 = i\frac{E}{c}$$

Now, as concluded from our discussion above in order to conserve momentum, $\frac{i}{c}\sum_{k=1}^{N} E_{kI}$ must be conserved and this in turn conserves $\frac{i}{c}\sum_{k=1}^{N} E'_{kI}$ before and after an interaction. Therefore, as opposed to classical definition, momentum as well as energy, both must be conserved before and after an interaction. Since the momentum is intricately related to the energy and the four vector of momentum cannot be written without including the energy component, we call this new four vector as the **momentum-energy four vector**.

$$\begin{pmatrix} p_x' \\ p_y' \\ p_z' \\ i\frac{E'}{c} \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ p_z \\ i\frac{E}{c} \end{pmatrix}$$

9.6 Some Useful Relations

With our above treatment we have seen that momentum and energy are intricately interwined and not separable. Here, we shall write a new a definition for Kinetic Energy and see its approximation in the classical limit:

$$K = mc^2 - m_0c^2$$

In the classical limit ($u << c$):

$$K = \frac{m_0c^2}{\sqrt{1-\frac{u^2}{c^2}}} - m_0c^2 = m_0c^2\left(1 - \frac{u^2}{c^2}\right)^{-\frac{1}{2}} - m_0c^2 \approx m_0c^2\left(1 + \frac{u^2}{2c^2} - 1\right) = \frac{1}{2}m_0u^2$$

Thus we see that our new definition gives our usual expression for Kinetic Energy that we have been using in Newtonian Mechanics.

We know that the norm of the momentum-energy four vector is a scalar and hence constant, i.e.

$$\underset{\sim}{p}.\underset{\sim}{p} = \text{const.} = p_x^2 + p_y^2 + p_z^2 - \frac{E^2}{c^2} = p^2 - \frac{E^2}{c^2} = (m_0\gamma_u u)^2 - (m_0\gamma_u c)^2 = -m_0^2c^2$$

$$\Rightarrow p^2 - \frac{E^2}{c^2} = -m_0^2c^2$$

Rearranging, we get:

$$E^2 = p^2c^2 + m_0^2c^4$$

From this we get another beautiful insight $-$ if a particle has zero rest mass, then it does not necessarily mean that its momentum is zero. Putting $m_0 = 0$ in the above equation, we see that $p = \frac{E}{c}$! Further $\because m_0 = m\sqrt{1-\frac{v^2}{c^2}}$, a particle having zero rest mass must travel with speed of light! It completely justifies the notion that light itself is such a particle. We shall use this concept in the next section, namely the Compton Effect.

9.7 Force Four Vector

Just as the velocity four vector is the derivative of displacement four vector w.r.t. proper time (a four scalar), we define force four vector as the derivative of momentum four vector w.r.t. proper time. From the exact same procedure as used in section 9.3, we have

$$\underset{\sim}{F} = \frac{d\underset{\sim}{p}}{d\tau} = \gamma_u \frac{d\underset{\sim}{p}}{dt} = \gamma_u \left(\frac{dp_x}{dt}, \frac{dp_y}{dt}, \frac{dp_z}{dt}, \frac{i}{c}\frac{dE}{dt} \right) = \gamma_u (F_x, F_y, F_z, \frac{i}{c}\vec{F}.\vec{u})$$

Thus we can also write the transformation of force four vector as follows:

$$\begin{pmatrix} \gamma_{u'} F'_x \\ \gamma_{u'} F'_y \\ \gamma_{u'} F'_z \\ \gamma_{u'} \frac{i}{c}\vec{F}'.\vec{u}' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} \gamma_u F_x \\ \gamma_u F_y \\ \gamma_u F_z \\ \gamma_u \frac{i}{c}\vec{F}.\vec{u} \end{pmatrix}$$

Note that the expression $\vec{F}.\vec{u}$ is nothing but the expression of Power.

10 Compton Effect

10.1 Working Formula

Compton effect is an experiment that verifies that light has momentum. For that, the crude idea is to hit an electron with a photon of a specific energy and observe the deflection and change in energy of the photon. The diagrammatical representation is as follows:

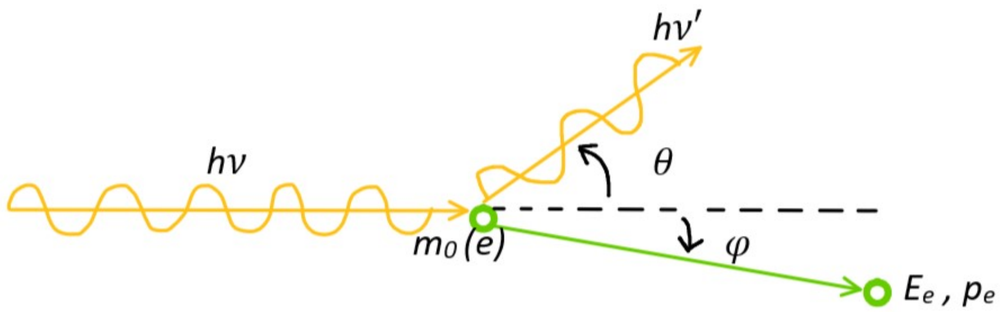

Figure 10: Photon-Electron Collision

By the time this experiment had been done, scientists were already aware of the particle nature of light (from photoelectric effect) and energy of photon was also known to be given by $E = h\nu$. Using this knowledge and the knowledge of the expression of momentum for a particle with zero rest mass (photon), we can conserve the momentum before and after the collision.

$$\frac{h\nu}{c} = \frac{h\nu'}{c}\cos\theta + p_e\cos\phi \tag{23}$$

$$\frac{h\nu'}{c}\sin\theta = p_e\sin\phi \tag{24}$$

$$m_0c^2 + h\nu = h\nu' + E_e \tag{25}$$

$$E_e^2 = p_e^2c^2 + m_0^2c^4 \tag{26}$$

From eqn.s (23) and (24), we eliminate ϕ by bringing the terms containing ϕ to one side and then squaring and adding.

$$p_e^2 = \left(\frac{h\nu}{c} - \frac{h\nu'}{c}\cos\theta\right)^2 + \left(\frac{h\nu'}{c}\sin\theta\right)^2$$

Simplifying, we get

$$p_e^2c^2 = (h\nu)^2 + (h\nu')^2 - 2(h\nu)(h\nu')\cos\theta \tag{27}$$

Substituting E_e and $p_e^2c^2$ from eqn.s (25) and (27) respectively in eqn. (26) and simplifying, we get

$$m_0c^2(h\nu - h\nu') = (h\nu)(h\nu')(1 - \cos\theta)$$

Putting $\nu = \frac{c}{\lambda}$ and $\nu' = \frac{c}{\lambda'}$ and simplifying, we obtain the working formula:

$$\lambda' - \lambda = \frac{h}{m_0c}(1 - \cos\theta)$$

10.2 Experimental Observations

A photon beam was hit on a thin metal sheet. It was observed that there were two peaks in intensity ditsribution curve for rays scattered at a specific angle θ. One was at λ, the original wavelength, and the other was at λ' the scattered wavelength. Apparently the peak corresponding to λ can be interpreted as those photons which were scattered after striking the nuclei. Since the mass m_0 of the nucleus is very very large compared to that of an electron, the change in wavelength is almost negligible. The graphical representation (not to scale) is as follows:

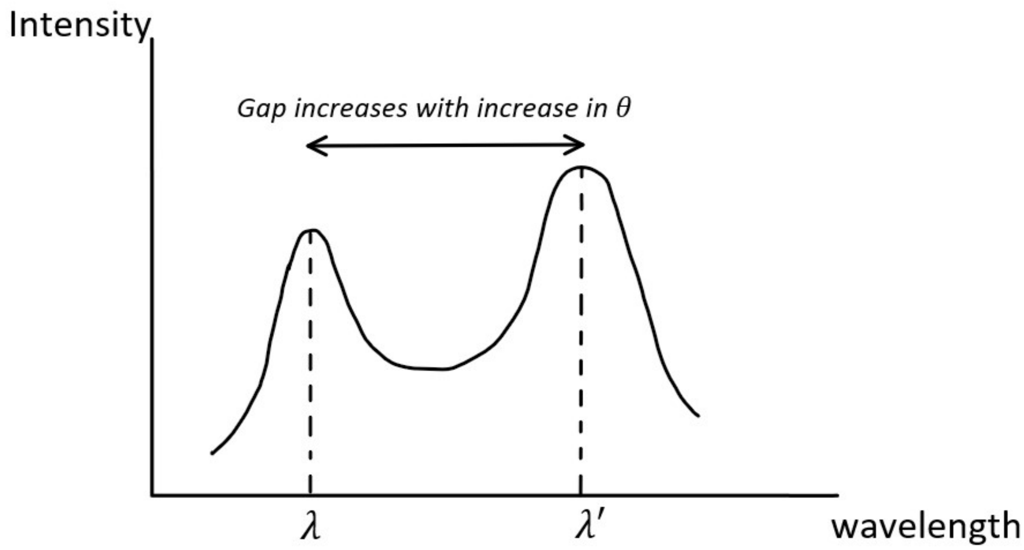

Figure 11: Intensity versus Wavelength Curve in Compton Scattering

11 Doppler Effect in Light

11.1 Longitudinal Doppler Effect

Before we move into the mathematical formulation, I would like to point out a vital difference between doppler effect in sound and light. In sound, the case of a source moving away from (or towards) an observer and an observer moving away from (or towards) source are treated differently because sound travels in a medium and we define the velocities with respect to the medium. This however does not happen for light, since light does not require any medium to travel. Additionally, the speed of light is a constant independent of inertial frames. Hence, an observer moving away from (or towards) source and a source moving away from (or towards) an observer are treated as the same case and we consider only the relative velocity between the two.

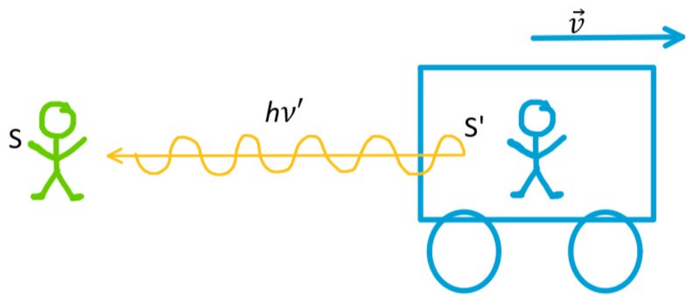

Figure 12: A Hypothetical Setting for Longitudinal Doppler Effect

From the inverse energy-momentum transformation, we have:

$$E = \gamma(E' + vp'_x)$$

Putting $E = h\nu$, $E' = h\nu'$ and $p'_x = -\frac{h\nu}{c}$ in the above equation, we get:

$$\nu = \nu' \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}}$$

We can further approximate this formula if $v << c$:

$$\nu = \nu' \left(1 - \frac{v}{c}\right)^{1/2} \left(1 + \frac{v}{c}\right)^{-1/2} \approx \nu' \left(1 - \frac{v}{2c}\right)^2 \approx \nu' \left(1 - \frac{v}{c}\right)$$

$$\text{or,} \qquad \nu_{ds} = \nu_0 \left(1 - \frac{v}{c}\right)$$

Here $\nu_0 = \nu'$ is the original frequency of light as emitted by the source (in S') and $\nu_{ds} = \nu$ is the doppler shifted frequency as seen by the observer.

11.2 Transverse Doppler Effect

Here, we shall consider a situation in which, the the light is emitted perpendicular to the motion of the source, as seen from the observer's frame of reference.

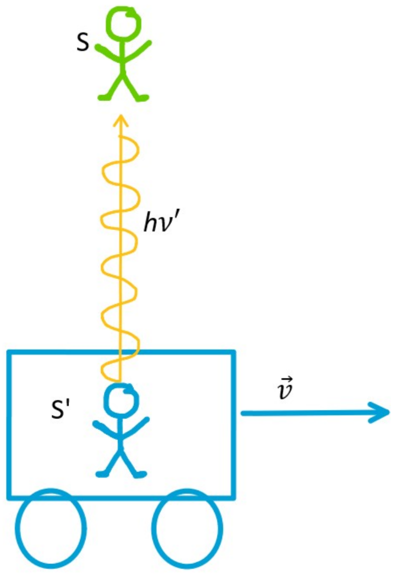

Figure 13: A Hypothetical Setting for Transverse Doppler Effect

From the energy-momentum transformation, we have:

$$E' = \gamma(E - vp_x)$$

Putting $E' = h\nu'$, $E = h\nu$ and $p_x = 0$ and simplifying:

$$\nu' = \nu \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\text{or,} \qquad \nu_{ds} = \nu_0 \sqrt{1 - \frac{v^2}{c^2}}$$

where $\nu_0 = \nu'$ is the original frequency of light as emitted by the source (in S') and $\nu_{ds} = \nu$ is the doppler shifted frequency as seen by the observer.

12 Bibliography

I have cited the references that helped me understand Special Theory of Relativity. It must be noted that I have used the reference to **study** and **understand** the topic **ONLY**.

References

[1] Albert Einstein. On the electrodynamics of moving bodies. *Annalen Phys.*, 17:891–921, 1905.

[2] R.P. Feynman, R.B. Leighton, and M.L. Sands. *The Feynman Lectures on Physics: Mainly mechanics, radiation and heat.* The Feynman Lectures on Physics. Pearson India Education Services Pvt Limited.

[3] D. Kleppner and R.J. Kolenkow. *An Introduction to Mechanics.* Cambridge University Press, 2010.

[4] R. Resnick. *Introduction to Special Relativity.* Wiley, 1968.