Module 2: Number Systems

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Email: pm21ms002@iiserkol.ac.in.These are my personal notes on Number Systems. We will consider $\mathbb{N}$ (Natural Numbers), $\mathbb{Z}$ (Integers), and $\mathbb{Q}$ (rational Numbers), but not $\mathbb{R}$ (Real Numbers).

§1 Natural Numbers

The natural numbers are $1, 2, 3, 4, \ldots$. The set of all natural numbers is denoted by $\mathbb{N}$.

Definition 1.1. We assume familiarity with the algebraic operations of addition and multiplication on the set $\mathbb{N}$ and also with the linear order relation $<$ on $\mathbb{N}$ defined by "$a < b$ if $a, b \in \mathbb{N}$ and a is less than b".

We discuss the following fundamental properties of the set $\mathbb{N}$.

1. Well Ordering Property
2. Principle of Induction

§1.1 Well Ordering Property

Definition 1.2. Every non-empty subset of $\mathbb{N}$ has a least element.

This means that if S is a non-empty subset of $\mathbb{N}$, then there is an element m in S such that $m \leq s$ for all $s \in S$.

In particular, $\mathbb{N}$ itself has the least element 1.

Proof. Let S be a non-empty subset of $\mathbb{N}$. Let k be an element of S. Then k is a natural number.

We define a subset T by $T = \{x \in S : x \leq k\}$. The T is a non-empty subset of $\{1, 2, 3, \ldots, k\}$.

Clearly, T is a finite subset of $\mathbb{N}$ and therefore it has a least element, say m. Then $1 \leq m \leq k$.

We now show that m is the least element of S. Let s be any element of S.

If $s > k$, then the inequality $m \leq k$ implies $m < s$.

If $s \leq k$, the $s \in T$; and m being the least element of T, we have $m \leq s$.

Thus m is the least element of S. $\square$

§1.2 Principle of Induction

Definition 1.3. Let S be a subset of $\mathbb{N}$ such that,

i) $1 \in S$ and,

ii) if $k \in S$, then $k+1 \in S$.

Then $S = \mathbb{N}$

Proof. Let $T = \mathbb{N} - S$. We prove that $T = \phi$.

Let T be non-empty. then by the *Well Ordering Property* of $\mathbb{N}$, the non-empty subset T has a least element, say m.

Since $1 \in S$ and 1 is the least element of $\mathbb{N}$, $m > 1$.

Hence, $m-1$ is a natural number and $m-1 \notin T$. So, $m-1 \in S$.

But by ii) $m-1 \in S \Rightarrow (m-1)+1 \in S$, i.e., $m \in S$.

This contradicts that m is the least element in T. Therefore, our assumption is wrong and $T = \phi$.

Therefore, $S = \mathbb{N}$. □

Theorem 1.4

Let $P(n)$ be a statement involving a natural number n. If,

i) $P(1)$ is true, and

ii) $P(k+1)$ is true whenever $P(k)$ is true,

then $P(n)$ is true for all $n \in \mathbb{N}$.

Proof. Let S be the set of those natural numbers for which the statement $P(n)$ is true.

Then S has the properties,

(a) $1 \in S$, by (i)

(b) $k \in S \Rightarrow k+1 \in S$ by (ii).

By the *Principle of Induction*, $S = \mathbb{N}$.

Therefore, $P(n)$ is true for all $n \in \mathbb{N}$. □

Remark 1.5. Let a statement $P(n)$ satisfies the conditions,

(i) for some $m \in \mathbb{N}$, $P(m)$ is true (m being the least possible); and

(ii) $P(k)$ is true $\Rightarrow P(k+1)$ is true for all $k \geq m$.

Then $P(n)$ is true for all natural numbers $\geq m$.

Worked Examples

> **Example 1.6**
> Prove that for each $n \in \mathbb{N}$, $1 + 2 + 3 + 4 + \cdots + n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

The statement is true for $n = 1$, because $1 = \dfrac{1(1+1)}{2}$.
Let the statement be true for some natural number k.
Then $1 + 2 + 3 + 4 + \cdots + k = \dfrac{(k+1)}{2}$ and therefore,
$(1 + 2 + \cdots + k) + (k+1) = \dfrac{k(k+1)}{2} + (k+1)$
or, $1 + 2 + 3 + \cdots + (k+1) = \dfrac{(k+1)(k+2)}{2}$.
This shows that the statement is true for the natural number $k + 1$ if it is true for k. By the principle of induction, the statement is true for all natural numbers.

> **Example 1.7**
> Prove that for each $n \geq 2$, $(n+1)! > 2^n$.

The equality holds for $n = 2$ since $(2+1)! > 2^2$.
Let the inequality hold for some natural number $k \geq 2$.
Then, $(k+1)! > 2^k$

$$\begin{aligned} \mathit{and}\ (k+2)! &= (k+2)(k+1)! \\ &> 2 \cdot 2^k, \mathit{since}\ k+2 > 2 \\ \mathit{or},\ \ (k+2)! &> 2^{k+1} \end{aligned}$$

This shows that if the inequality holds for $k(\geq 2)$ then it also holds for $k + 1$.
By the principle of induction, the inequality holds for all natural numbers ≥ 2.
[Note that the inequality does not hold for $n = 1$.]

§1.3 Second Principle of Induction (or, Principle of Strong Induction)

Definition 1.8. Let S be a subset of $\mathbb{N}$ such that

(i) $1 \in S$, and

(ii) if $\{1, 2, 3, 4, \ldots\} \subset S$, then $k + 1 \in S$.

Then $S = \mathbb{N}$

Proof. Let $T = \mathbb{N} - S$. We prove that $T = \phi$.
Let T be non-empty. Then T will have a least element, say m, by the WOP of $\mathbb{N}$. Since, $1 \in S$, $1 \notin T$.
As m is the least element in T and $1 \notin T$, $m > 1$.
By choice of m, all natural numbers less than m belong to S. That is $1, 2, \ldots, m-1$ all belong to S.
Then by (ii) $m \in S$ and consequently, $m \notin T$, a contradiction. It follows that $T = \phi$ and therefore, $S = \mathbb{N}$. □

Worked Examples(continued)

Example 1.9

Prove that for all $n \in \mathbb{N}$, $(3+\sqrt{5})^n + (3-\sqrt{5})^n$ is an even integer.

Let $P(n)$ be the statement "$(3+\sqrt{5})^n + (3-\sqrt{5})^n$ is an even integer".
$P(1)$ is true since $(3+\sqrt{5})^1 + (3-\sqrt{5})^1 = 6$, an even integer.
Let us assume that $P(n)$ is true for $n = 1, 2, \ldots, k$.

$$
\begin{aligned}
&(3+\sqrt{5})^{(k+1)} + (3-\sqrt{5})^{(k+1)} \\
&= a^{(k+1)} + b^{(k+1)} \; where \; a = 3+\sqrt{5},\; b = 3-\sqrt{5} \\
&= (a^k + b^k)(a+b) - (a^{k-1} + b^{k-1})ab \\
&= 6(a^k + b^k) - 4(a^{k-1} + b^{k-1}).
\end{aligned}
$$

It is an even integer, since $a^k + b^k$ and $a^{k-1} + b^{k-1}$ are even integers.
Hence, $P(k+1)$ is true whenever $P(n)$ is true for all $n = 1, 2, \ldots, k$.
By the second principle of induction, $P(n)$ is true for all natural numbers.

§2 Integers