Eigenvalues and eigenvectors

A scalar λ is called an eigenvalue of a square matrix A if there exists a non-zero vector v such that $Av = \lambda v$.

$$\begin{matrix} \lambda \rightarrow \textit{eigenvalue} \\ v \rightarrow \textit{eigenvector} \end{matrix} \implies \{\lambda, v\} \rightarrow \textit{eigenpair}$$

Example

$$A = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$$

$$v_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \qquad \therefore Av_1 = v_1 \implies \lambda_1 = 1$$

$$v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \therefore Av_2 = 4v_2 \implies \lambda_2 = 4$$

Observe $\{v_1, v_2\}$ are linearly independent and forms a basis in $\mathbb{R}^2$

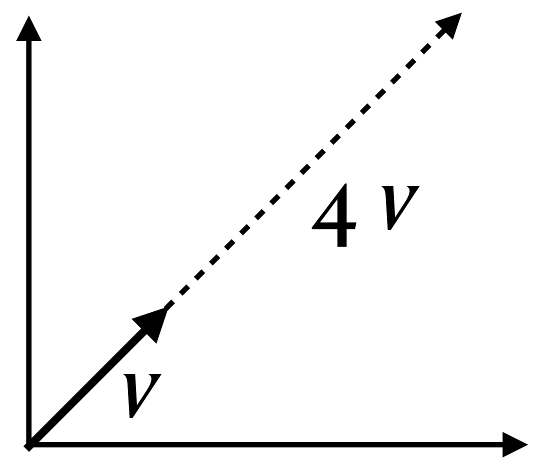
$$P = [v_1 \;\; v_2] = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix} \quad \text{and} \quad P^{-1} = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

Check $\quad D = P^{-1}AP = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$

Interpretation of eigenvalues and eigenvectors

$Av = \lambda v \implies v \neq 0$ but λ can be 0.

$A\vec{0} = \vec{0} \implies$ eigenvalues are not defined.

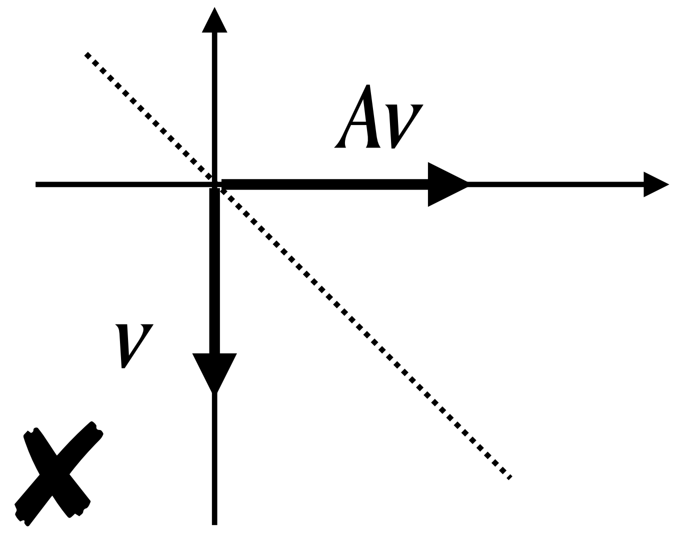


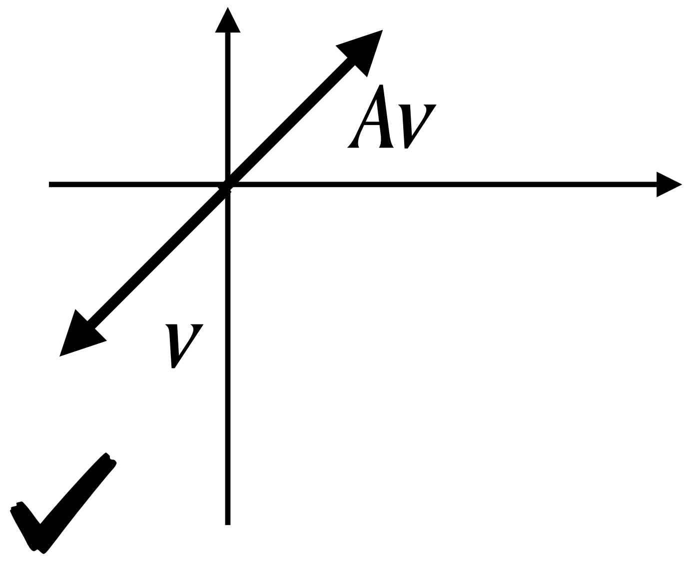
$(Av, \lambda v)$ are collinear with origin.

$$\star\ A = \begin{pmatrix} 2 & 2 \\ -4 & 8 \end{pmatrix} \;\&\; v = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \implies Av = \begin{pmatrix} 2 & 2 \\ -4 & 8 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} = 4v \implies v \to \textit{eigenvector}\ \&\ \textit{eigenvalue}\ \lambda = 4$$

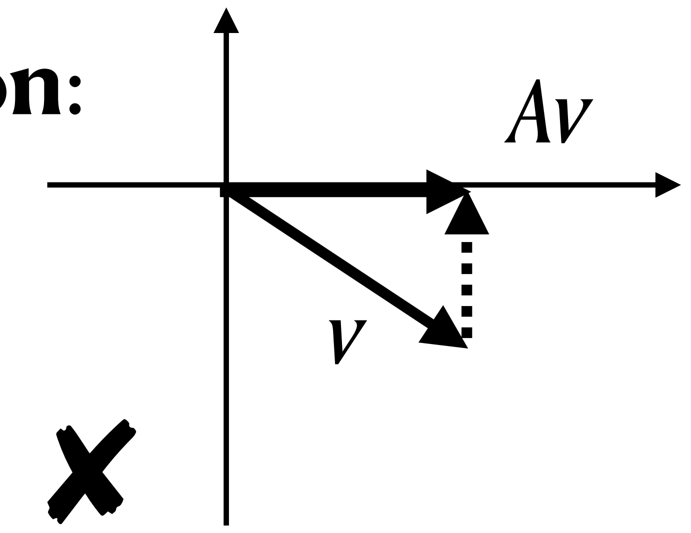
$$\&\ w = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \implies Aw = \begin{pmatrix} 2 & 2 \\ -4 & 8 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \end{pmatrix} \neq c\,w \implies w \ \textit{is not eigenvector}$$

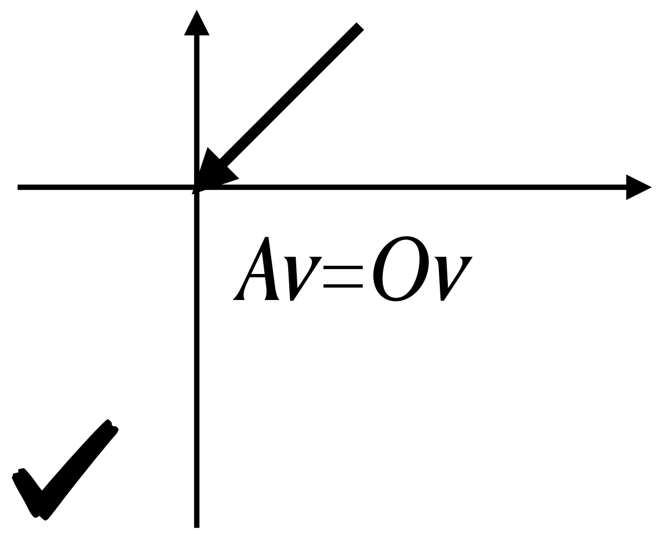
Reflection:





Projection:





Identity: $\mathbb{I}x = 1.x \implies$ Every non-zero vector is an eigenvector of $\mathbb{I}$ with eigenvalue 1.

Dilation: $d\,\mathbb{I}x = d\,.\,x \implies$ Every non-zero vector is an eigenvector of $\mathbb{I}$ with eigenvalue d.

In general shear and rotation have no eigenvectors.

Computing eigenvalues of a matrix

Characteristic polynomial $:= f(\lambda) = det(\lambda \mathbb{I} - A)$

Theorem: Eigenvalues are roots of the characteristic polynomial i.e. if λ is an eigenvalue $\implies Ax = \lambda x \implies f(\lambda) = 0.$

$$
\begin{aligned}
\text{Proof :} \quad Ax &= \lambda x \text{ has a solution such that } x \neq 0 \\
&\implies (A - \lambda \mathbb{I})x = 0 \\
&\implies (A - \lambda \mathbb{I}) \text{ is not invertible} \\
&\implies det(A - \lambda \mathbb{I}) = 0 \\
&\implies f(\lambda) = 0 \quad \square
\end{aligned}
$$

Example

$$A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \implies det(A - \lambda \mathbb{I}) = 0 \implies (5 - \lambda)(1 - \lambda) - 4 = 0 \implies \lambda^2 - 6\lambda + 1 = 0 \implies \lambda_{\pm} = 3 \pm 2\sqrt{2}$$

$$\lambda_+ + \lambda_- = tr(A) = 6$$

$$\lambda_+ \lambda_- = det(A) = 1$$

Computing eigenvectors

$$A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \implies det(A - \lambda\mathbb{I}) = 0 \implies (5-\lambda)(1-\lambda) - 4 = 0 \implies \lambda^2 - 6\lambda + 1 = 0 \implies \lambda_\pm = 3 \pm 2\sqrt{2}$$

★ Eigenvector corresponding to λ_+ : $Ax_+ = \lambda_+ x_+ \implies (A - \lambda_+\mathbb{I})x_+ = 0$

$$\begin{pmatrix} 2-2\sqrt{2} & 2 \\ 2 & -2-2\sqrt{2} \end{pmatrix} \begin{pmatrix} x_+^{(1)} \\ x_+^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies x_+ = \begin{pmatrix} x_+^{(1)} \\ x_+^{(2)} \end{pmatrix} = \begin{pmatrix} 1+\sqrt{2} \\ 1 \end{pmatrix}$$

★ Eigenvector corresponding to λ_- : $Ax_- = \lambda_- x_- \implies (A - \lambda_-\mathbb{I})x_- = 0$

$$\begin{pmatrix} 2+2\sqrt{2} & 2 \\ 2 & -2+2\sqrt{2} \end{pmatrix} \begin{pmatrix} x_-^{(1)} \\ x_-^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies x_- = \begin{pmatrix} x_-^{(1)} \\ x_-^{(2)} \end{pmatrix} = \begin{pmatrix} 1-\sqrt{2} \\ 1 \end{pmatrix}$$

Example

Eigenvalue:

$$A = \begin{pmatrix} 5 & -1 \\ 1 & 3 \end{pmatrix} \implies f(t) = det(t\mathbb{I} - A) = \left| \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} - \begin{pmatrix} 5 & -1 \\ 1 & 3 \end{pmatrix} \right| = t^2 - 8t + 16 = (t-4)^2 \implies t = 4 \text{ only eigenvalue}$$

$$= t^2 - tr(A)\ t + det(A)$$

Eigenvector:

$$(A - t\mathbb{I})\vec{x} = \vec{0} \implies \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies x_1 = x_2 \text{ is the only solution i.e.} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Only one vector does not span the vector space i.e. a basis does not exist hence A is **not** diagonalizable.

Example

$$A = \begin{pmatrix} 3 & -5 \\ 2 & -3 \end{pmatrix} \implies tr(A) = 0 \;\;\&\;\; det(A) = 1 \implies f(t) = t^2 + 1$$

If A is a matrix over real field i.e. $A \in M_n(\mathbb{R})$ then $f(t)$ has no real roots and A is not diagonalizable.

If $A \in M_n(\mathbb{C})$ then eigenvalues are $\lambda_\pm = \pm\, i$ and the matrix is diagonalizable if $P^{-1}AP = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$

$$(A - \lambda_+ \mathbb{I})\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \begin{pmatrix} 3-i & -5 \\ 2 & -3-i \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(3-i)x_1 - 5x_2 = 0 \longrightarrow$$

$$2x_1 - (3+i)x_2 = 0 \implies x_1 = \frac{3+i}{2}x_2 \longrightarrow$$

$$\left[\frac{(3-i)(3+i)}{2} - 5\right] x_2 = 0$$

$$\implies 0 \;\; x_2 = 0$$

Choose $x_2 \neq 0$ since $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ can not be an eigenvector. So let $x_2 = 1$.

$\lambda_+ = i : \begin{pmatrix} \frac{3+i}{2} \\ 1 \end{pmatrix}$ similarly $\lambda_- = -\, i : \begin{pmatrix} \frac{3-i}{2} \\ 1 \end{pmatrix}$

Theorem : Hermitian matrices have real eigenvalues.

Proof:
$$
\begin{array}{rccll}
 & Av & = & \lambda v & \\
\Longrightarrow & (Av)^\star & = & (\lambda v)^\star & \\
\Longrightarrow & v^\star A^\star & = & \lambda^\star v^\star & \because \lambda \;\; is \;\; a \;\; scalar \\
\Longrightarrow & v^\star A & = & \lambda^\star v^\star & \because A^\star = A \quad Hermitian \\
\Longrightarrow & v^\star A v & = & \lambda^\star v^\star v & \\
\Longrightarrow & v^\star \lambda v & = & \lambda^\star v^\star v & \\
\Longrightarrow & \lambda v^\star v & = & \lambda^\star v^\star v & \\
\Longrightarrow & \lambda & = & \lambda^\star & \qquad v \neq 0 \\
\Longrightarrow & \lambda \in \mathbb{R} & & &
\end{array}
$$

$\square$

★ Diagonalizable with same eigenvalues $\Longrightarrow$similarity

$$
\begin{array}{l} S^{-1}AS = \Lambda \\ P^{-1}BP = \Lambda \end{array} \Longrightarrow S^{-1}AS = P^{-1}BP \Longrightarrow PS^{-1}ASP^{-1} = B \Longrightarrow U^{-1}AU = B
$$

where $U = SP^{-1}$ & $U^{-1} = (SP^{-1})^{-1} = PS^{-1}$

Similarity implies that characteristic polynomial are identical but the converse is not true.

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ & $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \rightarrow (1-\lambda)^2$ Same characteristic polynomial but not similar matrices.

Example

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Characteristic polynomial:

$$|A - \lambda\mathbb{I}| = 0 \implies \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0 \implies -\lambda^3 + 2 + 3\lambda = 0 \implies -(\lambda - 2)(\lambda + 1)^2 = 0$$

Eigenvalues: $\{\lambda\} = \{-1, -1, 2\}$

Eigenvector for $\lambda = 2$:

$$(A - 2\mathbb{I})\vec{x} = \vec{0} \implies \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies \begin{aligned} -2x_1 + x_2 + x_3 &= 0 \\ x_1 - 2x_2 + x_3 &= 0 \\ x_1 + x_2 - 2x_3 &= 0 \end{aligned}$$

Only two eqns are linearly independent giving $x_2 = x_3$ & $x_1 = x_3$ such that we can choose $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Eigenvectors for $\lambda = -1$:

$$(A + \mathbb{I})\vec{y} = \vec{0} \implies \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies y_1 + y_2 + y_3 = 0$$

There are infinite possibilities choose any pair of independent vectors (v_2, v_3) lying on a plane passing through origin and orthogonal to v_1

$$v_2 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \ \& \ v_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

Check with the change of matrix $P = [v_1 \ \ v_2 \ \ v_3] \to P^{-1}AP = D$