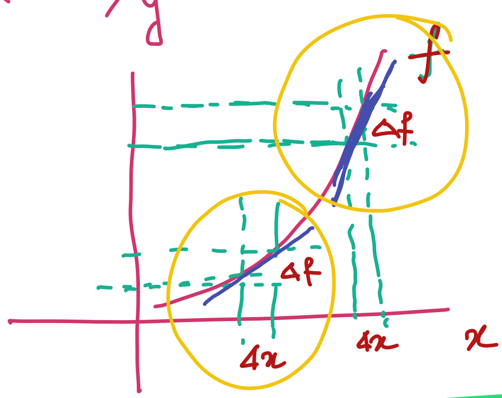
FEB, 2nd, 2022: Partial Differentiation

$z = f(x,y)$

$dz = ?$

$$\boxed{dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy}$$

$$\left(\frac{\partial z}{\partial x}\right)_y \qquad \left(\frac{\partial z}{\partial y}\right)_x$$



$$\left(\frac{\Delta f}{\Delta x}\right)$$

$$z(\underline{x},\underline{y}) \Rightarrow \boxed{dz = \left(\frac{\partial z}{\partial x}\right)_y \underline{dx} + \left(\frac{\partial z}{\partial y}\right)_x \underline{dy}}$$

Example:

$$y = \ln(\sin 2x)$$

$$\frac{dy}{dx} = \frac{1}{\sin 2x}\frac{d}{dx}(\sin 2x)\,\frac{d}{dx}(2x)$$

$$= \frac{1}{\sin 2x}\cos 2x \cdot 2 = 2\,\frac{\cos 2x}{\sin 2x}$$

$$= \underline{2\cot 2x}$$

$y = \ln \sin 2x$ consider $u = \sin 2x$

where

$y = \ln u$, $u = \sin v$

$v = 2x$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

Differentiation of a function of a funtion

Example: find $\frac{dz}{dt}$, where $z = 2t^2 \sin t$

$$\Rightarrow \frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$z = \underbrace{2t^2}_{x}\underbrace{\sin t}_{y}$

$z = xy$

$x = 2t^2$ | $2t$

$y = \sin t$ | $t \sin t$

$$\frac{\partial z}{\partial x} = 2t^2 \sin t$$

$$= \frac{\partial (xy)}{\partial x} = y$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(xy) = x$$

$$\frac{dz}{dt} = y\frac{dx}{dt} + x\frac{dy}{dt} \quad \text{--- (1)}$$

$x = 2t^2 \quad \frac{dx}{dt} = 4t$

$y = \sin t \quad \frac{dy}{dt} = \cos t$

Substitute these values in equation — (1)

$$\frac{dz}{dt} = \sin t \cdot 4t + 2t^2 \cos t$$

$$\boxed{\frac{dz}{dt} = 4t \sin t + 2t^2 \cos t}$$

$z = 2t^2 \sin t \Rightarrow \frac{dz}{dt} = 2 \frac{d}{dt}(t^2) \times \sin t + 2t^2 \frac{d}{dt}(\sin t)$

$= 2 \times 2t \sin t + 2t^2 \cos t$

$$\frac{dz}{dt} = 4t \sin t + 2t^2 \cos t$$

Example: find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ given $z = xy$.

where $x = \sin(s+t)$

$y = s - t$

$z = f(x, y)$ } are functions of different independent variables 't'

$$\boxed{dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy}$$

$z = xy \qquad \frac{\partial z}{\partial x} = y \,;\, \frac{\partial z}{\partial y} = x$

$$\boxed{dz = y\,dx + x\,dy}$$

$x = \sin(s+t)$

from the partial differentiation;

$\frac{\partial}{\partial s}\sin(s+t) \cdot \frac{\partial}{\partial s}(s+t)$

$\left.\frac{\partial x}{\partial s}\right)_t = \cos(s+t) \times 1$

$\left.\frac{\partial x}{\partial t}\right)_s = \cos(s+t) \times 1$

$y = s - t \qquad \left.\frac{\partial y}{\partial s}\right)_t = \frac{\partial}{\partial s}(s-t) = 1$

$\left.\frac{\partial y}{\partial t}\right)_s = \frac{\partial}{\partial t}(s-t) = -1$

$\frac{\partial z}{\partial s} \Rightarrow dz = y\,dx + x\,dy$

$\frac{\partial z}{\partial s} = y\frac{\partial x}{\partial s} + x\frac{\partial y}{\partial s}$

$= y\cos(s+t) + x \times 1$

$$\boxed{\frac{\partial z}{\partial s} = (s-t)\cos(s+t) + \sin(s+t)}$$

$\frac{\partial z}{\partial t} = y\frac{\partial x}{\partial t} + x\frac{\partial y}{\partial t}$

$= y\cos(s+t) + x(-1)$

$$\boxed{\frac{\partial z}{\partial t} = (s-t)\cos(s+t) - \sin(s+t)}$$

Simple point to remember: $\frac{\partial u}{\partial v}$, $\frac{\partial v}{\partial u}$

These are not usually 'reciprocals'

$$\frac{1}{\frac{\partial u}{\partial v}} \Rightarrow \frac{\partial v}{\partial u}$$

Unless the independent variables of u & v are same, then only these are reciprocals

What is the need of partial differentiation:-

$z = f(x,y)$

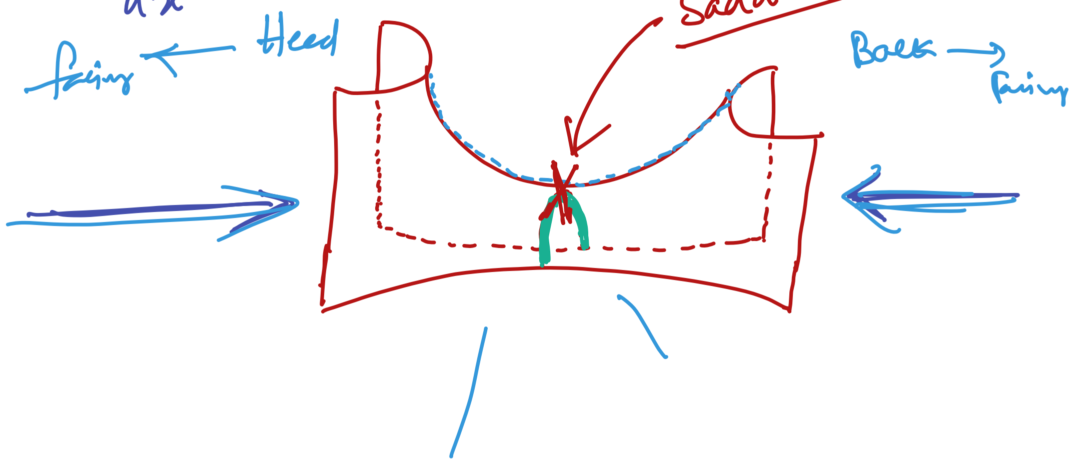
$\left(\frac{\partial z}{\partial x}\right)_y$ = slope, the rate of change of z w.r.t x when y is const

$\left(\frac{\partial z}{\partial y}\right)_x$ = slope, " w.r.t y x is const

$\left\{ \begin{array}{l} \frac{\partial z}{\partial x} = 0 \\ \frac{\partial z}{\partial y} = 0 \end{array} \right\}$ a maximum or minimum point } Saddle point

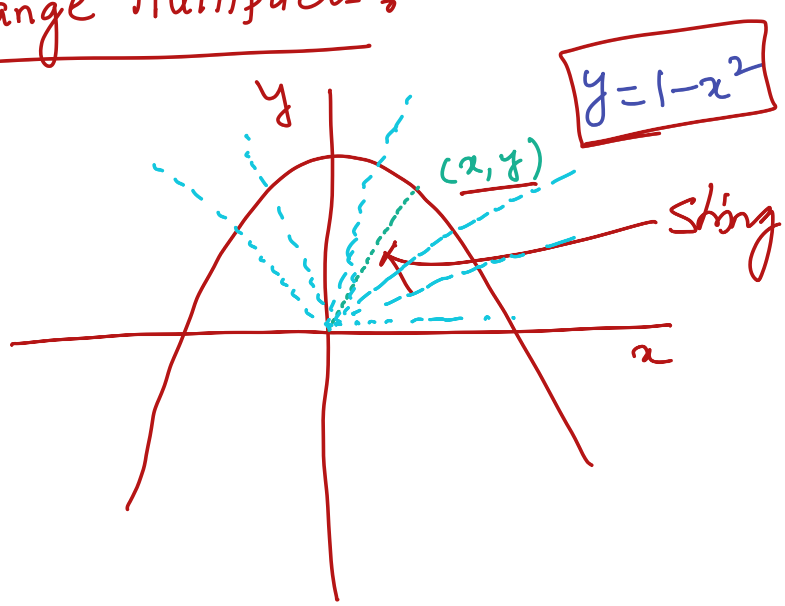
$\left\{ \frac{dy}{dx} = 0 \right\}$ it is a necessary condition for maximum point

$$\frac{d^2y}{dx^2} > 0 \text{ or } < 0$$



Using Constraints : One can identify the maximum & minimum points of a function.

Lagrange Multipliers :



Find (x, y) to minimize the length of the string

(A) Elimination
(B) Implicit Differentiation
(C) Lagrange multipliers

$(x, y) \Rightarrow d = \sqrt{x^2+y^2}$

$\underline{d^2 = x^2+y^2}$

$d = \sqrt{x^2+y^2}$

Elimination:

$\boxed{y = \sqrt{d^2-x^2}}$

or

$\boxed{y = 1-x^2}$

$f = d^2 = x^2+y^2$

eliminate y

$f = x^2 + (1-x^2)^2$

$f = x^2 + 1^2 + (x^2)^2 - 2x^2$

$f = 1 - x^2 + x^4$

$\frac{df}{dx} = 0 \quad -2x + 4x^3 = 0$

$4x^3 - 2x = 0$

$2x(2x^2-1) = 0 \Rightarrow \begin{array}{l} x=0 \\ x = \pm\sqrt{\frac{1}{2}} \end{array}$

$$\frac{d^2f}{dx^2} = 12x^2 - 2 = \begin{cases} -2 & \text{at } x=0 \\ 4 & \text{at } x = \pm\sqrt{1/2} \end{cases}$$

relative maximum

(Minimum)

$\left(x = \pm\sqrt{1/2} \Rightarrow y = 1/2 \right)$

where we have minimum

Implicit Differentiation: $f = x^2 + y^2$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2) = 2x \quad [y \text{ is constant}]$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + y^2) = 2y \quad [x \text{ is constant}]$$

$$df = 2x\,dx + 2y\,dy$$

$$y = 1 - x^2$$

$$x = 0, \quad \pm\sqrt{1/2}$$

$$\therefore \quad y = 1/2$$

Lagrange multipliers: $f(x, y)$

Constant $\Rightarrow \phi(x, y)$

$$\frac{df}{dx} = 0$$

$$\frac{df}{dy} = 0$$

$$\Rightarrow d\phi = 0 = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = 0$$

$$\Rightarrow df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

$$\left(\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x}\right) dx + \left(\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y}\right) dy = 0$$

where λ is lagrange multiplier.

$$\boxed{\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0} \qquad \boxed{\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0}$$

$$\boxed{F(x,y) = \underbrace{f(x,y)}_{} + \lambda \underbrace{\phi(x,y)}_{}}$$

$x^2 + y^2 = d^2$

$y = 1 - x^2 \Rightarrow \boxed{y + x^2 = 1} \rightarrow$ constant

$\phi(x,y)$

$$F(x,y) = f + \lambda \phi$$
$$= x^2 + y^2 + \lambda(y + x^2)$$

$$F = (x^2 + \lambda x^2) + y^2 + \lambda y$$

$$\left.\begin{aligned} \frac{\partial F}{\partial x} &= 2x + \lambda 2x = 0 \\ \frac{\partial F}{\partial y} &= 2y + \lambda = 0 \end{aligned}\right\} \quad \begin{aligned} x &= ? \\ y &= ? \\ \text{for } \lambda &= ! \end{aligned}$$

$y + x^2 = 1$

$2x + \lambda 2x = 0$

$2x(1+\lambda) = 0$

$\Rightarrow x = 0$ (or)

$\lambda = -1$

$\Downarrow$

$2y + \lambda = 0$

$y = \frac{1}{2}$

$1 - x^2 = y$

$1 - y = x^2$

$x = \sqrt{1-y}$

$x = \sqrt{1-\frac{1}{2}} = \sqrt{\frac{1}{2}}$

① $\boxed{x = \sqrt{\tfrac{1}{2}}, \; y = \tfrac{1}{2}}$ $(\lambda = -1)$

$x = 0 \cdot y = 1 - x^2$

② $\boxed{x = 0 \quad y = 1}$