Error Analysis (PH1102)

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Purpose of experimental error analysis

- In teaching lab course (PH1102)
- In research

➢ Experimental errors/ uncertainties

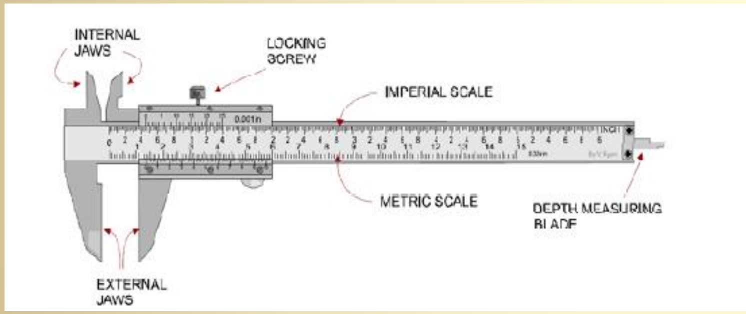
- Errors are inevitable: seek the best estimate
- x_{true}: true value of the physical quantity we measure
- x_{best}: observed value
- Error: $x_{best} - x_{true}$

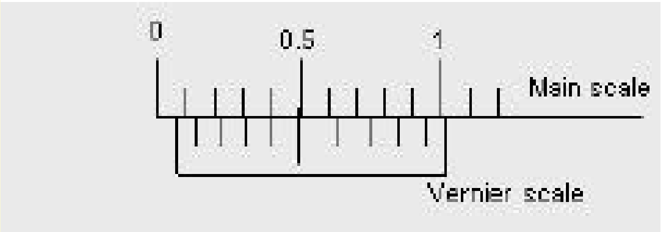
Sources of experimental errors

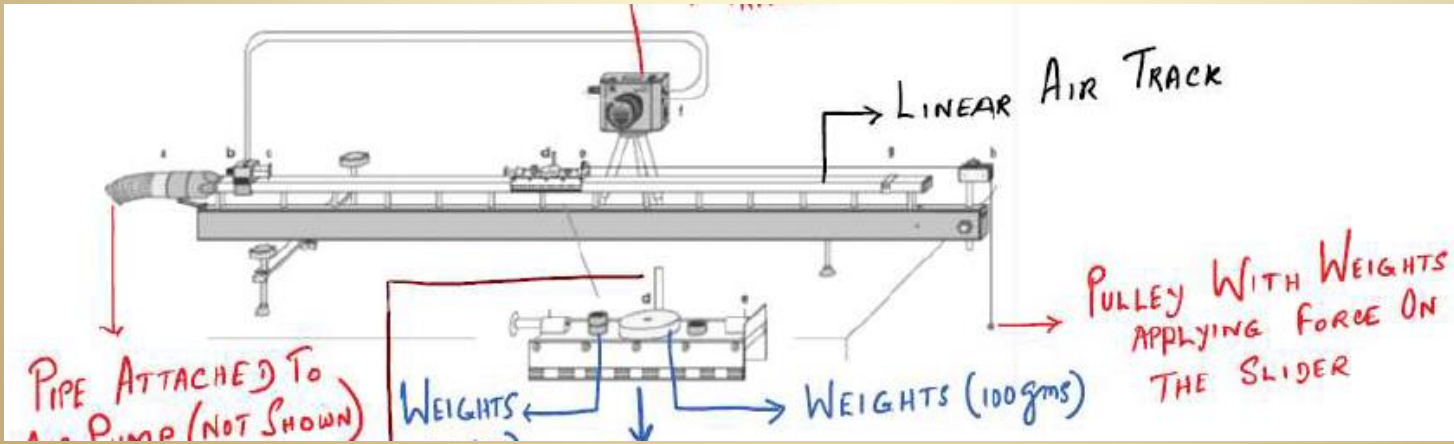
1. Real mistakes: wrong measurements

2. Systematic errors

zero error in vernier calipers



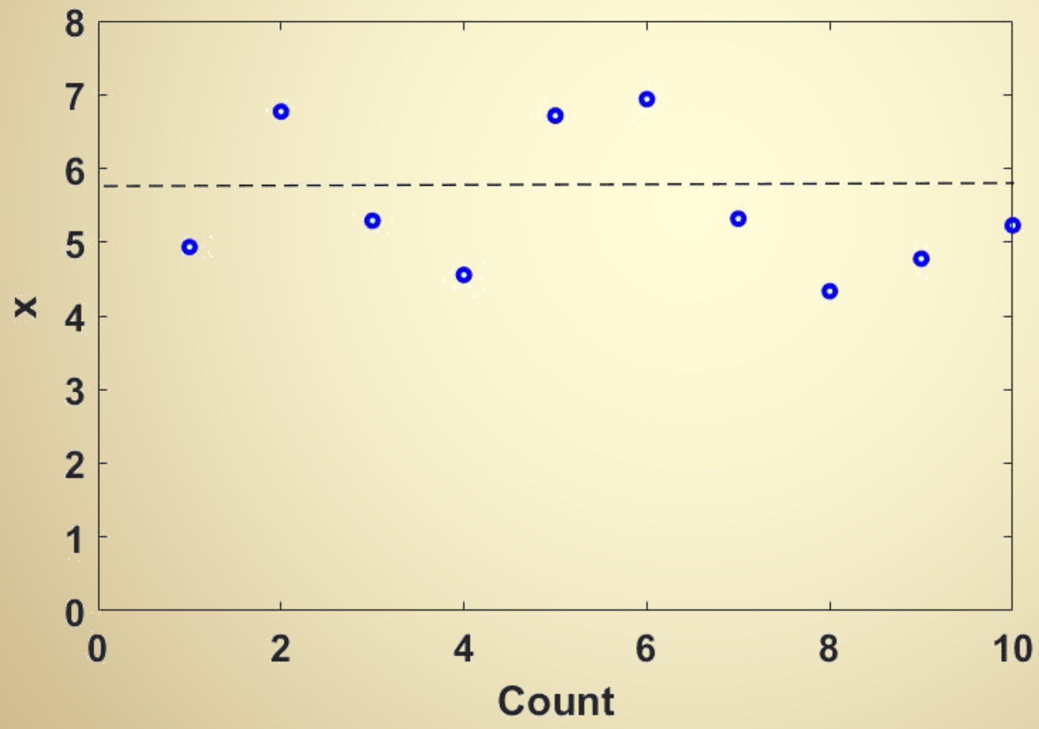




Air Track

Sources of experimental errors

3. Random errors:



Estimation of errors

- **Propagation method:**

  Estimation through propagation of primary errors

- **Statistical method:**

  Estimation through statistical analysis of data

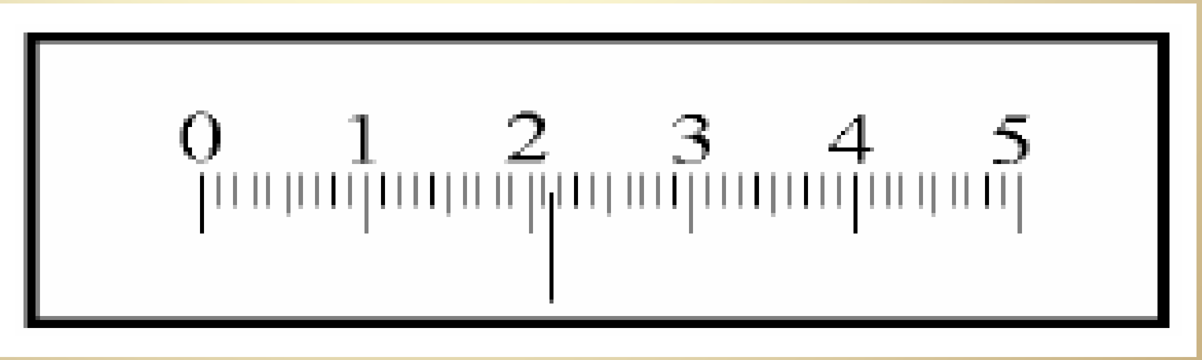
Estimation of errors

- **Estimating a primary error:**

  - Best estimate and least count

  - Estimation of error: $\delta x = x_{best} - x_{true}$

- Effective least count:



Reporting of results

$$x_{measured} = x_{best} \pm \delta x$$

- upper limit: $x_{best} + \delta x$

- lower limit: $x_{best} - \delta x$

Propagation of estimated errors

f = f (x, y); f = x + y

We measure: $x = x_{best} \pm \delta x$

$y = y_{best} \pm \delta y$

$f = x_{best} + y_{best}$

- error/ uncertainty in f : $\delta f = \delta x + \delta y$

- while reporting: $f \pm \delta f$

$$= (x_{best} + y_{best}) \pm (\delta x + \delta y)$$

Propagation of estimated errors

f = f (x, y);   f = x - y

We measure: $x = x_{best} \pm \delta x$

$y = y_{best} \pm \delta y$

$f = x_{best} - y_{best}$

- error/ uncertainty in f : $\delta f = \delta x + \delta y$

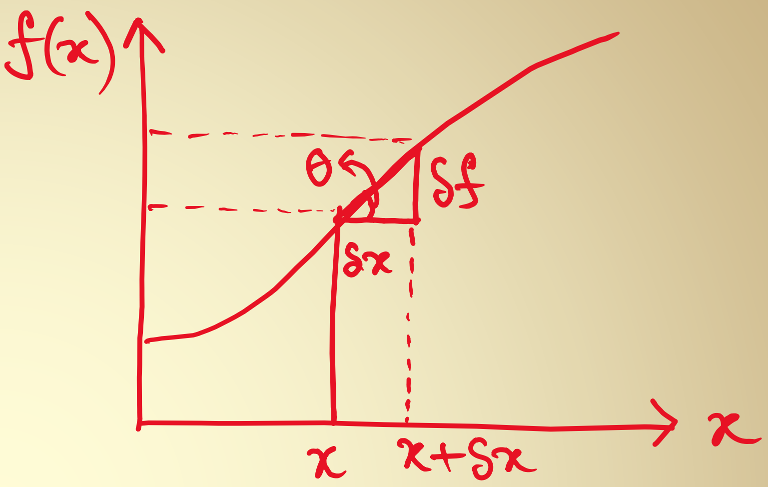
- while reporting: $f \pm \delta f$

$$= (x_{best} - y_{best}) \pm (\delta x + \delta y)$$

$\delta f = \sqrt{(\delta x)^2 + (\delta y)^2}$; $(\delta x + \delta y) \geq \sqrt{(\delta x)^2 + (\delta y)^2}$

Propagation of estimated errors

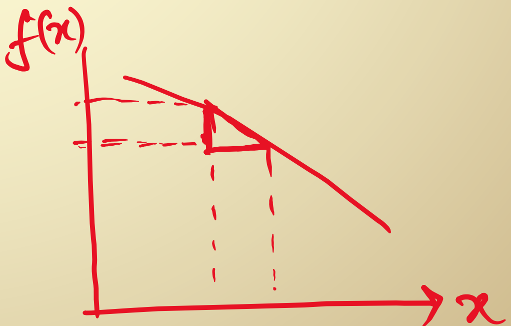
$$f = f(x);$$



$$\frac{\delta f}{\delta x} = \tan\theta = \frac{df}{dx}$$

$$\lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} = \frac{df}{dx}$$

$$\delta f = \frac{df}{dx}\,\delta x$$



$$\delta f = \left|\frac{df}{dx}\right|\delta x$$

Propagation of estimated errors

f = f (x, y, z);

$$df = \left|\frac{\partial f}{\partial x}\right| \delta x + \left|\frac{\partial f}{\partial y}\right| \delta y + \left|\frac{\partial f}{\partial z}\right| \delta z$$

Example (1): **f = x + y - z;**

$$\delta f = \delta x + \delta y + \delta z$$

Propagation of estimated errors

Example (2): $\mathbf{f\,(x) = x^3}$

$$\delta f = \left|\frac{\partial f}{\partial x}\right| \delta x = \left|3x^2\right| \delta x$$

Fractional error/ uncertainty:

$$\frac{\delta f}{|f|} = \left|\frac{3x^2}{x^3}\right| \delta x = \frac{3}{|x|} \delta x$$

$$\boxed{\frac{\delta f}{|f|} = 3\,\frac{\delta x}{|x|}}$$

Propagation of estimated errors

Example (3): $\mathbf{f\,(a, b, c) = \dfrac{a}{b} - c^{3/2}}$

$$\delta f = \left|\frac{\partial f}{\partial a}\right| \delta a + \left|\frac{\partial f}{\partial b}\right| \delta b + \left|\frac{\partial f}{\partial c}\right| \delta c$$

$$= \left|\frac{1}{b}\right| \delta a + \left|\frac{a}{b^2}\right| \delta b + \left|\frac{3}{2} c^{1/2}\right| \delta c$$

Propagation of estimated errors

Example (4): $\mathbf{f\,(a,b,c)} = \dfrac{ab}{c} - a^2$

$$\delta f = \frac{\partial f}{\partial a}\,\delta a + \frac{\partial f}{\partial b}\,\delta b + \frac{\partial f}{\partial c}\,\delta c$$

$$= \left(\frac{b}{c}\right)\delta a - (2a)\,\delta a + \left(\frac{a}{c}\right)\delta b + \left(\frac{ab}{-c^2}\right)\delta c$$

$$\delta f = \left|\frac{\partial f}{\partial a}\right|\delta a + \left|\frac{\partial f}{\partial b}\right|\delta b + \left|\frac{\partial f}{\partial c}\right|\delta c$$

$$\delta f = \left|\frac{b}{c} - 2a\right|\delta a + \left|\frac{a}{c}\right|\delta b + \left|\frac{ab}{c^2}\right|\delta c$$

Propagation of estimated errors

Example (5): $\mathbf{f\ (a,\ b,\ c)\ =\ \frac{\boldsymbol{ab}}{\boldsymbol{c}}}$

$$\delta f = \left|\frac{\partial f}{\partial a}\right| \delta a + \left|\frac{\partial f}{\partial b}\right| \delta b + \left|\frac{\partial f}{\partial c}\right| \delta c$$

$$= \left|\frac{b}{c}\right| \delta a + \left|\frac{a}{c}\right| \delta b + \left|\frac{ab}{c^2}\right| \delta c$$

Now, $\dfrac{\delta f}{|f|} = \left|\frac{1}{a}\right| \delta a + \left|\frac{1}{b}\right| \delta b + \left|\frac{1}{c}\right| \delta c$

Propagation of estimated errors

Example (6): $f\,(a, b, c) \;=\; \dfrac{ab}{c}$

Prescription: take a log and differentiate

$$\log f = \log a + \log b - \log c$$

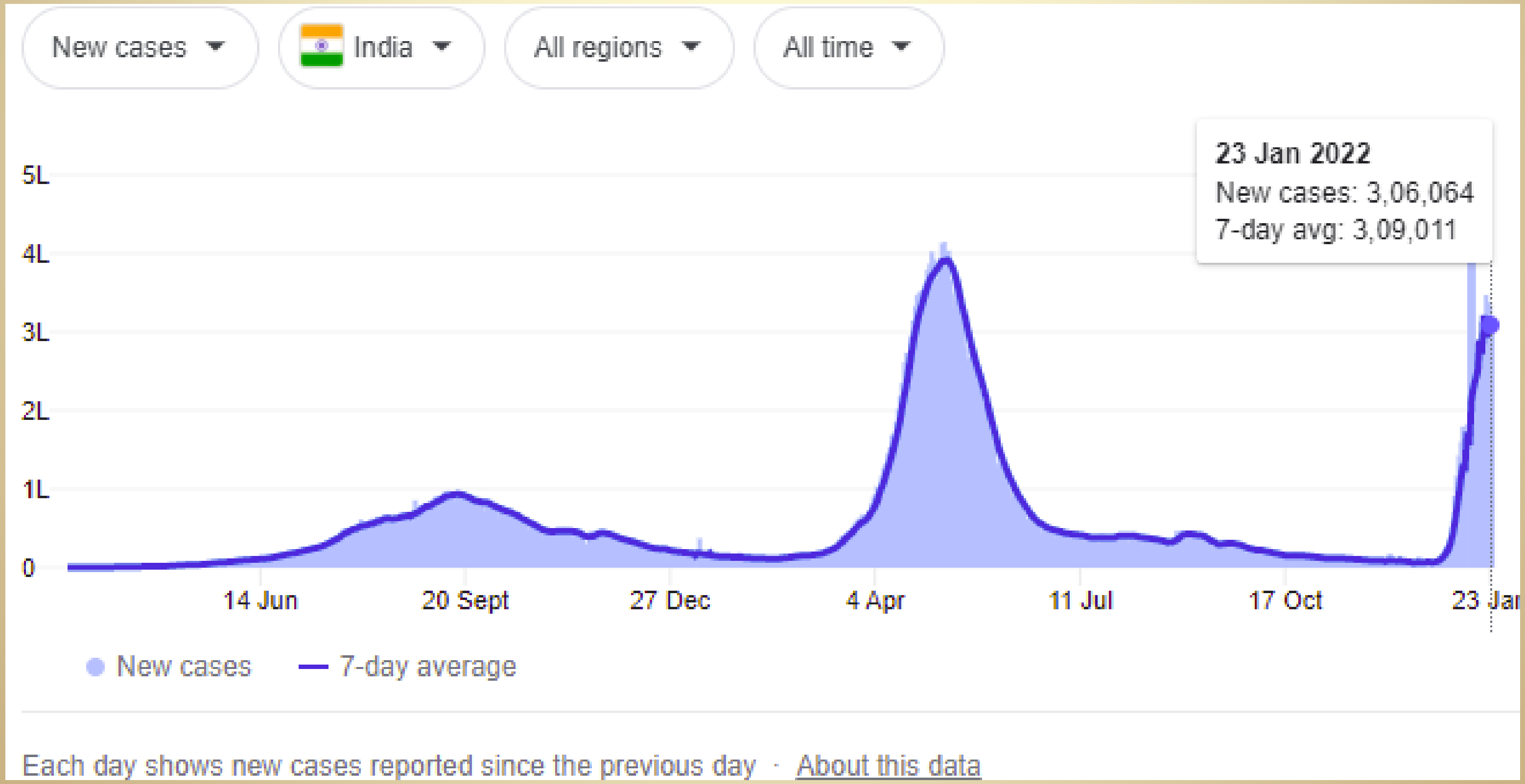
differentiate : $\dfrac{df}{f} = \dfrac{da}{a} + \dfrac{db}{b} - \dfrac{dc}{c}$

Error form : $\boxed{\dfrac{\delta f}{|f|} = \dfrac{\delta a}{|a|} + \dfrac{\delta b}{|b|} + \dfrac{\delta c}{c}}$

Random processes: Statistical analysis of errors

COVID-19: Statistics data: Number of new cases

(14 March 2020- 23 Jan 2022)

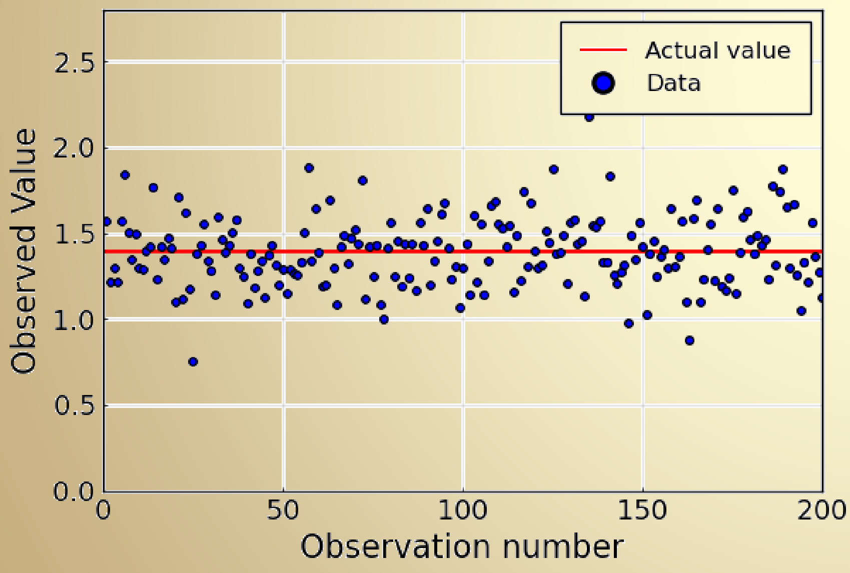


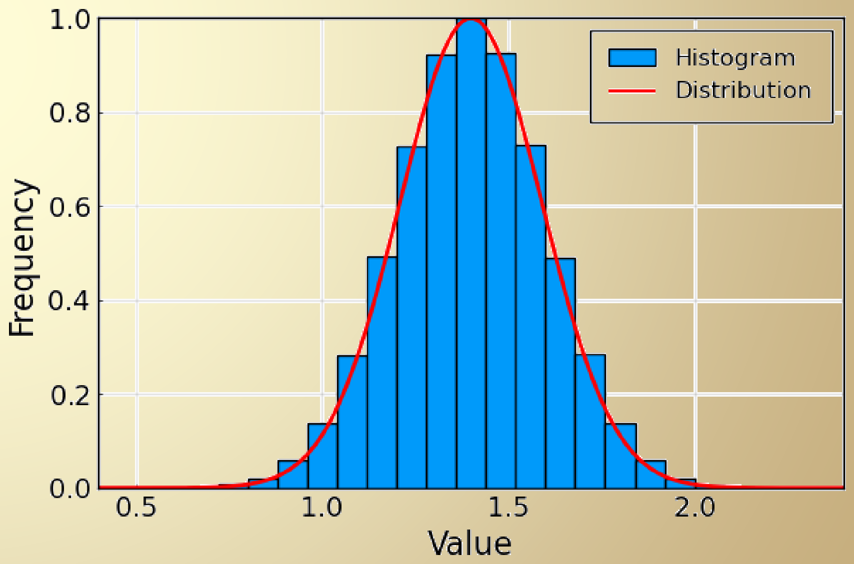
(Source: Google: Johns Hopkins University: https://systems.jhu.edu/research/public-health/ncov/)

Statistical analysis of errors

Measurement of the quantity x made N times:

$$x_1,\ x_2,\ x_3,\ \ldots\ldots\ldots\ x_{N-1},\ x_N$$





Statistical analysis of errors

Measurement of the quantity x made N times:

$$x_1,\ x_2,\ x_3,\ \ldots\ldots\ldots\ x_{N-1},\ x_N$$

Mean: $\bar{x} = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} x_i$

Deviation for single measurement : $d_i = x_i - \bar{x}$

Average deviation: $d = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} (x_i - \bar{x})$

$$= \lim_{N\to\infty} \frac{1}{N}\sum x_i - \lim_{N\to\infty} \frac{1}{N}\sum \bar{x}$$

$$= \bar{x} - \bar{x} = 0$$

Statistical analysis of errors

Variance: mean of deviation squared:

$$\sigma^2 = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})^2$$

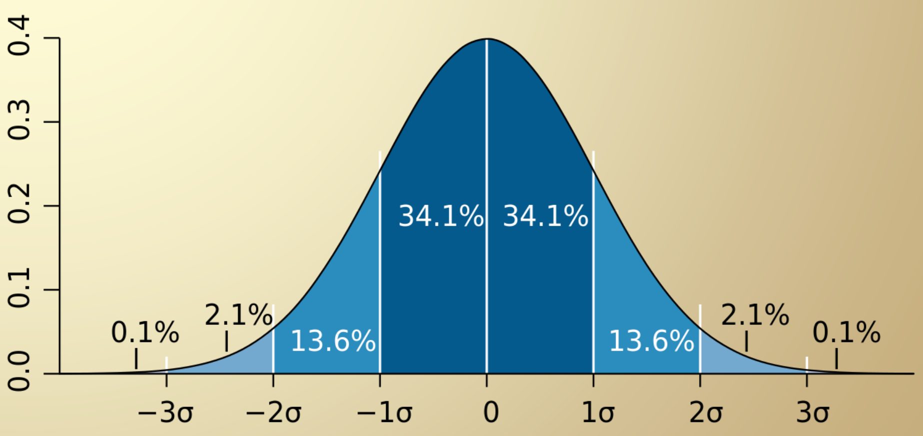
$$= \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} x_i^2 - (\bar{x})^2 = \overline{x_i^2} - (\bar{x})^2 = \langle x^2\rangle - \langle x\rangle^2$$

Standard deviation (s.d.): square root of the variance:

$$\sigma = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$$

- **while reporting:**

(mean $\pm$ s. d.)

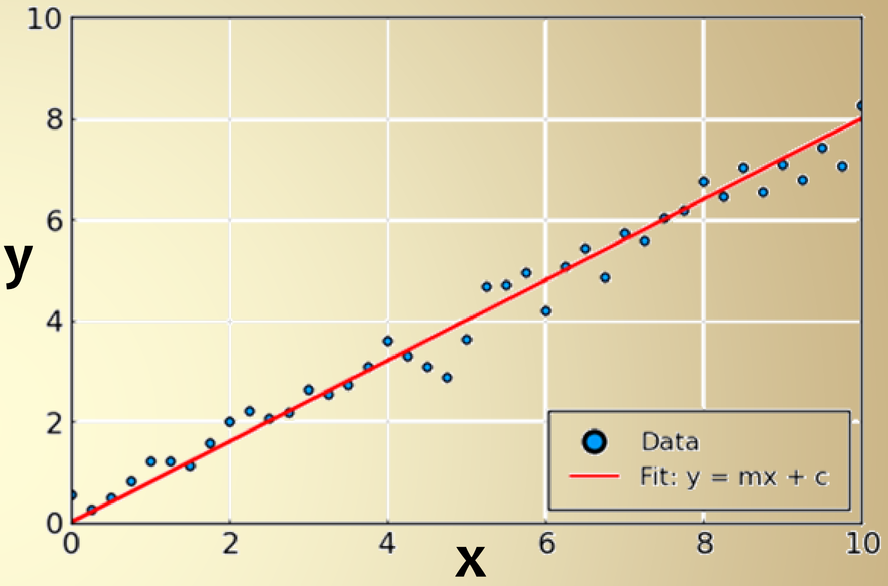


Normal Distribution (Source: Wikipedia)

Method of least square curve fitting

N data points: $(x_{i,} y_i)$

$$y = mx + c$$



Deviation between observed values and calculated values:

$$\delta y_i = y_i - (mx_i + c)$$

Minimizing the mean square deviation:

$$S = \frac{1}{n}\sum_{i=1}^{n}(y_i - mx_i - c)^2$$

Reporting the error in experiments

- **Accuracy:** how close to true/given value
- **Precision:** how well the value is determined (regardless of true value)
- **Example:** $x_{true} = 30$ (unit)

- $x_{measured} = 29 \pm 6$: accurate but imprecise (random/statistical error)

- $x_{measured} = 22 \pm 1$: precise but inaccurate (might have systematic error)

Reporting the error in experiments

- while reporting: $f \pm \delta f$ (with appropriate unit)

Significant digits: example: (a) 3.45 ± 0.22;

(b) 3.4587 ± 0.22; X

(c) 10.82 ± 0.01857; X

- Check precision and then round it to significant digit

- Compare with the given/true value; percentage error

- Discussion on sources of errors

Questions??