ASSIGNMENT -08

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Q1. The Damped Harmonic Oscillator :

We assume the effect of a viscous friction force $f_{fric} = -bv$.
The total force acting on the mass 'm' is :

$$F = F_{spring} + f_{fric}$$
$$= -kx - bv$$

The equation of motion becomes,

$$m\ddot{x} = -kx - b\dot{x}$$

which can be written in the standard form

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$$

where, $\gamma = \frac{b}{m}$ and, $\omega_0^2 = \frac{k}{m}$ as before.

If the friction were neglible the motion would be given by.

$$x = x_0 \cos(\omega_0 t + \varphi)$$

On the other hand, if the spring force were negligible, the mass would move according to,

$$v = v_0 e^{(-b/m)t}$$

We might therefore guess that the solution is of the form,

$$x = x_0 e^{-\alpha t} \cos(\omega_1 t + \varphi)$$

where, if our guess is correct, the constants α and ω_1 can be chosen to make this trial solution satisfy the eqn. x_0 and φ are arbitrary constants for satisfying the initial conditions. We find that the eqn is satisfied provided that

$$\alpha = \gamma/2$$

$$\omega_1 = \sqrt{\omega_0^2 - (\gamma/2)^2}$$

where $\gamma = \frac{b}{m}$ and $\omega_0 = \sqrt{\frac{k}{m}}$ as before.

This solution is valid when $\omega_0^2 - \gamma^2/4 > 0$.

We can rewrite the equation as,

$$x = x(t) \cos(\omega_1 t + \varphi)$$

$$\text{where, } x(t) = x_0 e^{-(\gamma/2)t}.$$

The essential features of the motion depend on the ratio ω_1/γ.
If $\omega_1/\gamma >> 1$, the amplitude decreases only slightly during the time the cosine makes many zero crossings; in this regime, the motion is called <u>lightly damped</u>.
If ω_1/γ is comparatively small, $x(t)$ tends rapidly to zero when the cosine makes a few oscillations. The motion is called <u>heavily damped</u>.

Energy Dissipation in the Damped Oscillator :-

$$v = -x_0 e^{-(\gamma/2)t} \left[\omega_1 \sin(\omega_1 t + \varphi) + \frac{\gamma}{2} \cos(\omega_1 t + \varphi) \right]$$

We will consider systems with <u>light damping</u> only where $\omega_1 >> \gamma/2$, so that $\omega_1 \approx \omega_0$. This allows us to make an approximation that simplifies the arithmetic and reveals some universal features

$$\omega_1^2 = \omega_0^2 - (\gamma/2)^2 \approx \omega_0^2$$

With our approximation that $\omega_1 >> \gamma/2$, the second term in the bracket can be neglected, giving,

$$v = v_0 e^{-(\gamma/2)t} \sin(\omega_0 t + \varphi)$$

where,

$$v_0 = \omega_0 x_0$$

In this case, the potential energy is,

$$V(t) = \frac{1}{2} k x_0^2 e^{-\gamma t} \cos^2(\omega_0 t + \varphi)$$

and the kinetic energy is,

$$K(t) = \frac{1}{2} m v_0^2 = \frac{1}{2} m \omega_0^2 x_0^2 e^{-\gamma t} \sin^2(\omega_0 t + \varphi)$$

$$= \frac{1}{2} k x_0^2 e^{-\gamma t} \sin^2(\omega_0 t + \varphi)$$

The total energy is,

$$E(t) = \frac{1}{2} k x_0^2 e^{-\gamma t}$$

The decay of the Total Energy is described by a simple differential equation :

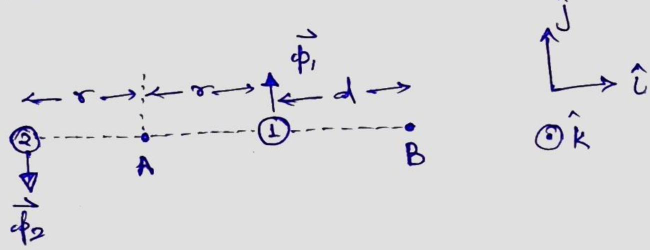
$$\frac{dE}{dt} = -\gamma E$$

which has the sol^n,

$$E = E_0 e^{-\gamma t}$$

where E_0 is the energy at time $t = 0$. The energy's decay is characterised by the time $\tau = 1/\gamma$ in which the energy decreases from its initial value by a factor of $e^{-1} \approx 0.368$. τ is often called the <u>damping time</u> of the system. In the limit of zero damping, $\gamma \to 0$, $\tau \to \infty$ and E is constant. The system then behaves like an undamped oscillator.

Q2. Consider a system where two identical particles form a system. At the instant shown in the figure below, the particles have equal and opposite momentums ($p_1 = p_2 = p$).



We calculate angular momentum from point A & B.

Angular Momentum about A :-

$$\vec{L}_{sys}(A) = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2$$

$$= \vec{r} \times \vec{p} + (-\vec{r}) \times (-\vec{p})$$

$$= 2(\vec{r} \times \vec{p})$$

Angular Momentum about B :-

$$\vec{L}_{sys}(B) = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2$$

$$= (-\vec{d}) \times \vec{p} + \{-(\vec{d} + 2\vec{r})\} \times (-\vec{p})$$

$$= -\vec{d} \times \vec{p} + (\vec{d} + 2\vec{r}) \times \vec{p}$$

$$= \cancel{-\vec{d} \times \vec{p}} + \cancel{\vec{d} \times \vec{p}} + 2\vec{r} \times \vec{p}$$

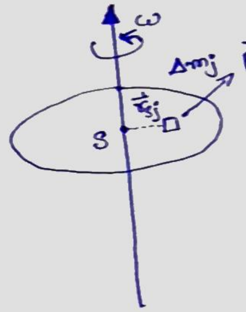
$$= 2(\vec{r} \times \vec{p})$$

Here, we see that angular momentum calculated from points A & B, both are same/equal. Thus, if the total linear momentum of a system is zero then the angular momentum is independent of position.

Q3. Rotational Eq^n of Motion. a.k.a.

Relationship between Torque and Angular Acceleration

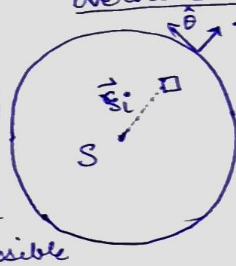
$$\vec{F}_j^{ext} = (F_{jr}\hat{r} + F_{j\theta}\hat{\theta} + F_{jz}\hat{k})$$



$\alpha = \alpha_z \hat{k}$

$$\boxed{\tau_s = \vec{r}_{sj} \times \vec{F}_j}$$

overhead View



$$\because \vec{r}_{sj} = (r_{sjr}\hat{r} + \cancel{r_{sj\theta}\hat{\theta}})$$

$$\vec{r}_{sj} \times \vec{F}_j^{ext} = (r_{sjr}\hat{r} + \cancel{r_{sj\theta}\hat{\theta}}) \times (F_{jr}\hat{r} + \cancel{F_{j\theta}\hat{\theta}} + \underbrace{F_{jz}\hat{k}}_{\Downarrow})$$

Dropping due to not possible component remaining

$$\left\{ \begin{array}{l|l} \hat{r}\times\hat{r} = 0 & \hat{r}\times\hat{\theta} = \hat{k} \\ \hat{\theta}\times\hat{\theta} = 0 & \hat{\theta}\times\hat{k} = \hat{r} \\ \hat{k}\times\hat{k} = 0 & \hat{k}\times\hat{r} = \hat{\theta} \end{array} \right\}$$

$$\vec{r}_{sj} \times \vec{F}_j^{ext} = (r_{sj}\hat{r} \times F_{sj}\hat{\theta})$$

$$\vec{\tau}_s = \sum_{j=1}^{N} r_{sj}\hat{r} \times F_{sj\theta}\hat{\theta} = \sum_{j=1}^{N} r_{sj} F_{sj\theta}\hat{k}$$

From the Newton's 2nd Law, we have $F_{j\theta} = \Delta m_j a_{j\theta}$

$$\Rightarrow a_{j\theta} = r_{sj}\alpha_z.$$

$$\Rightarrow \vec{\tau}_s = \sum_{j=1}^{N} r_{sj}\cdot r_{sj}\,\alpha_z\,\Delta m_j = \sum_{j=1}^{N} \Delta m_j r_{sj}^2 \alpha_z.$$

$$\Rightarrow \vec{\tau}_s \equiv \vec{\tau}_{s,z} = \underbrace{\sum_{j=1}^{N} \Delta m_j r_{sj}^2}\,\alpha_z.$$

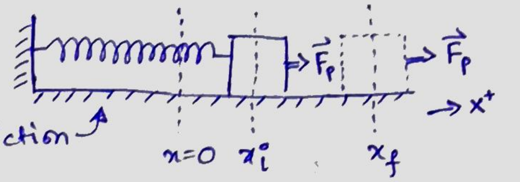
$$\lim_{\substack{N\to\infty \\ \Delta m\to 0}} \sum_{j=1}^{N} \Delta m_j r_{sj}^2 = I_s$$

$$\boxed{\vec{\tau}_{s,z} = I_s\,\vec{\alpha}_z}$$

$\vec{\tau}$ = Torque

I_s = Moment of Inertia about axis

$\vec{\alpha}$ = Angular Accel^n

Q4. 

(massless spring)

SYS: Mass & spring

$\Delta E_{sys} = W_p$

$\Rightarrow \Delta K + \Delta U_s = W_p$ — (I)

SYS: Mass only.

$\Delta E_{sys} = W_p + W_s$

$\Rightarrow \Delta K = W_p + W_s$

$\Rightarrow \Delta K + (-W_s) = W_p$. — (II)

From (I) & (II), we have,

$$\Delta U_s = -W_s$$

Now,

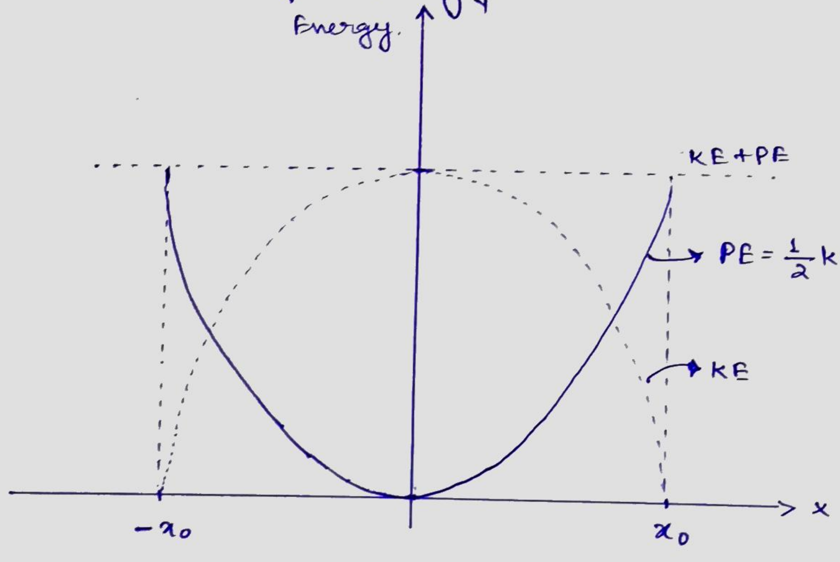
$$\Delta U_s = -W_s = -\int_{x_i}^{x_f} (-kx)\,dx \qquad [F_{s,x} = -kx]$$

$$\Rightarrow U_f - U_i = \frac{1}{2}kx^2 \Big|_{x_i}^{x_f} = \frac{1}{2}kx_f^2 - \frac{1}{2}kx_i^2$$

$$\therefore \boxed{U_s = \frac{1}{2}kx^2}$$

↳ where, k is the spring constant

Graph for KE and PE of a spring:—



Q5. Situation 1 (Frictionless surface, $F^{ext} = 0$)

→ The momentum is conserved ($p_f = p_i$)

$\because p_f = p_i$

$p_f = (3m + m) v_f$ — (I).

$p_i = 3m v_0 - m v_0 = 2 m v_0$ — (II)

Putting (I) & (II),

$\Rightarrow 2 m v_0 = 4 m v_f$

$\Rightarrow \boxed{v_f = v_0/2}$.

Situation 2. (Rough Surface)

$W = \Delta K$ (From Work-Energy Theorem)

$K_f = 0$ when $x = d$.

$K_i = \frac{1}{2}(m + 3m) v_f^2$ when $x = 0$

$\therefore \Delta K = 2 m v_f^2$ $\quad \because v_f = \frac{v_0}{2}$.

$W = \Delta K = 2m \frac{v_0^2}{4} = \frac{1}{2} m v_0^2$

$\Rightarrow \boxed{W = \frac{1}{2} m v_0^2}$ — $*_1$

Work done due to friction,

$F^k = \mu_k N \hat{i}$

$= bx^2 (3m + m) g \hat{i}$

$= 4bmg x^2 \hat{i}$

$$W = \int_i^f \vec{F} \cdot ds = \int_{x=0}^{x=d} 4bmg\, x^2 dx = 4bmg \cdot \frac{x^3}{3} \Big|_{x=0}^{x=d} = 4bmg \frac{d^3}{3}.$$

$\therefore \boxed{W = 4bmg \frac{d^3}{3}}$ — $*_2$

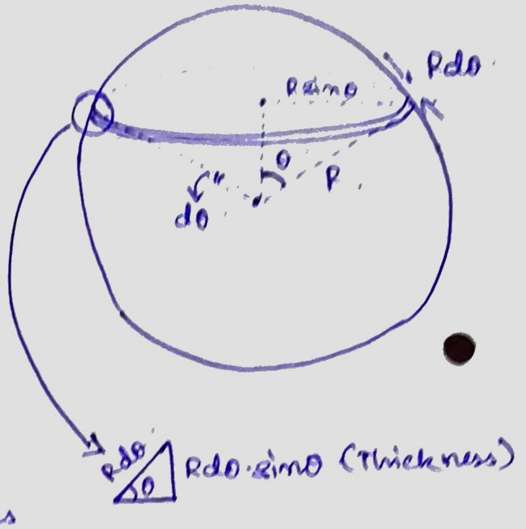
From (i) and (ii), we get,

$$\frac{1}{2} m v_0^2 = 4 b m g \frac{d^2}{3}$$

$$\Rightarrow \boxed{v_0 = \sqrt{\frac{8}{3} b g d^2}}$$

Q6. COM of a solid sphere :-

Consider an elementary solid disc subtending an angle 2θ at the origin (0,0,0) and having small axial thickness $Rd\theta \cdot \sin\theta$. Using Spherical Polar Coordinates, we have,



$$dm = \rho\pi (R\sin\theta)^2 R d\theta \cdot \sin\theta$$
$$= \pi\rho R^3 \sin^3\theta \, d\theta \quad (\rho : \text{density})$$

Hence, the distance of centre of mass of the sphere from the origin (0,0,0) is given as,

$$\bar{x} = \frac{\int x \, dm}{\int dm} = \frac{\int_0^\pi (r\cos\theta) \times \rho r^3 \sin^3\theta \, d\theta}{\int_0^\pi \pi \rho r^3 \sin^3\theta \, d\theta} = \underline{0}.$$

$\therefore$ The COM of a solid sphere lies at its centre.

Q7. Contact forces acting on the object :-

(i) Gravitational force → Acts towards the surface of the Earth and depends on mass.

(ii) Normal Force → Reaction force due to surface.

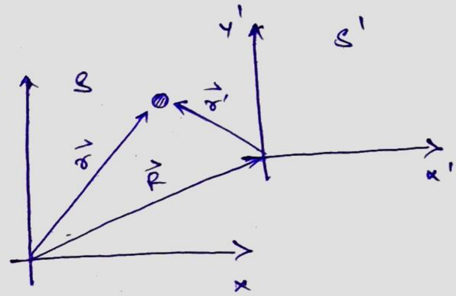
(iii) Frictional Force → Opposite to the direction of applied force. It acts $\perp^r$ to both Grav. & Normal Force.

Contact Forces are the forces which arise when two or more bodies come in contact to each other.

Q8. Reference Frames :-

① Inertial Reference Frame.

② Non-Inertial Reference Frame.



$$\vec{r} = \vec{R} + \vec{r}\,'$$

$$\vec{r}\,' = \vec{r} - \vec{R}$$

Case ①:

Let's assume that the frame S' is moving constantly with velocity $\vec{v}$. Relative distance between the two frames:

$$\boxed{\vec{R}(t) = \vec{R}_0 + \vec{v}t}$$

Velocity: $\dfrac{d\vec{r}\,'}{dt} = \dfrac{d\vec{r}}{dt} - \dfrac{d}{dt}(R_0 + \vec{v}t)$

$$\Rightarrow \boxed{\vec{v}\,' = \vec{u} - \vec{v}}$$

Accel^n: $\dfrac{d\vec{v}\,'}{dt} = \dfrac{d\vec{u}}{dt} - \cancel{\dfrac{d\vec{v}}{dt}}^{\,0} \Rightarrow \boxed{\vec{a}\,' = \vec{a}}$

Hence, $\boxed{\vec{F} = \vec{F}\,'}$

Case (2) :-

If frame S' is moving continuously, where $\vec{v} \neq$ constant,

$$\vec{v}' = \vec{v} + \vec{V}$$

Accelⁿ : $\frac{d\vec{v}'}{dt} = \frac{d\vec{v}}{dt} - \frac{d\vec{V}}{dt}$

$$\Rightarrow \boxed{\vec{a}' = \vec{a} - \vec{A}}$$

$$m\vec{a}' = m\vec{a} - m\vec{A}$$

$$\Rightarrow \boxed{\vec{F}' = \vec{F}_{physical} + \vec{F}_{pseudo}}$$

$\vec{F}_{pseudo}$ or $\vec{F}_{fictitious} = -m\vec{A}$

Q9. Taylor Series :

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{k=0}^{\infty} a_k \cdot x^k$$

where $a_0, a_1, a_2, \ldots$ are all arbitrary constants.

$x = 0 \Rightarrow f(0) = a_0. \Rightarrow f(0) = a_0 + \underbrace{a_1 \cdot 0 + a_2 \cdot 0 + \cdots}_{\text{zero.}}$

$$\frac{df}{dx} = f'(x) = \frac{d}{dx}(a_0 + a_1 x + \cdots)$$

$$\Rightarrow f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

$$\Rightarrow x = 0.$$

$$\therefore \boxed{f'(0) = a_1} \Rightarrow \boxed{a_1 = \frac{f'(0)}{1!}}$$

Similarly,

$$f''(0) = 2a_2 \Rightarrow \boxed{a_2 = \frac{f''(0)}{2!}}$$

Differentiating $f(x)$ for k times, we get,

$$f^{(k)}(0) = k!\,a_k \Rightarrow \boxed{a_k = \frac{1}{k!} f^{(k)}(0)}$$

~~Considering~~ Combining all the above results,

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

$$\Rightarrow \boxed{f(x) = f(0) + \frac{f'(0)}{1!} \cdot x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \cdots}$$

Q10. Kinematics Equations.

$$\frac{dv(t)}{dt} = a(t)$$

$$\Rightarrow \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j} + \frac{dv_z}{dt}\hat{k} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$$

so, $\frac{dv_x}{dt} = a_x$; $\frac{dv_y}{dt} = a_y$; $\frac{dv_z}{dt} = a_z$

Integrating $\frac{dv_x}{dt} = a_x$ we get,

$$\int_{t_0}^{t_1} dv_x = \int_{t_0}^{t_1} a_x(t)\,dt.$$

$$\Rightarrow v_x(t_1) - v_x(t_0) = \int_{t_0}^{t_1} a_x(t)\,dt$$

Similarly,

$$v_y(t_1) - v_y(t_0) = \int_{t_0}^{t_1} a_y(t)\,dt$$

$$v_z(t_1) - v_z(t_0) = \int_{t_0}^{t_1} a_z(t)\,dt$$

$$\Rightarrow v(t) = v_0 + \int_0^{t_1} a(t)\,dt.$$

$$\because \frac{dr(t)}{dt} = v(t)$$

$$\Rightarrow r(t) = r_0 + \int_0^{t_1} v(t)\,dt. \quad \text{— (i).}$$

Again,

$$\int_{t_0}^{t_1} v(t)\,dt = \int_{t_0}^{t_1} (v_0 + a(t))\,dt.$$

$$\Rightarrow v(t) = v_0 + \int_{t_0}^{t_1} a(t)\,dt$$

At $t=0$, $a=$ const. If we take accⁿ as const.,

$$v(t) = v_0 + a\int_0^t dt$$

$$\therefore \boxed{v(t) = v_0 + at} \quad \text{— (ii).}$$

Putting eqⁿ (ii) in (i),

$$r(t) = r_0 + \int_0^t (v_0 + at)\,dt.$$

$$\therefore \boxed{r(t) = r_0 + v_0 t + \tfrac{1}{2}at^2} \quad \text{— (iii)}$$