PH1101 : Mechanics I

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Q1. $F_{spring} = -kx$

$$\Rightarrow m\ddot{x} = -kx \Rightarrow m \cdot \frac{d^2x}{dt^2} = -kx$$

$$\Rightarrow \frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \quad \text{(2nd order ODE)}$$

Consider $\omega_0^2 = \frac{k}{m}$, $\left[D = \frac{d}{dt} \,;\, D^2 = \frac{d^2}{dt^2}\right]$

$$\Rightarrow D^2 x + \omega_0^2 x = 0$$

$$\Rightarrow (D^2 + \omega_0^2)x = 0$$

$$\Rightarrow (D + i\omega_0)(D - i\omega_0)x = 0$$

$(D + i\omega_0)x = 0$

$\Rightarrow Dx = -i\omega_0 x$

$\Rightarrow \frac{dx}{dt} = -i\omega_0 x$

$\Rightarrow \int \frac{dx}{x} = \int -i\omega_0 dt$

$\Rightarrow \ln x = -i\omega_0 t + c$

$\Rightarrow x = Ae^{-i\omega_0 t}$

$(D - i\omega_0)x = 0$

$\Rightarrow Dx = i\omega_0 x$

$\Rightarrow \frac{dx}{dt} = i\omega_0 x$

$\Rightarrow x = Be^{i\omega_0 t}$

$$\Rightarrow x = Ae^{-i\omega_0 t} + Be^{i\omega_0 t} \longrightarrow \begin{cases} e^{i\omega_0 t} = \cos\omega_0 t + i\sin\omega_0 t \\ e^{-i\omega_0 t} = \cos\omega_0 t - i\sin\omega_0 t \end{cases}$$

$$\Rightarrow x = (A+B)\cos\omega_0 t - i(A-B)\sin\omega_0 t$$

$$Re(x) = (A+B)\cos\omega_0 t$$
$$= x_0 \cos\omega_0 t$$

Generally,

$$\boxed{x = x_0 \cos(\omega_0 t + \varphi)}$$

x_0 : Amplitude
ω_0 : Frequency

Now, deriving total energy of a spring,

$$x = x_0 \cos \omega_0 t$$

$$\Rightarrow v = \dot{x} = -x_0 \omega_0 \sin \omega_0 t$$

$$\Rightarrow v^2 = x_0^2 \omega_0^2 \sin^2(\omega_0 t)$$

$$KE = \frac{1}{2} m v^2 = \frac{1}{2} m \omega_0^2 x_0^2 \sin^2 \omega_0 t$$

Now, deriving formula for PE of spring, we know,

$$dW = \vec{F} \cdot d\vec{x}$$

$$\Rightarrow \int dW = \int F dx$$

substituting, $F = -kx$, we get,

$$\Rightarrow W = \int_0^x -kx\, dx = -k \int_0^x x\, dx = -k \left[\frac{x^2}{2}\right]_0^x = -\frac{1}{2} k x^2$$

The work done is nothing but elastic potential energy of spring,

$$U = -\frac{1}{2} k x^2 .$$

$$\Rightarrow PE = U = \frac{1}{2} k x^2 = \frac{1}{2} k x_0^2 \cos^2 \omega_0 t$$

$$\because m\omega_0^2 = k \,;\, \frac{k}{m} = \omega_0^2$$

$$\Rightarrow E = K + V = \frac{1}{2} k x_0^2 (\sin^2 \omega_0 t + \cos^2 \omega_0 t) \Rightarrow \boxed{E = \frac{1}{2} k x_0^2}$$

Total energy is constant

Q2. Angular Momentum : $\vec{L} = \vec{r} \times \vec{p}$

Angular Momentum about point A,

$$\left.\begin{aligned} \vec{L}_{sys}(A) &= \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2 \\ &= \vec{r} \times \vec{p} + (-\vec{r}) \times (-\vec{p}) \\ &= 2(\vec{r} \times \vec{p}) \quad \text{— (1)} \end{aligned}\right\} \quad \begin{aligned} \vec{r} &= r\hat{i} \\ -\vec{r} &= r(-\hat{i}) \\ \vec{p}_1 &= p_1 \hat{j} \\ \vec{p}_2 &= p_2(-\hat{j}) \end{aligned}$$

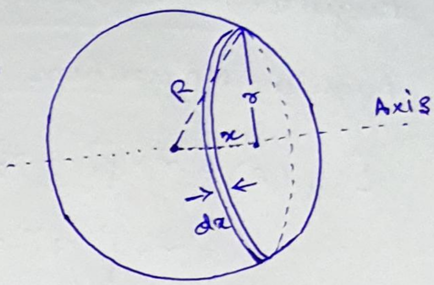
Angular Momentum about point B,

$$\left.\begin{aligned} \vec{L}_{sys}(B) &= \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2 \\ &= (-\vec{d}) \times \vec{p} + \{-(\vec{d} + 2\vec{r})\} \times (-\vec{p}) \\ &= -\vec{d} \times \vec{p} + (\vec{d} + 2\vec{r}) \times \vec{p} \\ &= \cancel{-\vec{d} \times \vec{p}} + \cancel{\vec{d} \times \vec{p}} + 2\vec{r} \times \vec{p} \\ &= 2(\vec{r} \times \vec{p}) \quad \text{— (11)} \end{aligned}\right\} \quad \begin{aligned} \vec{r} &= r\hat{i} \\ -\vec{r} &= r(-\hat{i}) \\ \vec{d} &= d\hat{i} \\ -\vec{d} &= d(-\hat{i}) \\ \vec{p}_1 &= p_1 \hat{j} \\ \vec{p}_2 &= p_2(-\hat{j}) \end{aligned}$$

Here, we see that angular momentum calculated about points A (1) and B (11) turn out to be the same, and hence, we can conclude that when linear momentum of a system adds up to zero (0), the angular momentum doesn't depend on the position from where it is calculated.

Q3. Moment of Inertia of a solid sphere.

We slice up the solid sphere into infinitesimally thin solid cylinders / discs.



Moment of Inertia of a solid cylinder / disc $= \frac{1}{2}MR^2$

Hence, for this problem,

$$dI = \frac{1}{2}r^2 dm$$

Now,

$$dm = \rho dV \qquad [\rho : \text{density}]$$

Finding dV,

$$dV = \pi r^2 dx$$

Substituting dV in dm,

$$dm = \rho \pi r^2 dx$$

Substituting dm in dI,

$$dI = \frac{1}{2}\rho \pi r^4 dx$$

Now, to introduce 'x' into the equation. Note that x, r and R makes a right triangle. Hence, using Pythagoras' Theorem,

$$r^2 = R^2 - x^2$$

Substituting,

$$dI = \frac{1}{2}\rho\pi (R^2 - x^2)^2 dx$$

Hence,

$$I = \frac{1}{2}\rho\pi \int_{-R}^{R} (R^2 - x^2)^2 dx$$

After expanding out and integrating, we get,

$$I = \frac{1}{2}\rho\pi\frac{16}{15}R^5$$

Now, we have to find the density of sphere,

$$\rho = \frac{M}{V} = \frac{M}{\frac{4}{3}\pi R^3}$$

substituting, we have,

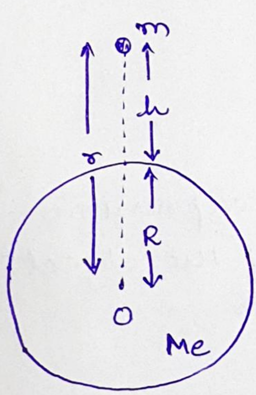
$$\boxed{I = \frac{2}{5}MR^2}$$

Q4. Gravitational Potential Energy near the surface of the Earth

$$PE = W$$
$$= \int_{\infty}^{R} \frac{GMm}{x^2}\,dx$$

$$U_{(p)} + V_{(\infty)} = GMm \int_{\infty}^{R} x^{-2}\,dx.$$

$$U_{(p)} = -\frac{GMm}{R}.$$ [$V_{(\infty)}$ is considered zero]



Consider the Earth-Mass system, with r, the distance between the mass 'm' and the Earth's centre. Then the gravitational potential energy,

$$U = -\frac{GM_em}{r}$$

Here, $r = R_e + h$, where R_e is the radius of the Earth, h is the height above the Earth's surface.

$$U = -G\frac{M_em}{(R_e+h)}$$

If $h << R_e$, the eqⁿ can be modified as,

$$U = -G \frac{M_e m}{R_e \left(1 + \frac{h}{R_e}\right)}$$

$$\Rightarrow U = -G \frac{M_e m}{R_e} \cdot \left(1 + \frac{h}{R_e}\right)^{-1}.$$

By using Binomial Expansion and neglecting higher order terms, we get,

$$U = -G \frac{M_e m}{R_e} \left(1 - \frac{h}{R_e}\right) \quad -(*_1)$$

We know that, for a mass 'm' on the Earth's surface,

$$G \frac{M_e m}{R_e} = mgR_e \quad -(*_2)$$

substituting $(*_2)$ in $(*_1)$, we get,

$$\boxed{U = -mgR_e + mgh} \text{ Ans}$$

It is clear that the first term in the above expression is independent of the height 'h'. For example, if the object is taken from height h_1 to h_2 then,

$$U(h_1) = -mgR_e + mgh_1$$
$$U(h_2) = -mgR_e + mgh_2$$

$$\Rightarrow \boxed{\Delta U = U(h_2) - U(h_1) = mg(h_2 - h_1)} \text{ Ans}$$

Q6. The centre of mass of a system of particles behaves as if all the mass were concentrated at that point and all external forces act on that point.

$$\Rightarrow \vec{F}_{ext} = M_{sys}\ddot{R}_{COM} = \frac{d}{dt}\vec{P}_{sys}$$

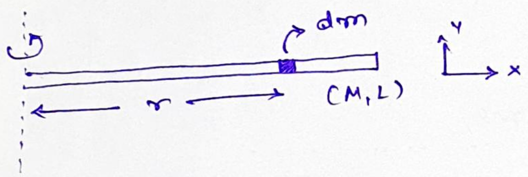
where, $\vec{P}_{sys} = \sum_j m_j \dot{r}_j$

$$\Rightarrow M_{sys} \cdot \ddot{\vec{R}}_{COM} = \sum_j m_j \vec{a}_j$$

$$\Rightarrow M_{sys} \cdot \vec{R}_{COM} = \sum_j m_j r_j$$

$$\Rightarrow \vec{R}_{COM} = \frac{1}{M_{sys}} \times \sum_j m_j r_j$$

Centre of mass of a uniform rod :



$$\Rightarrow \vec{R}_{COM} = \frac{1}{M}\sum_j m_j r_j$$

But $m_j \to 0$ and $j \to \infty$

$$= \frac{1}{M}\int_0^L dm \cdot r$$

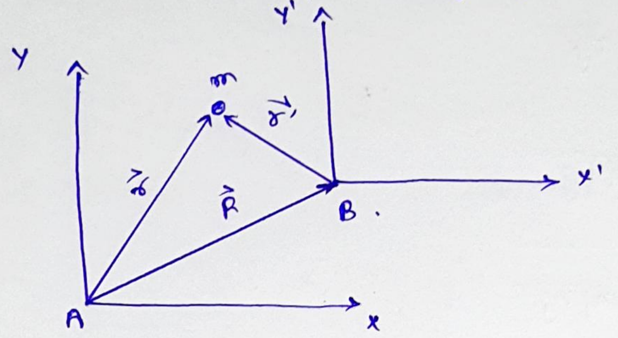
Linear Mass Density, $\lambda = \frac{M}{L} = \frac{dm}{dr}$

$$\Rightarrow R_{COM} = \frac{1}{M}\int_0^L \lambda r\, dr = \frac{1}{L}\left[\frac{r^2}{2}\right]_0^L = \frac{L}{2}.$$

Hence, R_{COM} for a rod of negligible thickness exists at $\frac{L}{2}$ w.r.t the chosen coordinates.

Q7. Inertial frames of References are those which either are at rest or moving with uniform velocity w.r.t. to other already classified inertial frame.

Non-Inertial frames of Reference on the other hand is accelerated w.r.t. to an inertial frame of reference.



From the diagram, it is clear that,

$$\vec{r} = \vec{r}' + \vec{R}$$

Case 1 : frame A moving with const. velocity ($\vec{v_0}$)

$$\Rightarrow \vec{r} = \vec{r}' + \vec{R}$$

Differentiating w.r.t. time,

$$\Rightarrow \frac{d\vec{r}}{dt} = \frac{d\vec{r}'}{dt} + \frac{d\vec{R}}{dt}$$

Let, $\frac{d\vec{r}}{dt} = \vec{v}$ be the velocity of particle w.r.t. frame A

$\frac{d\vec{r}'}{dt} = \vec{v}'$ be the velocity of particle w.r.t. frame B

$\frac{d\vec{R}}{dt} = \vec{v}_0$

Hence, we get,

$$\boxed{\vec{v} = \vec{v}' + \vec{v}_0}$$

Differentiating it again, and considering the particle to have variable $\vec{v}$ & $\vec{v}'$,

$$\Rightarrow \frac{d\vec{v}}{dt} = \frac{d\vec{v}'}{dt} + \cancelto{0}{\frac{d\vec{v_0}}{dt}} \qquad [\because \vec{v_0} \rightarrow \text{constant}]$$

$$\Rightarrow \boxed{\vec{a} = \vec{a}'}$$

Hence, the force on the particle w.r.t. frame A = the force on particle w.r.t. frame B.

So, both frames are inertial.

<u>Case 2</u>: If the frame A accelerates with $A = \frac{d\vec{v_0}}{dt}$

$$\Rightarrow \vec{r} = \vec{r}' + \vec{R}$$

Differentiating both the sides,

$$\vec{v} = \vec{v}' + \vec{v_0}$$

Differentiating again

$$\vec{a} = \vec{a}' + \vec{A}$$

$$\Rightarrow m\vec{a} = m\vec{a}' + m\vec{A}$$

Force on particle wrt. frame A = Force on particle wrt frame B + (- fictitious force)

$$\therefore \boxed{F_{fic} = -m\vec{A}}$$

Presence of fictitious force represents the presence of non-inertial frame of reference.