Matrix Multiplication

$$C = A * B \equiv c_{ij} = \sum_{k=1}^{K} a_{ik} b_{kj} \qquad \forall \;\; 1 \leq i \leq M \quad \& \quad 1 \leq j \leq N$$

$$C_{M,N} = A_{M,K} * B_{K,N}$$

Associative: $A(BC) = (AB)C$

Distributive: $A(B \pm C) = AB \pm AC$

$(A \pm B)C = AC \pm BC$

Identity: $A\mathbb{I} = \mathbb{I}A = A$

$c\,(A\,B) = (c\,A)\,B$

$A\,\mathbf{O} = \mathbf{O}\,A = \mathbf{O}$

$A^1 = A \qquad A^2 = A * A \qquad A^n = A * \cdots * A$

$AB \neq BA$

matrix multiplication is not commutative

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad \& \quad B = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$AB = \begin{pmatrix} -1 & 3 \\ -3 & 7 \end{pmatrix}$$

$$\neq BA = \begin{pmatrix} 2 & 2 \\ 3 & 4 \end{pmatrix}$$

$AC = BC \;\not\Longrightarrow\; A = B$

$$AC = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix}$$

$$BC = \begin{pmatrix} 1 & 3 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix}$$

Matrix Transpose

Switches rows and columns of a matrix

$a_{ij} \to a^T_{ji}$

$A \in M_{m,n}(F) \longrightarrow A^T \in M_{n,m}(F)$

$$A \equiv \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \vdots & & & \ddots \end{pmatrix}_{mXn} \qquad A^T \equiv \begin{pmatrix} a_{11} & a_{21} & a_{31} & \cdots \\ a_{12} & a_{22} & a_{32} & \cdots \\ a_{13} & a_{23} & a_{33} & \cdots \\ \vdots & & & \ddots \end{pmatrix}_{nXm}$$

$a_{ij} \to \bar{a}^T_{ji}$

$A \in M_{m,n}(\mathbb{C}) \longrightarrow A^\star \in M_{m,n}(\mathbb{C})$

$A^\star \longrightarrow$ Hermitian Adjoint
Matrix Adjoint
Conjugate transpose
Adjoint operator

If a square matrix is such that $A^T = A \to$ **Symmetric** matrix
$A^T = -A \to$ Skew-symmetric matrix
$A^\star = A \to$ **Hermitian** matrix
$A^\star = -A \to$ Skew-hermitian matrix

Laws of Transpose

- $\star \quad (A^T)^T = A$
- $\star \quad (A^\star)^\star = A$
- $\star \quad (A \pm B)^T = A^T \pm B^T$
- $\star \quad (c\ A)^T = c\ A^T$
- $\star \quad (AB)^T = B^T\ A^T$

Theorem: For any matrix $A \in M_N(F)$ $\;A - A^T \to$ skew-symmetric and $A + A^T \to$ symmetric

Proof: $\quad (A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T) \quad \square$

Inner product

$$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \qquad v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \qquad < u, v > = u^T v = (u_1 \;\; u_2 \;\; u_3) \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

$< u, v > = 0 \implies u, v$ are **orthogonal**

Norm of a vector: $||u|| = [u^T u]^{1/2} = [u_1^2 + u_2^2 + u_3^2]^{1/2}$ $\qquad ||u|| = 1 \implies$ **normalised**

Set of normalised vectors mutually orthogonal — **orthonormal**

Outer product

$$u \; v^T = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} (v_1 \;\; v_2 \;\; v_3) = \begin{pmatrix} u_1 v_1 & u_1 v_2 & u_1 v_3 \\ u_2 v_1 & u_2 v_2 & u_2 v_3 \\ u_3 v_1 & u_3 v_2 & u_3 v_3 \end{pmatrix}$$

Matrix Inverse

If for a square matrix A there exists A^{-1} such that $AA^{-1} = A^{-1}A = \mathbb{I}$

A is called **invertible**

Matrix A^{-1} is the **inverse** of matrix A

Matrix Inverse Calculation for $N = 2$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \Longrightarrow \begin{pmatrix} ax_1 + by_1 & ax_2 + by_2 \\ cx_1 + dy_1 & cx_2 + dy_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

If $ad - bc \neq 0 \implies A^{-1}$ exists

Theorem: Inverse of a matrix is unique.

Proof: Let B and C both inverses of A such that AB=BA=I and AC=CA=I.

$$B = B.I = B(AC) = (BA)C = I.C = C \quad \square$$

★ $(AB)^{-1} = B^{-1}A^{-1}$

★ $(A^T)^{-1} = (A^{-1})^T$

★ A is invertible $\implies$ A^T is invertible

★ $AC = BC = Z \implies A = B = ZC^{-1}$ if C^{-1} exists

Orthogonal Matrices

A square matrix $Q \in M_N(\mathbb{R})$ such that $Q^{-1} = Q^T \to QQ^T = Q^TQ = \mathbb{I}$

Row vectors and column vectors are orthonormal.

Orthogonal matrices preserve norm:

$$x \to Qx \implies ||Qx||^2 = (Qx)^T(Qx) = x^TQ^TQx = x^Tx = ||x||^2$$

Unitary Matrices: $U \in M_N(\mathbb{C})$ such that $U^{-1} = U^\star$

Examples:

$$\begin{pmatrix} \cos\theta & i\sin\theta \\ -i\sin\theta & -\cos\theta \end{pmatrix} \qquad \frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix} \qquad \begin{pmatrix} 1 & 1+i \\ 1-i & 2 \end{pmatrix}$$

Linear System of Equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \longrightarrow \begin{aligned} u_{11}x_1 + u_{12}x_2 + \cdots + u_{1n}x_n &= b'_1 \\ u_{22}x_2 + \cdots + u_{2n}x_n &= b'_2 \\ &\vdots \\ u_{nn}x_n &= b'_n \end{aligned}$$

m=n

$$A\,.\,\overrightarrow{x} = \overrightarrow{b} \qquad U\,.\,\overrightarrow{x} = \overrightarrow{b'}$$

Solve by back substitution

- Interchange two equations
- Multiply/scale any equation with a nonzero constant
- Add a multiple of one equation to another

Example

$$\begin{aligned} -3x_1 + 2x_2 - x_3 &= -1 \\ 6x_1 - 6x_2 + 7x_3 &= -7 \\ 3x_1 - 4x_2 + 4x_3 &= -6 \end{aligned} \implies \begin{pmatrix} -3 & 2 & -1 \\ 6 & -6 & 7 \\ 3 & -4 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ -7 \\ -6 \end{pmatrix}$$

Let us construct an **augmented** matrix

$$\left(\begin{array}{ccc|c} -3 & 2 & -1 & -1 \\ 6 & -6 & 7 & -7 \\ 3 & -4 & 4 & -6 \end{array}\right) \xrightarrow[\substack{R1=R1 \\ R2=2R1+R2 \\ R3=R1+R3}]{} \left(\begin{array}{ccc|c} -3 & 2 & -1 & -1 \\ 0 & -2 & 5 & -9 \\ 0 & -2 & 3 & -7 \end{array}\right) \xrightarrow[R3=(-1)R2+R3]{} \left(\begin{array}{ccc|c} -3 & 2 & -1 & -1 \\ 0 & -2 & 5 & -9 \\ 0 & 0 & -2 & 2 \end{array}\right)$$

Solved by back substitution $$\begin{aligned} -3x_1 + 2x_2 - x_3 &= -1 \\ -2x_2 + 5x_3 &= -9 \\ -2x_3 &= 2 \end{aligned} \implies \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned} \xrightarrow[R2 \to R2 - R1\frac{a_{21}}{a_{11}}]{} \begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ 0 \quad + a_{22}^{(2)}x_2 + \cdots + a_{2n}^{(2)}x_n &= b_2^{(2)} \\ &\vdots \end{aligned}$$

2nd generation

$$\forall i, j = 2,3,\cdots, n \qquad m_{i1} = \frac{a_{i1}^{(1)}}{a_{11}^{(1)}}$$

$$a_{ij}^{(2)} = a_{ij}^{(1)} - m_{i1}a_{ij}^{(1)}$$

$$b_i^{(2)} = b_i^{(1)} - m_{i1}b_i^{(1)}$$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ 0 \quad + a_{22}^{(2)}x_2 + \cdots + a_{2n}^{(2)}x_n &= b_2^{(2)} \\ &\vdots \\ 0 \quad + a_{n2}^{(2)}x_2 + \cdots + a_{nn}^{(2)}x_n &= b_n^{(2)} \end{aligned}$$

(k+1) generation

$$\forall i, j = k+1, k+2, \cdots, n \qquad m_{ik} = \frac{a_{i1}^{(k)}}{a_{kk}^{(k)}}$$

$$a_{ij}^{(k+1)} = a_{ij}^{(k)} - m_{ik}a_{kj}^{(k)}$$

$$b_i^{(k+1)} = b_i^{(k)} - m_{ik}b_k^{(k)}$$

$$\begin{aligned} u_{11}x_1 + u_{12}x_2 + \cdots + u_{1n}x_n &= g_1 \\ u_{22}x_2 + \cdots + u_{2n}x_n &= g_2 \\ &\vdots \\ u_{n-1,n-1}x_{n-1} + u_{n-1,n}x_n &= g_{n-1} \\ u_{nn}x_n &= g_n \end{aligned}$$

$$u_{ij} = a_{ij}^{(i)}$$

$$g_i = b_i^{(i)}$$

Solution by back substitution: $$x_i = \frac{g_i - \sum_{j=i+1}^{n} u_{ij}x_j}{u_{ii}}$$

Pivot $u_{kk} \neq 0$

Example contd.

$$\begin{aligned} -3x_1 + 2x_2 - x_3 &= -1 \\ 6x_1 - 6x_2 + 7x_3 &= -7 \\ 3x_1 - 4x_2 + 4x_3 &= -6 \end{aligned} \Longrightarrow \begin{pmatrix} -3 & 2 & -1 \\ 6 & -6 & 7 \\ 3 & -4 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ -7 \\ -6 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} -3 & 2 & -1 & -1 \\ 6 & -6 & 7 & -7 \\ 3 & -4 & 4 & -6 \end{array}\right) \xrightarrow[\substack{R1=R1 \\ R2=2R1+R2 \\ R3=R1+R3}]{} \left(\begin{array}{ccc|c} -3 & 2 & -1 & -1 \\ 0 & -2 & 5 & -9 \\ 0 & -2 & 3 & -7 \end{array}\right) \xrightarrow[R3=(-1)R2+R3]{} \left(\begin{array}{ccc|c} -3 & 2 & -1 & -1 \\ 0 & -2 & 5 & -9 \\ 0 & 0 & -2 & 2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} -3 & 2 & -1 & -1 \\ 0 & -2 & 5 & -9 \\ 0 & 0 & -2 & 2 \end{array}\right) \xrightarrow[R3=R3/-2]{} \left(\begin{array}{ccc|c} -3 & 2 & -1 & -1 \\ 0 & -2 & 5 & -9 \\ 0 & 0 & 1 & -1 \end{array}\right) \xrightarrow[\substack{R1=R1+R3 \\ R2=R2-5R3}]{} \left(\begin{array}{ccc|c} -3 & 2 & 0 & -2 \\ 0 & -2 & 0 & -4 \\ 0 & 0 & 1 & -1 \end{array}\right) \xrightarrow[R2=R2/-2]{} \left(\begin{array}{ccc|c} -3 & 2 & 0 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} -3 & 2 & 0 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right) \xrightarrow[R1=R1-2R2]{} \left(\begin{array}{ccc|c} -3 & 0 & 0 & -6 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right) \xrightarrow[R1=R1/-3]{} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

Row Reduced Echelon Form (RREF)

- Each non-zero row has 1 as the first entry = leading entry=**pivot**
- All column entries above and below the pivot is zero
- Pivot(=1) is the only entry in its column
- The leading element of a row is the leftmost of all leading elements in rows below
- All zero rows are at the bottom

$$\begin{pmatrix} 1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 5 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Row Reduced Echelon Form (RREF) - Gauss Jordan Algorithm

- **Start** j=1, k=1 equivalently j=N, k=N
- If $a_{jk} = 0$ swap j row with some other row such that $a_{jk} \neq 0$
- Divide j row by a_{jk} such that $a_{jk} = 1$ (pivot)
- If all entries in the column are zero k=k+1
- Else eliminate all entries in the k column by suitable multiples of j row from other rows
- j=j+1 k=k+1. **Repeat**

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{pmatrix} \xrightarrow[\substack{R2=R2-4R1 \\ R3=R3-6R1}]{} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 0 & -5 & -10 & -15 \end{pmatrix} \xrightarrow[\substack{R2=R2/-3 \\ R3=R3/-5}]{} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{pmatrix} \xrightarrow[\substack{R1=R1-2R2 \\ R3=R3-R2}]{} \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Every matrix has an unique Row Reduced Echelon Form (RREF)