Computer Graphics Written Assignment 2

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Given a set of finite points, we want to generate a surface passing through these points by writing an implicit function. We use **Poisson Surface Reconstruction** method for doing this

1 General Idea

We approach the problem of surface reconstruction using an implicit function framework. The input data is a set of points S ($p_i \in$ S) and their associated inward facing normal n_i. We compute a 3D indicator function χ, which is defined as 1 at points inside the surface and 0 at the points outside. Mathematically,

$$\chi_M(p) = \begin{cases} 1 & p \in M \\ 0 & p \notin M \end{cases} \tag{1}$$

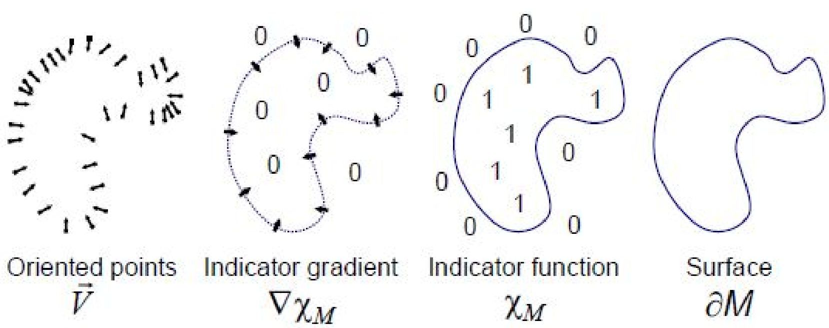

Figure 1: Intuitive illustration of Poisson reconstruction in 2D.

There is an integral relationship between oriented points sampled from the surface of a model and the indicator function of the model. Specifically, the gradient of the indicator function is a vector field that is zero almost everywhere (since the indicator function is constant almost everywhere), except at points near the surface, where it is equal to the inward surface normal. Thus, the oriented point samples can be viewed as samples of the gradient of the model's indicator function (Figure 1).

The problem of computing the indicator function thus reduces to finding the function

χ whose gradient best approximates a vector field $\vec{V}$ defined by the sample normals n_i's, i.e. $min_\chi \left\| \nabla\chi - \vec{V} \right\|$.

2 Implementation Method

One of the main challenges is that the vector field $\vec{V}$ is generally not analytically integrable, so finding an exact solution is not possible. To find the best least-squares approximate solution, we apply the divergence operator to transform this into a Poisson Problem by computing scalar function χ whose Laplacian (divergence of gradient) equals the divergence of $\vec{V}$,

$$\Delta\chi = \nabla.\nabla\chi = \nabla.\vec{V} \tag{2}$$

where

$$Laplacian\Delta\chi = \frac{\partial^2\chi}{\partial x^2} + \frac{\partial^2\chi}{\partial y^2} + \frac{\partial^2\chi}{\partial z^2} \tag{3}$$

$$Divergence\nabla.V = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \tag{4}$$

Solving the Poisson problem We define a space of functions with high resolution near the surface of the model and coarser resolution away from it, express the vector field $\vec{V}$ as a linear sum of functions in this space, set up and solve the Poisson equation, and extract an isosurface of the resulting indicator function.

We use an adaptive octree(An octree is a tree data structure in which each internal node has exactly eight children) both to represent the implicit function and to solve the Poisson system. Specifically, we use the positions of the sample points to define an octree O and associate a function F_o to each node $o \in O$ of the tree, choosing the tree and the functions so that the following conditions are satisfied:

1. The vector field $\vec{V}$ can be represented as the linear sum of the F_o.
2. The matrix representation of the Poisson equation, expressed in terms of the F_o's can be solved efficiently.
3. A representation of the indicator function as the sum of the F_o's can be precisely and efficiently evaluated near the surface of the model.

Defining the function space Given a set of point samples and a maximum tree depth D, we define the octree O to be the minimal octree with the property that every point sample falls into a leaf node at depth D.
Next, we define a space of functions obtained as the span of translates and scales of a

fixed, unit-integral, base function $F : R^3 \rightarrow R$. For every node $o \in O$, we set F_o to be the unit-integral "node function" centered about the node o and stretched by the size of o -

$$F_o(q) = F(\frac{q - c_o}{w_o})(\frac{1}{{w_o}^3}) \tag{5}$$

where c_o and w_o are the center and width of node o.
This space of functions $\zeta_{O,F} \equiv Span\{F_o\}$ has a multiresolution structure similar to that of traditional wavelet representations. Finer nodes are associated with higher-frequency functions, and the function representation becomes more precise as we near the surface.

Selecting a base function In selecting a base function F, our goal is to choose a function so that the vector field $\vec{V}$, can be precisely and efficiently represented as the linear sum of the node functions F_o. Thus, We set F to be the n-th convolution of a box filter with itself resulting in the base function F as:

$$F(x, y, z) = (B(x)B(y)B(z))^{*n} \tag{6}$$

where,

$$B(t) = \begin{cases} 1 & |t| < 0.5 \\ 0 & otherwise \end{cases} \tag{7}$$

Note that as n is increased, F more closely approximates a Gaussian and its support grows larger.

Vector Field Definition We define our approximation to the gradient field of the indicator function as:

$$\vec{V}(q) = \sum_{p_i \in S} \sum_{o \in Nbr_D(p_i)} \alpha_{o,s} F_o(q) n_i \tag{8}$$

where $Nbr_D(s)$ are the eight depth-D nodes closest to p_i and $\{\alpha_{o,s}\}$ are the trilinear interpolation weights.

Poisson Solution Having defined the vector field $\vec{V}$, we would like to solve for the function $\hat{\chi} \in \zeta_{O,F}$ such that the gradient of $\hat{\chi}$ is closest to $\vec{V}$, i.e. a solution to the Poisson equation $\Delta\hat{\chi} = \nabla.\vec{V}$
We can simplify the problem by solving for the function $\hat{\chi}$ minimizing:

$$\sum_{o \in O} \left\| \langle \Delta\hat{\chi} - \nabla.\vec{V}, F_o \rangle \right\|^2 = \sum_{o \in O} \left\| \langle \Delta\hat{\chi}, F_o \rangle - \langle \nabla.\vec{V}, F_o \rangle \right\|^2 \tag{9}$$

To express this in matrix form, let $\hat{\chi} = \sum_o o x_o F_o$, so that we are solving for the vector $x \in \mathbb{R}^{|O|}$. Then, let us define the $|O| \times |O|$ matrix L such that Lx returns the dot product of the Laplacian with each of the F_o. Specifically, for all $o, o' \in O$, the (o, o')-th entry of L is set to:

$$L_{o,o'} = \langle \frac{\partial^2 F_o}{\partial x^2}, F'_o \rangle + \langle \frac{\partial^2 F_o}{\partial y^2}, F'_o \rangle + \langle \frac{\partial^2 F_o}{\partial z^2}, F'_o \rangle \tag{10}$$

Thus, solving for $\hat{\chi}$ amounts to finding

$$\min_{x \in \mathbb{R}^{|O|}} \|Lx - v\|^2 \tag{11}$$

In order to obtain a reconstructed surface $\partial\hat{M}$, it is necessary to first select an isovalue and then extract the corresponding isosurface from the computed indicator function. We choose the isovalue so that the extracted surface closely approximates the positions of the input samples. We do this by evaluating $\hat{\chi}$ at the sample positions and use the average of the values for isosurface extraction:

$$\partial\hat{M} = \{q \in \mathbb{R}^3 | \hat{\chi}(q) = \gamma\}, \gamma = \frac{1}{|S|} \sum_{p_i \in S} \hat{\chi}(p_i) \tag{12}$$

This choice of isovalue has the property that scaling $\hat{\chi}$ does not change the isosurface. Thus, knowing the vector field $\vec{V}$ up to a multiplicative constant provides sufficient information for reconstructing the surface.

3 Algorithm and Examples

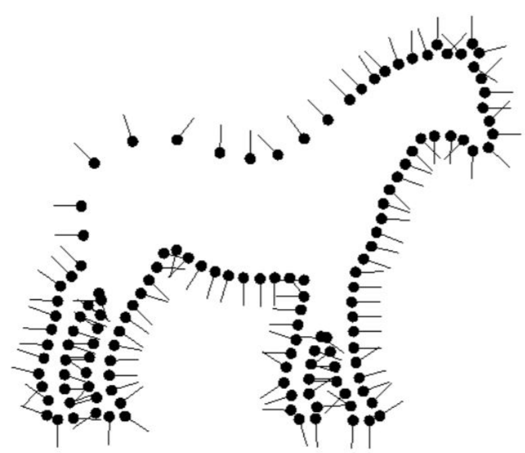

Figure 2: Given points with their normals

Given the points:

1. Set octree

2. Compute vector field

- Define a function space
- Splat the samples

3. Compute indicator function
4. Extract iso-surface

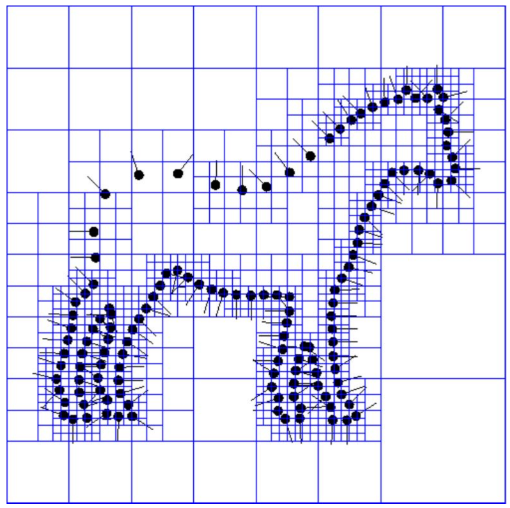

Figure 3: After setting Octree

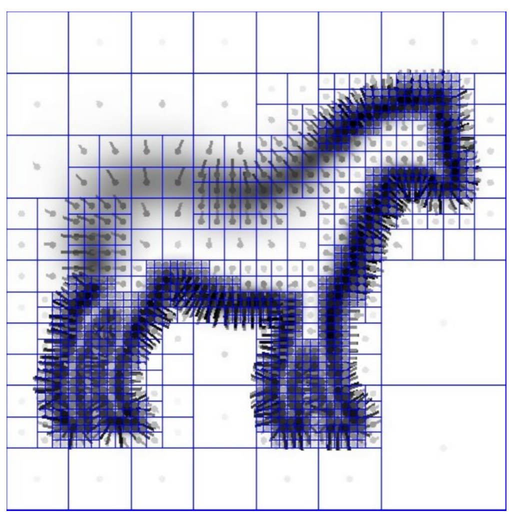

Figure 4: After computing the vector field

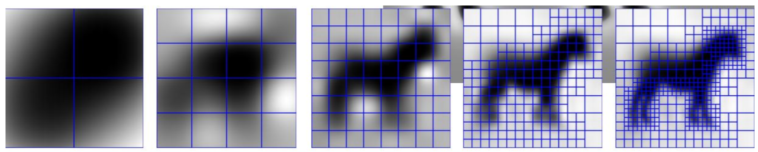

Figure 5: Steps for computing indicator function using Poisson equation

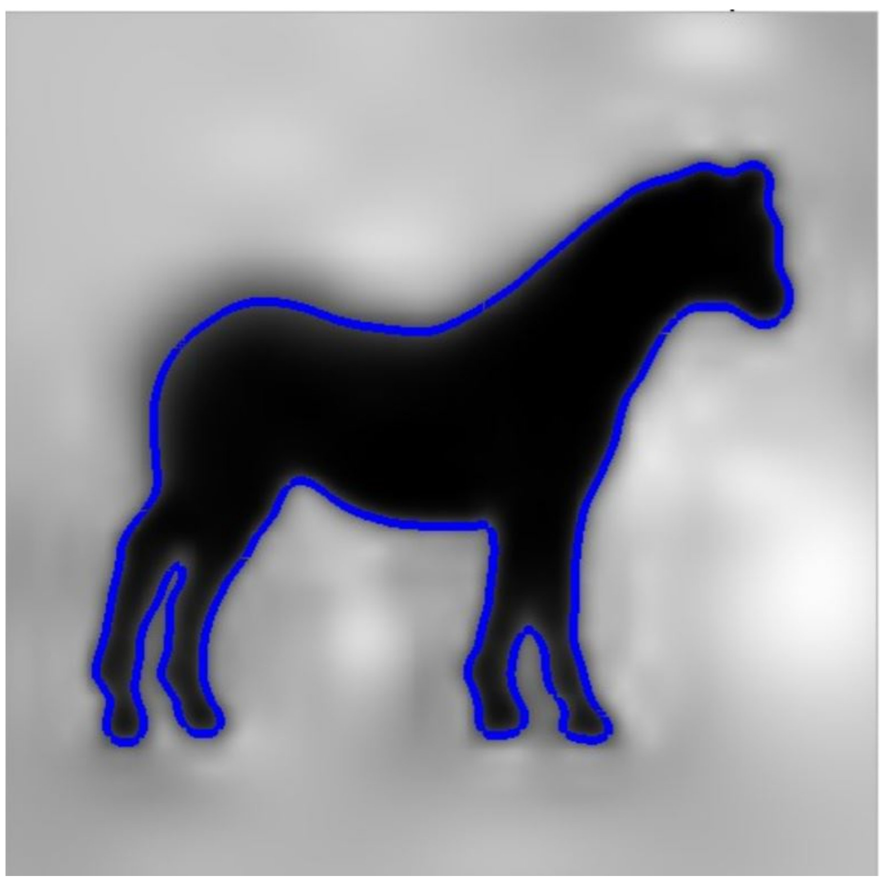

Figure 6: Final surface produced after applying all the steps of the algorithm