Quaternions

Computer Graphics Written Assignment 3

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1 Introduction

The quaternions are members of a noncommutative division algebra first invented by William Rowan Hamilton on 16 October 1843 in Dublin, Ireland. The story behind it is an interesting one. Hamilton was on his way to the Royal Irish Academy with his wife and as he was passing over the Royal Canal on the Brougham Bridge he made a dramatic realization that he immediately carved this equation into the stone of the bridge.

$$i^2 = j^2 = k^2 = ijk = -1 \tag{1}$$

Simply speaking, the quaternions are a number system that extends the complex numbers.

Quaternions form an interesting algebra where each object contains 4 scalar variables (sometimes known as Euler Parameters not to be confused with Euler angles), these objects can be added and multiplied as a single unit in a similar way to the usual algebra of numbers. However, there is a difference, unlike the algebra of scalar numbers qa * qb is not necessarily equal to qb * qa (where qa and qb are quaternions). In mathematical terms, quaternion multiplication is not commutative.

Quaternions have 4 dimensions (each quaternion consists of 4 scalar numbers), one real dimension and 3 imaginary dimensions. Each of these imaginary dimensions has a unit value of the square root of -1, but they are different square roots of -1 all mutually perpendicular to each other, known as i,j and k.

2 Definition

The general form to express quaternions is

$$q = \{s + xi + yj + zk : s, x, y, z \in \mathbb{R}\} \tag{2}$$

Where, according to Hamilton's famous expression:

$$i^2 = j^2 = k^2 = ijk = -1 \tag{3}$$

and,

$$ij = k = -ji \tag{4a}$$
$$jk = i = -kj \tag{4b}$$
$$ki = j = -ik \tag{4c}$$

The properties of i, j and k are very similar to the cross product rules and can be visualized from the image below:

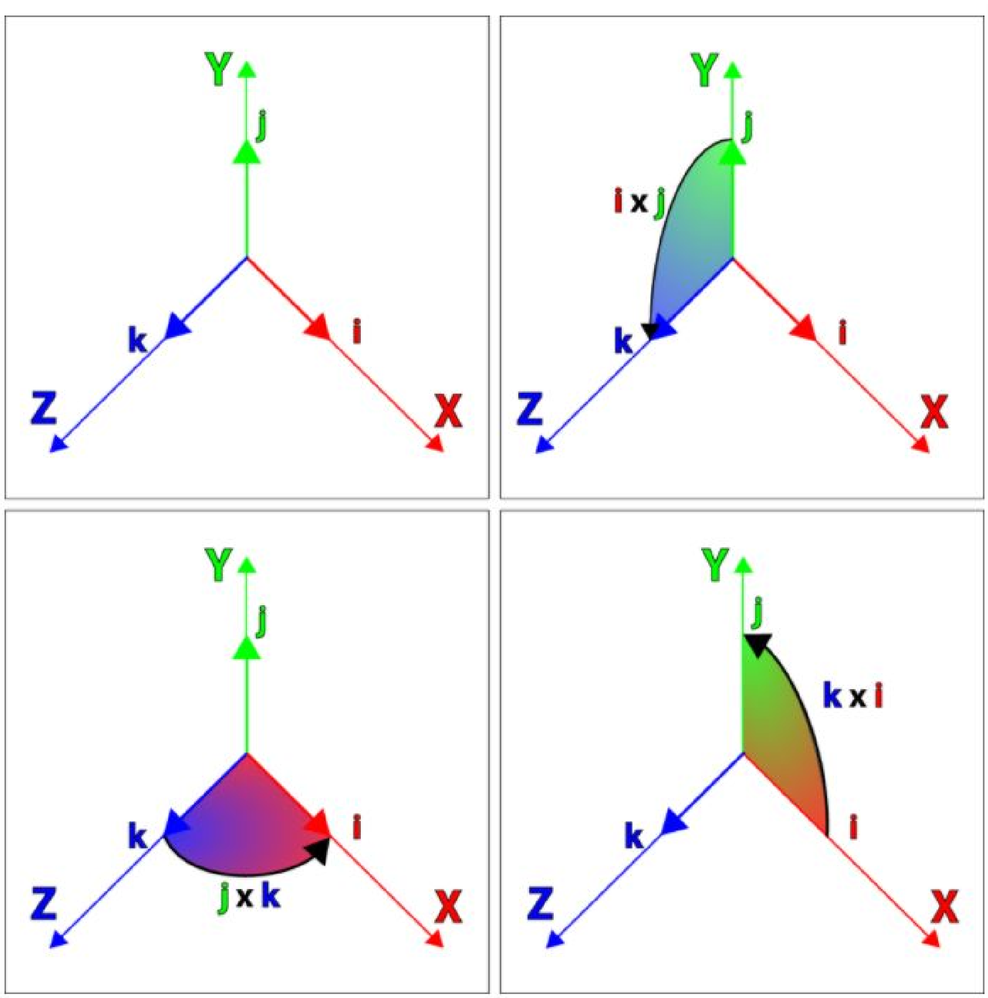


Figure 1: Visualizing the Properties of ij, jk, ki

The quaternion is generally written in the form of $q = s + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}$, as consisting of a scalar part and a vector part. The quaternion $b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}$ is called the vector part (sometimes imaginary part) of q, and s is the scalar part (sometimes real part) of q.

Thus quternion can also be expressed as:

$$q = s + \vec{v} \tag{5}$$

where s is a scalar number and $\vec{v}$ is a vector representing an axis.

We can also represent quaternions as an ordered pair:

$$q = [s, \mathbf{v}] : s \in \mathbb{R}, \mathbf{v} \in \mathbb{R}^3 \tag{6}$$

where $\mathbf{v}$ can also be represented by its individual components:

$$q = [s, x\mathbf{i} + y\mathbf{j} + z\mathbf{k}] : s, x, y, z \in \mathbb{R} \tag{7}$$

3 Quaternion Algebra

3.1 Real Quaternion

A Real Quaternion is a quaternion with a vector term of $\mathbf{0}$:

$$q = s + \vec{0} \tag{8}$$

3.2 Pure Quaternion

Similar to Real Quaterions, Hamilton also defined the Pure Quaternion as a quaternion that has a zero scalar term:

$$q = 0 + \vec{v} \tag{9}$$

3.3 Adding and Subtracting Quaternions

The quaternions are added and subtracted similar to complex numbers i.e the real part and the imaginary part are added and subtracted separately.

$$q_a = s_a + \vec{a} \tag{10}$$

$$q_b = s_b + \vec{b} \tag{11}$$

$$q_a \pm q_b = (s_a \pm s_b) + (\vec{a} \pm \vec{b}) \tag{12}$$

3.4 Quaternion Products

Quaternions can be multiplied among themselves. Again, the process is quite similar to the multiplication in complex numbers.

$$\begin{aligned} q_a q_b &= (s_a + \vec{a})(s_b + \vec{b}) \\ &= (s_a s_b - \vec{a}.\vec{b}) + (s_a\vec{b} + s_b\vec{a} + \vec{a} \times \vec{b}) \end{aligned} \tag{13}$$

3.5 Scalar Multiplication

Just like vectors and matrices, a quaternion can be multiplied by a scalar. Mathematically, this is represented as follows:

$$q = s + \vec{v} \tag{14}$$

$$kq = k(s + \vec{v}) \tag{15}$$

$$kq = ks + k\vec{v} \tag{16}$$

where k is a scalar real number.

3.6 Normalization

Mathematically, the norm of a quaternion is represented as follows:

$$\|q\| = \sqrt{s^2 + v^2} \tag{17}$$

3.7 Conjugate

The conjugate of a quaternion is calculated in the same way as it is done for complex numbers. The conjugate of a quaternion is very important in computing the inverse of a quaternion. Mathematically, the conjugate of quaternion q is calculated as follows: Given quaternion q as

$$q = s + \vec{v} \tag{18}$$

its conjugate is given as

$$q' = s - \vec{v} \tag{19}$$

4 Quaternions in Computer Graphics

Quaternions are mainly used in computer graphics when a 3D character rotation is involved. Quaternions allows a character to rotate about multiple axis simultaneously, instead of sequentially as matrix rotation allows. For example, to rotate 45 degrees about the xy-axis using matrix rotations, the character must first rotate about the x-axis and then rotate about the y-axis. With quaternions this sequential process is not necessary. There are two reasons why Quaternions are preferred in computer graphics:

1. Matrix rotations suffer from what is known as **Gimbal Lock**. Gimbal lock is the loss of one degree of freedom in a three-dimensional, three-gimbal mechanism that occurs when the axes of two of the three gimbals are driven into a parallel configuration, ”locking” the system into rotation in a degenerate two-dimensional space.

2. Quaternions consume less memory and are faster to compute than matrices.

4.1 Rotations

We will be using the representation of quaternions as ordered pair in this section for convenience. In complex numbers, we defined a special form of the complex number called a **Rotor** that could be used to rotate a point through the 2D complex plane as:

$$q = \cos\theta + i\sin\theta \tag{20}$$

Then by its similarities to complex numbers, it should be possible to express a quaternion that can be used to rotate a point in 3D-space as such:

$$q = [\cos\theta, v\sin\theta] \tag{21}$$

Let's test if this theory holds by computing the product of the quaternion q and the vector $\mathbf{p}$. First, we can express $\mathbf{p}$ as a Pure quaternion in the form:

$$p = [0, \mathbf{p}] \tag{22}$$

And q is a unit-norm quaternion in the form:

$$q = [s, \lambda\hat{\mathbf{v}}] \tag{23}$$

Then,

$$\begin{aligned} p' &= qp \\ &= [s, \lambda\hat{\mathbf{v}}][0, \mathbf{p}] \\ &= [-\lambda\hat{\mathbf{v}} \cdot \mathbf{p}, s\mathbf{p} + \lambda\hat{\mathbf{v}} \times \mathbf{p}] \end{aligned} \tag{24}$$

Let's first consider the "special" case where $\mathbf{p}$ is perpendicular to $\hat{\mathbf{v}}$ in which case, the dot-product term $-\lambda\hat{\mathbf{v}} \cdot \mathbf{p} = 0$ and the result becomes the Pure quaternion:

$$p' = [0, s\mathbf{p} + \lambda\hat{\mathbf{v}} \times \mathbf{p}] \tag{25}$$

In this case, to rotate $\mathbf{p}$ about $\hat{\mathbf{v}}$ we just substitute $s = \cos\theta$ and $\lambda = \sin\theta$.

$$p' = [0, \cos\theta\mathbf{p} + \sin\theta\hat{\mathbf{v}} \times \mathbf{p}] \tag{26}$$

As an example, let's rotate a vector $\mathbf{p}$ 45° about the z-axis then our quaternion q is:

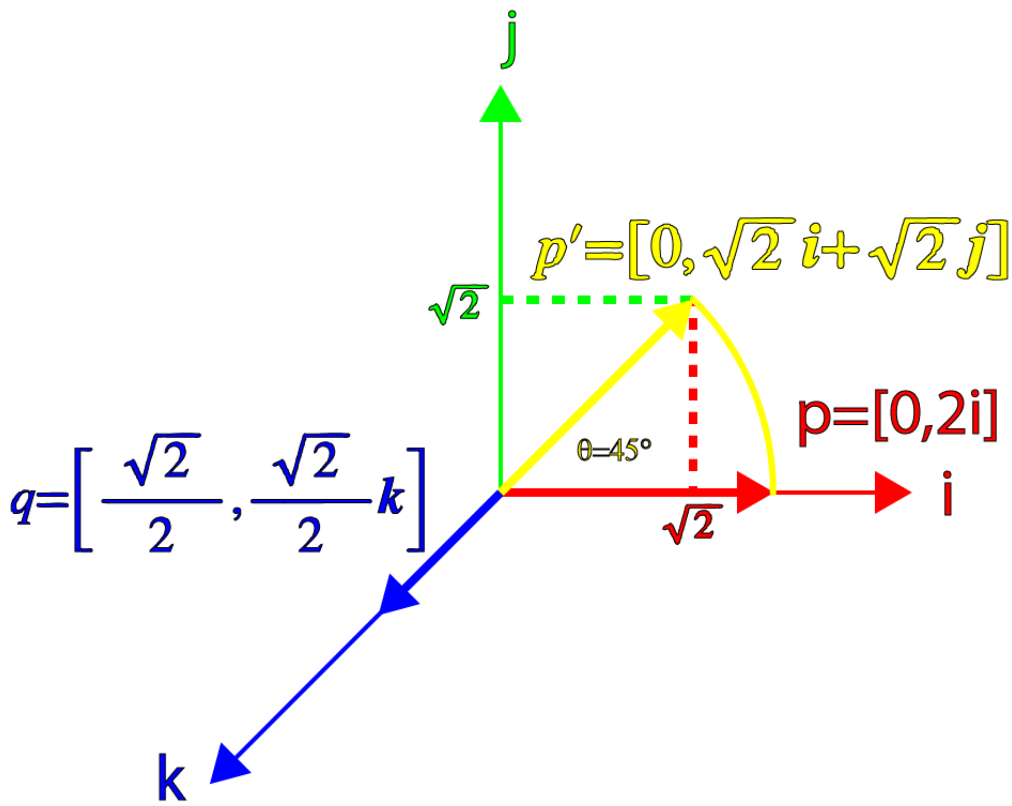
$$\begin{aligned} q &= [\cos\theta, \sin\theta\mathbf{k}] \\ &= \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\mathbf{k}\right] \end{aligned} \tag{27}$$

And let's take a vector $\mathbf{p}$ that adheres to the special case that $\mathbf{p}$ is perpendicular to $\mathbf{k}$:

$$p = [0, 2\mathbf{i}] \tag{28}$$

Now let's find the product of qp:

$$\begin{aligned} p' &= qp \\ &= \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\mathbf{k}\right][0, 2\mathbf{i}] \\ &= \left[0, 2\frac{\sqrt{2}}{2}\mathbf{i} + 2\frac{\sqrt{2}}{2}\mathbf{k} \times \mathbf{i}\right] \\ &= [0, \sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}] \end{aligned} \tag{29}$$



which results in a Pure quaternion that is rotated 45° about the **k** axis, which can be seen in the above image.

Now let's consider a quaternion that is not orthogonal to **p**. If we specify the vector part of our quaternion to 45° offset from **p** we get:

$$\begin{aligned} \hat{\mathbf{v}} &= \tfrac{\sqrt{2}}{2}\mathbf{i} + \tfrac{\sqrt{2}}{2}\mathbf{k} \\ \mathbf{p} &= 2\mathbf{i} \\ q &= [\cos\theta, \sin\theta\hat{\mathbf{v}}] \\ p &= [0, \mathbf{p}] \end{aligned} \tag{30}$$

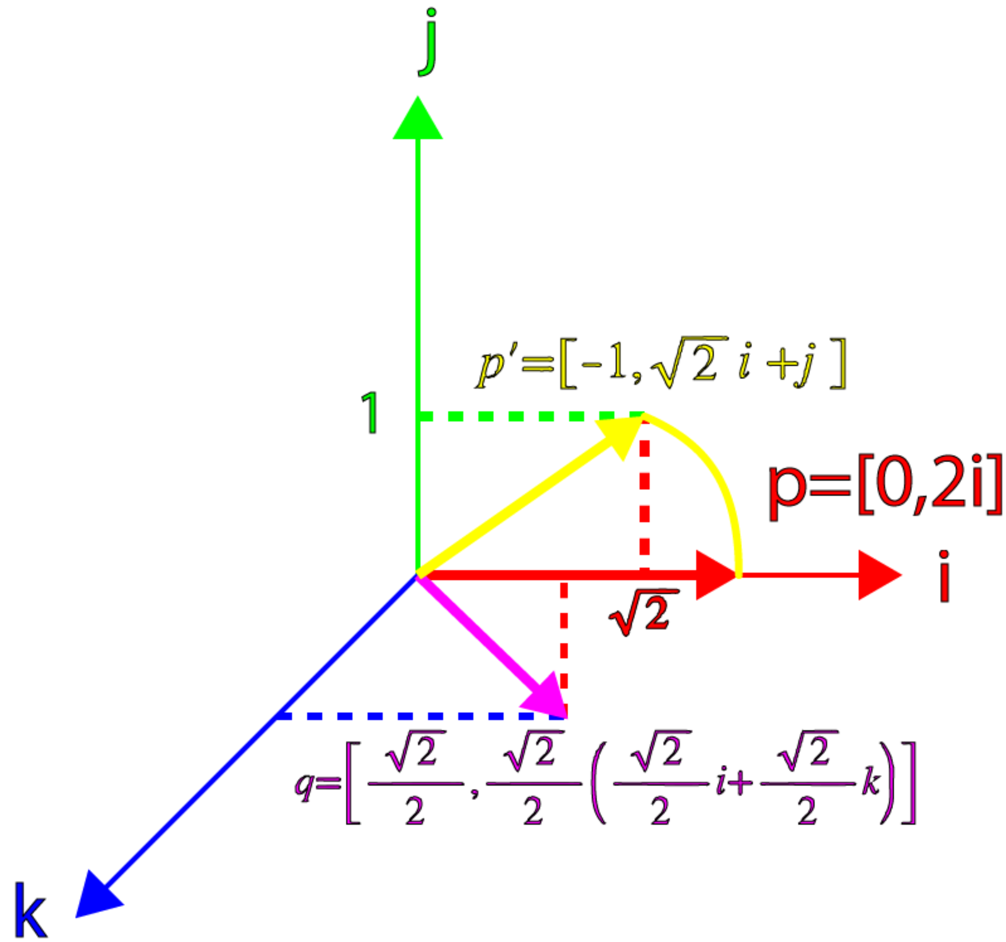
And multiplying our vector **p** by q we get:

$$\begin{aligned} p' &= qp \\ &= [\cos\theta, sin\theta\hat{\mathbf{v}}][0, \mathbf{p}] \\ &= [-\sin\theta\hat{\mathbf{v}}\cdot\mathbf{p}, \cos\theta\mathbf{p} + \sin\theta\hat{\mathbf{v}}\times\mathbf{p}] \end{aligned} \tag{31}$$

And substituting $\hat{\mathbf{v}}$, **p** and $\theta = 45°$ gives:

$$\begin{aligned} p' &= \left[-\tfrac{\sqrt{2}}{2}\left(\tfrac{\sqrt{2}}{2}\mathbf{i} + \tfrac{\sqrt{2}}{2}\mathbf{k}\right)\cdot(2\mathbf{i}), \tfrac{\sqrt{2}}{2}2\mathbf{i} + \tfrac{\sqrt{2}}{2}\left(\tfrac{\sqrt{2}}{2}\mathbf{i} + \tfrac{\sqrt{2}}{2}\mathbf{k}\right)\times 2\mathbf{i}\right] \\ &= [-1, \sqrt{2}\mathbf{i} + \mathbf{j}] \end{aligned} \tag{32}$$

which is no longer a pure quaternion, and it has not been rotated 45° and the vector's norm is no longer 2 (instead it has been reduced to $\sqrt{3}$). This result can be visualized by the image below:



4.2 Use of inverses

After rotation in the previous example, the new quaternion p' no longer was a pure quaternion. Hamilton recognized (but didn't publish) that if we post-multiply the result of qp by the inverse of q then the result is a pure quaternion and the norm of the vector component is maintained.

First, let's compute q^{-1}:

$$\begin{aligned} q &= \left[\cos\theta, \sin\theta\left(\tfrac{\sqrt{2}}{2}\mathbf{i} + \tfrac{\sqrt{2}}{2}\mathbf{k}\right)\right] \\ q^{-1} &= \left[\cos\theta, -\sin\theta\left(\tfrac{\sqrt{2}}{2}\mathbf{i} + \tfrac{\sqrt{2}}{2}\mathbf{k}\right)\right] \end{aligned} \tag{33}$$

For $\theta = 45^\circ$ we have:

$$\begin{aligned} q^{-1} &= \left[\tfrac{\sqrt{2}}{2}, -\tfrac{\sqrt{2}}{2}\left(\tfrac{\sqrt{2}}{2}\mathbf{i} + \tfrac{\sqrt{2}}{2}\mathbf{k}\right)\right] \\ &= \tfrac{1}{2}\left[\sqrt{2}, -\mathbf{i} - \mathbf{k}\right] \end{aligned} \tag{34}$$

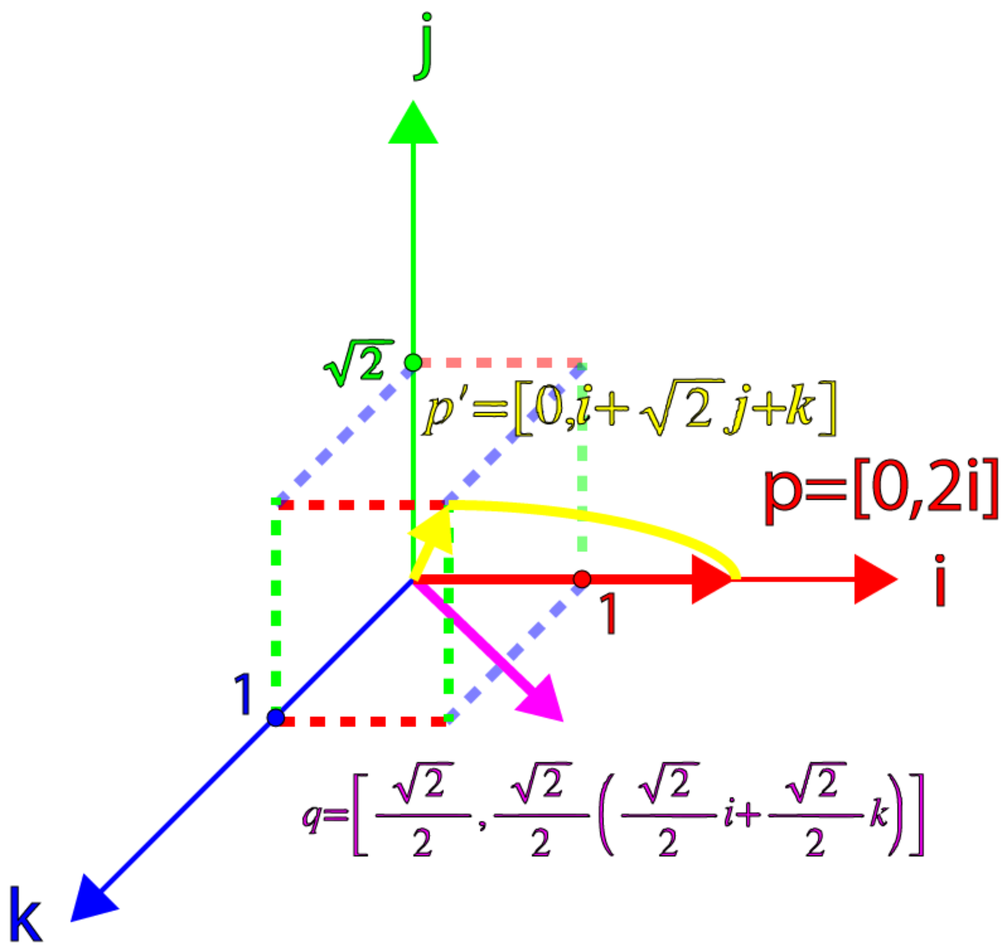
And combining the previous value of qp and q^{-1} gives:

$$\begin{aligned}
qp &= \left[-1, \sqrt{2}\mathbf{i}+\mathbf{j}\right] \\
qpq^{-1} &= \left[-1, \sqrt{2}\mathbf{i}+\mathbf{j}\right] \tfrac{1}{2}\left[\sqrt{2}, -\mathbf{i}-\mathbf{k}\right] \\
&= \tfrac{1}{2}\left[-\sqrt{2}-\left(\sqrt{2}\mathbf{i}+\mathbf{j}\right)\cdot(-\mathbf{i}-\mathbf{k}), \mathbf{i}+\mathbf{k}+\sqrt{2}\left(\sqrt{2}\mathbf{i}+\mathbf{j}\right)-\mathbf{i}+\sqrt{2}\mathbf{j}+\mathbf{k}\right] \\
&= \tfrac{1}{2}\left[-\sqrt{2}+\sqrt{2}, \mathbf{i}+\mathbf{k}+2\mathbf{i}+\sqrt{2}\mathbf{j}-\mathbf{i}+\sqrt{2}\mathbf{j}+\mathbf{k}\right] \\
&= \left[0, \mathbf{i}+\sqrt{2}\mathbf{j}+\mathbf{k}\right]
\end{aligned} \tag{35}$$

which is a pure quaternion and the norm of the result is:

$$\begin{aligned}
|p'| &= \sqrt{1^2+\sqrt{2}^2+1^2} \\
&= \sqrt{4} \\
&= 2
\end{aligned} \tag{36}$$

which is the same as $\mathbf{p}$ so the norm of the vector is maintained. The image below visualizes the result of rotation. So we can see that the result is a pure quaternion and that



the norm of the initial vector is maintained, but the vector has been rotated 90° rather than 45° which is twice as much as desired! So in order to correctly rotate a vector $\mathbf{p}$ by an angle θ about an arbitrary axis $\hat{\mathbf{v}}$, we must consider the half-angle and construct the following quaternion:

$$q = \left[\cos\frac{1}{2}\theta, \sin\frac{1}{2}\theta\hat{\mathbf{v}}\right] \tag{37}$$

which is the general form of a rotation quaternion!

5 Summary

Quaternions are often used in computer graphics (and associated geometric analysis) to represent rotations and orientations of objects in three-dimensional space. They are smaller than other representations such as matrices, and operations on them such as composition can be computed more efficiently. Quaternions also see use in control theory, signal processing, attitude control, physics, and orbital mechanics, mainly for representing rotations/orientations in three dimensions.