INCENTIVES IN COMPUTER SCIENCE

FAIR DIVISION

A. Cake Cutting

- Two-person Cake Cutting

Protocol:

> 1. Player 1 splits the cake into two pieces A and B, such that the player's value for each is exactly half that of the entire cake.
>
> 2. Player 2 picks whichever of A, B she likes better.

Let us assume the cake is the interval [0,1] (in general we can assume the interval to be [a,b] but to make our work simpler we take a=0 and b=1). Now each player 'i' has its valuation function v_i, which specifies the value $v_i(S)$ of the subset S of the cake. For example, let player 1 divide the cake in 4 equal parts, so the subset S can be: [0,0.25], [0.25,0.5], [0.5,0.75], [0.75,1]. Note that the subsets are disjoint (no element common) and their union is equal to the interval [0,1]. Now we make two assumptions about each valuation v_i:

> 1. v_i is normalized. That is, $v_i [0,1] = 1$. In the above example let the 4 subsets be S_1, S_2, S_3 and S_4. Then $v_i [S_1] + v_i [S_2] + v_i [S_3] + v_i [S_4] = 1$.
>
> 2. v_i is additive on disjoint sets. That is, in the above example since all the subsets are disjoint, then $v_i [S_1] + v_i [S_2] = v_i [S_1 \cup S_2]$.

Now let's check whether the cut and choose protocol is strategyproof and Pareto optimal.

> ***Strategyproof (synonym: truthful) (adj.)****: the property of a mechanism that honesty is always the best policy, meaning that lying about your preferences cannot make you better off.*

- Since player 1 splits the cake into two halves A and B so player 2 cannot affect splitting. She has to choose the piece she likes better so she has no incentive to deviate.
- Now let's assume the cake has a strawberry, the first player likes all parts of the cake equally, while the second player really cares about the strawberry. If player 1 has some information about the second's player valuation function, the first player could split the cake into strawberry and the rest, knowing that the second player would take the strawberry, leaving a very valuable piece for the first player.
- If the first player doesn't know anything about what the second player wants, and assumes that the second player will always leave the piece that is worse for the first player, then the first player is incentivized to follow the protocol (to guarantee herself a piece with value ½).
- In any case, the cut and choose protocol is not strategyproof.

> ***Pareto optimal (adj.)****: the property of an outcome that you can't make anyone better off without making someone else worse off.*

- Now let's assume the cake is on the x-y plane with centre at origin and the first player likes the cake in first and second quadrant while the second players like the third and fourth quadrant of the cake.
- Splitting the cake about the y-axis will result in both players getting a piece valued at ½ whereas splitting it around the x-axis will result in both players getting a piece valued at one.

  So, we conclude that the cut and choose protocol is neither strategyproof nor Pareto optimal.

Can we say something about "fairness"? First of all, let us understand what fairness is.

One possible definition of fairness would be that both players wind up equally happy. But this property is also not satisfied by the cut and choose protocol: the first player is guaranteed to get a piece that she values at ½, while the second player might well end up with a piece that she values at greater than ½.

> ***Definition 1.1*** *An allocation A1, A2, . . ., An of cake to n players is proportional if*
>
> $$v_i(Ai) \geq 1/n$$
>
> *for every player 'i'.*

The cut and choose protocol satisfies this property as the first player gets the piece that she values at ½ and the second player will always choose the better piece from the two and so $v_2(A_2) \geq ½$.

> ***Defintion 1.2*** *An allocation A1, A2, . . ., An of cake to n players is envy-free if*
>
> $$v_i(Ai) \geq v_i(Aj)$$
>
> *for every pair 'i', 'j' of players.*

This means every player likes her piece more than others piece, that is, no player wants to trade pieces with any other player.

The second definition itself implies the first one. To see this, recall our first assumption i.e., for any player i, the summation over j=1 to j=n i.e., $\sum_j v_i(A_j) = v_i([0,1]) = 1$. Now if player 'i' likes A_i better than any other piece it must imply $v_i(A_i) \geq 1/n$.

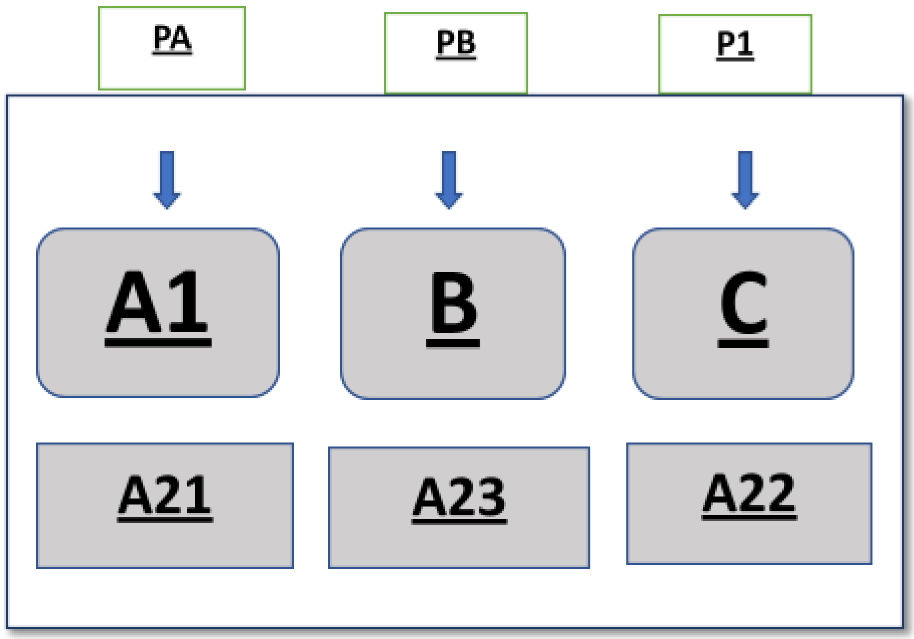
Cake cutting beyond two players

- Selfridge-Conway protocol (for three players):

  1. Suppose we have three players P1, P2 and P3. P1 divides the cake into three pieces he considers of equal piece.
  2. Let's call A the largest piece according to P2.
  3. P2 cuts off a bit of A to make it the same size as the second largest. Now A is divided into: the trimmed piece A1 and the trimmings A2. Leave the trimmings A2 to the side for now.
     - If P2 thinks that the two largest parts are equal (such that no trimming is needed), then each player chooses a part in this order: P3, P2 and finally P1 and we are done here.
  4. P3 chooses a piece among A1 and the two other pieces.
  5. P2 chooses a piece with the limitation that if P3 didn't choose A1, P2 must choose it.
  6. P1 chooses the last piece leaving just the trimmings A2 to be divided.

It remains to divide the trimmings A2. The trimmed piece A1 has been chosen by either P2 or P3; let's call the player who chose it PA and the other player PB.

1. PB cuts A2 into three equal pieces.
2. PA chooses a piece of A2 - we name it A21.
3. P1 chooses a piece of A2 - we name it A22.
4. PB chooses the last remaining piece of A2 - we name it A23.



B. Rent Division: Fair division in practice

- One place where fair division protocols are used in practice is the rent division problem, where there are n people, n rooms, and a rent of R. The goal is to assign people and rents to rooms, with one person per room and with the sum of rents equal to R, in the "best" way possible.
- We assume that each person i has a value v_{ij} for each room j, and normalize these values so that $\sum_j v_{ij}$ = R. (In effect, we force each player to acknowledge the constraint that the entire rent must get paid). We assume that each player 'i' wants to maximize v_{ij} minus the rent paid for her room j. It enables envy-free solutions and is reasonable in this context.
- A solution to a rent division problem is envy-free if

$$\boxed{v_{i\,f(i)} - r_{f(i)} \geq v_{i\,f(j)} - r_{f(j)}}$$

for every pair i, j of players, where f(i) denotes the room to which i is assigned and r_j denotes the rent assigned to the room j. That is, no one wants to trade places with anyone else (where trading places means swapping both rooms and rents).

- The good news is that an envy-free solution is guaranteed to exist, and that one can be computed efficiently.
- The bad news is that there can be many envy-free solutions, and not all of them are reasonable.
- For example, let there be 3 rooms and 3 persons and that the total rent is 1500. Now if 1st player only wants 1st room, 2nd person only wants room 2 and 3rd person only wants room 3, i.e.
    1. $V_{1\,1} = 1500$, $V_{1\,2} = 0$, $V_{1\,3} = 0$.
    2. $V_{21} = 0$, $V_{2\,2} = 1500$, $V_{2\,3} = 0$.
    3. $V_{3\,1} = 0$, $V_{3\,2} = 0$, $V_{3\,3} = 1500$.
- The only reasonable room assignment is to give each person the room that they want. Intuitively, by symmetry, each person should pay 500 in rent. But every division of the rent is **envy-free**! Because $v_{i\,f(i)} - r_{f(i)} \geq 0$ whereas $v_{i\,f(j)} - r_{f(j)} < 0$.
- So even if you make the first person pay almost 1500 for her room, she still doesn't want to swap with the other person.
- So, we need a method for selecting one out of the many envy-free solutions. One of the methods is discussed below:
    1. Choose the room assignment f to maximize $\sum_i V_{i\,f(i)}$.
    2. Set the room rents so that envy-freeness holds, and subject to this, maximize the minimum utility:

$$\boxed{\max\left(\min\left(v_{i\,f(i)} - r_{f(i)}\right)\right)}$$