DATA SCIENCE & ARTIFICIAL INTELLIGENCE & CS & IT

Linear Algebra

Lecture No. 01

By- Dr. Puneet Sharma Sir

Topics to be Covered

Topic

→ BASICS of Determinants

Start: 10th March, END - 31st MAY, 7:00 - 9:30 AM (SUNDAY off)

Syllabus	ESE	GATE EC	EE	IN	CE	ME	CH	PI	XE
1 Linear Algebra	✓	✓							
2 Calculus	✓	✓							
3 V. Calculus	✓	✓							
4 Complex	✓	Not for CE							
5 D. Equ^n	✓	✓							
6 N Tech	✓	✓							
7 LT & FS	✓	Not for EC/EE							
8 Prob & Stats	✓	✓							

Tel: dr puneet sir pw

Strategy:

1. Live Class.
2. Revision
3. Short Notes (in Later Phase)
4. D.P.P
5. Chapterwise test (Sund)
6. P.Y.Q. → Judge

Don't Judge yourself before solving P.Y.Q.

Book: → No Book is Needed. only PYQ Book is Required.

eg. L-Algebra: Class (150-200Q) | DPP (70-90) | WT (13Q) | OTS 100Q | PYQ (200Q) → 700-800 Q

Doubts:
- → Conceptual Doubts → you can ask anytime.
- → Generic Doubt → will be discussed after class.

Parachute Landing → Conceptual Doubts are also not allowed.

PREREQUISITE of Engg Maths: → ✓ (25 Lectures)

(Foundation Series of Engg Maths)

* Engg Maths — CS: 7-8/100 — DSAI: 40+/100

* Maths: is the Language of Engg.

* Maths
  - Language → understanding of symbols.
  - Concept
    - Information Based Concept (No Brain should be used)
    - Analysis Based Concept (Pure Maths) (useful to develop Neurons)
    - Application Based Concept (Applied Maths in LIVE Class)

* M.M. Imp Point → Try to have Patience as much as possible

Determinants

$$\begin{vmatrix} + & - \\ - & + \end{vmatrix}, \quad \begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}, \quad \begin{vmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{vmatrix}$$

$$\boxed{\text{sign of } a_{ij} = (-1)^{i+j}} \quad \forall i \,\&\, j$$

eg sign of $a_{23} = (-1)^{2+3} = -ve$

eg sign of $a_{42} = (-1)^{4+2} = +ve$

(*) $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

eg $|A| = \begin{vmatrix} 2 & -4 \\ 5 & 3 \end{vmatrix} = (2)(3) - (5)(-4) = 6 + 20 = 26$

eg $|A| = \begin{vmatrix} 2+3i & -i \\ i & 2-3i \end{vmatrix} \Rightarrow = (2+3i)(2-3i) - (i)(-i)$

$= 2^2 - (3i)^2 + i^2$

$= 4 - 9i^2 + i^2 = 4 - 9(-1) + (-1) = 4 + 9 - 1 = 12$

where $i = \sqrt{-1}$

eg $|A| = \begin{vmatrix} 2 & -3 & 4 \\ -1 & 2 & 0 \\ 3 & 4 & 1 \end{vmatrix} = ? = -(-1)\begin{vmatrix} -3 & 4 \\ 4 & 1 \end{vmatrix} + (2)\begin{vmatrix} 2 & 4 \\ 3 & 1 \end{vmatrix} - (0)\begin{vmatrix} 2 & -3 \\ 3 & 4 \end{vmatrix}$

$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$

$= 1[-3-16] + 2[2-12] - 0 = -39$ Ans

Shortcut (SARRUS Method) →

$$\begin{matrix} 2 & -3 & 4 & 2 & -3 \\ -1 & 2 & 0 & -1 & 2 \\ 3 & 4 & 1 & 3 & 4 \end{matrix}$$

$|A| = (4+0+(-16)) - (24+0+3) = -39$

eg $|A| = \begin{vmatrix} 1 & 4 & 16 \\ 1 & 3 & 9 \\ 1 & 2 & 4 \end{vmatrix} = ? \Rightarrow \begin{matrix} 1 & 4 & 16 & 1 & 4 \\ 1 & 3 & 9 & 1 & 3 \\ 1 & 2 & 4 & 1 & 2 \end{matrix} \Rightarrow |A| = [12+36+32] - [48+18+16]$

$= 80 - 82 = -2$

M-II $|A| = \begin{vmatrix} 1 & 4 & 4^2 \\ 1 & 3 & 3^2 \\ 1 & 2 & 2^2 \end{vmatrix} = ? = (4-3)(3-2)(2-4) = (1)(+1)(-2) = -2$

Q. $|A| = \begin{vmatrix} 1 & -2 & 3 & 0 \\ 0 & 1 & 2 & 4 \\ -1 & 2 & 0 & 3 \\ 0 & 2 & -2 & 1 \end{vmatrix} = ? = (+1)\begin{vmatrix} 1 & 2 & 4 \\ 2 & 0 & 3 \\ 2 & -2 & 1 \end{vmatrix} - (0)\Big| \; ? \; \Big| + (-1)\begin{vmatrix} -2 & 3 & 0 \\ 1 & 2 & 4 \\ 2 & -2 & 1 \end{vmatrix} - (0)\Big| \; ? \; \Big|$

$$\begin{vmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{vmatrix}$$

$$= 1\left[-2\{2+8\} - 3\{-2-4\}\right] - 0 - (1)\left[(-2)\{2+8\} - 3\{1-8\}\right] - 0$$

$$= -20 + 18 - [-20 + 21] = -2 - 1 = -3$$

Q. $|A| = \begin{vmatrix} -3 & 1 & 1 & 2 \\ 1 & -3 & 1 & 2 \\ 1 & 1 & -3 & 2 \\ 2 & 1 & 1 & -3 \end{vmatrix} = ?$

M-I Directly open it using conventional Approach.

(TIME CONSUMING APP)

M-II $\xrightarrow{C_1 \to C_1 + (C_2 + C_3 + C_4)} \begin{vmatrix} 1 & 1 & 1 & 2 \\ 1 & -3 & 1 & 2 \\ 1 & 1 & -3 & 2 \\ 1 & 1 & 1 & -3 \end{vmatrix} \xrightarrow[\substack{R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1}]{R_2 \to R_2 - R_1} \begin{vmatrix} 1 & 1 & 1 & 2 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -5 \end{vmatrix} = 1\left[(-4)(-4)(-5)\right]$

$= -80$

The determinant value of the matrix

$$\begin{vmatrix} 13 & 2 & 1 & 3 \\ 31 & 4 & 5 & 6 \\ 26 & 3 & 7 & 4 \\ 10 & 1 & 3 & 2 \end{vmatrix}$$ is _____.

(a) 55 (b) 101

(c) 126 (d) -10

$$\xrightarrow{C_2 \to C_2 - C_4} \begin{vmatrix} 13 & -1 & 1 & 3 \\ 31 & -2 & 5 & 6 \\ 26 & -1 & 7 & 4 \\ 10 & -1 & 3 & 2 \end{vmatrix} \xrightarrow[\substack{R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1}]{R_2 \to R_2 - 2R_1} \begin{vmatrix} 13 & -1 & 1 & 3 \\ 5 & 0 & 3 & 0 \\ 13 & 0 & 6 & 1 \\ -3 & 0 & 2 & -1 \end{vmatrix}$$

$$= -(-1) \begin{vmatrix} 5 & 3 & 0 \\ 13 & 6 & 1 \\ -3 & 2 & -1 \end{vmatrix}$$

$$= (+1)\left[+5\{-6-2\} - 3\{-13+3\}\right]$$

$$= +\left[-40+30\right] = -10$$

Q. $|A| = \begin{vmatrix} 3 & -3 & 0 & 0 & 0 \\ 0 & 3 & -3 & 0 & 0 \\ 0 & 0 & 3 & -3 & 0 \\ 0 & 0 & 0 & 3 & -3 \\ -3 & 0 & 0 & 0 & 3 \end{vmatrix}_{5\times 5} = \underline{?=0}$

$\xrightarrow{C_1 \to C_1 + (C_2 + C_3 + C_4 + C_5)} \begin{vmatrix} 0 & - & - & - & - \\ 0 & - & - & - & - \\ 0 & - & - & - & - \\ 0 & - & - & - & - \\ 0 & - & - & - & - \end{vmatrix}_{5\times 5}$ = Expanding along C_1

$= 0$

Q. $|A| = \begin{vmatrix} 2 & -1 & 2 & 3 \\ -3 & 2 & -3 & 2 \\ 4 & 1 & 4 & 1 \\ 0 & 4 & 0 & 4 \end{vmatrix}_{4\times 4} = ? \xrightarrow{C_1 \to C_1 - C_3} \begin{vmatrix} 0 & - & - & - \\ 0 & - & - & - \\ 0 & - & - & - \\ 0 & - & - & - \end{vmatrix} = 0$

Elementary Operations →

E-Row Operations

(i) $R_i \leftrightarrow R_j$

(ii) $R_i \to kR_i$

(iii) $R_i \to R_i + kR_j$

E-Column Operations

(i) $C_i \leftrightarrow C_j$

(ii) $C_i \to kC_i$

(iii) $C_i \to C_i + kC_j$

$R_3 \to R_3 + 5R_2$ (E-operation of 3rd type)

$R_3 \to 5R_3 + R_2$ (it is not an E-operation, it is Mixed operation)

$R_3 \to 5R_3$ & then $R_3 \to R_3 + R_2$

Note: ① we are free to apply only 3rd E-operations while solving Determinant

② If we interchange any two Rows (or any two Columns) in a Matrix then value of it's Determinant changes by -ve sign GATE

③ If we multiply with constant K in any Row (or in any Column) in a Mat then value of it's determinant changes by K times

i.e. we are not free to Multiply by any constant in Determinant

But we can take out common Rowwise or Columnwise

PROPERTIES of Determinants →

① If in a Mat all the elements in a Row (or in a Column) are zero then it's Det = 0

② If " " any two Rows (or any two Columns) are identical then it's Det = 0

③ $|ABC| = |A| \cdot |B| \cdot |C|$

④ $|A^m| = |A|^m$, $m \in N$.

⑤ $|A^T| = |A|$

⑥ $|(A+B+C)| \neq |A| + |B| + |C|$

No formula exist, Calculate Manually

⑦ $|A|^{-1} = \frac{1}{|A|}$ for eg, if $|A| = 5$ then $|A|^{-1} = ? = \frac{1}{5}$

⑧ $|A^{-1}| = \frac{1}{|A|}$ or $\boxed{\det(A^{-1}) = \frac{1}{\det(A)}}$

where $A^{-1} = ? = \frac{1}{A}$ → $A^{-1} = \frac{\text{adj}\,A}{|A|} = \frac{(\text{Cof}\,A)^T}{|A|}$

PAAP.

(9) Area of $\triangle ABC$ formed by points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$:→

$$\text{Area} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

eg Area of $\triangle$ formed $(1,0), (2,2), (4,3)$?

$$A = \frac{1}{2}\begin{vmatrix} 1 & 0 & 1 \\ 2 & 2 & 1 \\ 4 & 3 & 1 \end{vmatrix} = \cdots\cdots = -3/2$$

So Req Area $= \frac{3}{2}$ Ans

(10) Differentiation of Determinant :

$$\frac{d}{dx}\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = ? = \begin{vmatrix} a' & b' & c' \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} a & b & c \\ d' & e' & f' \\ g & h & i \end{vmatrix} + \begin{vmatrix} a & b & c \\ d & e & f \\ g' & h' & i' \end{vmatrix}$$

(11) $\begin{vmatrix} a+l & b & c \\ d+m & e & f \\ g+n & h & i \end{vmatrix} = ? = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} l & b & c \\ m & e & f \\ n & h & i \end{vmatrix}$

& $\begin{vmatrix} (a+l) & (b+m) & (c+n) \\ d & e & f \\ g & h & i \end{vmatrix} = ? = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} l & m & n \\ d & e & f \\ g & h & i \end{vmatrix}$

eg $\begin{vmatrix} 1+1 & -3 & 0 \\ 2+3 & 1 & 2 \\ 2+2 & 0 & 3 \end{vmatrix} = ? = \begin{vmatrix} 1 & -3 & 0 \\ 2 & 1 & 2 \\ 2 & 0 & 3 \end{vmatrix} + \begin{vmatrix} 1 & -3 & 0 \\ 3 & 1 & 2 \\ 2 & 0 & 3 \end{vmatrix}$

$= 27$ $\quad = 9$ $\quad = 18$

(12) $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 4 \\ 0 & -1 & 2 \end{bmatrix}$ & $2A = \begin{bmatrix} 2 & -4 & 6 \\ 4 & 2 & 8 \\ 0 & -2 & 4 \end{bmatrix} = 2 \cdot \begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 4 \\ 0 & -1 & 2 \end{bmatrix}$

$\Rightarrow |A| = 8$

$|2A| = 64 = 8 \cdot 8 = 2^3 \cdot |A|$

ie $|2A| = \begin{vmatrix} 2 & -4 & 6 \\ 4 & 2 & 8 \\ 0 & -2 & 4 \end{vmatrix} = 2 \times 2 \times 2 \begin{vmatrix} 1 & -2 & 6 \\ 2 & 1 & 4 \\ 0 & -1 & 2 \end{vmatrix} = 2^3 \cdot |A|$

M. Imp

(*) if $A_{n \times n}$ & K is any scalar then

$[KA] = K[A]$

$\boxed{|KA| = K^n \cdot |A|}$

(13) Max Number of terms in the General Expansion of $|A|_{n\times n} = ? = n!$ learn.

Gate.

(a) n

(b) n^2

(c) $2n$

(d) $n!$ ✓

eg $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}_{2\times 2} = ad - bc$ i.e Max term = 2 terms = 2!

eg $|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}_{3\times 3} = a(ei-hf) - b(di-gf) + c(dh-eg)$

i.e Max term = 6 terms = 3! terms.

eg $|A| = \begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{vmatrix}_{4\times 4} = a\begin{vmatrix} - & - & - \\ - & - & - \\ - & - & - \end{vmatrix} - b\begin{vmatrix} - & - & - \\ - & - & - \\ - & - & - \end{vmatrix} + c\begin{vmatrix} - & - & - \\ - & - & - \\ - & - & - \end{vmatrix} - d\begin{vmatrix} - & - & - \\ - & - & - \\ - & - & - \end{vmatrix}$

$= (6 \text{ terms}) + (6 \text{ terms}) + (6 \text{ term}) + (6 \text{ terms})$

i.e Max term = 24 terms = 4! & so on.....

(*) Total Number of terms in Mat $A_{n\times n} = ? = n^2$

HWQ ① $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = ? = (a-b)(b-c)(c-a)$ learn

HWQ ② $\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = ? = (a-b)(b-c)(c-a)(a+b+c)$

CW ③ $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = ? = -(a+b+c)[a^2+b^2+c^2-ab-bc-ca]$

OR

$= -(a^3+b^3+c^3-3abc)$

OR

$= -\frac{1}{2}(a+b+c)[(a-b)^2+(b-c)^2+(c-a)^2]$

Prob. (3) $|A| = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a+b \end{vmatrix} \xrightarrow{C_1 \to C_1 + (C_2 + C_3)} \begin{vmatrix} (a+b+c) & b & c \\ (b+c+a) & c & a \\ (c+a+b) & a & b \end{vmatrix} = (a+b+c) \begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}$

$\xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} (a+b+c) \begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix} = (a+b+c)\left[(c-b)(b-c) - (a-b)(a-c)\right]$

$|A| = -\left[a^3+b^3+c^3-3abc\right] = -\left[(a+b+c)(a^2+b^2+c^2-ab-bc-ca)\right]$ Ans Ans

$= -\frac{1}{2}\left[(a+b+c)(2a^2+2b^2+2c^2-2ab-2bc-2ca)\right]$

$= -\frac{1}{2}(a+b+c)\left[(a^2+b^2-2ab)+(b^2+c^2-2bc)+(c^2+a^2-2ca)\right]$

$= -\frac{1}{2}(a+b+c)\left[(a-b)^2+(b-c)^2+(c-a)^2\right]$ Ans

If $\Delta = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$ then which of the following is a factor of Δ.

(a) $a + b$ (b) $a - b$

(c) abc (d) $a + b + c$

$R_2 \to R_2 - R_1$ & $R_3 \to R_3 - R_1$

$$\Delta = \begin{vmatrix} 1 & a & bc \\ 0 & b-a & c(a-b) \\ 0 & c-a & (a-c)b \end{vmatrix} = (a-b)(a-c)\begin{vmatrix} 1 & a & bc \\ 0 & -1 & c \\ 0 & -1 & b \end{vmatrix} \xrightarrow{R_3 - R_2} (a-b)(a-c)\begin{vmatrix} 1 & a & bc \\ 0 & -1 & c \\ 0 & 0 & b-c \end{vmatrix}$$

$$= (a-b)(b-c)(c-a)$$

Q: if $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$ then $|A^3| = ?$

Sol: $|A| = (3)(3) - (2)(2) = 5$

so $|A^3| = |A|^3 = (5)^3 = 125$

(using Prop 4)

Q: If $A = \begin{bmatrix} \alpha & 2 \\ 2 & \alpha \end{bmatrix}$ s.t $|A^3| = 125$

then $\alpha = ?$

(a) 3 (b) -3 (c) ± 3 (d) 0

Sol: $|A^3| = 125$

$|A|^3 = (5)^3$

$|A| = 5$

$\because |A| = 5$

$\alpha^2 - 4 = 5$

$\alpha^2 = 9 \Rightarrow \alpha = \pm 3$

Q: if $A = \begin{bmatrix} 1 & -\tan x & -\sec x \\ \tan x & 0 & 3 \\ \sec x & 3 & \sin x \end{bmatrix}_{3\times 3}$

then $|A^T \bar{A}^{-1}| = ?$

$= |A^T| \cdot |\bar{A}^{-1}|$ (using Prop 3)

$= |A| \cdot \frac{1}{|A|}$ (using Prop 5 & 8)

$= 1$

Q: if $|A|_{3\times3} = 6$ then ① $|(2A)^{-1}| = ?$ ② $|2\cdot A^{-1}| = ?$

$\because A_{3\times3} \Rightarrow (A^{-1})_{3\times3}$ so ① $|(2A)^{-1}| = \frac{1}{|2A|} = \frac{1}{2^3\cdot|A|} = \frac{1}{8\times6} = \frac{1}{48}$

Sol ② $|2\cdot A^{-1}| = 2^3\cdot|A^{-1}| = 2^3\cdot\frac{1}{|A|} = \frac{8}{6} = \frac{4}{3}$

Q: if $|A|_{3\times3} = -1$, $|B|_{3\times3} = 4$ then $|5AB| = ?$ (a) 320 (b) −20 (c) −500 (d) −320

Sol: $\because A_{3\times3}$ & $B_{3\times3} \Rightarrow (AB)_{3\times3}$

$|5AB| = 5^3\cdot|A.B|$

$= 125\ |A|.|B|$

$= 125(-1)(4) = -500$

If A and B are two matrices of order 3 × 5 and 5 × 3 respectively then determinant of the matrix 4BA equals

(a) $4|B||A|$ (b) $4^3|B||A|$

(c) $4^5|A||B|$ (d) None $= 4^5|BA|$

The concept of Determinant is valid only for Sq Mat

& A & B are Rectangular

So $|A| =$ DNE

$|B| =$ DNE

$\because B_{5\times3} A_{3\times5} = (BA)_{5\times5}$

So $|4BA| = 4^5|BA| = 4^5 \cdot |B| \cdot |A|$

(d)

Let a 3 × 3 matrix A have determinant value 5. If $B = 4A^2$ then the determinant valueo f B is equal to _____.

(a) 20 (b) 100

(c) 320 (d) 1600

$|A| = 5$

$|A^2| = |A|^2 = 5^2 = 25$

& $(A^2)_{3\times3}$

$B = 4A^2$ then

$|B| = |4A^2| = 4^3|A^2|$

$= 64 \cdot |A|^2 = 64(5)^2 = 6400.$

Q. Find the area of the Δ formed by $(1,0)$, $(2\ 2)$, $(4\ 3)$?

Sol: $\text{Area} = \frac{1}{2}\begin{vmatrix} 1 & 0 & 1 \\ 2 & 2 & 1 \\ 4 & 3 & 1 \end{vmatrix} = \cdots = -3/2$ so Req Area $= +\frac{3}{2}$.

Q. Show that $\underset{A}{(a, b+c)}$, $\underset{B}{(b, c+a)}$, $\underset{C}{(c, a+b)}$ are collinear.

Sol:

Points A, B, C are collinear if Area of the $\Delta ABC = 0$

So $\text{Area} = \frac{1}{2}\begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} (a+b+c) & b+c & 1 \\ (a+b+c) & c+a & 1 \\ (a+b+c) & a+b & 1 \end{vmatrix} = \frac{(a+b+c)}{2}\begin{vmatrix} 1 & b+c & 1 \\ 1 & c+a & 1 \\ 1 & a+b & 1 \end{vmatrix}$

$= 0$ Hence proved

Q: if $\Delta = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2 & 3x \\ 0 & 2 & 5x \end{vmatrix}$ then $\frac{d\Delta}{dx} = ?$

Sol: $\frac{d\Delta}{dx} = \begin{vmatrix} 1 & 2x & 3x^2 \\ 1 & 2 & 3x \\ 0 & 2 & 5x \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ 0 & 0 & 3 \\ 0 & 2 & 5x \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2 & 3x \\ 0 & 0 & 5 \end{vmatrix}$

$= 1[10x-6x] - 1[10x^2-6x^2] + (-3)[2x] + 5[2x-x^2]$

$= 4x - 4x^2 - 6x + 10x - 5x^2$

$-9x^2+8x$

Q (MSQ) if $f(x) = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2 & 3x \\ 0 & 2 & 5x \end{vmatrix}$ s.t $f'(x) = -1$ then $x = ?$

(a) 1 (b) -1

(c) $-1/9$ (d) 0

$-9x^2+8x = -1$

$9x^2-8x-1=0$

$(9x+1)(x-1)=0$

$x=1$

$x=-1/9$

Which one of the following does NOT equal

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} ? = |A| = (x-y)(y-z)(z-x)$$

(a) $\begin{vmatrix} 1 & x(x+1) & x+1 \\ 1 & y(y+1) & y+1 \\ 1 & z(z+1) & z+1 \end{vmatrix} = -|A|$ ✓

(b) $\begin{vmatrix} 1 & x+1 & x^2+1 \\ 1 & y+1 & y^2+1 \\ 1 & z+1 & z^2+1 \end{vmatrix} = |A|$ ✗

(c) $\begin{vmatrix} 0 & x-y & x^2-y^2 \\ 0 & y-z & y^2-z^2 \\ 1 & z & z^2 \end{vmatrix} = |A|$ ✗

(d) $\begin{vmatrix} 2 & x+y & x^2+y^2 \\ 2 & y+z & y^2+z^2 \\ 1 & z & z^2 \end{vmatrix} = |A|$ ✗

Taking (a):

$$\begin{vmatrix} 1 & x^2+x & x+1 \\ 1 & y^2+y & y+1 \\ 1 & z^2+z & z+1 \end{vmatrix} = \begin{vmatrix} 1 & x^2 & x+1 \\ 1 & y^2 & y+1 \\ 1 & z^2 & z+1 \end{vmatrix} + \begin{vmatrix} 1 & x & x+1 \\ 1 & y & y+1 \\ 1 & z & z+1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & x^2 & x \\ 1 & y^2 & y \\ 1 & z^2 & z \end{vmatrix} + \begin{vmatrix} 1 & x^2 & 1 \\ 1 & y^2 & 1 \\ 1 & z^2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & x & x \\ 1 & y & y \\ 1 & z & z \end{vmatrix} + \begin{vmatrix} 1 & x & 1 \\ 1 & y & 1 \\ 1 & z & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & x^2 & x \\ 1 & y^2 & y \\ 1 & z^2 & z \end{vmatrix} + 0 + 0 + 0$$

$$= -\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = -|A| \text{ ie } (a)$$

Taking (d): $|A| = \begin{vmatrix} 2 & x+y & x^2+y^2 \\ 2 & y+z & y^2+z^2 \\ 1 & z+0 & z^2+0 \end{vmatrix} = \begin{vmatrix} 2 & x & x^2+y^2 \\ 2 & y & y^2+z^2 \\ 1 & z & z^2+0 \end{vmatrix} + \begin{vmatrix} 2 & y & x^2+y^2 \\ 2 & z & y^2+z^2 \\ 1 & 0 & z^2+0 \end{vmatrix}$

$$= \begin{vmatrix} 2 & x & x^2 \\ 2 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + \begin{vmatrix} 2 & x & y^2 \\ 2 & y & z^2 \\ 1 & z & 0 \end{vmatrix} + \begin{vmatrix} 2 & y & x^2 \\ 2 & z & y^2 \\ 1 & 0 & z^2 \end{vmatrix} + \begin{vmatrix} 2 & y & y^2 \\ 2 & z & z^2 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & x-2y & x^2-2y^2 \\ 0 & y-2z & y^2-2z^2 \\ 1 & z & z^2 \end{vmatrix} + \begin{vmatrix} 0 & x-2z & y^2 \\ 0 & y-2z & z^2 \\ 1 & z & 0 \end{vmatrix} + \begin{vmatrix} 0 & y & x^2-2z^2 \\ 0 & z & y^2-2z^2 \\ 1 & 0 & z^2 \end{vmatrix} + \begin{vmatrix} 0 & y & y^2 \\ 0 & z & z^2 \\ 1 & 0 & 0 \end{vmatrix}$$

$= \cdots = |A|$ for (d) ✗

(M-II) $|A| = \begin{vmatrix} 2 & x+y & x^2+y^2 \\ 2 & y+z & y^2+z^2 \\ 1 & z & z^2 \end{vmatrix} \xrightarrow{\text{use E operation}} \cdots = |A|$ for (d) ✗

The value of the determinant $\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix}$ will be _____.

(a) $2a^2b^2c^2$ (b) $-2a^2b^2c^2$

(c) $4a^2b^2c^2$ (d) $-4a^2b^2c^2$

$$= (abc)\begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix} \quad \begin{matrix} R_2 \to R_2 + R_1 \\ R_3 \to R_3 + R_1 \end{matrix}$$

$$= (abc)\begin{vmatrix} -a & b & c \\ 0 & 0 & 2c \\ 0 & 2b & 0 \end{vmatrix}$$

$$= (abc)\left[(-a)\{0 - 4bc\}\right]$$

$$= (abc)(4abc) = 4a^2b^2c^2$$

M-II

$$|A| = abc\begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix}$$

$$= (abc)(abc)\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= (a^2b^2c^2)(4)$$

If A and B are matrices of determinant 1 then

(a) Determinant of A + B is 2

(b) Determinant of A + B is 1

(c) Determinant of A + B is 0

(d) Nothing can be concluded about the determinant of A + B

ATQ, $|A| = 1$ & $|B| = 1$ (given)

So $|A+B| = ? \neq |A| + |B|$

while $|A+B| \leq |A| + |B|$

$|A+B| \leq 1+1$

or $|A+B| \leq 2$

The determinant of a matrix has 720 terms (in the unsimplified form). The order of the matrix is ____

(a) 5 (b) 6 (c) 7 (d) 8

Let A is of order $n \times n$

then A.T.Q.,

$n! = 720$

$n! = 6!$

$n = 6$

So $A_{6\times 6}$

Tel:

dr puneet Sir pw

Thank you

Keep Hustling!

DS & AI
CS & IT

Linear Algebra

Lecture No. 02

By- Dr. Puneet Sharma Sir

Recap of previous lecture

Topic

BASICS of Determinants

Topics to be Covered

Topic

BASICS of MATRICES

Remaining Portion of Det –

Analysis: if $A_{n\times n}$ then $\boxed{A^{-1} = \frac{\operatorname{adj} A}{|A|}}$ where $(\operatorname{adj} A)_{n\times n}$ & $|A|$ is any const.

(PODCAST)

Now, $AA^{-1} = \frac{A\operatorname{adj} A}{|A|} \Rightarrow I = \frac{A\operatorname{adj} A}{|A|} \Rightarrow A\operatorname{adj} A = |A|.I_n$

Now $A(\operatorname{adj} A) = |A|.I_n = |A| \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}_{n\times n} = \begin{bmatrix} |A| & 0 & 0 & \cdots & 0 \\ 0 & |A| & 0 & \cdots & 0 \\ 0 & 0 & |A| & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & |A| \end{bmatrix}_{n\times n}$ = Scalar Mat.

So $|A\operatorname{adj} A| = \underbrace{|A|.|A|.|A|.\cdots\cdots|A|}_{n \text{ times}} = |A|^n$ learn

& $Tr(A\operatorname{adj} A) = |A| + |A| + |A| + \cdots + |A| = n|A|$ learn

Ex) If $A_{n\times n}$ then Evaluate $|\text{adj}A|$, $|\text{Cof}A|$, $|A^{-1}|$ = ?

<u>Proof</u>: w.k.that $|A.(\text{adj}A)| = |A|^n$

$$|A|.|\text{adj}A| = |A|^n$$

$$\boxed{|\text{adj}A| = |A|^{n-1}}$$

Now again w.k.that, $|B| = |B^T|$

So $|\text{Cof}A| = |(\text{Cof}A)^T| = |\text{adj}A| = |A|^{n-1}$

ie $\boxed{|\text{Cof}A| = |A|^{n-1}}$

Now, $A^{-1} = \dfrac{\text{adj}A}{|A|}$

$$\Rightarrow |A^{-1}| = \left|\frac{(\text{adj}A)}{|A|}\right| = \frac{1}{|A|^n}.|\text{adj}A|$$

$$|A^{-1}| = \frac{|A|^{n-1}}{|A|^n} = \frac{1}{|A|}$$

Hence $\det A^{-1} = \dfrac{1}{\det A}$

Imp point: if $A_{n\times n}$ then $\boxed{|\underbrace{\text{adj adj adj} \cdots\cdots \text{adj}}_{r \text{ times}} A| = |A|^{(n-1)^r}}$ Learn.

eg if $|A|_{4\times 4} = 5$ then evaluate $|\text{adj adj adj} A| = ?$

(a) 5^9

(b) 5^{27} ✓

(c) 5^4

(d) 5^3

$n = 4,\ r = 3$

$\therefore |\text{adj adj adj} A| = |A|^{(n-1)^r}$

$= (5)^{(4-1)^3} = (5)^{3^3} = 5^{27}$

Note: $(a^b)^c = a^{bc}$, $(a)^{b^c} = (a)^{\underbrace{b\times b\times b\times \cdots \times b}_{c \text{ times}}}$

MATRIX

→ "It is a Rectangular arrangement of m·n numbers"

$$A = [a_{ij}]_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

$1 \le i \le m$

$1 \le j \le n$

H-lines ≃ Rows

V-lines ≃ Columns.

→ Sq. Mat → Def^n ① No. of Rows = No. of Columns.

Def^n ② If in a Mat, Diag exist, then it Must be Square

Def^n ③ if ∀ element, ∃ Corresponding element then Mat is Square Mat.

eg $A = [a_{ij}]_{n\times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}_{n\times n}$

$1 \le i \le n$

$1 \le j \le n$

Sq Mat

① for Diagonal elements, $i = j \ \forall i \ \& \ j$

② for upper Diag elements, $i < j \ \forall i \ \& \ j$

③ for Lower " " , $i > j$ " "

④ for off Diag elements, $i \ne j$ " "

⑤ Corresponding elements are a_{ij} & a_{ji}

$$\boxed{\text{Trace(Sq Mat)} = \sum a_{ii}}$$

ie $Tr(A)$ = sum of diag elements

eg $A = \begin{bmatrix} 2 & -3 & 1 \\ 4 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

$Tr(A) = 3$ & $|A| = -16$

Some Special types of Sq. Matrices →

eg

$$\begin{bmatrix} 2 & 4 & -1 & 3 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -4 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 2 & -1 & 3 & -3 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

U.T.M	L.T.M	Diag. Mat	Scalar Mat	Identity Mat
$Tr = -2$, $Det = 8$	$Tr = 0$, $Det = 0$	$Tr = 4$, $Det = 0$	$Tr = 20$, $Det = 5^4$	$Tr = 4$, $Det = 1$

(*) Diag Mat: $A = [a_{ij}]_{n \times n}$ s.t. $a_{ij} = \begin{cases} 0, & i \neq j \\ \text{at least one Diag element should be Non Zero}, & i = j \end{cases}$

(*) To Find Determinant of U.T.M, L.T.M, Diag Mat, Scalar Mat, Identity Mat,

Imp

just Multiply the Diag elements ie $\boxed{\text{Det} = \text{Product of Diag elements}}$

Q. $|A| = \begin{vmatrix} 0 & 1 & 2 & 0 \\ -1 & 0 & 4 & 0 \\ -2 & -4 & 0 & 6 \\ 0 & 0 & -6 & 0 \end{vmatrix}_{4\times 4} = ?$

(a) 0

(b) 16

(c) 36 ✓

(d) 21

Given Mat is neither U.T.M, Nor L.T.M, not a Diag Mat so we can not use Shortcut Method as discussed in Previous Slide.

Hence we will follow Conventional approach.

$|A| = \ldots\ldots \text{(HW)} \ldots\ldots = 36$

A matrix $A = [a_{ij}]_{n\times n}$ is said to be lower triangular if

(a) $a_{ij} = 0$ for $i > j$

(b) $a_{ij} = 0$ for $i < j$

(c) $a_{ij} = 0$ for $i \geq j$

(d) $a_{ij} = 0$ for $i \leq j$

eg $A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3} = L.T.M.$

$a_{12} = a_{13} = a_{23} = 0.$

ie $a_{ij} = 0 \ \forall \ i < j$ ie L.T.M

Q. If $A = [a_{ij}]_{n\times n}$ s.t $a_{ij} = 0 \ \forall \ i > j$ then A is U.T.M

Singular Mat: If $|A| = 0$ then A is called Singular Mat

Non Singular Mat: If $|A| \neq 0$ then A is called Non Singular Mat

Invertible Mat → If A^{-1} exist then A is called Invertible Mat

& it is defined as $\boxed{A^{-1} = \frac{\text{adj } A}{|A|}}$ or $\frac{(\text{Cof } A)^T}{|A|}$

N. Condition for a Matrix to be invertible is $|A| \neq 0$ i.e A must be Non Sing.

Sp. Conclusion: → w.k.that, $AA^{-1} = A^{-1}A = I$

So if we have, $\boxed{AB = BA = I}$ ⟹ Inverse of $A = B$ i.e $A^{-1} = B$

Inverse of $B = A$ i.e $B^{-1} = A$

i.e Both are the Inverses of each other.

eg If $A_{n\times n}$ s.t $A^2 = I$ then $A^{-1} = ?$

(a) I (b) A ✓

(c) O (d) Not possible

Sol: Given, $A.A = I \Longrightarrow A^{-1} = A$ i.e A is self Invertible

or A is inverse of itself.

on Comparision with,

$(Mat)(Mat)^{-1} = I$

eg: If $M^4 = I$ then Various Conclusions are ?

$M^{-1} = ?$

$(M^2)^{-1} = ?$

$(M^3)^{-1} = ?$

(i) $M.M^3 = I \Rightarrow M^{-1} = M^3$

(ii) $M^2.M^2 = I \Rightarrow (M^2)^{-1} = M^2$

(iii) $M^3.M = I \Rightarrow (M^3)^{-1} = M$

Shortcut Method of finding Inverse of 2×2 Mat →

if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A^{-1} = \dfrac{\text{adj}A}{|A|} = \dfrac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

eg $A = \begin{bmatrix} 2 & 3 \\ -4 & 1 \end{bmatrix}$, then $A^{-1} = ?$ $= \dfrac{1}{(14)}\begin{bmatrix} 1 & -3 \\ 4 & 2 \end{bmatrix}$

eg $A = \begin{bmatrix} 3+4i & -i \\ i & 3-4i \end{bmatrix}$ then $A^{-1} = ?$ $= \dfrac{1}{24}\begin{bmatrix} 3-4i & +i \\ -i & 3+4i \end{bmatrix}$

Sol: $|A| = (3+4i)(3-4i) - (i)(-i)$

$= \ldots\ldots\ldots$

$= 24 \because |A| \neq 0 \Rightarrow A^{-1}$ exist

Shortcut Method of finding Inverse of 3×3 Mat →

eg $A = \begin{bmatrix} 1 & -2 & 2 \\ 2 & 0 & 3 \\ -3 & 1 & 4 \end{bmatrix}$ then $A^{-1} = ? = \frac{\text{adj}A}{|A|} = \frac{(\text{Cof}A)^T}{|A|} = \frac{1}{35}\begin{bmatrix} -3 & 10 & -6 \\ -17 & 10 & 1 \\ 2 & 5 & 4 \end{bmatrix}$

Sol: $|A| = \text{------} = 35$

$\because |A| \neq 0 \Rightarrow A^{-1}$ exist.

	1	-2	2	1	-2
2		0	3	2	0
-3		1	4	-3	1
1		-2	2	1	-2
2		0	3	2	0

$\text{Cof}A = ? = (\text{adj}A)^T = \begin{bmatrix} -3 & -17 & 2 \\ 10 & 10 & 5 \\ -6 & 1 & 4 \end{bmatrix}$

Top Row of $A^{-1} = ? = \begin{bmatrix} -3 & 10 & -6 \end{bmatrix}$ ✗

$= \begin{bmatrix} \frac{-3}{35} & \frac{10}{35} & \frac{-6}{35} \end{bmatrix}$ ✓

If $R = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{bmatrix}$. Then top row of R^{-1} is

(a) $[5 \quad 6 \quad 4]$

(b) $[5 \quad -3 \quad 1]$

(c) $[2 \quad 0 \quad -1]$

(d) $\begin{bmatrix} 2 & -1 & \frac{1}{2} \end{bmatrix}$

$|R| = 1[2+3] + (-1)[6-2]$
$= 5 - 4 = 1.$

$R^{-1} = \frac{\text{adj}\,R}{|R|} = \frac{1}{(1)}\begin{bmatrix} 5 & -3 & 1 \\ - & - & - \\ - & - & - \end{bmatrix}$

1	0	−1	1	0
2	1	−1	2	1
2	3	2	2	3
1	0	−1	1	0
2	1	−1	2	1

(ii) Cof of $a_{21} = ? = -3$ Ans

$\because \text{Cof}\,A = \begin{bmatrix} 5 & - & - \\ -3 & - & - \\ 1 & - & - \end{bmatrix}$

P8 $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & -6 \\ -10 & -14 & 6 \end{bmatrix}$ then $A^{-1} = ?$, $\text{adj}A = ?$

Sol $\because |A| = \cdots\cdots = 0$ ☹

is A is Singular so $A^{-1} = \text{DNE}$

(ii)

1	2	3	1	2
4	5	-6	4	5
-10	-14	6	-10	-14
1	2	3	1	2
4	5	-6	4	5

$\text{adj}A = (\text{Cof}A)^T = \begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}$

HW

Methods of finding Inverse of 4×4, 5×5, 6×6 ..... Matrices →

Not in a Syllabus by Conventional Approach.

Procedure: write $A = I \cdot A$

using E-operation

$$I = BA \Rightarrow A^{-1} = B = \text{Ans.}$$

(8) Addition & Subtraction in Matrix →

this is defined only when Matrices are of Same order

for eg $A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 0 & 2 \end{bmatrix}_{2\times3}$, $B = \begin{bmatrix} -1 & 3 & 0 \\ 1 & 2 & 5 \end{bmatrix}_{2\times3}$, $C = \begin{bmatrix} 2 & -1 & 2 \\ 1 & 3 & 1 \\ 0 & 0 & 4 \end{bmatrix}_{3\times3}$

$A+B = B+A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 2 & 7 \end{bmatrix}_{2\times3}$

$A-B = \begin{bmatrix} 3 & -4 & 3 \\ 3 & -2 & -3 \end{bmatrix}$ i.e. $A-B \neq B-A$

$B-A = \begin{bmatrix} -3 & 4 & -3 \\ -3 & 2 & 3 \end{bmatrix}$

$A+C =$ N.D & $B+C =$ N.D.

"Matrix Addition is Commutative But Mat Subtraction is not Commutative"

Matrix Multiplication → is defined when

"Number of Columns in 1st Mat = Number of Rows in 2nd Matrix"

for eg, consider $A_{m\times n}$ & $B_{n\times p}$ then

$AB = [\ ]_{m\times p}$

$BA = B_{n\times p} A_{m\times n} = N.D.$

eg: $A = \begin{bmatrix} 1 & 2 & -4 \end{bmatrix}_{1\times 3}$ & $B = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}_{3\times 1}$ then $AB = ?$ & $BA = ?$

Sol: $AB = \begin{bmatrix} 1 & 2 & -4 \end{bmatrix}_{1\times 3} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}_{3\times 1} = \left[(2 + 2 + (-12)) \right] = \begin{bmatrix} -8 \end{bmatrix}_{1\times 1} = -8$

Multi = 3 times

Add = 2 times

$BA = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}_{3\times 1} \begin{bmatrix} 1 & 2 & -4 \end{bmatrix}_{1\times 3} = \begin{bmatrix} 2 & 4 & -8 \\ 1 & 2 & -4 \\ 3 & 6 & -12 \end{bmatrix}_{3\times 3}$

No. of Multi used = 9 times

No. of Addition used = 0 times

The value of $\underset{A}{\begin{bmatrix}1\\2\\3\end{bmatrix}} \times \underset{B}{\begin{bmatrix}3 & 2 & 4\end{bmatrix}} \times \underset{C}{\begin{bmatrix}4\\6\\7\end{bmatrix}}$ equals

PYQ

(a) $\begin{bmatrix}52\\-104\\156\end{bmatrix}$ (b) $\begin{bmatrix}52 & -104 & 15\end{bmatrix}$

(c) $\begin{bmatrix}52\\104\\156\end{bmatrix}$ (d) None of these

$$AB = \begin{bmatrix}1\\2\\3\end{bmatrix}_{3\times1}\begin{bmatrix}3 & 2 & 4\end{bmatrix}_{1\times3} = \begin{bmatrix}3 & 2 & 4\\6 & 4 & 8\\9 & 6 & 12\end{bmatrix}_{3\times3}$$

$$(AB)C = \begin{bmatrix}3 & 2 & 4\\6 & 4 & 8\\9 & 6 & 12\end{bmatrix}_{3\times3}\begin{bmatrix}4\\6\\7\end{bmatrix}_{3\times1}$$

$$= \begin{bmatrix}(12+12+28)\\(24+24+56)\\(36+36+84)\end{bmatrix}_{3\times1} = \begin{bmatrix}52\\104\\156\end{bmatrix}_{3\times1}$$

M-II $BC = [\ \ ]_{1\times1}$

$$A(BC) = \begin{bmatrix}-\\-\\-\end{bmatrix}_{3\times1}[-]_{1\times1} = \begin{bmatrix}52\\104\\156\end{bmatrix}_{3\times1}$$

eg: $A = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 1 & 2 \end{bmatrix}_{2\times3}$ & $B = \begin{bmatrix} 2 & -1 & 3 \\ -1 & 0 & 4 \\ 1 & 2 & -2 \end{bmatrix}_{3\times3}$ then

$AB = ?$

$BA = ? = \text{N.D.}$

$$AB = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 1 & 2 \end{bmatrix}_{2\times3} \begin{bmatrix} 2 & -1 & 3 \\ -1 & 0 & 4 \\ 1 & 2 & -2 \end{bmatrix}_{3\times3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}_{2\times3} = \begin{bmatrix} 7 & 5 & -11 \\ 9 & 0 & 12 \end{bmatrix}$$

$a_{11} = (1).(2) + (-2).(-1) + (3).(1) = 7$

$a_{12} = (1).(-1) + (-2).(0) + (3).(2) = 5$

$a_{13} = (1).(3) + (-2).(4) + (3).(-2) = -11$

$a_{21} = (4).(2) + (1).(-1) + (2).(1) = 9$

$a_{22} = (4).(-1) + (1).(0) + (2).(2) = 0$

$a_{23} = (4).(3) + (1).(4) + (2).(-2) = 12$

Number of times, symbol of Multiplication is used to find AB = 18 times

Number of times, symbol of addition is used to find AB = 12 times

Shortcut: Consider $A_{m\times n}$ & $B_{n\times p}$ then to find $(AB)_{m\times p}$

Number of Multiplications Required $= m.n.p$ & Number of Additions Req $= m(n-1).p$.

eg if $A_{2\times 3}$ & $B_{3\times 3}$ then to find AB

$m=2, n=3, p=3$

- No. of Multi Req $= 2\times 3\times 3 = 18$ times
- No. of Additions Req $= 2(3-1)3 = 12$ times

eg if $A_{3\times 4}$ & $B_{4\times 3}$ then to find AB

$m=3, n=4, p=3$

- No. of Multi Req $= 3\times 4\times 3 = 36$ times
- No. of Additions Req $= 3(4-1)3 = 27$ times

& to find BA

$B_{4\times 3}A_{3\times 4}$ = Defined. $m=4, n=3$, $p=4$

- No. of Multi Req $= 4\times 3\times 4 = 48$
- No. of Additions Req $= 4(3-1)4 = 32$

MAJEDAAR QUESTION → Consider $A_{2\times3}$, $B_{3\times4}$ & $C_{4\times2}$ then find the minimum number of Multiplications & Additions that will be Required To find Matrix Product (ABC) ?

Sol: we can find (ABC) either by using

- Case I: (AB) C
  - Multi = 40
  - Additions = 28
- Case II: A(BC)
  - Multi = 36
  - Addition = 26

i.e. Min. Multi Required = 36

Min Addition Required = 26 (obtained by Case II)

Commutative Law

$$A+B=B+A$$

But $AB \neq BA$

Associative Law

$$A+(B+C)=(A+B)+C$$

& $A(BC)=(AB)C$

Matrix Addition is Commutative as well as Associative.

while Matrix Multiplication is associative but not commutative in general.

Note Although $AB \neq BA$ in general But $\boxed{Tr(AB)=Tr(BA)}$ always.

eg $A=\begin{bmatrix}1 & -2 & 3\\ 1 & 4 & 0\end{bmatrix}$

& $B=\begin{bmatrix}2 & 1\\ 0 & -2\\ 1 & 4\end{bmatrix}$

then

$AB=\begin{bmatrix}5 & 17\\ 2 & -7\end{bmatrix}$ $Tr(AB)=-2$

$BA=\begin{bmatrix}3 & 0 & 6\\ -2 & -8 & 0\\ 5 & 14 & 3\end{bmatrix}$ $Tr(BA)=-2$

hence verified

If A & B are two matrices of same order then which of the following is true.

$\Downarrow$

AB & BA both defined

(a) $(A + B)^2 = A^2 + 2AB + B^2$

(b) $(A - B)^2 = A^2 - 2AB + B^2$

(c) $(A + B)^2 + (A - B)^2 = 2A^2 + 2B^2$

(d) $(A + B)(A - B) = A^2 - B^2$

(a) $(A+B)^2 = (A+B)(A+B)$

$= A^2 + \underbrace{AB + BA}_{(AB \neq BA)} + B^2$

(b) $(A-B)^2 = A^2 - \underbrace{AB - BA}_{?} + B^2$

(d) $(A+B)(A-B) = A^2 \underbrace{- AB + BA}_{} + B^2$

(c) $(A+B)^2 + (A-B)^2$

$= (A^2 + B^2 + AB + BA)$

$+ (A^2 + B^2 - AB - BA)$

$= 2A^2 + 2B^2$

Thank you

Keep Hustling!

DS & AI
CS & IT

Linear Algebra

Lecture No. 03

By- Dr. Puneet Sharma Sir

Recap of previous lecture

Topic

Algebra of MATRICES

Topics to be Covered

Topic

TYPES of MATRICES

MAJEDAAR QUESTION → Consider $A_{2\times3}$, $B_{3\times4}$ & $C_{4\times2}$ then find the minimum number of Multiplications & Additions that will be Required to find Matrix Product (ABC) ?

Sol: ABC → Case I: (AB)C
→ Case II: A(BC)

Case II: $B_{3\times4}\, C_{4\times2} = (BC)_{3\times2}$ → No. of Multi $= 3\times4\times2 = 24$
→ No. of Additions $= 3(4-1)2 = 18$

Now $A_{2\times3}(BC)_{3\times2} = (ABC)_{2\times2}$ → No. of Multi $= 2\times3\times2 = 12$
→ No of Additions $= 2(3-1)2 = 8$

overall to find ABC using Case II, Multi $= 24+12 = 36$ & Additions $= 18+8 = 26$

PQ: if $E = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$ & $G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ s.t multi of E & F is G then $F = ?$

Sol: ATQ, $EF = G \Rightarrow EF = I$

$E^{-1} = F$ ie $F = E^{-1} = ?$ (Ans)

$F^{-1} = E$

$\because |E| = \ldots\ldots = 1$ ie E is Non sing so E^{-1} will exist.

	$\cos\theta$	$-\sin\theta$	0
$\sin\theta$	$\cos\theta$	0	$\sin\theta$
0	0	1	0
$\cos\theta$	$-\sin\theta$	0	$\cos\theta$
$\sin\theta$	$\cos\theta$	0	$\sin\theta$

$\cos\theta \quad -\sin\theta \quad 0 \quad \cos\theta \quad -\sin\theta$

$$E^{-1} = \frac{\text{adj}E}{|E|} = \frac{1}{(1)} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = E^T$$

ie $F = E^{-1} = E^T =$ Ans

RECAP

1. $A_{m\times n}$ = R. Mat (Diag DNE)
2. $A_{n\times n}$ = Sq. Mat (Diag exist)
3. for upper diag elements $i < j$
4. for lower " " $i > j$
5. for diag elements $i = j$
6. Addition & Subtraction are possible if A & B are of <u>same</u> order
7. $A_{m\times n} B_{n\times p}$ = Possible
8. $A_{m\times n} B_{n\times p}$ — Multi $= m \cdot n \cdot p$; Add $= m(n-1)p$.
9. <u>Commutative law</u>:

   $A+B = B+A$ But $AB \neq BA$
10. <u>Associative law</u>:-

    $A+(B+C) = (A+B)+C$

    & $A(BC) = (AB)C$

(*) In Matrix Algebra, Concept of Division is not defined.

ie If $AB = C$ then $B = \cancel{\frac{C}{A}}$ Blunder

But in Number System
$ab = c \Rightarrow b = \frac{c}{a}$ (Valid if $a \neq 0$)

if A is Nonsingular ie A^{-1} exist then we can proceed as follows;

$$A^{-1}(AB) = A^{-1}C$$
$$I B = A^{-1}C$$
$$\text{or } B = A^{-1}C \checkmark$$

Note: In Number System
Cancellation Law Holds when $\boxed{No \neq 0}$

ie if $a^2 = a$
$\Rightarrow \cancel{a}a = \cancel{a} \Rightarrow a = 1, (a \neq 0)$

Similarly in Matrix Algebra,
Cancellation Law Holds only when $\boxed{A \text{ is Non Sing.}}$
if $|A| \neq 0$ & $A^2 = A \Rightarrow A = I$

(*) In Matrix Algebra,

"Cancellation Law Holds only when Matrix is Non Singular."

eg if A is a Non Sing Mat s.t

$\cancel{A}B = \cancel{A}$ then $B = I$

Proof: $\because$ A is Non Sing so A^{-1} exist.

so $AB = A$

$A^{-1}(AB) = A^{-1}A$

$IB = I$ ie $B = I$

eg if $\cancel{A}B = \cancel{A}$ then $B = I$ (T/F)?

False $\because$ it is not given that A is Non Singular

Doubt: If $(ABC = AC) \not\Rightarrow (B = I)$

Q. If $\boxed{XY = Y}$ ① & $\boxed{YX = X}$ ② then $X^2 + Y^2 = ?$

MS8

(a) $X+Y$

(b) $2I$

(c) O

(d) I

Case I: if X & Y are Non Singular Matrices i.e X^{-1} = exist, Y^{-1} = exist.

i.e $X\cancel{Y} = \cancel{Y} \Rightarrow X = I$

& $Y\cancel{X} = \cancel{X} \Rightarrow Y = I$

So $X^2 + Y^2 = I^2 + I^2$

$= I + I$

$= 2I$

Case II: if X & Y are Singular then

$X^2 + Y^2 = XX + YY$

$= X(YX) + Y(XY)$

$= (XY)X + (YX)Y$ (By Associative law)

$= (Y)X + (X)Y$

$= X + Y$

If X and Y are two singular matrices such that $XY = Y$ and $YX = X$ then $X^2 + Y^2$ equals

(a) $X + Y$

(b) XY

(c) YX

(d) $2(X + Y)$

Complex Conjugate : $z = x + iy$ then $\bar{z} = x - iy$ eg $z_1 = 3-2i$, $\bar{z}_1 = 3+2i$

$z_2 = 4i$, $\bar{z}_2 = -4i$

$z_3 = 2$, $\bar{z}_2 = 2$

Conjugate Mat if $A = \begin{bmatrix} 2 & 4-i & 2i \\ -2+i & 0 & -5 \end{bmatrix}_{2\times 3}$

then $A^T = \begin{bmatrix} 2 & -2+i \\ 4-i & 0 \\ 2i & -5 \end{bmatrix}_{3\times 2}$ $\bar{A} = \begin{bmatrix} 2 & 4+i & -2i \\ -2-i & 0 & -5 \end{bmatrix}_{2\times 3}$

Tranjugate Mat (Transposed Conjugate of Mat)

$A^\theta = (\overline{A^T})$ or $(\bar{A})^T$ $\quad$ $A^\theta = (\overline{A^T}) = \begin{bmatrix} 2 & -2-i \\ 4+i & 0 \\ -2i & -5 \end{bmatrix}$

(*) The Concept of Trace, Det, Inverse are defined only for Square Matrices while the " of Transpose, Conjugate & Transjugate & Rank are also defined for Rectangular Matrices

Real Mat : If in a Mat all the elements are Real numbers then Mat is called Real Matrix

ie if A is a Real Mat then $\boxed{\bar{A} = A}$ or $A^{\theta} = A^{T}$ Learn.

Complex Mat : If in a Mat, at least one element is Complex Number then Mat is called Complex Mat ie $\boxed{\bar{A} \neq A}$

eg $A = \begin{bmatrix} 2 & -5 & 4 \\ 1 & 2 & 0 \end{bmatrix}$ = Real Mat, $B = \begin{bmatrix} 2 & -5 & 0 \\ 1 & 2 & 4i \end{bmatrix}$ = Complex Mat

Sp. Points:

① $Tr(A+B+C) = Tr(A) + Tr(B) + Tr(C)$

② $(A+B+C)^{\theta} = A^{\theta} + B^{\theta} + C^{\theta}$

③ $(A+B+C)^{T} = A^{T} + B^{T} + C^{T}$

④ $|A+B+C| \leq |A| + |B| + |C|$

⑤ $(A+B+C)^{-1} \neq A^{-1} + B^{-1} + C^{-1}$

But $(A+B+C)^{-1} = \dfrac{adj(A+B+C)}{|A+B+C|}$

⑥ $|ABC| = |A| \cdot |B| \cdot |C|$

REVERSAL LAW:-

$$(ABC)^{T} = C^{T} B^{T} A^{T}$$

$$(ABC)^{\theta} = C^{\theta} B^{\theta} A^{\theta}$$

$$(ABC)^{-1} = C^{-1} B^{-1} A^{-1}$$

⊛ we know that, $AB \neq BA$ in general

But $Tr(AB) = Tr(BA)$ always.

Reversal Law $\rightarrow$ $\boxed{(ABC)^T = C^T B^T A^T}$, $\boxed{(ABC)^\theta = C^\theta B^\theta A^\theta}$, $\boxed{(ABC)^{-1} = C^{-1} B^{-1} A^{-1}}$

& these Results can also be generalized.

Q: if A, B, C, D are Non Singular Matrices of Same order s.t $\boxed{ABCD = I}$ then evaluate B & $B^{-1} = ?$

Sol: $\cancel{ABCD = I \Rightarrow B = \frac{I}{ACD} = A^{-1}C^{-1}D^{-1}}$ Blunder

Correct Appr:

$$ABCD = I$$

$$A^{-1}(ABCD)D^{-1} = A^{-1} \cdot I D^{-1}$$

$$IBCI = A^{-1} \cdot I \cdot D^{-1}$$

$$BC = A^{-1}D^{-1}$$

$$(BC)C^{-1} = A^{-1}D^{-1}C^{-1} \Rightarrow B = A^{-1}D^{-1}C^{-1}$$

(ii) $B^{-1} = (A^{-1}D^{-1}C^{-1})^{-1}$

$$= (C^{-1})^{-1}(D^{-1})^{-1}(A^{-1})^{-1}$$

$$B^{-1} = C \cdot D \cdot A$$ Ans

Some More Special Matrices

eg

$$\begin{bmatrix} a & b & c \\ b & e & d \\ c & d & f \end{bmatrix}, \quad \begin{bmatrix} 0 & b & -c \\ -b & 0 & d \\ c & -d & 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & 5+i & -7 \\ 5-i & 0 & 2i \\ -7 & -2i & -3 \end{bmatrix}, \quad \begin{bmatrix} 2i & 5+i & -7 \\ -5+i & 0 & 2i \\ 7 & 2i & -3i \end{bmatrix}$$

$\Rightarrow |A| = 0$

$3+3$

Symmetric Mat. $(\because A^T = A)$

Skew Symm Mat (Anti Symm Mat) $(\because A^T = -A)$

Hermitian Mat $(\because A^\theta = A)$

Skew Herm. Mat. $(\because A^\theta = -A)$

(*) Imp. if $A_{n\times n}$ s.t. A is skew symm. then $|A| = \begin{cases} 0 & , n = \text{odd order} \\ \text{Perfect sq.} & , n = \text{even order} \end{cases}$

i.e. Det of Skew Symm Mat of odd order is always zero.

Note ① Symm Mat is symmetrical about leading diagonal
i.e. Corresponding elements are same & $A^T = A$

② Skew symm Mat : Diagonal elements are all zero. and corresponding elements are of opposite sign & $A^T = -A$

③ Hermitian Mat : Diagonal elements are either zero or purely Real Nos & Corresponding elements are Conjugates of each other & $A^\theta = A$.

④ Skew Hermitian Mat : Diagonal elements are either zero or Purely Imaginary numbers and Corresponding elements are −ve Conjugates of each other. & $A^\theta = -A$

PODCAST

Consider Sq Mat $A_{n\times n}$ then we can write (using common sense).

① $A = \left(\frac{A+A^T}{2}\right) + \left(\frac{A-A^T}{2}\right)$

$A = P + Q$

where $P = \frac{A+A^T}{2}$, Here $P^T = P$ i.e. P is symm.

$Q = \frac{A-A^T}{2}$, Here $Q^T = -Q$ i.e. Q is skew symm

i.e. $\boxed{\text{Sq Mat} = \text{Symm} + \text{Skew symm}}$

② $A = \left(\frac{A+A^\theta}{2}\right) + \left(\frac{A-A^\theta}{2}\right)$

$A = R + S$

$R = \frac{A+A^\theta}{2}$, Here $R^\theta = R$ i.e. R is Herm.

$S = \frac{A-A^\theta}{2}$, Here $S^\theta = -S$ i.e. S is Skew Hermitian.

$\boxed{\text{Sq Mat} = \text{Herm} + \text{Skew Herm.}}$

(5) Every sq. Mat can be expressed as the sum of symm & skew symm Mat. as follows; Consider sq Mat $A_{n\times n}$ then by common sense we can write.

$$\boxed{A = \left(\frac{A+A^T}{2}\right) + \left(\frac{A-A^T}{2}\right)}$$

where $P = \frac{A+A^T}{2}$ = symm Mat

ie $A = P + Q$

& $Q = \frac{A-A^T}{2}$ = skew symm Mat

for, $P^T = \left(\frac{A+A^T}{2}\right)^T = \frac{A^T + (A^T)^T}{2} = \frac{A^T + A}{2} = P$ Hence symm.

& $Q^T = \left(\frac{A-A^T}{2}\right)^T = \frac{A^T - (A^T)^T}{2} = \frac{A^T - A}{2} = -\left(\frac{A-A^T}{2}\right) = -Q$ Hence skew symm.

(5) Every sq. Mat can be expressed as the sum of Hermitian & Skew Herm Mat. as follows; Consider sq Mat $A_{n\times n}$ Then by common sense we can write.

$$\boxed{A = \left(\frac{A+A^{\theta}}{2}\right) + \left(\frac{A-A^{\theta}}{2}\right)}$$

$$A = R + S$$

$R = \frac{A+A^{\theta}}{2}$ = Herm. Mat

$S = \frac{A-A^{\theta}}{2}$ = Skew Herm. Mat.

So, $R^{\theta} = \left(\frac{A+A^{\theta}}{2}\right)^{\theta} = \ldots\ldots = R$ Hence Proved

$S^{\theta} = \left(\frac{A-A^{\theta}}{2}\right)^{\theta} = \ldots\ldots = -S$ " ;

Doubt:

$(A+B)^{-1} \neq A^{-1} + B^{-1}$

$(A+B)^{-1} = \frac{adj(A+B)}{|A+B|}$

If A and B are two symmetric matrices. Then consider the following staements.

(i) A + B is symmetric

(ii) AB is symmetric

(iii) AB + BA is symmetric

(a) Only (i) is true

(b) (i) and (ii) are true

(c) (i) and (iii) are true

(d) (i), (ii) and (iii) are true.

$A+B$, $A^T = A$

$B^T = B$

(i) $(A+B)^T = A^T + B^T = (A+B)$

i.e. $(A+B)$ is symmetric

(ii) $(AB)^T = B^T A^T = BA \neq AB$?

i.e. AB is Not symmetric

(iii) $(AB+BA)^T = (AB)^T + (BA)^T$

$= B^T A^T + A^T B^T$

$= BA + AB = (AB+BA)$ So $(AB+BA)$ is symm.

⊛ If $A_{n\times n}$ s.t A is skew symm then $|A| = \begin{cases} 0 & , n = \text{odd} \\ \text{perfect sq} & , n = \text{even} \end{cases}$

eg $|A| = \begin{vmatrix} 0 & -1 & -2 & 0 & 4 \\ 1 & 0 & 5 & 0 & -7 \\ 2 & -5 & 0 & 2 & 1 \\ 0 & 0 & -2 & 0 & -8 \\ -4 & 7 & -1 & 8 & 0 \end{vmatrix}_{5\times 5} = ? = 0$, eg $|A| = \begin{vmatrix} 0 & 3 \\ -3 & 0 \end{vmatrix}_{2\times 2} = 9 = (3)^2$ = perfect sq.

⊛ Determinant of skew symm Mat of odd order is always ZERO

eg $|A| = \begin{vmatrix} 0 & 1 & 2 & 0 \\ -1 & 0 & 4 & 0 \\ -2 & -4 & 0 & 6 \\ 0 & 0 & -6 & 0 \end{vmatrix}_{4\times 4} = ? = 36 = (6)^2$ = perfect sq.

(*) Max Number of Different elements that are required to construct a Symm Mat $A_{n\times n} = \frac{n(n+1)}{2}$

M-I

$$A = \begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ x_2 & y_1 & y_2 & \cdots & y_{n-1} \\ x_3 & y_2 & z_1 & \cdots & z_{n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ x_n & y_{n-1} & z_{n-2} & & \alpha \end{bmatrix}_{n\times n} = \text{Symm.}$$

Total elements $= n^2$

Total diff elements $= \frac{(n^2-n)+2n}{2} = \frac{n^2+n}{2} = \frac{n(n+1)}{2}$

M-II

$A = \begin{bmatrix} a & c \\ c & b \end{bmatrix} \Rightarrow$ Max Diff elements $= 3$

$A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \Rightarrow$ " " " $= 6$

$A = \begin{bmatrix} a & b & c & d \\ b & e & f & g \\ c & f & h & i \\ d & g & i & j \end{bmatrix} \Rightarrow$ " " " $= 10$

& so on ---

(*) Max Number of Ind. entries that are required to construct skew symm Mat $A_{n\times n} = \frac{n(n-1)}{2}$ Learn.

$$A = \begin{bmatrix} 0 & x_1 & x_2 & x_3 & \cdots & x_n \\ -x_1 & 0 & y_1 & y_2 & & y_{n-2} \\ -x_2 & -y_1 & 0 & z_1 & & z_{n-3} \\ \vdots & \vdots & \vdots & 0 & \ddots & \vdots \\ -x_n & -y_{n-2} & -z_{n-3} & \cdots & \cdots & 0 \end{bmatrix} = \text{skew symm.}$$

Lower Diag elements are Dependent

Diag elements are also Dependent.

i.e. only upper diag elements are Ind

No. of Ind. entries

= No. of upper diag elements

$$= \frac{n^2 - n}{2} = \frac{n(n-1)}{2}$$

$$= \frac{\text{Total elements} - \text{Diag elements}}{2}$$

Q. If $A=[a_{ij}]_{n\times n}$ s.t $a_{ij}=i^2-j^2$ $\forall i$ & j then for $n\geq 3$, $n=$ odd, $A^{-1}=?$

Sol: Let $n=3$, $A=[a_{ij}]_{3\times 3}=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}=\begin{bmatrix} 0 & -3 & -8 \\ 3 & 0 & -5 \\ 8 & 5 & 0 \end{bmatrix}_{3\times 3}$

$=$ Skew symm of odd order.

$\because |A|=0$

$\Rightarrow A^{-1}=$ DNE

Q. If $A=[a_{ij}]_{n\times n}$ s.t. $\forall i \& j$; $a_{ij}=-a_{ji}$ then the value of

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(a_{ij}) = \underline{\ ?\ }$$

Sol: Given $a_{ij}=-a_{ji}\ \forall i \& j$

For Diag elements: $i=j$ then

$$a_{ij}=-a_{ji}$$

$$\Rightarrow a_{ii}=-a_{ii} \Rightarrow 2a_{ii}=0$$

or $a_{ii}=0\ \forall i$ i.e. Diag elements are all zero.

For off Diag elements:

$a_{ij}=-a_{ji} \Rightarrow$ corresponding elements are of opposite sign

i.e. A is skew symm. so

$\boxed{\text{sum of all the elements in skew symm Mat} = \underline{0}}$

Some MORE SPECIAL MATRICES.

(*) if $A^2 = A$ then A is Idempotent Mat

if [A is Idempotent] $\Longrightarrow$ [$|A| = 0$ or 1]
$\Longleftarrow$

$|A^2| = |A|$
$|A|^2 = |A| = 0$
$|A|(|A|-1) = 0$
$|A| = 0$ or 1

(*) if $A^2 = I$ then A is Involutary Mat

if [A is Involutary] $\Longrightarrow$ [$|A| = 1$ or -1]
$\Longleftarrow$

$|A^2| = |I|$
$|A|^2 = 1 \Rightarrow |A| = 1$ or -1

(*) if $A^k = O$ then A is Nilpotent Mat of Power k where k is least +ve int.

eg if A is s.t $A^2 \neq 0$, $A^3 \neq 0$ but $A^4 = 0$ then $A^5 = A^6 = A^7 = \dots = O$

i.e A is Nilpotent Mat of index 4.

Idempotent Mat → If $\boxed{A^2 = A}$ then A is called Idempotent Mat.

$|A^2| = |A| \Rightarrow |A|^2 = |A| \Rightarrow |A|.(|A|-1) = 0 \Rightarrow |A| = 0$ or $|A| = 1$.

ie if A is an Idempotent Mat $\underset{\not\Leftarrow}{\Longrightarrow}$ $|A| = 0$ or 1.

Involutary Mat → If $\boxed{A^2 = I}$ then A is called Involutary Mat

$|A^2| = |I| \Rightarrow |A|^2 = 1 \Rightarrow |A| = \pm 1$.

ie if A is an Involutary Mat $\underset{\not\Leftarrow}{\Longrightarrow}$ $|A| = 1$ or -1

Nilpotent Mat → if $\boxed{A^K = O}$ then A is called Nilpotent Mat of power K.

where K is least +ve integer.

if A is s.t $A \neq O$, $A^2 \neq O$, $A^3 \neq O$ but $A^4 = O$ then $A^5 = A^6 = A^7 = \dots = O$

Here A is Nilpotent Mat of power 4. $|A^K| = |O| \Rightarrow |A|^K = 0 \Rightarrow |A| = 0$.

Q. if $A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$ then A is _____ mat

$$A^2 = A \cdot A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix} \cdot \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

$\because A^2 = O$ so A is Nilpotent Mat of index 2.

eg $A = \begin{bmatrix} 6 & 9 \\ -4 & -6 \end{bmatrix}$ then A is ? N-Mat of index 2

$(\because a = 2, b = 3)$ $(\because A^2 = O)$

If T is Idempdotent then $T^k = T$ for

(a) $k = 2$

(b) All integer k

(c) All positive integer $k \geq 2$

(d) All of the above

MAG

$\Rightarrow k = 2, 3, 4, \ldots$

W.K.that,

for an Idempotent Mat A

$A^2 = A$

i.e $T^2 = T$

$(T^k = T) \Rightarrow k = 2$

But wait, wait, wait....

$A^3 = A^4 = A^5 = \ldots = A$

i.e If T is Idempotent then $T^2 = T, T^3 = T, T^4 = T \ldots$

(*) if $\boxed{A.A^T = I}$ then A is called $\boxed{\text{orthogonal Mat.}}$

or $A^{-1} = A^T$ i.e orthogonal Mat is always Invertible

(*) if $\boxed{A.A^\theta = I}$ then A is called $\boxed{\text{Unitary Mat}}$

or $A^{-1} = A^\theta$ i.e unitary Mat is always invertible.

(*) $\underbrace{\text{Unitary Mat}}_{AA^\theta = I \text{ — (1)}}$ formed by $\underbrace{\text{Real Nos}}_{\text{Real Mat}}$ is <u>orthogonal</u> also.

i.e $A^\theta = A^T$ — (2)

By (1) & (2), $A.A^T = I$

which is Condition of orthogonal Mat

Orthogonal Mat → if $\boxed{A.A^T = I}$ then A is called Orthogonal Mat

or $(A^{-1} = A^T)$

Unitary Mat → if $\boxed{AA^\theta = I}$ then A is called Unitary Mat.

or $(A^{-1} = A^\theta)$

Note ① Det of O-Mat is either 1 or −1 (i.e. O-Mat can't be singular i.e. always Invertible)

Proof: w.k.that $AA^T = I$

$$|AA^T| = |I|$$

$$|A|.|A^T| = 1$$

$$|A|.|A| = 1 \Rightarrow |A|^2 = 1 \Rightarrow |A| = \pm 1$$

Sp Note:

if (A is an O-Mat) $\underset{\not\Leftarrow}{\Longrightarrow}$ $(|A| = 1 \text{ or } -1)$

If $A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is an orthogonal Mat. ? True

Sol: $AA^T = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

So A is orthogonal Mat & $A^{-1} = A^T = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

For the given orthogonal matrix Q.

$$Q = \begin{bmatrix} 3/7 & 2/7 & 6/7 \\ -6/7 & 3/7 & 2/7 \\ 2/7 & 6/7 & -3/7 \end{bmatrix}$$

The inverse is

is $Q^{-1} = ? = Q^T =$ (c)

(a) $\begin{bmatrix} 3/7 & 2/7 & 6/7 \\ -6/7 & 3/7 & 2/7 \\ 2/7 & 6/7 & -3/7 \end{bmatrix}$

(b) $\begin{bmatrix} -3/7 & -2/7 & -6/7 \\ 6/7 & -3/7 & -2/7 \\ -2/7 & -6/7 & 3/7 \end{bmatrix}$

(c) $\begin{bmatrix} 3/7 & -6/7 & 2/7 \\ 2/7 & 3/7 & 6/7 \\ 6/7 & 2/7 & -3/7 \end{bmatrix}$

(d) $\begin{bmatrix} -3/7 & -6/7 & -2/7 \\ -2/7 & -3/7 & -6/7 \\ -6/7 & -2/7 & 3/7 \end{bmatrix}$

Q.8 MS8 HW

if $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ then A is

(a) orthogonal

(b) Involutary

(c) unitary

(d) Hermitian

Q. if $A = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix}_{4\times4}$ & A is an orthogonal mat then $(AA')^{-1} = ?$

MCQ

$(AA^T = I)$

(a) $I_2 = I_{2\times2}$

(b) $I_3 = I_{3\times3}$

(c) $I_4 = I_{4\times4}$

(d) $I = I_{n\times n}$

Now $(AA')^{-1} = (I)^{-1} = I = I_{4\times4} = I_4$

THANK - YOU

Tel: dr puneet sir pw

DS & AI
CS & IT

Linear Algebra

Lecture No. 04

By- Dr. Puneet Sharma Sir

Recap of previous lecture

Topic

TYPES of MATRICES

Topics to be Covered

Topic

→ RANK OF MATRIX

→ Nature of Vectors.

Q. if $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ then A is

MS8

HW

(a) orthogonal

(b) Involutary

(c) unitary

(d) Hermitian

By observation, A is Hermitian ($\because A^{\theta} = A$)

Now, $A^2 = A \cdot A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} \cdot \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

$= \frac{1}{3}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

i.e. A is Involutary also.

Again, $AA^{\theta} = A(A) = A^2 = I$ i.e. A is unitary also.

QUICK RECAP :-

1. Singular Mat if $|A| = 0$
2. Non Sing Mat if $|A| \neq 0$
3. Invertible Mat if A^{-1} exist & $A^{-1} = \frac{\text{adj}A}{|A|}$
4. Real Mat if $\bar{A} = A$ or $A^{\theta} = A^{T}$
5. Complex Mat if $\bar{A} \neq A$
6. Symm Mat if $A^{T} = A$
7. Skew Symm Mat if $A^{T} = -A$
8. Hermitian Mat if $A^{\theta} = A$
9. Skew Herm Mat if $A^{\theta} = -A$
10. Idempotent if $A^{2} = A$
11. Involutary if $A^{2} = I$
12. Nilpotent if $A^{k} = O$
13. orthogonal Mat if $AA^{T} = I$ or $A^{-1} = A^{T}$
14. unitary Mat if $AA^{\theta} = I$ or $A^{-1} = A^{\theta}$
15. U.T.M $A = [a_{ij}]_{n \times n}$ where $a_{ij} = 0 \; \forall \; i > j$
16. L.T.M $A = [a_{ij}]_{n \times n}$ where $a_{ij} = 0 \; \forall \; i < j$
17. Diag Mat $A = [a_{ij}]_{n \times n}$ where

$$a_{ij} = \begin{cases} 0 & , i \neq j \\ \text{at least one element is Non zero} & , i = j \end{cases}$$

VECTORS & their Properties

Ordered n-tuple (n-dim vector) → Any ordered set of n numbers is called Ordered n-tuple. Generally it is represented in the form of Column Matrix (But we can also represent it in the form of Row Mat).

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} \text{ or } \begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_n \end{bmatrix}_{1\times n}$$

for eg $A = \begin{array}{c} \begin{bmatrix} - & - & - & - \\ - & - & - & - \\ - & - & - & - \end{bmatrix}_{3\times 4} \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \\ \begin{matrix} C_1 & C_2 & C_3 & C_4 \end{matrix} \end{array}$ has 3 Row vectors & 4 Column vectors

ordered pair $= (x_1, x_2) = x_1\hat{i} + x_2\hat{j} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$ 2 dim vector

(Point in 2D)

ordered Triplet $= (x_1, x_2, x_3) = x_1\hat{i} + x_2\hat{j} + x_3\hat{k} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} =$ 3-dim vector

(Point in 3D)

ordered Quadruple $= (x_1, x_2, x_3, x_4) = ?? = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} =$ 4 dim vector

$\vdots$

ordered n-tuple $= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} =$ n dim vector.

Consider $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$ & $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix}$ as two given vectors then

Dot product :→ $\boxed{X \cdot Y = X^T Y} = (x_1 y_1 + x_2 y_2 + x_3 y_3 + \ldots\ldots + x_n y_n)$

Proof: $X \cdot Y = X^T Y = [x_1\ x_2\ x_3 \ldots x_n]_{1\times n} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix}_{n\times 1} = [(x_1 y_1 + x_2 y_2 + \ldots + x_n y_n)]_{1\times 1}$

Also Calculate $XY = ? = [\ ]_{n\times 1} [\ ]_{n\times 1} = \text{N.D}$

Norm of Vector (Length) → $\boxed{\|X\| = \sqrt{X^T X}} = \sqrt{x_1^2 + x_2^2 + x_3^2 + \ldots\ldots + x_n^2}$

Normalised Vector (unit vector) → if $\|X\| = 1$ then X is called Normalised vector.

orthogonal vectors → if $[X \cdot Y = 0]$ then $[X$ & Y are called orthogonal vectors$]$

(while if $AA^T = I$ then A is called orthogonal Matrix)

three vectors X, Y, Z are called orthogonal if they are pairwise orthogonal.

i.e. $X \cdot Y = Y \cdot Z = Z \cdot X = 0$

orthonormal vectors → if $[X \cdot Y = 0, \|X\| = 1, \|Y\| = 1]$ then $[X$ & Y are called orthonormal vectors.$]$

Note: $\because X \cdot Y = Y \cdot X$ i.e. Dot product is commutative

Q: Check the Nature of following vectors given in following set

$$A = \{(1\ 2\ 1)', (2\ 1\ -4)', (3\ -2\ 1)'\} = \{X_1, X_2, X_3\}$$

$$\underbrace{X_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, X_2 = \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}, X_3 = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}}_{\text{orthogonal vectors}}$$

$\because X_1 \cdot X_2 = 1\times 2 + 2\times 1 + 1\times(-4) = 0.$

$X_2 \cdot X_3 = 2\times 3 + 1\times(-2) + (-4)\times 1 = 0$

$X_3 \cdot X_1 = 3\times 1 + (-2)(2) + 1\times 1 = 0$

$\|X_1\| = \sqrt{1^2+2^2+1^2} = \sqrt{6}$

$\|X_2\| = \sqrt{2^2+1^2+(-4)^2} = \sqrt{21}$

$\|X_3\| = \sqrt{3^2+(-2)^2+1^2} = \sqrt{14}$

ie Not orthonormal.

Note: $\because X_1, X_2, X_3$ are orthogonal vectors $\Rightarrow$ they are L.I also.

2009 / LS

MS8

The vectors $(1,1,1)$ & $(1, a, a^2)$ (where $a = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$) are ?

(a) Linearly Ind.

(b) L. Dep.

(c) orthogonal.

(d) orthonormal.

$X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, Y = \begin{bmatrix} 1 \\ a \\ a^2 \end{bmatrix} = \begin{bmatrix} 1 \\ \omega \\ \omega^2 \end{bmatrix}$

$= \omega =$ cube root of unity

(i.e. $\omega^3 = 1$, $1 + \omega + \omega^2 = 0$)

$X \cdot Y = 1 \times 1 + 1 \times \omega + 1 \times \omega^2 = 1 + \omega + \omega^2 = 0$.

Hence X & Y are orthogonal $\Longrightarrow$ LI also.

$\because \|X\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$ i.e. Not orthonormal.

& $\|Y\| = \sqrt{1^2 + \omega^2 + (\omega^2)^2} = \sqrt{1 + \omega^2 + \omega^4} = \sqrt{1 + \omega^2 + \omega} = 0$

$\because X \neq KY$ for any Non Zero Value of K

So vectors are LI

Note: $z^3-1=0 \Rightarrow \boxed{z=(1)^{1/3}} \begin{cases} 1 \\ \omega \\ \omega^2 \end{cases}$ where $\omega = -\frac{1}{2}+i\frac{\sqrt{3}}{2}$

& $\omega^2 = -\frac{1}{2}-i\frac{\sqrt{3}}{2}$

where $\omega^3=1$, $1+\omega+\omega^2=0$, $\bar{\omega}=\omega^2$, $\overline{\omega^2}=\omega$, $\frac{1}{\omega}=\omega^2$

$$\frac{1}{\omega} = ? = \frac{\omega^2}{\omega^3} = \frac{\omega^2}{1} = \omega^2$$

Proof:

$$z^3-1=0$$

$$(z-1)(z^2+z+1)=0$$

$$z=1 \ \& \ z^2+z+1=0$$

$$z=1 \ \& \ z=\frac{-1\pm\sqrt{-3}}{2}=\frac{-1\pm i\sqrt{3}}{2}=\underbrace{-\frac{1}{2}+\frac{i\sqrt{3}}{2}}_{=\omega},\ \underbrace{-\frac{1}{2}-\frac{i\sqrt{3}}{2}}_{=\omega^2}$$

TWO GAJAB KI PROPERTIES →

① "Column vectors of an orthogonal Matrix are orthonormal vectors"

eg $A = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$ is an orthogonal mat ($\because AA^T = I$)

$C_1 \quad C_2 \quad C_3$

$C_1 \cdot C_2 = C_2 \cdot C_3 = C_3 \cdot C_1 = 0$

& $\|C_1\| = \|C_2\| = \|C_3\| = 1$ Hence verified

② If [vectors are orthogonal] $\Longrightarrow$ then [these are L.I also.] (True)

$\not\Longleftarrow$

ie LI set of vectors are also orthogonal vectors (F)

Q $A = \begin{bmatrix} 1/9 & -4/9 & 8/9 \\ 8/9 & 4/9 & 1/9 \\ \alpha/9 & -7/9 & \beta/9 \end{bmatrix}$ is an orthogonal Matrix then $\alpha+\beta = ?$

$C_1 \quad C_2 \quad C_3$

$\Downarrow$

Column vectors are orthogonal vectors as well as of unit Norm.

(a) 0

(b) 4

(c) -4

(d) 8

ie $C_1 \cdot C_2 = 0$ & $C_2 \cdot C_3 = 0$

$\frac{1}{81}[-4+32-7\alpha]=0$ & $\frac{1}{81}[-32+4-7\beta]=0$

$28-7\alpha=0$ $\quad$ $-28-7\beta=0$

$\alpha=4$ $\quad$ $\beta=-4$

So $\alpha+4=0$

$\|C_1\| = 1$, $\|C_2\| = 1$, $\underbrace{\|C_3\| = 1}$

$\frac{1}{9}\sqrt{1^2+8^2+\alpha^2} = 1$

$65+\alpha^2 = 81$

$\alpha^2 = 16$

$\alpha = \pm 4$

$\frac{1}{9}\sqrt{64+1+\beta^2} = 1$

$65+\beta^2 = 81$

$\beta^2 = 16$

$\beta = \pm 4$

But $C_1 \cdot C_2 = 0 \Rightarrow \alpha = 4$

$C_2 \cdot C_3 = 0 \Rightarrow \beta = -4$

i.e. $\alpha + \beta = 0$.

RANK

(*) **Submatrix** → By deleting some Rows or Some Columns or Both, the Matrix obtained is called Submatrix.

Def^n of Rank: "It is the order of Non singular Submatrix of Highest order that can exist in a given Mat"

Def^n In Books:

if $\boxed{\rho(A_{6\times7}) = 4}$ then

- ∃ at least one Non singular Submatrix of order 4×4
- Every square Submatrix of order 5×5 & 6×6 are singular

Eg: $A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 5 & 25 \\ 1 & 3 & 9 \end{bmatrix}_{3\times3}$ $\because |A| = (2-5)(5-3)(3-2) = (-3)(2)(1) = -6$

i.e. $|A| \neq 0 \Rightarrow A$ is Non singular of $3\times3 \Rightarrow \rho(A) =$ three

Eg $A = \begin{bmatrix} 2 & 3 & 5 \\ 3 & 5 & 2 \\ -5 & -8 & -7 \end{bmatrix}_{3\times3}$ $\because |A| = 0$ i.e. A is singular then $\rho(A) \neq$ three

Consider a submatrix of the type

$A_1 = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}_{2\times2}$ (By deleting R_3 & C_3)

$\because |A_1| = +1$ i.e. $|A_1| \neq 0$ or A_1 is (Non sing)$_{2\times2} \Rightarrow \rho(A) =$ order of $A_1 =$ Two

Eg $A = \begin{bmatrix} 2 & -3 & 4 \\ -4 & 6 & -8 \\ 4 & -6 & 7 \end{bmatrix}_{3\times3}$ $\because |A| = 0$ i.e. A is singular, $\rho(A) \neq 3$

Consider $A_1 = \begin{bmatrix} 6 & -8 \\ -6 & 7 \end{bmatrix}$ By deleting R_1 & C_1

$\because A_1$ is (Non sing)$_{2\times2} \Rightarrow \rho(A) =$ order of $A_1 =$ Two

Q(4) $A = \begin{bmatrix} 1 & 2 & 4 & 8 & 16 \\ 1 & 2 & 4 & 8 & 16 \\ 1 & 2 & 4 & 8 & 16 \\ 1 & 2 & 4 & 8 & 16 \end{bmatrix}_{4\times 5}$

$\rho(A) \neq 5$ $\because$ it is not possible to find any submatrix of 5×5. ☹

$\rho(A) \neq 4$ or 3 or 2 $\because$ all the submatrices of 4×4, 3×3, 2×2 are sing.

So $\rho(A) =$ one Ans.

Q(5) $A = \begin{bmatrix} 2 & & & & \\ & -1 & & & \\ & & 0 & & \\ & & & 5 & \\ & & & & 3 \end{bmatrix}_{5\times 5}$ Zero Zero

or diag$(2,-1,0,5,3)$ then $\rho(A) = ?$

$\because |A| = (2)(-1)(0)(5)(3) = 0$ i.e. $\rho(A) \neq 5$

Consider $A_1 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}_{4\times 4}$ By deleting R_3 & C_3

$\because A_1$ is Non sing. so

$\rho(A) =$ order of $A_1 =$ Four

Sol: $A = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}_{6\times 6}$

$\rho(A) \neq 6, 5, 4$ $\because$ all the submatrices of 6×6, 5×5, 4×4 are singular

Consider $A_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$

By deleting $R_2, R_4, R_5, C_4, C_5, C_6$

$\because A_1$ is $(\text{Non Sing})_{3\times 3} \Rightarrow \rho(A) = \text{three}$.

M-II

$A = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_6} \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{6\times 6} = A_1$

$\because (\text{Non Sing Mat})_{3\times 3}$ exist so $\rho(A_1) = 3$

$\because A \sim A_1 \Rightarrow \rho(A) = 3$.

Note: ① $\boxed{\text{E-operations do not alter the Rank of Matrix}}$ i.e

we are free to apply all three E-operations while Calculating Rank.

That's why RANK is called INVARIANT property of Mat.

② Equivalent Matrices :→ Matrix obtained by applying E-operations are equivalent to each other.

"Equivalent Matrices have same Rank"

P8 (PYQ)

If $A = \begin{bmatrix} 1 & a & a^2 & a^3 & \cdots & a^n \\ 1 & a & a^2 & a^3 & \cdots & a^n \\ 1 & a & a^2 & a^3 & \cdots & a^n \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & a & a^2 & a^3 & \cdots & a^n \end{bmatrix}_{(n+1)\times(n+1)}$ then Rank of this $(n+1)\times(n+1)$ Matrix ?

$$\xrightarrow[\substack{R_3 \to R_3 - R_1 \\ \vdots \\ R_{n+1} \to R_{n+1} - R_1}]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & a & a^2 & \cdots & a^n \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{(n+1)\times(n+1)} \Rightarrow \rho(A) = \text{one}.$$

Q. (2015) if $A = [\alpha_{ij}]_{n\times n}$ s.t $\alpha_{ij} = i\cdot j \ \forall i \ \& \ j$ then for $n \geq 4$, $\rho(A) = ?$

Let $n = 4$, $A = [\alpha_{ij}]_{4\times 4} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \\ 4 & 8 & 12 & 16 \end{bmatrix}_{4\times 4}$

$$\xrightarrow[\substack{R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - 4R_1}]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{4\times 4} \quad \rho(A) = \text{one}$$

Q. If $A = [a_{ij}]_{n\times n}$; $a_{ij} = i - j$ $\forall$ i & j then for $n = 4$, $\rho(A) = ?$

$$A = [a_{ij}]_{4\times 4} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} 0 & -1 & -2 & -3 \\ 1 & 0 & -1 & -2 \\ 2 & 1 & 0 & -1 \\ 3 & 2 & 1 & 0 \end{bmatrix}_{4\times 4}$$

$\because |A| = \cdots = 0$ i.e. $\rho(A) \neq 4$

& all the submatrices of order 3×3 are also singular. So $\rho(A) \neq 3$

So By observation, $\rho(A) = 2$

= skew symm Mat of even order

$\Rightarrow |A| =$ Perfect Sq.

But 0 is also Treated as Perfect Sq.

So Directly we <u>can not</u> say that $|A| \neq 0$.

Q $|A| = \begin{vmatrix} 0 & -1 & -2 & -3 \\ 1 & 0 & -1 & -2 \\ 2 & 1 & 0 & -1 \\ 3 & 2 & 1 & 0 \end{vmatrix} \xrightarrow[R_4 \to R_4 - 3R_2]{R_3 \to R_3 - 2R_2} \begin{vmatrix} 0 & -1 & -2 & -3 \\ 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 6 \end{vmatrix}$

$= -(1) \begin{vmatrix} -1 & -2 & -3 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \end{vmatrix} \xrightarrow{R_2 \to R_2 + R_1} - \begin{vmatrix} -1 & -2 & -3 \\ 0 & 0 & 0 \\ 2 & 4 & 6 \end{vmatrix} = 0 = (0)^2$

= Perfect sq.

M-II Convert $A = \begin{bmatrix} 0 & -1 & -2 & -3 \\ 1 & 0 & -1 & -2 \\ 2 & 1 & 0 & -1 \\ 3 & 2 & 1 & 0 \end{bmatrix}$ into an Echelon Form & Hence Find Rank

HW

See Next Slide.

Sol:

$$A = \begin{bmatrix} 0 & -1 & -2 & -3 \\ 1 & 0 & -1 & -2 \\ 2 & 1 & 0 & -1 \\ 3 & 2 & 1 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & -1 & -2 & -3 \\ 2 & 1 & 0 & -1 \\ 3 & 2 & 1 & 0 \end{bmatrix} \xrightarrow[R_4 - 3R_1]{R_3 - 2R_1} \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & +6 \end{bmatrix}$$

$$\xrightarrow[R_4 + 2R_2]{R_3 + R_2} \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \text{Echelon form}$$

So $\rho(A) = \text{Two}$.

Q: $A = \begin{bmatrix} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 44 \end{bmatrix}$ then $\rho(A) = ?$

$$\xrightarrow[\substack{R_3 \to R_3 - R_2 \\ R_2 \to R_2 - R_1}]{R_4 \to R_4 - R_3} \begin{bmatrix} 11 & 12 & 13 & 14 \\ 10 & 10 & 10 & 10 \\ 10 & 10 & 10 & 10 \\ 10 & 10 & 10 & 10 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_2]{R_4 \to R_4 - R_2} \begin{bmatrix} 11 & 12 & 13 & 14 \\ 10 & 10 & 10 & 10 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{4\times 4} = A_1$$

By observation, $\rho(A_1) = \text{two}$

$\therefore A \sim A_1 \Rightarrow \rho(A) = \text{two}$ Ans

Echelon Form → (Triangular Form) →

Any Mat $A_{m\times n}$ is said to be in Echelon Form if

① Number of Zeros before the 1st Non Zero element in a Row should be in an Increasing order in the subsequent Rows.

② Every Zero Row (if exist) should occur at the bottom of a Mat.

Note: ① $\boxed{\rho(\text{Echelon form}) = \text{Number of Non Zero Rows.}}$

② Any Mat can be converted into an E-form by using E-operations.

③ It is advisable to apply only E-Row operations while converting given Mat into an E-Form. (as per our syllabus)

Flowchart of Converting given Mat into an E-Form →

① Make a_{11} unity (Not compulsory but advisable)

② Make all the elements of C_1 (that lies below a_{11}) zero by using E-Row operation

③ Make a_{22} unity (Not compulsory but advisable)

④ Make all the elements of C_2 (that lies below a_{22}) zero by " " "

⑤ Make a_{33} unity & so on - - - - - -

Note: Take care, In E-Form, a_{21} = Zero.

Eg:

$$\begin{bmatrix} 2 & -1 & 4 & 0 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 4 & 7 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 2 & 1 & 4 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & -3 & 1 & 4 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 7 & 5 \\ 0 & 0 & 0 & 2 \end{bmatrix},$$

✓ $\rho(A) = 4$ ✓ $\rho(A) = 3$ ✓ $\rho(A) = 4$

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 4 \\ 0 & 0 & 7 & 5 \end{bmatrix}, \quad \begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & 4 & -1 & 3 \\ 0 & 2 & 1 & -3 \\ 0 & 0 & 7 & 5 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

✗ ✓ $\rho(A) = 3$ ✗

THANK - YOU

Tel: dr puneet sir pw

DS & AI
CS & IT

Linear Algebra

Lecture No. 05

By- Dr. Puneet Sharma Sir

Recap of previous lecture

Topic

RANK (part 1)

Topics to be Covered

Topic

→ RANK (Part 2)

→ LD/LI vectors.

QUICK RECAP :-

1. Singular Mat if $|A| = 0$
2. Non Sing Mat if $|A| \neq 0$
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17. Diag Mat $A = [a_{ij}]_{n \times n}$ where

$$a_{ij} = \begin{cases} 0 & , i \neq j \\ \text{at least one element is Non zero} & , i = j \end{cases}$$

RECAP

RANK

(*) **Submatrix** → By deleting some Rows or Some Columns or Both, the Matrix obtained is called Submatrix.

Def^n of Rank: "It is the order of Non singular Submatrix of Highest order that can exist in a given Mat"

Def^n In Books:

if $\boxed{\rho(A_{6\times7}) = 4}$ then

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- Every Square Submatrix of order 5×5 & 6×6 are Singular

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Flowchart of converting given Mat into an E-Form → (RECAP)

① Make a_{11} unity (Not compulsory but advisable)

② Make all the elements of C_1 (that lies below a_{11}) zero by using E-Row operation

③ Make a_{22} unity (Not compulsory but advisable)

④ Make all the elements of C_2 (that lies below a_{22}) zero by " " "

⑤ Make a_{33} unity & so on - - - - - -

Note: Take care, In E-Form, a_{21} = Zero.

Q: $A = \begin{bmatrix} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 44 \end{bmatrix}$ then $\rho(A) = ?$

$$\xrightarrow[\substack{R_3 \to R_3 - R_2 \\ R_2 \to R_2 - R_1}]{R_4 \to R_4 - R_3} \begin{bmatrix} 11 & 12 & 13 & 14 \\ 10 & 10 & 10 & 10 \\ 10 & 10 & 10 & 10 \\ 10 & 10 & 10 & 10 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_2]{R_4 \to R_4 - R_2} \underset{4\times 4}{\begin{bmatrix} 11 & 12 & 13 & 14 \\ 10 & 10 & 10 & 10 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}} = A_1$$

By observation, $\rho(A_1) = \text{two}$

$\therefore A \sim A_1 \Rightarrow \rho(A) = \text{two}$ Ans

Sol:

$$A = \begin{bmatrix} 0 & -1 & -2 & -3 \\ 1 & 0 & -1 & -2 \\ 2 & 1 & 0 & -1 \\ 3 & 2 & 1 & 0 \end{bmatrix}_{4\times 4} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & -1 & -2 & -3 \\ 2 & 1 & 0 & -1 \\ 3 & 2 & 1 & 0 \end{bmatrix} \xrightarrow[R_4 - 3R_1]{R_3 - 2R_1} \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & +6 \end{bmatrix}$$

$$\xrightarrow[R_4 + 2R_2]{R_3 + R_2} \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \text{Echelon form}$$

So $\rho(A) = \text{Two}$.

Q. NET $A = \begin{bmatrix} 2 & & & & & \\ & -1 & & & & \\ & & 0 & & & \\ & & & 3 & & \\ & & & & 4 & \\ & & & & & 0 \end{bmatrix}_{6\times 6}$ Zero Zero

or $A = \text{diag}(2, -1, 0, 3, 4, 0)$ then $\rho(A) = ?$

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{6\times 6} \xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_4 \leftrightarrow R_5} \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{6\times 6}$$

So $\rho(A)$ = No. of Non Zero Rows $= 4$.

(M-I) $A_1 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}_{4\times 4}$ = Diag Mat

= Non Singular $\Rightarrow \rho(A)$ = Four.

Q. $A = \begin{bmatrix} 2 & -1 & 4 & 2 \\ 1 & 3 & 0 & -2 \\ -3 & -2 & -4 & 0 \\ 0 & 2 & 1 & 3 \end{bmatrix}_{4\times 4}$

(a) 1

(b) 2

(c) 3

(d) 4

$$\xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 3 & 0 & -2 \\ 2 & -1 & 4 & 2 \\ -3 & -2 & -4 & 0 \\ 0 & 2 & 1 & 3 \end{bmatrix} \xrightarrow[R_3+3R_1]{R_2-2R_1} \begin{bmatrix} 1 & 3 & 0 & -2 \\ 0 & -7 & 4 & 6 \\ 0 & 7 & -4 & -6 \\ 0 & 2 & 1 & 3 \end{bmatrix}$$

$$\xrightarrow{R_3+R_2} \begin{bmatrix} 1 & 3 & 0 & -2 \\ 0 & -7 & 4 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 3 \end{bmatrix} \xrightarrow{R_2+3R_4} \begin{bmatrix} 1 & 3 & 0 & -2 \\ 0 & -1 & 7 & 15 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 3 \end{bmatrix}$$

$$\xrightarrow{R_4+2R_2} \begin{bmatrix} 1 & 3 & 0 & -2 \\ 0 & -1 & 7 & 15 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 15 & 33 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix} 1 & 3 & 0 & -2 \\ 0 & -1 & 7 & 15 \\ 0 & 0 & 15 & 33 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \text{Echelon Form} = B$$

$\rho(B) = 3 \because A \sim B \Rightarrow \rho(A) = 3.$

The rank of the matrix $\begin{bmatrix} 1 & 4 & 8 & 7 \\ 0 & 0 & 3 & 0 \\ 4 & 2 & 3 & 1 \\ 3 & 12 & 24 & 2 \end{bmatrix}$

(a) 3 (b) 1

(c) 2 (d) 4

$$\xrightarrow[R_4 - 3R_1]{R_3 - 4R_1} \begin{bmatrix} 1 & 4 & 8 & 7 \\ 0 & 0 & 3 & 0 \\ 0 & -14 & -29 & -27 \\ 0 & 0 & 0 & -19 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 4 & 8 & 7 \\ 0 & -14 & -29 & -27 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -19 \end{bmatrix} = \text{E-Form.}$$

$\rho(A) =$ Four.

Ist II

$$|A| = (-3)\begin{vmatrix} 1 & 4 & 7 \\ 4 & 2 & 1 \\ 3 & 12 & 2 \end{vmatrix} = (-3)\begin{vmatrix} 1 & 4 & 7 \\ 4 & 2 & 1 \\ 0 & 0 & -19 \end{vmatrix}$$

$$= (-3)\left[(-19)(2-16)\right] \neq 0 \text{ i.e. } (A \text{ is Non Sing})_{4\times 4} \Rightarrow \rho(A) = 4$$

2015 / 2M

If $A = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}_{5\times 5} \xrightarrow[R_4 \leftrightarrow R_5]{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_5 \to R_5 + R_1 + R_2 + R_3 + R_4} \sim \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \rho(A) = 4$

(a) 5

(b) 4

(c) 3

(d) 2

M-II

$\xrightarrow{R_4 + R_1} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \sim \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{R_4 + R_2} \sim \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$

$\xrightarrow{R_4 + R_3} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{R_5 + R_4} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ = E-Form

so $\rho(A) = 4$

M III

WRONG APP

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}_{5\times 5} \xrightarrow{R_2 \leftrightarrow R_4} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \Rightarrow \rho(A) = 5$$

But $a_{21} \neq 0$??

Applying elementary transform to a matrix its rank _____.

(a) increases
(b) decreases
(c) does not change
(d) None of the above

Properties of Rank :→

① $\rho(A_{m\times n}) \geq 1$ & $\rho(A_{m\times n}) \leq \min\{m,n\}$ eg $\rho(A_{6\times 4}) \leq 4$

② $\rho(\text{Null Mat}) = 0$ (defined) ie $\boxed{1 \leq \rho(A_{m\times n}) \leq \min\{m,n\}}$

③ if $A_{n\times n}$ & A is Non Singular then $\rho(A) = n$

④ $\rho(A+B) = \rho(A) + \rho(B)$ (F) $\left[\because \rho(A+B) \leq \rho(A) + \rho(B)\ (T)\right]$

⑤ $\rho(A) = \rho(A^T) = \rho(A^\theta) = \rho(A^{-1}) = \rho(AA^T) = \rho(AA^\theta)$

$\Rightarrow$ ~~$\rho(A)$ & ρ(orthogonal Mat) are equal.~~ ISSEY BADA PAAP NAHI HOO SAKTA.

($\because$ If $AA^T = I$ then A is called O-Mat)

⑥ if A & B are two Matrices s.t AB is defined then

$$\boxed{\rho(AB) \le \min\{\rho(A), \rho(B)\}}$$

eg if $\rho(A_{5\times6}) = 4$ & $\rho(B_{6\times7}) = 3$

then $\rho(AB)_{5\times7} \le \min\{4, 3\} \Rightarrow \rho(AB) \le 3$

ie

"RANK of the product can never exceed their individual Ranks."

⑦ $\rho(\text{Row Mat}) = \rho(A_{1\times n}) = 1$

$\rho(\text{Column Mat}) = \rho(B_{n\times1}) = 1$

$\rho(\text{Row} \times \text{Column}) = \rho(AB)_{1\times1} = 1 \text{ or } 0$

$\rho(\text{Column} \times \text{Row}) = \rho(BA)_{n\times n} = 1 \text{ or } 0$ (using prop. 6).

Q1: if $A = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}_{4\times 4}$ then $\rho(A) = ?$

$\because |A| = \cdots\cdots = 5 \neq 0$.

is A is Non Singular of 4×4

$\Rightarrow \rho(A) = 4$ **Ans** (using Prop 3)

$$|A| = \begin{vmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{vmatrix} \xrightarrow{C_1 + C_1 + (C_2 + C_3 + C_4)} \begin{vmatrix} 5 & 1 & 1 & 1 \\ 5 & 2 & 1 & 1 \\ 5 & 1 & 2 & 1 \\ 5 & 1 & 1 & 2 \end{vmatrix} = 5 \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{vmatrix}$$

$$\xrightarrow[\substack{R_3 - R_1 \\ R_4 - R_1}]{R_2 - R_1} = 5 \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 5[(1)] = 5$$

Q. if $\rho(A)=3$ where $A=\begin{bmatrix} \mu-1 & 0 & 0 \\ 0 & \mu-1 & 0 \\ 0 & 0 & \mu-1 \\ -6 & 11 & -6 & 1 \end{bmatrix}_{4\times4}$

then No. of different values of μ will be?

(a) 1 = one

(b) 2 = two

(c) 3 = three

(d) All of above

$\rho(A)=3 \Rightarrow |A|_{4\times4}=0$

$$\begin{vmatrix} \mu-1 & 0 & 0 & 0 \\ 0 & \mu-1 & 0 & 0 \\ 0 & 0 & \mu-1 & 0 \\ -6 & 11 & -6 & 1 \end{vmatrix}=0$$

$\mu^3-6\mu^2+11\mu-6=0$

$\mu^3-6\mu^2+11\mu-6=0$

$(\mu-1)(\mu-2)(\mu-3)=0$

$\mu=1,2,3$

Q: if $\underbrace{\rho(A_{m\times n})=n}_{n\le m}$, $\underbrace{\rho(B_{n\times p})=p}_{p\le n}$ then $\rho(AB)=?$ (a) m (b) n (c) p (d) $n+p$

$$\boxed{p\le n\le m} \Rightarrow \rho(AB)_{m\times p}\le p$$

Let $P = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$ and $Q = \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$ be two matrices. Then the rank of P + Q is ____2____.

$$P+Q = \begin{bmatrix} 0 & -1 & -2 \\ 8 & 9 & 10 \\ 8 & 8 & 8 \end{bmatrix}_{3\times 3}$$

$\Rightarrow$ Consider $A_1 = \begin{bmatrix} 0 & -1 \\ 8 & 9 \end{bmatrix}_{2\times 2} \Rightarrow A_1$ is non singular.

So $\rho(A) =$ two.

$$|P+Q| = 0-(-1)[64-80]+(-2)[64-72]$$

$$= +(1)[-16] - 2[-8]$$

$$= -16+16 = 0$$

i.e. $(P+Q)$ is singular $\Rightarrow \rho(P+Q) \neq 3$

Q. (2015) If $A=\begin{bmatrix} p & q \\ r & s \end{bmatrix}$, $B=\begin{bmatrix} p^2+q^2 & rp+sq \\ pr+qs & r^2+s^2 \end{bmatrix}$ s.t. $\rho(A)=N$ then $\rho(B)=$?

(a) N ✓

(b) $N-1$

(c) $2N$

(d) N^2

$\because A.A^T = \begin{bmatrix} p & q \\ r & s \end{bmatrix}\begin{bmatrix} p & r \\ q & s \end{bmatrix} = \begin{bmatrix} - & - \\ - & - \end{bmatrix} = B$

w.k.that $\rho(A)=N$

$\Rightarrow \rho(AA^T)=N$

or $\rho(B)=N$

M-II let $p=2, q=-3, r=1, s=4$

$A=\begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}$, $B=\begin{bmatrix} 13 & -10 \\ -10 & 17 \end{bmatrix}$

$|A|\neq 0$, $|B|\neq 0$

$\rho(A)=2=N$ $\rho(B)=2=N$

again, $p=2, q=0, r=4, s=0$.

$A=\begin{bmatrix} 2 & 0 \\ 4 & 0 \end{bmatrix}$, $B=\begin{bmatrix} 4 & 8 \\ 8 & 16 \end{bmatrix} \Rightarrow \rho(A)=1=N$

$\rho(B)=1=N$ Ans

Q: if $A = \begin{bmatrix} 2 & -1 & 3 \end{bmatrix}_{1\times 3}$, $B = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}_{3\times 1}$ then $\rho(AB) = ?$ & $\rho(BA) = ?$

$$AB = \begin{bmatrix} 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} (8-2+3) \end{bmatrix} = \begin{bmatrix} 9 \end{bmatrix}_{1\times 1} \Rightarrow \rho(AB) = \text{one}$$

$$BA = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}\begin{bmatrix} 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 8 & -4 & 12 \\ 4 & -2 & 6 \\ 2 & -1 & 3 \end{bmatrix}_{3\times 3} \neq O \text{ so } \rho(BA) = \text{one}$$

$$\because BA = \begin{bmatrix} 2 & -1 & 3 \\ 4 & -2 & 6 \\ 8 & -4 & 12 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & -1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3} \Rightarrow \rho(BA) = \text{one}$$

Q. (CE) if $X = [x_1\ x_2\ x_3 \cdots x_n]'$ is an ordered n-tuple Non Zero vector

s.t $XX' = V$ then $\rho(V) = ?$

(a) n

(b) $n-1$

(c) 1 ✓

(d) n^2

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} = \text{Column Mat}, \quad X' = [x_1\ x_2\ x_3 \cdots x_n]_{1\times n} = \text{Row Mat}$$

$$\rho(V) = \rho(XX')_{n\times n} = \rho[\text{Column Mat} \times \text{Row Mat}]_{n\times n} = 1.$$

(using prop 7)

A be 3 × 3 matrix and Rank of A^3 is 2. Then rank of A^6 will be :

(a) 2 (b) 3

(c) 1 (d) $\cancel{1 \le 2}$, $1 \le \rho(A^6) \le 2$

$\because A_{3\times3} \Rightarrow (A^3)_{3\times3}$ & $(A^6)_{3\times3}$

$\because \rho(A^3) = 2 \Rightarrow |A^3|_{3\times3} = 0 \Rightarrow |A| = 0 \Rightarrow |A|^6 = 0 \Rightarrow |A^6| = 0$

ie A, A^3, A^6 all are singular $\Rightarrow \rho(A^6) \le 2$

Rank of a skew symmetric matrix cannot be

MCQ.

(a) 1 (b) 2

(c) 4 (d) 0

VERY GOOD QUESTION.

$$A = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}_{3\times 3} \Rightarrow \rho(A) \neq 3 \quad \text{But } \rho(A) = 2$$

$$A = \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}_{2\times 2} \Rightarrow \rho(A) = 2$$

$$A = \begin{bmatrix} 0 \end{bmatrix}_{1\times 1} = \text{Null Mat}$$

i.e skew symm Mat of order 1×1 = DNE

$$\text{eg } A = \begin{bmatrix} 0 & -1 & -2 & 0 \\ 1 & 0 & -4 & 0 \\ 2 & 4 & 0 & 6 \\ 0 & 0 & -6 & 0 \end{bmatrix}_{4\times 4}$$

$\because |A| = 36 \Rightarrow \rho(A) = 4$

$$\text{eg } A = \begin{bmatrix} 0 & -1 & -2 & -3 \\ 1 & 0 & -1 & -2 \\ 2 & 1 & 0 & -1 \\ 3 & 2 & 1 & 0 \end{bmatrix}_{4\times 4}$$

$\because |A| = 0 \Rightarrow \rho(A) = 2$

& $\rho(\text{skew symm})_{n\times n} \geq 2$

Linearly Dependent & Linearly Independent Vectors →

L.D vectors → vectors are called L.D if ∃ Linear Relation b/n them.

L.I Vectors → " " " L.I if there DNE any Linear Relation b/n them.

Linear Combination of vectors → Let $X_1, X_2, X_3, \ldots, X_r$ are the given Vectors
(Linear Relationship) & $k_1, k_2, k_3 \ldots, k_r$ are the Scalars (Const)
then the Relationship of the type $\boxed{k_1X_1 + k_2X_2 + k_3X_3 + \ldots + k_rX_r = 0}$ — (1)
is called Linear Combination of vectors.

Sp. Note $X = \begin{bmatrix} \\ \\ \end{bmatrix}_{n\times 1}$ then $X^2 = X \cdot X = \begin{bmatrix} - \\ - \\ \vdots \\ - \end{bmatrix}_{n\times 1} \begin{bmatrix} - \\ - \\ \vdots \\ - \end{bmatrix}_{n\times 1} = N.D$ i.e if Relationship exist it is always in Linear Form.

Methods of Checking the Nature of Vectors →

Consider the given vectors are $X_1, X_2, X_3, \ldots, X_r$

then Construct a Matrix A as follows; $A = [X_1 X_2 X_3 \ldots X_r] \neq$ Row Mat

(M-I) General Method (always applicable) →

(i) if $\rho(A) =$ No. of vectors $\Rightarrow$ vectors are LI

(ii) if $\rho(A) <$ " " " $\Rightarrow$ " " LD

(M-II) Tricky Method (applicable only when A is Sq Mat) →

(i) if $|A| \neq 0 \Rightarrow$ vectors are LI

(ii) if $|A| = 0 \Rightarrow$ " " LD

Note: If there are two vectors X_1 & X_2 then No Need to use G. Method or T. Method, only use observation method

i.e Consider X_1 & X_2 are given vectors.

if $\boxed{X_1 = KX_2}$ for any K then vectors are LD (for Non Zero K)

& if $X_1 \neq KX_2$ (i.e K = DNE) then " " LI

eg $X_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, X_2 = \begin{bmatrix} -4 \\ 2 \\ -6 \end{bmatrix}$

$\because -2X_1 = X_2$

i.e LD

eg $X_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, X_2 = \begin{bmatrix} -4 \\ 2 \\ 0 \end{bmatrix}$

$\because X_1 \neq KX_2$ for any K i.e K = DNE

Hence LI

eg Check the Nature of vectors, $\begin{bmatrix}2\\-1\\0\\3\end{bmatrix}, \begin{bmatrix}-1\\2\\3\\1\end{bmatrix}, \begin{bmatrix}-6\\6\\6\\-4\end{bmatrix}$

Sol: $X_1=\begin{bmatrix}2\\-1\\0\\3\end{bmatrix}, X_2=\begin{bmatrix}-1\\2\\3\\1\end{bmatrix}, X_3=\begin{bmatrix}-6\\6\\6\\-4\end{bmatrix}$

M-I

By observation, $X_1-X_2=\begin{bmatrix}3\\-3\\-3\\2\end{bmatrix}=-\frac{1}{2}X_3$

i.e. $\boxed{2X_1-2X_2+X_3=0}$

Hence X_1, X_2, X_3 are LD

M-II $A=[X_1\ X_2\ X_3]$

$$A=\begin{bmatrix}2&-1&-6\\-1&2&6\\0&3&6\\3&1&-4\end{bmatrix}_{4\times3}$$

$$\xrightarrow{R_1\leftrightarrow R_2}\begin{bmatrix}-1&2&6\\2&-1&-6\\0&3&6\\3&1&-4\end{bmatrix}$$

$$\xrightarrow{\text{H.W.}}\begin{bmatrix}-1&2&6\\0&a_{21}&a_{22}\\0&0&0\\0&0&0\end{bmatrix}$$

$\rho(A)=$ two $<$ No. of vectors (3)

So vectors are LD

PYQ The vectors $(1\ 2\ -1)'$, $(2\ 3\ 4)'$, $(0\ 1\ 2)'$, $(4\ -3\ 2)'$ are ? = LD.

Sol: $X_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, X_2 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, X_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, X_3 = \begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix}$

Construct $A = [X_1\ X_2\ X_3\ X_4]$

$$= \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 3 & 1 & -3 \\ -1 & 4 & 2 & 2 \end{bmatrix}_{3\times 4}$$

$\because \rho(A) \le 3$ (By Property of Rank)

ie $\rho(A) < 4$ definitely (By Common Sense)

or $\rho(A) <$ No. of vectors $\Rightarrow$ LD

$\Rightarrow$ L.D vectors.

Q: Find λ for which $\exists$ a linear combination b/n the vectors;

$\hat{i}+2\hat{j}+3\hat{k}$, $4\hat{i}+5\hat{j}+6\hat{k}$, $7\hat{i}+\lambda\hat{j}+9\hat{k}$ then $\lambda = \underline{?}$

Sol: $X_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $X_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, $X_3 = \begin{bmatrix} 7 \\ \lambda \\ 9 \end{bmatrix}$

$$A = [X_1 X_2 X_3] = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & \lambda \\ 3 & 6 & 9 \end{bmatrix}_{3\times 3} = \text{Sq. Mat.}$$

For (LD) vectors, $|A| = 0 \Rightarrow \begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & \lambda \\ 3 & 6 & 9 \end{vmatrix} = 0 \cdots\cdots \Rightarrow \lambda = 8$

Ch: No. of different values of λ for which above vectors are (LI) will be $\underline{? = \infty}$

for $\lambda \neq 8 \Rightarrow$ vectors are (LI)

ie $\lambda \in R - \{8\}$.

Q. the vectors $(1\ 2\ 1)'$, $(2\ 1\ -4)'$, $(3\ -2\ 1)'$ are orthogonal & L.I.

(M-I) $X_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $X_2 = \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}$, $X_3 = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$

$X_1 \cdot X_2 = X_2 \cdot X_3 = X_3 \cdot X_1 = 0$

Hence orthogonal.

w.k. that $\boxed{\text{orthogonal vectors} \underset{\not\Leftarrow}{\Rightarrow} \text{LI also.}}$

So these vectors are LI also.

VARIOUS Defn of RANK :-

Defn $\rho(A_{6\times7}) = 4$

- Mat A will have at most 4 LI Row vectors.
- Mat A " " at most 4 LI Column vectors.

Defn of Rank: "It is the order of Non singular submatrix of Highest order that can exist in a given Mat"

Defn In Books:

if $\boxed{\rho(A_{6\times7}) = 4}$ then

- $\exists$ at least one Non singular submatrix of order 4×4
- Every square submatrix of order 5×5 & 6×6 are singular

THANK - YOU

Tel: dr puneet sir pw

DS & AI
CS & IT

Linear Algebra

Lecture No. 06

By- Dr. Puneet Sharma Sir

Recap of previous lecture

Topic

→ RANK of MATRIX

→ VECTORS & their PROPERTIES

Topics to be Covered

Topic

System of Equations

(Non Homogeneous)

Methods of Checking the Nature of Vectors →

Consider the given vectors are $X_1, X_2, X_3, \ldots, X_r$

then Construct a Matrix A as follows; $A = [X_1 X_2 X_3 \ldots X_r] \neq$ Row Mat

M-I General Method (always applicable) →

(i) if $\rho(A) =$ No. of vectors $\Rightarrow$ vectors are LI

(ii) if $\rho(A) <$ " " " $\Rightarrow$ " " LD

M-II Tricky Method (applicable only when A is Sq Mat) →

(i) if $|A| \neq 0 \Rightarrow$ vectors are LI

(ii) if $|A| = 0 \Rightarrow$ " " LD

Linear System of Equations

① Non Homogeneous System $(AX=B)$

② Homogeneous System $(AX=0)$

eg
$$\left.\begin{aligned} 2x-y+4z&=0\\ x-2+3z&=0\\ -2x+2y-z&=0\\ 4x-3y+2z&=0 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} 2x-y+4z&=0\\ x+0y+3z&=2\\ -2x+2y-z&=0\\ 4x-3y+2z&=0 \end{aligned}\right\} \Rightarrow \begin{bmatrix} 2 & -1 & 4\\ 1 & 0 & 3\\ -2 & 2 & -1\\ 4 & -3 & 2 \end{bmatrix}_{4\times 3} \begin{bmatrix} x\\ y\\ z \end{bmatrix}_{3\times 1} = \begin{bmatrix} 0\\ 2\\ 0\\ 0 \end{bmatrix}_{4\times 1}$$

$$A_{4\times 3} X_{3\times 1} = B_{4\times 1}$$

Here $[A:B] = \left[\begin{array}{ccc|c} 2 & -1 & 4 & 0\\ 1 & 0 & 3 & 2\\ -2 & 2 & -1 & 0\\ 4 & -3 & 2 & 0 \end{array}\right]_{4\times 4}$ = Augumented Mat.

(1)

Coefficient Mat

Constant Mat

$$\boxed{A_{m\times n} X_{n\times 1} = B_{m\times 1}}$$

, $[A:B]$ = Augmented Mat
$m\times(n+1)$

No. of eqns

No. of Variables

Variable Mat
Solution of system
unknown vector

where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$

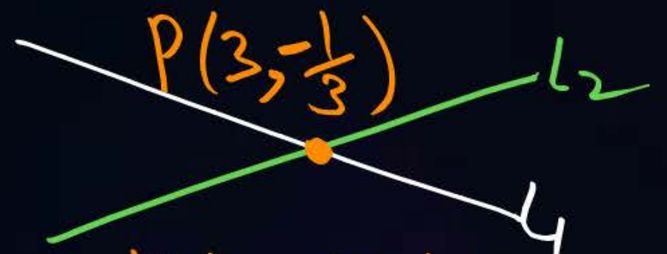
(2) if $m > n$ then system is overdetermined (tough)
if $m = n$ then " " equally determined
if $m < n$ then " " underdetermined. (Easy)

③ Nature of solution →

eg $2x+3y=5$
$x-3y=4$

$X=\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}3\\-1/3\end{bmatrix}$ = unique V.

= unique soln.



unique point of intersection

eg $2x+3y=5$
$4x+6y=10$

$X=\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}2\\1/3\end{bmatrix},\begin{bmatrix}1/2\\4/3\end{bmatrix},\ldots\ldots$

= Multiple vectors exist

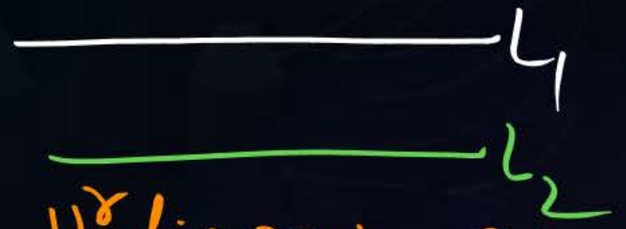
= ∞ soln exist.

overlapping lines

eg $2x+3y=5$
$4x+6y=9$

$X=\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}?\\?\end{bmatrix}$ = No soln.

∵ Both the equ^n contradict each other.



$\parallel^{el}$ lines ≃ Never intersect

Methods of Solving Non Homog System → Consider $\boxed{A_{m\times n} X_{n\times 1} = B_{m\times 1}}$

RANK Method (always applicable)
(i.e. for $m>n$, $m=n$, $m<n$)

Matrix Method
(applicable only when $m=n$)

① if $\boxed{\rho(A)=\rho(A:B)}$ = No. of Variables ⟹ unique sol exist ⟸ if $|A|\neq 0$

② if $\boxed{\rho(A)=\rho(A:B)}$ < No. of Variables ⟹ ∞ sol. exist ⟸ if $|A|=0$ & $(\text{adj}A)B=0$

③ if $\rho(A)\neq\rho(A:B)$ ⟹ No sol exist ⟸ if $|A|=0$ & $(\text{adj}A)B\neq 0$

(*) **Consistent system** → System is called consistent if $\exists$ solution.
(whether unique or ∞ sol.)

Inconsistent system → System is called Inconsistent if we have No sol.

(*) Necessary condition for a system $\boxed{AX = B}$ to be consistent is ?

$$\boxed{\rho(A) = \rho(A \vdots B)}$$

Gate → write the condition for the existence of $\underbrace{\text{at least one solution}}_{\text{Consistent}}$ of an $\underbrace{\text{unknown vector}}$ in the system $\boxed{PY = Q}$

$$\rho(P) = \rho(P \vdots Q)$$

(*) **Another form of N. Cond^n for consistency** :→ B must be L.D on columns of A.

(*) $\boxed{\rho(A) \ngtr \rho(A:B)}$ i.e Rank of Coeff Mat Can never exceed Rank of Augmented Mat.

eg $[A:B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 5 & 14 \end{array}\right]$, $[A:B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 10 \end{array}\right]$, $[A:B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\rho(A) = 3$, $\rho(A:B) = 3$ (unique sol)

$\rho(A) = 2$, $\rho(A:B) = 3$ (No sol.)

$= C_1 \; C_2 \; C_3 \; B$

$\rho(A) = 2 = \rho(A:B)$ (∞ sol)

$\Downarrow$

$\rho(A:B) = 2 <$ No of vectors

i.e C_1, C_2, C_3, B are (LD)

$[A:B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 5 & 0 \end{array}\right]$

$\rho(A) = 3$, $\rho(A:B) = 3$ (unique sol)

i.e for consistency B must be L.D on columns of A.

Q. For the system $AX = B$ which of the following can be taken as condition for No soln.

(a) $\rho(A) = \rho(A:B)$

(b) $\rho(A) < \rho(A:B)$

(c) $\rho(A) > \rho(A:B)$

(d) Both (b) & (c)

$$\rho(A) \neq \rho(A:B) \Rightarrow \begin{cases} \rho(A) < \rho(A:B) \\ \rho(A) > \rho(A:B) \end{cases}$$

(2) underdetermined System can not have unique solution (Learn)

Note: underdetermined Non Homog system if Consistent or $\rho(A) = \rho(A:B)$ always Consist Infinite sol.

The system of equations :

$$\begin{cases} 2x + y = 5 \\ x - 3y = -1 \end{cases}$$

$$3x + 4y = k$$

is consistent when k is _____

(a) 1 (b) 2

(c) 5 (d) 10

$$[A:B] = \left[\begin{array}{cc|c} 2 & 1 & 5 \\ 1 & -3 & -1 \\ 3 & 4 & k \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & -3 & -1 \\ 2 & 1 & 5 \\ 3 & 4 & k \end{array}\right]$$

$$\xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & -3 & -1 \\ 0 & 7 & 7 \\ 0 & 13 & k+3 \end{array}\right]$$

$$\xrightarrow{R_2/7} \left[\begin{array}{cc|c} 1 & 3 & -1 \\ 0 & 1 & 1 \\ 0 & 13 & k+3 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & k-10 \end{array}\right]$$

$\rho(A) = 2$ so for consistency

$\rho(A) = \rho(A:B) = 2 \Rightarrow k = 10$

& Here we have, unique soln.

(M-II) Solving (1) & (2), $x = 2, y = 1$

So By (3), $3x + 4y = k$

$3(2) + 4(1) = k$

$k = 10$

Q: The solution of the system:

$$\begin{cases} x+y+z=6 \\ x+2y+3z=10 \\ x+2y+5z=14 \end{cases}$$

will be.

(a) $\begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$ (c) $\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$ (d) $\begin{bmatrix} -4 \\ 0 \\ -2 \end{bmatrix}$

M-I (Using option ellimination Appr)

→ (c) is correct.

M-II → Solve it simultaneously as discussed in class 8th..... Do yourself

M-III Using Gauss Ellimination Method →

$$[A:B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & 5 & 14 \end{array}\right] \xrightarrow[R_3-R_1]{R_2-R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & 4 & 8 \end{array}\right]$$

$$\xrightarrow{R_3-R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{array}\right], \quad \rho(A)=3, \quad \rho(A:B)=3$$

$\because \rho(A) = \rho(A:B) =$ No. of Variables (3)

$\Rightarrow$ unique sol. exist.

Process of finding sol. → See Next Slide.

Process of Finding Sol^n:-

$$AX = B$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ 14 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} (x+y+z) \\ (y+2z) \\ 2z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 4 \end{bmatrix}$$

$x+y+z=6 \Rightarrow x+0+2=6 \Rightarrow x=4$

$y+2z=4 \Rightarrow y+2(2)=4 \Rightarrow y=0$

$2z=4 \rightarrow z=2$

i.e. Sol^n is $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$ i.e. unique sol exist.

Note- this method of solving Linear system of Equ^n is called ECHELON FORM method.

or Gauss-Elimination method.

or Backward substitution Method.

or RANK method.

Q. Find the values of λ & μ for which following system has

① ∞ sol. ② No sol ③ unique sol.

$$x+y+z=6$$
$$x+2y+3z=10$$
$$x+2y+\lambda z=\mu$$

$$[A:B]=\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \boxed{(\lambda-3)} & (\mu-10) \end{array}\right]_{3\times 4}$$

3×3

For ∞ sol:- $\rho(A)=\rho(A:B)<3$

Let $\underbrace{\rho(A)=2}_{\lambda=3}=\underbrace{\rho(A:B)}_{\mu=10}$ Ans

For No sol: $\rho(A)\neq\rho(A:B) \Longrightarrow \rho(A)<\rho(A:B)$

Let $\rho(A)=2$ & $\rho(A:B)=3$

$\Downarrow$

$\lambda=3$ & $\mu\neq 10$ Ans

$\because x\neq y \begin{cases} x>y \\ x<y \end{cases}$, $\rho(A)\neq\rho(A:B) \begin{cases} \rho(A)>\rho(A:B) \\ \rho(A)<\rho(A:B) \end{cases}$

③ For unique sol →

$$[A:B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \boxed{(\lambda-3)} & (\mu-10) \end{array}\right]$$

for unique sol $\Rightarrow \rho(A) = \rho(A:B) = 3$

$\rho(A) = 3$ & $\rho(A:B) = 3$

$\Downarrow$ $\lambda \neq 3$

$\Downarrow$ μ can take any value.

i.e. $\lambda \neq 3$ & $\mu \in R$ Ans

Note: for $\lambda = 5$, $\mu = 14$, sol of above system is unique

& for $\lambda = 11$, $\mu = 10$, sol of .. .. is unique

Q. Solve $3x+3y+2z=1$

$x+2y \quad =4$

$10y+3z=-2$

$2x-3y-z=5$

unique sol

Q Find the Nature of soln of

MSQ

$x+y+z=1$

$2x+y+4z=1$

$4x+y+10z=1$

(a) No soln.

(b) unique soln.

(c) ∞ soln^n

(d) Finite No. of Ind soln.

$$[A:B]=\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 1 & 4 & 1 \\ 4 & 1 & 10 & 1 \end{array}\right] \xrightarrow[R_3-4R_1]{R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & -1 \\ 0 & -3 & +6 & -3 \end{array}\right]$$

$$\xrightarrow{R_3-3R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & +2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\rho(A)=2=\rho(A:B)$

$\Rightarrow \infty$ soln^n exist.

$$AX=B \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ 4 & 1 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\cdots\cdots \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & +2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} (x+y+z) \\ (-y+2z) \\ 0 \end{bmatrix}=\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$x+y+z=1$

$-y+2z=-1$

let $z=k$, $-y+2k=-1 \Rightarrow y=2k+1$

& $x+y+z=1 \Rightarrow x+2k+1+k=1 \Rightarrow x=-3k$

Sol is $X=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-3k\\2k+1\\k\end{bmatrix}=\infty$ sol. $=\begin{bmatrix}0-3k\\1+2k\\0+k\end{bmatrix}=\begin{bmatrix}0\\1\\0\end{bmatrix}+\begin{bmatrix}-3k\\2k\\k\end{bmatrix}=\begin{bmatrix}0\\1\\0\end{bmatrix}+k\begin{bmatrix}-3\\2\\1\end{bmatrix}$

($\because k$ = arbitrary const.)

$=\alpha+k\beta.$

$X=\underset{(k=0)}{\begin{bmatrix}0\\1\\0\end{bmatrix}}=X_1,\ \underset{(k=1)}{\begin{bmatrix}-3\\3\\1\end{bmatrix}}=X_2,\ \underset{k=-1}{\begin{bmatrix}3\\-1\\-1\end{bmatrix}}=X_3,\ \underset{k=2}{\begin{bmatrix}-6\\5\\2\end{bmatrix}}=X_4,$ ..... ∞ sol. exist.

$X_3=2X_1-X_2$ $\quad X_4=-X_1+2X_2$ & so on....

ie only X_1 & X_2 are LI & Rest are LD on them.

eg the nature of the soln of following system:

(1) $x_1 - 2x_2 + 4x_3 = 5$

$2x_1 - 4x_2 + 8x_3 = 7$

$$[A:B] = \left[\begin{array}{ccc|c} 1 & -2 & 4 & 5 \\ 2 & -4 & 8 & 7 \end{array}\right]_{2\times 4}$$

$$= \left[\begin{array}{ccc|c} 1 & -2 & 4 & 5 \\ 0 & 0 & 0 & -3 \end{array}\right]_{2\times 4}$$

$\rho(A) = 1$, $\rho(A:B) = 2$

No soln

(2) $x_1 - 2x_2 + 4x_3 = 5$

$2x_1 - 4x_2 + 8x_3 = 10$

$$[A:B] = \left[\begin{array}{ccc|c} 1 & -2 & 4 & 5 \\ 2 & -4 & 8 & 10 \end{array}\right]_{2\times 4}$$

$$= \left[\begin{array}{ccc|c} 1 & -2 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]_{2\times 4}$$

$\rho(A) = 1 = \rho(A:B) <$ No of variables (3)

∞ soln

Given a system of equations :

$$x + 2y + 2z = b_1$$
$$5x + y + 3z = b_2$$

underdetermined

Which of the following is true regarding its solution ?

(a) The system has a unique solution for any given b_1 and b_2 ✗

(b) The system will have infinitely many solutions for any ~~given~~ b_1 and b_2 ✓

(c) Whether or not a solution exists depends on the given b_1 and b_2 ✗

(d) The system would have no solution for any values of b_1 and b_2 ✗

$$[A:B] = \left[\begin{array}{ccc|c} 1 & 2 & 2 & b_1 \\ 5 & 1 & 3 & b_2 \end{array}\right]_{2\times 4}$$

2×2

$\rho(A) = 2 = \rho(A:B)$

Consistent $\Rightarrow \infty$ soln

Not depends upon the values of b_1 & b_2

Q. if $A_{n\times n}$ s.t $A^2 = I$ then $AX = B$ has _____ sol.

(a) unique sol.

(b) ∞ sol.

(c) No sol.

(d) More than one but finite No of sol.

Sol: $A^2 = I$

$|A^2| = |I|$

$|A|^2 = 1$

$|A| = \pm 1$

i.e $|A| \neq 0$

So By Matrix method unique sol exist

M-II (w/o using property) →

$AX = B$

$A(AX) = AB$

$A^2X = A_{n\times n} B_{n\times 1}$

$IX = (AB)_{n\times 1}$

$X = \begin{bmatrix} - \\ - \\ - \\ \vdots \\ - \end{bmatrix}_{n\times 1}$ = unique vector exist.

So unique sol exist.

THANK - YOU

Tel: dr puneet sir pw

DS & AI
CS & IT

Linear Algebra

Lecture No. 07

By- Dr. Puneet Sharma Sir

Recap of previous lecture

Topic

Non Homogeneous System of Eqn

Topics to be Covered

Topic

→ Homogeneous System of Equ^n

→ Basics of Eigen Values.

Homogeneous system of linear Equ^n $AX=O$

eg
$$\left.\begin{aligned} 2x-y+4z&=0\\ x+2y-2z&=0\\ -x-y+3z&=0\\ x+2y-4z&=0 \end{aligned}\right\} \Rightarrow \begin{bmatrix} 2 & -1 & 4\\ 1 & 2 & -2\\ -1 & -1 & 3\\ 1 & 2 & -4 \end{bmatrix}\begin{bmatrix} x\\ y\\ z \end{bmatrix}=\begin{bmatrix} 0\\ 0\\ 0\\ 0 \end{bmatrix} \Rightarrow \boxed{A_{4\times3}X_{3\times1}=O_{4\times1}}$$

By observation, we can see that $X=\begin{bmatrix} x\\ y\\ z \end{bmatrix}=\begin{bmatrix} 0\\ 0\\ 0 \end{bmatrix}$ is a solution of above system.
because these values of x, y, z satisfying all 4 equ^n at a time.
& this type of solution always exist & this sol. is known as TRIVIAL sol.

(*) Homog system is always consistent OR Homog system Never Inconsistent.
because Reason 1: Trivial sol always exist.
Reason 2: $\rho(A)=\rho(A \vdots O)$ always. Hence there is No Need to write Aug Mat further.

Why zero vector is treated as LD vector →

vectors $x_1, x_2, x_3 \cdots\cdots x_r$ are called LD if $\exists$ a relationship of the type $k_1x_1 + k_2x_2 + \cdots\cdots + k_rx_r = 0$ ——— (1)

where $k_1, k_2, k_3 \cdots\cdots k_r$ should not all zero simultaneously.

eg(i) Let us consider $\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}-2\\2\\0\end{bmatrix}$ and $\begin{bmatrix}0\\0\\0\end{bmatrix}$ then we have

$2x_1 + x_2 + x_3 = 0$ i.e x_1, x_2, x_3 are LD

eg(ii) Let us consider $\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}2\\0\\1\end{bmatrix}$ and $\begin{bmatrix}0\\0\\0\end{bmatrix}$ then we have

$0x_1 + 0x_2 + kx_3 = 0$ i.e relationship exist where not all constants are zero Hence x_1, x_2, x_3 are LD

eg(iii) Let us consider $x = \begin{bmatrix}0\\0\\0\end{bmatrix}$ then we can write

$kx = 0$ where ~~k is non zero~~ Hence relationship again exist so x is LD

Note: ① $\underbrace{\text{underdetermined}}_{\text{unique sol}^n = \text{DNE}}$, $\underbrace{\text{Homog system}}_{\text{No sol}^n = \text{DNE}}$ always consist $\underline{\infty}$ solutions

② unique sol^n = Trivial sol^n = ZERO solution (always exist)

③ ∞ sol^n = Non Trivial sol^{n}s also exist = Non zero sol^n also exist.

④ ZERO sol^n $\neq$ No sol^n

$\because X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\qquad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$

Methods of solving Homog. system

$$(A_{m\times n} X_{n\times 1} = O_{m\times 1})$$

RANK Method

$(m>n, m=n, m<n)$

① if $\rho(A)$ = No. of Variables $\Rightarrow$ unique sol.

② if $\rho(A) <$ " " $\Rightarrow \infty$ sol.

MATRIX Method

(only for $m=n$)

① if $|A| \neq 0 \Rightarrow$ unique sol exist

② if $|A| = 0 \Rightarrow \infty$ sol exist

Note - ① unique sol = Trivial sol. = zero sol. always exist.

② ∞ sol = Non Trivial sol = Non zero sol also exist

Q. Find k for which

MSQ

$$(3k-8)x+3y+3z=0$$
$$3x+(3k-8)y+3z=0$$
$$3x+3y+(3k-8)z=0$$

has Non Trivial Sol.

(a) $\frac{2}{3}$ (b) $\frac{11}{3}$ (c) $\frac{4}{3}$ (d) $\frac{8}{3}$

for Non Trivial Sol (∞ Sol) $\rightarrow |A|=0$

$$\begin{vmatrix} (3k-8) & 3 & 3 \\ 3 & (3k-8) & 3 \\ 3 & 3 & (3k-8) \end{vmatrix}=0 \longrightarrow \begin{vmatrix} (3k-2) & 3 & 3 \\ (3k-2) & (3k-8) & 3 \\ (3k-2) & 3 & (3k-8) \end{vmatrix}=0$$

$$(3k-2)\begin{vmatrix} 1 & 3 & 3 \\ 1 & (3k-8) & 3 \\ 1 & 3 & (3k-8) \end{vmatrix}=0$$

$$(3k-2)\begin{vmatrix} 1 & 3 & 3 \\ 0 & (3k-11) & 0 \\ 0 & 0 & (3k-11) \end{vmatrix}=0 \Rightarrow (3k-2)(3k-11)^2=0$$

$$k=\frac{2}{3},\frac{11}{3},\frac{11}{3}$$

The value of α for which the system of equation

PYQ

$$x + y + z = 0$$
$$y + 2z = 0$$
$$\alpha x + z = 0$$

has more than one solution is

(a) -1 (b) 0

(c) $\frac{1}{2}$ (d) 1

$|A| = 0$

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ \alpha & 0 & 1 \end{vmatrix} = 0$$

$1[1-0] - 0 + \alpha[2-1] = 0$

$1 + \alpha = 0$

$\alpha = -1$

M-II Expanding along R_3

$\alpha[2-1] - 0 + 1[1-0] = 0$

$\alpha + 1 = 0 \Rightarrow \alpha = -1$

HW 8 For the System to have Infinite Sol, which of the following is/are True

MS 8

$px+qy+rz=0$

$qx+ry+pz=0$

$rx+py+qz=0$

(a) $p=q=r$ (By observation)

(b) $p+q+r=0$

(c) $p^3+q^3+r^3-3pqr=0$

(d) None

$|A|=0 \Rightarrow \begin{vmatrix} p & q & r \\ q & r & p \\ r & p & q \end{vmatrix}=0$

$\Rightarrow \boxed{p^3+q^3+r^3-3pqr=0}$ Ans

$(p+q+r)\cdot\left[(p-q)^2+(q-r)^2+(r-p)^2\right]=0$

is $\boxed{p+q+r=0}$ or $\boxed{p=q=r}$ Ans

Q. Consider $\boxed{A_{m\times n} X_{n\times 1} = B_{m\times 1}}$ then which one is false ?

(a) if $m>n$, $B\neq 0$, $\rho(A) < \rho(A:B)$ then system has NO sol. (T)

(b) if $m=n$, $B=0$, $|A|\neq 0$ then system has only Trivial sol. (T)

(c) if $m=3$, $n=5$, $B=0$ then system has also Non Zero sol. (T)

(d) if $m=5$, $n=3$, $B=0$ s.t $\rho(A)=3$ then system has only zero sol (T)

(e) if $m=n$, $B=0$, then system has sol. (T) (obviously True)

(f) if $m>n$, $B=0$ then System has Multiple sol (False)

(g) if $m>n$, $B=0$, $\boxed{\rho(A)<n}$ then system has Multiple sol. (True)

(h) Sun Rises from east (T) (obviously True)

Analysis (PODCAST) → Consider $\boxed{x+y+z=0}$ — (1) $\begin{bmatrix}1&1&1\end{bmatrix}_{1\times3}\begin{bmatrix}x\\y\\z\end{bmatrix}_{3\times1}=\begin{bmatrix}0\end{bmatrix}_{1\times1}$

ie $A_{1\times3}X_{3\times1}=O_{1\times1}$ then it's soln are;

$$X=\begin{bmatrix}x\\y\\z\end{bmatrix}=\underbrace{\begin{bmatrix}0\\0\\0\end{bmatrix}}_{\text{Trivial soln.}},\underbrace{\begin{bmatrix}1\\-1\\0\end{bmatrix},\begin{bmatrix}1\\0\\-1\end{bmatrix},\begin{bmatrix}2\\-1\\-1\end{bmatrix},\begin{bmatrix}4\\-4\\0\end{bmatrix},\begin{bmatrix}-7/2\\0\\7/2\end{bmatrix},\begin{bmatrix}3\\-2\\-1\end{bmatrix},\ldots\ldots\ldots}_{\text{Non Trivial soln's}}\ \infty\ \text{soln.}$$

Null space.

X

$X_1=\begin{bmatrix}1\\-1\\0\end{bmatrix}\longrightarrow\begin{bmatrix}2\\-2\\0\end{bmatrix},\begin{bmatrix}1/3\\-1/3\\0\end{bmatrix},\begin{bmatrix}-0.2\\+0.2\\0\end{bmatrix}\ldots\ldots$ But all are (LD) on X_1

$X_2=\begin{bmatrix}1\\0\\-1\end{bmatrix}\longrightarrow\begin{bmatrix}4\\0\\-4\end{bmatrix},\begin{bmatrix}-\sqrt5\\0\\\sqrt5\end{bmatrix},\ldots\ldots\ldots$ But all are (LD) on X_2

$X_3=\begin{bmatrix}2\\-1\\1\end{bmatrix}=X_1+X_2,\ X_4=\begin{bmatrix}3\\-2\\-1\end{bmatrix}=2X_1+X_2\ldots$ ie again all are (LD)

ie out of ∞ soln only X_1 & X_2 are LI & Rest are LD on them.

Nullity (Dimension of Nullspace) →

"the counting of L-I soln in any Homog system $\boxed{A_{m\times n} X_{n\times 1} = O_{m\times 1}}$ is called Nullity" & it is given as $N(A)$.

Rank-Nullity Th: Nullity = Number of columns − Rank(A)

ie $\boxed{N(A) = N(\text{columns}) - \rho(A)}$

Q Find Nullity of $\boxed{x+y+z=0}$? Here $A = [1\ 1\ 1]_{1\times 3}$ so $\rho(A) = 1$.

Hence Nullity$(A) = 3 - 1 = 2$

ie above system has two LI soln & Rest are LD on them.

PODCAST: Consider $\boxed{A_{m\times n} X_{n\times 1} = O_{m\times 1}}$

Row space: set of all Linear Combinations of Row vectors of A is called Row space

& Dimension of Row Space = Max No. of LI Row vectors = $\rho(A)$

Column space set of all linear combinations of Column vectors of A is called Column space

(Range space)

& Dimension of Column space = Max No of L-I Column vectors = $\rho(A)$

Null space set of all solutions of $\boxed{AX=0}$ is called Null space

& Dimension of Null space = Nullity = No. of LI solution in terms of X.

(*) $N(A) \perp R(A)$ i.e Row space is $\perp^r$ to Null space (Sol^n space)

"Every vector in Null space is orthogonal to every vector in Row space"

$$\because AX = O \Rightarrow \underset{m\times n}{\begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ b_1 & b_2 & b_3 & \cdots & b_n \\ c_1 & c_2 & c_3 & \cdots & c_n \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_n \end{bmatrix}} \underset{n\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}} = \underset{m\times 1}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}$$

$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$ i.e R_1 & X are orthogonal

again $b_1x_1 + b_2x_2 + \cdots + b_nx_n = 0$ i.e R_2 & X " "

& so on .... i.e Hence proved.

Q: If every soln of system $\boxed{A_{4\times 5} X = 0}$ (1) is a scalar multiplication of $\begin{bmatrix} 2 \\ -3 \\ 4 \\ 1 \\ 2 \end{bmatrix}$ then $\rho(A) = ?$

ATQ, No. of LI soln of (1) = one = $N(A)$

By Rank-Nullity Th: $\boxed{N(A) = N(C) - \rho(A)}$

$1 = 5 - \rho(A) \Rightarrow \rho(A) = 4$ Ans

Solution set $X = \begin{bmatrix} 2 \\ -3 \\ 4 \\ 1 \\ 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 4 \\ -6 \\ 8 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ -4 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1.5 \\ 2 \\ 0.5 \\ 1 \end{bmatrix} \cdots\cdots \infty$ soln!

LI

LD

The nullity of system of quations:

$$\begin{cases} x_1 + x_2 - x_3 + x_4 = 0 \\ 2x_1 + 3x_2 + x_3 + 4x_4 = 0 \\ 3x_1 + 2x_2 - 6x_3 + x_4 = 0 \end{cases}$$

(a) 1 (b) 2

(c) 3 (d) 4

$$A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 2 & 3 & 1 & 4 \\ 3 & 2 & -6 & 1 \end{bmatrix}_{3\times 4} \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1}$$

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & -1 & -3 & -2 \end{bmatrix} \xrightarrow{R_3 + R_2} \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 2$ & $N(\text{Columns}) = 4$

So $N(A) = N(C) - \rho(A) = 4 - 2 = 2$

$$A_{3\times 4} X_{4\times 1} = [0]_{3\times 1} \Rightarrow X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} =$$

only Two soln are LI & Rest are LD on them.

The number of linearly independent solutions of the system of equations

$$\begin{bmatrix} 1 & 0 & 2 \\ 1 & -1 & 0 \\ 2 & -2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = O \text{ is equal to}$$

(a) 1 (b) 2

(c) 3 (d) 0

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & -1 & 0 \\ 2 & -2 & 0 \end{bmatrix}$$

$\because |A| = 0 \Rightarrow \rho(A) \neq 3$

ie $\rho(A) = 2$

$N(A) = N(C) - \rho(A)$

$= 3 - 2 = 1.$

Q. Solve:

$$\left.\begin{aligned} 2x+y+z&=0\\ x+y&=0\\ y-z&=0 \end{aligned}\right\}$$

$$\Downarrow$$

$$A_{3\times3}X_{3\times1}=O_{3\times1}$$

$$A=\begin{bmatrix}2&1&1\\1&1&0\\0&1&-1\end{bmatrix}\longrightarrow\begin{bmatrix}1&1&0\\2&1&1\\0&1&-1\end{bmatrix}\longrightarrow\begin{bmatrix}1&1&0\\0&-1&1\\0&1&-1\end{bmatrix}$$

$$\longrightarrow\begin{bmatrix}1&1&0\\0&-1&1\\0&0&0\end{bmatrix}$$

$\rho(A)=2<$ No of variables

$\Rightarrow \infty$ sol exist.

& $N(A)=3-\rho(A)=3-2=1$.

$$\begin{bmatrix}2&1&1\\1&1&0\\0&1&-1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\simeq\begin{bmatrix}1&1&0\\0&-1&1\\0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}(x+y)\\(-y+z)\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$x+y=0\Rightarrow x=-k$

$-y+z=0\Rightarrow y=z$ let $z=k$, then $y=k$.

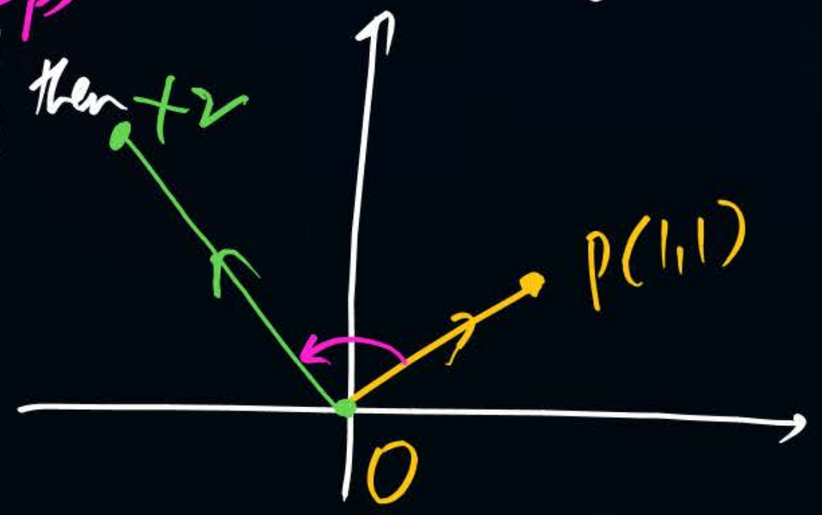
So $X=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-k\\k\\k\end{bmatrix}=k\begin{bmatrix}-1\\1\\1\end{bmatrix}$, $k=$ arbitrary const.

ie ∞ sol exist in which only $\begin{bmatrix}-1\\1\\1\end{bmatrix}$ is LI

Eigen Values / Eigen Vectors (100% they will ask Q.)

PODCAST: Consider $A = \begin{bmatrix} -1 & -2 \\ 0 & 3 \end{bmatrix}$ (sp) & $X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (sp) then

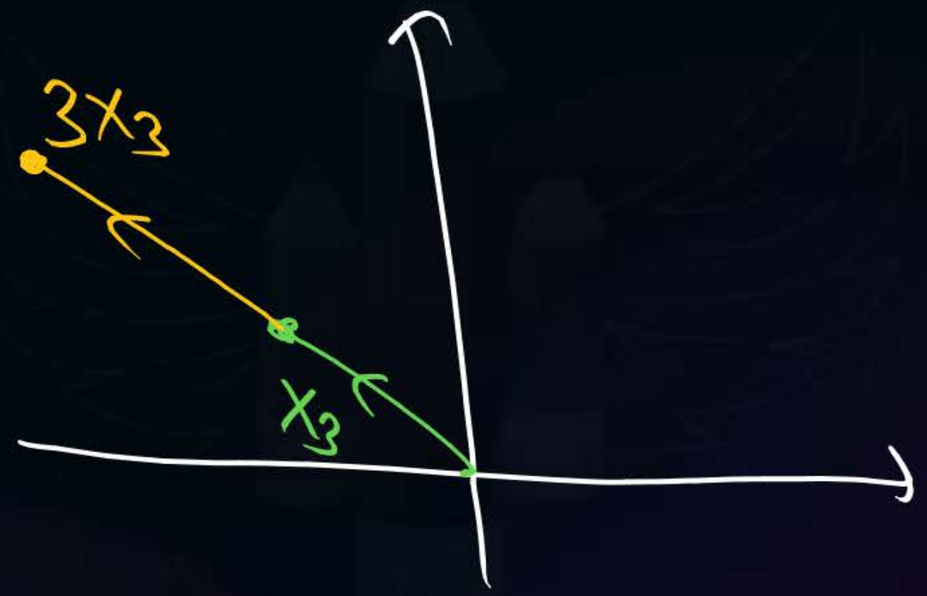
$$AX_1 = \begin{bmatrix} -1 & -2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix} = X_2$$



again Consider $X_3 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

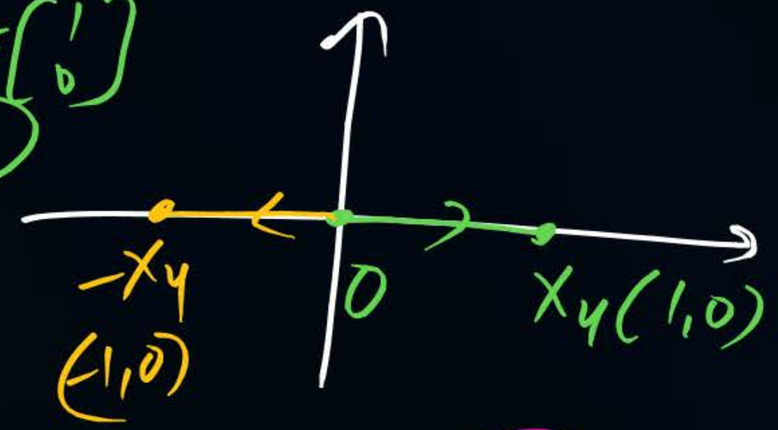
$$AX_3 = \begin{bmatrix} -1 & -2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \end{bmatrix} = 3 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = 3X_3$$

$X_3 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ = Sp. vector & Sp Value = 3



Now consider $A = \begin{bmatrix} -1 & -2 \\ 0 & 3 \end{bmatrix}$, $X_4 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ Sp. vector $= X_4 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & Sp value $= -1$

$$AX_4 = \begin{bmatrix} -1 & -2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} = -1 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = -X_4$$



Consider $A = \begin{bmatrix} -1 & -2 \\ 0 & 3 \end{bmatrix}$, $X_5 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow AX_5 = \begin{bmatrix} -1 & -2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = X_5$

But X_5 is not an eigen vector $\because$ it is a zero vector.

ie E vector can not be zero vector.

Consider $A = \begin{bmatrix} 0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $X_6 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ then $AX_6 = \begin{bmatrix} 0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0 \cdot X_6$

So X_6 is also special vector whose sp. value $= 0$

ie E value can be zero also.

Def^n: Consider Sq. Mat $A_{n\times n}$ then Non Zero Vector X is called Eigen vector, corresponding to Eigen value λ (Real/Complex/Zero) if we are able to find a relationship of the type,

$$\boxed{AX = \lambda X}$$

λ = Eigen value
X = Eigen vector.

LHS is the Multi of Two Matrices = RHS is the Scalar Multi in a Mat
(Tough) (Easy)

(*) Here we are considering Homogeneous system as follows

$$AX = \lambda X \Longrightarrow AX - \lambda X = 0 \Longrightarrow \boxed{(A - \lambda I)X = 0}$$

So it will satisfy all the prop of Homog system.

(*) Consider $AX = \lambda X$

$$\boxed{(A-\lambda I)X = 0} \quad \text{--- (1)}$$

$$MX = 0$$

$\downarrow$

Non Zero Eigen vector

Non Zero solution

$= \infty$ soln.

$$\rho(M) < n \text{ or } |M| = 0$$

$$\Rightarrow \rho(A-\lambda I) < n \text{ or } |A-\lambda I| = 0$$

So Necessary condition for the existence of Non Zero Eigen vector is

$$\rho(A-\lambda I) < n \text{ or } \boxed{|A-\lambda I| = 0} \quad \text{--- (1)}$$

Characteristic Equ^n of A $\rightarrow$

equ^n (1) is called C Eq of A & Roots of this equ^n is values of λ are called E. Values / E. Roots / Char Values / Char Roots / Latent Roots / Sp. Values.

eg: Find the E-Values of ① $A=\begin{bmatrix}2&1&1\\1&2&1\\0&0&1\end{bmatrix}$ ② $A=\begin{bmatrix}0&-1\\1&0\end{bmatrix}$ ③ $A=\begin{bmatrix}8&-6&2\\-6&7&-4\\2&-4&3\end{bmatrix}$

Sol: ① $A-\lambda I=\begin{bmatrix}2&1&1\\1&2&1\\0&0&1\end{bmatrix}-\lambda\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

$$=\begin{bmatrix}(2-\lambda)&1&1\\1&(2-\lambda)&1\\0&0&(1-\lambda)\end{bmatrix}$$

$\neq$ C-Equ^n But Char Mat.

Now C-Equ^n is $|A-\lambda I|=0$

$$\begin{vmatrix}(2-\lambda)&1&1\\1&(2-\lambda)&1\\0&0&(1-\lambda)\end{vmatrix}=0$$

$(1-\lambda)\left[(2-\lambda)^2-1\right]=0$

$(1-\lambda)\left[(2-\lambda+1)(2-\lambda-1)\right]=0$

$\boxed{(\lambda-3)(\lambda-1)^2=0}$ — C-Equ^n of A.

& $\lambda=3,1,1$ ⟶ Char Values.

(2) $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ then

C. Equ^n is $|A - \lambda I| = 0$.

$$\begin{vmatrix} (0-\lambda) & -1 \\ 1 & (0-\lambda) \end{vmatrix} = 0$$

$\lambda^2 + 1 = 0$ → C. Equ^n

$\lambda = i, -i$ E Values.

(3) $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

C. Equ^n is $|A - \lambda I| = 0$

$$\begin{vmatrix} (8-\lambda) & 6 & 2 \\ -6 & (7-\lambda) & -4 \\ 2 & -4 & (3-\lambda) \end{vmatrix} = 0$$

HW

$\lambda(\lambda - 3)(\lambda - 15) = 0$ — C. Equ^n

$\lambda = 0, 3, 15$ — E Values

THANK - YOU

Tel: dr puneet sir pw

DS & AI
CS & IT

Linear Algebra

Lecture No. 08

By- Dr. Puneet Sharma Sir

Recap of previous lecture

Topic

1. Homogeneous System
2. Basics of Eigen Values

Topics to be Covered

Topic

PROPERTIES of EIGEN VALUES

Def^n: Consider Sq. Mat $A_{n\times n}$ then Non Zero Vector X is called Eigen vector, corresponding to Eigen value λ (Real/Complex/Zero) if we are able to find a relationship of the type,

$$\boxed{AX = \lambda X}$$

λ = Eigen value
X = Eigen vector.

LHS is the Multi of Two Matrices = RHS is the Scalar Multi in a Mat
(Tough) (Easy)

(*) Here we are considering Homogeneous system as follows

$$AX = \lambda X \Longrightarrow AX - \lambda X = 0 \Longrightarrow \boxed{(A - \lambda I)X = 0}$$

So it will satisfy all the prop of Homog system.

(*) Consider $AX = \lambda X$

$$\boxed{(A-\lambda I)X = 0} \quad \text{--- (1)}$$

$$MX = 0$$

$\downarrow$

Non Zero Eigen vector

Non Zero solution

$= \infty$ soln.

$$\rho(M) < n \text{ or } |M| = 0$$

$$\Rightarrow \rho(A-\lambda I) < n \text{ or } |A-\lambda I| = 0$$

So Necessary condition for the existence of Non Zero Eigen vector is

$$\rho(A-\lambda I) < n \text{ or } \boxed{|A-\lambda I| = 0} \quad \text{--- (1)}$$

Characteristic Equ^n of A $\rightarrow$

equ^n (1) is called C.Eq of A & Roots of this equ^n is values of λ are called E. Values / E. Roots / Char Values / Char Roots / Latent Roots / Sp. Values.

PROPERTIES of E. Values → Let $A_{n\times n}$ having Eigen Values $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$

① T. Number of E-Values of A = order of A (whether diff or Repeated)

② Sum of E-Values = Trace(A) ie $\lambda_1 + \lambda_2 + \cdots + \lambda_n = Tr(A)$

③ Product of E Values = Det(A) ie $\lambda_1 \cdot \lambda_2 \lambda_3 \cdots \lambda_n = |A|$

④ (Zero is an E. Value of A) iff (A is Singular)

⑤ Number of Non Zero E.Values of A $\leq \rho(A)$ ie we must have at least two eigen Values as 0, 0

eg if $\rho(A_{6\times 6}) = 4$ then No of Non Zero E Values ≤ 4

⑥ If Sum of all the elements in each Row (or each Column) is unique Constant K then that Constant K will be one of the E Value of A.

(7) Don't use E-Operations in a given Mat while Calculating E-values

But we can apply 3rd type of E-operations in it's C-Equ^n.

(8) E-values of U.T.M, L.T.M, Diag Mat, Scalar Mat, Identity Mat are just the diagonal elements.

eg $A = \begin{bmatrix} 2 & 4 & -1 & 3 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$, $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$, $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\lambda = 2, 0, -3, -1$ $\quad$ $\lambda = 2, -1, 1$ $\quad$ $\lambda = 2, -3, 4$ $\quad$ $\lambda = 1, 1, 1$

(9) If λ is an Eigen Value of A then to find Eigen Value of any algebraic expression formed by A, we can Replace A with λ in that expression. i.e $A \to \lambda$

(*) Equivalent Matrices have Same Rank but may be different E-Values

(10) Let $\lambda_1, \lambda_2, \lambda_3 \ldots \lambda_n$ are the E Values of $A_{n\times n}$ then

(i) E Values of A^T are $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ i.e. A & A^T have Same E Values

(ii) E Values of A^m are $\lambda_1^m, \lambda_2^m, \lambda_3^m, \ldots, \lambda_n^m$; $(m \in N)$

(iii) E Values of A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3} \ldots \frac{1}{\lambda_n}$ (Provided $|A| \neq 0$)

(iv) E-Values of (adj A) are $\frac{|A|}{\lambda_1}, \frac{|A|}{\lambda_2}, \frac{|A|}{\lambda_3}, \ldots, \frac{|A|}{\lambda_n}$ (when $|A| \neq 0$)

Proof 10(ii): $AX = \lambda X$ — (1)

$$A(AX) = A(\lambda X)$$
$$A^2X = \lambda(AX)$$
$$A^2X = \lambda(\lambda X)$$
$$A^2X = \lambda^2 X$$

ie λ^2 is an E Value of A^2

Similarly λ^3 " " " A^3 . . . .

Proof 10(iii): $AX = \lambda X$ — (1)

if $|A| \neq 0 \Rightarrow A^{-1}$ exist

$$A^{-1}(AX) = A^{-1}(\lambda X)$$
$$IX = \lambda(A^{-1}X)$$
$$\lambda(A^{-1}X) = IX$$
$$A^{-1}X = \left(\frac{1}{\lambda}\right)X$$

$\frac{1}{\lambda}$ is an E Value of A^{-1}.

Proof of 10(iv): $AX = \lambda X$ — (1)

Now $A^{-1}X = \left(\frac{\text{adj}A}{|A|}\right)X$

$$\left(\frac{1}{\lambda}\right)X = \left(\frac{\text{adj}A}{|A|}\right)X$$
$$\left(\frac{|A|}{\lambda}\right)X = (\text{adj}A)X$$

or $(\text{adj}A)X = \left(\frac{|A|}{\lambda}\right)X$

ie $\frac{|A|}{\lambda}$ is an E Value of (adj A)

this Prop Holds when A is Non Sing.

eg $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $(\text{adj}\,A) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -6 \end{bmatrix}$

$\lambda = 2, -3, 0$

$\lambda = -6, 0, 0.$

	2	0	0	2	0
0	-3	0	0	-3	
0	0	0	0	0	
2	0	0	2	0	
0	-3	0	0	-3	

Evalue of A are $2, -3, 0$

" , adj A are $\cancel{\frac{|A|}{\lambda_1}, \frac{|A|}{\lambda_2}, \frac{|A|}{\lambda_3}}$ Not applicable

$\cancel{= \frac{0}{2}, \frac{0}{-3}, \frac{0}{0}}$

eg $A = \begin{bmatrix} -2 & -1 & 3 & 4 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

U.T.M.

$|A| = -24$

then
- EValues of A are $\lambda = -2, 1, 4, 3$
- " " A^T are same
- " " A^3 are $-8, 1, 64, 27$
- " " A^{-1} are $\frac{1}{-2}, 1, \frac{1}{4}, \frac{1}{3}$
- " " adj A are $\frac{-24}{-2}, \frac{-24}{1}, \frac{-24}{4}, \frac{-24}{3}$

eg $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

Diag Mat.

- EValues of A are $1, -2, 0, 4$.
- " of A^T are $1, -2, 0, 4$
- " of A^2 are $1, 4, 0, 16$
- " of A^{-1} are $1, \cancel{\frac{-1}{2}, \text{ND}}, \frac{1}{4}$ (Blunder)

$\because A^{-1} = \text{DNE}$ so Case of EValues will not arise

eg $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$

$\lambda = 3, 1, 1$

$Tr(A) = 5$

$|A| = 3$

eg $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}_{2\times 2}$

$\lambda = i, -i$

$Tr = 0$

$|A| = 1$

eg $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}_{3\times 3}$

$\lambda = 0, 3, 15$

$Tr = 18$

$|A| = 0$

$\rho(A) = 2 =$ No. of Non Zero E Values

(Prop 5)

eg $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix}$

$= 10 \quad 10 \quad 10 \quad 10$

is one E value is $\underline{\underline{\lambda = 10}}$

(Prop 6)

Q: if E-values of $A = \begin{bmatrix} 2 & 3 \\ x & y \end{bmatrix}$ are 4 & 8 then $x+y = ?$

(a) -4 (b) 6 (c) 10 (d) 14

Let λ_1 & λ_2 are the E-Values of A

then $\lambda_1+\lambda_2 = Tr(A)$ & $\lambda_1\lambda_2 = |A|$

$4+8 = 2+y$ & $(4)(8) = 2y-3x$

$y = 10$

$32 = 2(10)-3x$

$3x = -12$

$x = -4$

$x+y = 6$ Ans

Q: if Trace & Det of $A_{2\times2}$ are -2 & -35 resp then $\lambda_1+\lambda_2 = ?$

(a) 12 (b) -12 (c) 2 (d) -2

W.K.that $\lambda_1+\lambda_2 = Tr(A) = -2$

(ii) Also Find E Values

W.K.that,

$\lambda_1+\lambda_2 = Tr(A)$ & $\lambda_1\cdot\lambda_2 = |A|$

$\lambda_1+\lambda_2 = -2$ & $\lambda_1\cdot\lambda_2 = -35$

$\lambda_1 = 5$, $\lambda_2 = -7$

Consider a 2 × 2 square matrix

$$A = \begin{bmatrix} \sigma & x \\ \omega & \sigma \end{bmatrix}$$

where x is unknown. If the eigen values of the matrix A are $(\sigma + j\omega)$ and $(\sigma - j\omega)$, then x is equal to

(a) $+j\omega$ (b) $-j\omega$

(c) $+\omega$ (d) $-\omega$

ATQ, $\lambda_1 = \sigma + i\omega$ & $\lambda_2 = \sigma - i\omega$

$\lambda_1 + \lambda_2 = Tr(A)$

$2\sigma = 2\sigma$

(identity)

$\lambda_1 \cdot \lambda_2 = |A|$

$\sigma^2 - i^2\omega^2 = \sigma^2 - \omega x$

$\omega x = i^2\omega^2$

$x = -\frac{\omega^2}{\omega} = -\omega$

$i = \sqrt{-1} = j$

$i^2 = -1 = j^2$

Q. If $A_{3\times3}$ & $|A-I|=0$, $Tr(A)=13$, $|A|=32$ then $\lambda_1^2+\lambda_2^2+\lambda_3^2=$?

(a) 12

(b) 13

(c) 81

(d) 80

$\because A_{3\times3}$ so let it's E Values are $\lambda_1, \lambda_2, \lambda_3$

ATQ, $|A-1I|=0$, $Tr(A)=13$, $|A|=32$

$|A-\lambda I|=0$

$\Rightarrow \lambda_1=1$

$\lambda_1+\lambda_2+\lambda_3=13$, $\lambda_1\cdot\lambda_2\cdot\lambda_3=32$

$1+\lambda_2+\lambda_3=13$, $1(\lambda_2)(\lambda_3)=32$

$\lambda_2+\lambda_3=12$, $\lambda_2\lambda_3=32$

$\lambda_2=8, \lambda_3=4$

So $\lambda_1^2+\lambda_2^2+\lambda_3^2=1^2+8^2+4^2=81$

Q. If P and Q are two Sq. Matrices of same order s.t $\boxed{PQ = QP = I}$ then 0 is an eigen value of ?

(a) P But Not Q

(b) Q But not P

(c) Both P & Q

(d) Neither P Nor Q

$\because PQ = QP = I \Rightarrow \begin{cases} P^{-1} = Q \\ Q^{-1} = P \end{cases}$

i.e. Both P & Q are Invertible.

or $|P| \neq 0$ & $|Q| \neq 0$.

i.e. Both are Non Singular

So Zero Can't be eigen value of any one Mat.

Q: if one of the EValue of $A = \begin{bmatrix} 40 & -16 & -24 \\ -11 & 30 & -19 \\ 26 & 24 & -50 \end{bmatrix}$ is λ then other EValues will be ?

MS8

(a) $\lambda - 20$

(b) $20 - \lambda$

(c) $20 + \lambda$

(d) 0

So By prop (6), one EValue is $\lambda = 0$

Now $\lambda_1 + \lambda_2 + \lambda_3 = Tr(A)$

$0 + \lambda + \lambda_3 = 20$

$\lambda_3 = 20 - \lambda$

M-II

$$|A| = \begin{vmatrix} 40 & -16 & -24 \\ -11 & 30 & -19 \\ 26 & 24 & -50 \end{vmatrix}$$

$C_1 \to C_1 + C_2 + C_3$

$$= \begin{vmatrix} 0 & -16 & -24 \\ 0 & 30 & -19 \\ 0 & 24 & -50 \end{vmatrix}$$

$|A| = 0 \Rightarrow \lambda = 0$

Prop (4)

Q: one of the E-Value of $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix}$ $\begin{matrix} =10 \\ =10 \\ =10 \\ =10 \end{matrix}$ 4×4 will be ? (a) 5 (b) 0 (c) 10 (d) −1

$= 10 \quad 10 \quad 10 \quad 10$

M-I using prop (6), one of the E Value of A is $= 10$ Ans

M-II C. Eqn of A, $|A| - \lambda I| = 0$

$$\begin{vmatrix} (1-\lambda) & 2 & 3 & 4 \\ 2 & (3-\lambda) & 4 & 1 \\ 3 & 4 & (1-\lambda) & 2 \\ 4 & 1 & 3 & (2-\lambda) \end{vmatrix} = 0$$

$C_1 \to C_1 + (C_2 + C_3 + C_4)$

$$\begin{vmatrix} (10-\lambda) & 2 & 3 & 4 \\ (10-\lambda) & (3-\lambda) & 4 & 1 \\ (10-\lambda) & 4 & (1-\lambda) & 2 \\ (10-\lambda) & 1 & 3 & (2-\lambda) \end{vmatrix} = 0$$

$$(10-\lambda) \begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 3-\lambda & 4 & 1 \\ 1 & 4 & 1-\lambda & 2 \\ 1 & 1 & 3 & 2-\lambda \end{vmatrix} = 0 \Rightarrow \lambda = 10$$

Q: the E-Values of $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}_{4\times 4}$ are ? $\lambda = 4, ?, ?, ?$

$= 4 \quad 4 \quad 4 \quad 4$

By prop (6) one E-Value is $\lambda = 4$.

$\because \rho(A) =$ one & Number of Non Zero E Values of $A \le \rho(A)$ (Prop 5)

i.e. No of Non Zero E-Values of $A \le 1$ ($\Leftarrow$ one)

A has only one Non Zero E Value which is $\lambda = 4$

So Remaining E Values are $\lambda = 0, 0, 0$.

overall for A, $\lambda = 4, 0, 0, 0$.

(M-II) Also find E. Values by conventional approach ? $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

C. Equ^n is $|A - \lambda I| = 0$

$$\begin{vmatrix} (1-\lambda) & 1 & 1 & 1 \\ 1 & (1-\lambda) & 1 & 1 \\ 1 & 1 & (1-\lambda) & 1 \\ 1 & 1 & 1 & (1-\lambda) \end{vmatrix} = 0$$

$C_1 \to C_1 + (C_2 + C_3 + C_4)$

$$\begin{vmatrix} (4-\lambda) & 1 & 1 & 1 \\ (4-\lambda) & (1-\lambda) & 1 & 1 \\ (4-\lambda) & 1 & (1-\lambda) & 1 \\ (4-\lambda) & 1 & 1 & (1-\lambda) \end{vmatrix} = 0$$

$$(4-\lambda)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & (1-\lambda) & 1 & 1 \\ 1 & 1 & (1-\lambda) & 1 \\ 1 & 1 & 1 & (1-\lambda) \end{vmatrix} = 0$$

$R_2 - R_1$, $R_3 - R_1$, $R_4 - R_1$

$$(4-\lambda)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{vmatrix} = 0 \Rightarrow (4-\lambda)(-\lambda^3) = 0$$

$\lambda = 4, 0, 0, 0.$

M-III (WRONG APP) →

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \xrightarrow[\substack{R_3 - R_1 \\ R_4 - R_1}]{R_2 - R_1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = A_1$$

$\lambda = 4, 0, 0, 0$ $\qquad$ $\lambda = 1, 0, 0, 0$

ie we have justified that equivalent Matrices have different E Values

Q: If $A_{2\times 2}$ s.t. $a_{11}=a_{12}=a_{21}=1$ & $a_{22}=-1$ then E. values of A^{19} are ?

(a) $\pm\sqrt{2}$

(b) ± 2

(c) ± 1

(d) $\pm 512\sqrt{2}$

$$A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}=\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

C.Eq is $|A-\lambda I|=0$

$$\boxed{\lambda^2-2=0} \Rightarrow \lambda=\pm\sqrt{2}$$

E. values of A are $\sqrt{2}$ & $-\sqrt{2}$

" " A^{19} are $(\sqrt{2})^{19}$ & $(-\sqrt{2})^{19}$

$$=(\sqrt{2})^{18}(\sqrt{2}) \;\&\; (-\sqrt{2})^{18}(-\sqrt{2})$$

$$=2^9\sqrt{2} \;\&\; 2^9(-\sqrt{2})$$

$$=512\sqrt{2} \;\&\; -512\sqrt{2}$$

Q. $A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & \frac{-1+i\sqrt{3}}{2} & 0 \\ 3 & 4 & \frac{-1-i\sqrt{3}}{2} \end{bmatrix}$ then $Tr(A^{102}) = ?$

Sol: $A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & \omega & 0 \\ 3 & 4 & \omega^2 \end{bmatrix} = L.T.M$

$\omega = \frac{-1}{2} + \frac{i\sqrt{3}}{2}, \omega^2 = \frac{-1}{2} - \frac{i\sqrt{3}}{2}$

$\omega^3 = 1, 1+\omega+\omega^2 = 0, \bar{\omega} = \omega^2$

$\bar{\omega} = \omega^2$ $\quad \bar{\omega^2} = \omega$

E Values of A are $1, \omega, \omega^2$

" " A^{102} are $1^{102}, \omega^{102}, (\omega^2)^{102}$

$= 1, \omega^{102}, \omega^{204}$

$= 1, (\omega^3)^{34}, (\omega^3)^{68} = 1, 1^{34}, 1^{68}$

E Values of A^{102} are $1, 1, 1$

$Tr(A^{102}) = 1+1+1 = 3$

Let A be a 3 × 3 matrix with Eigen values –1, 1, 0. Then $|\underbrace{A^{100}+I}_{B}|$ is _____.

E Values of A are $-1, 1, 0$

" " A^{100} are $(-1)^{100}, (1)^{100}, (0)^{100}$

" " A^{100} are $= 1, 1, 0$

Let $B = A^{100} + I$

$1+1=2$

$1+1=2$

$0+1=1$

Using prop (9)

E Values of B are $2, 2, 1$

So $|B| = (2)(2)(1) = 4$

Q. If $A=\begin{bmatrix}4&2\\2&4\end{bmatrix}$ & $B=A^2-2A+3I$ then Find the Product of the Eigen Values of B?

(a) 81

(b) 30

(c) 12

(d) 144

M-I

$A=\begin{bmatrix}4&2\\2&4\end{bmatrix}$ $\begin{matrix}\to 6\\ \to 6\end{matrix}$ $\lambda_A=6$, $\lambda_A=2$ (Do yourselfs/HW).

Using prop (9):

$B=(A^2-2A+3I)$ $\lambda_B=(6)^2-2(6)+3(1)=27$

$\lambda_B=(2)^2-2(2)+3(1)=3$

So Req. Product $=27\times3=81$

Verification:- $B=A^2-2A+3I=\begin{bmatrix}4&2\\2&4\end{bmatrix}\begin{bmatrix}4&2\\2&4\end{bmatrix}-2\begin{bmatrix}4&2\\2&4\end{bmatrix}+3\begin{bmatrix}1&0\\0&1\end{bmatrix}=\begin{bmatrix}15&12\\12&15\end{bmatrix}$ $\begin{matrix}=27\\=27\end{matrix}$

So E-Values of B are 27 & 3 & Product $=|B|=81$

Let A be a 4 × 4 matrix with real entries such that –1, 1, 2, –2 are its Eigen values. If $B = A^4 - 5A^2 + 5I$ then trace of A + B is _____.

$A \to \lambda_A = -1$

$\lambda_A = 1$

$\lambda_A = -2$

$\lambda_A = 2$

$B = A^4 - 5A^2 + 5I$

$(-1)^4 - 5(-1)^2 + 5(1) = 1 - 5 + 5 = 1$

$(1)^4 - 5(1)^2 + 5(1) = 1 - 5 + 5 = 1$

$(-2)^4 - 5(-2)^2 + 5(1) = 16 - 20 + 5 = 1$

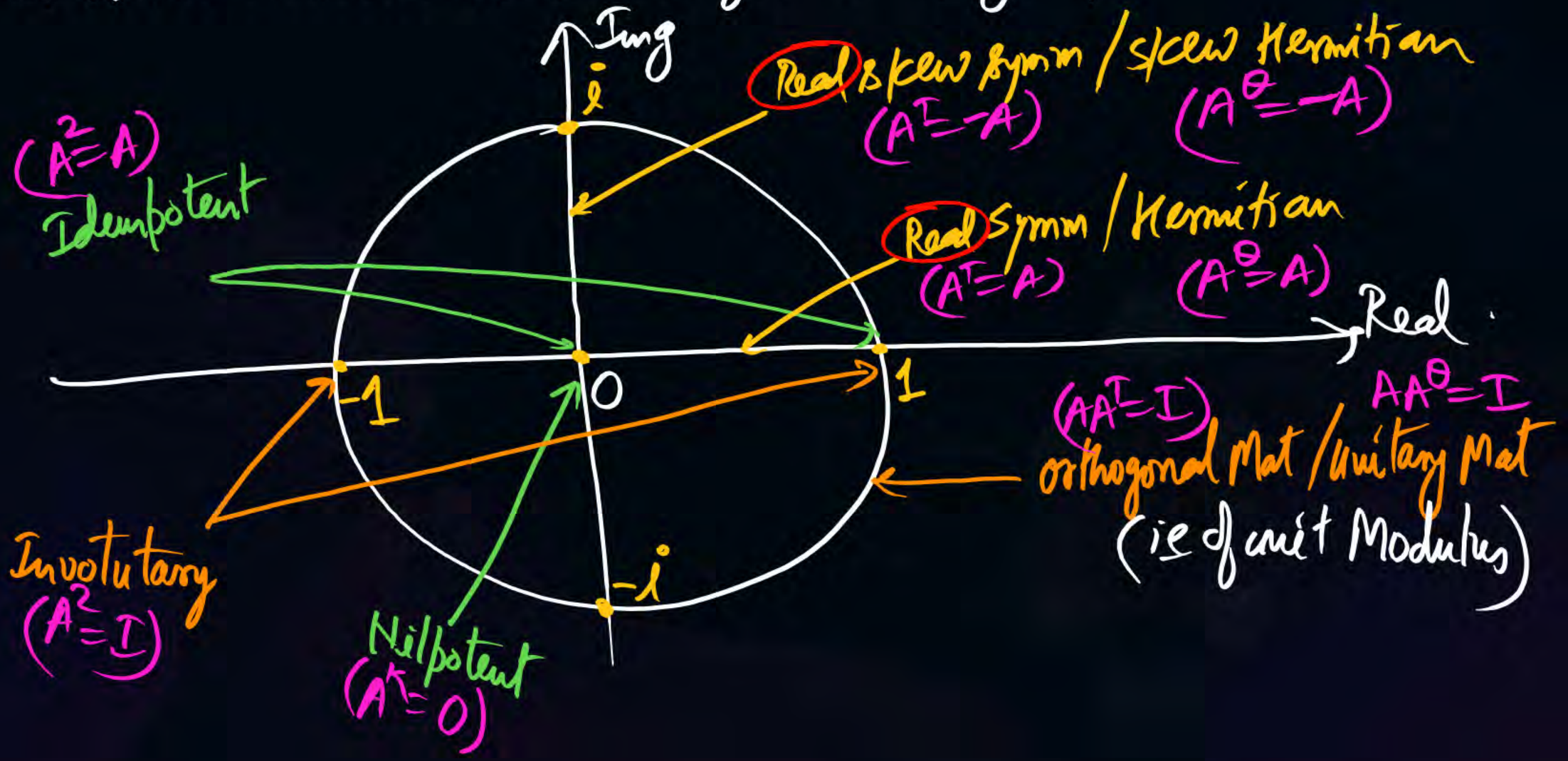
$(2)^4 - 5(2)^2 + 5(1) = 16 - 20 + 5 = 1$

$Tr(A+B) = Tr(A) + Tr(B)$

$= (-1+1-2+2) + (1+1+1+1) = 4$

(11) Shortcut Method to Learn Various Th based on E Values →

Consider a unit circle centered at origin i.e. $x^2+y^2=1$



eg $A = \begin{bmatrix} 2 & -3 & 4 \\ -3 & 5 & 1 \\ 4 & 1 & 0 \end{bmatrix}$, $A = \begin{bmatrix} 2 & -3 & 4+i \\ -3 & 5 & 1 \\ 4+i & 1 & 0 \end{bmatrix}$

= Real Symm. = Complex Symm.

$\lambda \in R$ it is not Necessary that λ is Real.

eg $A = \begin{bmatrix} 0 & -3 & 4 \\ 3 & 0 & 1 \\ -4 & -1 & 0 \end{bmatrix}$, $A = \begin{bmatrix} 0 & -3 & 4i \\ 3 & 0 & 2-i \\ -4i & -2+i & 0 \end{bmatrix}$

= Real skew symm = Comp skew symm.

$\lambda \in$ Purely Img Number it is not Necessary that λ is Purely Img.

Q If E Values of $A = \begin{bmatrix} 2 & 5+i & -3 \\ x & -1 & 4 \\ -3 & 4 & 3 \end{bmatrix}$ are all Real Nos then $x = ?$

(a) $5+i$

(b) $5-i$

(c) 0

(d) Both a & b

Real E-Values $\Longrightarrow$ Either A is Real Symm. ✗ ($\because a_{12}$ = Complex No)

or A is Hermitian ✓

$\Downarrow$

Corresponding elements are Conjugates of each other.

$\Downarrow$

$x = 5-i$

Q. if $A = \begin{bmatrix} - & - & - & - \\ - & - & - & - \\ - & - & - & - \\ - & - & - & - \end{bmatrix}_{4\times 4}$ is an O-Mat then $(AA^T)^{-1} = ? = (I)^{-1} = I = I_{4\times 4}$

(ii) which of the following can not be the E value of A ?

(a) $-\frac{1}{2} + i\frac{\sqrt{3}}{2} \Rightarrow |\lambda| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{1} = 1$ ✓

(b) $-1 \Rightarrow |\lambda| = 1$ ✓

(c) $\frac{\sqrt{3}}{2} + \frac{i}{2} \Rightarrow |\lambda| = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4} + \frac{1}{4}} = \sqrt{1} = 1$ ✓

(d) $\frac{1}{4} + i\frac{\sqrt{3}}{4} \Rightarrow |\lambda| = \sqrt{\left(\frac{1}{4}\right)^2 + \left(\frac{\sqrt{3}}{4}\right)^2} = \sqrt{\frac{1}{16} + \frac{3}{16}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$ ✓

(e) None

(*) If $\{\lambda_i, X_i\}$ is an E-pair of $A_{n\times n}$ then which is false ?

(a) $\exists$ Non Zero E vector of A if $\rho(A-\lambda I) < n$ T ($\because$ it is not Condition)

(b) if $A^{\theta} = A$ then $\lambda_i \in R \ \forall i$ (T) $\because$ A is Hermitian

(c) if $\boxed{A^{-1} = A^T}$ then $|\lambda_i| = 1 \ \forall i$ (T) $\because$ A is O-Mat

(d) $\{\lambda_i^m, X_i^m\}$ is an E-pair of A^m (F) $\because X^m =$ D'NE

T F

(e) If $\boxed{A = A^{-1}}$ then the Eigen Value of A is 1 (T) $\because$ A is Involutary Mat

$\because X = \begin{bmatrix} = \\ = \\ = \\ \vdots \\ = \end{bmatrix}_{n\times 1}$ so $X^2 = X_{n\times 1} X_{n\times 1} =$ N.D.

E-Value of A is λ & A^m is λ^m (T)

E-Vector of A & A^m are same (T)

Explanation: For Involutory Mat A,

$$A^2 = I$$

$$\Rightarrow \lambda^2 = 1 \Rightarrow \lambda = \pm 1$$

Note: for Idempotent Mat A,

$$A^2 = A$$

$$\Rightarrow \lambda^2 = \lambda$$

$$\lambda(\lambda - 1) = 0 \Rightarrow \lambda = 0 \text{ or } 1$$

Note: Real Evalues of an orthogonal Mat are 1 or -1 (only)

For any O-Mat A, we have

$$AA^T = I$$

$$(\lambda)(\lambda) = 1$$

$$\lambda^2 = 1 \Rightarrow \lambda = \pm 1$$

$\because |\lambda| = 1$ (justified)

& for complex Evalues, $|\lambda| = 1$.

The eigen values of $A = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$ are _____.

(a) Purely imaginary (b) Zero

(c) Real (d) None of the above

M-IV

$|A| = 0 - (i)(-i)$

$= +i^2 = -1$

$\Rightarrow \lambda = -1, 1$

(Real)

M-I

$\because A = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} =$ Skew symm.

(But Comp. Skew symm)

$\Rightarrow \lambda =$ Purely Imag.

M-II Now $A = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} =$ Hermitian also.

$\Downarrow$

$\lambda \in R$.

M-III C. Equ^n is $|A - \lambda I| = 0$

$$\begin{vmatrix} 0-\lambda & +i \\ -i & 0-\lambda \end{vmatrix} = 0$$

$\lambda^2 - (i)(-i) = 0$

$\lambda^2 + i^2 = 0$

$\lambda^2 - 1 = 0$

$\lambda = \pm 1$ (Real Nos)

Q. A & B are Non zero sq. Matrices then $AB = O \Rightarrow$

Famous Doubt

(a) B is singular

(b) A is singular

(c) A & B are orthogonal

(d) A & B are singular

Case I: let A is Non singular then A^{-1} exist

$AB = O$

$A^{-1}(AB) = A^{-1}O$

$B = O$ which is false ATQ.

So our assumption is wrong ie A must be sing

Case II: let B is Non sing then B^{-1} exist.

$AB = O$

$(AB)B^{-1} = O.B^{-1}$

$A = O$ (false)

So again our assumption is wrong Hence B must be sing

THANK - YOU

Tel: dr puneet sir pw

DS & AI
CS & IT

Linear Algebra

Lecture No. 09

By- Dr. Puneet Sharma Sir

Recap of previous lecture

Topic

EIGEN VALUES & their properties

Topics to be Covered

Topic

(1) CAYLEY-HAMILTON THEOREM

(2) Methods of finding Eigen Vectors

Cayley-Hamilton Theorem → "Every square Mat satisfies it's <u>own</u> C.Eq."

ie we can Replace $\lambda \to A$ in C.Equ^n.

Consider $A_{n\times n}$ then it's C.Equ^n is $|A-\lambda I|=0$

$$\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots\cdots + a_{n-1}\lambda + a_n = 0 \longrightarrow \text{Algebraic Equ}^n$$

Using C.H.T, $\lambda \to A$

$$1.A^n + a_1A^{n-1} + a_2A^{n-2} + \cdots\cdots + a_{n-1}A + a_nI = O \longrightarrow \text{Matrix Equ}^n.$$

<u>Application</u>: with the Help of C.H.T we can Calculate $Tr(A)$, $|A|$, A^{-1} & $A^m = ?$

① $Tr(A) = -(a_1)$

② $|A| = (-1)^n a_n$

③ A^{-1} = will be discussed in Questions

④ A^m = " " " " "

<u>Take Care</u>:- Above Results are Valid when Coeff of $A^n = 1$

Some More Conclusions from C.H.T. :-

① Constant term in the C.Eq of $A_{n\times n}$ = $\boxed{a_n = \begin{cases} -|A|, & n = \text{odd} \\ +|A|, & n = \text{even} \end{cases}}$

Proof: we know that (by C.H.T)

$$|A| = (-1)^n a_n$$

$$(-1)^n |A| = (-1)^n \cdot (-1)^n a_n$$

$$(-1)^n |A| = (-1)^{2n} \cdot a_n$$

$$(-1)^{2n} a_n = (-1)^n |A|$$

i.e. $a_n = (-1)^n |A| = \begin{cases} -|A|, & n = \text{odd} \\ +|A|, & n = \text{even} \end{cases}$

② Shortcut Method of finding C.Eq of $A_{2\times 2}$ →

C.Equ^n is $|A - \lambda I| = 0$

$$\lambda^2 + a_1\lambda + a_2 = 0$$

$$\lambda^2 - (-a_1)\lambda + \{(-1)^2 a_2\} = 0$$

$$\boxed{\lambda^2 - (\text{Tr}A)\lambda + (|A|) = 0}$$

Q. for the Matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ Evaluate $Tr(A)$, $|A|$, & A^{-1}? using C.H.Th.

Sol: C-Equ^n of A is $|A - \lambda I| = 0$

$$(\lambda-3)(\lambda-1)^2 = 0$$

$$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

By C.H.Th, $\lambda \to A$.

$$\boxed{A^3 - 5A^2 + 7A - 3I = 0} \quad \text{--- (1)}$$

on Comparision with,

$$A^3 + a_1 A^2 + a_2 A + a_3 I = 0.$$

(1) $Tr(A) = -(a_1) = -(-5) = +5$ ✓

(2) $|A| = (-1)^3 a_3 = -(-3) = +3$ ✓

(3) $\because |A| \neq 0$ ie A^{-1} exist. So By (1),

$$A^{-1}(A^3 - 5A^2 + 7A - 3I) = A^{-1} \cdot O$$

$$A^2 - 5A + 7I - 3A^{-1} \cdot I = O$$

$$-3A^{-1} = -(A^2 - 5A + 7I)$$

$$A^{-1} = \frac{1}{3}(A^2 - 5A + 7I)$$ Ans

(iv) Evaluate $B = \underbrace{A^8 - 5A^7 + 7A^6 - 3A^5} + \underbrace{A^4 - 5A^3 + 7A^2 + 4A} + 2I = ?$

By (1) we know that

$$\boxed{A^3 - 5A^2 + 7A - 3I = 0} \quad \text{—— (1)}$$

$$B = A^5(A^3 - 5A^2 + 7A - 3I) + A(A^3 - 5A^2 + 7A - 3I) + 7A + 2I$$

$$= A^5(0) + A(0) + 7A + 2I$$

$$B = 7A + 2I$$

(a) $A + 2I$ (b) $2I$

(c) $7A + 2I$ (d) 0

(ii) Also Evaluate $|B| = ?$ $= |7A + 2I| = \cancel{7|A| + 2|I|} = \cancel{7(3) + 2(1) = 23}$

$$B = 7A + 2I = 7\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 16 & 7 & 7 \\ 7 & 16 & 7 \\ 0 & 0 & 9 \end{bmatrix}$$

$$|B| = 9[256 - 49] = 9 \times 207 = 1863$$

Q: If $P_{3\times3}$ Mat s.t its C.Eq is $a(\lambda) = |\lambda I - P| = \boxed{\lambda^3+\lambda^2+\lambda+1=0}$ $f(\lambda)$

then evaluate $Tr(P)$, $|P|$, P^{-1}

Sol: C.Equ of P is $|P-\lambda I|_{3\times3}=0$

$$(-1)^3|\lambda I-P|=0$$

$$|\lambda I-P|=0$$

$$\lambda^3+\lambda^2+\lambda+1=0$$

By C.H.T, $\lambda \to P$

$$P^3+1P^2+1P+1I=0 \quad \text{--- (1)}$$

on Comparision with, $(P^3+a_1P^2+a_2P+a_3I=0)$

(i) $Tr(P) = -(a_1) = -(1) = -1$

(ii) $|P| = (-1)^3 a_3 = -(1) = -1$

(iii) By (1), $P^{-1}(P^3+P^2+P+I) = P^{-1}\cdot 0$

$$P^2+P+I+P^{-1}=0$$

$$P^{-1} = -(P^2+P+I)$$

(iv) If $B = \underbrace{P^8+P^7+P^6+P^5}\underbrace{+P^4+P^3+P^2+P}+4I$ then $|B| = ?$

(a) 1

(b) −1

(c) 4

(d) 64

$= P^5(P^3+P^2+P+I)+P(P^3+P^2+P+I)+4I$

$B = P^5(O)+P(O)+4I$

$\boxed{B = 4I} \Rightarrow |B| = |4I|_{3\times3} = 4^3|I| = 64(1) = 64$

Q: Evaluate the Constant term in the C.Equ^n of $A = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}_{4\times 4}$

M-I C.Equ^n is $|A - \lambda I| = 0$

$$\begin{vmatrix} (2-\lambda) & 1 & 1 & 1 \\ 1 & (2-\lambda) & 1 & 1 \\ 1 & 1 & (2-\lambda) & 1 \\ 1 & 1 & 1 & (2-\lambda) \end{vmatrix} = 0$$

- - - - - - - - -

$$\lambda^4 - 8\lambda^3 + a_2\lambda^2 + a_3\lambda + (5) = 0$$

So Constant term $= 5$ **Ans**

M-II $|A| = \ldots\ldots\ldots = 5$

$\because$ A is of Even order so

$$a_n = (-1)^4 |A| = +|A| = +5 \quad \textbf{Ans}$$

Q: Constant term in the C.Eq of $\begin{bmatrix} 0 & 1 & 2 & -3 & 4 \\ -1 & 0 & 4 & -1 & 2 \\ -2 & -4 & 0 & 3 & 0 \\ 3 & 1 & -3 & 0 & 2 \\ -4 & -2 & 0 & -2 & 0 \end{bmatrix}_{5\times 5}$

$\because$ A is Skew symm Mat of odd order

So $|A| = 0 \Rightarrow$ Const term $a_5 = -|A| = 0$

Q. if $A = \begin{bmatrix} -1 & -2 \\ 0 & 3 \end{bmatrix}$ then A will satisfy ?

(a) $A^2 - 2A - 3I$

(b) $(A+I)(A-3I) = 0$

(c) $A - 2I - 3A^{-1} = 0$

(d) $A^2 - 3A + 2I = 0$

C.Equ of A is $|A - \lambda I| = 0$

$$\lambda^2 - (Tr(A))\lambda + |A| = 0$$

$$\lambda^2 - (2)\lambda + (-3) = 0$$

By C.H.T, $\boxed{A^2 - 2A - 3I = 0}$ Ans

A^{-1} | factor

$\boxed{A - 2I - 3A^{-1} = 0}$ Ans

$\boxed{(A+I)(A-3I) = 0}$ Ans

TAG DAA QUEST:-

Q. if $A = \begin{bmatrix} -1 & -2 \\ 0 & 3 \end{bmatrix}$ then which is/are True

(MS8)

(a) $A^2 - 2A - 3I$

(b) $(A - 3I)(A + I) = 0$

(c) $A - 2I - 3A^{-1}$

(d) $A^2 - 3A - 2I$

Q. If E-Values of $A_{4\times4}$ are ±1 & $\pm i$ then $A^4 = ?$

(a) A

(b) O

(c) I_3

(d) I_4

$\because \lambda = 1, -1, i, -i$

so C-Equ^n is

$$(\lambda-1)(\lambda+1)(\lambda-i)(\lambda+i)=0$$

$$(\lambda^2-1^2)(\lambda^2-i^2)=0$$

$$(\lambda^2-1)(\lambda^2+1)=0$$

$$(\lambda^4-1^2)=0$$

$$\lambda^4-1=0$$

By C.H.Th, $\lambda \to A$

$$A^4-I=O$$

$$A^4=I=I_{4\times4}=I_4$$

eg: if CE of $A_{4\times4}$ is $2\lambda^4-6\lambda^3+4\lambda^2-8\lambda+12=0$ then $|A|=$?

Sol: CEqu^n is $\lambda^4-3\lambda^3+2\lambda^2-4\lambda+6=0$

so $|A|=(-1)^4 a_4=(+1)(6)=6$

Qe constant term in the CEq of $A=\begin{bmatrix}0&1&2&3\\1&0&3&0\\2&3&0&1\\3&0&1&2\end{bmatrix}$ will be ?

HW

$\because |A|=\ldots\ldots\ldots=88$

so constant term in C.Equ^n $=a_n=(-1)^4|A|=+(88)$

EIGEN VECTORS

Def^n: Consider Sq. Mat $A_{n\times n}$ then Non Zero Vector X is called Eigen vector, corresponding to Eigen value λ (Real/Complex/Zero) if we are able to find a relationship of the type,

$$\boxed{AX = \lambda X}$$

λ = Eigen value
X = Eigen vector.

LHS is the Multi of Two Matrices = RHS is the Scalar Multi in a Mat
(Tough) (Easy)

(*) Here we are considering Homogeneous system as follows

$$AX = \lambda X \Longrightarrow AX - \lambda X = 0 \Longrightarrow \boxed{(A - \lambda I)X = 0}$$

So it will satisfy all the prop of Homog System.

PROPERTIES of E.VECTORS →

① If X is an E-vector of A then (KX) will also be an E-vector

$$AX = \lambda X \qquad\qquad A(KX) = \lambda(KX)$$

$$\boxed{(A-\lambda I)X = 0}$$ E.vector

$$K(A-\lambda I)X = K\cdot 0$$

$$\boxed{(A-\lambda I)(KX) = 0}$$ E vector

i.e "we are free to multiply or divide with any constant in case of E vector.

② E-Vectors corresponding to different E Values of symm Mat are orthogonal

Q (P-48) if $A_{3\times3}$ s.t $A^T = A$ = symm. and a, b are the E-Values & Corresponding E-vectors are $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ & $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ then for $a \neq b$ (different EValues) the value of

$x_1y_1 + x_2y_2 + x_3y_3 = \underline{?}$

By above property $X \cdot Y = 0 \Rightarrow X^T Y = 0$

$\Rightarrow x_1y_1 + x_2y_2 + x_3y_3 = 0$

③ For Different E-Values, Corresponding E-Vectors are also LI

④ If E-Value Repeats (then Headache will start) then Corresponding E Vectors may be LI or may be LD.

⑤ MODAL MATRIX :→ Matrix formed by E-Vectors is Called Modal Mat

Let $A_{3\times3}$ is the given Mat & X_1, X_2, X_3 are the E-Vectors then

Modal Mat is $P = [X_1 \; X_2 \; X_3]$

eg: for $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$

$\lambda = 6, \; X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \because AX_1 = 6X_1$

$\lambda = 2, \; X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \because AX_2 = 2X_2$

Now we try to understand the Procedure of finding E.vectors.

Q: Find the E-Values & E.Vectors of $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$

Sol: C-Equ^n is $|A-\lambda I| = 0 \Rightarrow \lambda^2 - (8)\lambda + (12) = 0 \Rightarrow (\lambda-6)(\lambda-2) = 0 \Rightarrow \lambda = 6 \,\&\, 2$

Eigen Vector for ($\lambda = 6$) $\rightarrow$

Consider $AX = \lambda X$

$$(A - \lambda I)X = 0$$

$$(A - 6I)X = 0$$

$$\begin{bmatrix} (4-6) & 2 \\ 2 & (4-6) \end{bmatrix} X = 0$$

$$\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} X = 0$$

$R_2 \rightarrow R_2 + R_1$

$$\underset{2\times 2}{\begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \underset{2\times 1}{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} \Rightarrow \begin{bmatrix} (-2x_1 + 2x_2) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

ie $\left.\begin{array}{l} -2x_1 + 2x_2 = 0 \\ \& \quad 0 = 0 \end{array}\right\} \Rightarrow x_1 = x_2 = K$ (let)

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} K \\ K \end{bmatrix} = K \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

E. Vector for $\lambda=2$ →

Consider $AX=\lambda X$

$$(A-\lambda I)X=0$$

$$(A-2I)X=0$$

$$\begin{bmatrix}(4-2) & 2\\ 2 & (4-2)\end{bmatrix}X=0$$

$$\begin{bmatrix}2 & 2\\ 2 & 2\end{bmatrix}X=0$$

$R_2\to R_2-R_1$ $\begin{bmatrix}2 & 2\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$

$$\begin{bmatrix}(2x_1+2x_2)\\ 0\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\Rightarrow \left.\begin{matrix}2x_1+2x_2=0\\ 0=0\end{matrix}\right\}\Rightarrow x_1=-x_2$$

Let $x_2=K$ then $x_1=-K$ so

$$X=\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}-K\\ K\end{bmatrix}=K\begin{bmatrix}-1\\ 1\end{bmatrix}$$

Finally, $A=\begin{bmatrix}4 & 2\\ 2 & 4\end{bmatrix}$ → $\lambda=6,\ X=\begin{bmatrix}1\\ 1\end{bmatrix}$

→ $\lambda=2,\ X=\begin{bmatrix}-1\\ 1\end{bmatrix}$

Analysis (PODCAST) →

① $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$

$\lambda = 6,\ X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ -3 \end{bmatrix}, \begin{bmatrix} 1/4 \\ 1/4 \end{bmatrix}$ ---- ∞ Evectors exist.

But all are LD on X_1

$\lambda = 2,\ X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \longrightarrow \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1/5 \\ -1/5 \end{bmatrix}$ --- ∞ Evectors exist.

But all are LD on X_2

② Here X_1 & X_2 are LI (By observation)

Actually we are getting two Ind families of E.vectors. & if we are taking one member from F_1 & one member from F_2 then these are LI

i.e for Mat A, $\exists$ two LI eigenvectors at a time

(3) $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$, $\begin{cases} \lambda = 6, & X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \lambda = 2, & X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{cases}$

= Symm Mat.

$\because X_1 \cdot X_2 = (1)(-1) + (1)(1)$
$= 0$

i.e. X_1 & X_2 are orthogonal vectors $\Rightarrow$ Hence they are LI also.

i.e. for different eigen values of symm Mat, corresponding E. vectors are <u>orthogonal</u> as well as LI

(4) In this Mat, $X_1 \cdot X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = (-1)(1) + (1)(1) = 0$

is X_1 & X_2 are orthogonal vectors.

(5) one pair of L.I E vectors of $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$ will be

(HW)

(a) $\begin{bmatrix} -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -4 \end{bmatrix}$

(b) $\begin{bmatrix} 3 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

(c) $\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

(d) $\begin{bmatrix} 3 \\ -3 \end{bmatrix}, \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$

Q: find E. Values & E-vectors of $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$

Sol: $\because$ A is U.T.M so $\lambda = 1$ & 2

E-vector for $\lambda = 1$ $\longrightarrow$

$$AX = \lambda X$$

$$(A - \lambda I)X = 0$$

$$(A - I)X = 0$$

$$\begin{bmatrix} (1-1) & 2 \\ 0 & (2-1) \end{bmatrix} X = 0$$

$$\begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} X = 0$$

$R_2 \to R_2 - \frac{1}{2} R_1$; $\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 2x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_2 = 0$, Let $x_1 = K$.

$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} K \\ 0 \end{bmatrix} = K \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ i.e for $\lambda = 1$, $X = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

E. Vector for $\lambda=2$ →

Consider $AX=\lambda X$

$$(A-\lambda I)X=0$$

$$(A-2I)X=0$$

$$\begin{bmatrix}(1-2) & 2\\ 0 & (2-2)\end{bmatrix}X=0$$

$$\begin{bmatrix}-1 & 2\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}(-x_1+2x_2)\\ 0\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\left.\begin{aligned}-x_1+2x_2&=0\\ 0&=0\end{aligned}\right\}\Rightarrow x_1=2x_2$$

Let $x_2=K$ then $x_1=2K$ so

$$X=\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}2K\\ K\end{bmatrix}=K\begin{bmatrix}2\\ 1\end{bmatrix}$$

so for $\lambda=2$, $X=\begin{bmatrix}2\\ 1\end{bmatrix}$

Finally, $A=\begin{bmatrix}1 & 2\\ 0 & 2\end{bmatrix}$ $\begin{cases}\lambda=1,\ X=\begin{bmatrix}1\\ 0\end{bmatrix}\\ \lambda=2,\ X=\begin{bmatrix}2\\ 1\end{bmatrix}\end{cases}$

Q: Find the E-Values and E-Vectors of $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Sol: $\because$ A is U.T.M so $\lambda = 2, 2, 3$

E-Vector for $(\lambda = 3) \rightarrow$

$$AX = \lambda X$$

$$(A - \lambda I)X = 0$$

$$(A - 3I)X = 0$$

$$\begin{bmatrix} (2-3) & 1 & 0 \\ 0 & (2-3) & 0 \\ 0 & 0 & (3-3) \end{bmatrix} X = 0$$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} X = 0$$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} (-x_1 + x_2) \\ -x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} -x_1 + x_2 = 0 \\ -x_2 = 0 \\ 0 = 0 \end{array}$$

ie $x_2 = 0$, $x_1 = x_2 = 0$ so $x_1 = 0$

let $x_3 = K$ so

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ K \end{bmatrix} = K \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

ie for $\lambda = 3$, $X = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

E. Vector for $\lambda = 2$ →

Consider $AX = \lambda X$

$$(A - \lambda I)X = 0$$

$$(A - 2I)X = 0$$

$$\begin{bmatrix} (2-2) & 1 & 0 \\ 0 & (2-2) & 0 \\ 0 & 0 & (3-2) \end{bmatrix} X = 0$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} X = 0$$

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_2 \\ x_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x_2 = 0, x_3 = 0$$

Let $x_1 = K$.

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} K \\ 0 \\ 0 \end{bmatrix} = K \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

So for $\lambda = 2$, $X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Finally:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$(\lambda = 3, 2, 2)$

$\lambda = 3, X = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\lambda = 2, X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Q. Find E.Values & E.vectors of $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Sol: C.Equ^n is $|A - \lambda I| = 0 \longrightarrow (\lambda-3)(\lambda-1)^2 = 0 \Rightarrow \lambda = 3, 1, 1$

E.vector for $\lambda = 3$

$$AX = \lambda X$$

$$(A - 3I)X_1 = 0$$

HW

for $\lambda = 3$, $X_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

E.vector for $\lambda = 1$ $\rightarrow$

$$AX = \lambda X$$

$$(A - \lambda I)X = 0$$

$$(A - I)X = 0$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} X = 0$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} (x_1 + x_2 + x_3) \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x_1 + x_2 + x_3 = 0$$

Let $x_3 = K_1$, $x_2 = K_2$

$x_1 = -(K_2 + K_1)$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -(K_2 + K_1) \\ K_2 \\ K_1 \end{bmatrix}$$

$$X = \begin{bmatrix} -(k_2+k_1) \\ k_2 \\ k_1 \end{bmatrix} = \begin{bmatrix} -k_2 \\ k_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -k_1 \\ 0 \\ k_1 \end{bmatrix} = k_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + k_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

ie for $\lambda = 1$ we are getting two E-vectors

$$X_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \because AX_2 = 1 \cdot X_2$$

$$X_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \because AX_3 = 1 \cdot X_3$$

THANK - YOU

Tel: dr puneet sir pw

DS & AI
CS & IT

Linear Algebra

Lecture No. 10

By- Dr. Puneet Sharma Sir

Recap of previous lecture

Topic

BASICS of EIGEN VECTORS

Topics to be Covered

Topic

- EIGEN VECTORS (Continued)
- Diagonalisation
- Miscellaneous

EIGEN VECTORS

Def^n: Consider Sq. Mat $A_{n\times n}$ then Non Zero Vector X is called Eigen vector, corresponding to Eigen value λ (Real/Complex/Zero) if we are able to find a relationship of the type,

$$\boxed{AX = \lambda X}$$

λ = Eigen value
X = Eigen vector.

LHS is the Multi of Two Matrices (Tough) = RHS is the Scalar Multi in a Mat (Easy)

(*) Here we are considering Homogeneous system as follows

$$AX = \lambda X \Longrightarrow AX - \lambda X = 0 \Longrightarrow \boxed{(A - \lambda I)X = 0}$$

So it will satisfy all the prop of Homog System.

PROPERTIES of E.VECTORS →

① If X is an E-vector of A then (KX) will also be an E-vector

$$AX = \lambda X \qquad \Longleftrightarrow \qquad A(KX) = \lambda(KX)$$

$$\boxed{(A-\lambda I)X = 0}$$ E.vector

$$K(A-\lambda I)X = K\cdot 0$$

$$\boxed{(A-\lambda I)(KX) = 0}$$ E vector

i.e "we are free to multiply or divide with any constant in case of E vector.

② E-Vectors corresponding to different E Values of Symm Mat are orthogonal

$\Downarrow$

Hence L.I also.

Q if $A_{3\times3}$ s.t $A^T = A$ (= Symm.) and a, b are the E-Values & Corresponding E-vectors are $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ & $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ then for $a \neq b$ (different EValues) the value of

(PYQ)

$x_1y_1 + x_2y_2 + x_3y_3 = \underline{?}$

By above property $X \cdot Y = 0 \Rightarrow X^T Y = 0$

$\Rightarrow x_1y_1 + x_2y_2 + x_3y_3 = 0$

③ For Different E-Values, Corresponding E-Vectors are also LI

④ If E-Value Repeats (then Headache will start) then Corresponding E Vectors may be LI or may be LD.

⑤ <u>MODAL MATRIX</u> :→ Matrix formed by E-Vectors is Called Modal Mat

Let $A_{3\times3}$ is the given Mat & X_1, X_2, X_3 are the E-Vectors then

Modal Mat is $P = [X_1 \; X_2 \; X_3]$

⑥ **Algebraic Multiplicity (A.M)** Number of times particular eigen value repeats is called it's A.M

⑦ **Geometric Multiplicity (G.M)** Number of L.I E vectors for particular E value λ is called it's G.M

⑧ **Shortcut of Finding G.M (for λ)** → let λ is the Repeated E value then

$$\underbrace{\text{No. of L.I E. Vectors for } \lambda}_{\text{GM of } \lambda} = \boxed{\text{order} - \rho(A - \lambda I)}$$

(9) Short cut method of finding E. Vectors (if options are given) →

" λ should be unique in all comparisions & there should not be any Invalid Result "

(10) $\boxed{\text{E. Vectors of } A \text{ \& } A^m \text{ would be Same}}$

i.e if (λ, X) is an Eigen pair of A

then (λ^m, X) is an Eigen Pair of A^m.

(11) **Positive Definite Mat** → If All the E Values are +ve then A is Called +ve Definite.

ie $\lambda > 0$

Negative Definite → If all the E Values are −ve then A is Called −ve Definite.

Idefinite Mat → If E Values are +ve as well as −ve then A is Called Indefinite

Semi +ve Definite → If E Values are 0 and +ve then A is semi +ve definite

ie $\lambda \geq 0$

Spectral Radius → Largest Magnitude of Eigen Values is Called Spectral Radius.

Analysis (PODCAST) →.

① $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$

$\lambda = 6,\ X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ -3 \end{bmatrix}, \begin{bmatrix} 1/4 \\ 1/4 \end{bmatrix}$ ---- ∞ Evectors exist.

But all are LD on X_1

$\lambda = 2,\ X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \longrightarrow \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1/5 \\ -1/5 \end{bmatrix}$ --- ∞ Evectors exist.

But all are LD on X_2

② Here X_1 & X_2 are LI (By observation)

Actually we are getting two Ind families of E. vectors. & if we are taking one member from F_1 & one member from F_2 then these are LI

i.e for Mat A, $\exists$ two LI eigen vectors at a time

③ $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$, $\begin{cases} \lambda = 6, & X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \lambda = 2, & X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{cases}$

= Symm Mat.

$\because X_1 \cdot X_2 = (1)(-1) + (1)(1)$
$= 0$

i.e. X_1 & X_2 are orthogonal vectors $\Longrightarrow$ Hence they are LI also.

i.e. for different eigen values of symm Mat, corresponding E. vectors are <u>orthogonal</u> as well as LI

(4) In this Mat, $X_1 \cdot X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = (-1)(1) + (1)(1) = 0$

is X_1 & X_2 are orthogonal Vectors.

(5) one pair of L.I, E Vectors of $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$ will be

(HW)

(a) $\begin{bmatrix} -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -4 \end{bmatrix}$ these are E. Vector But L.D

(b) $\begin{bmatrix} 3 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ these are E. Vector but L.D

(c) $\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ these are L.I but 2nd is not an E. Vector.

(d) ✓ $\begin{bmatrix} 3 \\ -3 \end{bmatrix}, \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$ these are eigen Vector as well as L.I

$\because AX = \lambda X$

$\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \lambda \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$\begin{bmatrix} 14 \\ 14 \end{bmatrix} = \begin{bmatrix} 2\lambda \\ 3\lambda \end{bmatrix}$

$\lambda = 7, \lambda = \frac{14}{3}$ (not unique)

As it is not an E. Vector

eg: For $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

(1) $\lambda = 3, 2, 2$

(Already solved).

$\lambda = 3,\ X = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ i \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1/3 \end{bmatrix} \ldots\ldots \infty$ E.Vexist

$\lambda = 2,\ X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} i \\ 0 \\ 0 \end{bmatrix} \ldots\ldots \infty$ E.vector exist

(2) X_1 & X_2 are LI (By observation)

(3) AMof '3' = one, GMof '3' = one

AMof '2' = two, GMof '2' = one

(M-II) $GMof(\lambda = 2) = order - \rho(A - \lambda I)$

$= 3 - \rho(A - 2I)$

$GMof(\lambda = 2) = 3 - \rho \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= 3 - 2 =$ one

If $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

① $(\lambda = 3, 1, 1)$

(Already Solved)

$\lambda = 3$, $X_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/3 \\ 1/3 \\ 0 \end{bmatrix}$ ---- ∞ EV exist.

$\lambda = 1$

$X_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} -3 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ - - - - - - " "

$X_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -4 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$ - - - - - " "

② Here X_1, X_2, X_3 are LI $\because$ Modal Mat is Non singular where $P = [X_1 X_2 X_3]$

③ AM of '3' = one, GM of '3' = one

AM of '1' = two, GM of '1' = two

M-II $GM\,of\,(\lambda = 1) = order - \rho(A - \lambda I)$

$= 3 - \rho(A - I)$

$GM\,of\,(\lambda = 1) = 3 - \rho \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $(\because |P| \neq 0)$

$= 3 - 1 = 2$ Ans

Q. Find AM & GM of λ for $A = \begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3}$, $\alpha \neq 0$.

Sol: $\because$ A is U.T.M so $\lambda = 0, 0, 0$ Now, AM of '0' = three.

$$GM \text{ of } (\lambda = 0) = \boxed{\text{order} - \rho(A - \lambda I)}$$

$$= 3 - \rho(A - 0I) = 3 - \rho(A) = 3 - 1 = 2 \text{ Ans}$$

Verification :- E. Vector for $(\lambda = 0) \rightarrow$

$$AX = \lambda X$$
$$(A - \lambda I)X = 0$$
$$(A - 0I)X = 0$$
$$AX = 0$$

$$\begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \alpha x_3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\because \alpha \neq 0$, $x_3 = 0$

let $x_1 = k_1$, $x_2 = k_2$ then

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ 0 \end{bmatrix} = \begin{bmatrix} k_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ k_2 \\ 0 \end{bmatrix}$$

$$X = k_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

i.e. for $\lambda = 0$, X_1 & X_2 are E. Vector

Note:

① $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ has <u>three</u> LI Evectors, <u>one</u> for $\lambda = 3$ & <u>two</u> for $\lambda = 1$

② $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ has <u>two</u> LI Evectors, <u>one</u> for $\lambda = 3$ & <u>one</u> for $\lambda = 2$

③ $A = \begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ has <u>two</u> LI Evectors & <u>Both</u> for $\lambda = 0$

$(\alpha \neq 0)$

$\because$ for $\lambda = 0$

$X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix} \cdots \infty$ Evectors exist

$X_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix} \cdots \infty$ Evectors exist.

(1993) The E. vectors of $A = \begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\alpha \neq 0$ is/are $(\lambda = 0, 0, 0)$

MS8

(a) $(\alpha\ 0\ 0)'$ (b) $(0\ 0\ 1)'$ (c) $(0\ \alpha\ 0)'$ (d) $(0\ 0\ \alpha)'$

Sol: Taking (a): $AX = \lambda X$

$$\begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda\alpha \\ 0 \\ 0 \end{bmatrix}$$

$\lambda\alpha = 0$, $0 = 0$, $0 = 0$

$\lambda = 0$ unique, Valid

(a) ✓

Taking (b), $AX = \lambda X$

$$\begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \lambda \end{bmatrix}$$

$\alpha = 0$, $0 = 0$, $\lambda = 0$

Invalid, Valid, unique

So (b) ✗

Taking (c) ✓

Taking (d) ✗

Q. one of the EVector of $A=\begin{bmatrix}5&0&0&0\\0&5&5&0\\0&0&2&1\\0&0&3&1\end{bmatrix}$ will be ?

(a) $\begin{bmatrix}1\\-2\\3\\0\end{bmatrix}$ (b) $\begin{bmatrix}1\\-2\\0\\0\end{bmatrix}$, (c) $\begin{bmatrix}2\\3\\0\\-1\end{bmatrix}$, (d) $\begin{bmatrix}1\\-1\\2\\3\end{bmatrix}$

(ii) Also find Corresponding E. Value.

Taking (a): $AX=\lambda X$

$$\begin{bmatrix}5&0&0&0\\0&5&5&0\\0&0&2&1\\0&0&3&1\end{bmatrix}\begin{bmatrix}1\\-2\\3\\0\end{bmatrix}=\lambda\begin{bmatrix}1\\-2\\3\\0\end{bmatrix}$$

$$\begin{bmatrix}5\\5\\6\\9\end{bmatrix}=\begin{bmatrix}\lambda\\-2\lambda\\3\lambda\\0\end{bmatrix}\Rightarrow \underbrace{\lambda=5,\ \lambda=-\frac{5}{2},\ \lambda=2}_{\text{Not unique}},\ \underbrace{9=0}_{\text{Invalid.}}$$

Taking (b) $AX=\lambda X$

$$\begin{bmatrix}5&0&0&0\\0&5&5&0\\0&0&2&1\\0&0&3&1\end{bmatrix}\begin{bmatrix}1\\-2\\0\\0\end{bmatrix}=\lambda\begin{bmatrix}1\\-2\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}5\\-10\\0\\0\end{bmatrix}=\begin{bmatrix}\lambda\\-2\lambda\\0\\0\end{bmatrix}$$

$$\underbrace{\lambda=5,\ \lambda=5}_{\text{unique}},\ \underbrace{0=0,\ 0=0}_{\text{Valid}}$$

so (b) ✓

(ii) Corresponding E Value is $\lambda=5$

HW8 ① Find the E Values & E Vectors of $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ $\lambda = 1, x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; $\lambda = 2, x = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Gate / ESE

OR

② If one pair of LI E.vectors of $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ is $\begin{bmatrix} 1 \\ a \end{bmatrix}$ & $\begin{bmatrix} 1 \\ b \end{bmatrix}$ then $a+b = ?$

OR

③ one pair of L.I, E vectors of $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$?

ⓐ × $\begin{bmatrix} -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \end{bmatrix}$ these are LD

ⓑ × $\begin{bmatrix} -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}$ " " LD

ⓒ × $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ these are LI but $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ is not an E Vector $\because AX \neq \lambda X$

✓ ⓓ $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ " " LI as well as E Vectors

(ii) one pair of L.I. E.Vectors of $A=\begin{bmatrix}1&2\\0&2\end{bmatrix}$ is $\begin{bmatrix}1\\a\end{bmatrix}$ & $\begin{bmatrix}1\\b\end{bmatrix}$ then $a+b=?$

(a) 1

(b) 2

(c) 0.5

(d) 0.6

M-I Separately find E. Vectors using conventional Approach & then Compare.

$A=\begin{bmatrix}1&2\\0&2\end{bmatrix}$

$\lambda=1,\ X=\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}1\\a\end{bmatrix}\Rightarrow a=0$

$\lambda=2,\ X=\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}1\\b\end{bmatrix}=\begin{bmatrix}1\\1/2\end{bmatrix}\Rightarrow b=\frac{1}{2}$

ie $a+b=\frac{1}{2}$

M-II

$AX_1=\lambda_1X_1$ & $AX_2=\lambda_2X_2$

$\begin{bmatrix}1&2\\0&2\end{bmatrix}\begin{bmatrix}1\\a\end{bmatrix}=1.\begin{bmatrix}1\\a\end{bmatrix}$ & $\begin{bmatrix}1&2\\0&2\end{bmatrix}\begin{bmatrix}1\\b\end{bmatrix}=2\begin{bmatrix}1\\b\end{bmatrix}$

$\begin{bmatrix}1+2a\\2a\end{bmatrix}=\begin{bmatrix}1\\a\end{bmatrix}$ & $\begin{bmatrix}(1+2b)\\2b\end{bmatrix}=\begin{bmatrix}2\\2b\end{bmatrix}$

$\Rightarrow \left.\begin{matrix}1+2a=1\\2a=a\end{matrix}\right\}\Rightarrow a=0$ & $1+2b=2\Rightarrow b=\frac{1}{2}$

ie $a+b=\frac{1}{2}$

Analysis:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$$

$\lambda = 1,\ X = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix} \cdots\cdots \infty$

$= \begin{bmatrix} 1 \\ a \end{bmatrix}$

$\lambda = 2,\ X \begin{bmatrix} 2 \\ 1 \end{bmatrix} \longrightarrow \begin{bmatrix} -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \end{bmatrix} \cdots\cdots \infty$

$= \begin{bmatrix} 1 \\ b \end{bmatrix}$

$\begin{bmatrix} 1 \\ a \end{bmatrix} \sim \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow a = 0$

$\begin{bmatrix} 1 \\ b \end{bmatrix} \sim \begin{bmatrix} 1 \\ 1/2 \end{bmatrix} \Rightarrow b = \frac{1}{2}$

For the matrix $A = \begin{bmatrix} 3 & -2 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ one of the eigen values is equal to -2 which of the following is an eigen vector?

(a) $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ (d) $\begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$

Talig (a) $\begin{bmatrix} 3 & -2 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = -2\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$

$\begin{bmatrix} \\ \\ 1 \end{bmatrix} \neq \begin{bmatrix} \\ \\ -2 \end{bmatrix}$ (a) ✗

(b) $\begin{bmatrix} \\ \\ 0 \ 0 \ 1 \end{bmatrix}\begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix} = -2\begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix} \Rightarrow \begin{bmatrix} \\ \\ -1 \end{bmatrix} \neq \begin{bmatrix} \\ \\ 2 \end{bmatrix}$ (b) ✗

(c) $\begin{bmatrix} \\ \\ 0 \ 0 \ 1 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = -2\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} \\ \\ 3 \end{bmatrix} \neq \begin{bmatrix} \\ \\ -6 \end{bmatrix}$ (c) ✗

The vector $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ is an eigen vector of

$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ then corresponding eigen value of A is

(a) 1 (b) 2

(c) 5 (d) –1

$$AX = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ - \\ - \end{bmatrix} = 5 \begin{bmatrix} 1 \\ - \\ - \end{bmatrix}$$

$$AX = 5X \Rightarrow \lambda = 5$$

Similar Matrices :- Two Matrices A & B are said to similar if

$\exists$ a relationship of the type, $\boxed{P^{-1}AP = B}$

where P is any invertible Matrix

& this Transformation from A to B is called Similarity Transformation

Note ① Similar Matrices have same Trace same Det, same C-Eq.

same E.Values and same Rank

But Equivalent Matrices have only same Rank & other properties may be different

② Similarity Relation b/n Matrices is an Equivalence relation

i.e Reflexive, Symmetric as well as Transitive.

Diagonalisation → Mat A is said to be diagonalisable if $\boxed{\text{it is similar to Diag Mat}}$

& this process of converting A into a Diag Mat is called Diagonalisation.

Consider $A_{3\times 3}$ & let it's E-Values are $\lambda_1, \lambda_2, \lambda_3$

& E-vectors are X_1, X_2, X_3

then Modal Mat is $P = [X_1\ X_2\ X_3]$ & let $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

& let $|P| \neq 0$.

Let us try to Calculate,

$$P^{-1}AP = [P^{-1}][A][P] = \cdots\cdots = D$$

i.e. $\boxed{P^{-1}AP = \text{Diag. Mat.}}$ i.e. A is similar to Diag Mat

this is very special Property of Mat & this property is called Diagonalisation

Application ① with the help of this concept we can find A as follows.

$$\boxed{P^{-1}AP = D} \Rightarrow P(P^{-1}AP)P^{-1} = P.D.P^{-1} \Rightarrow A = P.D.P^{-1} = ?$$

② Necessary Condition for Diagonalisation →

"No. of L.I E.Vectors of A = order of A"

eg Consider $A_{3\times3}$ & let it has 3.E vector X_1, X_2, X_3

& let these are L.D then Modal Mat $P = [X_1 X_2 X_3]$ will

become Singular $(|P| = 0)$ & P^{-1} = DNE

ie whole concept of Diagonalisation will become meaningless.

Verification: for $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$ $\begin{cases} \lambda = 6, & X = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \lambda = 2, & X = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{cases}$

let $P = [X_1 \; X_2] = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $P^{-1} = \dfrac{\text{adj } A}{|A|} = \dfrac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

Now $P^{-1}AP = \dfrac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \ldots\ldots = \begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix} = D$

ie we have verified that $P^{-1}AP = D$ ie A is similar to Diag. Mat.

Use: $\because P^{-1}AP = D$

$P(P^{-1}AP)P^{-1} = PDP^{-1} \Longrightarrow A = P.D.P^{-1} = ?$

Q: if $A_{2\times2}$ s.t $A\begin{bmatrix}-1\\1\end{bmatrix}=\begin{bmatrix}-2\\2\end{bmatrix}$ & $A\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}6\\6\end{bmatrix}$ then $A=?$

(a) $\begin{bmatrix}-1&1\\1&1\end{bmatrix}$ $\because Tr\neq 8$ ✗

(b) $\begin{bmatrix}4&2\\2&4\end{bmatrix}$ ✓

(c) $\begin{bmatrix}-2&6\\2&6\end{bmatrix}$ $\because Tr\neq 8$ ✗

(d) $\begin{bmatrix}2&0\\0&6\end{bmatrix}$

$A\begin{bmatrix}-1\\1\end{bmatrix}=2\begin{bmatrix}-1\\1\end{bmatrix}$ & $A\begin{bmatrix}1\\1\end{bmatrix}=6\begin{bmatrix}1\\1\end{bmatrix}$

ie $\lambda=2$, $X_1=\begin{bmatrix}-1\\1\end{bmatrix}$ ie $\lambda=6$, $X_2=\begin{bmatrix}1\\1\end{bmatrix}$

$Tr(A)=2+6=8$

$|A|=(2)(6)=12$

$D=\begin{bmatrix}2&0\\0&6\end{bmatrix}$, $P=[X_1,X_2]=\begin{bmatrix}-1&1\\1&1\end{bmatrix}$, $P^{-1}=\frac{-1}{2}\begin{bmatrix}1&-1\\-1&-1\end{bmatrix}$

Now using the concept of Diagonalisation,

$P^{-1}AP=D \Rightarrow A=P.DP^{-1}=\begin{bmatrix}-1&1\\1&1\end{bmatrix}\begin{bmatrix}2&0\\0&6\end{bmatrix}\frac{-1}{2}\begin{bmatrix}1&-1\\-1&-1\end{bmatrix}=\begin{bmatrix}4&2\\2&4\end{bmatrix}$

Q: If $A_{2\times 2}$ s.t. $A\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$ & $A\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}4\\2\end{bmatrix}$ then find $A=$?

(a) $\begin{bmatrix}1&2\\0&1\end{bmatrix}$ $\because Tr\neq 3$

(b) $\begin{bmatrix}1&0\\0&2\end{bmatrix}$

(c) $\begin{bmatrix}1&4\\0&2\end{bmatrix}$

(d) $\begin{bmatrix}1&2\\0&2\end{bmatrix}$

$A\begin{bmatrix}1\\0\end{bmatrix}=1\cdot\begin{bmatrix}1\\0\end{bmatrix}$ & $A\begin{bmatrix}2\\1\end{bmatrix}=2\begin{bmatrix}2\\1\end{bmatrix}$

$\lambda=1,\ X_1=\begin{bmatrix}1\\0\end{bmatrix}$, $\lambda=2,\ X_2=\begin{bmatrix}2\\1\end{bmatrix}$

$Tr(A)=1+2=3$

$|A|=1\times 2=2$

$D=\begin{bmatrix}1&0\\0&2\end{bmatrix}$, $P=[X_1, X_2]=\begin{bmatrix}1&2\\0&1\end{bmatrix}$, $P^{-1}=\frac{1}{(1)}\begin{bmatrix}1&-2\\0&1\end{bmatrix}$

Now, $P^{-1}AP=D \Rightarrow A=P\cdot DP^{-1}=\begin{bmatrix}1&2\\0&1\end{bmatrix}\cdot\begin{bmatrix}1&0\\0&2\end{bmatrix}\begin{bmatrix}1&-2\\0&1\end{bmatrix}$

$=\ldots\ldots=\begin{bmatrix}1&2\\0&2\end{bmatrix}$ //

For a given 2 × 2 matrix A, it is observed that

$A\begin{bmatrix}1\\-1\end{bmatrix} = -1\begin{bmatrix}1\\-1\end{bmatrix}$ and $A\begin{bmatrix}1\\-2\end{bmatrix} = -2\begin{bmatrix}1\\-2\end{bmatrix}$ then the matrix A is

$\lambda = -1$ $\lambda = -2$

(a) $A = \begin{bmatrix}2 & 1\\-1 & -1\end{bmatrix}\begin{bmatrix}-1 & 0\\0 & -2\end{bmatrix}\begin{bmatrix}1 & 1\\-1 & -2\end{bmatrix}$

(b) $A = \begin{bmatrix}1 & 1\\-1 & -2\end{bmatrix}\begin{bmatrix}1 & 0\\1 & 2\end{bmatrix}\begin{bmatrix}2 & 1\\-1 & -1\end{bmatrix}$

(c) $A = \begin{bmatrix}1 & 1\\-1 & -2\end{bmatrix}\begin{bmatrix}-1 & 0\\0 & -2\end{bmatrix}\begin{bmatrix}2 & 1\\-1 & -1\end{bmatrix}$

(d) $A = \begin{bmatrix}1 & -2\\1 & -3\end{bmatrix}$ $Tr \neq -3$

$Tr(A) = (-1) + (-2) = -3$

$|A| = (-1)(-2) = 2$

$\because P^{-1}AP = D \Rightarrow A = P \cdot D \cdot P^{-1}$

$A = \begin{bmatrix}1 & 1\\-1 & -2\end{bmatrix}\begin{bmatrix}-1 & 0\\0 & -2\end{bmatrix}\begin{bmatrix} ? \end{bmatrix}$

(c)

Some Confusions →

① No. of L.I Row vectors in $A = \rho(A)$

② No. of L.I Column vectors in $A = \rho(A)$

③ No. of $\underbrace{\text{L.I } \boxed{\text{Solution}}}_{\text{Nullity}}$ in Homog. System $(AX=0) = \boxed{\text{No. of Column} - \rho(A)}$

④ No. of $\underbrace{\text{LI } \boxed{\text{Eigen Vector}}}_{\text{GM of } \lambda}$ for $\lambda = \boxed{\text{order} - \rho(A-\lambda I)}$

∵ Here we have Homog system of the type $(A-\lambda I)X=0$

L-U Decomposition

If we want to factorize any square mat into the product of L.T.M & U.T.M then we have following Methods;

① Dolittle Method

write $A = (\text{unit LTM})(\text{U.TM})$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} \begin{bmatrix} d & e & f \\ 0 & g & h \\ 0 & 0 & i \end{bmatrix}$$

② CROUT Method

$A = (\text{L.T.M})(\text{unit U.T.M})$

$$A = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix} \begin{bmatrix} 1 & g & h \\ 0 & 1 & i \\ 0 & 0 & 1 \end{bmatrix}$$

③ Cholesky Method

$$A = LL^{\theta}$$

$$A = L(\overline{L^T})$$

Application

By using L-U Decomposition Concept we can solve S.O.Eq.

Consider $\boxed{AX=B}$ is to be solved i.e our aim is to find $X=\begin{bmatrix} \\ \\ \\ \vdots \\ \end{bmatrix}=?$

write $A=LU$ — (1) → D. Method, C. Method.

$$(LU)X=B$$

$$L(UX)=B$$

$$LY=B \quad \text{where } UX=Y \quad \text{— (3)}$$

— (2)

$$\begin{bmatrix} L.T.M \end{bmatrix}_{3\times3}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}=\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \Longrightarrow Y=\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}=$$ Possible to solve easily using Forward substitution.

Now By (3),

$$UX=Y$$

$$\begin{bmatrix} U.T.M \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Possible to solve x_3, x_2, x_1 easily using Backward substitution.

Q. If $A = \begin{bmatrix} 1 & 2 \\ 2 & 13 \end{bmatrix}$ is to be decomposed into the product of L.T.M & U.T.M then properly decomposed L & U Matrices are ?

MSQ

(a) $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 13 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 \\ 2 & 9 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 13 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 13 \end{bmatrix}$

(d) None

(M-I) Using Dollitle Method

$A = (\text{Unit LTM})(\text{U.T.M})$

$A = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \begin{bmatrix} b & c \\ 0 & d \end{bmatrix}$ — (1)

$\begin{bmatrix} 1 & 2 \\ 2 & 13 \end{bmatrix} = \begin{bmatrix} b & c \\ ab & ac+d \end{bmatrix}$

$b = 1$, $c = 2$, $ab = 2 \Rightarrow a = 2$

$ac + d = 13$

$(2)(2) + d = 13 \Rightarrow d = 9$

So by (1)

$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 9 \end{bmatrix}$

(a) ✓

M-II Using Crout M :-

$$A = (L.T.M)(\text{unit U.T.M})$$

$$= \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \quad \text{--- (1)}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 13 \end{bmatrix} = \begin{bmatrix} a & ad \\ b & bd+c \end{bmatrix}$$

$$\Rightarrow a=1,\ b=2,\ ad=2 \Rightarrow d=2$$

$$bd+c=13 \Rightarrow (2)(2)+c=13 \Rightarrow c=9$$

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

(b) ✓

M-III Using Cholesky Method :-

write $A = LL^{\theta}$ or $L(\bar{L})^T$

$$A = LL^T \quad (\because A \text{ is Real Mat so will } L \text{ also i.e. } L^{\theta} = L^T)$$

$$A = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \quad \text{--- (1)}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 13 \end{bmatrix} = \begin{bmatrix} a^2 & ab \\ ab & b^2+c^2 \end{bmatrix}$$

$$a^2=1,\ ab=2,\ b^2+c^2=13$$

$$a=1 \qquad b=2 \qquad 4+c^2=13$$

$$c^2=9 \Rightarrow c=3$$

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \text{ so (c)}$$ ✓

In the LU decomposition of the matrix $\begin{bmatrix} 2 & 2 \\ 4 & 9 \end{bmatrix}$, if the diagonal elements of U are both 1, then the lower diagonal entry l_{22} of L is _____.

Crout Method.

$$A = (L.T.M)(\text{unit } U.T.M)$$

$$\begin{bmatrix} 2 & 2 \\ 4 & 9 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} 1 & u_{12} \\ 0 & 1 \end{bmatrix}$$

$$\longrightarrow l_{22} = 5$$

Reflection Matrix → with angle $\theta = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$ Learn.

(i) About X axis $(\theta = 0)$

$RM = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(ii) About Y axis $(\theta = \pi/2)$

$RM = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

(iii) About $y = x$ $\left(\theta = \frac{\pi}{4}\right)$

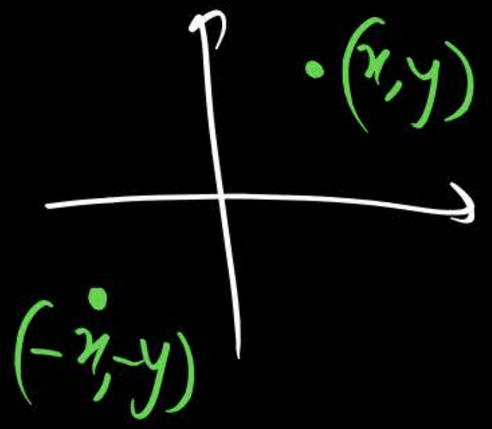
$RF = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(iv) About $y = -x$ $\left(\theta = \frac{3\pi}{4}\right)$

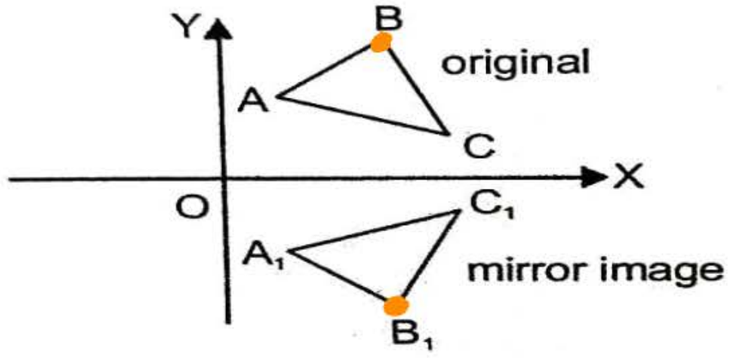
$RF = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

(v) Special Case: (About origin) → $R.M = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

(Above shortcut is not applicable)



The figure shows a shape ABC and its mirror image $A_1B_1C_1$ across the horizontal axis (X –axis). The coordinate transformation matrix that maps ABC to $A_1B_1C_1$ is



(a) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Let $A(x, y)$

$\Downarrow$

$A_1(x, -y)$

Taking (a)

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix} \neq \begin{bmatrix} x \\ -y \end{bmatrix}$$

Taking (b)

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ -x \end{bmatrix} \neq \begin{bmatrix} x \\ -y \end{bmatrix}$$

Taking (d)

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$$

Taking (c)

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix} \neq \begin{bmatrix} x \\ -y \end{bmatrix}$$

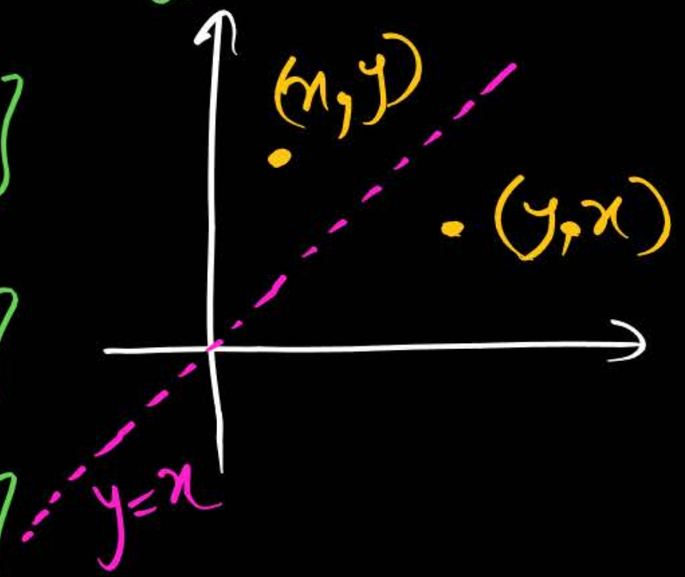
Q. The Transformation Matrix to find mirror image of any point $P\begin{bmatrix} x \\ y \end{bmatrix}$ in x-y plane about the line $y=x$ is ?

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$



(a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \neq \begin{bmatrix} y \\ x \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$

THANK - YOU

Tel: dr puneet sir pw

DATA SCIENCE & ARTIFICIAL INTELLIGENCE

Linear Algebra

Lecture No. 11

By- Dr. Puneet Sharma Sir

Recap of previous lecture

Topic

EIGEN VALUES & EIGEN VECTORS

Topics to be Covered

Topic

QUADRATIC FORM

LINEAR ALGEBRA (PART-2) only DA specific

1. QUADRATIC FORM
2. Singular Value Decomposition (S.V.D)
3. VECTOR SPACE (BASIS, SUBSPACE, SPAN, DIMENSION)
4. PROJECTION MATRIX
5. PARTITION Mat & ROTATION MAT

Quadratic Form →

eg (1) Consider $A_{3\times3}$ & consider a vector $X = \begin{bmatrix} - \\ - \\ - \end{bmatrix}_{3\times1}$

then $X^TAX = [- - -]_{1\times3} \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}_{3\times3} \begin{bmatrix} - \\ - \\ - \end{bmatrix}_{3\times1}$

$= [- - -]_{1\times3} \begin{bmatrix} - \\ - \\ - \end{bmatrix}_{3\times1} = [\ \ ]_{1\times1}$ = scalar value / constant.

eg (2) Consider $A = [\ \ \ \ ]_{4\times4}$, $X = [\ \ \ \ ]_{4\times1}$ then $X^T = [\ \ \ \ \ \ \ \ ]_{1\times4}$

Now $X^TAX = [- - - -]_{1\times4} [\ \ \ \ \ \ \ \ ]_{4\times4} \begin{bmatrix} - \\ - \\ - \\ - \end{bmatrix}_{4\times1} = [- - - -]_{1\times4} \begin{bmatrix} - \\ - \\ - \\ - \end{bmatrix}_{4\times1} = [\ \ ]_{1\times1}$ = scalar / constant

this scalar value is finally called quadratic form of vector X & it is denoted as $Q(X)$ ie $\boxed{Q(X) = X^TAX}$

Quadratic Form → "It is a process of assigning value to any Column Mat"

& it is denoted as $Q(X)$ and is defined as,

$$\boxed{Q(X) = X^T A X} = [\ \ ]_{1\times 1} = \text{Constant/scalar Value.}$$

where A is called Matrix associated with Quad. Form.

Note ① Here A is always a Symmetric Mat.

② if $A = I$ then $Q(X) = ? = X^T A X = X^T \cdot I \cdot X = X^T X = \|X\|^2 = (\text{Norm } X)^2$

is whenever $A = I$, Quadratic form of X is nothing but Sq. of Norm of X

Relation b/w Quadratic form & Norm of X →

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ then $\|X\| = \sqrt{X^T X} = \sqrt{(x_1^2 + x_2^2 + x_3^2)}$ i.e $\boxed{X^T X = \|X\|^2}$

Let us take a Symm Mat of the type $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3} = I$

& we will try to calculate $Q(X) = ?$

$$Q(X) = X^T A X = X^T . I . X = X^T X = \|X\|^2$$

Conclusion :- If $A = I$ then $\boxed{Q(X) = \|X\|^2}$.

eg if $A = I$ & $X = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ then $Q(X) = ? = 3^2 + (-2)^2 + 1^2 = 14$ Ans

eg if $A = \begin{bmatrix} 7 & 4 \\ 4 & 5 \end{bmatrix}_{2\times 2}$ then find Quad Form of $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2\times 1}$

$$Q(X) = X^T A X = [x_1 \ x_2] \begin{bmatrix} 7 & 4 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [(7x_1 + 4x_2) \ (4x_1 + 5x_2)] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= [(7x_1^2 + 4x_2x_1) + (4x_1x_2 + 5x_2^2)]_{1\times 1}$$

$$= 7x_1^2 + 8x_1x_2 + 5x_2^2$$

(iii) if $Q = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ then $Q(X) = ?$ $= 7(2)^2 + 8(2)(3) + 5(3)^2 = 28 + 48 + 45 = 121$

(ii) if $X = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ then $Q(X) = ?$ $= 7(3)^2 + 8(3)(-1) + 5(-1)^2 = 63 - 24 + 5 = 94$ Ans

Type I

Q. Find the Quadratic Forms of ① $A=\begin{bmatrix}1&0\\0&-2\end{bmatrix}$, ② $A=\begin{bmatrix}3&-1\\-1&4\end{bmatrix}$

Sol: Let $X=\begin{bmatrix}x_1\\x_2\end{bmatrix}$ be any given vector

① $Q(X)=X^TAX=[x_1\ x_2]\begin{bmatrix}1&0\\0&-2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$

$=[x_1\ -2x_2]\begin{bmatrix}x_1\\x_2\end{bmatrix}$

$=\left[(x_1^2+(-2x_2^2))\right]_{1\times1}$

$=1x_1^2-2x_2^2$

② $Q(X)=X^TAX=[x_1\ x_2]\begin{bmatrix}3&-1\\-1&4\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$

$=[(3x_1-x_2)\ (-x_1+4x_2)]\begin{bmatrix}x_1\\x_2\end{bmatrix}$

$=[(3x_1^2-x_2x_1)+(-x_1x_2+4x_2^2)]$

$=3x_1^2-2x_1x_2+4x_2^2$

Q. Find the Quadratic Forms of ① $A=\begin{bmatrix}2&0&-1\\0&4&3\\-1&3&1\end{bmatrix}$, ② $A=\begin{bmatrix}1&-3&0&1\\-3&0&2&4\\0&2&3&5\\1&4&5&-3\end{bmatrix}$

Sol: Let $X=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ is the given vector

① $Q(X)=X^TAX$

$$=[x_1\ x_2\ x_3]\begin{bmatrix}2&0&-1\\0&4&3\\-1&3&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

$$=[(2x_1-x_3)\ (4x_2+3x_3)\ (-x_1+3x_2+x_3)]\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

$$=[(2x_1^2-x_1x_3+4x_2^2+3x_3x_2-x_1x_3+3x_2x_3+x_3^2)]$$

$$=2x_1^2+4x_2^2+1x_3^2-2x_1x_3+6x_2x_3+0.x_1x_2$$

② $Q(X)=X^TAX$

$$=[x_1\ x_2\ x_3\ x_4]\begin{bmatrix}1&-3&0&1\\-3&0&2&4\\0&2&3&5\\1&4&5&-3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}$$

$$=\text{- - - - - - - - - -}$$

$$=1.x_1^2+0.x_2^2+3x_3^2-3x_4^2$$
$$-6x_1x_2+0x_1x_3+2x_1x_4$$
$$+4x_2x_3+8x_2x_4+10x_3x_4$$

PQ: if $A = \begin{bmatrix} 2 & -1 & 3 & 4 \\ -1 & 1 & 5 & 0 \\ 3 & 5 & 3 & 1.5 \\ 4 & 0 & 1.5 & -4 \end{bmatrix}_{4\times 4}$ & $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ then $Q(X) = ?$

$Q(X) = X^T A X = \text{- - - - - - - - - - - - - -}$

$= 2x_1^2 + x_2^2 + 3x_3^2 - 4x_4^2 - 2x_1x_2 + 6x_1x_3 + 8x_1x_4 + 10x_2x_3 + 3x_3x_4$

(ii) if $X = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \end{bmatrix}$ then $Q(X) = ?$ $= 2(1)^2 + (-2)^2 + 3(0)^2 - 4(3)^2 - 2(1)(-2) + 6(1)(0)$

$+ 10(-2)(0) + 3(0)(3)$

$= -2$

Type 2: Q. Find the Symmetric Matrix that is associated with the following Quadratic forms;

① $x_1^2 - 5x_1x_2 + 3x_2^2$

② $x_1^2 + 4x_3^2 - 2x_1x_2 + 5x_2x_4$

③ $x^2 + 2y^2 - 3z^2 + 4xy - 5yz + 3zx$

Sol ① $Q(X) = x_1^2 - 5x_1x_2 + 3x_2^2$

$= 1.x_1^2 - \frac{5}{2}x_1x_2 - \frac{5}{2}x_2x_1 + 3x_2^2$

So $A = \begin{bmatrix} 1 & -5/2 \\ -5/2 & 3 \end{bmatrix}$ //

③ $Q(X) = x^2 + 2y^2 - 3z^2 + 4xy - 5yz + 3zx$

Let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$Q(X) = 1x_1^2 + 2y_1^2 - 3z_1^2 + 4x_1x_2 - 5x_2x_3 + 3x_3x_1$

$= \begin{bmatrix} 1 & 2 & 1.5 \\ 2 & 2 & -2.5 \\ 1.5 & -2.5 & -3 \end{bmatrix}$ //

(2) $Q(X) = x_1^2 + 4x_3^2 - 2x_1x_2 + 5x_2x_4$

$$= 1\cdot x_1^2 + 0\cdot x_2^2 + 4x_3^2 + 0x_4^2 + (-2)x_1x_2 + 0\cdot x_1x_3 + 0\cdot x_1x_4 + 0\cdot x_2x_3 + 5x_2x_4 + 0\cdot x_3x_4$$

$$= \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 2.5 \\ 0 & 0 & 4 & 0 \\ 0 & 2.5 & 0 & 0 \end{bmatrix}_{4\times 4}$$

Q: if Quad. form of X is given as $Q(X) = a^2 - 2b^2 + 3c^2 + d^2 - ab + 4bc - 2cd + 2ad$ then find Symm Mat associated with above Q. Form.

$$Q(X) = 1 \cdot x_1^2 - 2x_2^2 + 3x_3^2 + 1 \cdot x_4^2 - x_1x_2 + 4x_2x_3 - 2x_3x_4 + 2x_1x_4$$

$$= 1 \cdot x_1^2 - 2x_2^2 + 3x_3^2 + 1 \cdot x_4^2 - x_1x_2 + 0\, x_1x_3 + 2x_1x_4 + 4x_2x_3 + 0\, x_2x_4 - 2x_3x_4$$

$$A = \begin{bmatrix} 1 & -1/2 & 0 & 1 \\ -1/2 & -2 & 2 & 0 \\ 0 & 2 & 3 & -1 \\ 1 & 0 & -1 & 1 \end{bmatrix}_{4\times 4}$$

Ans

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be a column vector and consider the quadratic form $Q(x) = 2x_1^2 + 3x_2^2 + 1x_3^2 + 2x_1x_2 - 4x_2x_3$.

Which of the following matrices represents this quadratic forms?

(a) $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -2 \\ 0 & -2 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -2 \\ 0 & -1 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 2 & 0 \\ 2 & 3 & -2 \\ 0 & -2 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 6 & -2 \\ 0 & -2 & 2 \end{bmatrix}$

TYPE 3 **Q.** If Quadratic form of any vector x is given as;

① $x_1^2 - 4x_2^2 + 3x_1x_2$ then evaluate $Q\begin{bmatrix}1\\-3\end{bmatrix} = ? = (1)^2 - 4(-3)^2 + 3(1)(-3)$

$= 1 - 36 - 9 = -44$

② $x^2 + 4y^2 - z^2 + 2zx + 4xy$ then evaluate $Q\begin{bmatrix}1\\2\\-3\end{bmatrix} = ? = \begin{bmatrix}x\\y\\z\end{bmatrix}$

$Q(x) = 1^2 + 4(2)^2 - (-3)^2 + 2(-3)(1) + 4(1)(2)$

$= 1 + 16 - 9 - 6 + 8 = 10$

Maximum and Minimum Values of Quadratic form $Q(X)$ {when $\|X\|=1$}

Let $A = \begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}_{3\times 3}$ & let it's EValues are $\lambda_m, \lambda, \lambda_M$. where $\begin{cases} \text{Min EV} = \lambda_m \\ \text{Max EV} = \lambda_M \end{cases}$

then for, X (where $\|X\|=1$) $\begin{cases} \text{Min } Q(X) = \text{min EValue} \\ \text{Max } Q(X) = \text{Max E.Value} \end{cases}$ i.e $\boxed{\lambda_{min} \le Q(X) \le \lambda_{Max}}$

& these min & Max Value occurs at unit E.Vectors for λ_m & λ_M Resp.

i.e $Q(X)$ will be min for X = unit EigenVector for λ_m

& $Q(X)$ will be Max for X = unit EigenVector for λ_m

Take Care :- this Property is Valid only when we are taking unit vectors.

for eg let $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$ & $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be any vector s.t $\|X\| = 1$

then min & Max value of Quad. form of $X = ?$

Sol: $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$ $\lambda = 2$, $X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ (already calculated in previous lectures)

$\lambda = 6$, $X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\because \|X\| = 1$ (given) so we can use our Theorem i.e $\min Q(X) = 2$, $\text{Max } Q(X) = 6$

i.e $\boxed{2 \le Q(X) \le 6}$

and this Min value will occur for $X = N_1 = \frac{X_1}{\|X_1\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

& " Max " " " for $X = N_2 = \frac{X_2}{\|X_2\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Verification:

$$Q(N_1) = N_1^T A N_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix}4 & 2\\ 2 & 4\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\ 1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}-2 & 2\end{bmatrix}\begin{bmatrix}-1\\ 1\end{bmatrix} = \frac{1}{2}\left[(2+2)\right] = [2] = 2$$

$$Q(N_2) = N_2^T A N_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}4 & 2\\ 2 & 4\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}6 & 6\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix} = \frac{1}{2}\left[(6+6)\right] = [6] = 6$$

Let us take another vector $X = \frac{1}{5}\begin{bmatrix}3\\ 4\end{bmatrix}$ (Here $\|X\| = 1$)

then $Q(X) = X^T A X = \frac{1}{5}\begin{bmatrix}3 & 4\end{bmatrix}\begin{bmatrix}4 & 2\\ 2 & 4\end{bmatrix}\frac{1}{5}\begin{bmatrix}3\\ 4\end{bmatrix} = \frac{1}{25}\begin{bmatrix}20 & 22\end{bmatrix}\begin{bmatrix}3\\ 4\end{bmatrix} = \frac{1}{25}\left[(60+88)\right]$

$$= \frac{148}{25} = 5.92$$

Hence we have verified that $2 \leq Q(X) \leq 6$

TYPE-4

Q. for the Matrix $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$, the Max and Minimum Value of $X^T A X$ will be? subject to the constraint $X^T X = 1$

Already solved. $\text{Max}(X^T A X) = 6$

& $\min(X^T A X) = 2$

Q: For the Matrix $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$, Max and minimum values of $X^T A X$ lies in the interval ? (Provided $\|X\| = 1$)

MSQ

(a) $[1\ 6]$ (b) $[-5\ 5]$ (c) $[1\ 7]$ (d) $[2\ 3]$

$\min Q(X) = 2$ & $\text{Max } Q(X) = 6$

Q: if for any vector X, Quadratic form associated with Matrix A is given as $Q(X) = \boxed{2x^2 - 2xy + 2y^2}$ then sum of the squares of minimum & Maximum Value of $Q(X)$ will be _____ subject to the constraints $X^T X = 1$

Sol: Associated Mat is $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

C-Equ^n is $\lambda^2 - (Tr(A))\lambda + (|A|) = 0$

$$\lambda^2 - (4)\lambda + (3) = 0$$

$$(\lambda - 3)(\lambda - 1) = 0$$

$$\lambda = 3 \ \& \ 1$$

$\min Q(X) = 1$

$\text{Max } Q(X) = 3$

Req Ans $= 1^2 + 3^2 = 10$ Ans

Q: If $Q(X) = 4x_1^2 + 4x_2^2 + x_3^2 - 2x_1x_2$ in the Quadratic form of any unit vector X then sum of minimum & Maximum Value of $Q(X)$ is ?

Sol:

$$A = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$Tr(A) = 9$

$|A| = 15$

C.Equ is $|A - \lambda I| = 0$

$(\lambda-1)(\lambda-3)(\lambda-5) = 0$

$\lambda = 1, 3, 5$

$\min Q(X) = 1$, $\max Q(X) = 5$

So Req sum $= 1 + 5 = 6$

Q. Min & Max values of $Q(x)$ for $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ lies in b/n ? where $\|x\| = 1$

$= U.T.M.$

MS 8

(a) $(-1\ 1)$

(b) $(1\ 5)$

(c) $[1\ 2]$

(d) $(3\ 5)$

(e) None

WRONG APP

~~E. Values of A are $\lambda = 1$ & $\lambda = 2$~~

~~$\min Q(x) = 1$~~

~~$\text{Max } Q(x) = 2$~~

ie $1 \le Q(x) \le 2$

$\because$ A is Not Symm Mat so Quest is senseless

Data Inadequate

Relation b/n Quadratic form and Eigen Values →

Positive Definite Mat → If (all the E. Values are +ve) then (A is called +ve definite.)

Negative Definite Mat → If all the E Values are −ve then A is called −ve definite.

INDEFINITE Mat → If E Values are +ve as well as −ve then A is Indefinite.

Semi +ve Definite → If E Values are 0 or +ve then A is called semi +ve Def.

Now, if A is Mat associated with Quad. Form then Q(X) is called.

(i) +ve definite (i.e. $Q(X)>0$) if All the E Values are +ve

(ii) −ve definite ($Q(X)<0$) if " " " " " −ve

(iii) Indefinite ($Q(X)>0$ or <0) if all the " " are +ve as well as −ve.

Type 5 Q. The Nature of $Q(X) = 3x_1^2 + 2x_2^2 + x_3^2 + 4x_1x_2 + 4x_2x_3$ will be?

(a) +ve Definite

(b) −ve Definite

(c) Indefinite ✓

(d) Can't say anything

Associated Mat is $A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \end{bmatrix}$

C. Equ^n is $|A - \lambda I| = 0$

$(\lambda - 5)(\lambda - 2)(\lambda + 1) = 0 \Rightarrow \lambda = -1, 2, 5$

i.e. A is Indefinite so $Q(X)$ will also Indefinite.

Analysis: $|A| = -10$ (−ve) i.e. → Case I: either all E Values −ve → Not possible

→ Case II: or one −ve & two +ve E Values. (✓)

Now $Tr(A) = 6$ (+ve) $\Rightarrow$ all 3 E Values can not be −ve.

$\Downarrow$

A is Indefinite.

Verification :- $Q(X) = 3x_1^2 + 2x_2^2 + x_3^2 + 4x_1x_2 + 4x_2x_3$

Consider $X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ then

$$Q(X_1) = 3(-1)^2 + 2(0)^2 + (1)^2 + 4(-1)(0) + 4(0)(1) = +4 \text{ ie } Q(X) > 0$$

& for $X_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ then

$$Q(X_2) = 3(0)^2 + 2(1)^2 + (-1)^2 + 4(0)(1) + 4(1)(-1) = -1 \text{ ie } Q(X) < 0$$

Hence we have verified that $Q(X)$ is Indefinite.

Let $A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$. Which of the following statements is true regarding the quadratic form $Q(x) = X^T A X$?

(a) $Q(x)$ is positive definite.

(b) $Q(x)$ is negative definite

(c) $Q(x)$ is indefinite.

(d) $Q(x)$ is positive, semi-definite.

$\because |A| = 8$ (+ve)

$\because |A| \neq 0 \Rightarrow (\lambda \neq 0)$

Case I → Either all the E Values are +ve

Case II → or A has one +ve & 2-ve E Values

Now $Tr(A) = 8$ (+ve)

either all +ve

one-ve & two +ve

two-ve & one +ve

$\Rightarrow$ All the E Values can't be -ve

Not very easy to justify the Nature of E Values using this technique.

C. Equ'n $|A-\lambda I| = 0$

$$\begin{vmatrix} (3-\lambda) & 1 & 2 \\ 1 & (2-\lambda) & 1 \\ 2 & 1 & (3-\lambda) \end{vmatrix} = 0$$

- - - - - - - -

i.e. All the E Values are +ve

So A = +ve definite

$\Rightarrow Q(X) > 0$

$$(\lambda-1)(\lambda^2-7\lambda+8) = 0$$

$$\lambda = 1, \quad \lambda = \frac{7 \pm \sqrt{49-32}}{2}$$

$$\lambda = 1, \quad \lambda = \frac{7 \pm \sqrt{17}}{2} = \frac{7 \pm 4.\ldots}{2} = +ve, +ve$$

PRINCIPAL AXES THEOREM →

With the help of this theorem we can convert $Q(X)$ into $Q(Y)$ where $Q(X) = X^T A X$ contains cross product terms & $Q(Y) = Y^T D Y$ does not contain cross product term.

Consider $A = \begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}_{3\times 3}$ = Symm & let $\lambda_1, \lambda_2, \lambda_3$ are its E values & X_1, X_2, X_3 are their E vectors.

Now, consider N_1, N_2, N_3 as unit E vectors where $N_1 = \frac{X_1}{\|X_1\|}, N_2 = \frac{X_2}{\|X_2\|}, N_3 = \frac{X_3}{\|X_3\|}$

is $\underbrace{\text{orthonormal Modal Mat}}_{PP^T = I \Rightarrow P^T = P^{-1}}$ $P = [N_1\ N_2\ N_3]$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Let $X = P.Y$ then

$$\boxed{Q(X)} = X^T A X = (PY)^T A (PY)$$

$$= (Y^T P^T) A (PY) \quad \text{(R. Law)}$$

$$= Y^T (P^T A P) Y \quad \text{(Asso. Law)}$$

$$= Y^T (P^{-1} A P) Y \quad (\because P \text{ is O-Mat})$$

$$= Y^T . D . Y \quad \text{(using Diagonalization)}$$

$$= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2$$

$$= \boxed{Q(Y)}$$

Here Associated Mat with $X = A$

& " " " $Y = D$.

ie $\boxed{\begin{aligned} Q(X) &= X^T A X \\ &= Y^T D Y \\ &= Q(Y) \end{aligned}}$

FINAL Conclusion of (P.A.Th) $\Rightarrow$

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ & $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ & $A = \begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}_{3\times 3}$ $\to$ λ_1, X_1; λ_2, X_2; λ_3, X_3

$= \text{Symm.}$

Let P = orthonormal Modal Mat $= \begin{bmatrix} \frac{X_1}{\|X_1\|} & \frac{X_2}{\|X_2\|} & \frac{X_3}{\|X_3\|} \end{bmatrix}_{3\times 3} = [N_1\ N_2\ N_3]$

$\Rightarrow P^{-1} = P^T$. If we take $X = PY$ then

$$Q(X) = X^T A X = \ldots\ldots = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 = Y^T D Y = Q(Y)$$

which is free from Cross Product term. where $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

Type 6 Qs if $Q(X) = 2x_1^2 - 2x_1x_2 + 2x_2^2$ then write it's another Quadratic form which is free from cross product term?

× (a) $2y_1^2 + 2y_2^2$ ($\because |A| \neq 3$)

$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ $\lambda = 1$, $\lambda = 3$, $D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

(b) $1y_1^2 + 3y_2^2$

let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ & let $Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

(c) $1y_1^2 + 2y_2^2$

× ($\because Tr \neq 4$, $|A| \neq 3$)

$Q(X) = X^T A X = \text{using P.A.Th} \ldots = Y^T D Y$

$= (2x_1^2 - 2x_1x_2 + 2x_2^2) = \ldots = \boxed{\underset{\lambda_1}{1} \cdot y_1^2 + \underset{\lambda_2}{3} \cdot y_2^2}$

(d) $-y_1^2 + 5y_2^2$

× ($\because |A| \neq 3$)

Analysis: $\because Tr(A) = 4$, $|A| = 3$

So (a), (c), (d) can be elliminated directly.

Analysis :- Let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, & $\boxed{Q(X) = 2x_1^2 - 2x_1x_2 + 2x_2^2}$ then $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

(PODCAST)

Now we will calculate E. Values & E. vectors of A which are as follows,

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad \lambda_1 = 1, \; X_1 = \begin{bmatrix} +1 \\ 1 \end{bmatrix}, \; \|X_1\| = \sqrt{2}$$

$$\lambda_2 = 3, \; X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \; \|X_2\| = \sqrt{2}$$

Now

$$P = [N_1 \; N_2] = \left[\frac{X_1}{\|X_1\|} \; \frac{X_2}{\|X_2\|}\right] = \begin{bmatrix} +1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} +1 & -1 \\ 1 & 1 \end{bmatrix} = \text{orthonormal Modal Mat}$$

In P.A.T, $X = PY \Rightarrow Y = P^{-1}X = P^T X = \frac{1}{\sqrt{2}}\begin{bmatrix} +1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} (x_1 + x_2) \\ (-x_1 + x_2) \end{bmatrix}$

if $X = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$

$Q(X) = 2(2)^2 - 2(2)(-2) + 2(-2)^2$

$= 24$

$Y = \frac{1}{\sqrt{2}}\begin{bmatrix} 2-2 \\ -2-2 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -4 \end{bmatrix}$, $Q(Y) = \boxed{1 \cdot y_1^2 + 3 \cdot y_2^2}$

$= 1(0)^2 + 3\left(\frac{-4}{\sqrt{2}}\right)^2 = 24$

Q. The orthogonal Quadratic form of $Q(x) = 4x_1^2 + 4x_2^2 + x_3^2 - 2x_1x_2$ will be?

(2021)

(a) $-3y_1^2 + 2y_2^2 + 4y_3^2$ ($\because Tr \neq 9$) ✗

(b) $6y_1^2 + 5y_2^2 - 2y_3^2$ $\because |A| \neq 15$ ✗

(c) $3y_1^2 + 5y_2^2 + 1 \cdot y_3^2$ ✓

(d) $y_1^2 - y_2^2 + 6y_4^2$ $\because Tr \neq 9$ ✗

$Q(X) = X^T A X$ where $A = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$

$\because Tr(A) = 9, \ |A| = 15$

$Q(X) = X^T A X = Y^T D Y = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 = Q(Y)$

M-II

$A = \begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix} \rightarrow \lambda = 1, \ \lambda = 3, \ \lambda = 5$

So ans = (c)

The equation for the ellipsoid of inertia of a solid body is

$$P(x) \equiv 4x_1^2 + 4x_2^2 + x_3^2 - 2x_1x_2.$$

What is the standard from in terms of a new orthogonal set of axes $O\{y_1, y_2, y_3\}$?

(a) $y_1^2 - 3y_2^2 + 3y_3^3$ (b) $y_1^2 + 5y_2^2 + 3y_3^2$

(c) $y_1^2 - 5y_2^2 + 3y_3^2$ (d) $y_1^2 - 5y_2^2 - 3y_3^2$

$$\Rightarrow A = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \lambda = 1, 3, 5$$

So By P.A.Th,

$$Q(y) = Q(y) = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2$$

$$= 1 \cdot y_1^2 + 3y_2^2 + 5y_3^2$$

THANK - YOU

Tel: dr puneet sir pw

DATA SCIENCE & ARTIFICIAL INTELLIGENCE

Linear Algebra

Lecture No. 12

By- Dr. Puneet Sharma Sir

Recap of previous lecture

Topic QUADRATIC FORMS

Topics to be Covered

Topic SINGULAR VALUE DECOMPOSITION

(S.V.D)

SINGULAR VALUE DECOMPOSITION

Consider Sq. Mat $A_{3\times3}$ $\begin{cases} \lambda_1, & x_1 \\ \lambda_2, & x_2 \\ \lambda_3, & x_3 \end{cases}$ then $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$, $P = [x_1 x_2 x_3]$

Now using concept of Diagonalisation we can find A as follows,

$$P \cdot D \cdot P^{-1} = [x_1 x_2 x_3] \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} [x_1 x_2 x_3]^{-1} = \cdots = A.$$

unfortunately this concept is applicable only for Sq. Matrices, so if we want to find Rectangular $A_{m\times n}$ then what should be procedure ? Ans is S.V.D

ie $U \cdot \Sigma \cdot V^T = [\quad][\quad][\quad] = \cdots = A_{m\times n}$

First we will try to understand some Basic concepts.

Singular Values of A →

Consider $A_{m\times n}$ then $\boxed{\text{Sq. Roots of E-Values of } A^TA \text{ are called Singular Values of } A}$

$\because A_{m\times n}$ then $A^T_{n\times m}$

$(A^TA)_{n\times n}$ = Sq. Mat (Symm also)

$(AA^T)_{m\times m}$ = Sq. Mat (Symm also)

Let Eigen Values of $(A^TA)_{n\times n}$ are $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$

then Singular Values of $A_{m\times n}$ are $\sqrt{\lambda_1}, \sqrt{\lambda_2}, \sqrt{\lambda_3}, \ldots, \sqrt{\lambda_n}$

& these are denoted by σ_i i.e. Singular Values of $A_{m\times n}$ are $\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_n$

where $\boxed{\sigma_i = \sqrt{\lambda_i}}\ \forall i$ ← Generally these are written in Decreasing order.

$\Rightarrow \boxed{\sigma_i^2 = \lambda_i}\ \forall i$

Note (1)

① Consider $A_{m\times n}$, $A^T_{n\times m}$, Let $P=(A^TA)_{n\times n}$, $Q=(AA^T)_{m\times m}$

Now $P^T=(A^TA)^T=A^T(A^T)^T=A^TA=P$ i.e P is symm

& $Q^T=(AA^T)^T=(A^T)^TA^T=AA^T=Q$ i.e Q is symm

② $\boxed{Tr(A^TA)=Tr(AA^T)}$ In fact $AB\neq BA$ But $Tr(AB)=Tr(BA)$

③ Consider $A_{3\times 5}$, $A^T_{5\times 3}$, $(A^TA)_{5\times 5}$, $(AA^T)_{3\times 3}$

Non Zero E. Values of A^TA and AA^T are <u>Same</u>. In Fact, $\boxed{\rho(A^TA)=\rho(AA^T)}$

i.e $(A^TA)_{5\times 5}\longrightarrow \lambda_1,\lambda_2,\lambda_3,0,0$ $\because (AA^T)_{3\times 3}\longrightarrow \lambda_1,\lambda_2,\lambda_3$

④ <u>Relation b/n Left Sing Vector & Right Sing Vector</u> $\boxed{AV_i=\sigma_iU_i}$ i.e $AV_1=\sigma_1U_1$

$AV_2=\sigma_2U_2$ .....

(5) Let E. Value of A^TA is λ_i then Singular Value of A is $\boxed{\sigma_i = \sqrt{\lambda_i}}$ $\forall i$

for eg $A_{2\times3}$ then $(A^T)_{3\times2}$ & $(A^TA)_{3\times3}$ = Symm. Mat, let E Values are $\lambda_1, \lambda_2, \lambda_3$

then Singular Values of A are $\sqrt{\lambda_1}, \sqrt{\lambda_2}, \sqrt{\lambda_3}$

(6) If A is a Square Mat of $n\times n$ then $\boxed{|A| = |\lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_n| = \sigma_1 \cdot \sigma_2 \cdot \ldots \cdot \sigma_n}$

Here $\lambda_1, \lambda_2 \ldots \lambda_n$ are the EV of A

(7) If $A_{n\times n}$ s.t A is Symmetric then $\boxed{\text{Singular Values of } A = |\text{Eigen Values of } A|}$

ie $\sigma_i = |\lambda_i| \ \forall i$

(8) if $A_{m\times n}$ then $(A^TA)_{n\times n}$

Sum of the Squares of Sing Values of A = Sum of Eigen Values of A^TA = $Tr(A^TA)$

ie $\boxed{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \ldots + \sigma_n^2 = \lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_n = Tr(A^TA)}$

Q: Find the singular values of $A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}_{2\times 3}$

Sol: $A^TA = \begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix}_{3\times 2} \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}_{2\times 3} = \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix}_{3\times 3}$ = Symm Mat. (P)

C. Equ^n of P is $|P-\lambda I| = 0$

$$\begin{vmatrix} (80-\lambda) & 100 & 40 \\ 100 & (170-\lambda) & 140 \\ 40 & 140 & (200-\lambda) \end{vmatrix} = 0$$

- - - - - - - -

- - - - - - - -

$\lambda = 360, 90, 0$

i.e. EValues of $\boxed{A^TA}$ are $\lambda_1 = 360$, $\lambda_2 = 90$, $\lambda_3 = 0$

So singular values of A are $\sqrt{360}, \sqrt{90}, \sqrt{0}$

i.e. $\sigma_1 = \sqrt{360}$, $\sigma_2 = \sqrt{90}$, $\sigma_3 = 0$

these are the sing values of A & not of A^TA.

Conclusion:- $\sigma_1^2 + \sigma_2^2 + \sigma_3^2 = 450 = \lambda_1 + \lambda_2 + \lambda_3 = Tr(A^TA)$

Q: Find the singular values of $A = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix} = L.T.M$

M-I $\because$ E Values of A are $\lambda = 3$ & 5 (Blunder)

So singular values of A are $\sqrt{3}$ & $\sqrt{5}$

M-II $A^TA = \begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 25 & 20 \\ 20 & 25 \end{bmatrix} \begin{matrix} = 45 \\ = 45 \end{matrix}$ = Symm Mat.

By Property of E Value, $\lambda_1 = 45$, $\because Tr(A^TA) = 50$

$45 + \lambda_2 = 50 \Rightarrow \lambda_2 = 5$

i.e. E. Values of A^TA are 45 & 5

Hence Singular Values of A are $\sqrt{45}$ & $\sqrt{5}$

i.e. $\sigma_1 = \sqrt{45}$, $\sigma_2 = \sqrt{5}$

Conclusion: $|A| = 15$, $\lambda_1 \cdot \lambda_2 = 3 \times 5 = 15$

$\sigma_1 \cdot \sigma_2 = \sqrt{45} \cdot \sqrt{5} = \sqrt{225} = 15$

i.e. if A is Sq. Mat then

$|A| = \lambda_1 \cdot \lambda_2 \cdots \lambda_n = \sigma_1 \cdot \sigma_2 \cdot \sigma_3 \cdots \sigma_n$

Right and Left Singular Vectors of A →

$A_{3\times4}$ → $(A^TA)_{4\times4}$ → unit E vectors of (A^TA) are called R. Sing vector vectors of A

$V = [v_1\ v_2\ v_3\ v_4]_{4\times4}$

① $A_{3\times4}$ → $(AA^T)_{3\times3}$ → unit " " (AA^T) " " Left Sing " " " of A

$U = [u_1\ u_2\ u_3]_{3\times3}$

② ∵ A^TA & AA^T both are symm. so v_1, v_2, v_3, v_4 are orthonormal vectors & u_1, u_2, u_3 are also orthonormal vectors

ie $v_i \cdot v_j = 0$, $\|v_i\| = 1$, $u_i \cdot u_j = 0$, $\|u_i\| = 1$

③ $(A^TA)_{4\times4}$ → orthonormal Modal Mat is $V = [v_1\ v_2\ v_3\ v_4]$

$(AA^T)_{3\times3}$ → " " " is $U = [u_1\ u_2\ u_3]$

(4) $(A^TA)_{4\times 4}$ $\begin{cases} \lambda_1, x_1 \\ \lambda_2, x_2 \\ \lambda_3, x_3 \\ \lambda_4, x_4 \end{cases}$ then $v_1 = x_1/\|x_1\|$, $v_2 = x_2/\|x_2\|$, $v_3 = x_3/\|x_3\|$, $v_4 = x_4/\|x_4\|$ i.e. $v_i = \dfrac{x_i}{\|x_i\|}$ $\forall i$

& $V = [v_1\ v_2\ v_3\ v_4]$

& $(AA^T)_{3\times 3}$ $\begin{cases} \lambda'_1, y_1 \\ \lambda'_2, y_2 \\ \lambda'_3, y_3 \end{cases}$ then $u_i = \dfrac{y_i}{\|y_i\|}$ & $U = [u_1\ u_2\ u_3]$

(5) <u>Special Mat Associated with (S.V.D)</u> →

$\because$ E Values of $(A^TA)_{4\times 4}$ are $\lambda_1, \lambda_2, \lambda_3, \lambda_4 = 0$ & S. Values are $\sigma_1, \sigma_2, \sigma_3, \sigma_4$

then $D = \begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 \\ 0 & 0 & 0 & \sigma_4 = 0 \end{bmatrix}$ then special Mat is $\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$

⑥ <u>S.V.D</u> → $A_{3\times4}$ which is Required, then

$$U\Sigma V^T = [U]_{3\times3}\,\Sigma_{3\times4}\,[V^T]_{4\times4}$$

$$= [U]_{3\times3}\begin{bmatrix}\sigma_1 & 0 & 0 & 0\\ 0 & \sigma_2 & 0 & 0\\ 0 & 0 & \sigma_3 & 0\end{bmatrix}_{3\times4}[V^T]_{4\times4}$$

$$= \cdots\cdots$$

$$= [A]_{3\times4}$$

ie S.V.D of A is as follows

$$\boxed{A = U\Sigma V^T}$$

Conclusions

(*) Let E. Value of A^TA is λ_i then Singular Value of A is $\boxed{\sigma_i = \sqrt{\lambda_i}}$ $\forall i$

for eg $A_{2\times3}$ then $(A^T)_{3\times2}$ & $(A^TA)_{3\times3}$ = symm. Mat, let EValues are $\lambda_1, \lambda_2, \lambda_3$

then Singular Values of A are $\sqrt{\lambda_1}, \sqrt{\lambda_2}, \sqrt{\lambda_3}$

(*) Σ is special Mat, includes D, which is formed by Non Zero Sing Values of A

let Sing Values of $A_{2\times3}$ are $\sigma_1, \sigma_2, 0$ then $\Sigma = \begin{bmatrix} D & O \\ O & O \end{bmatrix} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{bmatrix}$

$A_{2\times3}, (A^T)_{3\times2}$ → $(A^TA)_{3\times3} \to (V)_{3\times3}$, $(AA^T)_{2\times2} \to (U)_{2\times2}$ so $\boxed{A_{2\times3} = U_{2\times2}\,\Sigma_{2\times3}\,V^T_{3\times3}}$

Q: Find the S.V.D of $A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$?

Sol:

$$A^TA = \begin{bmatrix} 80 & 140 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix}, \quad AA^T = \begin{bmatrix} 333 & 81 \\ 81 & 117 \end{bmatrix}$$

Sol A^TA:

$\lambda_1 = 360, \; X_1 = [1 \;\; 2 \;\; 2]'$

$\lambda_2 = 90, \; X_2 = [-2 \;\; -1 \;\; 2]'$

$\lambda_3 = 0, \; X_3 = [2 \;\; -2 \;\; 1]'$

$$V_1 = \frac{X_1}{\|X_1\|} = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \quad V_2 = \frac{X_2}{\|X_2\|} = \frac{1}{3}\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}, \quad V_3 = \frac{X_3}{\|X_3\|} = \frac{1}{3}\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

So $V = [V_1 \; V_2 \; V_3] = \frac{1}{3}\begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{bmatrix}_{3\times 3}$

$\because \boxed{AV_i = \sigma_i U_i} \Rightarrow$

$AV_1 = \sigma_1 U_1 \Rightarrow U_1 = \frac{1}{\sigma_1} AV_1$

$$U_1 = \frac{1}{\sqrt{360}} \cdot \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}_{2\times 3} \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}_{3\times 1}$$

$$U_1 = \frac{1}{3\sqrt{360}}\begin{bmatrix} 54 \\ 18 \end{bmatrix} = \frac{1}{\sqrt{10}}\begin{bmatrix} 3 \\ 1 \end{bmatrix} \checkmark$$

$$U_2 = \frac{1}{\sigma_2} AV_2 = \frac{1}{\sqrt{90}}\begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} \cdot \frac{1}{3}\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{3\sqrt{90}}\begin{bmatrix} 9 \\ -27 \end{bmatrix} = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ -3 \end{bmatrix} \checkmark$$

So $U = [u_1, u_2] = \frac{1}{\sqrt{10}}\begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix}_{2\times 2}$, Also we have $V = \frac{1}{3}\begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{bmatrix}_{3\times 3}$

$D =$ formed by Non Zero sing values of $A = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} = \begin{bmatrix} \sqrt{360} & 0 \\ 0 & \sqrt{90} \end{bmatrix}_{2\times 2}$

Hence S.V.D of A will be,

$$\boxed{A_{2\times 3} = U \Sigma V^T} = U_{2\times 2} \Sigma_{2\times 3} V^T_{3\times 3}$$

$$= U_{2\times 2} \begin{bmatrix} D & 0 \end{bmatrix} V^T_{3\times 3}$$

$$= \frac{1}{\sqrt{10}}\begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} \sqrt{360} & 0 & 0 \\ 0 & \sqrt{90} & 0 \end{bmatrix} \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & 2 \\ 2 & -2 & 1 \end{bmatrix}$$

Q: Obtain the S.V.D of $A = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}_{2\times 2}$

Ans: $A = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} \sqrt{45} & 0 \\ 0 & \sqrt{5} \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

Sol: $A^TA = \begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 25 & 20 \\ 20 & 25 \end{bmatrix}$

$\lambda = 45$ & 5 (using prop of E.V.)

So $\sigma = \sqrt{45}$ & $\sqrt{5}$

for $\lambda_1 = 45$, $X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (HW)

& for $\lambda_2 = 5$, $X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ (HW)

$V_1 = \frac{X_1}{\|X_1\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ & $V_2 = \frac{X_2}{\|X_2\|} = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

So $V = [V_1\ V_2] = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ //

Now $AV_i = \sigma_i U_i \Rightarrow U_i = \frac{1}{\sigma_i} AV_i$

$U_1 = \frac{1}{\sqrt{45}}\begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{90}}\begin{bmatrix} 3 \\ 9 \end{bmatrix} = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$U_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{10}}\begin{bmatrix} -3 \\ 1 \end{bmatrix}$

So $U = [U_1\ U_2] = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}$ //

$$D = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} = \begin{bmatrix} \sqrt{45} & 0 \\ 0 & \sqrt{5} \end{bmatrix}, \quad \& \quad \Sigma = [D]_{2\times 2}$$

$$\underset{2\times 2}{A} = U \Sigma V^T = [U]_{2\times 2} [\Sigma]_{2\times 2} [V^T]_{2\times 2}$$

$$= \frac{1}{\sqrt{10}} \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{45} & 0 \\ 0 & \sqrt{5} \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \cdots\cdots$$

$$= \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}$$

Q. If $A_{2\times 2}$ s.t $M = A^TA$ and $N = AA^T$ and also we have,

$$M\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}45\\45\end{bmatrix},\ M\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}-5\\5\end{bmatrix},\ N\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}45\\135\end{bmatrix},\ N\begin{bmatrix}-3\\1\end{bmatrix} = \begin{bmatrix}-15\\5\end{bmatrix}$$ then Find $A = ?$

Sol: $M\begin{bmatrix}1\\1\end{bmatrix} = 45\begin{bmatrix}1\\1\end{bmatrix},\ M\begin{bmatrix}-1\\1\end{bmatrix} = 5\begin{bmatrix}-1\\1\end{bmatrix},\ N\begin{bmatrix}1\\3\end{bmatrix} = 45\begin{bmatrix}1\\3\end{bmatrix},\ N\begin{bmatrix}-3\\1\end{bmatrix} = 5\begin{bmatrix}-3\\1\end{bmatrix}$

$M = A^TA$: $\lambda_1 = 45,\ X_1 = \begin{bmatrix}1\\1\end{bmatrix}$; $\lambda_2 = 5,\ X_2 = \begin{bmatrix}-1\\1\end{bmatrix}$ & $N = AA^T$: $\lambda_1 = 45,\ Y_1 = \begin{bmatrix}1\\3\end{bmatrix}$; $\lambda_2 = 5,\ Y_2 = \begin{bmatrix}-3\\1\end{bmatrix}$

$$V = \left[\frac{X_1}{\|X_1\|}, \frac{X_2}{\|X_2\|}\right] = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix},\quad U = \left[\frac{Y_1}{\|Y_1\|}, \frac{Y_2}{\|Y_2\|}\right] = \frac{1}{\sqrt{10}}\begin{bmatrix}1 & -3\\3 & 1\end{bmatrix}$$

Q80 $A = U \Sigma V^T = \ldots\ldots\ldots = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}$

(a) $\begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}$ (c) $\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$ (d) $\begin{bmatrix} 45 & 0 \\ 0 & 5 \end{bmatrix}$

Given a $\boxed{3 \times 3}$ matrix E with SVD = $U\Sigma V^T$, if E has exactly one zero singular value, what can be said about determinant of E? is $|E| = ?$

(a) The determinant of E is zero.

(b) The determinant of E is the product of its non-zero singular values.

(c) The determinant of E is equal to the sum of its singular values.

(d) det $(E) = 1$

For any Square Mat,

$$|E| = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdots \lambda_n \quad (\text{E-Values of } E)$$

$$= \sigma_1 \cdot \sigma_2 \cdot \sigma_3 \cdots \sigma_n \quad (\text{S-Values of } E)$$

$$= 0 \quad (\because \text{one } \sigma_i = 0).$$

Consider a matrix A where $A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}$. The matrix A^TA is :

MCQ

3×3

= sq. Mat as well as symm Mat $\Rightarrow \boxed{\sigma = |\lambda_A|}$

$$A^TA = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}^T \begin{bmatrix} 4 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

$\Rightarrow$ let E Values of A^TA are $\lambda_1, \lambda_2, \lambda_3$

then singular values of A are $\sigma_1, \sigma_2, \sigma_3$

where $\sigma_1 = \sqrt{\lambda_1}$, $\sigma_2 = \sqrt{\lambda_2}$, $\sigma_3 = \sqrt{\lambda_3}$

$\sigma_1^2 = \lambda_1$, $\sigma_2^2 = \lambda_2$, $\sigma_3^2 = \lambda_3$

i.e. $\lambda_i = \sigma_i^2 \ \forall i$

Eigen Value = (Sing Value)^2

Which of the following is correct about the eigen values of A^TA?

(a) They are the squares of the singular values of A.

(b) They are the sums of the squares of the singular values of A.

(c) They are the absolute values of singular values of A. $\rightarrow$?

(d) They are one less than the singular values of A.

Note: ① Sum of the E-Values of A^TA = (b)

$\because \lambda_1 + \lambda_2 + \lambda_3 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2$

② $\lambda_A = |\sigma_A|$ (False) $\because \sigma_A = |\lambda_A|$

Let A be a 2×2 matrix given by :

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$ = Sq. Mat But not symm i.e. $\sigma_i \neq |\lambda_A|$

The singular value decomposition (SVD) of A is given by $A = U\Sigma V^T$, where U and V are orthogonal matrices and Σ is a diagonal matrix with non-negative entries. If the singular values of A are σ_1 and σ_2, where $\sigma_1 \geq \sigma_2 \geq 0$.

What is the sum of the square of singular values of A?

(a) 5.24 (b) 6.37

(c) 30 (d) 5

$|A| = \lambda_1 \cdot \lambda_2 = \sigma_1 \cdot \sigma_2$

($\sigma \neq |\lambda_A|$ $\because$ A is No Symm)

$$A^T A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 14 \\ 14 & 20 \end{bmatrix}$$

$\sigma_1^2 + \sigma_2^2 = \lambda_1 + \lambda_2 = Tr(A^T A)$

$= 5 + 25$

$= 30$

Analysis:- $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \Rightarrow |A| = -2$

C.Eq of A is $\lambda^2 - (Tr(A))\lambda + |A| = 0$

$$\lambda^2 - (5)\lambda + (-2) = 0$$

$$\lambda = \frac{5 \pm \sqrt{25+8}}{2}$$

$$= \frac{5 \pm \sqrt{33}}{2} = 2.5 \pm \frac{\sqrt{33}}{2}$$

$$|A| = \lambda_1 \cdot \lambda_2 = \left(2.5 + \frac{\sqrt{33}}{2}\right)\left(2.5 - \frac{\sqrt{33}}{2}\right)$$

$$= 6.25 - \frac{33}{4} = -2$$

Now $M = A^T A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$= \begin{bmatrix} 10 & 14 \\ 14 & 20 \end{bmatrix} \Rightarrow |A^T A| = 4.$$

C.Eq is $\lambda^2 - (30)\lambda + 4 = 0$

$$\lambda = 15 \pm \sqrt{221}$$

$$\sigma_1 = \sqrt{15 + \sqrt{221}} \ \& \ \sigma_2 = \sqrt{15 - \sqrt{221}}$$

$$\sigma_1 \cdot \sigma_2 = \sqrt{15 + \sqrt{221}}\sqrt{15 - \sqrt{221}}$$

$$= \sqrt{225 - 221} = \sqrt{4} = 2$$

i.e. $|A| = \sigma_1 \cdot \sigma_2 = |\lambda_1 \cdot \lambda_2|$

$$\lambda^2 - 30\lambda + 4 = 0$$

$$\lambda = \frac{30 \pm \sqrt{900 - 16}}{2}$$

$$= 15 \pm \frac{\sqrt{884}}{2}$$

$$= 15 \pm \sqrt{\frac{884}{4}} = 15 \pm \sqrt{221}$$

Consider a 3×3 matrix B given by :

$$B = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$ = Sq. Mat as well as symm.

The singular value decomposition (SVD) of B is given by $B = U\Sigma V^T$, where Σ is a diagonal matrix with singular values on the diagonal. What is the sum of squares of the singular values of B?

(a) 29 (b) 30

(c) 31 (d) 32

$$B^T B = \begin{bmatrix} 16 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$\because \sigma_1^2 + \sigma_2^2 + \sigma_3^2 = \lambda_1 + \lambda_2 + \lambda_3$

$= Tr(B^T B) = 16 + 9 + 4 = 29$

E Values of $B^T B = 16, 9, 4$

S. Values of $B = 4, 3, 2$ = E. Values of B

$\sigma_1 = \sqrt{16} = |4|$ i.e. $\boxed{\sigma_i = |\lambda_i|}$ $\forall i$ if A is symm.

$\sigma_2 = \sqrt{9} = |3|$

$\sigma_3 = \sqrt{4} = |2|$ i.e. Sing Value = | E. Value |.

MSQ

Given a 5×5 matrix B with SVD $U\Sigma V^T$ which of the following statements is correct regarding the relationship between B and its SVD components.

(a) The matrix $U\Sigma V^T$ has the same rank as B. (T)

(b) The product U^TU equals the identity matrix of size 5×5. (T)

(c) The matrix Σ contains the eigen values of B on its diagonal. (F) ∵ it contains Sing values.

(d) The columns of V are the right singular vectors of B and the rows of V are the left singular vector of B.

True False.

Ans (a)

$B = U\Sigma V^T$

$\rho(B) = \rho(U\Sigma V^T)$ ✓

(b) $B_{5\times5} \to (B^TB)_{5\times5} \to V_{5\times5}$

$B_{5\times5} \to (BB^T)_{5\times5} \to U_{5\times5}$

⇒ ∵ U is orthonormal Mat so $\boxed{U^TU = I_{5\times5}}$

(c) $\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \sigma_1 & 0 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 & 0 \\ 0 & 0 & 0 & \sigma_4 & 0 \\ 0 & 0 & 0 & 0 & \sigma_5 \end{bmatrix}$

Let A be a 4×3 matrix. The singular value decomposition (SVD) of A is given by $A = U\Sigma V^T$. If V is a 4×4 orthogonal matrix, Σ is a 4×3 diagonal matrix and V is a 3×3 orthogonal matrix. Which of the following statements is true?

(a) Σ has exactly 3 non-zero singular values. (F) ∵ A may have zero sing value

(b) The columns of U form an orthonormal basis for R^4. (T)

(c) The columns of V form an orthonormal basis for R^4. (F)

(d) The rows of U form an orthonormal basis for R^3. (F)

$A_{4\times3} \to (A^TA)_{3\times3} \to V_{3\times3}$

$A_{4\times3} \to (AA^T)_{4\times4} \to U_{4\times4}$

$U\Sigma V^T = A_{4\times3}$

$[4\times4][4\times3][3\times3] = A_{4\times3}$

Sing values of $A = \sqrt{\text{E. values of } A^TA}$

$= \sigma_1, \sigma_2, \sigma_3$ But may be zero also.

The mathematician plays a game in which he himself invents the rules while the physicist plays a game in which the rules are provided by nature, but as time goes on it becomes increasingly evident that the rules which the mathematician finds interesting are the same as those which nature has chosen. Paul Dirac

THANK - YOU

DATA SCIENCE & ARTIFICIAL INTELLIGENCE

Linear Algebra

Lecture No. 13

By- Dr. Puneet Sharma Sir

Recap of previous lecture

Topic

S.V.D

Topics to be Covered

Topic

1. VECTOR SPACE
2. SUBSPACE
3. BASIS
4. DIMENSION of VECTOR SPACE
5. Concept of SPANNING in Vector Space.

VECTOR SPACE

Def^n: → Any set of vectors V is called Vector space if it satisfies following conditions

① if $X, Y \in V \Rightarrow \boxed{X+Y \in V}$ i.e V is closed under addition (closure property)

② if $X, Y \in V$ then $\boxed{X+Y=Y+X}$ i.e Commutative prop of v. addition.

③ if $X, Y, Z \in V$ then $\boxed{X+(Y+Z)=(X+Y)+Z}$ i.e Associative prop of v. addition.

④ if $X \in V$ then $\exists\ O \in V$ s.t $\boxed{X+O=X}$ i.e Additive identity exist

⑤ if $X \in V$ then $-X \in V$ i.e $\boxed{X+(-X)=O}$ i.e Additive inverse exist

⑥ if $X \in V$ & c is any scalar then $\boxed{cX \in V}$ i.e closure property for Multiplication

⑦ if $X \in V$ & λ, μ are any scalar then $\boxed{\lambda(\mu X) = (\lambda\mu)X}$ i.e Asso. prop of "

⑧ if $X \in V$ then $\boxed{1.X \in V}$ i.e Multiplicative identity exist.

⑨ if $X, Y \in V$ & λ is any scalar then $\boxed{\lambda(X+Y) = \lambda X + \lambda Y}$, Distributive Property

i.e scalar Distributes over vector addition.

⑩ if $X \in V$ & λ, μ are scalars then $\boxed{(\lambda+\mu)X = \lambda X + \mu X}$, Distributive Property

i.e vector Distributes over scalar Addition.

Any set of vectors satisfying above 10 properties is called V-Space.

Examples of Vector space →

eg $V = R^2, R^3, R^4 \ldots$ are the examples of V-space.

Let $V = R^2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix}, x, y \in R \right\}$ = set of vectors.

① $\begin{bmatrix} 2 \\ 3 \end{bmatrix} \in R^2, \begin{bmatrix} -1 \\ 2 \end{bmatrix} \in R^2 \Rightarrow \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \in R^2$ ie closure prop for addition Holds.

② $\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ ie Commutative prop for V. Addition Holds

③ $\because \underbrace{\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix}} + \begin{bmatrix} 4 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} + \begin{bmatrix} 4 \\ -3 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ ie Associative prop for V. Addition Holds.

while $\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \underbrace{\begin{bmatrix} -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ -3 \end{bmatrix}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

④ $\because \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in R^2 \Rightarrow \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \in R^2 \Rightarrow$ Additive identity exist.

(5) $\because \begin{bmatrix}-2\\-3\end{bmatrix} \in R^2$ i.e. $\begin{bmatrix}2\\3\end{bmatrix}+\begin{bmatrix}-2\\-3\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \Rightarrow$ Additive inverse exist

(6) $\because \begin{bmatrix}2\\3\end{bmatrix} \in R^2$ & let s is any scalar then $s\begin{bmatrix}2\\3\end{bmatrix}=\begin{bmatrix}10\\15\end{bmatrix} \in R^2$ i.e. $\boxed{cx \in R^2}$

i.e. closure prop for Multiplication Holds.

(7) $\because \begin{bmatrix}2\\3\end{bmatrix} \in R^2$, 3 & 5 are two scalars then $(3\times 5)\begin{bmatrix}2\\3\end{bmatrix}=15\begin{bmatrix}2\\3\end{bmatrix}=\begin{bmatrix}30\\45\end{bmatrix}$

& $3\left\{5\begin{bmatrix}2\\3\end{bmatrix}\right\}=3\begin{bmatrix}10\\15\end{bmatrix}=\begin{bmatrix}30\\45\end{bmatrix}$

So Associative prop for Multi Holds.

(8) $\because 1.\begin{bmatrix}2\\3\end{bmatrix}=\begin{bmatrix}2\\3\end{bmatrix} \in R^2$ i.e. Multi Identity exist.

(9) $\because (2+3)\begin{bmatrix}2\\3\end{bmatrix}=\begin{bmatrix}10\\15\end{bmatrix}=2\begin{bmatrix}2\\3\end{bmatrix}+3\begin{bmatrix}2\\3\end{bmatrix}$ i.e. Dist. prop Holds.

(10) $5\left[\begin{bmatrix}2\\3\end{bmatrix}+\begin{bmatrix}-1\\2\end{bmatrix}\right]=5\begin{bmatrix}1\\5\end{bmatrix}=\begin{bmatrix}5\\25\end{bmatrix}=5\begin{bmatrix}2\\3\end{bmatrix}+5\begin{bmatrix}-1\\2\end{bmatrix}$ i.e. Dist. Prop Holds.

Subspace → Let V is any v.space & let $\boxed{W \subset V}$ (i.e W is a subset of V) then

→ W is called subspace if it is itself a vector space.

→ W is called subspace if it also satisfy all 10 properties of v-space.

Shortcut – W is called subspace of any vector space V if.

$\forall X, Y \in W \Longrightarrow$ $\boxed{X+Y \in W}$ ie W is closed under vector addition

$\boxed{CX \in W}$ ie W is also closed under scalar Multi

OR

$\forall X, Y \in W$ & C is any scalar $\Longrightarrow$ $\boxed{CX+Y \in W}$ then W is called subspace.

SPANNING in Vector space → let W is subspace of any vector space V.

& let S is any subset of W ie $\boxed{S \subset W}$

Linear span: set of all linear combinations of vectors (that lies in S) is called linear span of S and it is denoted as $L(S)$

eg let $S = \{x_1, x_2, x_3 \ldots x_r\}$ then $L\{S\} = \left\{ (c_1x_1 + c_2x_2 + \ldots + c_rx_r);\ c_i = \text{sclars},\ x_i = \text{vectors} \right\}$

Spanning: if $\boxed{L\{S\} = W}$ then $\boxed{W \text{ is spanned by } S}$ (generated) or $\boxed{S \text{ spans } W}$ (generates)

ie "Any Random Vector in W can be generated by linear combinations of vectors in S"

ie Different-Different linear comb of vectors in S provides diff-diff vectors of W

Let us take an example, $S = \{X_1, X_2\} = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ where $X_1, X_2 \in R^2$ i.e $S \subset R^2$

then S spans R^2 i.e $\boxed{L(S) = R^2}$

Let $X = \begin{bmatrix}2\\-3\end{bmatrix} \in R^2$ then $\begin{bmatrix}2\\-3\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix} + \begin{bmatrix}0\\-3\end{bmatrix} = 2\begin{bmatrix}1\\0\end{bmatrix} - 3\begin{bmatrix}0\\1\end{bmatrix}$

$$X = 2X_1 - 3X_2$$

Let $Y = \begin{bmatrix}\lambda\\\mu\end{bmatrix} \in R^2$ then $Y = \begin{bmatrix}\lambda\\\mu\end{bmatrix} = \begin{bmatrix}\lambda\\0\end{bmatrix} + \begin{bmatrix}0\\\mu\end{bmatrix} = \lambda\begin{bmatrix}1\\0\end{bmatrix} + \mu\begin{bmatrix}0\\1\end{bmatrix} = \lambda X_1 + \mu X_2$

i.e we have justified that Any Random Member of R^2 can be expressed as a linear combination of X_1 & X_2 i.e $L\{S\} = R^2 \Rightarrow S$ generates R^2

or S spans R^2

Similarly if we take $S = \{X_1\ X_2\ X_3\} = \left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\} = \{X_1, X_2, X_3\}$ ie $S \subset R^3$

then S will generate R^3 i.e $\boxed{S \text{ spans } R^3}$ or $L\{S\} = R^3$

Cross check, let $X = \begin{bmatrix}\alpha\\\beta\\\gamma\end{bmatrix} \in R^3$ then $X = \begin{bmatrix}\alpha\\\beta\\\gamma\end{bmatrix} = \begin{bmatrix}\alpha\\0\\0\end{bmatrix} + \begin{bmatrix}0\\\beta\\0\end{bmatrix} + \begin{bmatrix}0\\0\\\gamma\end{bmatrix}$

$$X = \alpha\begin{bmatrix}1\\0\\0\end{bmatrix} + \beta\begin{bmatrix}0\\1\\0\end{bmatrix} + \gamma\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$X = \alpha X_1 + \beta X_2 + \gamma X_3$$

ie any Random Member X of R^3 can be expressed as a L-Comb of X_1, X_2, X_3

$\Longrightarrow S$ Spans $R^3 \Rightarrow L\{S\} = R^3$

Ex to understand BASIS of any V.Space

R^2

$X_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \end{bmatrix}$ - - - - on x axis

$X_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \end{bmatrix}$ - - - - on Y axis

$X = c_1 X_1 + c_2 X_2$ (elements of I, II, III, IV Quad)

So to form R^2 we need two L.I set of vectors that generates R^2

this New set is called BASIS of R^2

Spans.

$S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\} = \{X_1, X_2, X_3, X_4\}$ then

- S Spans R^2
- S is not LI

BASIS of Vector space → A set of vectors S is called Basis for V.space V

if ① S spans V ② S must consist L.I vectors.

eg: ∵ $V = R^2$ is vector space then $S = \{X_1, X_2\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

is called Basis for R^2 ∵ (i) S spans R^2

(ii) X_1 & X_2 are LI (By observation)

eg: ∵ $V = R^3$ is vector space then $S = \{X_1 X_2 X_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

is called Basis for R^3 ∵ (i) S spans R^3

(ii) X_1, X_2, X_3 are LI $(\because |S| \neq 0)$

RECAP: Let $X_1, X_2, X_3 \ldots X_r$ are the given set of vectors

& Consider $A = [X_1\ X_2\ X_3 \ldots X_r]$

① General Method

(i) if $\rho(A)$ = No. of vectors $\Rightarrow$ LI

(ii) if $\rho(A) <$ " " $\Rightarrow$ LD

② Tricky Method (if A is Sq. Mat)

(i) if $|A| \neq 0 \Rightarrow$ vectors are LI

(ii) if $|A| = 0 \Rightarrow$ " " LD

Note: In case of two vectors X_1 & X_2, there is no Need to follow G. Method or Tricky Method, only follow Observation Method.

Dimension of vector space →

$$\boxed{\dim(\text{V-Space}) = \text{Counting of vectors in it's Basis}}$$

for eg, $\dim(R^2) = \text{two}$ $\because S = \{X_1, X_2\} = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ contains two vectors

$\dim(R^3) = \text{three}$ $\because S = \{X_1 X_2 X_3\} = \left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$,, three vectors

Let W is subspace of R^3 then W is also a V-space Hence we have following three cases: ① $\dim(W)$ may be 3, ② $\dim(W)$ may be 2 ③ $\dim(W)$ may be 1.

Although vectors of W are 3-dimensional vectors.

eg for $V = R^2$, Basis is $S = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ But X axis is subspace of R^2 ie $X = \left\{\begin{bmatrix}a\\0\end{bmatrix}, a \in R\right\}$

& Basis for X axis is $S = \left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$ so $\dim(X) = \text{one}$.

PROPERTIES of BASIS & DIMENSION →

① Basis of any vector space is $\boxed{\text{Not unique}}$

② if $\text{Dim}(V) = n$ then Any set of n, LI vectors forms Basis.

eg $\because \dim(R^3) = 3$ then any set of 3, LI vectors will be Basis for R^3

eg $\because \dim(R^4) = 4$ then any set of 4, LI vectors will form Basis for R^4.

Q: Prove that $S = \{(1\,2\,1)', (2\,1\,-4)', (3\,-2\,1)'\}$ is also Basis for R^3

Note: Standard Basis for $R^3 = \{\epsilon_1, \epsilon_2, \epsilon_3\} = \left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$

Sol: $S = \{X_1, X_2, X_3\} = \left\{\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\1\\-4\end{bmatrix}, \begin{bmatrix}3\\-2\\1\end{bmatrix}\right\}$ $\because |S| = \cdots\cdots = -42$ (HW)

$\therefore |S| \neq 0$ so X_1, X_2, X_3 are LI (2nd condition for Basis Holds)

Now for spanning, Consider any Random Vector of R^3 say $X = \begin{bmatrix}2\\1\\3\end{bmatrix}$

Let us Consider $\boxed{k_1X_1 + k_2X_2 + k_3X_3 = X}$

$$k_1\begin{bmatrix}1\\2\\1\end{bmatrix} + k_2\begin{bmatrix}2\\1\\-4\end{bmatrix} + k_3\begin{bmatrix}3\\-2\\1\end{bmatrix} = \begin{bmatrix}2\\1\\3\end{bmatrix} \quad (1)$$

$$k_1 + 2k_2 + 3k_3 = 2$$
$$2k_1 + k_2 - 2k_3 = 1$$
$$k_1 - 4k_2 + k_3 = 3$$

$$\begin{bmatrix}1 & 2 & 3\\2 & 1 & -2\\1 & -4 & 1\end{bmatrix}\begin{bmatrix}k_1\\k_2\\k_3\end{bmatrix} = \begin{bmatrix}2\\1\\3\end{bmatrix} \Rightarrow \boxed{SX = B}$$

$\because |S| \neq 0$ so By Matrix Method, unique sol. will exist. i.e unique Values of k_1, k_2, k_3 exist.

i.e. $X = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}$ = unique values exist. i.e. eqn (1) is justified.

$\Rightarrow x_1, x_2, x_3$ together generates X i.e. S generates $R^3 \Rightarrow$ $\boxed{S \text{ spans } R^3}$

$\because$ S contains LI set of vectors & S spans $R^3 \Rightarrow S$ is also a Basis for R^3.

<u>Conclusion</u>: BASIS is not unique for any V. Space.

(2) Here we have also verified that,

$\dim(R^3) = 3 \Rightarrow$ Any set of 3 LI vectors forms Basis of R^3.

Ex: Standard Basis for R^4 is $S_1 = \{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\} = \left\{ \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \right\}$

& $\dim(R^4) = ? = 4$ ($\because$ Basis contains $\underline{4}$ vectors)

Ex P.T. $S_2 = \{X_1, X_2, X_3, X_4\} = \left\{ \begin{bmatrix}0\\1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\0\\3\\0\end{bmatrix}, \begin{bmatrix}2\\3\\0\\1\end{bmatrix}, \begin{bmatrix}3\\0\\1\\2\end{bmatrix} \right\}$ forms Basis for R^4.

$$S_2 = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{bmatrix}_{4\times 4}$$

$\because |S_2| = \cdots\cdots\cdots = 88$ (HW)

i.e. $|S_2| \neq 0 \Rightarrow X_1, X_2, X_3, X_4$ are LI.

$\therefore \dim(R^4)$ is also 4 so these vectors will form Basis for R^4 (using prop 2)

eg (PT) $S_3 = [\gamma_1\ \gamma_2\ \gamma_3\ \gamma_4] = \left\{ \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\\0\end{bmatrix} \right\}$ is not Basis for R^4.

(M-I) $\because \gamma_4 = \gamma_1 + \gamma_2 + \gamma_3$ i.e. these 4 vectors are LD vectors so they will not form Basis.

(M-II) $\because |S_3| = \begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0$ i.e. vectors are LD

Q: Consider the set of column vectors defined by $X = \{(x_1\ x_2\ x_3)^T ;\ x_1 + x_2 + x_3 = 0\}$

then $S = \{(1\ -1\ 0)',\ (1\ 0\ -1)'\}$ is Basis for subspace X ?

Sol: Here X is subspace of R^3 i.e $\boxed{X \subset R^3}$

& $S = \{X_1, X_2\} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$ Here X_1 & X_2 are LI (By observation)

Now we will try to check whether $\boxed{L(S) = X}$?? i.e S generates X ??

Let us take Random member of X say $Y = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ then

$$Y = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = 3X_1 - X_2 = 3\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$ i.e S spans X

Hence S is Basis for subspace X & $\dim(X)$ = No. of vectors in Basis = Two

Analysis :- $V = R^3 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} ; x_1 \in R, x_2 \in R, x_3 \in R \right\} = 3\text{-D space}$.

ie V is a vector space.

Let us take $X = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} ; x_1 + x_2 + x_3 = 0 \right\}$ obviously $X \subset R^3$

$\because$ X is also satisfying all the 10 properties of V. Space

So X is also a vector space & it (X) is called SubSpace of R^3

& $\dim(R^3) = 3$ But $\dim(X) = 2$

Consider the set of (column) vectors defined by $X = \{x \in R^3 \mid x_1 + x_2 + x_3 = 0$, where $x^T = [x_1, x_2, x_3]^T\}$. Which of the following is TRUE?

(a) $\{[1, -1, 0]^T, [1, 0, -1]^T\}$ is a basis for the subspace X. (T)

(b) $\{[1, -1, 0]^T, [1, 0, -1]^T\}$ is a linearly independent set, but it does not span X and therefore is not a basis of X (F)

(c) X is not a subspace for R^3 (F) (see previous slide)

(d) None of the above

If the vectors, $e_1 = (1,0,2), e_2 = (0,1,0)$ and $e_3 = (-2,0,1)$ from an orthogonal basis of the three dimensional real space $\mathbb{R}^3$, then the vector $\boxed{u = (4,3-3)} \in \mathbb{R}^3$ can be expressed as

(a) $u = -\frac{2}{5}e_1 - 3e_2 - \frac{11}{5}e_3$

(b) $u = -\frac{2}{5}e_1 - 3e_2 + \frac{11}{5}e_3$

(c) $u = -\frac{2}{5}e_1 + 3e_2 + \frac{11}{5}e_3$

(d) $u = -\frac{2}{5}e_1 + 3e_2 - \frac{11}{5}e_3$

$u = \begin{bmatrix} 4 \\ 3 \\ -3 \end{bmatrix}$, Basis $S = \{e_1 e_2 e_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$

S will generates R^3 i.e $\boxed{S \text{ Spans } R^3}$

Any Member of R^3 can be expressed as a linear comb of vectors in S.

(a) $4 = -\frac{2}{5}(1) - 3(0) - \frac{11}{5}(-2) = 4$

$3 = -\frac{2}{5}(0) - 3(1) - \frac{11}{5}(0) \neq 3$

(b) X (c) X (d) ✓

$k_1 e_1 + k_2 e_2 + k_3 e_3 = u$

$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ -3 \end{bmatrix} \Rightarrow X = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} -2/5 \\ 3 \\ -11/5 \end{bmatrix}$ Ans (d)

$\boxed{SX = u}$

$$\begin{bmatrix}4\\3\\-3\end{bmatrix} = k_1\begin{bmatrix}1\\0\\2\end{bmatrix} + k_2\begin{bmatrix}0\\1\\0\end{bmatrix} + k_3\begin{bmatrix}-2\\0\\1\end{bmatrix}$$

$$k_1\begin{bmatrix}1\\0\\2\end{bmatrix} + k_2\begin{bmatrix}0\\1\\0\end{bmatrix} + k_3\begin{bmatrix}-2\\0\\1\end{bmatrix} = \begin{bmatrix}4\\3\\-3\end{bmatrix}$$

$$\begin{cases} k_1 + 0k_2 - 2k_3 = 4 \\ 0k_1 + k_2 + 0k_3 = 3 \\ 2k_1 + 0k_2 + k_3 = -3 \end{cases}$$

$$\begin{bmatrix}1 & 0 & -2\\0 & 1 & 0\\2 & 0 & 1\end{bmatrix}\begin{bmatrix}k_1\\k_2\\k_3\end{bmatrix} = \begin{bmatrix}4\\3\\-3\end{bmatrix}$$

$$AX = B.$$

$\because |A| = \cdots = 5$ i.e. $|A| \neq 0$

so unique sol exist.

$$[A:B] = \left[\begin{array}{ccc|c}1 & 0 & -2 & 4\\0 & 1 & 0 & 3\\2 & 0 & 1 & -3\end{array}\right] \longrightarrow \left[\begin{array}{ccc|c}1 & 0 & -2 & 4\\0 & 1 & 0 & 3\\0 & 0 & 5 & -11\end{array}\right]$$

$$\begin{bmatrix}1 & 0 & -2\\0 & 1 & 0\\0 & 0 & 5\end{bmatrix}\begin{bmatrix}k_1\\k_2\\k_3\end{bmatrix} = \begin{bmatrix}4\\3\\-11\end{bmatrix}$$

$$\left.\begin{aligned} 5k_3 &= -11 \\ k_2 &= 3 \\ k_1 - 2k_3 &= 4 \end{aligned}\right\} \quad \begin{aligned} k_3 &= -11/5 \\ k_2 &= 3 \\ k_1 &= 4 - \frac{22}{5} = -\frac{2}{5} \end{aligned}$$

By (1) $u = -\frac{2}{5}e_1 + 3e_2 - \frac{11}{5}e_3$

It is given that $X_1, X_2, \ldots X_M$ are M non-zero, orthogonal vectors. The dimension of the vector space spanned by the 2M vectors $X_1, X_2, \ldots X_M, -X_1, -X_2, \ldots -X_M$ is

(a) 2M

(b) M + 1

(c) M

(d) dependent on the choice of $X_1, X_2, \ldots X_M$

$L(S) = \{x_1, x_2, \ldots x_M, -x_1, -x_2, \ldots, -x_M\}$

Basis $= \{x_1, x_2, x_3 \ldots x_M\}$ orthogonal vectors $\Downarrow$ LI vectors.

So dim = M

LD. $\because \dim(R^4) = 4$.

) If $v_1, v_2, \ldots, v_6$ are six vectors in R^4, which one of the following statements is False?

(a) If $\{v_1, v_3, v_5, v_6\}$ spans R^4, then it forms a basis for R^4. (T) $\because \dim(R^4) = 4$, so minimal spanning set contain 4 vectors.

(b) These vectors are not linearly independent. (T)

(c) It is not necessary that these vectors span R^4. (True)

(d) Any four of these vectors form a basis for R^4. (False) If we are taking all 6 vectors from single family then they will not span R^4 Hence this option is False.

Q) Let M_n denote the vector space of all $n \times n$ real matrices. Consider the following subsets of M_n

MSQ

1. $W_1 = \{A \in M_n \mid A^2 = I\}$, where I is the identity matrix.
2. $W_2 = \{A \in M_n \mid \text{rank}(A) = 1\}$
3. $W_3 = \{A \in M_n \mid \text{trace}(A) = 0\}$
4. $W_4 = \{A+B \mid A \in M_n, B \text{ is a fixed matrix in } M_n\}$

Which of the following statements is correct?

(a) W_1 is a linear subspace of M_n ✗

(b) W_2 is a linear subspace of M_n ✗

c) W_3 is a linear subspace of M_n (T)

d) W_4 is a linear subspace of M_n only if B is a zero matrix.

(b) $W_2 = \{A \in M_n ; \rho(A) = 1\}$

let $A, B \in M_n$ then $\rho(A) = 1$ & $\rho(B) = 1$

Now $\rho(A+B) \neq \rho(A) + \rho(B)$

$\because \rho(A+B) \leq \rho(A) + \rho(B)$

ie $A+B \notin W_2$

So W_2 is not subspace.

(a) $W_1 = \{A \in M_n : A^2 = I\}$, let $A, B \in W_1$ then $A^2 = I$ & $B^2 = I$ (given)

$(A+B)^2 = A^2 + AB + BA + B^2 = I + AB + BA + I$

$\neq I$ so $(A+B) \notin W_1$ so W_1 is not subspace.

(c) $W_3 = \{A \in M_n : Tr(A) = 0\}$

let $A, B \in M_n \Rightarrow Tr(A) = 0 = Tr(B)$

Now $Tr(A+B) = Tr(A) + Tr(B) = 0 + 0 = 0$

i.e $\boxed{A+B \in M_n}$ Hence

M_n is closed under vector Addition.

Now $Tr(cA) = c \cdot Tr(A) = c(0) = 0$

i.e $\boxed{c \cdot A \in M_n}$ Hence

M_n is closed under Scalar Multiplication.

So M_n is a vector space or subspace.

(d) $W_4 = \{A+B : A \in M_n \ \& \ B = \text{fixed Mat}\}$

we have two possibilities,

(i) Either $B =$ Zero Mat.

then W_4 has additive identity.

then W_4 will become subspace.

(ii) OR $B \neq$ ZERO Mat.

then W_4 does not have additive identity

So W_4 will not be a subspace

i.e (d) ✓

Q) Let S_1, S_2, S_3 be sets of real-valued functions defined as:

HW

$S_1 = \{ f \mid f(3) = 0 \}$

$S_2 = \{ g \mid g(x) = x+1 \text{ for all } x \in R \}$

$S_3 = \{ h \mid h(x) = c \text{ for some instant } c \in K \}$

Which of the following statements is correct?

a) S_1 is a vector space & S_2 is not a vector space.

b) S_2 is a vector space & S_3 is not a vector space.

c) S_1 & S_3 are vector spaces, but S_2 is not a vector space

d) S_2 & S_3 are vector spaces but S_1 is not a vector space.

$P = \begin{bmatrix} -10 \\ -1 \\ 3 \end{bmatrix}^T$, $Q = \begin{bmatrix} -2 \\ -5 \\ 9 \end{bmatrix}^T$ and $R = \begin{bmatrix} 2 \\ -7 \\ 12 \end{bmatrix}^T$ are three vectors. An orthogonal set of vectors having a span that contains P, Q, R is

(a) $\begin{bmatrix} -6 \\ -3 \\ 6 \end{bmatrix}$ $\begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}$ $\Rightarrow x_1 \cdot x_2 = 0$

(b) $\begin{bmatrix} -4 \\ 2 \\ 4 \end{bmatrix}$ $\begin{bmatrix} 5 \\ 7 \\ -11 \end{bmatrix}$ $\begin{bmatrix} 8 \\ 2 \\ -3 \end{bmatrix}$ $\Rightarrow$ Not orthogonal.

(c) $\begin{bmatrix} 6 \\ 7 \\ -1 \end{bmatrix}$ $\begin{bmatrix} -3 \\ 2 \\ -2 \end{bmatrix}$ $\begin{bmatrix} 3 \\ 9 \\ -4 \end{bmatrix}$ $\Rightarrow$ " "

(d) $\begin{bmatrix} 4 \\ 3 \\ 11 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 31 \\ 3 \end{bmatrix}$ $\begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}$ $\Rightarrow$ " "

M-II

KW.

M-I

The following vector is linearly dependent upon the solution to the previous problem

HW

(a) $\begin{bmatrix} 8 \\ 9 \\ 3 \end{bmatrix}$

(b) $\begin{bmatrix} -2 \\ -17 \\ 30 \end{bmatrix}$

(c) $\begin{bmatrix} 4 \\ 4 \\ 5 \end{bmatrix}$

(d) $\begin{bmatrix} 13 \\ 2 \\ -3 \end{bmatrix}$

THANK - YOU

DATA SCIENCE & ARTIFICIAL INTELLIGENCE

Linear Algebra

Lecture No. 14

By- Dr. Puneet Sharma Sir

Recap of previous lecture

Topic

VECTOR SPACE

Topics to be Covered

Topic

1. REFLECTION MATRIX
2. ROTATION MATRIX
3. PARTITION MATRIX
4. PROJECTION MATRIX (Part 1)

Q) Let S_1, S_2, S_3 be sets of real-valued functions defined as:

$S_1 = \{ f \mid f(3) = 0 \}$

$S_2 = \{ g \mid g(x) = x+1 \text{ for all } x \in R \}$

$S_3 = \{ h \mid h(x) = c \text{ for some instant } c \}$

Which of the following statements is correct?

a) S_1 is a vector space & S_2 is not a vector space.

b) S_2 is a vector space & S_3 is not a vector space ✗

c) ✓ S_1 & S_3 are vector spaces, but S_2 is not a vector space

d) S_2 & S_3 are vector spaces but S_1 is not a vector space. ✗

$S_3 = \{ h(x) : h(x) = c \}$

let $h, g \in S_3$ ie $h(x) = c$ & $g(x) = c$

$(h+g)(x) = h(x) + g(x) = c + c$

$= 2c = c_1 \Rightarrow h+g \in S_3$

$(\alpha h)(x) = \alpha(h(x)) = \alpha(c) = c_2$

$\Rightarrow \alpha h \in S_3$

So S_3 is also V. Space.

$S_1 = \{ f(x) : f(3) = 0 \}$

let f & $g \in S_1$ ie $f(3) = 0$, $g(3) = 0$

Now $(f+g)(x) = f(x) + g(x) = 0 + 0 = 0$ ie $f+g \in S_1$

& $(cf)(x) = c f(x) = c(0) = 0$ ie $cf \in S_1$

$\Rightarrow S_1$ is a V. Space

let $S_2 = \{ g(x) : g(x) = x+1 \}$

let $g_1(x), g_2(x) \in S_2 \Rightarrow g_1(x) = x+1$ & $g_2(x) = x+1$

$$(g_1+g_2)(x) = g_1(x) + g_2(x)$$
$$= (x+1)+(x+1) = 2x+2 = 2(x+1)$$
$$\neq (x+1) \text{ i.e. } g_1+g_2 \notin S_2$$

so S_2 is Not V.Space.

$P = \begin{bmatrix} -10 \\ -1 \\ 3 \end{bmatrix}^T$, $Q = \begin{bmatrix} -2 \\ -5 \\ 9 \end{bmatrix}^T$ and $R = \begin{bmatrix} 2 \\ -7 \\ 12 \end{bmatrix}^T$ are three vectors. An orthogonal set of vectors having a span that contains P, Q, R is

(a) $\begin{bmatrix} -6 \\ -3 \\ 6 \end{bmatrix}$ $\begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}$ $X_1 \cdot X_2 = 0$ ✓ orthogonal

(b) $\begin{bmatrix} -4 \\ 2 \\ 4 \end{bmatrix}$ $\begin{bmatrix} 5 \\ 7 \\ -11 \end{bmatrix}$ $\begin{bmatrix} 8 \\ 2 \\ -3 \end{bmatrix}$ Not orthogonal

(c) $\begin{bmatrix} 6 \\ 7 \\ -1 \end{bmatrix}$ $\begin{bmatrix} -3 \\ 2 \\ -2 \end{bmatrix}$ $\begin{bmatrix} 3 \\ 9 \\ -4 \end{bmatrix}$ " "

(d) $\begin{bmatrix} 4 \\ 3 \\ 11 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 31 \\ 3 \end{bmatrix}$ $\begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}$ " "

M-II At Q, $P, Q, R \in L\{S\}$

Let us take (a): i.e

$S = \{X_1, X_2\} = \left\{ \begin{bmatrix} -6 \\ -3 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix} \right\}$

$P = X_1 - X_2$, $Q = X_1 + X_2$

& $R = X_1 + 2X_2$

i.e $P, Q, R \in L\{S\}$ where S = option (a) ✓

The following vector is linearly dependent upon the solution to the previous problem

(a) $\begin{bmatrix} 8 \\ 9 \\ 3 \end{bmatrix} = x_3$

(b) $\begin{bmatrix} -2 \\ -17 \\ 30 \end{bmatrix} = x_4$

(c) $\begin{bmatrix} 4 \\ 4 \\ 5 \end{bmatrix} = x_5$

(d) $\begin{bmatrix} 13 \\ 2 \\ -3 \end{bmatrix} = x_6$

$S = \{x_1 x_2\} = \left\{ \begin{bmatrix} -6 \\ -3 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix} \right\}$

$S_a = \{x_1 x_2 x_3\} \Rightarrow |S_a| \neq 0$ ie LI

$S_b = \{x_1 x_2 x_4\} \Rightarrow |S_b| = 0$ ie LD

M-II : In option (b),

$x_4 = 3x_1 + 4x_2$

$= \begin{bmatrix} -18 \\ -9 \\ 18 \end{bmatrix} + \begin{bmatrix} 16 \\ -8 \\ 12 \end{bmatrix}$

Reflection Matrix → with angle $\theta = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$

(i) About X axis $(\theta = 0)$

$RM = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(ii) About Y axis $\left(\theta = \frac{\pi}{2}\right)$

$RM = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

(iii) About $y = x$ $\left(\theta = \frac{\pi}{4}\right)$

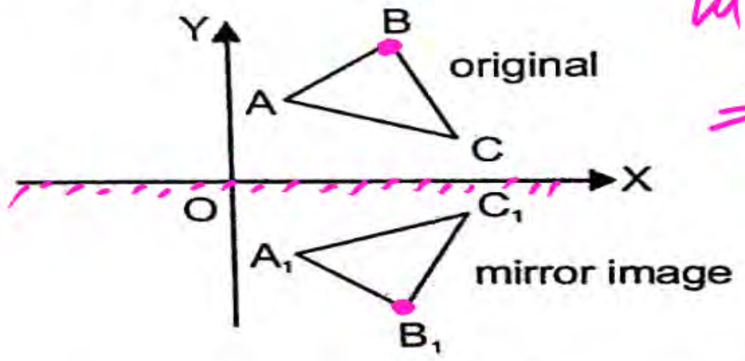
$RF = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(iv) About $y = -x$ $\left(\theta = -\frac{\pi}{4}\right)$

$RF = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

(v) Special Case: (About origin) → $R.M = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ $\because A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}$

The figure shows a shape ABC and its mirror image $A_1B_1C_1$ across the horizontal axis (X –axis). The coordinate transformation matrix that maps ABC to $A_1B_1C_1$ is



(a) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

let $B(x, y)$

$\Rightarrow B_1(x, -y)$

(a) $AB = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix} \neq B_1$

(b) $AB = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ -x \end{bmatrix} \neq B_1$

(c) $AB = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix} \neq B_1$

(d) $AB = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix} = B_1$ ✓

Q. The transformation Matrix to find mirror image of any point $P\begin{bmatrix} x \\ y \end{bmatrix}$ in x-y plane about the line $y=x$ is ?

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ ✓

(c) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

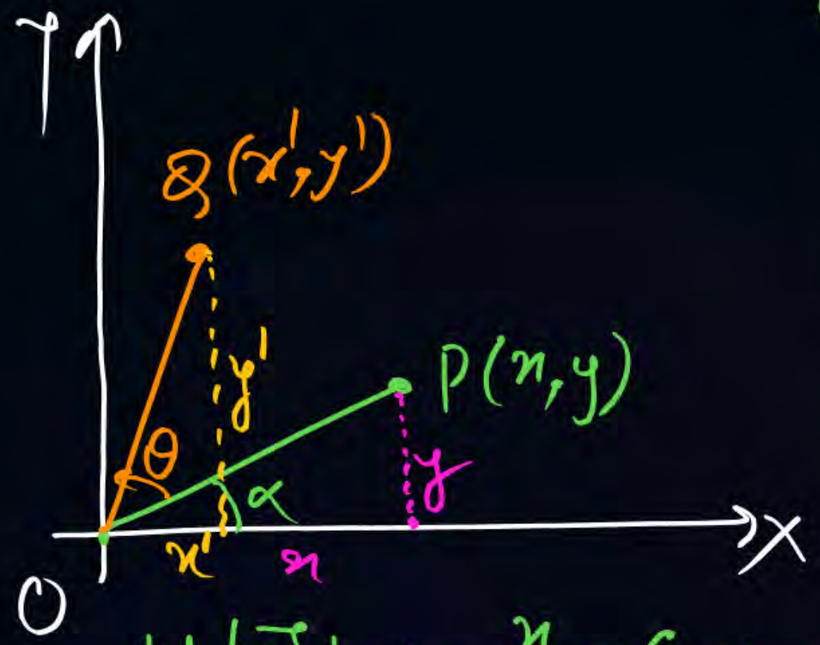
(d) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

(b) $AB = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$ ✓

ROTATION MATRIX (in 2D in ACW)

Let $\begin{bmatrix} x \\ y \end{bmatrix}$ is to be Rotated in ACW by angle θ

then Rot Mat $= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ & New vector is $\begin{bmatrix} x' \\ y' \end{bmatrix} = (\text{Rot Mat}) \begin{bmatrix} x \\ y \end{bmatrix} = ?$



let $|\vec{OP}| = r$, $\frac{x}{r} = \cos\alpha \Rightarrow x = r\cos\alpha$

$\frac{y}{r} = \sin\alpha \Rightarrow y = r\sin\alpha$.

$\frac{x'}{r} = \cos(\alpha+\theta) = \cos\alpha\cos\theta - \sin\alpha\sin\theta$

$\frac{y'}{r} = \sin(\alpha+\theta) = \sin\alpha\cos\theta + \cos\alpha\sin\theta$

$x' = (r\cos\alpha)\cos\theta - (r\sin\alpha)\sin\theta = x\cos\theta - y\sin\theta$

$y' = (r\cos\alpha)\sin\theta + (r\sin\alpha)\cos\theta = x\sin\theta + y\cos\theta$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{\text{Rot Mat}} \begin{bmatrix} x \\ y \end{bmatrix} = \underline{\underline{\text{Ans}}}$$

Note: If vector is Rotated by angle θ in C.W sense then Replace $\theta \to -\theta$ in Rot Mat.

ie $R = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ & $\begin{bmatrix} x' \\ y' \end{bmatrix} = R_{CW} \begin{bmatrix} x \\ y \end{bmatrix}$

(*) Rotation Mat in 3D :→ (in ACW sense by an angle θ) →

Case I → (If vector is Rotating) →

(1) About X axis

$$R_X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = R_X \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

(2) About Y axis

$$R_Y = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = R_Y \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

(3) About Z axis

$$R_Z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = R_Z \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

② If Coordinate system is Rotating → (in ACW sense) [Take Transpose of I]

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}, \quad R_y = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}, \quad R_z = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Analysis: → ① Rotation Matrices are orthogonal matrix

② $Tr(\text{Rot Mat}) = 1 + 2\cos\theta$.

Q. If the vector $\begin{bmatrix}1\\1\end{bmatrix}$ is to be rotated by 45° in an ACW sense then what are the coordinates of new vector?

(a) $\begin{bmatrix}1\\1\end{bmatrix}$ (b) $\begin{bmatrix}0\\1\end{bmatrix}$

(c) $\begin{bmatrix}0\\2\end{bmatrix}$ (d) $\begin{bmatrix}0\\\sqrt{2}\end{bmatrix}$

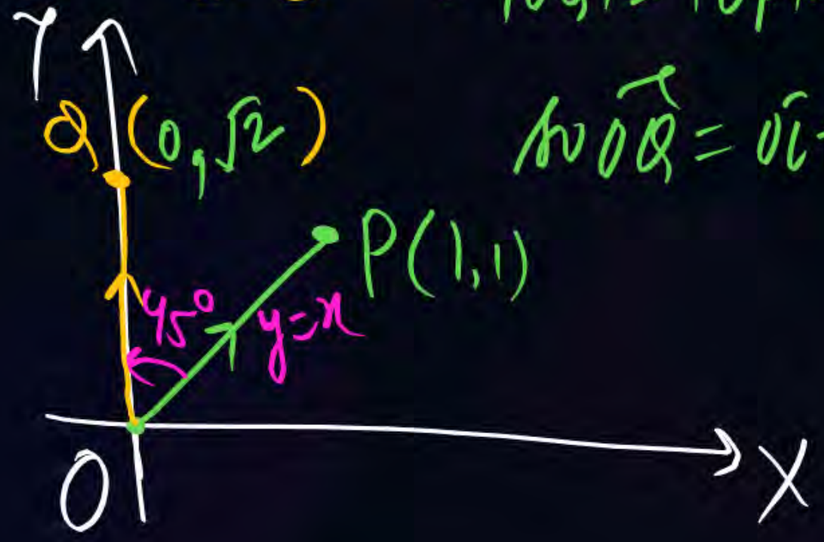
$\vec{OP} = \hat{i}+\hat{j} = \begin{bmatrix}1\\1\end{bmatrix}$

$|\vec{OP}| = \sqrt{1^2+1^2} = \sqrt{2}$

$\because |\vec{OQ}| = |\vec{OP}| = \sqrt{2}$

so $\vec{OQ} = 0\hat{i} + \sqrt{2}\hat{j} = \begin{bmatrix}0\\\sqrt{2}\end{bmatrix}$

M-I



M-II $\theta = 45^\circ$,

$$\text{Rot Mat} = \begin{pmatrix}\cos 45^\circ & -\sin 45^\circ\\ \sin 45^\circ & \cos 45^\circ\end{pmatrix} = \begin{pmatrix}1/\sqrt{2} & -1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}$$

so $\begin{bmatrix}x'\\y'\end{bmatrix} = R.M \begin{bmatrix}x\\y\end{bmatrix}$

$$= \begin{bmatrix}1/\sqrt{2} & -1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}0\\ \frac{2}{\sqrt{2}}\end{bmatrix}$$

$$\begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}0\\\sqrt{2}\end{bmatrix}$$ (d)

Q: If the vector $\begin{bmatrix}1\\1\\1\end{bmatrix}$ is to be Rotated by 30° in ACW about x axis then find New vector ?

Sol: ∵ vector is Rotating so it is Based on Case I :-

$$\theta = 30^\circ, \quad (R_x)_{ACW} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 30^\circ & -\sin 30^\circ \\ 0 & \sin 30^\circ & \cos 30^\circ \end{bmatrix}$$

$$\text{So } \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = R_x \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & -1/2 \\ 0 & 1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \left(\frac{\sqrt{3}-1}{2}\right) \\ \left(\frac{\sqrt{3}+1}{2}\right) \end{bmatrix}$$

Q: The Matrix $P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ rotates any vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ about the line $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ by an angle ?

(a) 90

(b) 45°

(c) 120°

(d) 180°

w.k that for Rotation Mat,

$$\boxed{Tr(\text{Rot Mat}) = 1 + 2\cos\theta}$$

$$0 = 1 + 2\cos\theta$$

$$\cos\theta = \frac{-1}{2}$$

$$\theta = 120^\circ$$

PYQ

The Cartesian coordinates of a point P in a right-handed coordinate system are (1, 1, 1). The transformed coordinates of P due to a 45° clockwise rotation of the coordinate system about the positive x-axis are

***

(a) $(1, 0, \sqrt{2})$

(b) $(1, 0, -\sqrt{2})$

(c) $(-1, 0, \sqrt{2})$

(d) $(-1, 0, -\sqrt{2})$

(d) $(1, \sqrt{2}, 0)$

When vector is Rotating (In CW) $\Longrightarrow$ When Coordinate system is Rotating (CW)

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \xrightarrow[\text{Transpose}]{\text{Taking}} R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

Hence $\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = R_x \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\because \theta = 45^\circ$ 故

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = R_x \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 45^\circ & -\sin 45 \\ 0 & \sin 45 & \cos 45 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \sqrt{2} \end{bmatrix}$$

ⓐ

PARTITION MATRIX (BLOCK MATRIX)

eg
$$\left[\begin{array}{cc|ccc} 1 & 2 & -3 & 0 & 1 \\ 2 & 0 & 1 & 2 & -1 \\ 3 & 1 & 2 & -1 & 0 \\ \hline 1 & 2 & 1 & 3 & 4 \end{array}\right]_{4\times 5} \rightarrow \left[\begin{array}{c|cccc} 1 & 2 & -3 & 0 & 1 \\ \hline 2 & 0 & 1 & 2 & -1 \\ 3 & 1 & 2 & -1 & 0 \\ 1 & 2 & 1 & 3 & 4 \end{array}\right]_{4\times 5} \rightarrow \left[\begin{array}{cc|ccc} 2 & -1 & 3 & 4 & 0 \\ 1 & 2 & 0 & -1 & 2 \\ \hline 3 & -1 & 2 & 1 & 4 \\ 0 & 1 & 3 & 1 & -3 \\ \hline 1 & 2 & -1 & 2 & 1 \end{array}\right]_{5\times 5}$$

these types of Matrices are called Partition Matrices

Def^n:- with the help of Horizontal & Vertical lines, we can convert given Mat into a Partition Mat. (Block Mat)

Square Block Matrix →

eg $\left[\begin{array}{cc|cc} 1 & -2 & 4 & 3 \\ 2 & 0 & 1 & 4 \\ \hline 0 & 1 & -1 & 7 \\ 3 & 1 & 0 & 5 \end{array}\right]_{4\times 4}$, eg $\left[\begin{array}{cc|ccc|c} 1 & 3 & 1 & 6 & & -2 \\ 0 & -1 & 5 & 7 & & 0 \\ \hline 0 & 0 & 2 & 4 & & 1 \\ 0 & 0 & 1 & -1 & & 3 \\ \hline 0 & 0 & 0 & 0 & & -3 \end{array}\right]_{5\times 5}$, eg $\left[\begin{array}{ccc|ccc} 1 & -2 & 4 & 1 & 3 & 2 \\ 2 & 4 & -1 & 0 & 1 & 4 \\ 1 & 1 & 2 & -1 & 0 & 3 \\ \hline -2 & 3 & 4 & 1 & 2 & 7 \\ 7 & 3 & 4 & -1 & 2 & 3 \\ 0 & 7 & 5 & 2 & -1 & 4 \end{array}\right]_{6\times 6}$

eg $\left[\begin{array}{cc|cc|cc} 1 & 2 & 4 & -5 & 0 & 7 \\ 2 & 4 & 1 & -1 & 2 & 7 \\ \hline 0 & 7 & 5 & 1 & 2 & 3 \\ 4 & 1 & 0 & 2 & -1 & 4 \\ \hline 1 & 2 & -1 & 4 & 2 & 1 \\ 7 & 2 & 4 & -1 & 2 & 0 \end{array}\right]_{6\times 6}$

Note ① No. of Horizontal lines = No. of Vertical lines

② Diagonal Blocks must be sq. Matrices.
(Not Necessarily of same order).

Block Diagonal Matrix →

$$A = \begin{bmatrix} A_1 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ 0 & 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & A_n \end{bmatrix}_{n \times n}$$

eg $A = \begin{bmatrix} 4 & -2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 2 \end{bmatrix}_{4\times 4} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \Rightarrow |A| = (8)(2) = 16$

eg $A = \begin{bmatrix} 4 & 2 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 3 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 7 & 1 \\ 0 & 0 & 0 & 0 & -5 & 2 \end{bmatrix}_{6\times 6} = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix}$

$|A| = (-6)(-10)(19)$
$= 1140$

eg $A = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 8 & 0 \\ 0 & 0 & 3 & 2 & 7 & 0 \\ 0 & 0 & 0 & 4 & -6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 8 \end{bmatrix}_{6\times 6}$

$A = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix}$

$|A| = (-2)(136)(8)$
$= -2176$

Note:

① Blocks exist only at Diag position.

② Off Diagonal Blocks must be Null Mat

③ $\boxed{|A| = |A_1| \cdot |A_2| \cdot |A_3| \cdots\cdots |A_n|}$

= Product of Det of Diag Blocks.

④ A^{-1} = Mat formed by Inverse of Diag Blocks.

BLOCK U.T.M

① Blocks exist only at upper Diag positions

② All the lower Diag Blocks must be Null Mat

③ $|A| = |A_1| \cdot |A_4| \cdot |A_6|$

= Product of Det of Diag Blocks.

eg $A = \left[\begin{array}{cc|cc|c} 1 & 3 & 1 & 6 & -2 \\ 0 & -1 & 5 & 7 & 0 \\ \hline 0 & 0 & 2 & 4 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ \hline 0 & 0 & 0 & 0 & -3 \end{array}\right] = \begin{pmatrix} A_1 & A_2 & A_3 \\ 0 & A_4 & A_5 \\ 0 & 0 & A_6 \end{pmatrix}$

$|A| = (-1)(-6)(-3) = -18$

BLOCK L.T.M

① Block exist only at lower Diag positions

② All the upper diag Blocks must be Null Mat

③ $|A| = |A_1| \cdot |A_3| \cdot |A_6|$

= Product of Det of Diag Blocks.

eg $A = \left[\begin{array}{cc|cc|c} 1 & 3 & 0 & 0 & 0 \\ 5 & -1 & 0 & 0 & 0 \\ \hline 0 & 7 & 2 & 4 & 0 \\ 3 & 6 & 1 & -1 & 0 \\ \hline -2 & 4 & 1 & 3 & -3 \end{array}\right] = \begin{pmatrix} A_1 & 0 & 0 \\ A_2 & A_3 & 0 \\ A_4 & A_5 & A_6 \end{pmatrix}$

$|A| = (-16)(-6)(-3) = -288$

Q. if $A = \left[\begin{array}{cc|cc} 1 & 2 & 0 & 0 \\ -1 & 5 & 0 & 0 \\ \hline 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & 3 \end{array}\right]$ then $|A| = ?$

PYQ

$= (5+2)(6+0)$

$= 7 \times 6 = 42$ Ans

Qus) Consider the block diagonal matrix A given.

$$A = \begin{bmatrix} 3 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

Tell what is A^{-1}?

a) $\begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & -2 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1/5 \end{bmatrix}$

(b) $\begin{bmatrix} \frac{2}{3} & -\frac{1}{5} & 0 & 0 & 0 \\ -\frac{1}{5} & \frac{3}{5} & 0 & 0 & 0 \\ 0 & 0 & 3 & -2 & 0 \\ 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$

c) $\begin{bmatrix} \frac{2}{5} & -\frac{1}{5} & 0 & 0 & 0 \\ -\frac{1}{5} & \frac{3}{5} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & -\frac{1}{6} & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{5} \end{bmatrix}$

(d) A is invertible

$$A = \left[\begin{array}{cc|cc|c} 3 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ \hline 0 & 0 & 4 & 2 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ \hline 0 & 0 & 0 & 0 & 5 \end{array}\right] = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix}$$

5×5

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow M^{-1} = \frac{1}{|M|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A_1^{-1} = \frac{1}{5}\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}, \quad A_2^{-1} = \frac{1}{12}\begin{bmatrix} 3 & -2 \\ 0 & 4 \end{bmatrix}$$

$$A_3^{-1} = \frac{1}{5}$$

$$A^{-1} = \begin{bmatrix} A_1^{-1} & 0 & 0 \\ 0 & A_2^{-1} & 0 \\ 0 & 0 & A_3^{-1} \end{bmatrix} = C$$

Q5) Given a 3×3 block matrix

$$A = \begin{bmatrix} A_{11} & A_{12} & 0 \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix} = \text{Block U.T.M} \Rightarrow A^{-1} = \begin{pmatrix} A_{11}^{-1} & ? & ? \\ 0 & A_{22}^{-1} & ? \\ 0 & 0 & A_{33}^{-1} \end{pmatrix}$$

where A_{11}, A_{22}, A_{33} are square matrices. If A is upper triangular & invertible, which of the following must be correct

(a) ✓ A_{11}, A_{22} & A_{33} must be invertible

(b) ✗ A_{11}, A_{22}, A_{33} & A_{23} must be invertible

(c) ✗ $\det(A_{12}) = \det(A_{23}) = 0$

(d) $A_{12}, A_{23}, A_{11}, A_{22}, A_{33}$ must be invertible

$$|A| \neq 0 \Rightarrow |A_{11}| \neq 0, |A_{22}| \neq 0, |A_{33}| \neq 0$$

N. Cond^n for any Block U.T.M / Block LTM / Block Diag Mat to be invertible is "Diag Blocks must be invertible"

Qus) Given the block matrix A as

MSQ

$$A = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}$$

where A_{11} is a $k \times k$ matrix & A_{22} is an $(n-k) \times (n-k)$ matrix, which of the following statements is INCORRECT?

(a) If A_{11} is invertible, then A is invertible (F) ∵ it also depends on Invertibility of A_{22}

(b) $\det(A) = \det(A_{11}) \times \det(A_{22})$ (T)

c) If A is positive definite, then A_{22} must be positive definite independent of the choice of A_{11} (F)

d) $A^{-1} = \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & A_{22}^{-1} \end{bmatrix}$, if A_{11} & A_{22} are invertible (T)

For A to be +ve definite, Both A_{11} & A_{22} must be +ve definite.

Q: If $A = \left[\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & 2 & 0 \\ \hline 0 & 0 & 1 \end{array}\right]$, $B = \left[\begin{array}{ccc|c} -1 & 1 & 2 & 5 \\ 3 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & -1 \end{array}\right]$ then evaluate AB using Indicated Partitioning.

Sol:

$$= \left[\begin{array}{c|c} P_{22} & Q_{21} \\ \hline O_{12} & R_{11} \end{array}\right] \qquad = \left[\begin{array}{c|c} S_{23} & T_{21} \\ \hline O_{13} & U_{11} \end{array}\right]$$

$$AB = \begin{bmatrix} P & Q \\ O_{12} & R \end{bmatrix}_{3\times 3} \cdot \begin{bmatrix} S & T \\ O_{13} & U \end{bmatrix}_{3\times 4} = \begin{bmatrix} PS & PT+QU \\ O_{13} & RU \end{bmatrix}_{3\times 4} = \left[\begin{array}{ccc|c} -4 & 1 & 1 & 2 \\ 6 & 0 & 2 & 0 \\ \hline 0 & 0 & 0 & -1 \end{array}\right]_{3\times 4}$$

Ans.

$$PS = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 & 2 \\ 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -4 & 1 & 1 \\ 6 & 0 & 2 \end{bmatrix}_{2\times 3}$$

$$RU = [1][-1] = [-1]_{1\times 1}$$

$$PT = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}_{2\times 1}$$

$$QU = \begin{bmatrix} 3 \\ 0 \end{bmatrix} [-1] = \begin{bmatrix} -3 \\ 0 \end{bmatrix}_{2\times 1} \Rightarrow PT + QU = \begin{bmatrix} 5 \\ 0 \end{bmatrix} + \begin{bmatrix} -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Note: $O_{12} S_{23} + R_{11} O_{13}$

$$= O_{13} + O_{13} = O_{13}$$

PROJECTION MATRIX

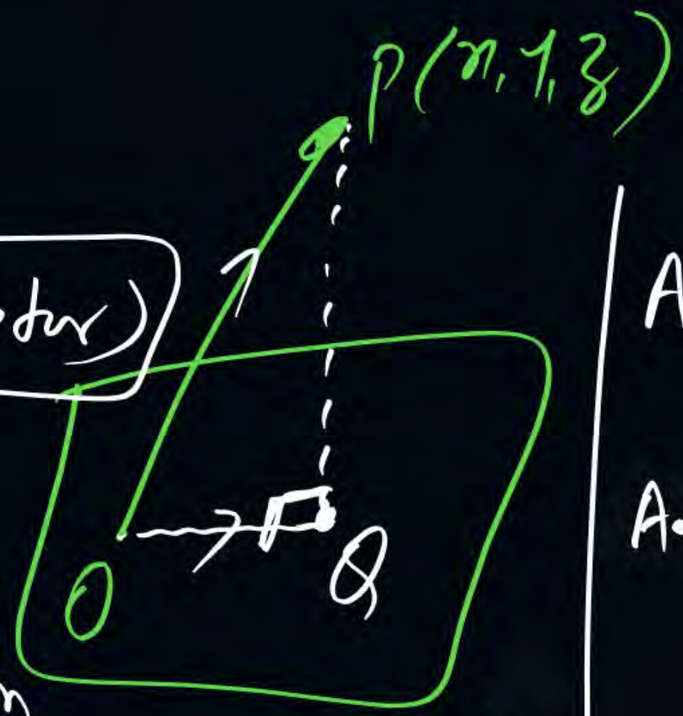
① $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|} \Rightarrow \vec{a} = \boxed{|\vec{a}|\,\hat{a}}$

$\boxed{\text{Any vector} = (\text{Magnitude})\,(\text{unit vector})}$

② $A_{n\times 1} = \begin{bmatrix} - \\ - \\ - \\ \vdots \\ - \end{bmatrix}_{n\times 1}$, $A^T = [\quad]_{1\times n}$

$(A^TA)_{1\times 1} = \text{Scalar}$, $(AA^T)_{n\times n} = \text{Mat (Symm)}$ $\vec{p} = ??$

$\|A\| = \sqrt{A^TA} \Rightarrow \|A\|^2 = (A^TA)_{1\times 1} = \text{Scalar}$



$A = [\quad]_{n\times 1}$, $B = [\quad]_{n\times 1}$

$A\cdot B = (A^TB)_{1\times 1}$

General Points:

① $\because \hat{a} = \frac{\vec{a}}{|\vec{a}|} \Rightarrow \boxed{\vec{a} = |\vec{a}| . \hat{a}}$

Any vector = (it's Magnitude) × (unit vector in it's direction)

② Let $A_{n\times 1}$ then $A^T_{1\times n}$

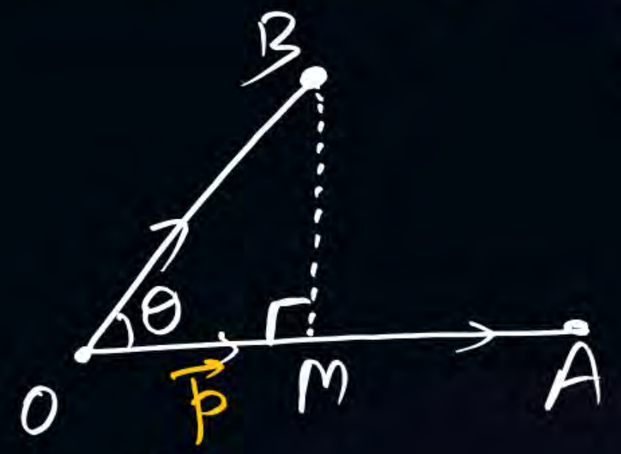
- $(A^T A)_{1\times 1}$ = symmetric & scalar
- $(A A^T)_{n\times n}$ = symmetric Mat

③ Let $A_{n\times 1}$ & $B_{n\times 1}$

- $(A^T B)_{1\times 1}$ = Scalar Quantity.
- $A \cdot B = A^T B$ = scalar Quantity
- $\|A\| = \sqrt{A^T A} \Rightarrow \|A\|^2 = A^T A$

④ if $A \cdot B = 0$ then A & B are called orthogonal vectors.

PROJECTION of VECTOR → Let B be any given vector & we want to find it's Projection vector onto the vector A then



$\vec{OM}$ = projection vector = ? = $\vec{p}$ (let)

$$\frac{OM}{OB} = \cos\theta \Rightarrow OM = |\vec{B}|\cos\theta$$

So $\vec{OM} = |\vec{OM}|\,\hat{OM}$

$$= |\vec{B}|\cos\theta\left(\frac{\vec{A}}{|\vec{A}|}\right)$$

$$= |B|\left(\frac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|}\right)\frac{\vec{A}}{|\vec{A}|}$$

$$\vec{OM} = \left(\frac{A\cdot B}{|A|}\right)\frac{\vec{A}}{|\vec{A}|} = \left(\frac{A\cdot B}{|\vec{A}|^2}\right)\vec{A} = \left(\frac{A^TB}{A^TA}\right)_{1\times1} A$$

i.e. $\boxed{\vec{p} = \left(\frac{A^TB}{A^TA}\right)_{1\times1}\vec{A}} \Rightarrow \vec{p} = \lambda\vec{A}$; λ = Scalar.

i.e. $\vec{p}$ & $\vec{A}$ are in same direction (Collinear Vectors)

PODCAST (ANALYSIS) $\longrightarrow$ $\vec{p} = \left(\frac{A^T B}{A^T A}\right)_{1\times 1} \vec{A}$

$$= \vec{A}\left(\frac{A^T B}{A^T A}\right)_{1\times 1} \quad \text{(using common sense)}$$

$$\vec{p} = \left(\frac{A A^T}{A^T A}\right)_{n\times n} \vec{B} \quad \text{(using Associative Law)}$$

$$\boxed{\vec{p} = P\vec{B}}, \text{ where } \boxed{P = \frac{A A^T}{A^T A}} = (\text{Projection Mat})_{n\times n}$$

PROJECTION MATRIX (along a line) → we want to find projection of $\vec{B}$ along a line (in vector form) A then

$$P = \left(\frac{AA^T}{A^TA}\right)_{n\times n} \quad \& \quad \boxed{\vec{p} = PB}$$

Q: Find the projection of $\begin{bmatrix}1\\1\end{bmatrix}$ onto the vector $\begin{bmatrix}-2\\1\end{bmatrix}$

Sol: (M-I) $B=\begin{bmatrix}1\\1\end{bmatrix}$, $A=\begin{bmatrix}-2\\1\end{bmatrix}$

$$A^TB=\begin{bmatrix}-2 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}-1\end{bmatrix}_{1\times 1}$$

$$A^TA=\|A\|^2=(-2)^2+1^2=5$$

So $\vec{p}=\left(\dfrac{A^TB}{A^TA}\right)\cdot\vec{A}=\dfrac{-1}{5}\cdot\begin{bmatrix}-2\\1\end{bmatrix}$

P. vector $=\begin{bmatrix}2/5\\-1/5\end{bmatrix}$

(M-II) $A^TA=5$ (already Calculated)

$$AA^T=\begin{bmatrix}-2\\1\end{bmatrix}\begin{bmatrix}-2 & 1\end{bmatrix}=\begin{bmatrix}4 & -2\\-2 & 1\end{bmatrix}=\text{Symm Mat}$$

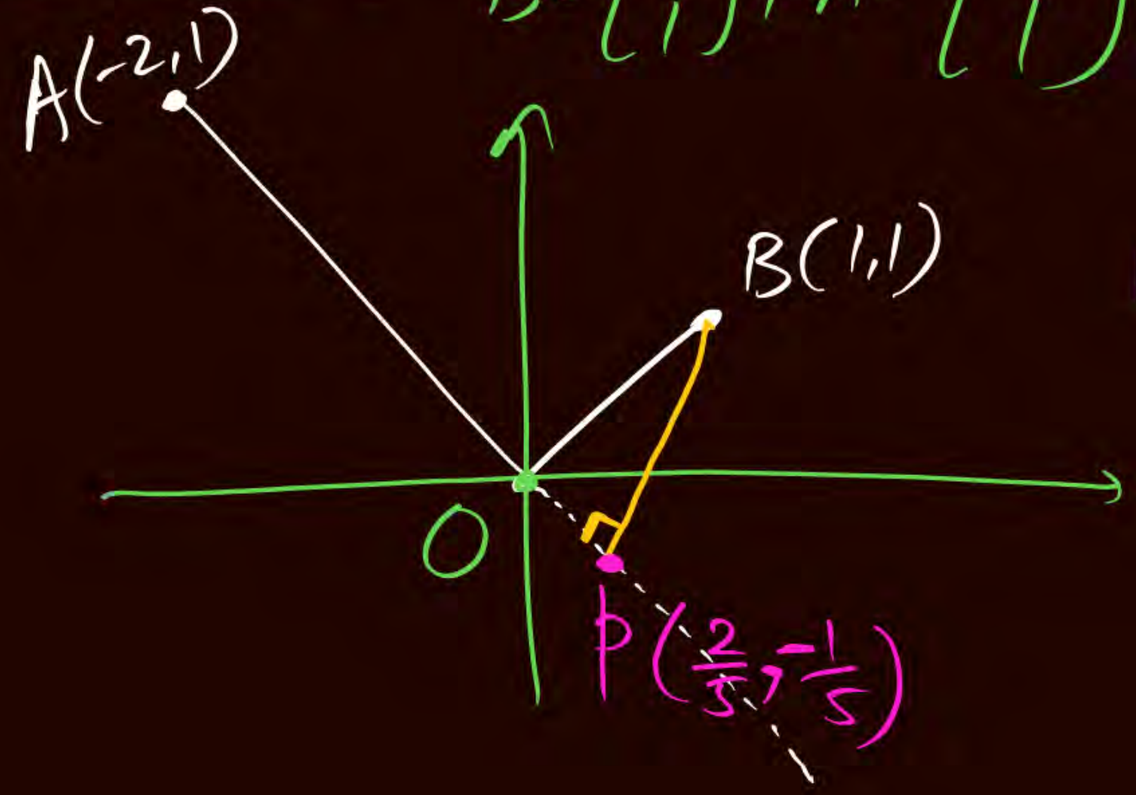
So Projection Mat $P=\dfrac{AA^T}{A^TA}=\dfrac{1}{5}\begin{bmatrix}4 & -2\\-2 & 1\end{bmatrix}$

& Projection vector $\vec{p}=PB=\dfrac{1}{5}\begin{bmatrix}4 & -2\\-2 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}$

$$=\begin{bmatrix}2/5\\-1/5\end{bmatrix}$$

ANALYSIS :-

$B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, A = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \vec{p} = \begin{bmatrix} 2/5 \\ -1/5 \end{bmatrix} = \frac{-1}{5}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \frac{-1}{5}A$



Here $P = \frac{1}{5}\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \neq$ Modal Mat

$Tr(P) = \frac{4}{5} + \frac{1}{5} = 1$

$|P| = 0$, E Values $\lambda = 0$ & 1

$P =$ Symm as well as Idempotent.

$P =$ Diagonalizable ($\because$ both E Values are diff so Corresponding E Vectors are LI)

THANK - YOU

DATA SCIENCE & ARTIFICIAL INTELLIGENCE

Linear Algebra

Lecture No. 15

By- Dr. Puneet Sharma Sir

Recap of previous lecture

Topic

1. Reflection Matrix
2. Rotation Matrix
3. Partition Matrix
4. Basics of projection Matrix

Topics to be Covered

Topic

PROJECTION MATRIX (In Detail)

→ Projection of vector ONTO the line

→ Projection of vector ONTO the SUBSPACE SPANNED By set of Vectors.

General Points:

① $\because \hat{a} = \frac{\vec{a}}{|\vec{a}|} \Rightarrow \boxed{\vec{a} = |\vec{a}| . \hat{a}}$

Any vector = (it's Magnitude) × (unit vector in it's direction)

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PROJECTION of VECTOR

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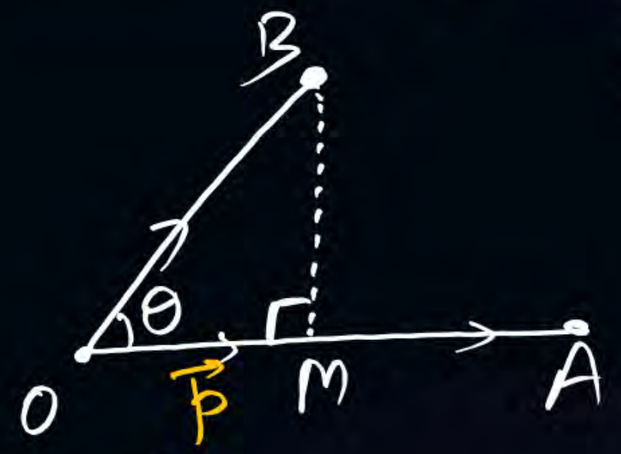

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So $\vec{OM} = |\vec{OM}|\,\hat{OM}$

$$= |\vec{B}|\cos\theta\left(\frac{\vec{A}}{|\vec{A}|}\right)$$

$$= |B|\left(\frac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|}\right)\frac{\vec{A}}{|\vec{A}|}$$

$$\vec{OM} = \left(\frac{A\cdot B}{|A|}\right)\frac{\vec{A}}{|\vec{A}|} = \left(\frac{A\cdot B}{|\vec{A}|^2}\right)\vec{A} = \left(\frac{A^TB}{A^TA}\right)_{1\times1} A$$

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PODCAST (ANALYSIS) $\longrightarrow$ $\vec{p} = \left(\frac{A^T B}{A^T A}\right)_{1\times 1} \vec{A}$

$$= \vec{A}\left(\frac{A^T B}{A^T A}\right)_{1\times 1} \quad \text{(using common sense)}$$

$$\vec{p} = \left(\frac{A A^T}{A^T A}\right)_{n\times n} \vec{B} \quad \text{(using Associative Law)}$$

$$\boxed{\vec{p} = P\vec{B}}, \text{ where } \boxed{P = \frac{A A^T}{A^T A}} = (\text{Projection Mat})_{n\times n}$$

PROJECTION MATRIX (along a line) → we want to find projection of $\vec{B}$ along a line (in vector form) A then

$$P = \left(\frac{AA^T}{A^TA}\right)_{n\times n} \quad \& \quad \boxed{\vec{p} = PB}$$

Q. Find the projection of $\begin{bmatrix}1\\1\end{bmatrix}$ onto the vector $\begin{bmatrix}-2\\1\end{bmatrix}$

Sol: (M-I) $B=\begin{bmatrix}1\\1\end{bmatrix}$, $A=\begin{bmatrix}-2\\1\end{bmatrix}$

$$A^TB=\begin{bmatrix}-2 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}-1\end{bmatrix}_{1\times 1}$$

$$A^TA=\|A\|^2=(-2)^2+1^2=5$$

So $\vec{p}=\left(\dfrac{A^TB}{A^TA}\right)\cdot\vec{A}=\dfrac{-1}{5}\cdot\begin{bmatrix}-2\\1\end{bmatrix}$

P. vector $=\begin{bmatrix}2/5\\-1/5\end{bmatrix}$

(M-II) $A^TA=5$ (already Calculated)

$$AA^T=\begin{bmatrix}-2\\1\end{bmatrix}\begin{bmatrix}-2 & 1\end{bmatrix}=\begin{bmatrix}4 & -2\\-2 & 1\end{bmatrix}=\text{Symm Mat}$$

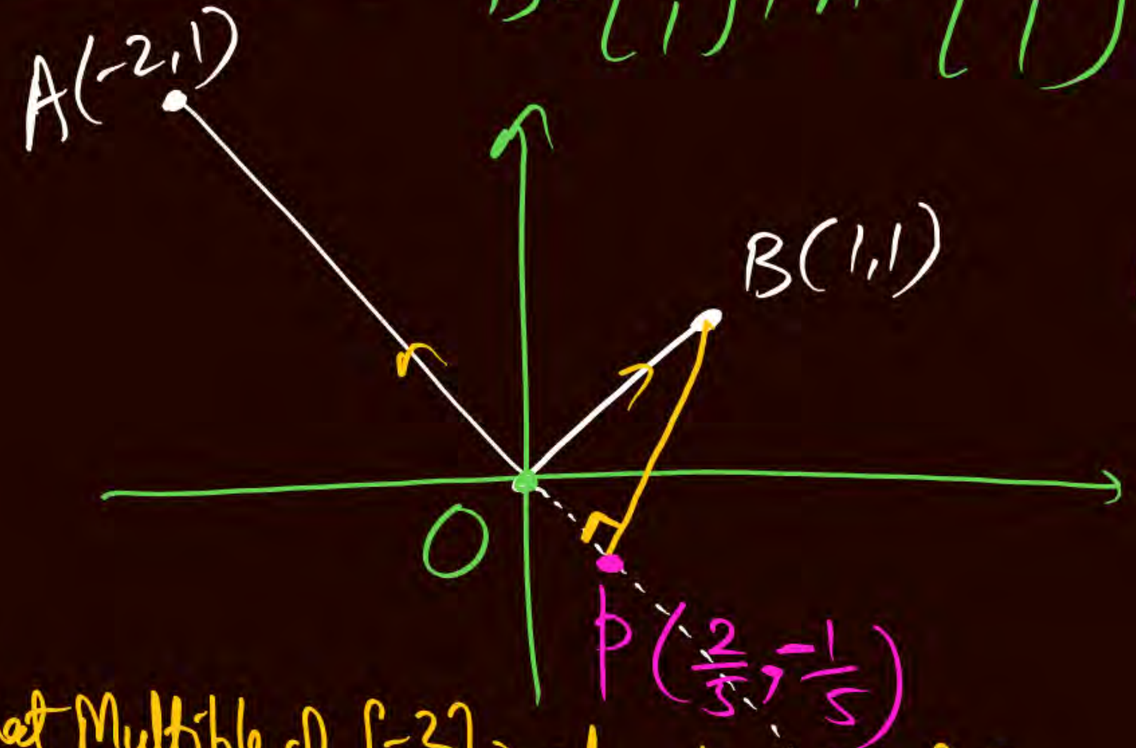
So Projection Mat $P=\dfrac{AA^T}{A^TA}=\dfrac{1}{5}\begin{bmatrix}4 & -2\\-2 & 1\end{bmatrix}$

& Projection vector $\vec{p}=PB=\dfrac{1}{5}\begin{bmatrix}4 & -2\\-2 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}$

$$=\begin{bmatrix}2/5\\-1/5\end{bmatrix}$$

ANALYSIS :-

$B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $A = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, $\vec{p} = \begin{bmatrix} 2/5 \\ -1/5 \end{bmatrix} = \frac{-1}{5}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \frac{-1}{5}A$



Here $P = \frac{1}{5}\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \neq$ Modal Mat

$Tr(P) = \frac{4}{5} + \frac{1}{5} = 1$

$|P| = 0$, E Values $\lambda = 0$ & 1

$P =$ Symm as well as Idempotent.

$P =$ Diagonalizable ($\because$ both E Values are diff so Corresponding E Vectors are LI)

(ii) what Multiple of $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is closest to the $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$?

(a) 5 (c) 1/5

(b) 3 (d) -1/5

$\vec{p} = \frac{-1}{5}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \frac{-1}{5}\vec{A}$

Q: Find a Matrix that projects every point in 2D plane onto the line $\boxed{x+2y=0}$
Also Find the projection of $\begin{bmatrix}1\\1\end{bmatrix}$ onto that line.

Sol: $B=\begin{bmatrix}1\\1\end{bmatrix}$, Given line is $\boxed{x+2y=0}$

So In vector form we have $A=\begin{bmatrix}-2\\1\end{bmatrix}$

So $P=\dfrac{AA^T}{A^TA}=\dfrac{1}{5}\begin{bmatrix}4 & -2\\-2 & 1\end{bmatrix}$ (already Calculated)

$$\vec{p}=PB=\frac{1}{5}\begin{bmatrix}4 & -2\\-2 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}2/5\\-1/5\end{bmatrix}$$

Q. The projection of $B=\begin{bmatrix}1\\2\\3\end{bmatrix}$ onto the line $A=\begin{bmatrix}1\\1\\1\end{bmatrix}$ will be ?

Sol: $A^TA=\begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}=[3]_{1\times1}=3$

$$AA^T=\begin{bmatrix}1\\1\\1\end{bmatrix}\begin{bmatrix}1 & 1 & 1\end{bmatrix}=\begin{bmatrix}1 & 1 & 1\\1 & 1 & 1\\1 & 1 & 1\end{bmatrix}_{3\times3}$$

So $P=\dfrac{AA^T}{A^TA}=\dfrac{1}{3}\begin{bmatrix}1 & 1 & 1\\1 & 1 & 1\\1 & 1 & 1\end{bmatrix}$

Hence $\hat{p}=PB=\dfrac{1}{3}\begin{bmatrix}1 & 1 & 1\\1 & 1 & 1\\1 & 1 & 1\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}=\dfrac{1}{3}\begin{bmatrix}6\\6\\6\end{bmatrix}=\begin{bmatrix}2\\2\\2\end{bmatrix}=2\begin{bmatrix}1\\1\\1\end{bmatrix}=2A$

(ii) what Multiple of $\begin{bmatrix}1\\1\\1\end{bmatrix}$ is closest to $\begin{bmatrix}1\\2\\3\end{bmatrix}$

(a) 3 (b) $\frac{1}{3}$ (c) 2 (d) $1/2$

Note: Trace $(P)=1$, $|P|=0$, $\rho(P)=1$

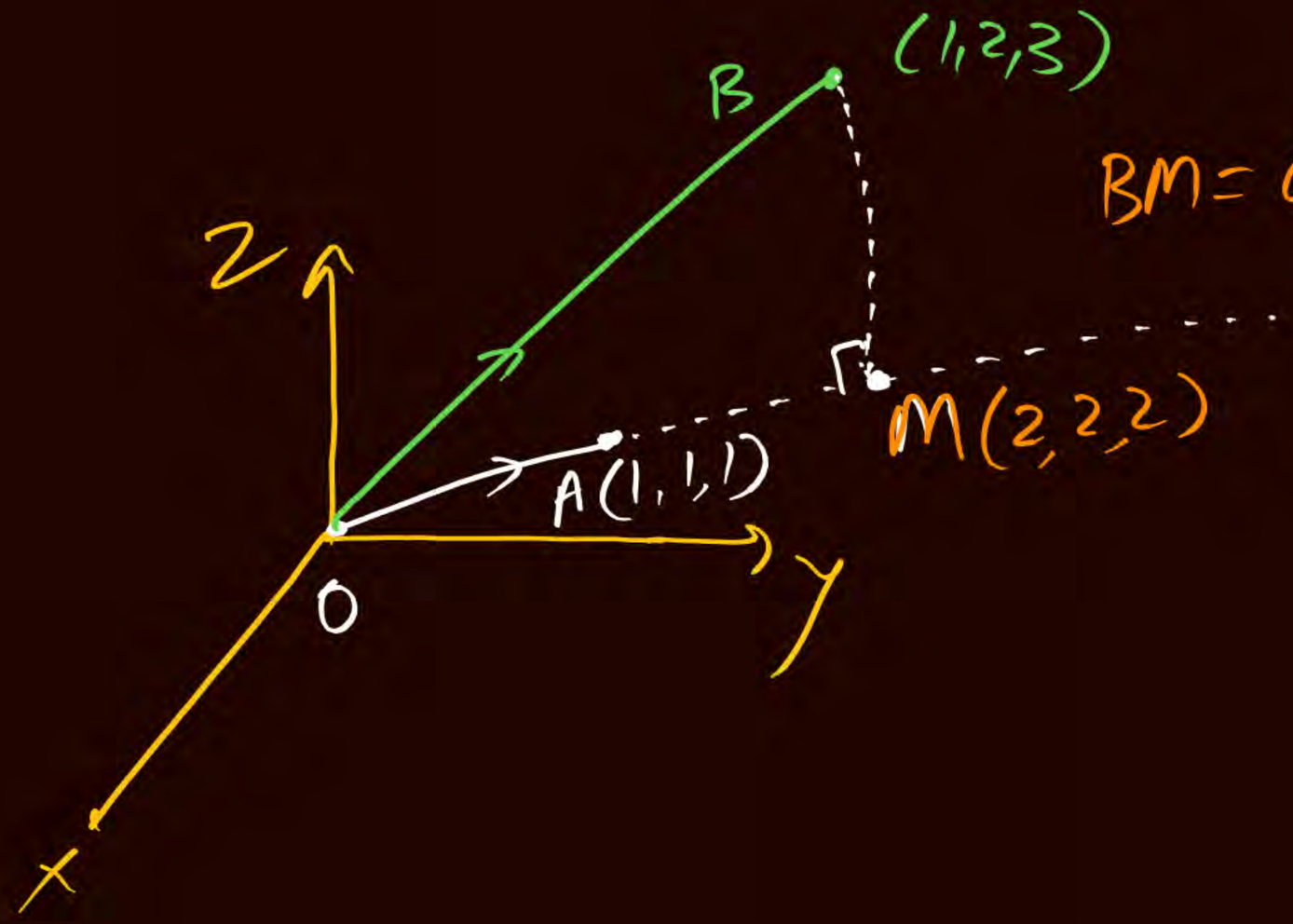
$\lambda=0,0,1$, $gmd(\lambda=0)=$ two

ie $\exists$ three LIE Vectors for P

Hence it is Diagonalizable

$\because P^2=P$ & $P^T=P$

Hence P is Idempotent & Symm.



BM = Closest Distance

M(2,2,2)

$\vec{B} = 2\vec{A}$

Q: what Multiple of (1 1 1) is closest to the point (2 4 4) ?

(a) 3

(b) 10

(c) 10/3

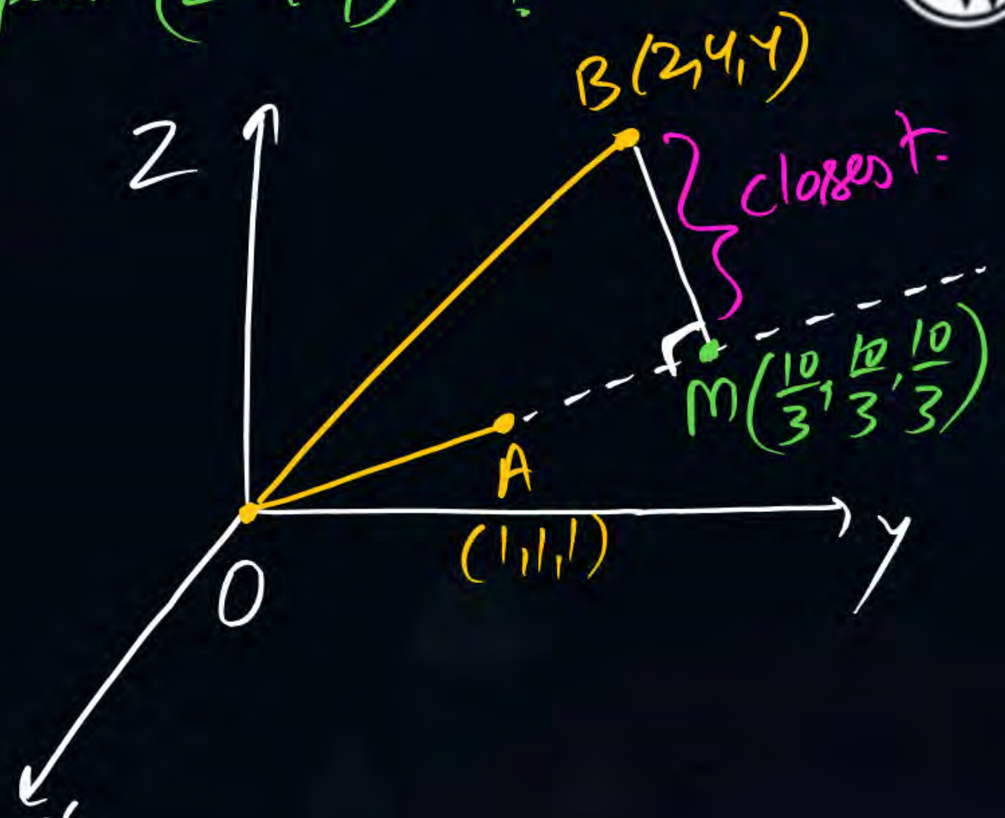
(d) 3/10

Here $A = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ & $B = \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}$

So $P = \dfrac{AA^T}{A^TA} = \dfrac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$\bar{p} = PB = \dfrac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}$

$= \dfrac{1}{3}\begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix} = \dfrac{10}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \dfrac{10}{3} \cdot A$



Q: Find the projection Matrix that projects every point in R^3 onto the line of Intersection of the planes $x+y+z=0$ & $x-z=0$.
& Hence find the projection of $B(1\,2\,3)$?

Sol: Given plane are $\left.\begin{array}{l} x+y+z=0 \\ x-z=0 \end{array}\right\} \Rightarrow \begin{array}{c} x+y+x=0 \\ \text{or} \\ 2x+y=0 \end{array}$

ie line of intersection of these two planes is given as,

$$2x+y+0z=0$$

ie in vector form it is $A=\begin{bmatrix}1\\-2\\0\end{bmatrix}$

$$A^TA=\begin{bmatrix}1 & -2 & 0\end{bmatrix}\begin{bmatrix}1\\-2\\0\end{bmatrix}=\underset{1\times 1}{\begin{bmatrix}5\end{bmatrix}}=5$$

$$AA^T=\begin{bmatrix}1\\-2\\0\end{bmatrix}\begin{bmatrix}1 & -2 & 0\end{bmatrix}=\underset{3\times 3}{\begin{bmatrix}1 & -2 & 0\\-2 & 4 & 0\\0 & 0 & 0\end{bmatrix}}$$

$$P=\frac{AA^T}{A^TA}=\frac{1}{5}\begin{bmatrix}1 & -2 & 0\\-2 & 4 & 0\\0 & 0 & 0\end{bmatrix} \text{ Req. P.Mat.}$$

(ii) $B=\begin{bmatrix}1\\2\\3\end{bmatrix}$ so $\vec{p}=PB$

$$\vec{p}=\frac{1}{5}\begin{bmatrix}1 & -2 & 0\\-2 & 4 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}=\begin{bmatrix}-3/5\\6/5\\0\end{bmatrix}=\frac{-3}{5}\begin{bmatrix}1\\-2\\0\end{bmatrix}$$

orthogonal Complement of B → (orthogonal complement vector)

$\vec{p}$ = projection vector of B onto A & P = Projection Mat

$\vec{q}$ = orthogonal complement of B & Q = Proj. Mat of orthogonal complement.

By vector Addition Law,

$$\vec{p}+\vec{q}=\overrightarrow{OB}$$

ie $\boxed{\vec{p}+\vec{q}=\vec{B}}$

$$PB+QB=B$$

$$(P+Q)B=B$$

$$\Rightarrow P+Q=I.$$

Again, $p+q=B$

$$PB+QB=B$$

$$(P+Q)B=B$$

$$\Rightarrow \boxed{P+Q=I}$$

Q: Find the Projection Matrix P onto the line through $A=\begin{bmatrix}1\\3\end{bmatrix}$ & hence Find the projection of $B=\begin{bmatrix}1\\1\end{bmatrix}$ onto A ?

(ii) Also Find the Matrix Q that projects onto the line $\perp^r$ to A & hence Find orthogonal Complement of B onto A

Sol (i) $A=\begin{bmatrix}1\\3\end{bmatrix}$, $A^T=\begin{bmatrix}1 & 3\end{bmatrix}$, $A^TA=10$

$$AA^T=\begin{bmatrix}1\\3\end{bmatrix}\begin{bmatrix}1 & 3\end{bmatrix}=\begin{bmatrix}1 & 3\\3 & 9\end{bmatrix}_{2\times 2}$$

$$P=\frac{AA^T}{A^TA}=\frac{1}{10}\begin{bmatrix}1 & 3\\3 & 9\end{bmatrix} \;\&\; B=\begin{bmatrix}1\\1\end{bmatrix}$$

So $\vec{p}=PB=\frac{1}{10}\begin{bmatrix}1 & 3\\3 & 9\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\frac{1}{10}\begin{bmatrix}4\\12\end{bmatrix}=\frac{4}{10}\begin{bmatrix}1\\3\end{bmatrix}=\frac{4}{10}A$

(ii) W.K.that $\boxed{P+Q=I} \Rightarrow Q=I-P$

$$Q=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}-\frac{1}{10}\begin{bmatrix}1 & 3\\3 & 9\end{bmatrix}=\begin{bmatrix}9/10 & -3/10\\-3/10 & 1/10\end{bmatrix}$$

& orthogonal Complement of $\vec{p}$ is $\vec{q}$ which is given as $\boxed{\vec{p}+\vec{q}=\vec{B}}$

$$\vec{q}=\vec{B}-\vec{p}=\begin{bmatrix}1\\1\end{bmatrix}-\begin{bmatrix}4/10\\12/10\end{bmatrix}=\begin{bmatrix}6/10\\-2/10\end{bmatrix}$$

Given a vector $V = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and a subspace W spanned by $W = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, what is the projection matrix P that projects any vector onto W, and what is the projection of V onto W?

(a) $P = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}$, Projection of $V = \begin{pmatrix} 2.5 \\ 2.5 \end{pmatrix}$

(b) $P = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}$, Projection of $V = \begin{pmatrix} 1.5 \\ 1.5 \end{pmatrix}$

(c) $P = \begin{pmatrix} 0.5 & 2.5 \\ 1.5 & 1.5 \end{pmatrix}$, Projection of $V = \begin{pmatrix} 1.5 \\ 1.5 \end{pmatrix}$

(d) $P = \begin{pmatrix} 0.5 & 0.25 \\ 1.5 & 1.25 \end{pmatrix}$, Projection of $V = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$

$\vec{B} = \vec{V} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad A = W = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$P = \frac{AA^T}{A^TA} = \frac{WW^T}{W^TW} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$\vec{P} = PB = PV = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 5 \\ 5 \end{bmatrix}$

11 In a 2-dimensional space R^2, consider the subspace W spanned by the vector $V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Let P be the projection matrix onto W. Whcih of the following vectors is the image of $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ under projection P?

(a) $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ (b) $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$ (c) $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ (d) $\begin{bmatrix} 7/2 \\ 7/2 \end{bmatrix}$

$B = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $A = V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow P = \frac{AA^T}{A^TA} = \frac{V_1V_1^T}{V_1^TV_1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$\therefore$ Hence Image of B = Projection of B onto A

$\vec{p} = PB = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 7/2 \\ 7/2 \end{bmatrix}$

Which of the following matrices represents a projection onto the line L with direction vector $d = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$?

(a) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

(b) $\begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{4}{5} \end{bmatrix}$

(c) $\begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{bmatrix}$

(d) $\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$

$A = d = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$P = \frac{AA^T}{A^TA} = \frac{dd^T}{d^Td} = ? = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

PROJECTION MATRIX (along a subspace spanned by set of vectors) →

Consider three vectors are given (say X_1, X_2, X_3) & we want to find Projection of vector $\vec{B}$ ONTO the subspace $\boxed{\text{spanned by } X_1, X_2, X_3}$?

Consider $A = [X_1\ X_2\ X_3]$ then, to find Projection vector of B onto the subspace spanned by columns of A, Projection Mat & Projection vector are given as;

$$\boxed{P = A(A^TA)^{-1}A^T} \quad \& \quad \boxed{\vec{P} = P\vec{B}}$$

Verification: if A contains of one vector then P = ?

$$A = \begin{bmatrix} \equiv \\ \equiv \\ \vdots \\ \equiv \end{bmatrix}_{n\times 1} \Rightarrow (A^TA)_{1\times 1} = \text{scalar} \text{ so } P = A(A^TA)^{-1}_{1\times 1}A^T = \frac{(AA^T)_{n\times n}}{(A^TA)_{1\times 1}} = \left(\frac{AA^T}{A^TA}\right)_{n\times n}$$

Q: The projection of $B = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ onto the column space of $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 0 & 0 \end{bmatrix}$ will be?

Sol: $A^TA = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 5 & 13 \end{bmatrix}_{2\times 2}$

$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow M^{-1} = \frac{1}{|M|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$(A^TA)^{-1} = \frac{1}{(1)} \begin{bmatrix} 13 & -5 \\ -5 & 2 \end{bmatrix}$

Now Projection Matrix $\boxed{P = A(A^TA)^{-1}A^T}$

$P = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 13 & -5 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 0 \end{bmatrix}$

$= \ldots\ldots\ldots\ldots = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

So Projection vector of B onto the subspace spanned by columns of A is

$\vec{p} = PB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix}$

Note: ① $Tr(P) = 2$, ② $|P| = 0$, ③ $\rho(P) = 2$

④ $\lambda = 0, 1, 1$, ⑤ Idempotent & symm also

⑥ GM of 1 = two & GM of 0 = one

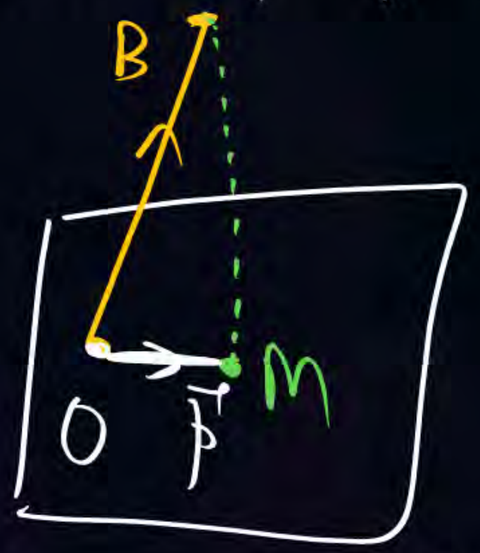
⑦ P is Diagonalizable

($\because$ No. of LI Evector = order of P)

PODCAST → (ANALYSIS)

(1) $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 0 & 0 \end{bmatrix} = [X_1 \; X_2]$, $B = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, $\vec{p} = \begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix}$

$\because X_1$ & X_2 are L.I so they are the Basis vectors for 2D space i.e X_1 & X_2 will generate (spans) XY plane



2D Plane

(2) Consider $\boxed{AX = B}$ — (1)

$$\begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 0 & 0 \end{bmatrix} X = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

i.e $[A \vdots B] = \left[\begin{array}{cc:c} 1 & 2 & 4 \\ 1 & 3 & 5 \\ 0 & 0 & 6 \end{array}\right]$ $\rho(A) = 2$, $\rho(A \vdots B) = 3$

i.e (1) is Inconsistent

Now we take the projection of B onto A

then $AX = B \Longrightarrow \boxed{AX = \vec{p}}$ — (2)

$[A \vdots \vec{p}] = \left[\begin{array}{cc:c} 1 & 2 & 4 \\ 1 & 3 & 5 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \rho(A) = 2 = \rho(A \vdots \vec{p})$

i.e (2) is Consistent

Heart Beat

$x = 104$

$x = 92$

$x = 116$

$x = 73$

$x = 144$

$x = 132$

$x = 75$

$\xrightarrow{\text{Project (Medicine)}}$ $\boxed{x = 98}$

$\downarrow$

$x =$ Consistent

$x =$ Inconsistent

Q: If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ then Find projection of B onto A ?

Also Find the orthogonal Vector (E) onto the Columns of A.

3×2

OR

If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$ & $B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ then split B into $\vec{p} + \vec{q}$ where $\vec{p}$ is in the Column space and $\vec{q}$ will be $\perp^r$ to that space.

Sol:

$$A^T A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$(A^T A)^{-1} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$P = A(A^T A)^{-1} A^T$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

So Projection of B onto A is $\vec{p} = PB$

$$\vec{p} = \frac{1}{3} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \\ 2/3 \end{bmatrix}.$$

(ii) $\vec{p} = \begin{bmatrix} 1/3 \\ 1/3 \\ 2/3 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $P = \frac{1}{3}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ $\quad Tr(P) = \frac{6}{3} = 2$

$\because \vec{p} + \vec{q} = \vec{B} \Rightarrow \vec{q} = \vec{B} - \vec{p}$

$\vec{E} = \vec{q} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1/3 \\ 1/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 2/3 \\ -2/3 \end{bmatrix}$

(iii) Also find Projection Mat for orthogonal Complement vector?

w.k.that $P + Q = I$

i.e. $Q = I - P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

$= \begin{bmatrix} HW \end{bmatrix}$ Ans

Note: Check the Nature of $P \cdot Q = ?$

$PQ = P(I-P) = PI - P^2 = P - (P) = 0$

PROPERTIES of Projection Mat →

① Trace of Projection Mat onto the line = 1

② Trace of Projection Mat onto the subspace is not Necessarily 1

③ Determinant of Projection Mat is always Zero except identity Mat

dosen't matter whether onto the line or onto the subspace.

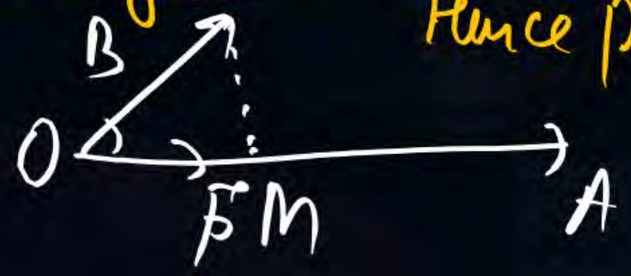
④ Projection Mat is also an Idempotent Matrix

if we take the projection of $\vec{p}$ onto A again then it will be vector itself

Hence P is Idempotent.

Let $P(B) = \vec{p}$ then $P(\vec{p}) = \vec{p} \Rightarrow P(PB) = PB$

$\Rightarrow P^2B = PB \Rightarrow P^2 = P$



is $P^2 = P$.

(5) $\because P$ is Idempotent (ie $P^2 = P$) $\Longrightarrow$ E.Values of P are 0 or 1 or Both.

(6) Projection Mat is Symm. Mat i.e $\boxed{P^T = P}$

Proof:

on Next Slide

(5) $\because$ P is an Idempotent Mat

i.e. $P^2 = P$

$\Rightarrow \lambda^2 = \lambda \Rightarrow \lambda(\lambda - 1) = 0$

or $\boxed{\lambda = 0 \text{ or } 1}$

i.e. E Values of Projection Mat are either 0 or 1 or Both

(6) Projection Mat is always symm

Proof: w.k.that $P = A(A^TA)^{-1}A^T$.

Now $P^T = \left(A\underbrace{(A^TA)^{-1}}A^T\right)^T$

$= \left(A \cdot A^{-1}(A^T)^{-1} \cdot A^T\right)^T$

$= \left(A \cdot A^{-1}(A^{-1})^T A^T\right)^T$

$= (A^T)^T((A^{-1})^T)^T \cdot (A^{-1})^T A^T$

$= A \cdot (A^{-1})(A^{-1})^T \cdot A^T$

$= A \cdot A^{-1}(A^T)^{-1} \cdot A^T$

$= A\underbrace{(A^TA)^{-1}} \cdot A^T$

$P^T = P \Rightarrow P$ is symmetric

⑦ If $A_{n\times n}$ (i.e A is a sq. Mat) s.t A^{-1} exist

$P = I$

$\vec{p} = B$ itself.

Proof: $P = A(A^TA)^{-1}A^T = A(A^{-1}(A^T)^{-1})A^T = (AA^{-1})((A^T)^{-1}A^T) = I \cdot I = I$

$\vec{p} = PB = IB = B$ Hence proved.

⑧ If B is orthogonal to the columns of A then

$P =$ No idea

$\vec{p} = \vec{0}$

i.e B is $\perp^r$ to the Column space of A Hence P. Vector = Null Vector.

let $A = [X_1 X_2 X_3]$ where $X_1 \cdot B = X_2 \cdot B = X_3 \cdot B = 0$

or $X_1^T B = X_2^T B = X_3^T B = 0$

i.e $[X_1 X_2 X_3] \cdot B = A \cdot B = 0 \Rightarrow A^T B = 0$

$\vec{p} = PB = (A(A^TA)^{-1}A^T)B$

$= (A(A^TA)^{-1})(A^TB) = A(A^TA)^{-1}0$

$= 0$.

(9) If B is L.D on Columns of A then

P = No idea (?)

$\vec{p}$ = B itself.

Proof: let $A = [x_1\ x_2\ x_3]$ & B is LD on x_1, x_2, x_3

i.e $k_1x_1 + k_2x_2 + k_3x_3 = B$

$$[x_1\ x_2\ x_3]\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = B$$

$$\boxed{AX = B} \quad \text{—— (1)}$$

So $\vec{p} = PB = \left(A(A^TA)^{-1}A^T\right)B$

$= \left(A(A^TA)^{-1}A^T\right)AX$

$\vec{p} = A(A^TA)^{-1}(A^TA)X$

$= AIX$

$= AX$

$= B$ Hence proved.

Q: If V is a subspace spanned by $(1,1,0,1)$ & $(0,0,1,0)$ then Find

① the projection Matrix P onto V

② The vector in V which is closest to the vector $B=(0\ 1\ 0\ -1)$

$$A = V = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}_{4\times 2} = [X_1\ X_2]$$

(ii) $\because X_1 \cdot B = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} = 0$

$X_2 \cdot B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} = 0$

i.e. B is orthogonal to the columns of A

Hence $\vec{p} = PB = \cdots\cdots = \vec{0}$ (By Prop 2)

① $P = A(A^TA)^{-1}A^T$

$= \cdots\cdots$ HW. $\cdots\cdots = \frac{1}{3}\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 3 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}_{4\times 4}$

Note: $\rho(P) = 2 = \rho(A) = \rho$(subspace spanned by A)

② $\lambda = 0, 0, 1, 1$ < GM of 0 = two, GM of 1 = two Diagonalizable

Q: Find the projection Matrix & projection of $B(b_1\ b_2\ b_3)$ onto the column space of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -2 \\ 1 & -4 & 1 \end{bmatrix}_{3\times 3}$

Sol: $\because |A| = \cdots\cdots = -42$

ie A Inverse exist

So By prop (7).

$$P = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \ \& \ \vec{p} = PB = IB$$

$$= \vec{B} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$Tr(P) = 3$, $|P| = 1$

$\lambda = 1, 1, 1$

$$\text{amd}(\lambda = 1) = \text{order} - \rho(P - \lambda I)$$

$$= 3 - \rho(P - I)$$

$$= 3 - \rho \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= 3 - 0 = 3$$

ie $P = I$ is also Diagonalizable.

Q: Find the projection of $B(3-21)$ onto the columns of $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & -4 \end{bmatrix}$

(a) $\vec{B}$

(b) $\vec{0}$

(c) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\because X_1 \cdot B = 0$ & $X_2 \cdot B = 0$. $= [X_1 \ X_2]$

ie B is orthogonal to the columns of A

So $P =$ No idea But $\boxed{\vec{P} = \vec{0}}$ By prop (8)

Q: Find the projection of $B(1\ 3\ 4)$ onto $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = [X_1\ X_2]$

Sol: (M-I) By observation,

$$\boxed{X_1 + 3X_2 = B}$$

i.e. B is LD on Columns of A

So By prop (9), $\vec{p} = B$ itself $= \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$

(M-II) Consider $M = [X_1\ X_2\ B]_{3\times 3}$

$\rho(M) = 2$

i.e. $\rho(M) < 3$

$\Downarrow$

X_1, X_2, B are LD

So By prop (9), $\vec{p} = B$ itself $= \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$

$|M| = 0$

$\Downarrow$

Columns are LD

Given a 3×3 projection matrix P that projects onto the plane spanned by the vectors $[1, 0, 1]^T$ and $[0, 1, 1]^T$. What will be the norm of $P\begin{bmatrix}1\\1\\1\end{bmatrix}$?

(a) 1

(b) $\sqrt{2}$

(c) $\sqrt{3}$

(d) $2\sqrt{\frac{2}{3}}$

$$A = [x_1\ x_2] = \begin{bmatrix}1 & 0\\0 & 1\\1 & 1\end{bmatrix} = \frac{1}{3}\begin{bmatrix}2 & -1 & 1\\-1 & 2 & 1\\1 & 1 & 2\end{bmatrix}$$

$$\vec{p} = P\begin{bmatrix}1\\1\\1\end{bmatrix} = \frac{1}{3}\begin{bmatrix}2 & -1 & 1\\-1 & 2 & 1\\1 & 1 & 2\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix}2\\2\\4\end{bmatrix} = \begin{bmatrix}2/3\\2/3\\4/3\end{bmatrix}$$

$$\|\vec{p}\| = \sqrt{\frac{4}{9}+\frac{4}{9}+\frac{16}{9}} = 2\sqrt{\frac{2}{3}}$$

In a data analysis scenario you have a data set represented by a 4×4 matrix X, where each row is a data point and each column is a feature. You want to project the data onto a 2-D subspace to visualise it. If the projection matrix P is a 4×4 matrix given by : $P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

Which of the following statements about the projected data is correct?

(a) The projected data will lie in the space spanned by vectors $[1\ 1\ 0\ 0]^T$ and $[0\ 0\ 1\ 1]^T$.

(b) The projection matrix P projects the data onto the x-axis and y-axis in the original feature space.

(c) The projected data will be identical to the original data matrix X.

(d) The projection matrix P effectively reduces the dimensionability of the data to 2 dimensions, and the basis vectors are aligned with the original features.

$$A = [x_1 \; x_2] = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}_{4\times 2} \Rightarrow P = A(A^TA)^{-1}A^T = \cdots\cdots = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}_{4\times 4}$$

<u>Analysis:-</u> $Tr(P) = 2$, $|P| = 0$, $\lambda = 0, 0, 1, 1$

$GMO\,0 = 2$ & $GMO\,1 = 2$

ie Diagonalizable, Idempotent, Symmetric.

ie (a)

$$\rho(P) = 2 = \rho(A) = \rho(\text{subspace / column space spanned by } A)$$

PROJECTION MATRIX (along a subspace spanned by set of vectors) →

Let W is the subspace spanned by set of vectors in A & we want to to find projection of B onto the subspace spanned by A then Projection Mat is given as.

$$\boxed{P = A(A^TA)^{-1}A^T}$$ & Projection vector is $\boxed{\vec{p} = PB}$

RECAP

PROJECTION MATRIX (along a vector A or ONTO the line A) →

$$P = \left(\frac{AA^T}{A^TA}\right)_{n\times n}$$ and Projection vector of B onto A is $\boxed{\vec{p} = PB}$

Note:

RECAP

$\vec{p}$ = Projection vector of B onto A

& it is given as $\boxed{\vec{p} = PB}$

Similarly $\vec{q}$ = orthogonal complement of B onto A

& it is given as $\boxed{\vec{q} = B - \vec{p}}$ Learn.

$\because \boxed{\vec{p} + \vec{q} = \vec{B}}$

or $P\vec{B} + Q\vec{B} = \vec{B}$

ie $\boxed{P + Q = I}$

ie orthogonal Projection Mat of B onto A is given as

$\boxed{Q = I - P}$ Learn.

Properties of Projection Matrix → $\boxed{P = A(A^TA)^{-1}A^T}$ & $\boxed{\bar{p} = PB}$

① $\boxed{\text{Trace of Proj. Mat (onto the line)} = 1}$

RECAP & Trace of Proj Mat (onto the subspace) is Not Necessarily one

② Projection Mat is always symmetric as well as Idempotent

(whether onto the line or onto the subspace)

③ Det of Projection Mat (except identity Matrix) is always $= 0$

(Doesn't matter, whether onto the line or onto the subspace)

eg $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ → $I^2 = I$ idempotent, $I^T = I$ symm

But $|I| \neq 0$

(4) If A is Invertible (i.e. $|A| \neq 0$ & A^{-1} exist) then $P = I$ & $\vec{p} = B$ itself

(5) If B is L-D on columns of A then $P =$ No idea & $\vec{p} = B$ itself

(6) If B is orthogonal to the columns of A then $P =$ No idea & $\vec{p} = \vec{0}$.

RECAP

For a projection matrix P that projects onto a subspace spanned by $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in R^2, what will be det (P)?

(a) 0

(b) 0.5

(c) 1

(d) 2

$|P| = 0$ (using prop of P. Mat)

If P is a 2×2 projection matrix such that $P\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}2\\4\end{bmatrix}$ and $P\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}1\\0.5\end{bmatrix}$, what is det (P)?

(a) 1.5

(b) 0.25

(c) 0.5

(d) No Valid Projection Matrix P

w.k.that for any Proj. Mat, $|P| = 0$ so (d)

Let P be a projection matrix. Which of the following properties does not necessarily hold?

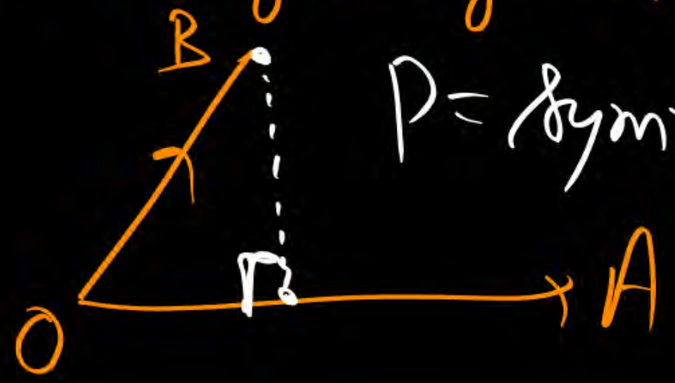
(a) P is idempotent. (T)

(b) The eigen values of P are either 0 or 1. (T)

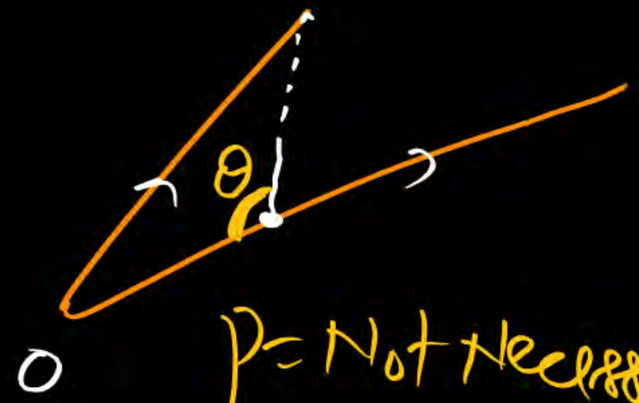
(c) P is always diagonalizable. (T)

(d) P is always symmetric. (e) All are Correct.

Projection Mat is always Symm in Case of orthogonal Projection



P = Symm.



P = Not Necessarily Symm.

Given a projection matrix P in R^n, how can you determine the dimension of the subspace onto which P projects?

(a) It is the number of non-zero columns in P.

(b) It is the rank of P.

(c) It is the number of zero eigen values of P.

(d) It is the trace of P.

Consider the matrix $P = A(A^TA)^{-1}A^T$, where A is an $m \times n$ matrix with full column rank. What can be said about the rank of P?

$\rho(A) = n$

(a) Rank of P is n

(b) Rank of P is m

(c) Rank of P is $m - n$

(d) Rank of P is $n - m$

THANK - YOU