VIJAY

DATA SCIENCE & ARTIFICIAL INTELLIGENCE

& CS

Calculus and Optimization

Lecture No. 01

By- Dr. Puneet Sharma Sir

Topics to be Covered

Topic FUNCTIONS & GRAPHS - 1

STRATEGY

1. Live Class
2. Class Notes — (3 times Revision)
3. DPP
4. W. Test
5. O.T.S
6. P.Y.Q

} Judge yourself

* PARACHUTE LANDING
  → Doubts Not allowed for 1 or 2 days.

* Doubts Engine → ✓

* Book: No Book Required.

* Tel: $\boxed{\text{drpuneetsirpw}}$

CALCULUS

300-400 Q Class
70-80 Q DPP
26 Q WT
300+ Q PYQ

1000 Q

Maths

DA (2025) → 43 Marks

CS/IT : 7-8 Marks.

1. Linear Algebra
   - Part 1 (for Both DA & CS)
   - Part 2 (only for DA)
2. $\boxed{\text{Calculus}}$
3. Prob & Stats
   - Part 1 (for Both DA & CS)
   - Part 2 (only for DA)

Types of functions

- ALGEBRAIC function
  1. Polynomial func^n
  2. Rational func^n
  3. Irrational func^n
  4. Piecewise func^n
     - Mod func^n
     - Signum func^n
     - G.I.F.
     - L.I.f
     - F.P.f
- TRANSCEDENTAL function
  1. Exponential func^n
  2. log function.
  3. Trigonometric func^n
  4. Inverse Trig. functions.

G.I.f = Greatest Integer func^n (Floor func^n)

L.I.f = Least Integer func^n (Ceiling func^n)

F.P.F = Fractional Part func^n.

Polynomial: It's Domain is $(-\infty, \infty)$ & Degree $= 0, 1, 2, 3, 4, 5 \ldots$
& it's Definition is same at all points in the Domain of $y = f(x)$

eg $y = K$ (Constant poly) $\simeq$ degree $= 0$

$y = ax + b$ (Linear poly) $\simeq$ Dg $= 1$

$y = ax^2 + bx + c$ (Quad. poly) $\simeq$ Dg $= 2$

$y = ax^3 + bx^2 + cx + d$ (Cubic poly) $\simeq$ Dg $= 3$

$\vdots$

Sp Note - $y = |x| = \begin{cases} -x, & x < 0 \\ +x, & x > 0 \end{cases}$, $D_f = (-\infty, \infty)$. It is not a poly bcoz it's Defn is not unique at all points in the Domain.

Even funcⁿ: if $f(-x) = f(x) \Rightarrow f(x)$ is called an Even funcⁿ & it's Graph is symmetrical about Y axis

odd funcⁿ: if $f(-x) = -f(x) \Rightarrow f(x)$ is called an odd funcⁿ. & it's Graph is symmetrical about origin i.e ($\mathrm{I} \Leftrightarrow \mathrm{III}$ & $\mathrm{II} \Leftrightarrow \mathrm{IV}$)

NENO funcⁿ: if $\left.\begin{array}{l} f(-x) \neq f(x) \\ \& \ f(-x) \neq -f(x) \end{array}\right\}$ then $f(x)$ is called NENO funcⁿ.

it's Graph is neither symmetrical about Y axis, nor about origin.

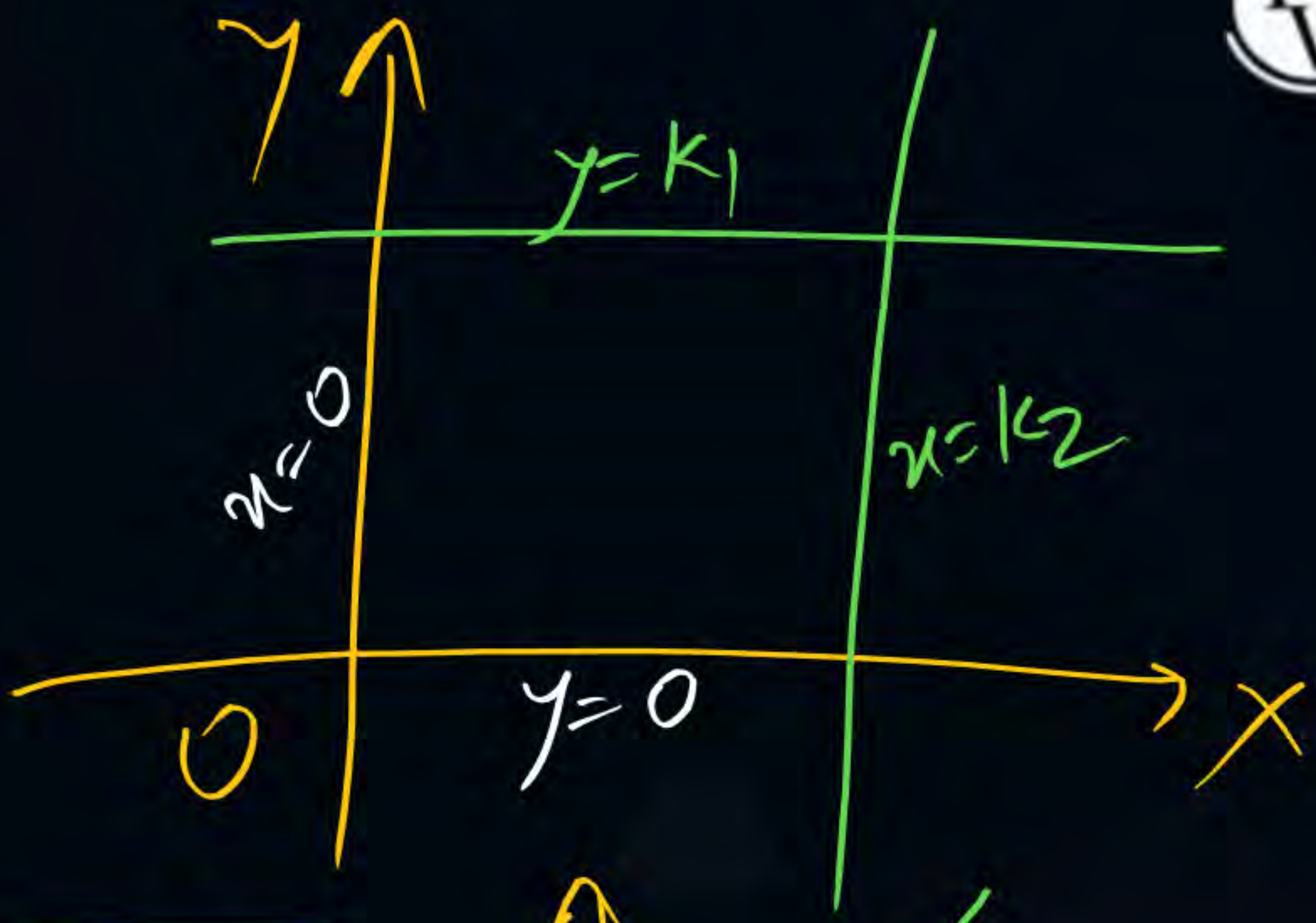
BASIC GRAPHS :→

① Equⁿ of X axis is $y=0$

② " of line $\parallel^r$ to X axis $y=K_1$

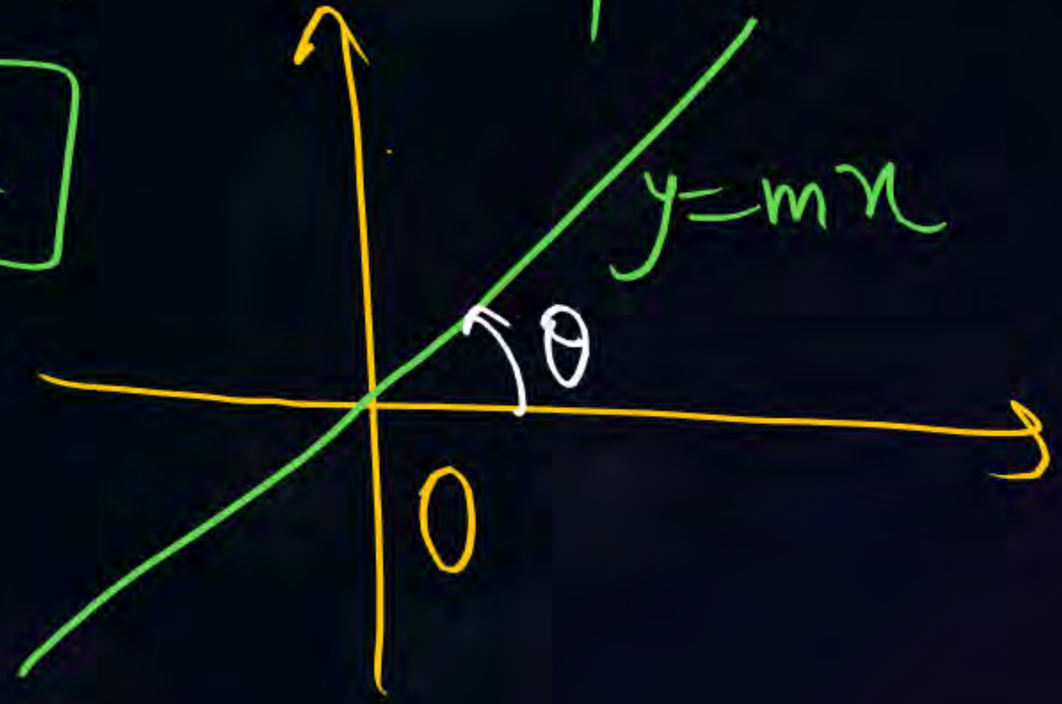
③ Equⁿ of Y axis $x=0$

④ Equⁿ of line $\parallel^r$ to Y axis $x=K_2$



⑤ Equⁿ of line passing through origin is $\boxed{y=mx}$

where m = slope of line = $\tan\theta$



(6) Slope Intercept Form of line:

$$y = mx + c$$

$m = \tan\theta = -ve$

$C = y$ Intercept.

M-II

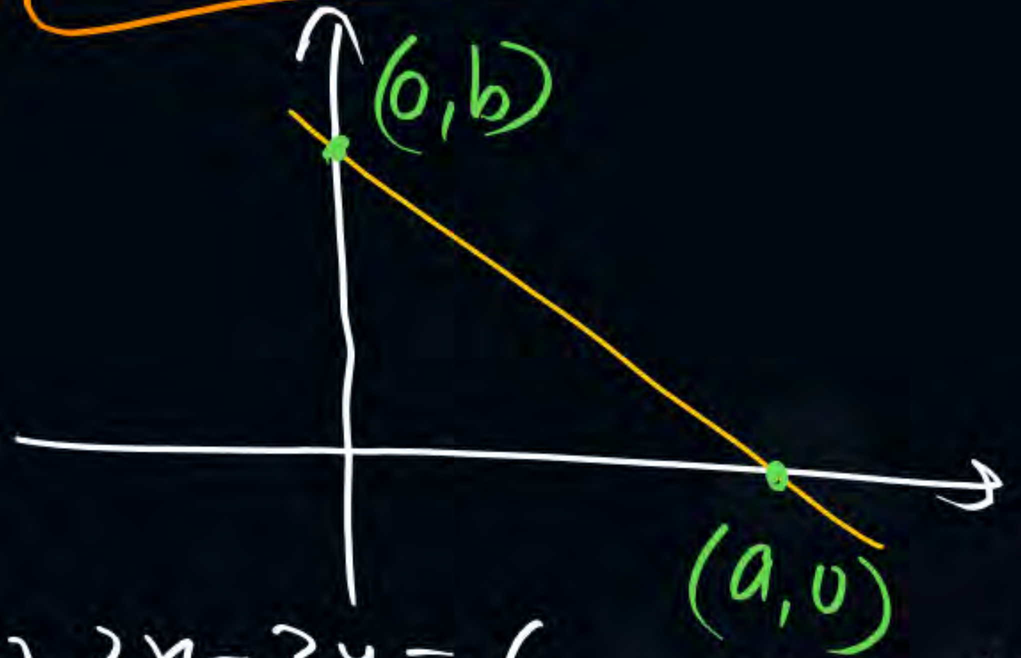
$$-mx + y = c$$

$$-\frac{mx}{c} + \frac{y}{c} = 1$$

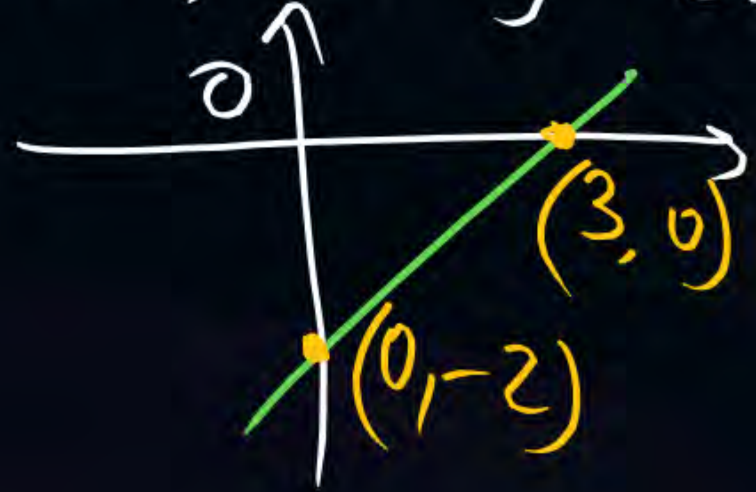
$$\frac{x}{-c/m} + \frac{y}{c} = 1$$

(7) Intercept Form of Line

$$\boxed{\frac{x}{a} + \frac{y}{b} = 1}$$



eg Draw, $2x - 3y = 6 \Rightarrow \frac{x}{3} + \frac{y}{-2} = 1$



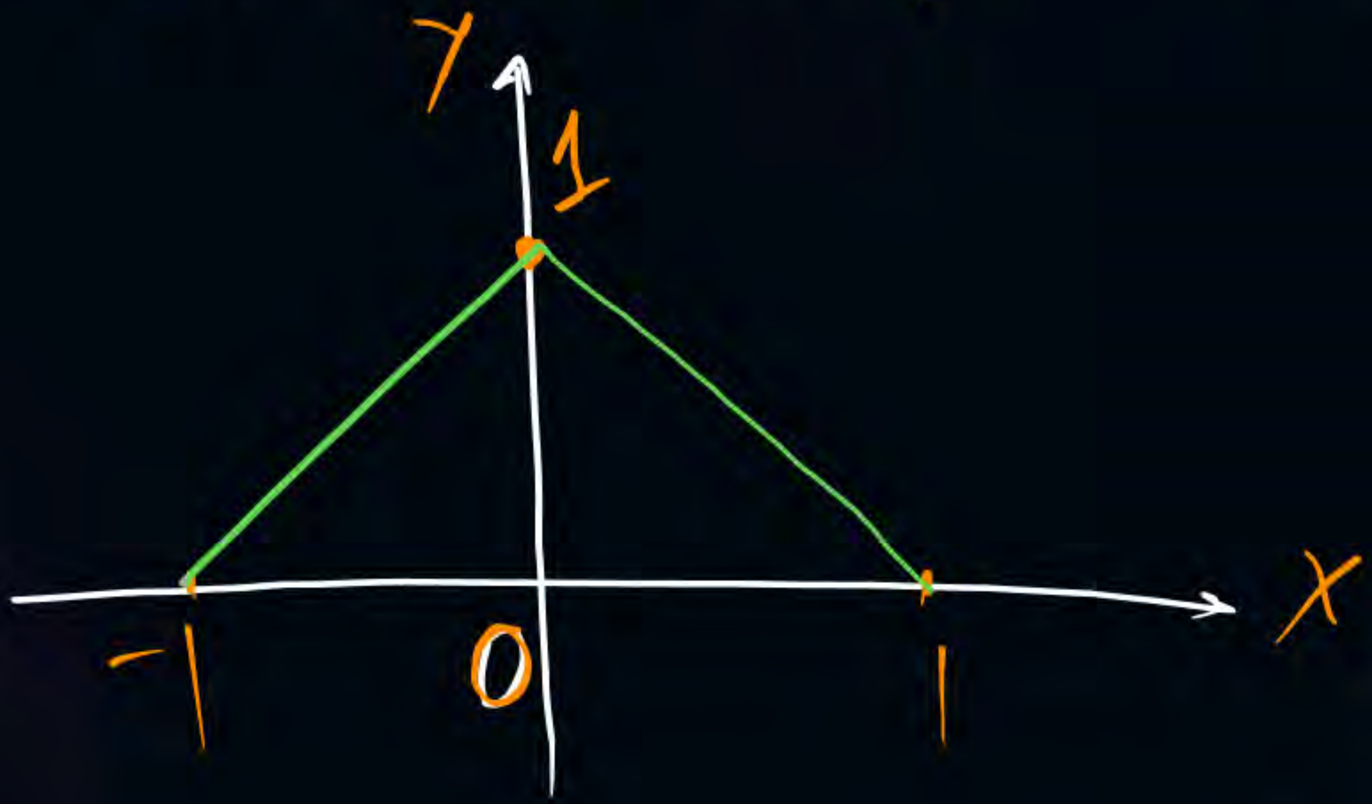
PQ: Draw the graph of $y = \begin{cases} 1+x & , -1 \le x \le 0 \\ 1-x & , 0 \le x \le 1 \end{cases}$ = Even func^n

Case I: $-1 \le x \le 0$

$$y = 1 + x$$

$$-x + y = 1$$

or $\boxed{\dfrac{x}{-1} + \dfrac{y}{1} = 1}$



Case II: $0 \le x \le 1$

$$y = 1 - x$$

$$x + y = 1$$

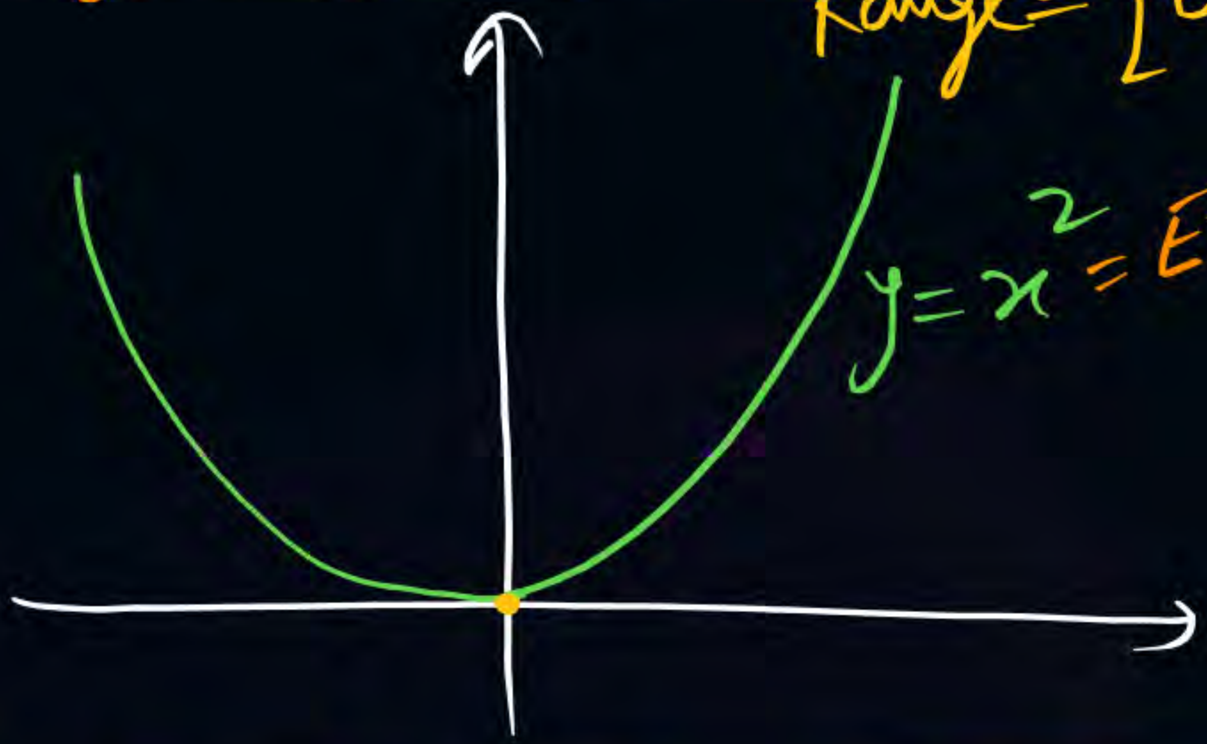
or $\boxed{\dfrac{x}{1} + \dfrac{y}{1} = 1}$

Poly of Even Degree

$y = x^2, x^4, x^6 \ldots\ldots$

Dom $= (-\infty, \infty)$

Range $= [0, \infty)$



$y = x^2 = $ Even func^n

x:	$\ldots$	-3	-2	-1	0	1	2	$3 \ldots$
y:	$\ldots$	9	4	1	0	1	4	$9 \ldots$

$y = x^2$

Poly of Odd Degree

Dom $= (-\infty, \infty)$

$y = x^3, x^5, x^7, \ldots$

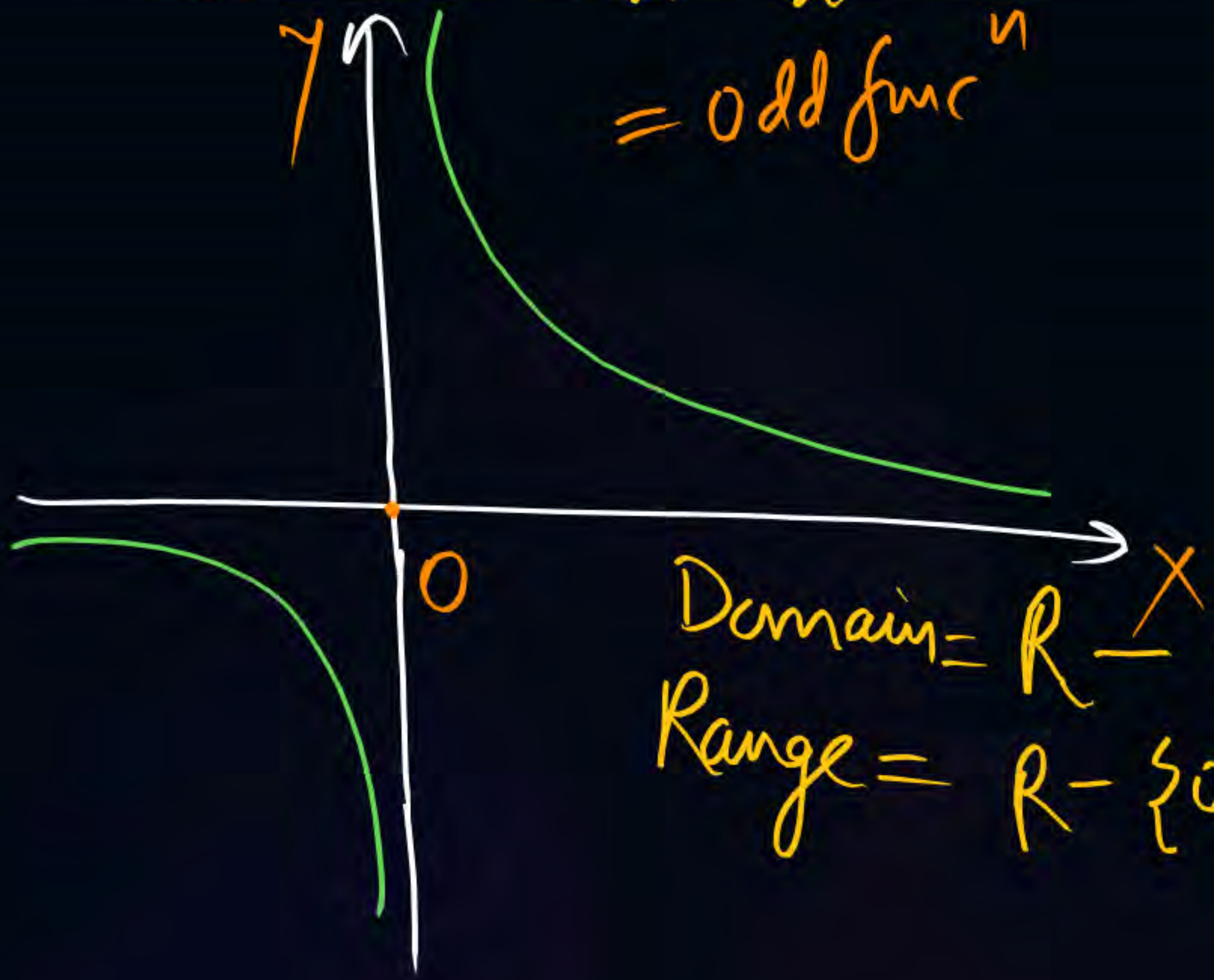
Range $= (-\infty, \infty)$

$y = x^3 = $ odd func^n

Rational Func^n

$y = \frac{1}{x}$, $\frac{1}{x^3}$, $\frac{1}{x^5}$ ......

$= \text{odd func}^n$

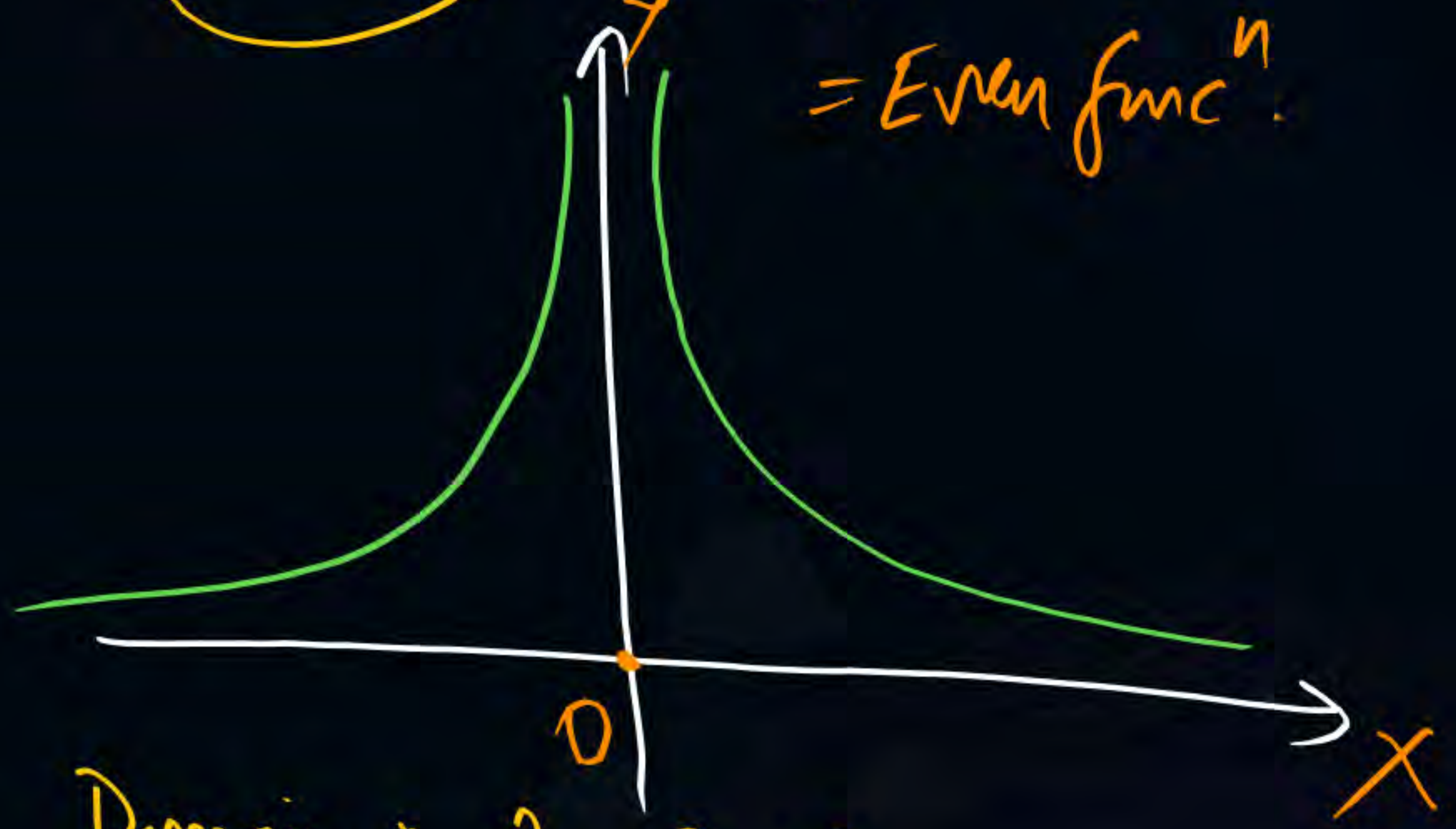


Domain $= R - \{0\}$

Range $= R - \{0\}$

$y = \frac{1}{x^2}$, $\frac{1}{x^4}$, $\frac{1}{x^6}$ ......

$= \text{Even func}^n$

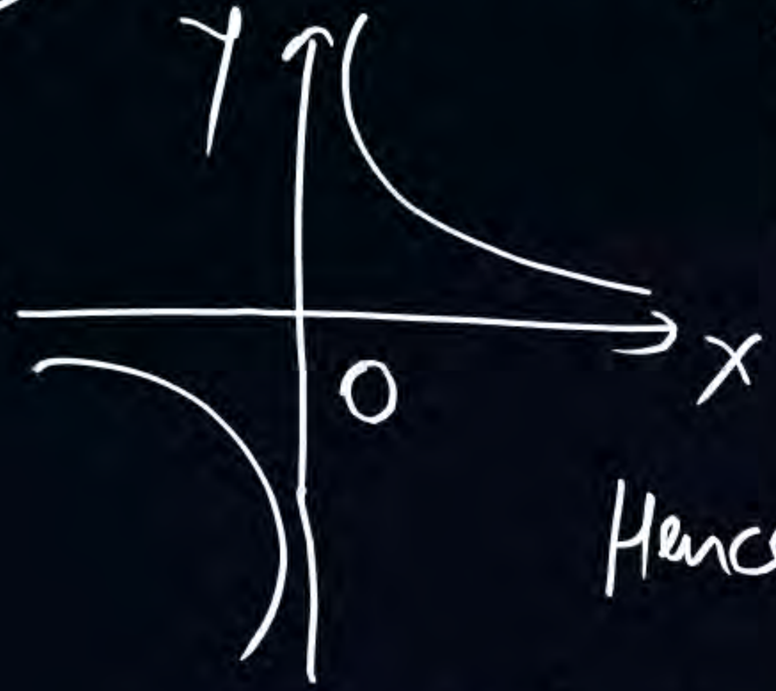


Domain : $R - \{0\}$

Range : $(0, \infty)$ or $y > 0$

ANALYSIS / PODCAST :→ $\boxed{\frac{\text{Something}}{0} = \text{N.D.}}$

(M-I) we know that, $y = \frac{1}{x}$



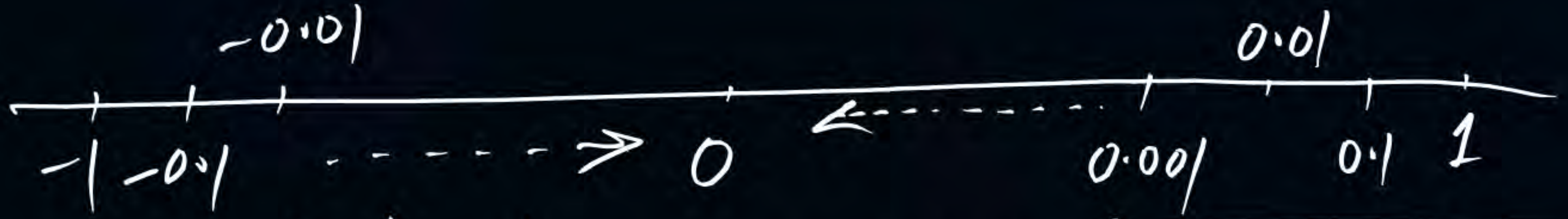
$y(0^-) = -\infty = \text{N.D}$

$y(0^+) = +\infty = \text{N.D}$

Hence $y(0) = \text{N.D.}$

Hence we have a liberty to choose $\frac{\text{Something}}{0} = \infty$ or $-\infty$ as Required.

M-II



$\frac{1}{1} = 1$

$\frac{1}{0.1} = 10$

$\frac{1}{0.01} = 100$

$\frac{1}{0.001} = 1000$

$\vdots$

$\frac{1}{0^+} = +\infty$

$\frac{1}{-1} = -1$

$\frac{1}{-0.1} = -10$

$\frac{1}{-0.01} = -100$

$\vdots$

$\frac{1}{0^-} = -\infty$

that's why $\boxed{\frac{\text{Something}}{0} = \text{N.D}}$

M-III $\frac{40}{8} = 5 \Rightarrow$ If we want to distribute 40 Mangoes among 8 children then each child will get $\underline{5}$ Mangoes

Similarly, $\frac{40}{0} = \text{N.D}$

Irrational Func^n

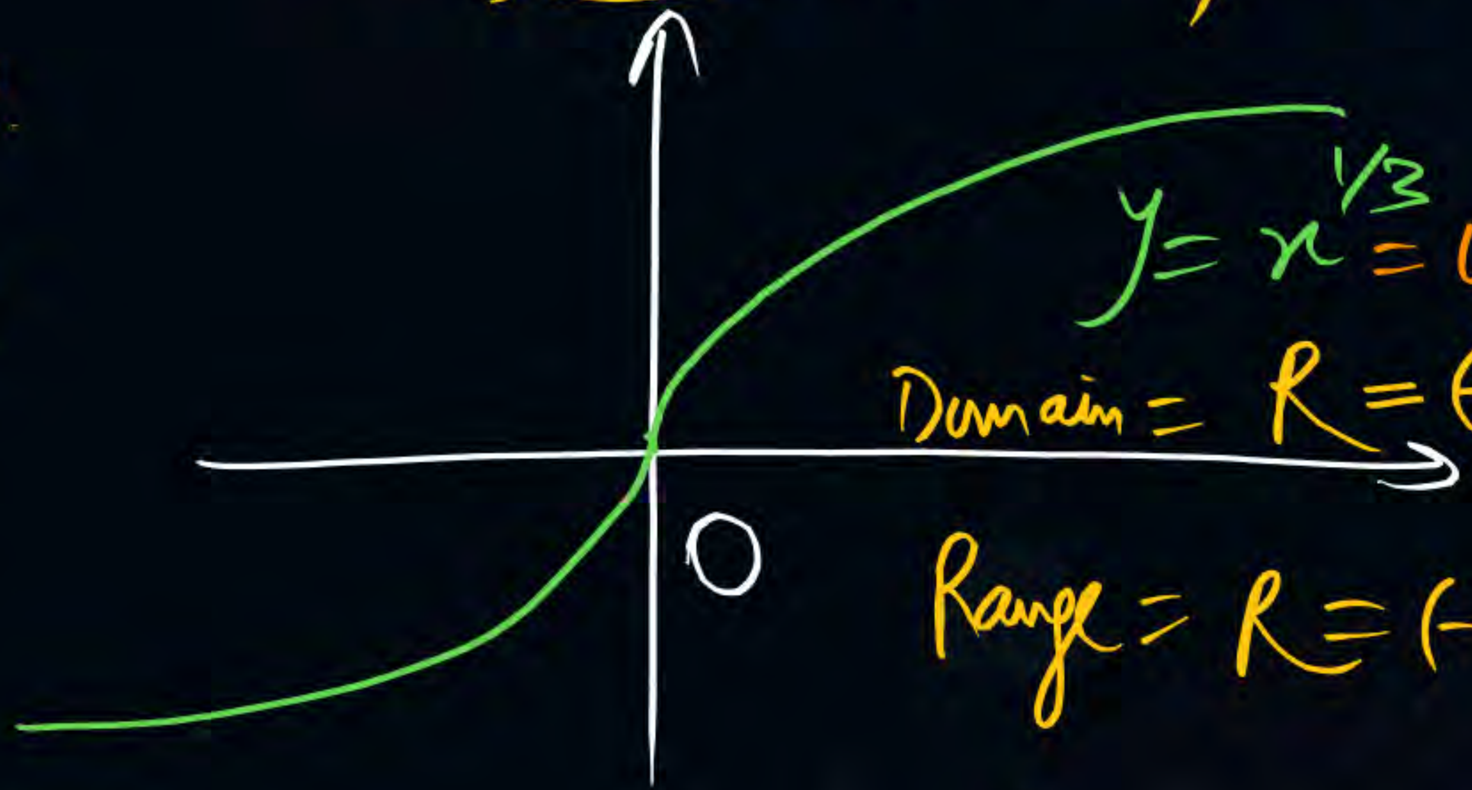
(1) $y = x^{1/2}$, $x^{1/4}$, $x^{1/6}$ ......

(2) $y = x^{1/3}$, $x^{1/5}$, $x^{1/7}$ ......

$y = \sqrt{x}$ = NENO

Domain $= [0, \infty)$

Range $= [0, \infty)$



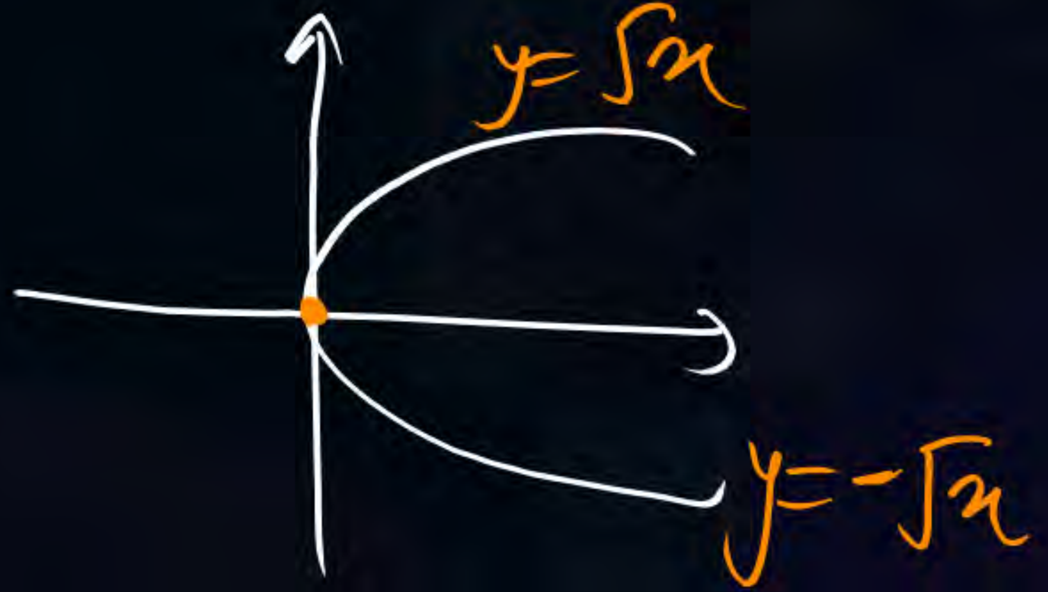
$y = x^{1/3}$ = odd func^n

Domain $= R = (-\infty, \infty)$

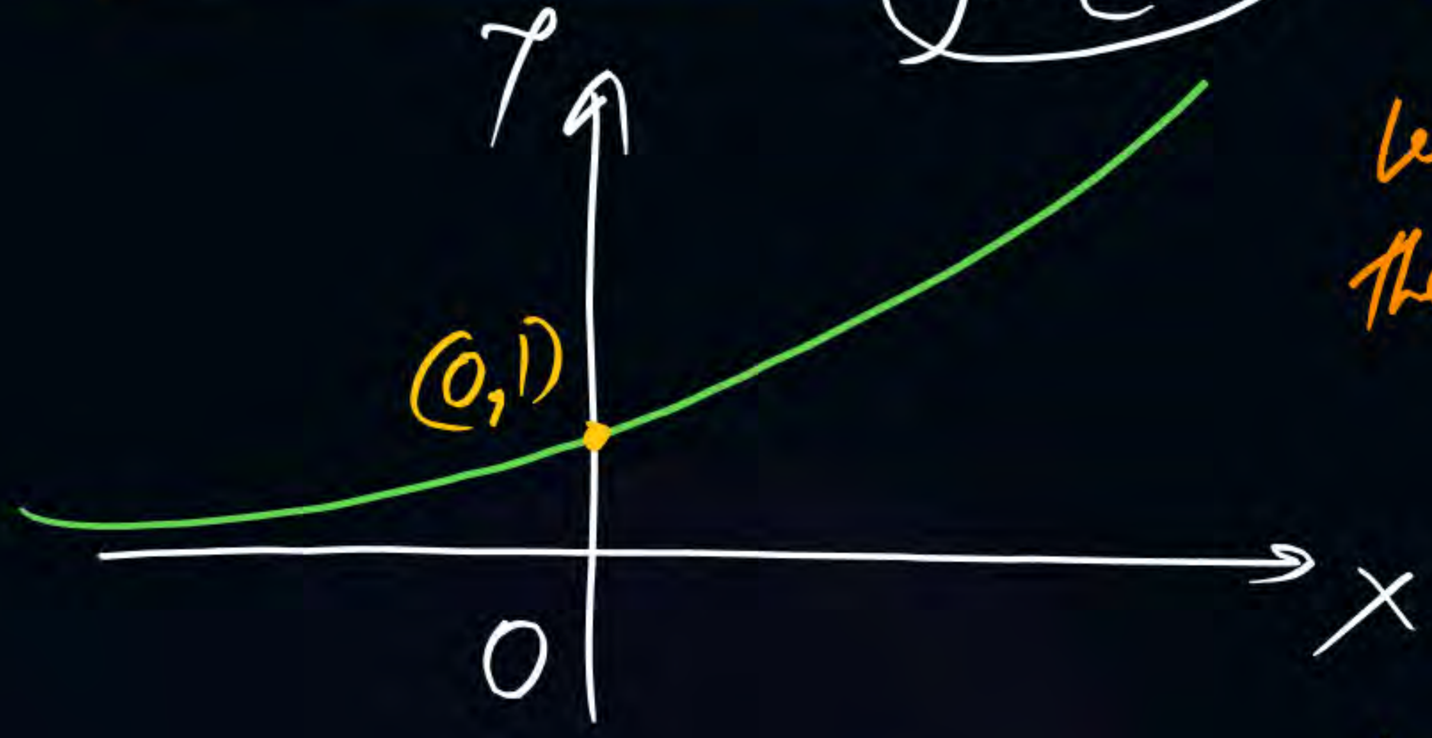
Range $= R = (-\infty, \infty)$

Analysis: Parabola: $y^2 = x$ $\rightarrow$ $y = \sqrt{x}$, $y = -\sqrt{x}$

it is Not a func^n



④ Exponential func^n $y = e^x$ = NENO func^n.



Let $f(x) = e^x$

then $f(-x) = e^{-x} = \frac{1}{e^x} = \frac{1}{f(x)}$

i.e. $f(-x) \neq f(x)$ or $-f(x)$

so $f(x) = e^x$ is NENO func^n.

$y(0) = e^0 = 1$

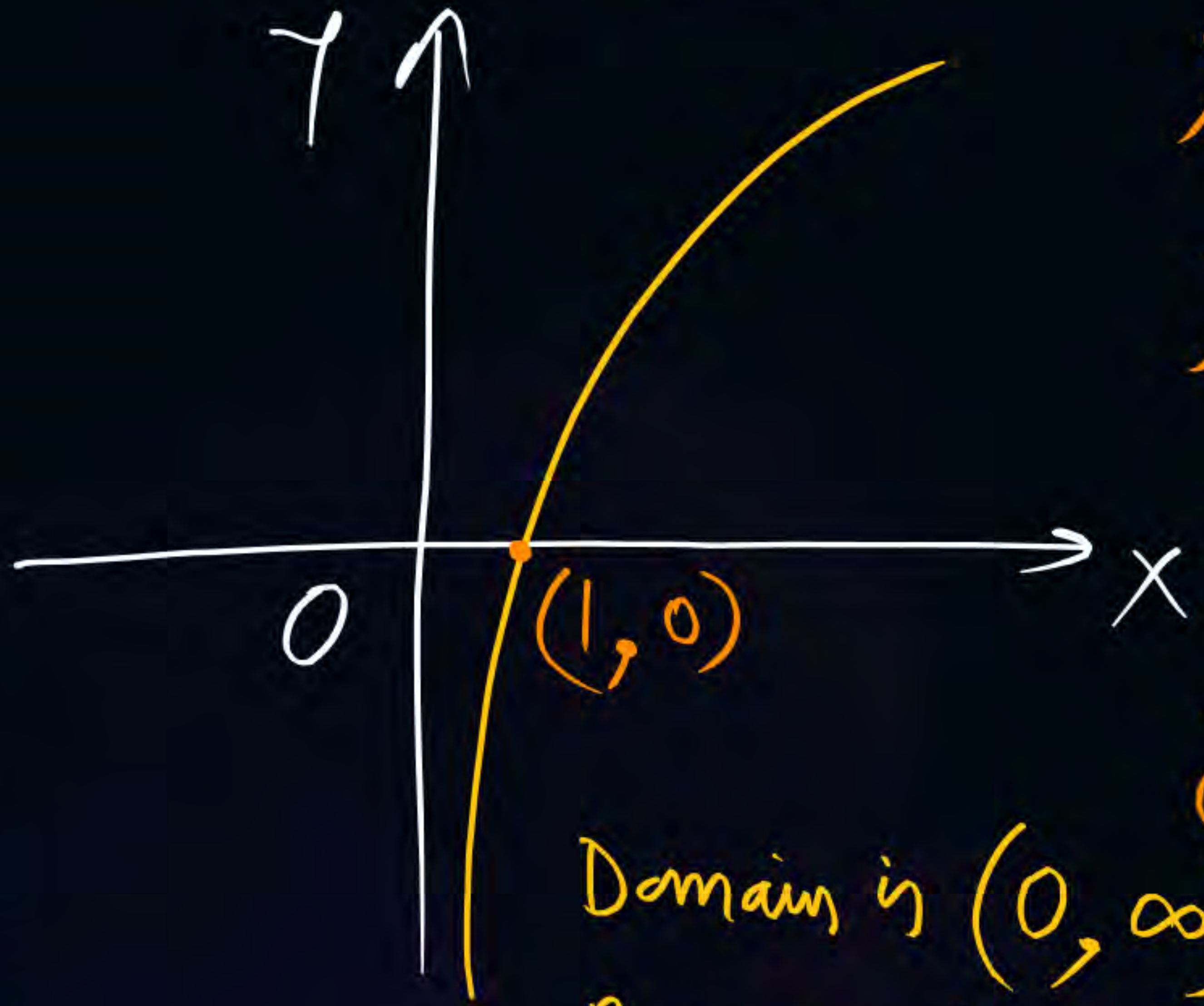
$y(-\infty) = \boxed{e^{-\infty} = 0}$

$y(\infty) = \boxed{e^{\infty} = +\infty}$

Domain $= (-\infty, \infty)$ i.e. $x \in (-\infty, \infty)$

Range $= (0, \infty)$ i.e. $y > 0$

(x) log fnc^n → $y = \log_e x$ or $y = \ln x$ = NEMO



$y(1) = \boxed{\log_e 1 = 0}$

$y(0) = \boxed{\log_e 0 = -\infty}$ = N.D

$y(\infty) = \boxed{\log_e \infty = +\infty}$ = N.D

$y(-\infty) = \log_e(-\infty)$ = N.D

Domain is $(0, \infty)$ i.e $y = \log_e x$ is defined for $x > 0$

Range is $(-\infty, \infty)$ i.e $y \in (-\infty, \infty)$

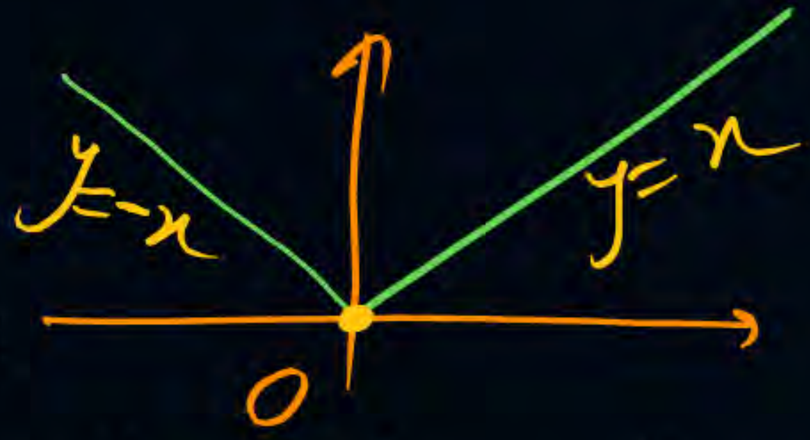
PIECEWISE func^n → If func^n is defined by Multiple subfunction s.t, Domain of each subfunction is different then function is called Piecewise func^n

for eg, Mod func^n, Signum func^n, G.I.F, L.I.F, Fractional part func^n etc

MODULUS funcⁿ →

Defⁿ-1 Def 2 Def 3

$$y = |x| = \begin{cases} -x, & x < 0 \\ +x, & x \geq 0 \end{cases} = \max(x, -x) = \sqrt{x^2} =$$



= Even funcⁿ → Domain $= (-\infty, \infty)$ & it is not a polynomial.
Range $= [0, \infty)$

eg $|3| = +(3) = +3$

$|-3| = -(-3) = +3$

eg $|3| = \max\{3, -3\} = +3$

$|-3| = \max\{-3, -(-3)\} = +3$

eg: $|3| = \sqrt{(3)^2} = \sqrt{9} = +3$

$|-3| = \sqrt{(-3)^2} = \sqrt{9} = +3$

$\because \sqrt{x^2} = |x| = +ve$

ie Sq. Root of any Real No is always +ve.

eg $x^2-9=0$

$x^2=9$

$x=\sqrt{9}$ WRONG step

$x=\pm 3$

$x^2-9=0$

$x^2=9$

$x=\pm\sqrt{9}$ ✓

$x=\pm(+3)$

$x=\pm 3$

$x^2-9=0$

$(x-3)(x+3)=0$

$x=3,-3$

Q. $\left[(-4)^2\right]^{\frac{1}{2}}=?$

(a) 4 ✓

(b) -4

(c) ± 4

(d) None

$\left[(-4)^{\cancel{2}}\right]^{\frac{1}{\cancel{2}}}=-4$ ✗

$\sqrt{(-4)^2}=\sqrt{16}=+4$ ✓

(*) Signum func^n →

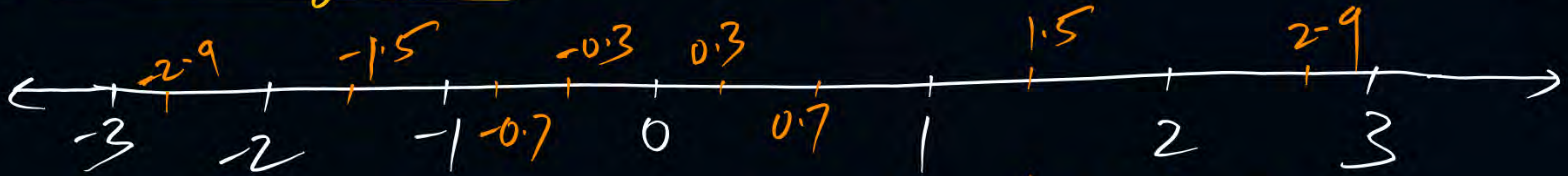
$$y = \operatorname{sgn}(x) = \begin{cases} \dfrac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} = \begin{cases} -1, & x < 0 \\ +1, & x > 0 \\ 0, & x = 0 \end{cases} =$$

= Odd func^n, Domain $= (-\infty, \infty)$, Range $= \{-1, 0, 1\}$

eg $\operatorname{sgn}(0) = 0$, $\operatorname{sgn}(1{\cdot}3) = 1$, $\operatorname{sgn}(1{\cdot}9) = 1$, $\operatorname{sgn}(2{\cdot}7) = 1$ . . . . .

& $\operatorname{sgn}(-3{\cdot}4) = -1$, $\operatorname{sgn}(-0{\cdot}2) = -1$, $\operatorname{sgn}(-4{\cdot}5) = -1$ . . . . .

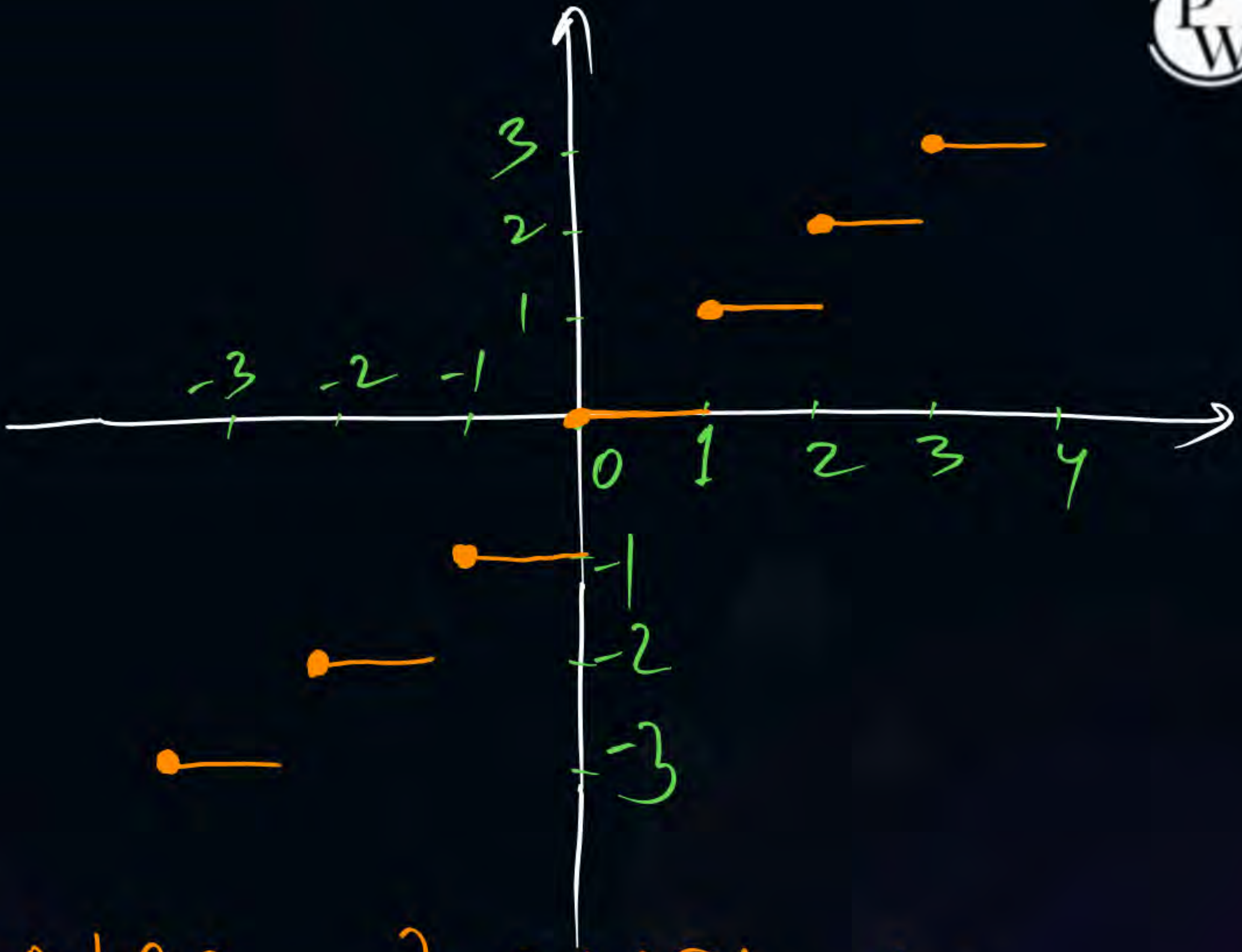
(*) Greatest Integer func^n (Floor Function) →



$\lfloor 0.3 \rfloor = 0$, $\lfloor 0.7 \rfloor = 0$, $\lfloor 1.5 \rfloor = 1$, $\lfloor 2.9 \rfloor = 2$

$\lfloor -0.3 \rfloor = -1$, $\lfloor -0.7 \rfloor = -1$, $\lfloor -1.5 \rfloor = -2$, $\lfloor -2.9 \rfloor = -3$

$$y=\lfloor x\rfloor=\begin{cases} \vdots & \\ -2, & -2\le x<-1\\ -1, & -1\le x<0 \\ 0, & 0\le x<1\\ 1, & 1\le x<2\\ 2, & 2\le x<3\\ 3, & 3\le x<4\\ \vdots & \end{cases}$$

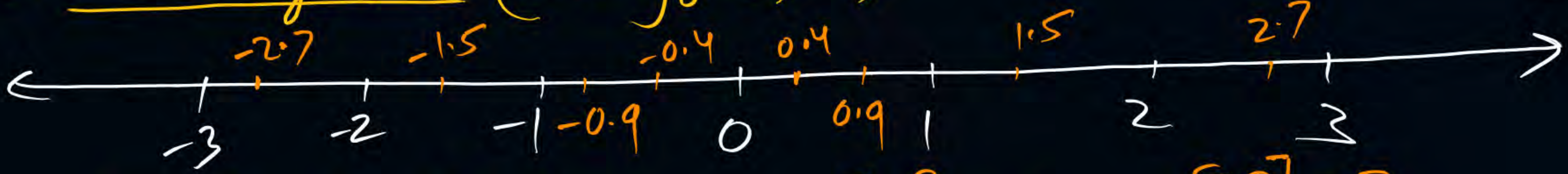


= NENO func^n

Domain = $(-\infty,\infty)$

Range = $\{\dots -3,-2,-1,0,1,2,3\dots\}$ = Set of Integers ($\mathbb{Z}$)

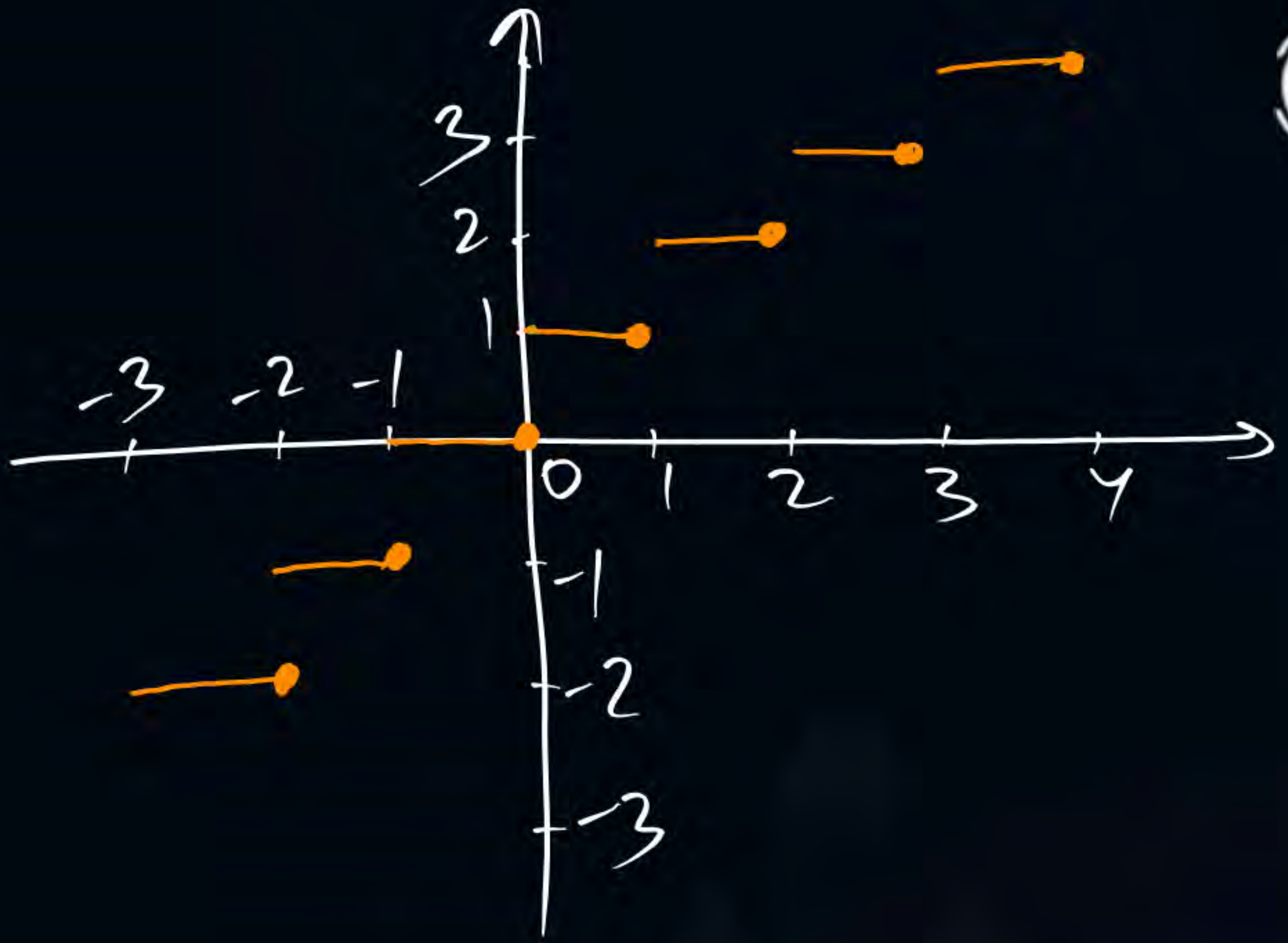
(*) Least Integer $func^n$ (Ceiling $func^n$) $\rightarrow$



$\lceil 0.4 \rceil = 1$, $\lceil 0.9 \rceil = 1$, $\lceil 1.5 \rceil = 2$, $\lceil 2.7 \rceil = 3$

$\lceil -0.4 \rceil = 0$, $\lceil -0.9 \rceil = 0$, $\lceil -1.5 \rceil = -1$, $\lceil -2.7 \rceil = -2$

$$y = \lceil x \rceil = \begin{cases} \vdots & \vdots \\ -1 & -2 < x \le -1 \\ 0 & -1 < x \le 0 \\ 1, & 0 < x \le 1 \\ 2, & 1 < x \le 2 \\ 3, & 2 < x \le 3 \\ 4 & 3 < x \le 4 \\ \vdots & \vdots \end{cases}$$



= NENO func^n

Domain $= (-\infty, \infty)$, Range $= \{\dots -2, -1, 0, 1, 2, 3 \dots\} = \mathbb{Z}$.

Fractional Part func^n: $y = \{x\} = \boxed{x - \lfloor x \rfloor}$

Tel: drpuneetsir pw

THANK you

VIJAY

DATA SCIENCE & ARTIFICIAL INTELLIGENCE

& CS

Calculus and Optimization

Lecture No. 02

By- Dr. Puneet Sharma Sir

Recap of previous lecture

Topic FUNCTIONS & GRAPHS - 1

Topics to be Covered

Topic

FUNCTIONS & GRAPHS - 2

RECAP

Types of functions

- ALGEBRAIC function
  1. Polynomial func^n
  2. Rational func^n
  3. Irrational func^n
  4. Piecewise func^n
     - Mod func^n
     - Signum func^n
     - G.I.F.
     - L.I.f
     - F.P.f
- TRANSCEDENTAL function
  1. Exponential func^n
  2. log function.
  3. Trigonometric func^n
  4. Inverse Trig. functions.

G.I.f = Greatest Integer func^n (Floor func^n)

L.I.f = Least Integer func^n (Ceiling func^n)

FPF = Fractional Part func^n.

Polynomial: It's Domain is $(-\infty, \infty)$ & Degree $= 0, 1, 2, 3, 4, 5 \ldots$
& it's Definition is same at all points in the Domain of $y = f(x)$

RECAP

eg $y = k$ (Constant poly) $\approx$ degree $= 0$

$y = ax + b$ (Linear poly) $\approx$ Dg $= 1$

$y = ax^2 + bx + c$ (Quad. poly) $\approx$ Dg $= 2$

$y = ax^3 + bx^2 + cx + d$ (Cubic poly) $\approx$ Dg $= 3$

$\vdots$

Sp Note - $y = |x| = \begin{cases} -x, & x < 0 \\ +x, & x > 0 \end{cases}$, $D_f = (-\infty, \infty)$. It's not a poly bcoz it's Def^n is not unique at all points in the Domain.

Even func^n: if $f(-x)=f(x) \Rightarrow f(x)$ is called an Even func^n & it's Graph is symmetrical about Y axis

RECAP

odd func^n: if $f(-x)=-f(x) \Rightarrow f(x)$ is called an odd func^n.
& it's Graph is symmetrical about origin i.e (I $\leftrightarrow$ III & II $\leftrightarrow$ IV)

NENO func^n: if $\left.\begin{array}{l} f(-x)\neq f(x) \\ \& \; f(-x)\neq -f(x) \end{array}\right\}$ then $f(x)$ is called NENO func^n.

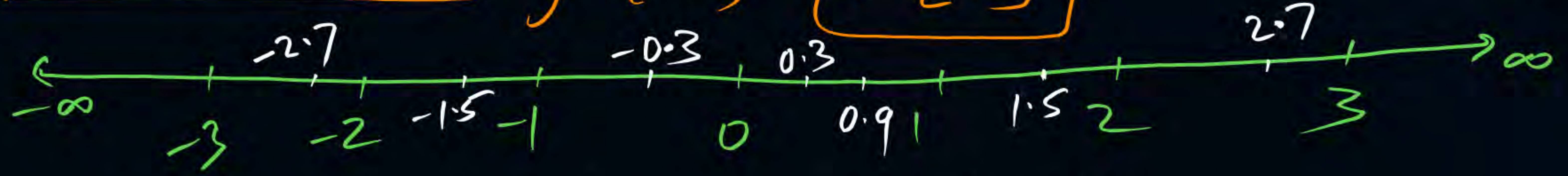
it's Graph is neither symmetrical about Y axis, nor about origin.

PIECEWISE func^n → If func^n is defined by Multiple subfunction s.t, Domain of each subfunction is different then function is called Piecewise func^n

for eg, Mod func^n, Signum func^n, G.I.F, L.I.F, Fractional Part func^n etc

RECAP

Fractional Part func^n : $y = \{x\} = \boxed{x - \lfloor x \rfloor}$



$\{0.3\} = 0.3$

$\{0.3\} = 0.3 - \lfloor 0.3 \rfloor$

$= 0.3 - 0$

$= 0.3$

$\{-0.3\} = -0.3 - \lfloor -0.3 \rfloor$

$= -0.3 - (-1)$

$= 0.7$

$\{1.5\} = 0.5$

$\{-1.5\} = 0.5$

$\{2\} = 0$

$\{-2\} = 0$

$\{2.7\} = 0.7$

$\{2.7\} = 2.7 - \lfloor 2.7 \rfloor$

$= 2.7 - 2$

$= 0.7$

$\{-2.7\} = -2.7 - \lfloor -2.7 \rfloor$

$= -2.7 - (-3)$

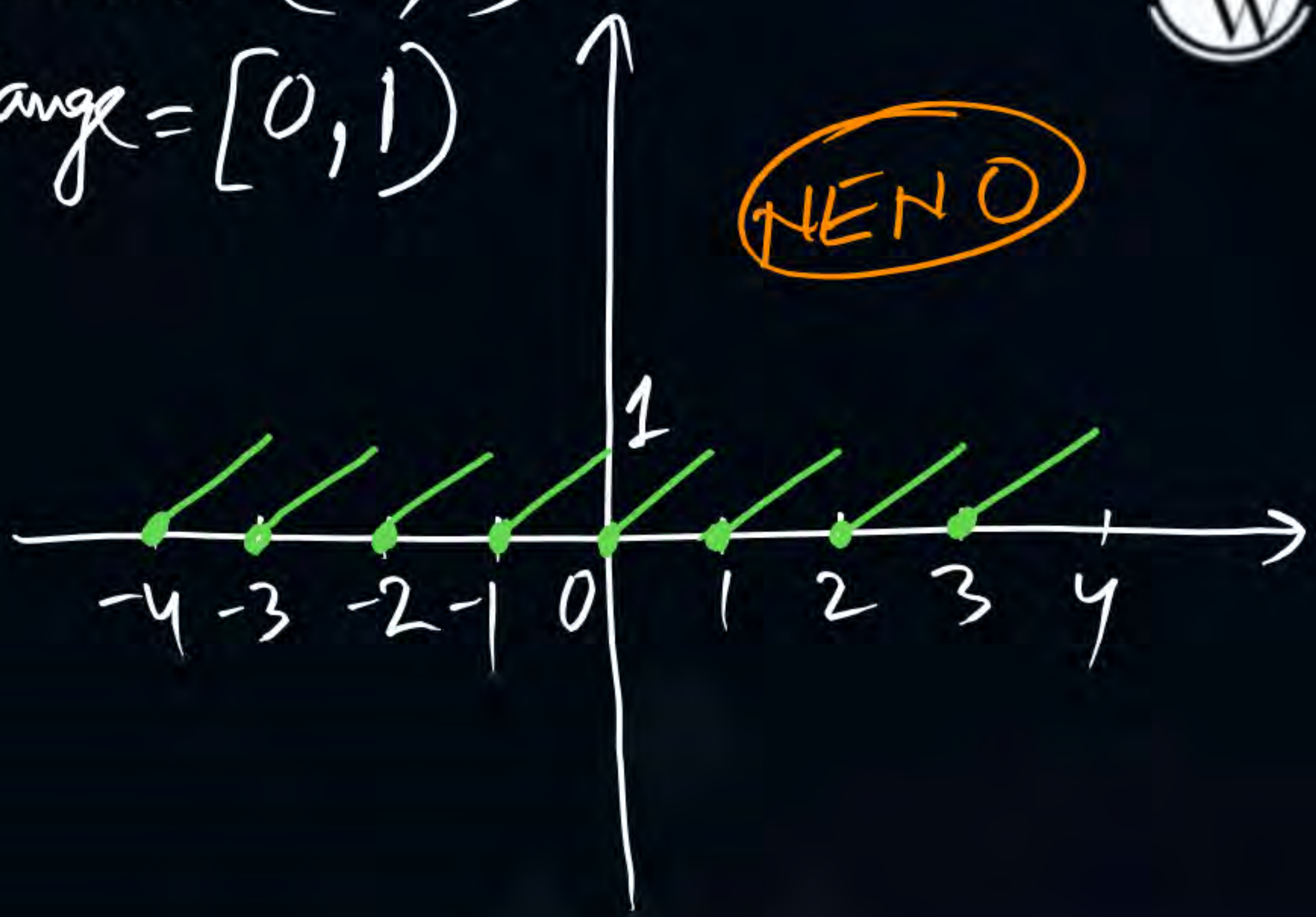
$= 0.3$

Def^n:

$$y = \{x\} = \boxed{x - [x]} = \begin{cases} \vdots \\ x+2, & -2 \le x < -1 \\ x+1, & -1 \le x < 0 \\ x, & 0 \le x < 1 \\ x-1, & 1 \le x < 2 \\ x-2, & 2 \le x < 3 \\ x-3, & 3 \le x < 4 \\ \vdots \end{cases}$$

Domain $= (-\infty, \infty)$

Range $= [0, 1)$



NENO

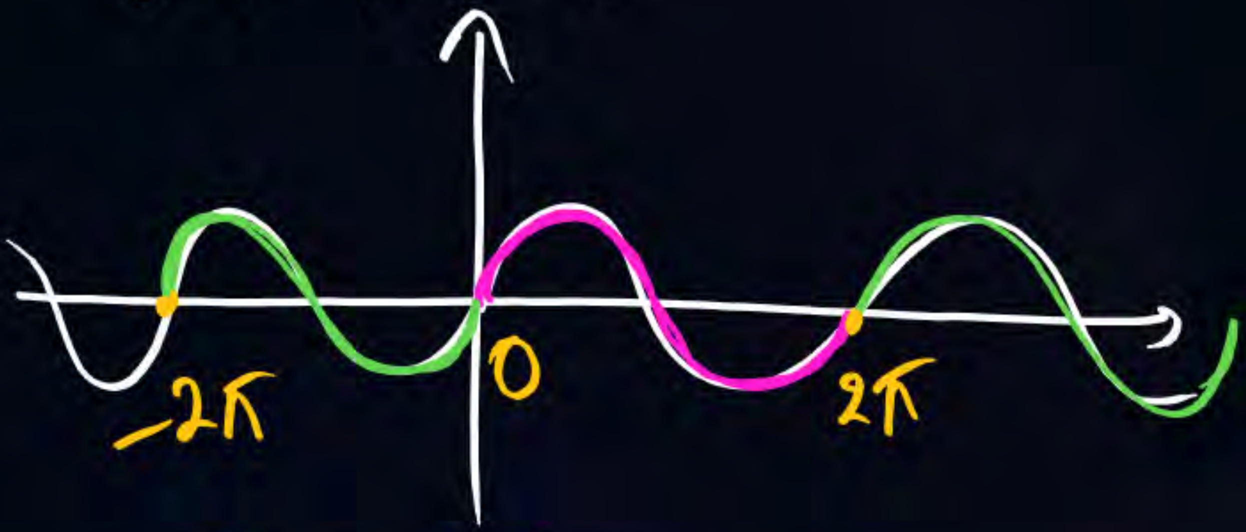
Periodic func^n $\rightarrow$ $f(x)$ is called periodic func^n of period T if

$$\boxed{f(x+T) = f(x)}$$

for eg $\sin(x+2\pi) = \sin x$ so period of $\sin x$ is 2π

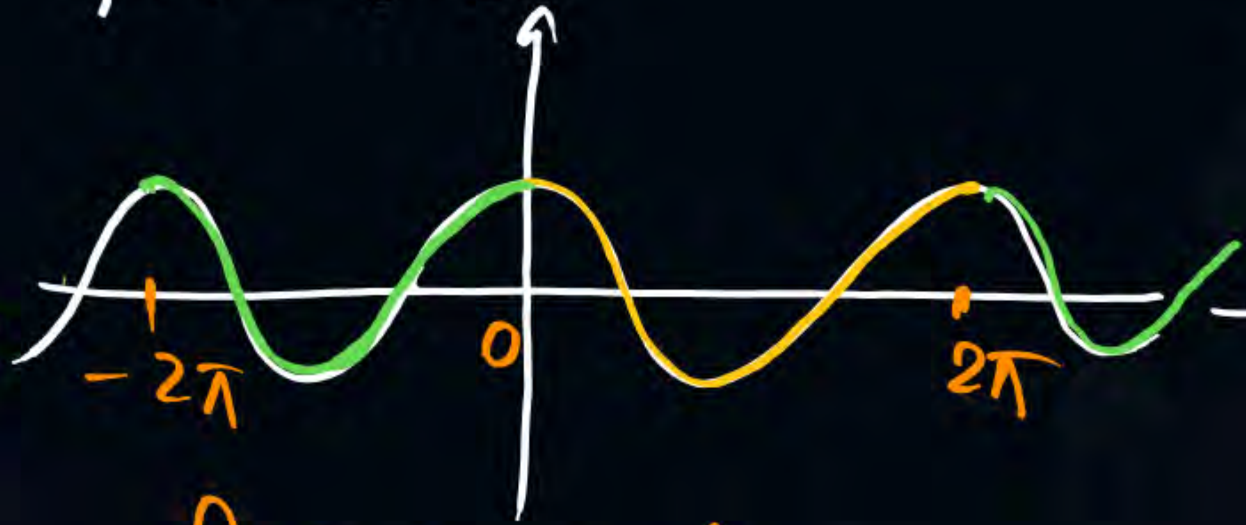
& $\tan(x+\pi) = \tan x$ is period in $\tan x$ is π

$f(x) = \sin x$



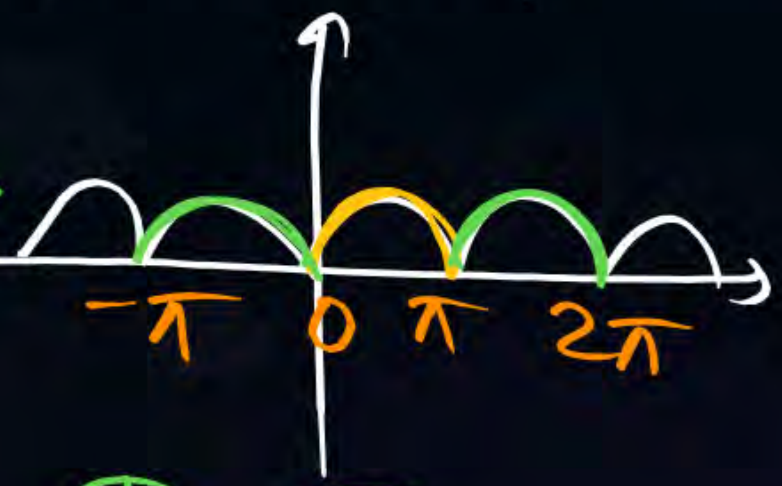
Period $(T) = 2\pi$

$f(x) = \cos x$



Periodic having $T = 2\pi$

$f(x) = |\sin x|$



$T = \pi$

Periodic.

* $y = \sin x$ = odd func^n

Domain $= (-\infty, \infty)$, Range $= [-1, 1]$, Period $= 2\pi$

* $y = |\sin x|$ = Even func^n

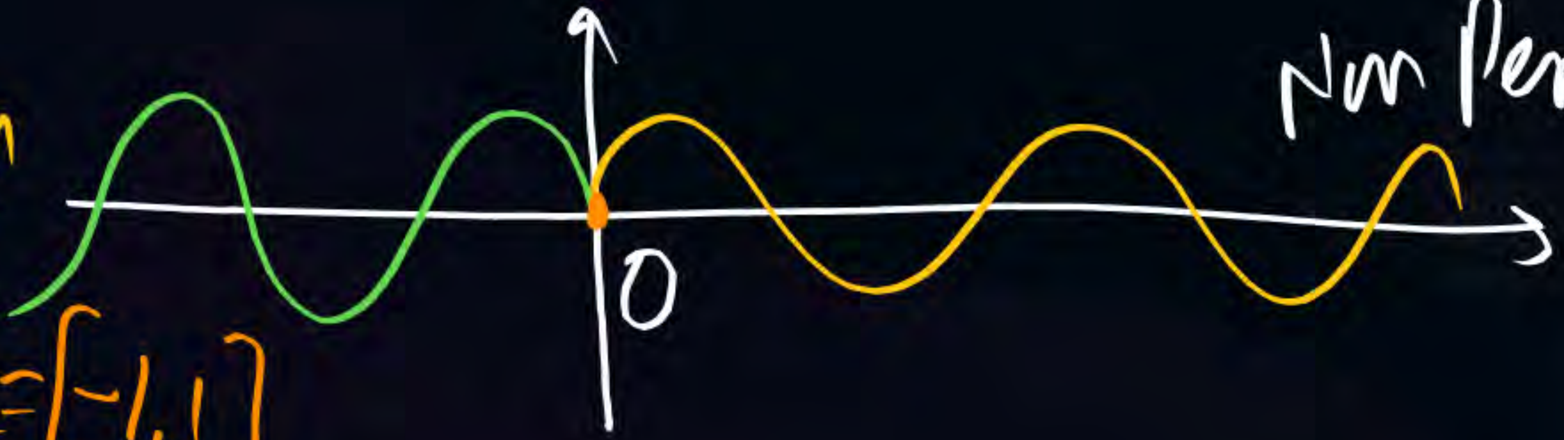
Dom $= (-\infty, \infty)$

Range $= [0, 1]$

Period $= \pi$

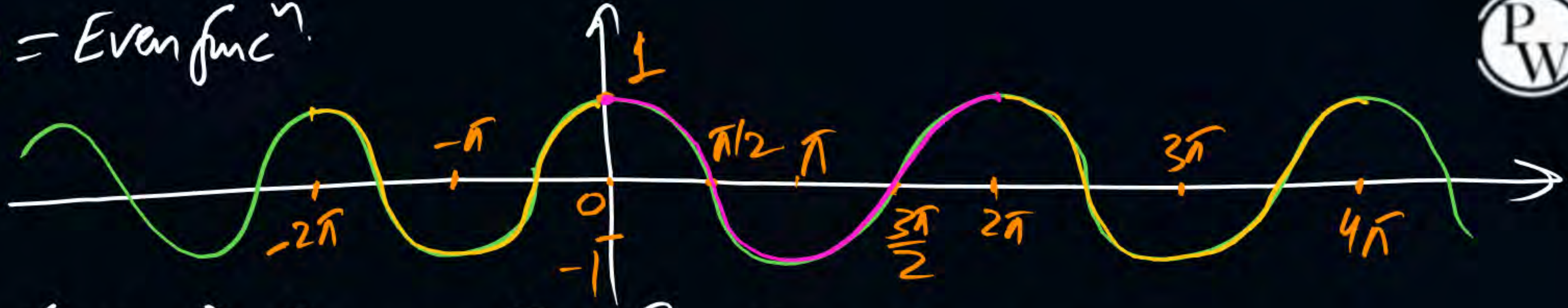
* $y = \sin|x|$ = Even func^n

Domain $= (-\infty, \infty)$, Range $= [-1, 1]$



Non Periodic

(*) $y = \cos x$ = Even func^n



Domain $= (-\infty, \infty)$, Range $= [-1, 1]$, Period $= 2\pi$

(*) $y = |\cos x|$ = Even func^n

Dom $= (-\infty, \infty)$

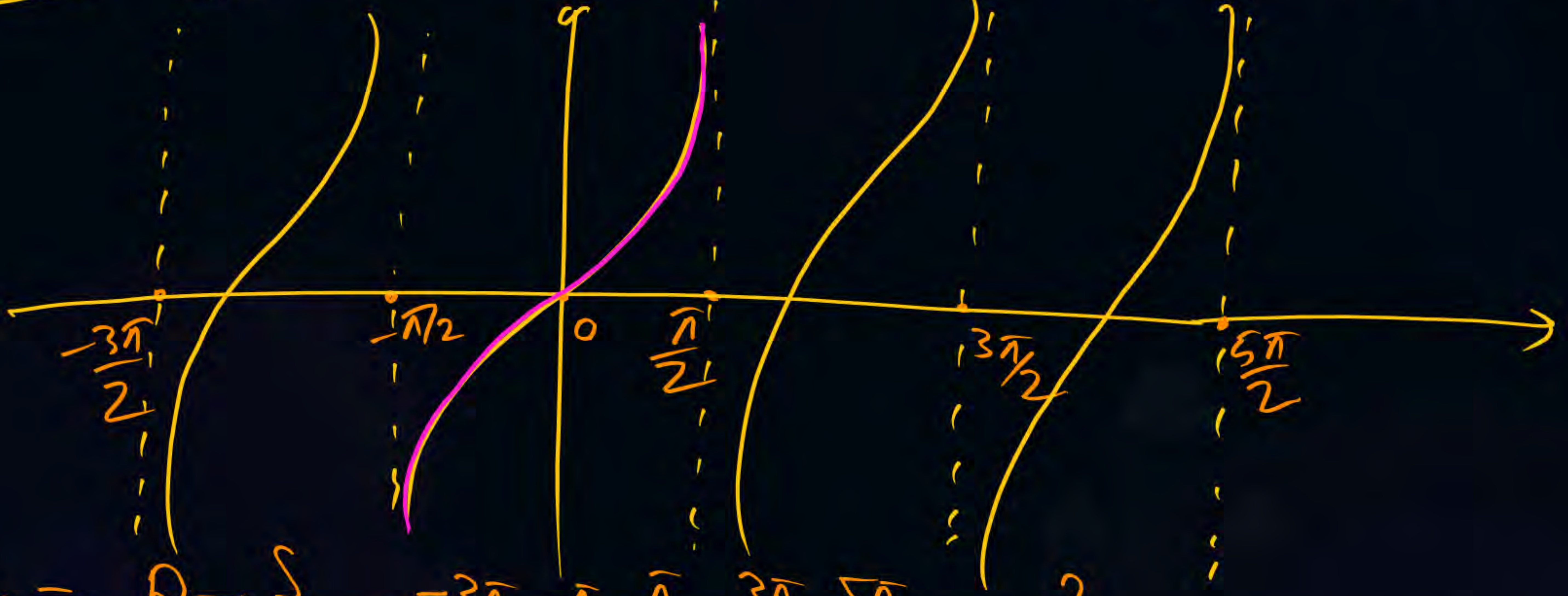
Range $= [0, 1]$

(*) $y = \cos|x|$ = Even func^n

Dom $= (-\infty, \infty)$

Range $= [-1, 1]$

(x) $y = \tan x = \frac{\sin x}{\cos x} = \frac{\text{odd}}{\text{Even}} = \text{odd func}^n$, Period $= \pi$, Range $= (-\infty, \infty)$



Domain $= R - \left\{ \ldots\ldots -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots\ldots \right\}$

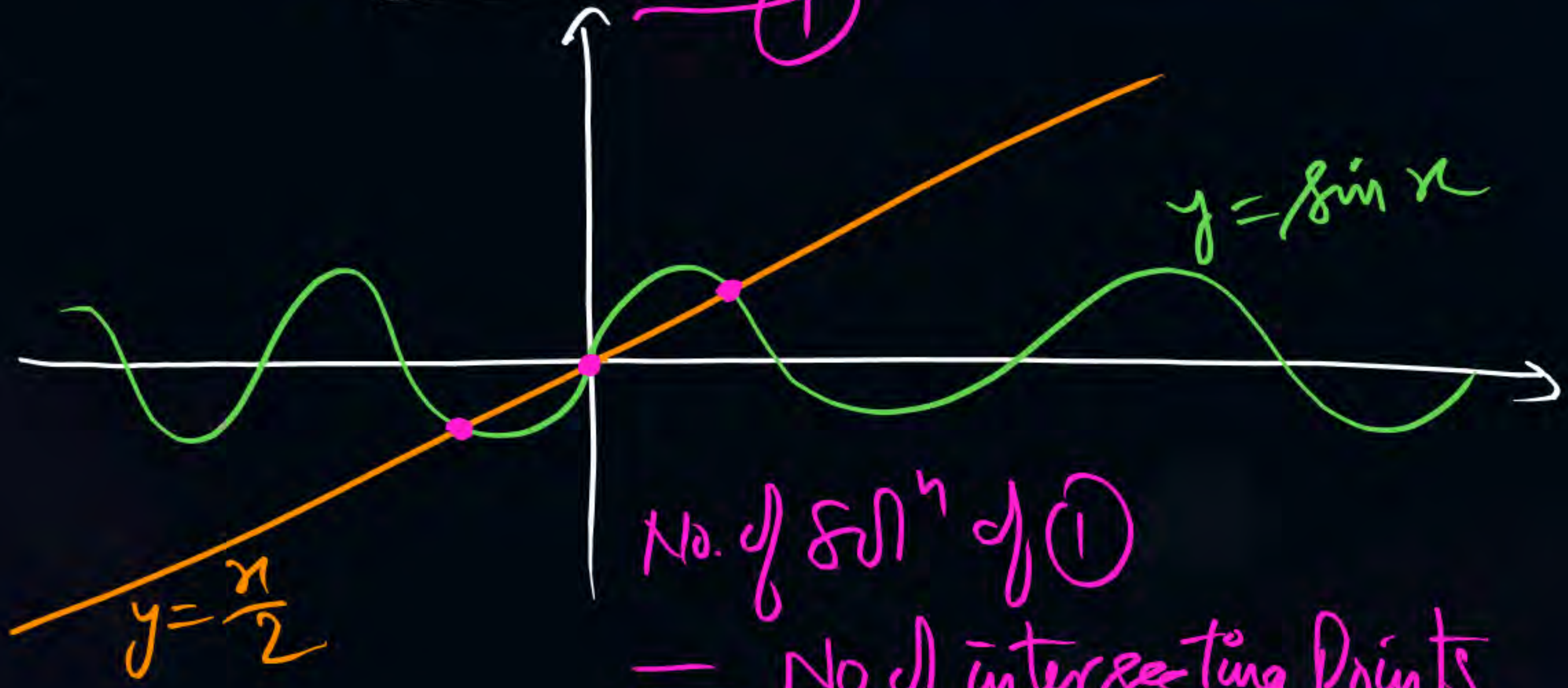
$= R - (2n+1)\frac{\pi}{2}, \ n \in I$

i.e. tan x is not defined at ODD multiples of $\frac{\pi}{2}$

Q. The Number of Solutions of $\boxed{\sin x = \frac{x}{2}}$ is/are ? = three —— (1)

Consider $y = \sin x$

$y = \frac{x}{2}$



No. of soln of (1)

= No. of intersecting Points

= three

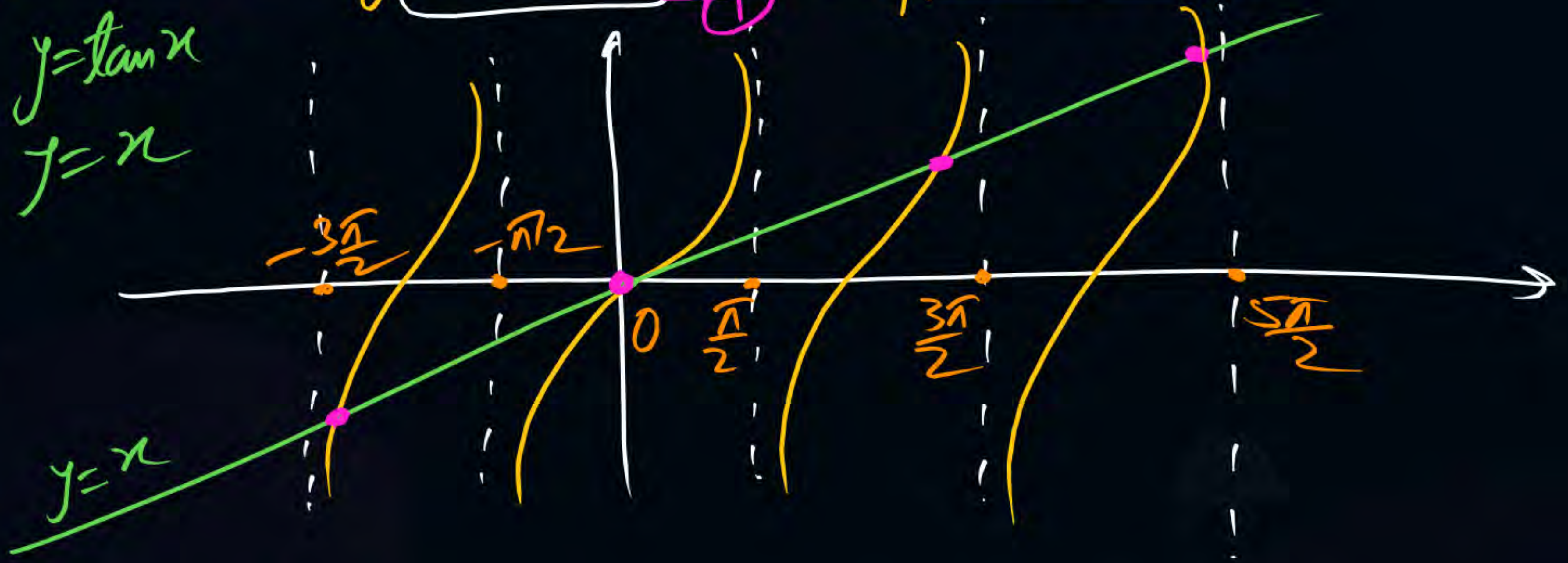
Q. one of the solution of $\boxed{\tan x = x}$ (1) can be approximated as

(a) 1.57

(b) 3.14

(c) 4.50 ✓

(d) None



(ii) No. of solⁿ's of (1) = ? = ∞

& solⁿ's are $x \simeq \ldots\ldots -\frac{3\pi}{2}, 0, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots\ldots$

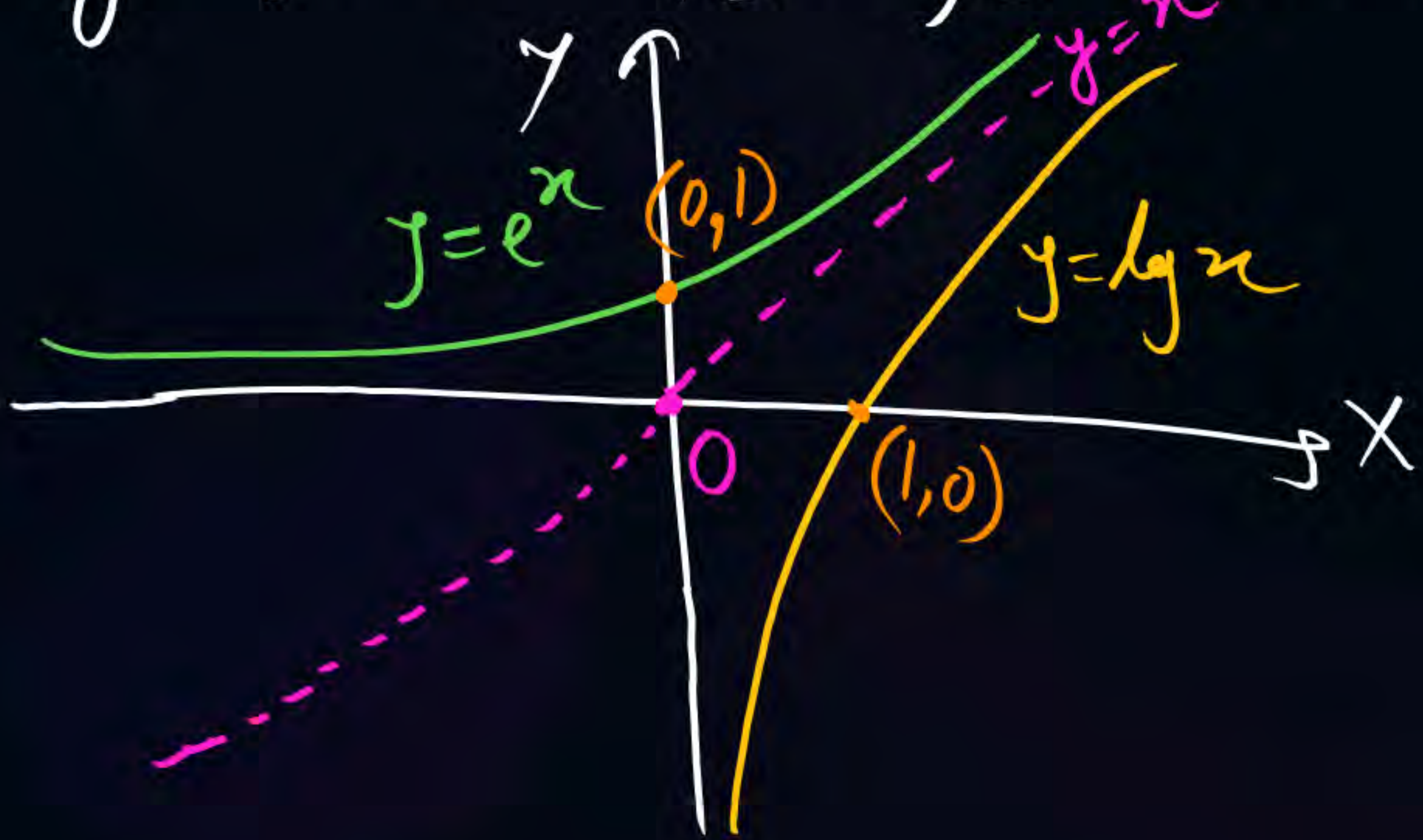
$x \approx \ldots\ldots -4.5, 0, 4.5, 7.5 \ldots\ldots$

INVERSE func^n → If $y = f(x)$ & $y = g(x)$ are Inverse func^n of each other

Then they are symmetrical about the line $y = x$

i.e $y = x$ will behave like a mirror for $f(x)$ & $g(x)$.

eg $y = e^x$ & $y = \log_e x$ are Inverse func^n of each other.

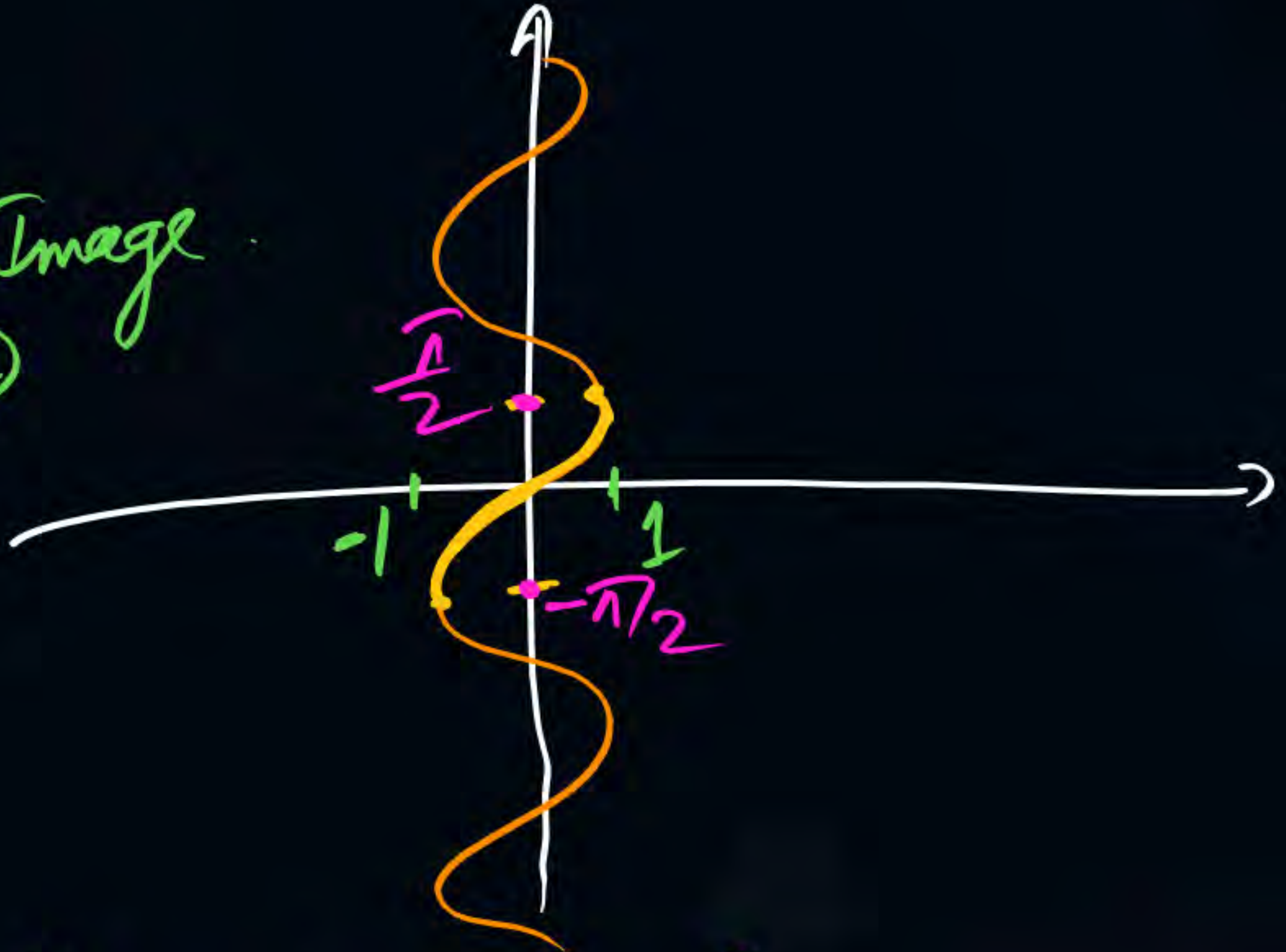


(*) $y = \sin^{-1}x$

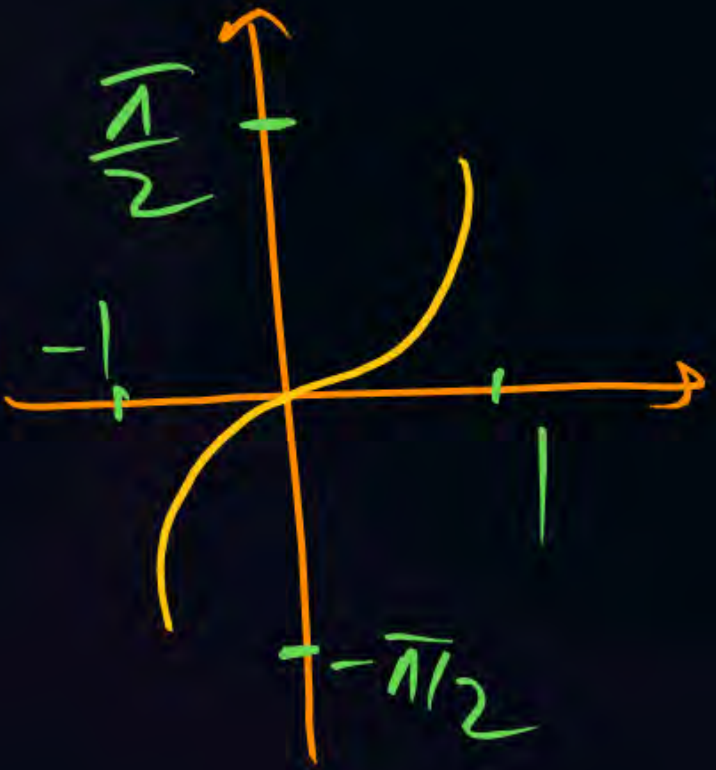
Dom $= (-\infty, \infty)$

Range $= [-1, 1]$

Mirror Image $\Longrightarrow$



$y = \sin^{-1}x$

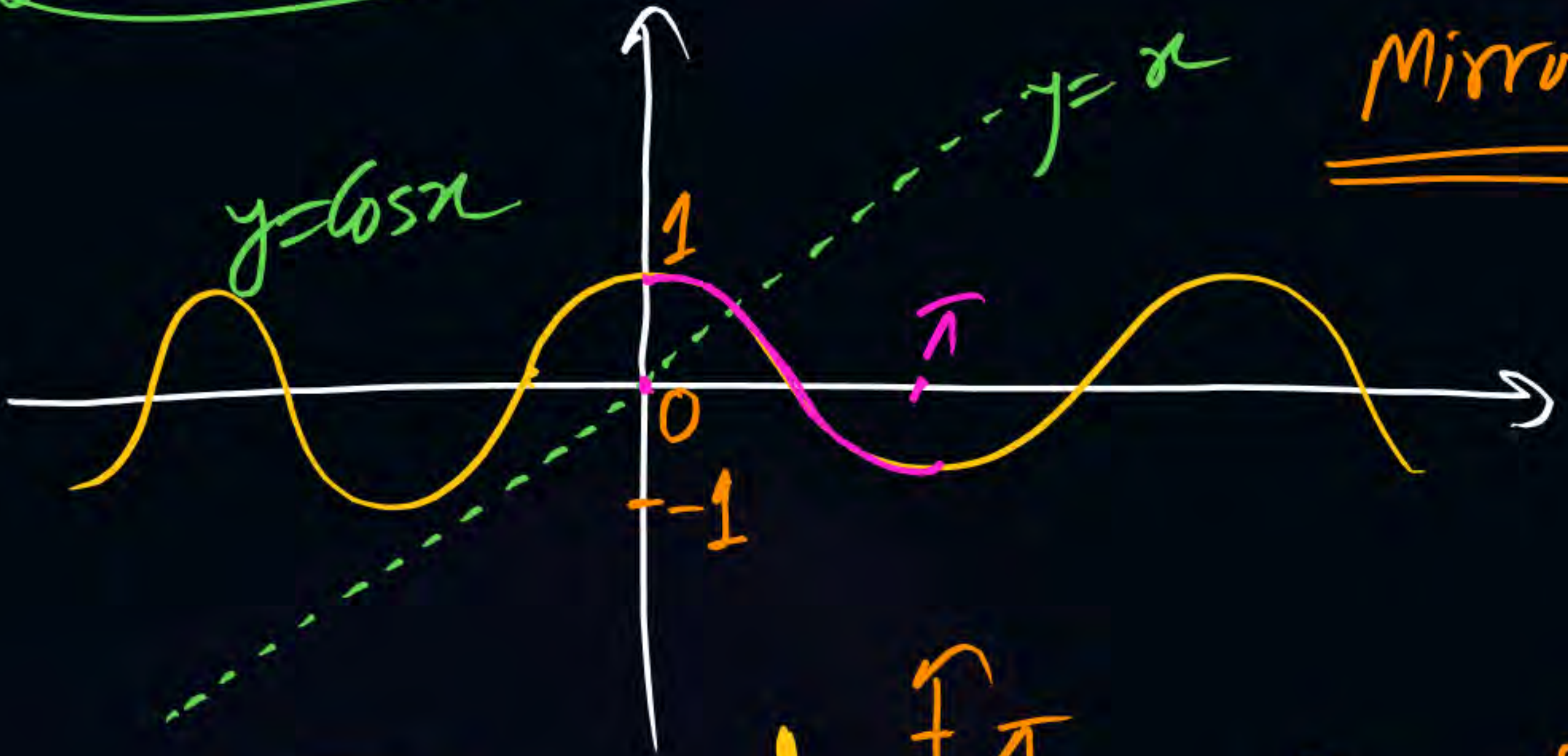


Domain $= [-1, 1]$

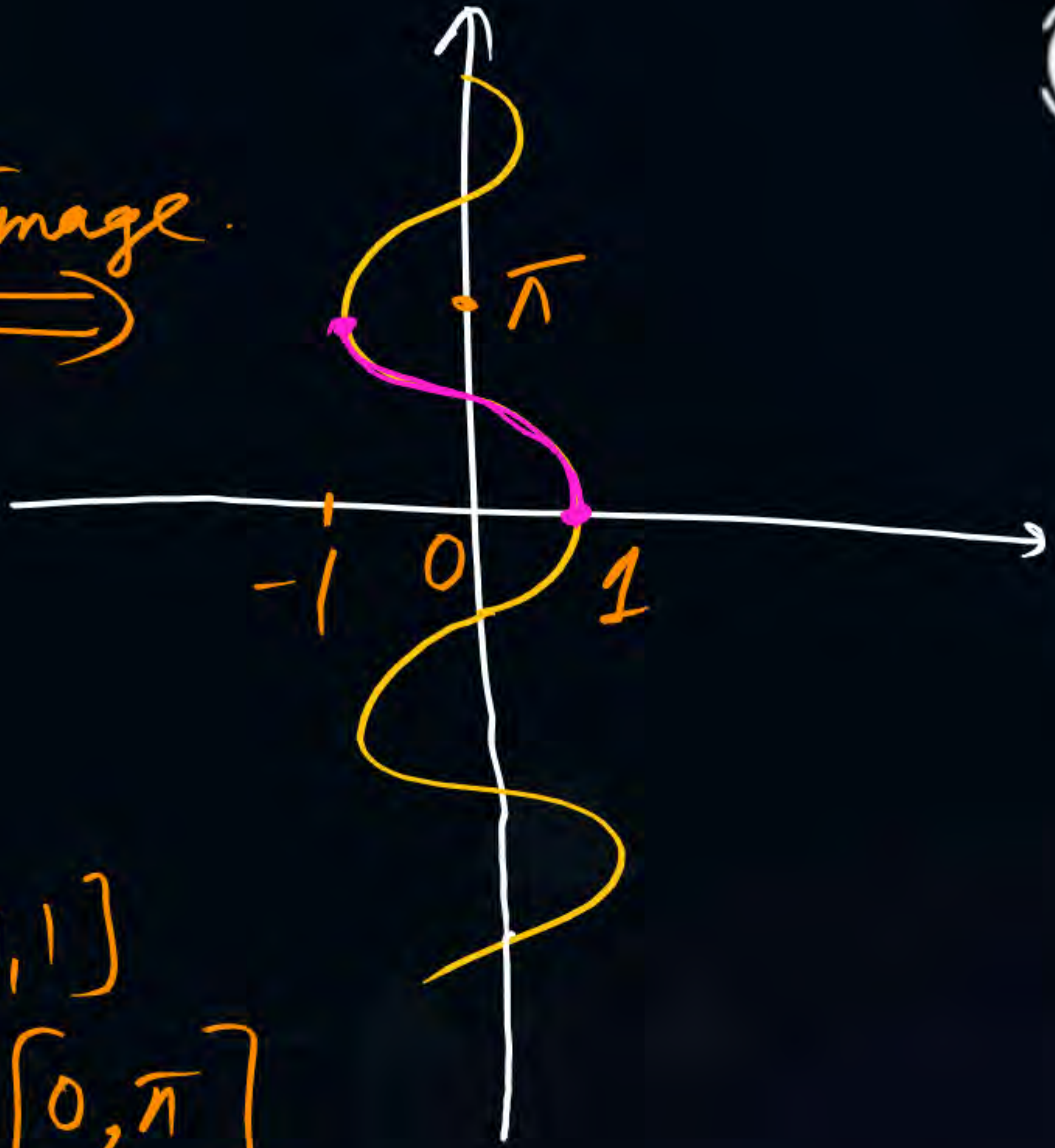
Range $= \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Odd func^n.

$y=\cos^{-1}x$



Mirror Image



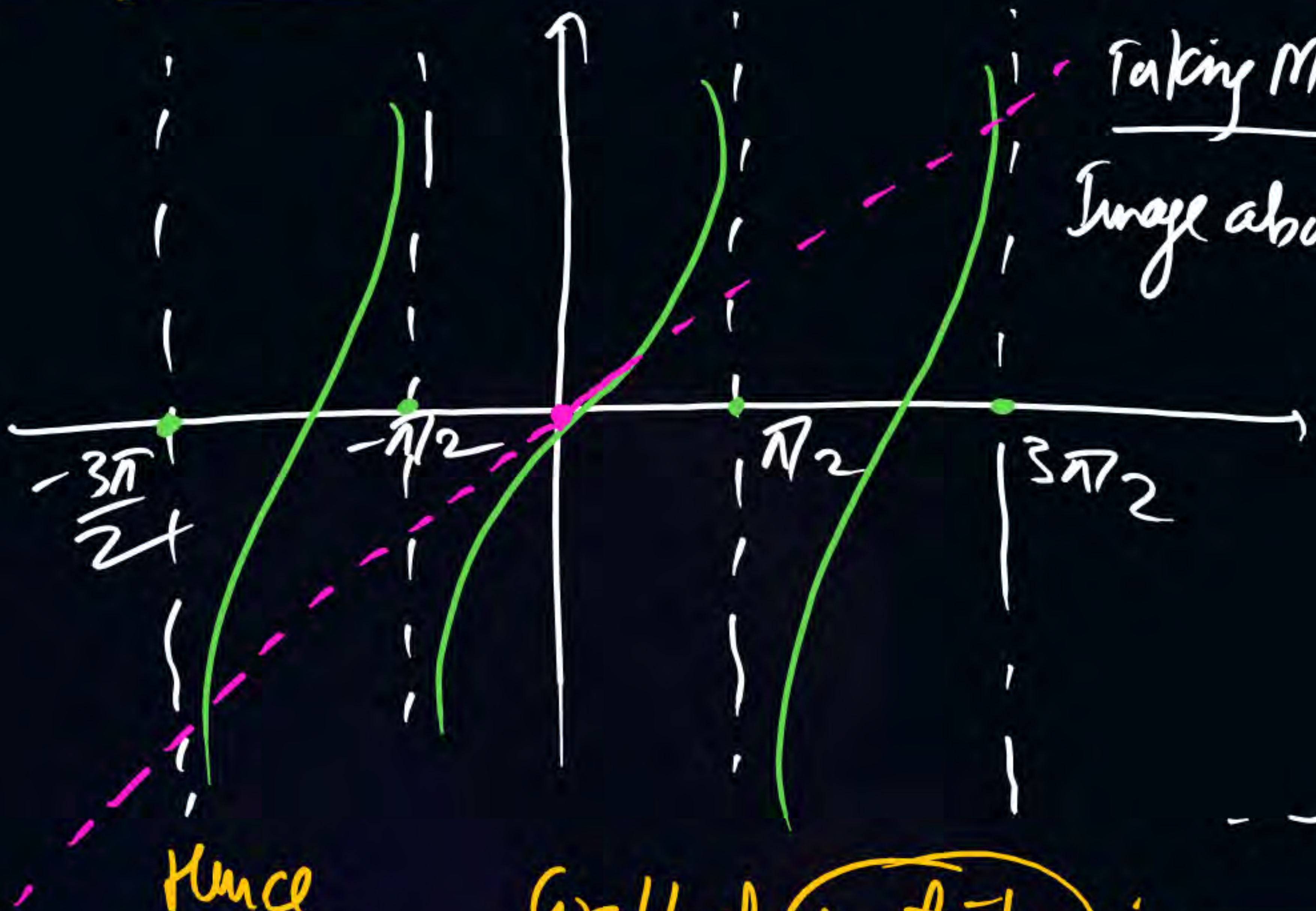
$y=\cos^{-1}x$

$\text{Dom} = [-1, 1]$

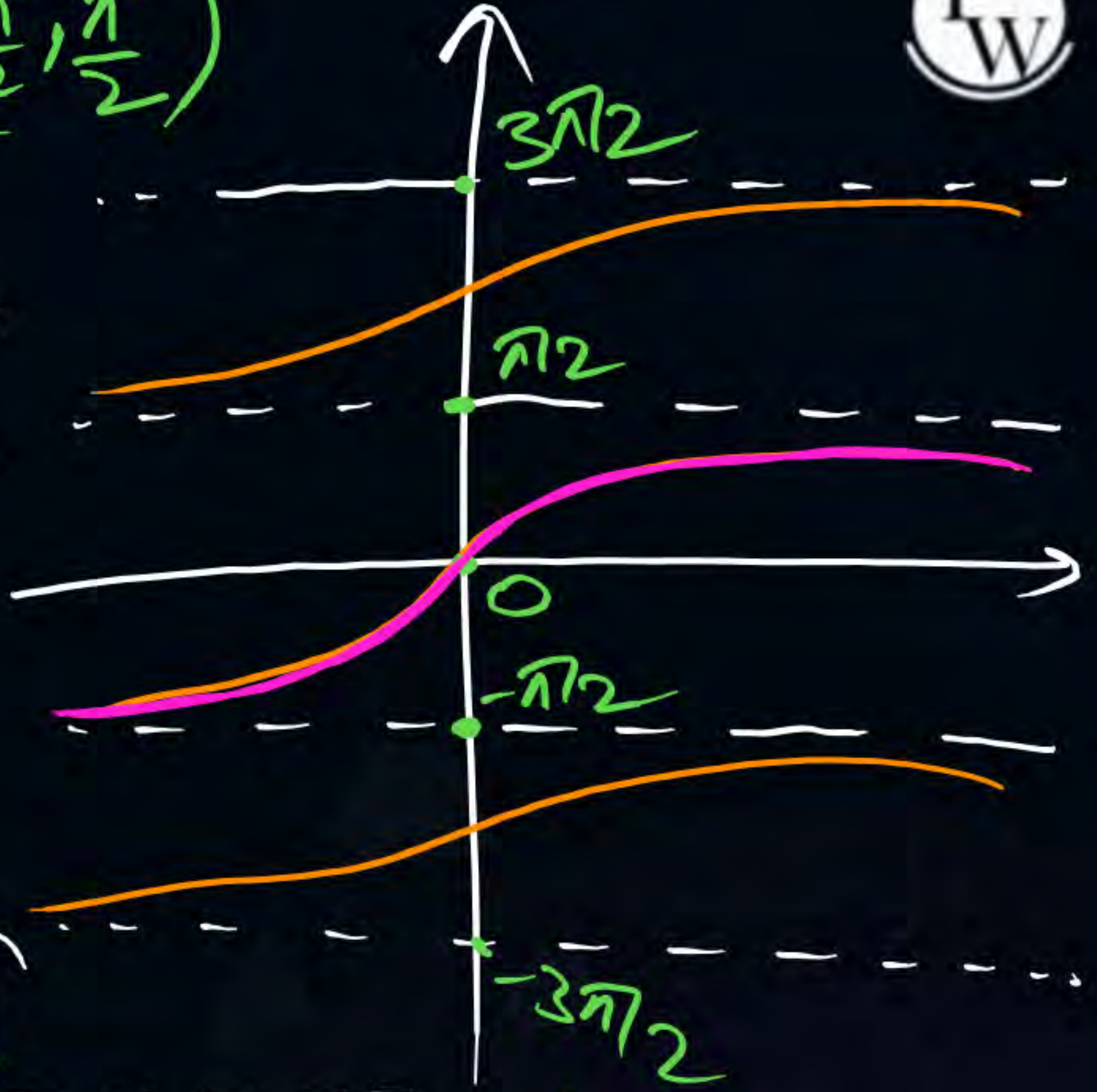
$\text{Range} = [0, \pi]$

NENO.

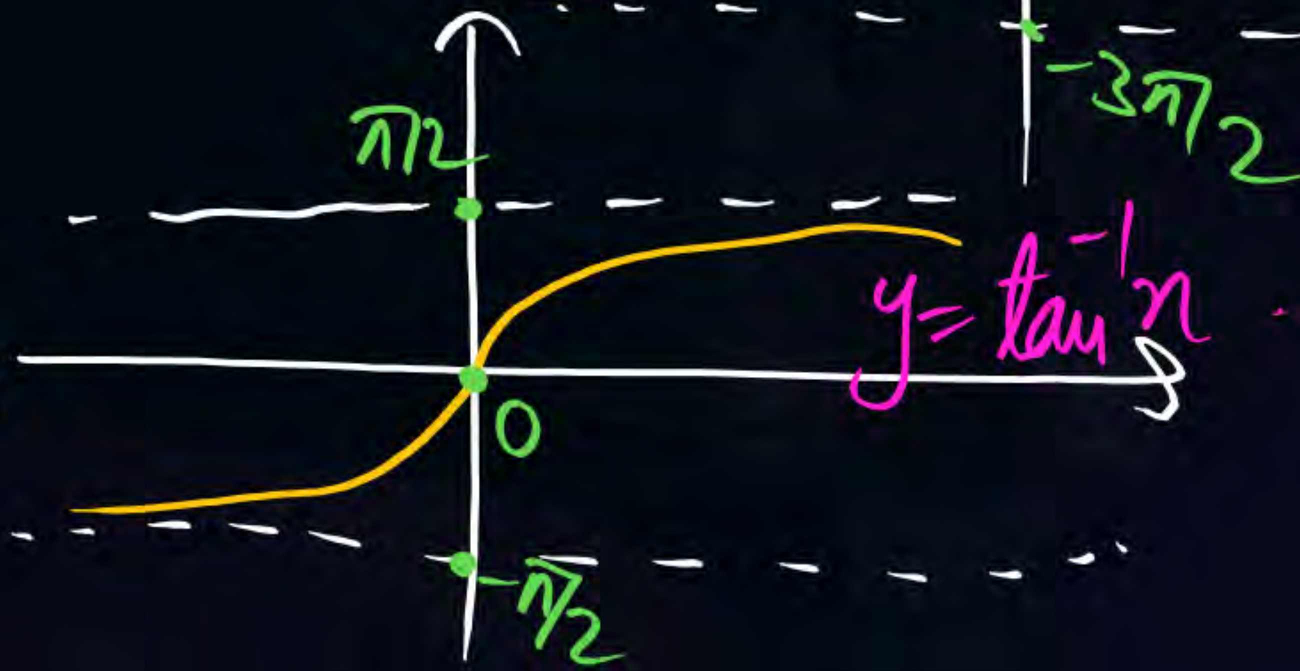
(x) $y = \tan^{-1}x$ Domain: $(-\infty, \infty)$, Range: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



Taking Mirror Image about $y = x$



Hence Graph of $y = \tan^{-1}x$ is



& it is an ODD func^n.

Function → If $\forall x \in A$ $\exists$ unique $y \in B$ s.t $f(x) = y$ then f is called func^n from A to B & it is denoted as $f : A \to B$

A → Domain

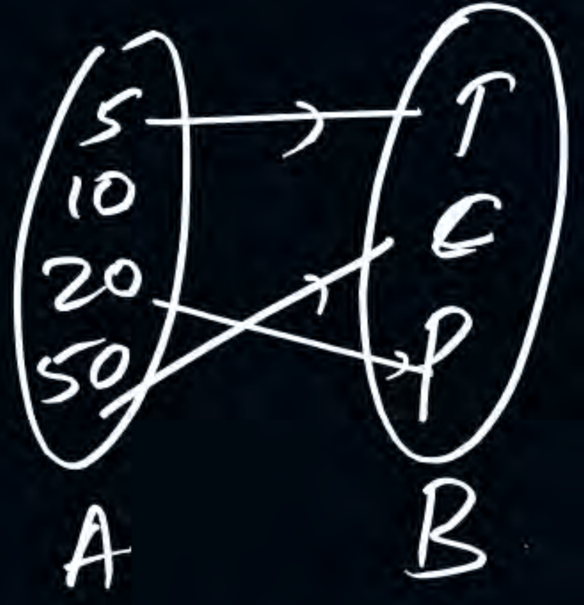
B → Codomain

(*) In $y = f(x)$

y → Dependent Variable (OUTPUT)

x → Independent Variable (INPUT)

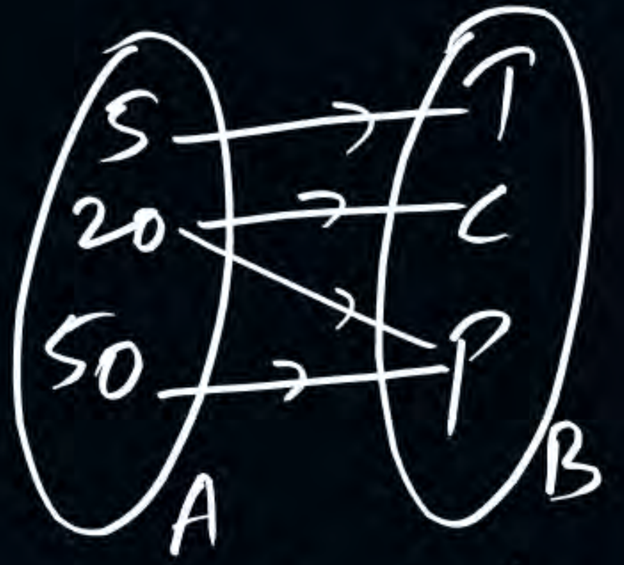
eg:



it is not a func^n

$\because$ $\forall$ is not satisfied

eg



it is not a func^n

$\because$ uniqueness is not satisfied

eg

it is func^n, Dom $= \{5, 20, 50\}$

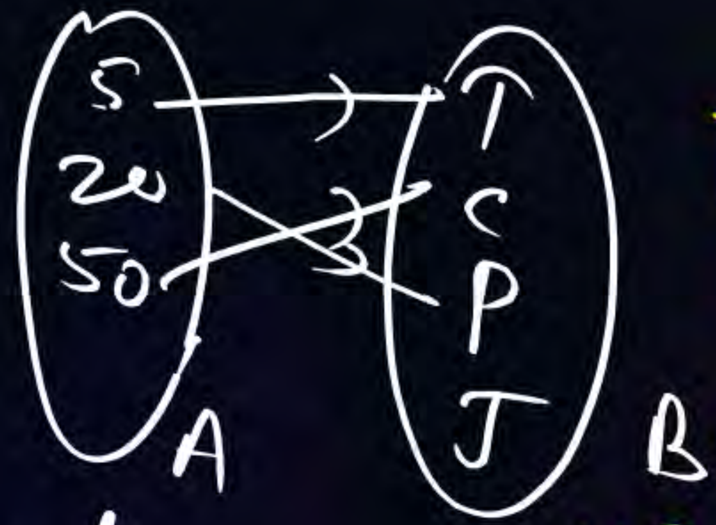
Range $= \{T, C, P\}$

Codomain $= \{T, C, P\}$

$\because$ Range $=$ Codomain

so f is ONTO

eg



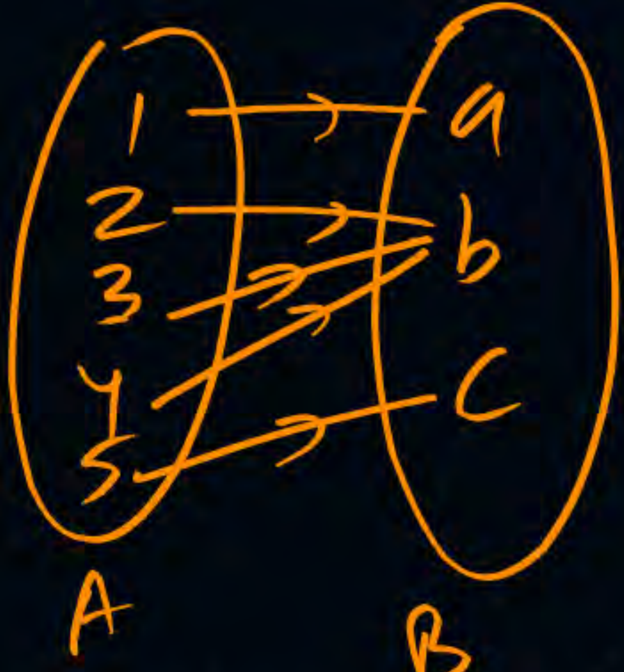
it is also func^n

Dom $= \{5, 20, 50\}$

Range $= \{T, C, P\}$

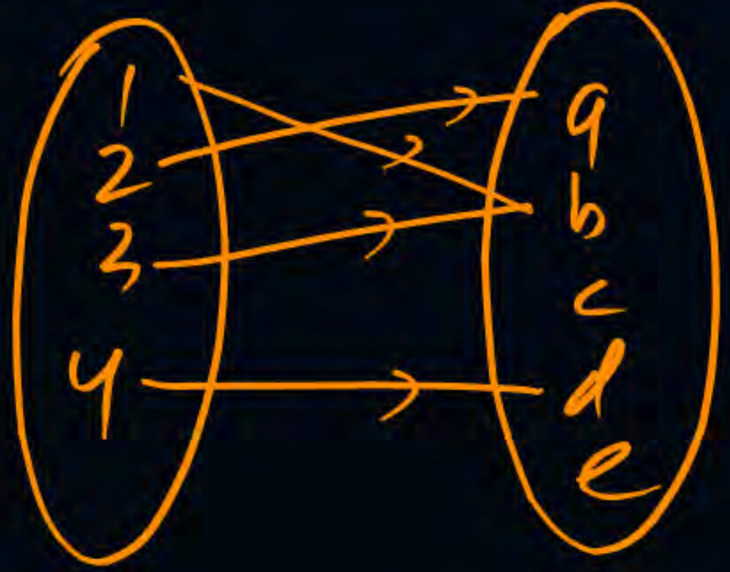
Codomain $= \{T, C, P, J\}$

$\because$ Range $\subseteq$ Codomain so f is INTO g



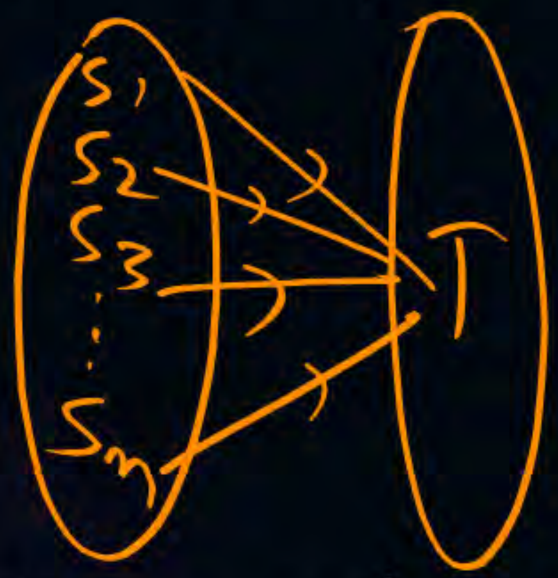
MANY ONE / ONTO

ONE-ONE / ONTO

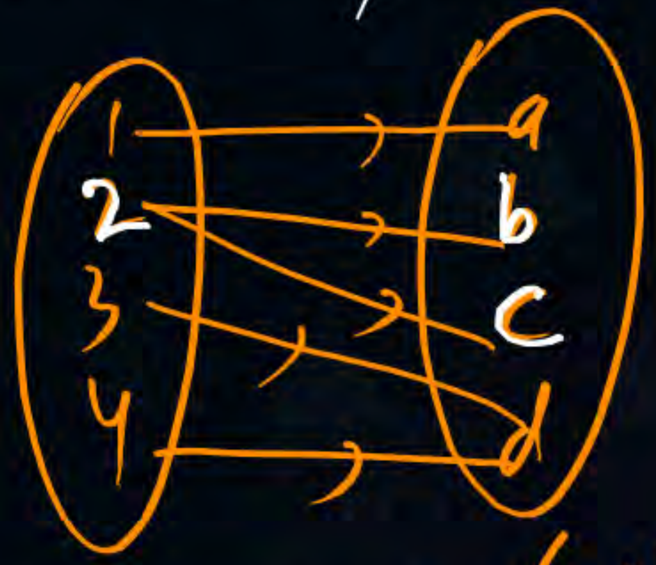


MANY ONE / INTO

ONE-ONE / INTO



MANY-ONE / ONTO



ONE-MANY (not a func^n)

Domain of $y=f(x) \Rightarrow$ Set of permissible values of x is called Domain

Range of $y=f(x) \Rightarrow$ Set of permissible values of y is called Range

i.e. Restrictions imposed on Inputs (x) is called Domain

& " " " outputs (y) " " Range.

Note: Vertical line Test $\rightarrow$ If any Random line $\parallel^r$ to y axis, cuts the graph only at one point, then it is a func^n.

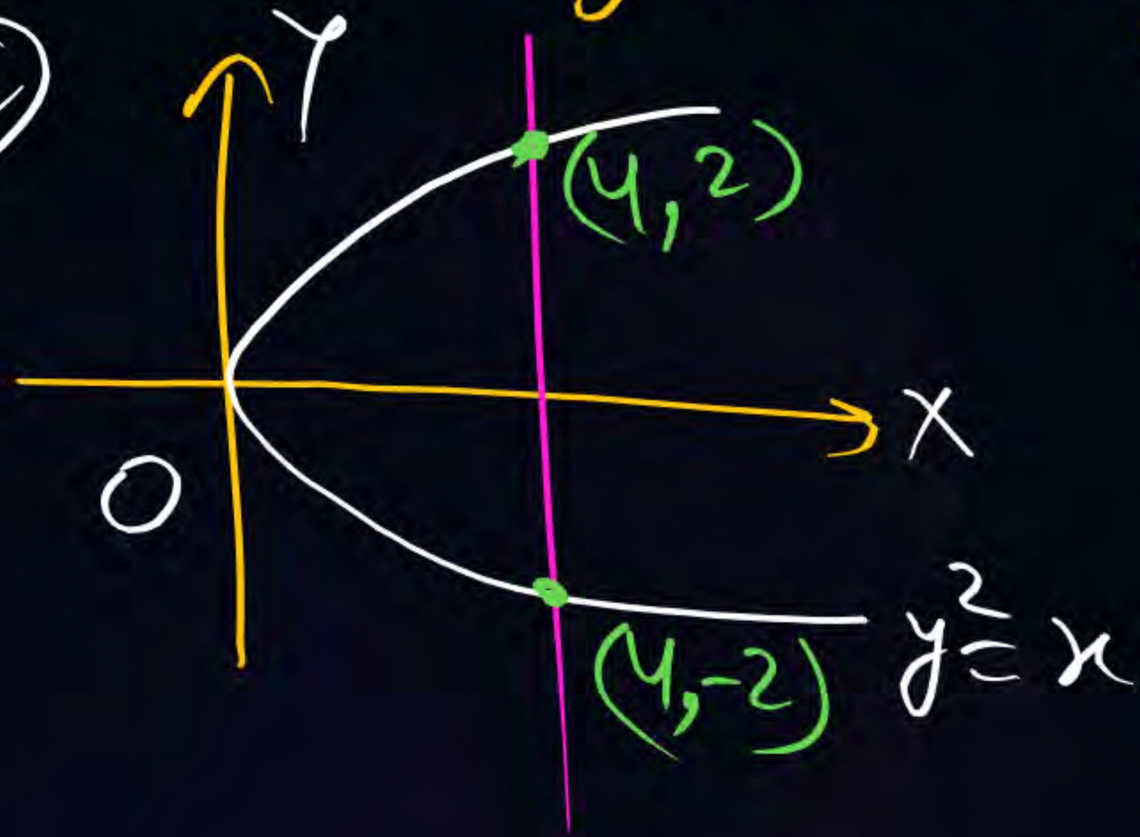
& if this line cuts the graph at more than one point, then it is not a func^n.

eg Take $y = x^2$ then At $x = 4$, $y = 16$, unique so it is func^n.

Take $y^2 = x$, then $y = \pm\sqrt{x}$ so at $x = 4$, $y = \pm\sqrt{4} = \pm 2$

i.e. y is not unique for $x = 4$ so it is not a func^n

M-II



for $y = f(x) \Rightarrow f(4) < \begin{matrix} +2 \\ -2 \end{matrix}$??

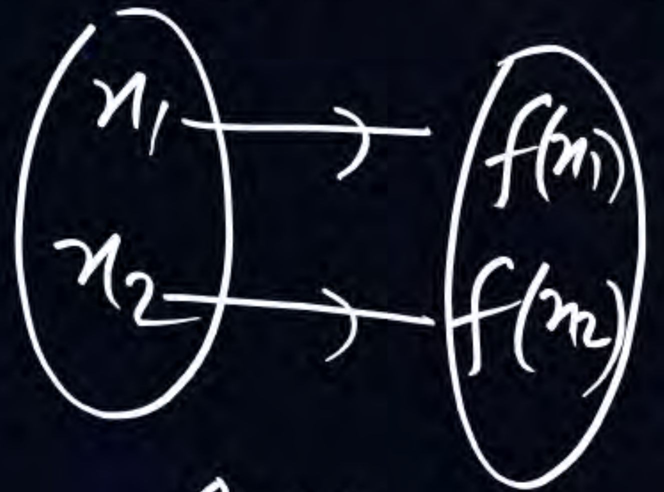
OR we can say that it is one to Many so Not a func^n.

one-one func^n: if $\boxed{x_1 \neq x_2 \Longleftrightarrow f(x_1) \neq f(x_2)}$ where $x_1, x_2 \in D_f$

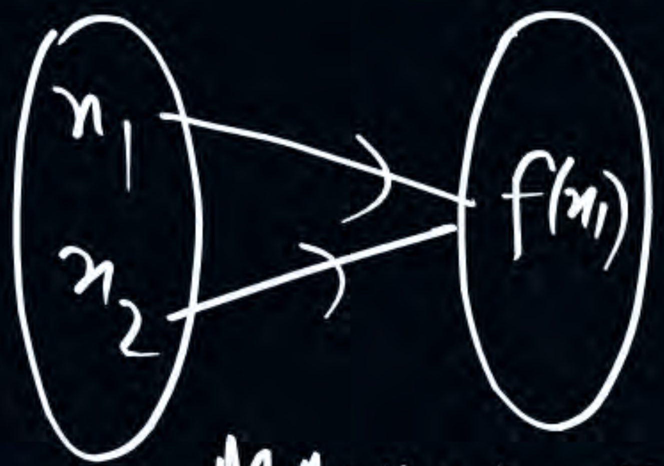
then f is called ONE-ONE i.e. Diff elements have Diff Images.

if $\boxed{x_1 \neq x_2 \Longleftrightarrow f(x_1) = f(x_2)}$ then it is MANY ONE

i.e. for MANY ONE func^n, Different elements Same Images.



ON-ONE



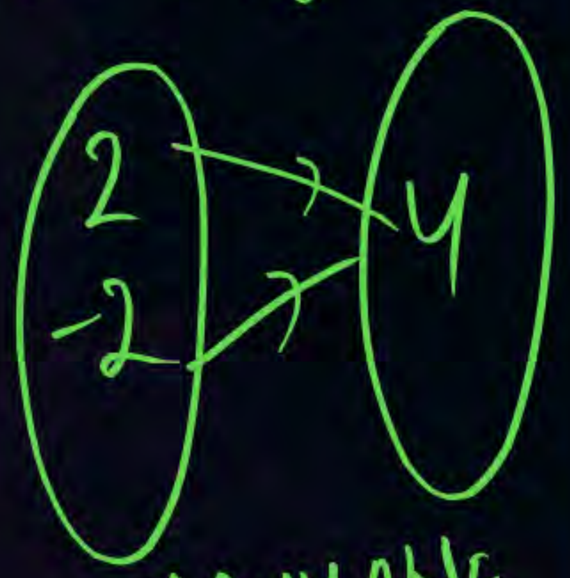
MANY ONE

Horizontal Line Test (Shortcut of checking ONE-ONE OR MANY ONE)

If Any Random line $\parallel^r$ to x axis cuts the graph only at one point then it is one-one otherwise MANY-ONE

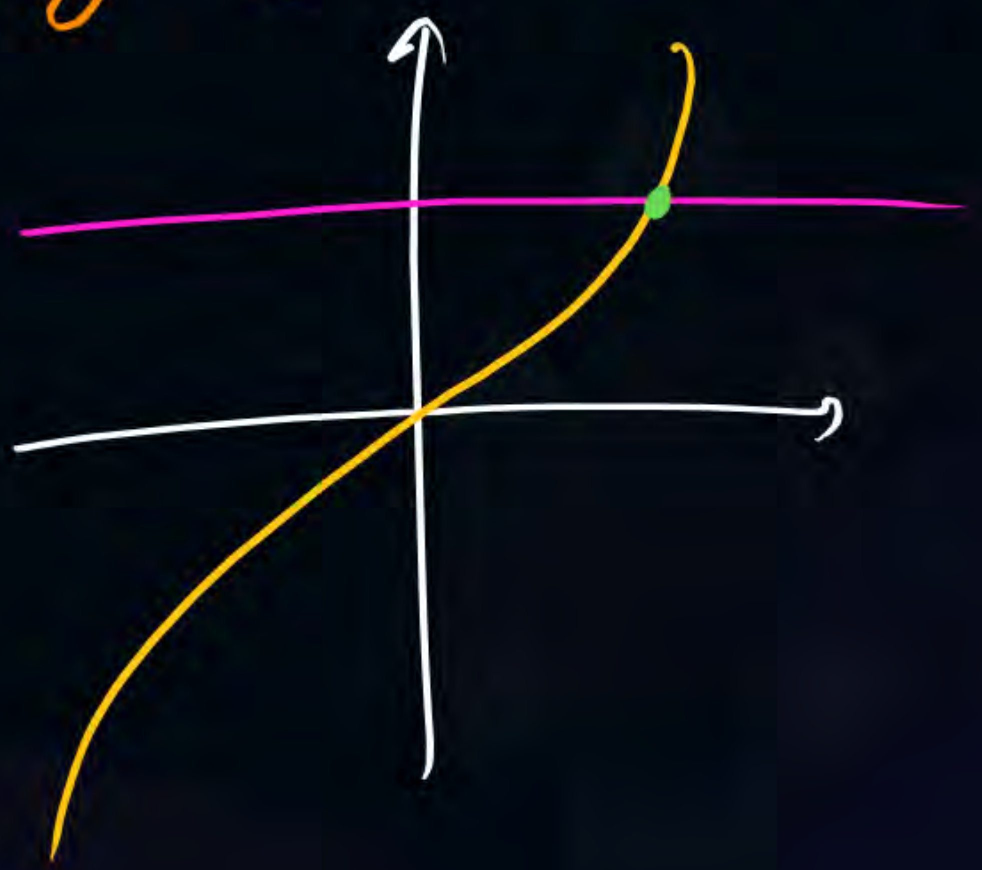
eg $y = x^2$ (M-I) : At $x = 2, y = 4$ & at $x = -2, y = 4$

So MANYONE.



MANYONE

eg $y = x^3$ = ONE-ONE



Conclusions:

① Vertical line test → To check validity of func^n.

② Horizontal line test → To check validity of one-one

③ If $\boxed{\text{Range} = \text{Codomain}}$ → Then func^n is ONTO.

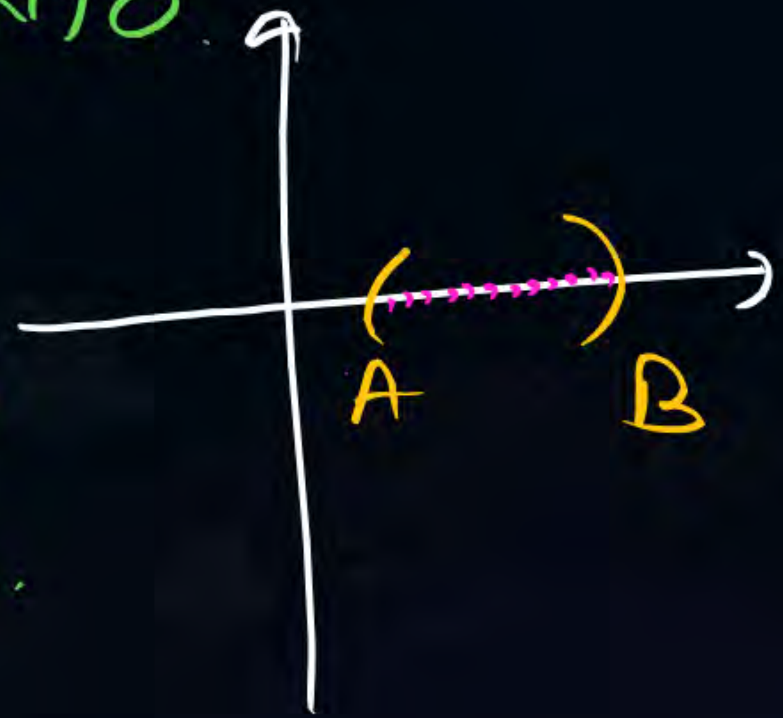
④ ONE-ONE func^n $\Longrightarrow$ INJECTIVE MAPPING

⑤ ONE-ONE/ONTO $\Longrightarrow$ BIJECTIVE MAPPING / one-one correspondence.

⑥ one one-one/ONTO func^n have Inverse.

i.e. if $f(x)$ is one one/ONTO then only $f^{-1}(x)$ exist

$\Longleftrightarrow$



Domain: Set of permissible values of x is called Domain of $y=f(x)$ & there is No shortcut Method to find Domain of given func^n; it can be calculated only by using common sense or by previous knowledge.

eg: find the Domain of following func^n;

① $y=f(x)=\dfrac{1}{x^2-5x+6}$

$y=\dfrac{1}{(x-2)(x-3)}$

At $x=2$ & $x=3$, $y=$ DNE

Dom $=R-\{2,3\}$

② $y=\sqrt{-x^2+5x-6}$

w.k.that, $-x^2+5x-6 \geq 0$

$x^2-5x+6 \leq 0$

$(x-2)(x-3) \leq 0$

$2 \leq x \leq 3$

So Dom $=[2,3]$

CROSS Check

eg $y = \sqrt{-x^2+5x-6}$, Dom $= [2,3]$

Let us take $x = 1$ then

$y = \sqrt{-1^2+5(1)-6} = \sqrt{-2} =$ Not Real

Let us take $x = 5$ then

$y = \sqrt{-5^2+5(5)-6} = \sqrt{-6} =$ Not Real.

ie Permissible values of x lies only in b/w 2 & 3

eg Let $x = 2.5$

$y = \sqrt{-(2.5)^2+5(2.5)-6}$

$= \sqrt{-6.25+12.5-6}$

$= \sqrt{-12.25+12.50}$

$= \sqrt{0.25} = 0.5$

ie y is also Real.

ie Valid

Common sense:

1. $\dfrac{1}{f(x)}$; $f(x) \neq 0$

2. $\sqrt{f(x)}$; $f(x) \geq 0$

3. $\dfrac{1}{\sqrt{f(x)}}$; $f(x) > 0$

4. $\log_e(f(x))$; $f(x) > 0$

5. $\sin(f(x))$; $-\infty < f(x) < \infty$

Q: $y = f(x) = \log(x-3)$,

w.k that, $(x-3) > 0 \Rightarrow x > 3$

So Dom $= (3, \infty)$

Q: $y = f(x) = \sin(x-3)$

w.k. that, $-\infty < (x-3) < \infty$

$-\infty < x < \infty$

i.e Dom $= (-\infty, \infty)$

Eg: $\boxed{y = \sin^{-1}(x-3)}$

or $\sin y = x-3$

w.k. that, $-1 \le \sin y \le 1$

$$-1 \le x-3 \le 1$$

$$-1+3 \le x \le 1+3$$

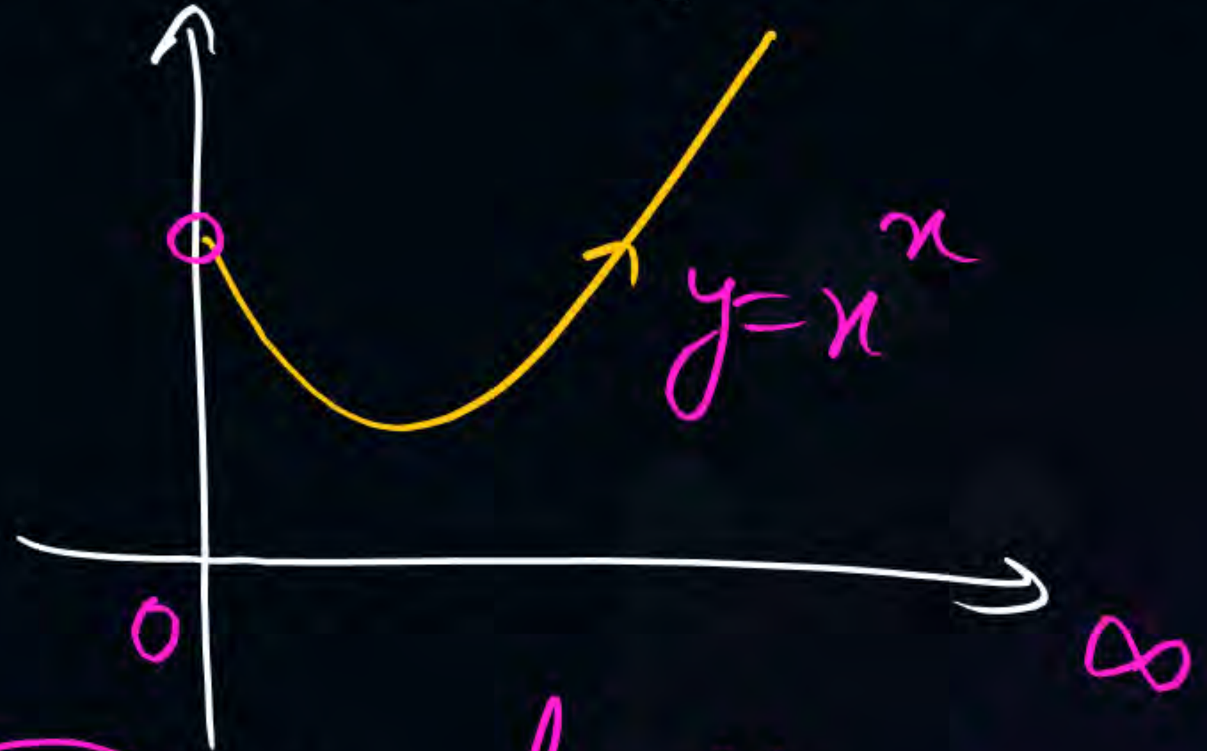
$$2 \le x \le 4$$

$$\text{Dom} = [2,4]$$

Q. $y = x^x$

$$y = e^{\log_e x^x} = e^{(x\log_e x)}$$

So Domain is $(0,\infty)$ or $x>0$



(*) $y = x \neq e^{\log_e x}$, $\text{Dom} = (-\infty,\infty)$

Doubts: (1) Evaluate $(-2)^{-2} = ? = \frac{1}{(-2)^2} = \frac{1}{4}$ ✓

(2) if $y = x^x$ then evaluate $y(-2) = ?$

$\because$ Domain $= (0, \infty)$ so $y(-2) = $ DNE

Defⁿ of funcⁿ in My Language → "it is special type of Relationship b/w two Variables x & y under certain Restrictions"

& these Restrictions on x are Called Domain of funcⁿ.

(*) one-one Mapping $\simeq$ INJECTIVE Mapping

(*) one-one / ONTO Mapping $\simeq$ $\underbrace{\text{BIJECTIVE Mapping}}_{\text{one-one correspondence}}$

(*) ONTO MAPPING $\rightleftharpoons$ SURJECTIVE

eg if, $y = f(x) = x^2$; $\underset{\text{Domain}}{R} \to \underset{\text{Codomain}}{R}$ then find Range = ? $= [0, \infty)$

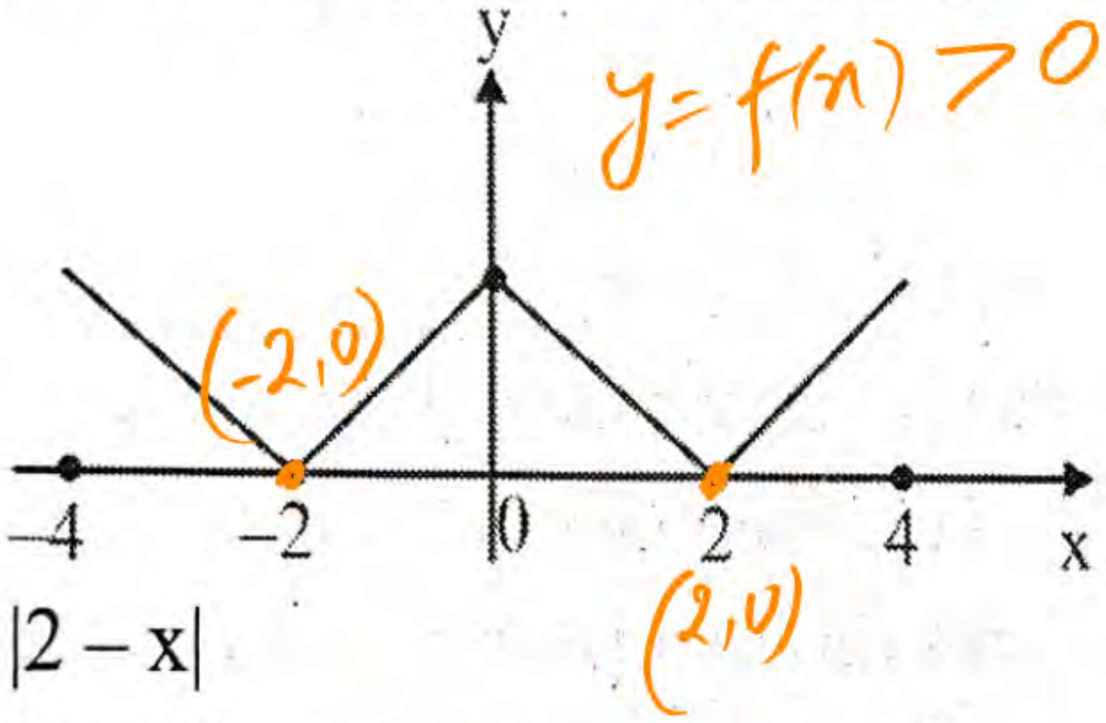
$\because$ Range $\subset$ Codomain $\Rightarrow$ INTO

eg if, $y = f(x) = x^2$; $\underset{\text{Dom}}{R} \to \underset{\text{Codom.}}{R^+}$ then find it's Nature?

& Range $= R^+$ Hence ONTO

[MCQ] [GATE-ME-2023: 1M]

The figure shows the plot of a function over the interval [–4, 4], which one of the options given CORRECTLY identifies the function?



(a) $|2 - x|$

(b) $|2 - |x||$

(c) $|2 + |x||$

(d) $2 - |x|$

At $x = -2$, $y = 0$

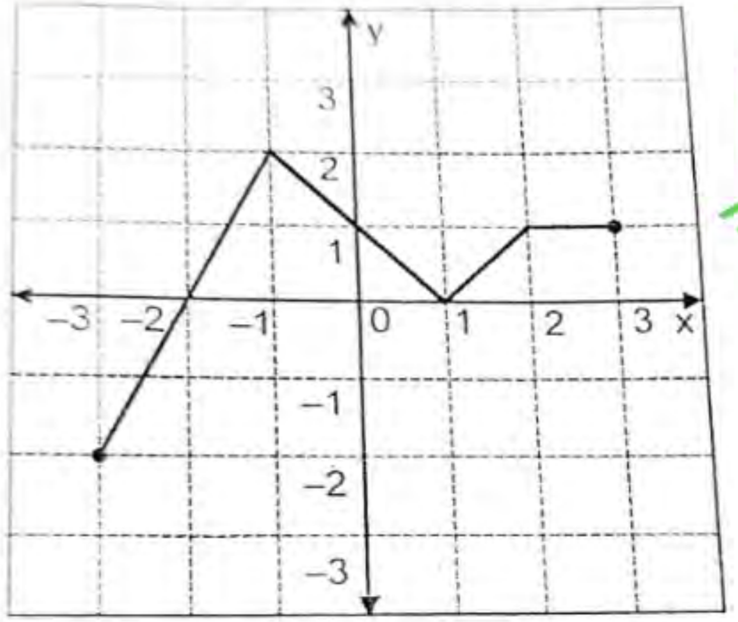
(a) $y = |2 - x| = |2 - (-2)| = |4| = 4$ ✗

(b) $y = |2 - |x|| = |2 - |-2||$

$= |2 - 2| = 0$

So (b) ✓

Which of the following function(s) is an accurate description of the graph for the range(s) indicated?



HW8

(i) $y = 2x + 4$ for $-3 \leq x \leq -1$

(ii) $y = |x - 1|$ for $-1 \leq x \leq 2$

(iii) $y = ||x| - 1|$ for $-1 \leq x \leq 2$

(iv) $y = 1$ for $2 \leq x \leq 3$

(a) (i), (ii) and (iii) only

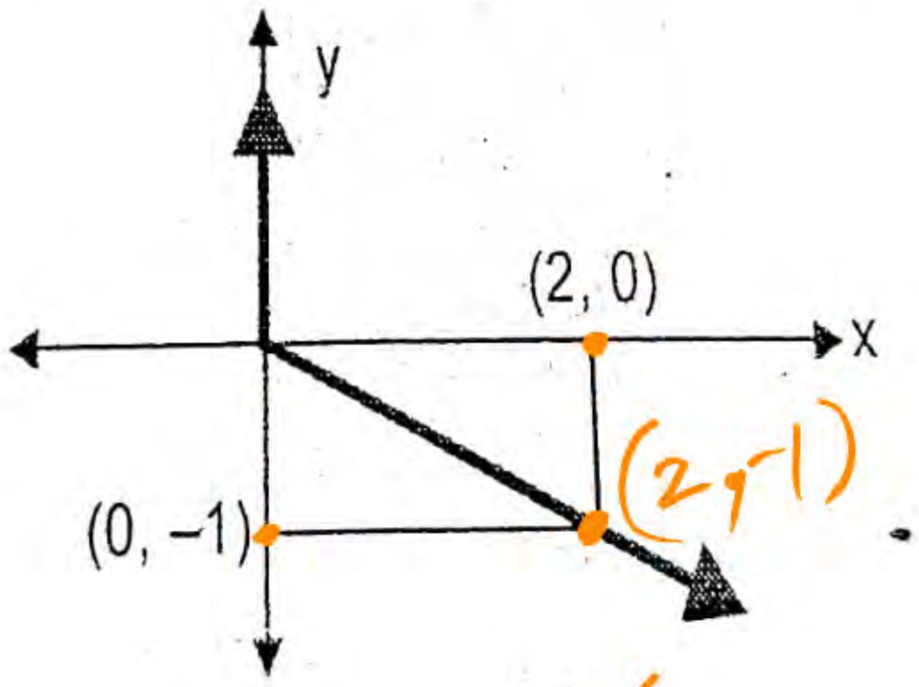
(b) (i), (ii) and (iv) only

(c) (i) and (iv) only

(d) (ii) and (iv) only

[GATE 2018 (CE-MorningSession

Choose the most appropriate equation for the function drawn as a thick line, in the plot below



(a) $x = y - |y|$

(b) $x = -(y - |y|)$

(c) $x = y + |y|$

(d) $x = -(y + |y|)$

[GATE-2015-CS-SET-3; 2 Marks]

$x = 2,\ y = -1$

Taking (a): $x = y - |y|$

$2 = -1 - |-1|$

$= -1 - (+1)$

$2 = -2$ Not Valid for (a)

Taking (b) $x = -(y - |y|)$

$2 = -(-2)$

$2 = 2$ i.e. Valid for (b)

$y > 0$, $x = -(y - |y|) = -(y - y) = 0$ i.e. y axis

$y < 0$, $x = -(y - |y|) = -(y - (-y)) = -2y$

or $y = -\frac{1}{2}x$

THANK you

Vijay

DATA SCIENCE & ARTIFICIAL INTELLIGENCE

& CS

Calculus and Optimization

Lecture No. 03

By- Dr. Puneet Sharma Sir

Recap of previous lecture

Topic FUNCTIONS & GRAPHS

Topics to be Covered

Topic

Limit_ Continuity & Differentiability

(PART-1)

RECAP

Types of functions

- **ALGEBRAIC function**
  1. Polynomial func^n
  2. Rational func^n
  3. Irrational func^n
  4. Piecewise func^n
     - Mod func^n
     - Signum func^n
     - G.I.F.
     - L.I.f
     - F.P.f
- **TRANSCEDENTAL function**
  1. Exponential func^n
  2. log function.
  3. Trigonometric func^n
  4. Inverse Trig. functions.

G.I.f = Greatest Integer func^n (Floor func^n)

L.I.f = Least Integer func^n (Ceiling func^n)

FPF = Fractional Part func^n.

Function $\to$ If $\forall x \in A$ $\exists$ unique $y \in B$ s.t $f(x) = y$ then f is called func^n from A to B & it is denoted as $f : A \to B$

A — Domain, B — Codomain

(*) In $y = f(x)$

y — Dependent Variable (OUTPUT)

x — Independent Variable (INPUT)

Domain of $y=f(x)$ $\Rightarrow$ Set of permissible values of x is called Domain

Range of $y=f(x)$ $\Rightarrow$ Set of permissible values of y is called Range

ie Restrictions imposed on Inputs (x) is called Domain

& " " " outputs (y) " " Range.

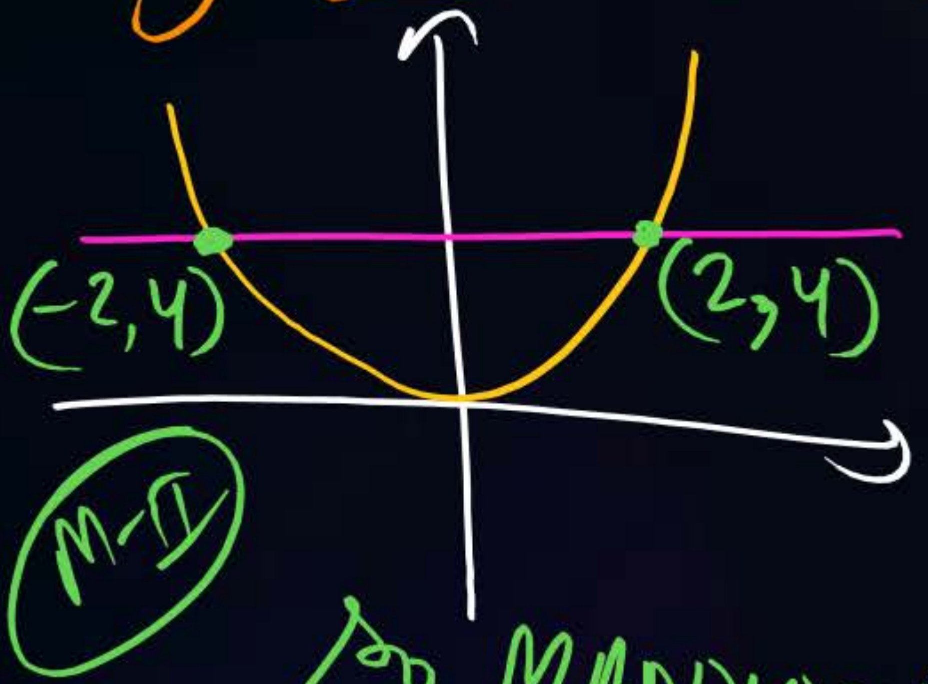
Note: Vertical line Test $\rightarrow$ If any Random line $\parallel^r$ to y axis, cuts the graph only at one point, then it is a func^n.

& if this line cuts the graph at more than one point, then it is not a func^n.

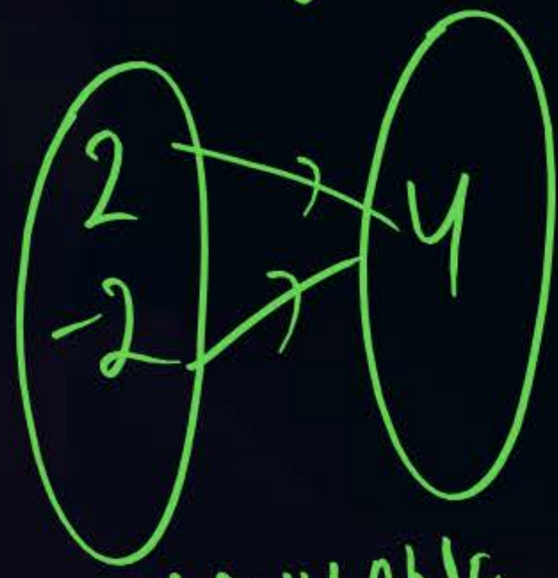
Horizontal Line Test (Shortcut of checking ONE-ONE OR MANY ONE).

If Any Random line $\parallel^r$ to x axis cuts the graph only at one point then it is one-one otherwise MANY-ONE

eg $y = x^2$ (M-I) : At $x=2$, $y=4$
& at $x=-2$, $y=4$



So MANYONE.



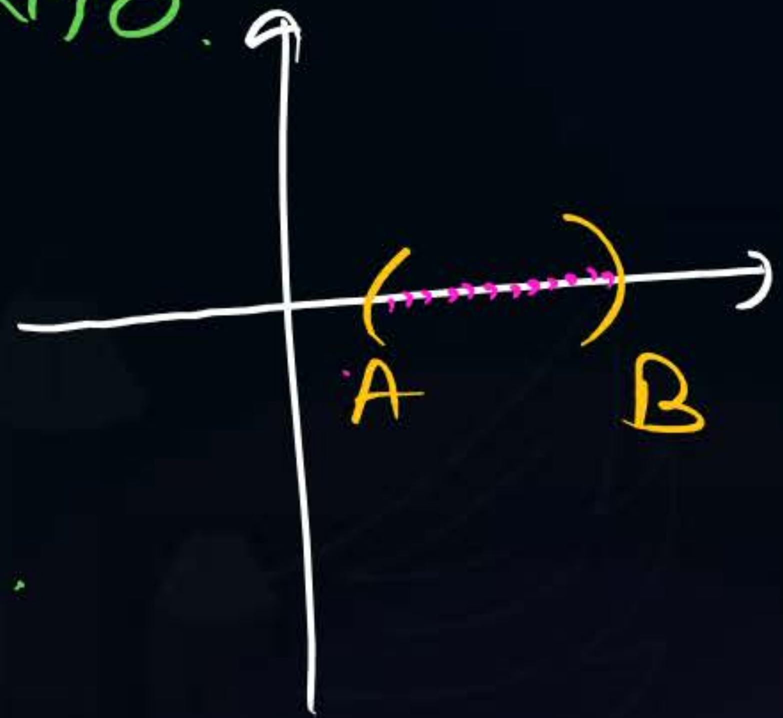
MANYONE

eg: $y = x^3$ = ONE-ONE

Conclusions: ① Vertical line test → To check validity of funcⁿ.

② Horizontal line test → To check validity of one-one

③ If $\boxed{\text{Range} = \text{Codomain}}$ → Then funcⁿ is ONTO.



④ ONE-ONE funcⁿ ⟹ INJECTIVE MAPPING

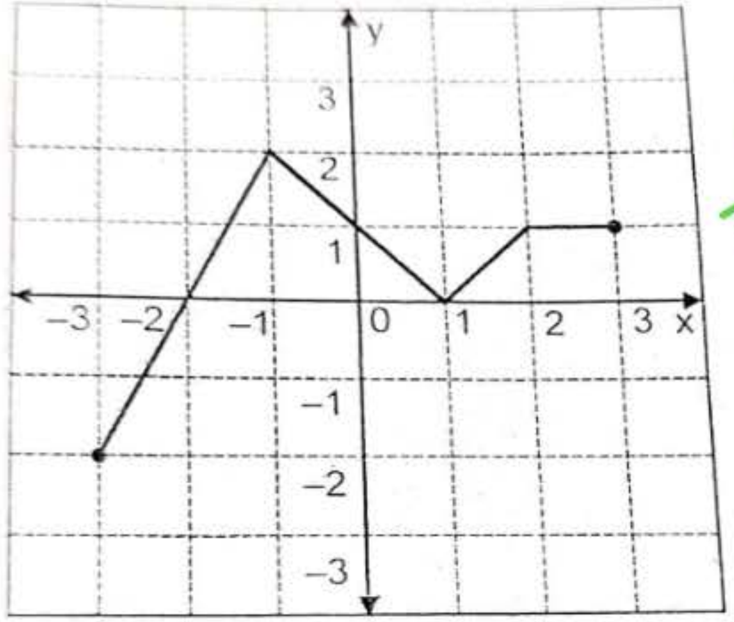
⑤ ONE-ONE/ONTO ⟹ BIJECTIVE MAPPING / one-one correspondence.

⑥ one one-one/ONTO funcⁿ have Inverse.

i.e. if $f(x)$ is one one/ONTO then only $f^{-1}(x)$ exist

⟺

Which of the following function(s) is an accurate description of the graph for the range(s) indicated?



(i) $y = 2x + 4$ for $-3 \le x \le -1$

(ii) $y = |x - 1|$ for $-1 \le x \le 2$

(iii) $y = ||x| - 1|$ for $-1 \le x \le 2$

(iv) $y = 1$ for $2 \le x \le 3$

(a) (i), (ii) and (iii) only

(b) (i), (ii) and (iv) only

(c) (i) and (iv) only

(d) (ii) and (iv) only

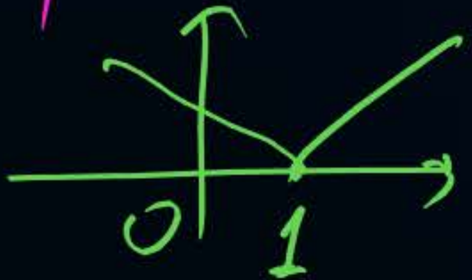
[GATE 2018 (CE-MorningSession

HWQ.

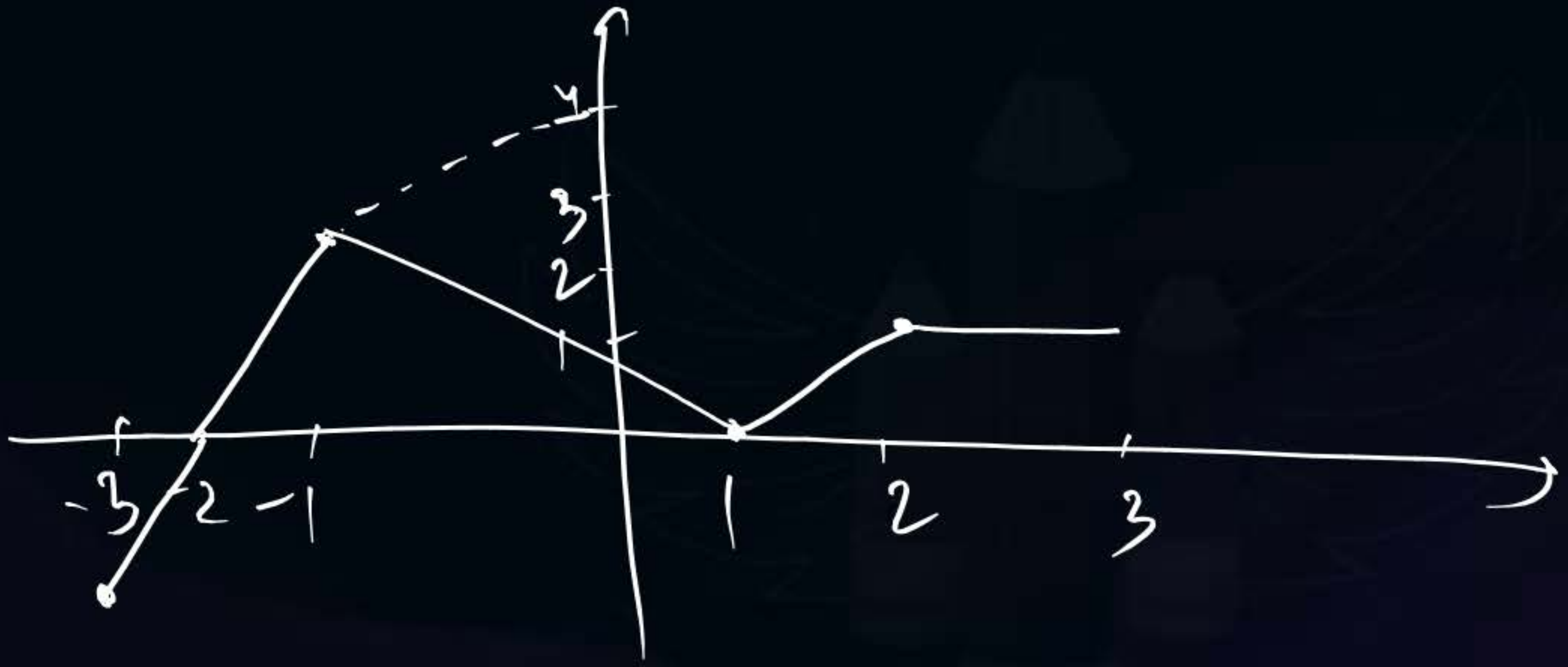
Case I: $y = 2x + 4 \Rightarrow -2x + y = 4 \Rightarrow \frac{x}{-2} + \frac{y}{4} = 1$

$\boxed{-3 \le x \le -1}$, $y(-3) = -3.5$, $y(0) = 4$

Case II: $\boxed{-1 \le x \le 2}$, $y = |x-1|$



Case IV: $\boxed{2 \le y \le 3}$, $y = 1$



MS8

Let max $\{a, b\}$ denote the maximum of two real numbers a and b. Which of the following statement(s) is/are TRUE about the function $f(x) = \max\{3 - x, x - 1\}$?

(a) It is continuous on its domain. ✓

(b) It has a local minimum at $x = 2$. ✓

(c) It has a local maximum at $x = 2$. ✗

(d) It is differentiable on its domain. ✗

$y = 3 - x$ — (1) $\quad y = x - 1$ — (2)

Solving, $3 - x = x - 1$

$2x = 4 \Rightarrow x = 2, \; y = 1$

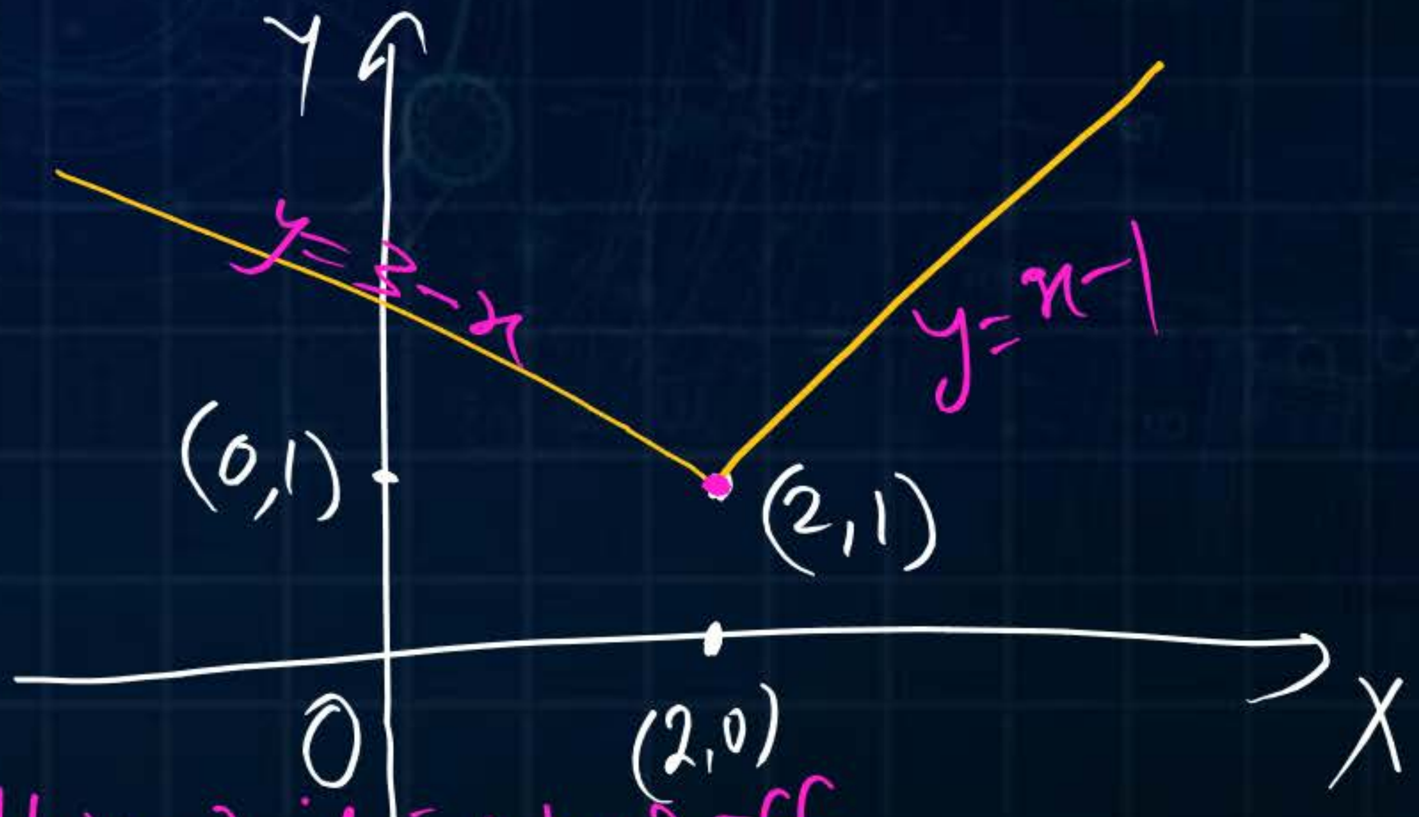
i.e. Intersecting Point $= (2, 1)$

$f(x) = \max\{3 - x, x - 1\}$, $D_f = (-\infty, \infty)$

for eg at $x = 1$, $\max(2, 0) = 2$

at $x = 3$, $\max(0, 2) = 2$

$$y = f(x) = \begin{cases} 3 - x, & x < 2 \\ x - 1, & x > 2 \end{cases}$$



At $x = 2$, it is Non Diff.

BASICS of limits & Continuity

Infinity → I don't know, Not unique,

$\boxed{\infty + \infty = \infty}$, $\dfrac{\text{Something}}{0} = \text{N.D}$, $\boxed{\dfrac{\text{Something}}{\infty} = 0}$

I you we

$\infty + \infty = \infty$

$\infty \times \infty = \infty$

$\infty^{\infty} = \infty$

N.D , $\underbrace{\frac{0}{0}, \frac{\infty}{\infty}, 0\times\infty, \infty-\infty, 0^0, \infty^0, 1^\infty}_{\text{IND Forms}}$

eg $\dfrac{10}{2} = 5$ Similarly, $\dfrac{0}{0} = \underbrace{1, 2, -\frac{1}{3}, -4, \sqrt{5}, 7, \ldots\ldots}_{\text{Multiple Ans exist.}}$

$$\frac{\infty}{\infty} = \frac{1/0}{1/0} = \frac{0}{0} = \text{IND Form.}$$

$$0\times\infty = 0\times\left(\frac{1}{0}\right) = \frac{0}{0} = \text{IND Form}$$

$\infty - \infty = ?$ (Have patience)

(it will be explained after 2 hrs)

WRONG App. :-

$$\infty-\infty = \frac{1}{0} - \frac{1}{0} = \frac{0-0}{0} = \frac{0}{0} = \text{IND}$$

if this app. is correct then why not

$$\infty+\infty = \frac{1}{0}+\frac{1}{0} = \frac{0+0}{0} = \frac{0}{0} \quad ???$$

(x) Let $\boxed{0^0 = K}$

or $K = 0^0$

$$\log_e K = \log_e(0^0)$$
$$= 0\times\log_e 0$$
$$= 0\times(-\infty)$$
$$= -0\times\left(\frac{1}{0}\right)$$
$$\log_e K = \frac{0}{0}$$
$$K = e^{\frac{0}{0}} = \text{IND Form}$$

(x) Let $\boxed{\infty^0 = K}$

or $K = \infty^0$

$\log_e K = \log(\infty^0)$

$= 0 \times \log \infty$

$= 0 \times (\infty)$

$= 0 \times \left(\frac{1}{0}\right)$

$\log_e K = \frac{0}{0}$

$K = e^{\frac{0}{0}} =$ IND Form

(x) $1^{20} = 1 \times 1 \times 1 \times \ldots \times 1 = 1$

$1^{50} = 1 \times 1 \times 1 \times \ldots \times 1 = 1$

$1^{\infty} \neq 1$

Let $1^{\infty} = K$

or $K = 1^{\infty}$

$\log_e K = \log(1^{\infty})$

$= \infty \times \log 1$

$= \infty \times 0$

$= \frac{1}{0} \times 0$

$\log_e K = \frac{0}{0}$

$K = e^{\frac{0}{0}} =$ IND Form

Check the Nature of 0^{∞}

Let $0^{\infty} = K$

or $K = 0^{\infty}$

$\log_e K = \log(0^{\infty})$

$= \infty \times \log(0)$

$= \infty \times (-\infty)$

$= -(\infty \times \infty)$

$\log_e K = -\infty$

$K = e^{-\infty} = 0$

ie $\boxed{0^{\infty} = 0}$ Hence Determinate

→ If we are getting ∞ or $-\infty$ or both, our conclusion is "It is N.D form".

→ If we are getting Multiple answers, " " is "It is I.N.D Form"

eg $\left(\frac{0}{0}, \frac{\infty}{\infty}, 0\times\infty, \infty-\infty, 0^0, \infty^0, 1^\infty\right)$.

(*) Check the Nature of ∞^∞ ?

Sol: let $K = \infty^\infty$

$\log_e K = \log(\infty^\infty) = \infty \times (\log \infty)$

$= \infty \times \infty$

$\log_e K = \infty$

$K = e^\infty = \infty$ i.e $\boxed{\infty^\infty = \infty}$ N.D.

(*) $\frac{\text{Something}}{\infty} = 0$, $\boxed{\frac{0}{\infty} = 0}$

$\because \frac{0}{\infty} = 0 \times \frac{1}{\infty} = 0 \times 0 = 0$

$\boxed{\infty \times \infty = \infty}$

let $K = \infty \times \infty$

$\log K = \log(\infty \times \infty) = \log(\infty)^2 = 2\log\infty$

$= 2 \times \infty = \infty$

$K = e^\infty = \infty$ i.e $\boxed{\infty \times \infty = \infty}$

Useful Concepts.

Infinity :- Infinity is Not a very large Number. it is the presentation of that Concept which is beyond Imagination.

→ & Infinity is Not unique, it depends upon the Imagination capacity of an Individual.

→ $\boxed{\dfrac{\text{Something}}{\infty} \cong 0}$, $\boxed{\dfrac{\text{Something}}{0} = \text{N.D.}}$ (FACT)

Assumption

GOD GOD GOD

$\rightarrow \boxed{\infty + \infty = \infty} \approx$ N.D

I don't know — you don't know — we don't know

$\rightarrow \boxed{\infty \times \infty = \infty} \approx$ N.D

$\rightarrow \boxed{\infty^{\infty} = \infty} \approx$ N.D

Note: if final ans is ∞ or $-\infty$ then we say that it is Not Defined

But if final ans is $\frac{0}{0}$ then we say that it is IND form

$\rightarrow \infty - \infty = ? =$ IND form.

$\rightarrow \frac{\infty}{\infty} = ? =$ IND form.

$\rightarrow \infty^{0} = ? =$ IND form.

$\rightarrow 1^{\infty} = ? =$ IND form.

$\rightarrow 0 \times \infty = ? =$ IND form.

$\rightarrow 0^{0} = ? =$ IND form.

$\rightarrow \frac{0}{0} = ? =$ IND form.

PROPERTIES of LOG → let $a > 1$

① $\boxed{\log_a x = b \Rightarrow x = a^b}$

eg $\log_4 64 = ? = 3 \Rightarrow 64 = 4^3$

eg $\log_5 625 = ? = 4 \Rightarrow 625 = 5^4$

eg $\log_a 1 = 0$, eg $\log_a 0 = -\infty$ where $a > 1$

$\left(\because a^{-\infty} = \frac{1}{a^{\infty}} = \frac{1}{\infty} = 0\right)$

eg $\log_a \infty = \infty$ eg $\boxed{\log_a a = 1}$, $a \neq 1$ & $\log_1 1 = \text{IND}$

② $\boxed{\log_a x > \log_a y \Rightarrow x > y}$

Provided $a > 1$ i.e when Base is greater than 1, $\log_a x$ is an Increasing func^n.

Domain $= (0, \infty)$

Range $= (-\infty, \infty)$

③ Product formula;

$$\log_a(x \cdot y) = \log_a x + \log_a y$$

④ Quotient formula →

$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

⑤ Power formula –

$$\log_a(x^y) = y \log_a x.$$

⑥ Base change formula →

$$\log_y x = \frac{\log_a x}{\log_a y},$$

eg $\log_\infty x = \frac{\log_e x}{\log_e \infty} = \frac{\log_e x}{\infty} = 0$

eg $\log_1 1 = \frac{\log_e 1}{\log_e 1} = \frac{0}{0} = ? = \text{IND}$ form

Note – $\log_1 x = \frac{\log_e x}{\log_e 1} = \frac{\log_e x}{0} = \text{N.D.}$ (i.e $\log_1 x$ is not defined)

(7) Reciprocal formula →

$$\log_y x = \frac{1}{\log_x y}$$

(8) $\log_{a^b}(x) = \frac{1}{b}\log_a x$

(9) $a = e^{\log_e a}$

(10) $a^n = e^{\log_e (a^n)} = e^{n(\log_e a)}$

(11) $a^{\log_e b} = b^{\log_e a}$

(3) Some Useful Results

(i) $a^3+b^3 = (a+b)(a^2+b^2-ab)$

(ii) $a^3-b^3 = (a-b)(a^2+b^2+ab)$

(iii) $(a+b)^3 = a^3+b^3+3ab(a+b)$

(iv) $(a-b)^3 = a^3-b^3-3ab(a-b)$

(V) $a^2-b^2 = (a-b)(a+b)$

Some Usefull Results :→

① $\boxed{|x-a|<l \Rightarrow a-l<x<a+l}$

eg $|x|<l \Rightarrow -l<x<l$

② $|x-a|>l \Rightarrow x<a-l$ or $x>a+l$

eg $|x|>l \Rightarrow x<-l$ or $x>l$

③ if $a<b$ s.t $(x-a)(x-b)<0$

$\Rightarrow a<x<b$

④ if $a<b$ s.t $(x-a)(x-b)>0$

$\Rightarrow x<a$ or $x>b$

Proof ① $|x-a|<l$

$\pm(x-a)<l$

$-(x-a)<l$ & $+(x-a)<l$

$(x-a)>-l$ & $(x-a)<l$

$x>a-l$ & $x<a+l$

$\boxed{a-l<x<a+l}$

INDETERMINATE form → There are exactly 7 Indeterminate forms for eg $\frac{0}{0}, \frac{\infty}{\infty}, 0\times\infty, \infty-\infty, 0^0, \infty^0, 1^\infty$

"If any Mathematical Expression has Multiple Answers then that expression is said to be in INDETERMINATE form"

L-Hospital Rule → this Rule provides unique answer out of ∞ answers & this Rule is applicable only for $\frac{0}{0}$ and $\frac{\infty}{\infty}$ form.

Rule: Separately Differentiate N^r & D^r, equal number of times, until we are free from $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form.

i.e. $\lim_{x \to a} \left\{ \frac{f(x)}{g(x)} \right\} \simeq \frac{0}{0} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \simeq \frac{0}{0} = \lim_{x \to a} \frac{f''(x)}{g''(x)} = \cdots\cdots = \text{unique ans}$

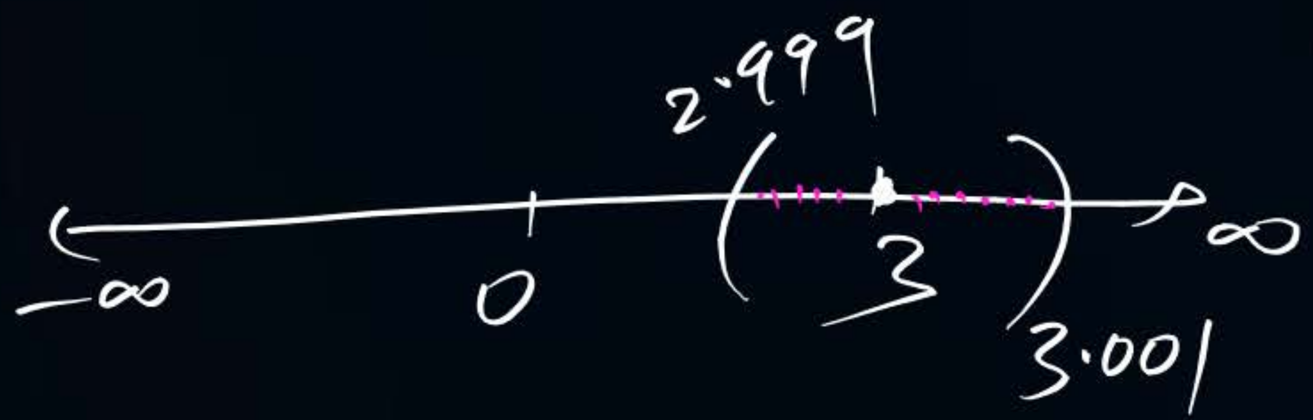
eg $\lim_{x \to \infty} \left(\frac{x^3}{e^x} \right) = ? = \frac{\infty}{\infty} = \lim_{x \to \infty} \left(\frac{3x^2}{e^x} \right) = \frac{\infty}{\infty} = \lim_{x \to \infty} \left(\frac{6x}{e^x} \right) = \frac{\infty}{\infty} = \lim_{x \to \infty} \left(\frac{6}{e^x} \right) = \frac{6}{e^\infty} = \frac{6}{\infty} = 0$

i.e. unique & constant. Hence exist

Note: limit exist means "There exist unique, constant value that must be free from x"

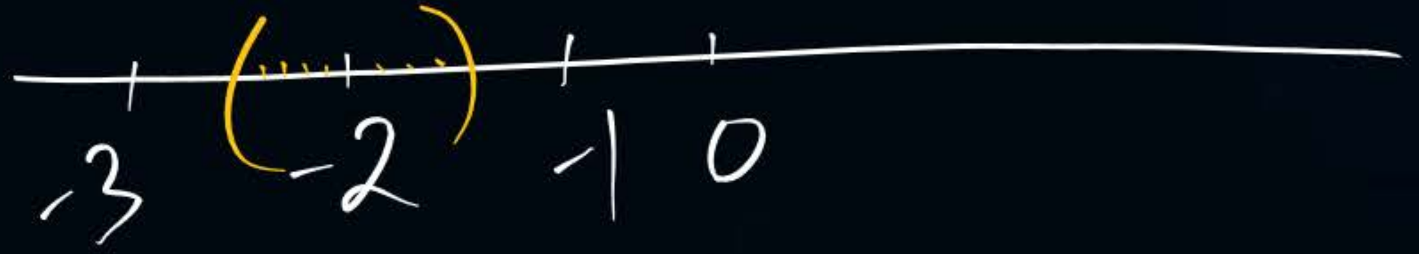
(*) Neighbourhood of Real No. 'a' $\rightarrow$

$$\text{Nbd of } 3 = (2.999, 3.001)$$



$$\text{Nbd of } 0 = (-0.001, 0.001)$$

$$\text{Nbd of } -2 = (-2.001, -1.999)$$



Neighbourhood of Real Number 'a' $\longrightarrow$ means we are considering an open interval of length $2h$ with centered at point 'a'

i.e. Nbd of $a = (a-h, a+h)$ where $h > 0$ & h depends on our choice. & h is very small +ve Number.

Note if $\boxed{x \text{ lies in the Nbd of } a} \Longrightarrow x \in (a-h, a+h)$

or

$a-h < x < a+h$

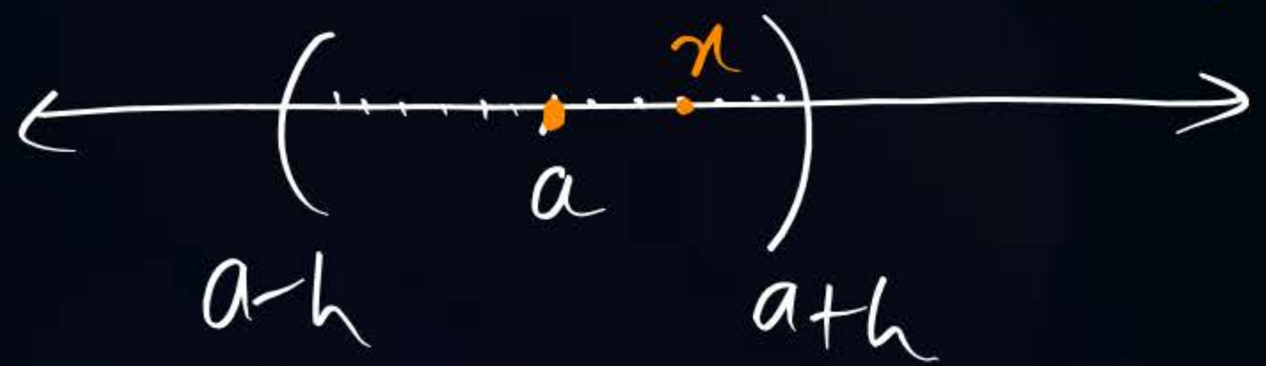
or

$|x-a| < h$

or

$x \to a$

or x is about a



Understanding of limit →

eg $f(x) = \dfrac{x^2-9}{x-3}$

$f(3) = DNE$ (i.e. exact Value $\underline{\text{at}}$ 3 DNE

$LHL = 6$ (i.e. App. Value in the $\underline{\text{Left Nbd}}$ of 3)

$RHL = 6$ (i.e. App. " " $\underline{\text{Right Nbd}}$ of 3)

$\lim\limits_{x\to 3} f(x) = 6$ (i.e. $\underbrace{\text{Approx Value}}$ in the $\underline{\text{Nbd}}$ of 3)

$$f(x) = \frac{(x-3)(x+3)}{(x-3)}$$

$$f(2.999) = \frac{-0.\cancel{00}1 \times 5.999}{-0.\cancel{00}1} = 5.999 \approx 6$$

$$f(3.001) = \frac{0.\cancel{00}1 \times 6.001}{0.\cancel{00}1} = 6.001 \approx 6$$

M-II $\lim\limits_{x\to 3}\left(\dfrac{x^2-9}{x-3}\right) = \lim\limits_{x\to 3} \dfrac{\cancel{(x-3)}(x+3)}{\cancel{(x-3)}}$

$\because x \to 3$

$\Rightarrow (x-3) \to 0$

i.e. $(x-3) \neq 0$

$= \lim\limits_{x\to 3} (x+3)$

$\simeq 3+3 = 6$

eg $\lim_{x\to 0}\left(\frac{\sin x}{x}\right) = ?$

$LHL = f(-0.001) = \frac{\sin(-0.001)}{-0.001} = ?$

$RHL = f(0.001) = \frac{\sin(0.001)}{0.001} = ?$

M-II $\lim_{x\to 0}\left(\frac{\sin x}{x}\right) = \frac{0}{0}$ form $= \lim_{x\to 0}\left(\frac{\cos x}{1}\right) = \frac{\cos(0)}{1} = \frac{1}{1} = 1$ ✓

eg: $\lim_{x\to \infty}\left(\frac{\sin x}{x}\right) = ?$

$LHL = f(\infty^-) = ?$

$RHL = f(\infty^+) = ?$

M-II $\lim_{x\to \infty}\left(\frac{\sin x}{x}\right) = \frac{\sin \infty}{\infty} = \frac{\text{Any No. b/w} -1 \text{ \& } 1}{\infty} = 0$ ie limit exist.

Continuity → if $\lim_{x\to a} f(x)$ exist & is equal to $f(a)$ then $f(x)$ is cont at a

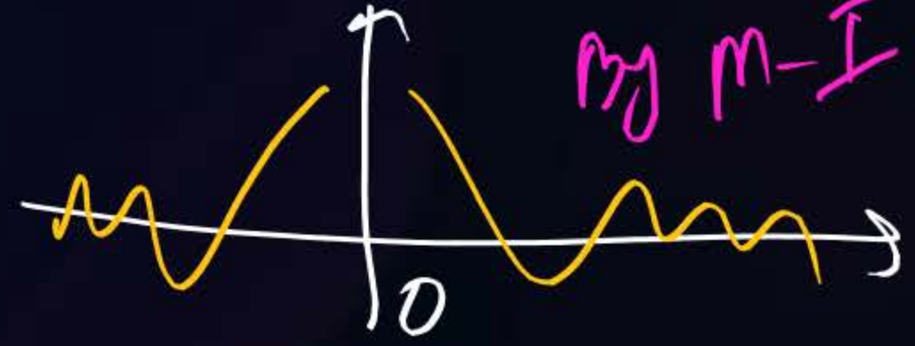
i.e. if $\boxed{\lim_{x\to a} f(x) = f(a)}$ $\Longleftrightarrow$ then $f(x)$ is cont at $x=a$

i.e. for cont. func^n, App. Value = Exact Value

eg Check the continuity of the following func^n's; at $x=0$

(1) $f(x) = \dfrac{\sin x}{x}$, $D_f = R-\{0\}$

Dis cont at $x=0$ ∵ Exact Value = DNE



By M-I

(2) $f(x) = \begin{cases} \dfrac{\sin x}{x}, & x\neq 0 \\ 1, & x=0 \end{cases}$ $D_f = R$

Cont. at $x=0$

Def^n of limit → $\boxed{\lim_{x \to a} f(x) = l}$ ⟹ whenever x lies in the Nbd of 'a'
$f(x)$ lies in the Nbd of l

eg $\lim_{x \to 0} \left(\frac{\sin x}{x}\right) = 1$ ⟹ whenever x lies in the Nbd of '0'
$\left(\frac{\sin x}{x}\right)$ lies in the Nbd of 1

Continuity: $f(x)$ is said to be continuous at $x = a$ if

$\boxed{\lim_{x \to a} f(x) = f(a)}$ ⟹ whenever x lies in the Nbd of a
$f(x)$ lies " " of $f(a)$

App Value = Exact Value

i.e $\boxed{(LHL = RHL) = f(a)}$

Sp. Note: To check Discontinuity of $f(x)$ at $x=a$ we have 3 methods;

(M-I) If Exact Value = DNE then $f(x)$ is Discontinuous

(M-II) if Approx Value = DNE then " " "

(M-III) if Both exist But not equal then $f(x)$ is Discont.

ie $\boxed{\lim_{x\to a} f(x) \neq f(a)} \Rightarrow f(x)$ is Discont at $x=a$

Method of Solving Questions →

(M-I) By Direct Substitution (Best Method).

(M-II) By factorisation

(M-III) By Rationatisation

(M-IV) By using I.N.D form Concept

(M-V) By using Standard Results

(M-VI) using Common sense.

Sp. formulaesi →

(1) $\sum_{N=1}^{N} N = 1+2+3+\ldots+N = \frac{N(N+1)}{2}$

(2) $\sum_{N=1}^{N} N^2 = 1^2+2^2+3^2+\ldots+N^2 = \frac{N(N+1)(2N+1)}{6}$

(3) $\sum_{N=1}^{N} N^3 = 1^3+2^3+3^3+\ldots+N^3 = \left(\frac{N(N+1)}{2}\right)^2$

(4) $\sum_{N=1}^{N} (a) = \underbrace{a+a+a+\ldots+a}_{N\ \text{times}} = N \cdot a$

Q1: $\lim_{x\to 2}\left(\frac{x^2-4}{x+3}\right)=?=\frac{2^2-4}{2+3}=\frac{4-4}{5}=\frac{0}{5}=0$ i.e unique so limit exist.

Q2: $\lim_{n\to\infty}\left(\frac{1+2+3+4+\cdots+n}{n^2}\right)=?=\lim_{n\to\infty}\left[\frac{\frac{n(n+1)}{2}}{n^2}\right]=\lim_{n\to\infty}\frac{1}{2}\left(1+\frac{1}{n}\right)$

PYQ

$=\frac{1}{2}(1+0)=\frac{1}{2}$

Q3: $\lim_{n\to\infty}\left[\frac{1^2+2^2+3^2+\cdots+n^2}{n^3}\right]$

$=\lim_{n\to\infty}\left[\frac{\frac{n(n+1)(2n+1)}{6}}{n^3}\right]=\lim_{n\to\infty}\frac{1}{6}\left[\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right]=\frac{1}{6}(1+0)(2+0)$

$=\frac{2}{6}=\frac{1}{3}$

Q. $\lim_{n\to\infty}\left[\dfrac{1-2+3-4+5-6+\dots\dots+(2n-1)-2n}{\sqrt{n^2+1}+\sqrt{n^2-1}}\right]=?$

(a) $1/2$

(b) $-1/2$

(c) 1

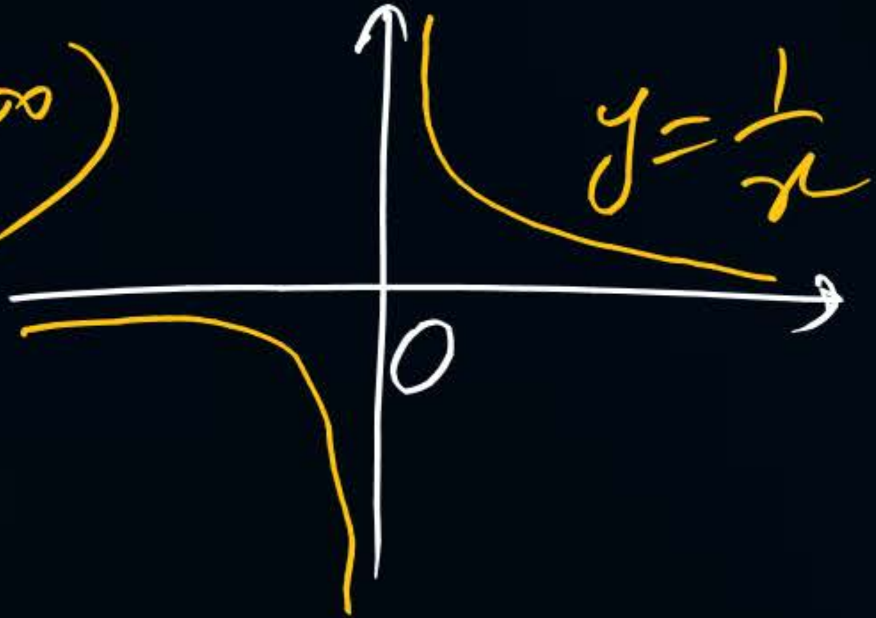
(d) D.N.E

$$=\lim_{n\to\infty}\left[\frac{(1-2)+(3-4)+(5-6)+\dots\dots+((2n-1)-2n)}{\sqrt{n^2+1}+\sqrt{n^2-1}}\right]$$

$$=\lim_{n\to\infty}\left[\frac{-1+(-1)+(-1)+\dots\dots+(-1)\ \ n\text{ times}}{\sqrt{n^2+1}+\sqrt{n^2-1}}\right]$$

$$=\lim_{n\to\infty}\left[\frac{-n}{n\left[\sqrt{1+\frac{1}{n^2}}+\sqrt{1-\frac{1}{n^2}}\right]}\right]=\frac{-1}{\sqrt{1+0}+\sqrt{1-0}}=-\frac{1}{2}$$

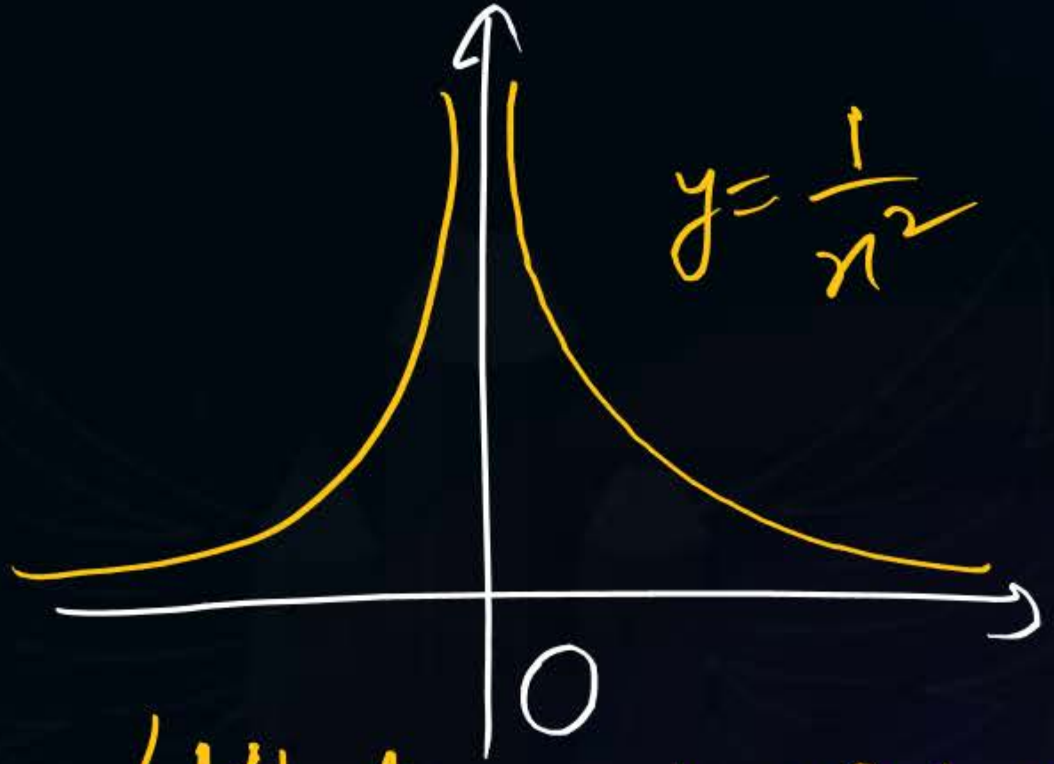
Q-1 $\lim_{x\to 0}\left(\frac{1}{x}\right)=\frac{1}{0}=$ N.D ie limit DNE $\left(\because \begin{array}{l} LHL=-\infty \\ RHL=+\infty \end{array}\right)$



Q② $\lim_{x\to\infty}\left(\frac{1}{x}\right)=\frac{1}{\infty}=0$ ie unique so limit exist

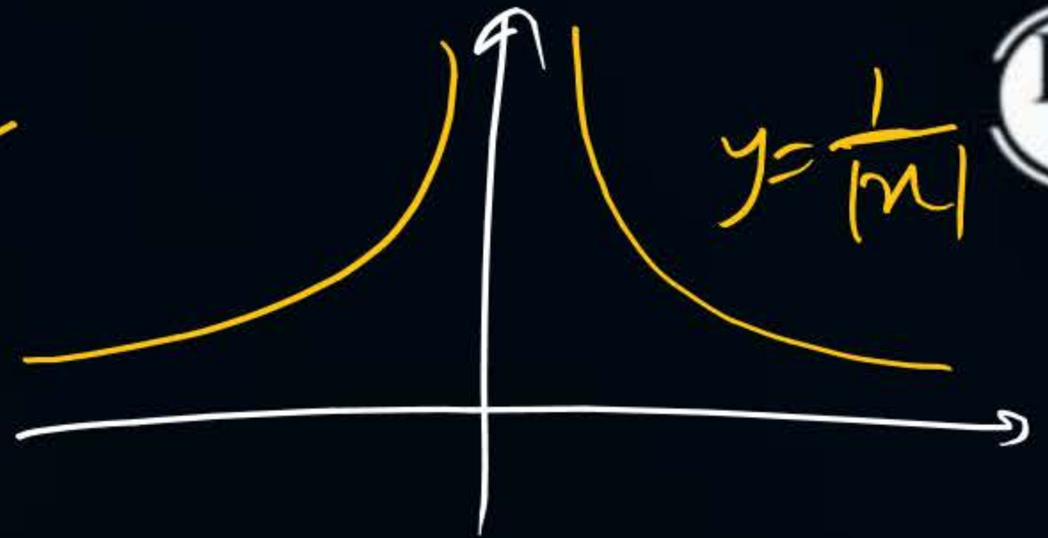
Q③ $\lim_{x\to 0}\left(\frac{1}{x^2}\right)=\frac{1}{0^2}=\frac{1}{0^+}=+\infty$ ie limit DNE

Q④ $\lim_{x\to\infty}\left(\frac{1}{x^2}\right)=\frac{1}{\infty^2}=\frac{1}{\infty}=0$ ie unique so limit exist.



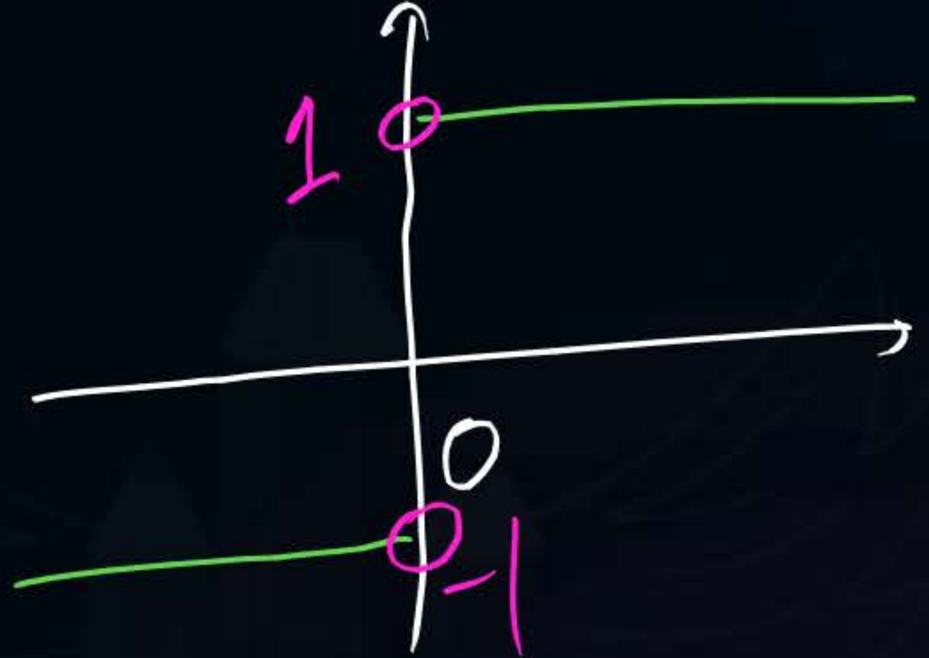
$\left.\begin{array}{l} \text{LHL at } 0=+\infty \\ \text{RHL at } =0=+\infty \end{array}\right\}$ limit DNE.

Q. $\lim_{x\to 0}\left(\frac{1}{|x|}\right) = \frac{1}{|0|} = \frac{1}{0^+} = +\infty$ i.e limit DNE



Q. $\lim_{x\to \infty}\left(\frac{1}{|x|}\right) = \frac{1}{|\infty|} = \frac{1}{\infty} = 0$ i.e limit exist

Q. $\lim_{x\to 0}\left(\frac{|x|}{x}\right) = \begin{cases} LHL = -1 \\ RHL = 1 \end{cases}$ $\because$ LHL $\neq$ RHL so limit DNE.



Q. $\lim_{x\to \infty}\left(\frac{|x|}{x}\right) = +1$ is unique so limit exist.

Draw the Graph of following func^n:

① $f(x) = \dfrac{|x|}{x}$, $D_f = R - \{0\}$

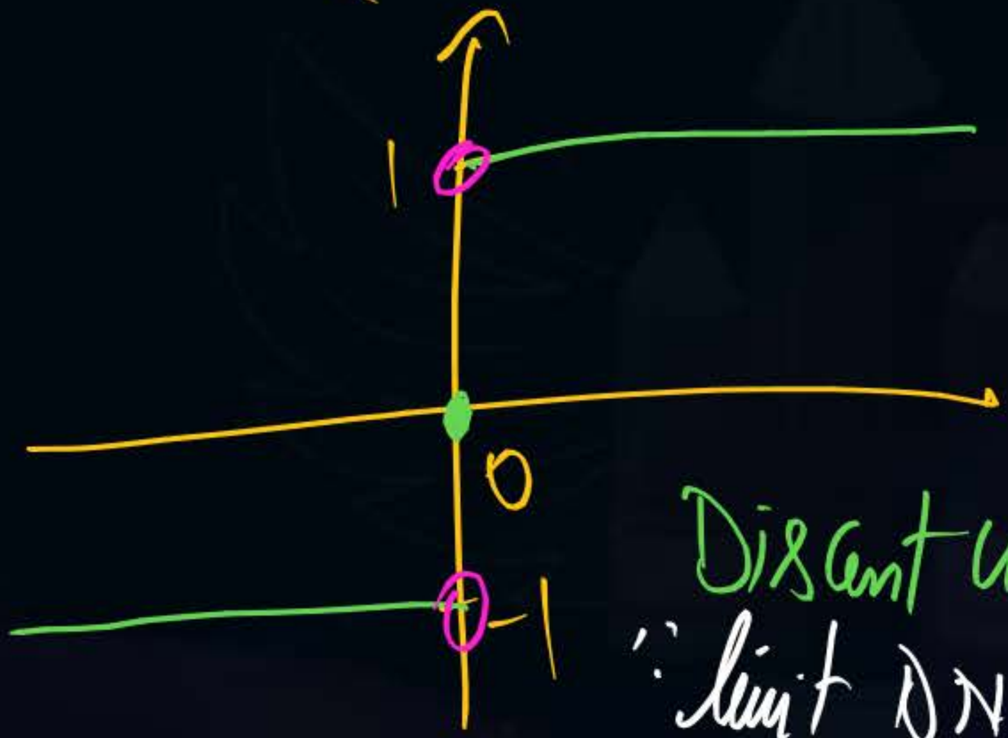
$$= \begin{cases} -1 & , x < 0 \\ 1 & , x > 0 \end{cases}$$

Discont using M-I

∵ exact value DNE

② $Sgn(x) = \begin{cases} \dfrac{|x|}{x} , & x \neq 0 \\ 0 , & x = 0 \end{cases}$

$$= \begin{cases} -1 & ; x < 0 \\ +1 & ; x > 0 \\ 0 & ; x = 0 \end{cases}$$



Discont using M-II

∵ limit DNE.

Type (2): FACTORISATION

Q. $\lim_{x\to 2}\left(\dfrac{x^2-5x+6}{x^2-4}\right) = ? = \dfrac{0}{0}$

(M-I) Using L-Hospital's Rule . . . .

(M-II) $\lim_{x\to 2}\dfrac{\cancel{(x-2)}(x-3)}{\cancel{(x-2)}(x+2)}$

$= \lim_{x\to 2}\left(\dfrac{x-3}{x+2}\right) = \dfrac{2-3}{2+2} = \dfrac{-1}{4}$

$x\to 2 \Rightarrow (x-2)\to 0$ i.e. $(x-2)\neq 0$

that's why we can cancel out $(x-2)$

Q. $\lim_{y\to 1}\left(\dfrac{y^3-1}{y-1}\right) = ? = \lim_{y\to 1}\dfrac{(y-1)(y^2+1+y)}{(y-1)}$

$= \lim_{y\to 1}(y^2+1+y) = 3$ Ans

Q. $\lim_{y\to 1}\left(\dfrac{y^{\frac{1}{4}}-1}{y^{\frac{1}{3}}-1}\right) = ? = \lim_{x\to 1}\left(\dfrac{x^3-1}{x^4-1}\right)$

Put $y = x^{12}$

when $y\to 1$, $x\to 1$

$= \lim_{x\to 1}\dfrac{(x-1)(x^2+1+x)}{(x^2-1)(x^2+1)}$

$= \lim_{x\to 1}\dfrac{x^2+1+x}{(x+1)(x^2+1)} = \dfrac{3}{4}$

Type 3 (Rationalisation) →

Q $\lim_{x \to \infty} \left(\sqrt{x^2+x+1} - \sqrt{x^2+1}\right) = ?$

Ans $= \frac{1}{2}$

Q. $\lim_{x \to \infty} \left[\sqrt{x^2+x} - x\right] = ? = (\infty - \infty)$ form.

DA 2025

$$= \lim_{x \to \infty} \left(\sqrt{x^2+x} - x\right) \frac{\left(\sqrt{x^2+x} + x\right)}{\left(\sqrt{x^2+x} + x\right)}$$

$$= \lim_{x \to \infty} \left[\frac{(x^2+x) - x^2}{\sqrt{x^2+x} + x}\right]$$

$$= \lim_{x \to \infty} \left[\frac{x}{\sqrt{x^2+x} + x}\right] = \frac{\infty}{\infty} \text{ form}$$

$$= \lim_{x \to \infty} \left[\frac{1}{\sqrt{1 + \frac{1}{x}} + 1}\right]$$

$$= \frac{1}{\sqrt{1+0} + 1} = \frac{1}{2}$$

Qs $\lim_{x\to 4}\left[\frac{3-\sqrt{5+x}}{1-\sqrt{5-x}}\right] = ? = \frac{0}{0}$ form

(M-I) using L'Hosp Rule → ......

(M-II) using Rationalisation → ....

(a) $1/2$

(b) $-1/2$

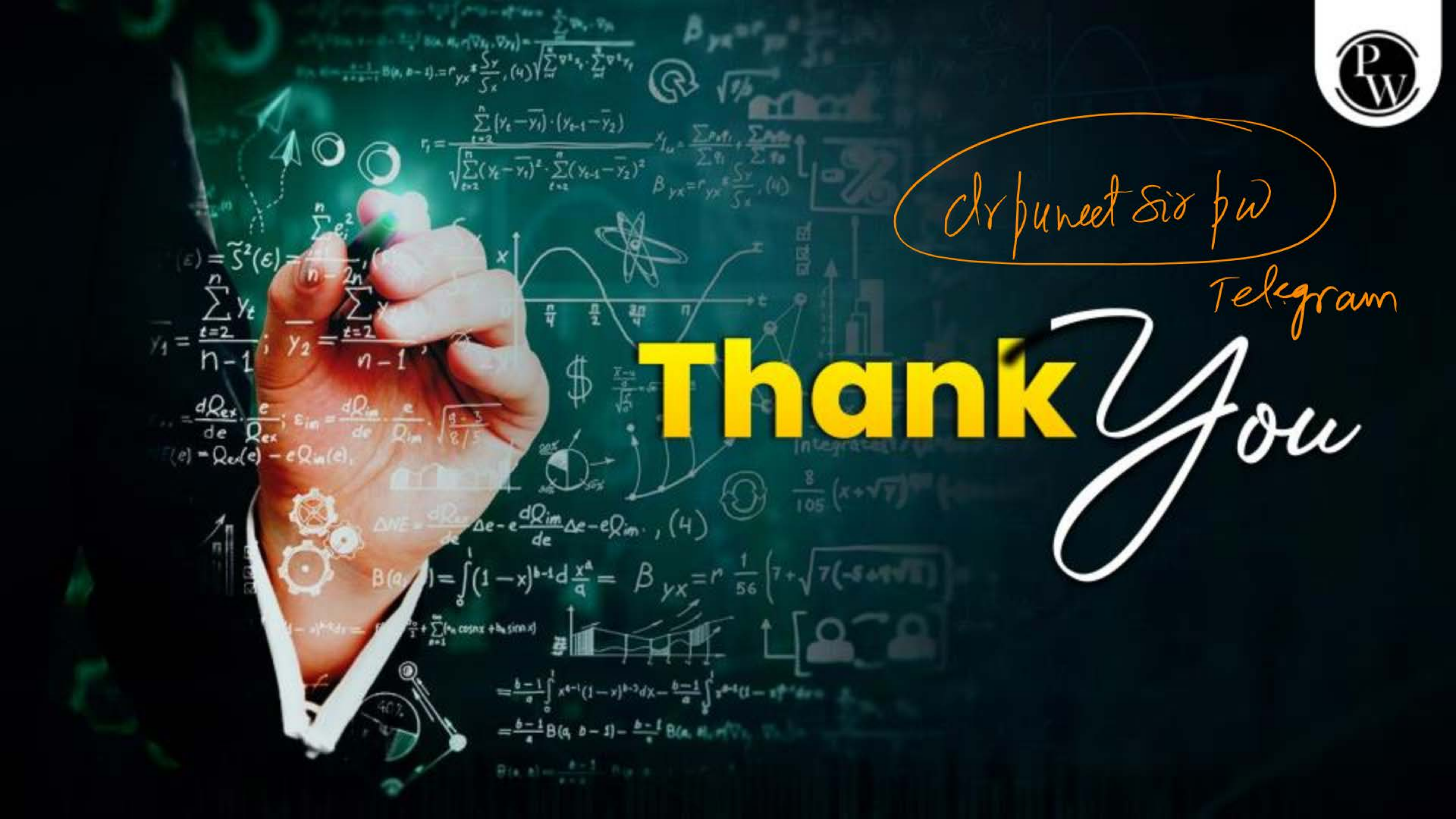
(c) $-1/3$ ✓

(d) 3

(M-II) $\lim_{x\to 4}\left(\frac{3-\sqrt{5+x}}{1-\sqrt{5-x}}\right)\times\left(\frac{1+\sqrt{5-x}}{1+\sqrt{5-x}}\right)\times\left(\frac{3+\sqrt{5+x}}{3+\sqrt{5+x}}\right)$

$= \lim_{x\to 4}\left[\frac{9-(5+x)}{1-(5-x)}\right]\times\left(\frac{1+\sqrt{5-x}}{3+\sqrt{5+x}}\right)$

$= \lim_{x\to 4}\left(\frac{4-x}{-4+x}\right)\times\left(\frac{1+\sqrt{5-x}}{3+\sqrt{5+x}}\right) = -1\times\frac{2}{6} = -\frac{1}{3}$

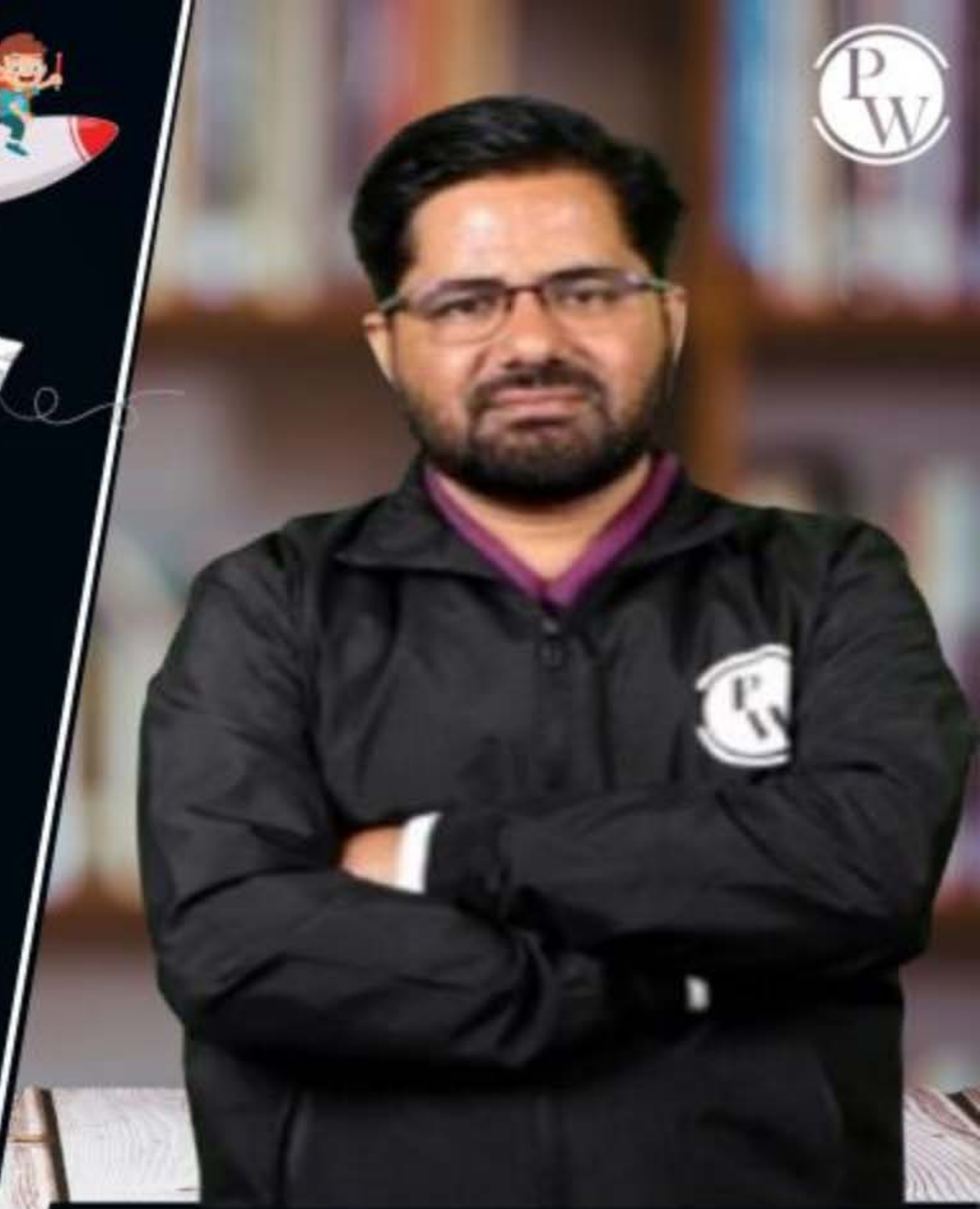

dr puneet sir pw
Telegram
Thank You

VIJAY

DATA SCIENCE
&
ARTIFICIAL INTELLIGENCE
& CS

Calculus and Optimization

Lecture No. 04



By- Dr. Puneet Sharma Sir

Recap of previous lecture

Topic

Limit / Continuity / Differantiability

(Part - 1)

Topics to be Covered

Topic

Limit, Continuity & Differentiability

(PART-2)

Method of Solving Questions →

(M-I) By Direct Substitution (Best Method).

(M-II) By factorisation

(M-III) By Rationalisation

(M-IV) By using IND form Concept

(M-V) By using Standard Results

(M-VI) using Common sense.

Sp. formulaes: →

(1) $\sum_{N=1}^{N} N = 1+2+3+\dots+N$
$= \frac{N(N+1)}{2}$

(2) $\sum_{N=1}^{N} N^2 = 1^2+2^2+3^2+\dots+N^2$
$= \frac{N(N+1)(2N+1)}{6}$

(3) $\sum_{N=1}^{N} N^3 = 1^3+2^3+3^3+\dots+N^3$
$= \left(\frac{N(N+1)}{2}\right)^2$

(4) $\sum_{N=1}^{N} (a) = \underbrace{a+a+a+\dots+a}_{N\ \text{times}} = N \cdot a$

Type 3 (Rationalisation) →

Q $\lim_{x \to \infty} \left(\sqrt{x^2+x+1} - \sqrt{x^2+1}\right) = ?$

$\boxed{\text{Ans} = \frac{1}{2}}$

$$= \lim_{x \to \infty} \left(\sqrt{x^2+x+1} - \sqrt{x^2+1}\right) \times \left(\frac{\sqrt{x^2+x+1} + \sqrt{x^2+1}}{\sqrt{x^2+x+1} + \sqrt{x^2+1}}\right)$$

$$= \lim_{x \to \infty} \left(\frac{(x^2+x+1) - (x^2+1)}{\sqrt{x^2+x+1} + \sqrt{x^2+1}}\right)$$

$$= \lim_{x \to \infty} \left[\frac{\cancel{x}}{\cancel{x}\left[\sqrt{1+\frac{1}{x}+\frac{1}{x^2}} + \sqrt{1+\frac{1}{x^2}}\right]}\right]$$

$$= \frac{1}{\sqrt{1+0+0} + \sqrt{1+0}} = \frac{1}{1+1} = \frac{1}{2}$$

Type IV (IND form Concept) $\longrightarrow$.

$$\underbrace{\frac{0}{0}, \frac{\infty}{\infty}}, \quad \underbrace{0\times\infty, \ \infty-\infty}, \quad \underbrace{0^0, \ \infty^0, \ 1^\infty}$$

Directly use L-Hospital Rule

use L-Hosp Rule after converting them into one of the 1st two forms.

use log Concept

Note: $\frac{d}{dx}(x^a) = a x^{a-1}$, $\boxed{\frac{d}{dx}(x^x) = x^x(1+\log x)}$

$\frac{d}{dx}(a^x) = a^x \log_e a$, $\frac{d}{dx}(a^a) = 0$

Prove that $\boxed{\frac{d}{dx}(x^x) = x^x(1+\log x)}$

Proof: Let $y = x^x \Rightarrow \frac{dy}{dx} = ?$

$$\log y = \log(x^x)$$

$$\log y = x\log x$$

$$\frac{d}{dx}(\log y) = \frac{d}{dx}(x\log x)$$

$$\frac{d}{dy}(\log y)\cdot\frac{dy}{dx} = x\frac{d}{dx}(\log x) + \log x\frac{d}{dx}(x)$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = x\left(\frac{1}{x}\right) + \log x(1)$$

$$\frac{dy}{dx} = y(1+\log x) = x^x(1+\log x)$$

Note:

$$\frac{d}{dx}(x^3) = 3x^2$$

$$\frac{d}{dy}(y^3) = 3y^2$$

$$\frac{d}{dx}(y^3) = \frac{d}{dy}(y^3)\cdot\frac{dy}{dx}$$

$$= (3y^2)\frac{dy}{dx}$$

Q1. $\lim_{x\to a}\left(\dfrac{x^x-a^a}{x^a-a^x}\right)=?\ \leftarrow \dfrac{0}{0}$ form $=\lim_{x\to a}\left(\dfrac{x^x(1+\log x)-0}{a x^{a-1}-a^x\log_e a}\right)=\dfrac{a^a(1+\log a)}{a a^{a-1}-a^a\log a}$

$=\dfrac{1+\log a}{1-\log a}$ **Ans**

Q $\lim_{x\to\infty}\left(\dfrac{x^n}{e^x}\right)=?\ \leftarrow \dfrac{\infty}{\infty}$ form $=\lim_{x\to\infty}\dfrac{D^n(x^n)}{D^n(e^x)}=\lim_{x\to\infty}\left(\dfrac{n!}{e^x}\right)=\dfrac{n!}{e^\infty}=\dfrac{n!}{\infty}=0$ **Ans**

$n\in$ +ve integer

w.k.that $\boxed{D^n(x^n)=n!\ \&\ D^{n+1}(x^n)=0}$

eg $D(x^2)=2x$, $D(x^3)=3x^2$

$\boxed{D^2(x^2)=2=2!}$, $D^2(x^3)=6x$

$\boxed{D^3(x^3)=6=3!}$

$D^4(x^3)=0$

$D(x^4)=4x^3$

$D^2(x^4)=12x^2$

$D^3(x^4)=24x$

$\boxed{D^4(x^4)=24=4!}$

$D^5(x^4)=0$

& so on....

Detailed Expl :-

$$\lim_{x\to\infty}\left(\frac{x^n}{e^x}\right) = \frac{\infty}{\infty} = \lim_{x\to\infty}\frac{n(x)^{n-1}}{e^x} = \frac{\infty}{\infty} = \lim_{x\to\infty}\frac{n(n-1)x^{n-2}}{e^x} = \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty}\frac{n(n-1)(n-2)\,x^{n-3}}{e^x} = \frac{\infty}{\infty} = \cdots\cdots = \lim_{x\to\infty}\left(\frac{n(n-1)\ldots 3.2.\,x^{1}}{e^x}\right)$$

$$= \frac{\infty}{\infty} = \lim_{x\to\infty}\frac{n!}{e^x} \neq \frac{\infty}{\infty} = \left(\frac{n!}{e^{\infty}}\right) = \frac{n!}{\infty} = 0$$

Note:

$$D(x) = 1 = 1!$$

$$D^2(x^2) = D(2x) = 2\times 1 = 2!$$

$$D^3(x^3) = D^2(3x^2) = D(3\times 2x) = 3\times 2\times 1 = 3!$$

$$D^4(x^4) = D^3(4x^3) = D^2(4\times 3x^2) = D(4\times 3\times 2x) = 4\times 3\times 2\times 1 = 4!$$

- - - - - - - - - -

In general $\boxed{D^n(x^n) = n!}$ & $D^{n+1}(x^n) = D\left(D^n(x^n)\right)$

$$= D(n!) = 0$$

ie $\boxed{D^{n+1}(x^n) = 0}$

Type V Standard Results of limits →

(1) $$\boxed{\lim_{x\to 0}(1+ax)^{\frac{1}{x}} = e^{a} = \lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{x}}$$

eg $$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e = \lim_{x\to 0}\left(1+\frac{1}{x}\right)^{x}$$

eg $$\lim_{x\to 0}(1-x^2)^{\frac{1}{x}} = ? = \lim_{x\to 0}\left[(1-x)(1+x)\right]^{\frac{1}{x}} = \lim_{x\to 0}(1-x)^{\frac{1}{x}} \cdot \lim_{x\to 0}(1+x)^{\frac{1}{x}}$$

$$= e^{-1}\times e^{1} = e^{0} = 1$$

eg $$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{2n} = ? = \lim_{n\to\infty}\left[\left(1-\frac{1}{n}\right)^{n}\right]^{2} = (e^{-1})^{2} = e^{-2}$$ Ans

Some More Standard Results:→

(1) $\lim_{x\to 0}\left(\frac{\sin x}{x}\right) = 1$ (2) $\lim_{x\to\infty}\left(\frac{\sin x}{x}\right) = \frac{\sin\infty}{\infty} = \frac{\text{Any No. b/n} -1 \,\&\, 1}{\infty} = 0$

(3) $\lim_{x\to 0}\left(\frac{\cos x}{x}\right) = \frac{\cos 0}{0} = \frac{1}{0} = \text{DNE}$, (4) $\lim_{x\to\infty}\left(\frac{\cos x}{x}\right) = \frac{\cos\infty}{\infty} = \frac{\text{Any No b/n} -1 \,\&\, 1}{\infty} = 0$

(5) $\lim_{x\to 0}\left[x.\sin\left(\frac{1}{x}\right)\right] =$

$= 0\times\sin\infty = 0\times(\text{Any No. b/n} -1 \,\&\, 1) = 0$

$= \lim_{x\to 0}\frac{\sin\left(\frac{1}{x}\right)}{\left(1/x\right)} = \lim_{y\to\infty}\left(\frac{\sin y}{y}\right) = 0$

Put $\left(\frac{1}{x} = y\right)$, when $x\to 0,\ y\to\infty$

(6) $\lim_{x\to 0}\sin\left(\frac{1}{x}\right) = ? = \sin(\infty) = \text{Any No b/n} -1 \,\&\, 1 = \text{Not unique} = \text{DNE}$

Explanation of (6) $\rightarrow$

$$\lim_{x\to 0}\left(\sin\frac{1}{x}\right) = ? = \lim_{x\to 0}\frac{\sin\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)}\times\left(\frac{1}{x}\right)$$

$$= \lim_{y\to\infty}\left(\frac{\sin y}{y}\right)\times(y),$$ Put $\frac{1}{x}=y$

$$= 0\times\infty$$

Still it is IND form.

So this method is not good enough.

(7) $\lim_{x\to 0}\left(\frac{\sin x}{x}\right) = \lim_{x\to 0}\left(\frac{x}{\sin x}\right) = \lim_{x\to 0}\left(\frac{\sin^{-1} x}{x}\right) = \lim_{x\to 0}\left(\frac{\tan x}{x}\right) = \lim_{(x-a)\to 0}\frac{\sin(x-a)}{(x-a)} = 1$

(8) $\lim_{x\to 0}\left(\frac{a^x-1}{x}\right) = \log_e a$, eg $\lim_{x\to 0}\left(\frac{e^x-1}{x}\right) = 1$

(9) $\lim_{x\to 0}\frac{\log(1+x)}{x} = 1$, (10) $\lim_{x\to 0}\frac{\log(1-x)}{x} = -1$

(11) $\lim_{x\to 0}\left(\frac{1-\cos x}{x^2}\right) = \frac{1}{2}$

Gate(P18):

$\lim_{a\to 0}\left(\frac{x^a-1}{a}\right) = ? = \log_e x$

using formula (8)

Note: All the above Results can be calculated using L-Hospital's Rule.

Q. $\lim_{x\to\infty}\left(\frac{3x-\sin x}{2x+5\cos x}\right)=?$ $=\lim_{x\to\infty}\frac{\cancel{x}}{\cancel{x}}\left[\frac{3-\left(\frac{\sin x}{x}\right)}{2+5\left(\frac{\cos x}{x}\right)}\right]$

(a) 3

(b) 1.5

(c) 1

(d) 0

$=\frac{3-0}{2+0}=1.5$

L-Hospital's Rule →

Q. $\lim_{x\to a}\left(\frac{x\sqrt{x}-a\sqrt{a}}{x-a}\right)=?$

(a) $\frac{3}{2}\sqrt{a}$ (b) $\frac{3}{2}$ (c) $\sqrt{a}$ (d) $\frac{3}{4}\sqrt{a}$

$$=\lim_{x\to a}\left(\frac{x^{3/2}-a^{3/2}}{x-a}\right)\simeq\frac{0}{0}$$

$$=\lim_{x\to a}\left(\frac{\frac{3}{2}x^{1/2}-0}{1-0}\right)$$

$$=\frac{3}{2}a^{1/2}=\frac{3}{2}\sqrt{a}$$

Q. $\lim_{x\to 0}\left[\frac{(1-x)^n-1}{x}\right]=?\simeq\frac{0}{0}$ form.

(a) 0 (b) n (c) $-n$ (d) $2n$

Put $1-x=t$

when $x\to 0,\ t\to 1$

$$=\lim_{t\to 1}\left(\frac{t^n-1}{1-t}\right)=-\lim_{t\to 1}\left(\frac{t^n-1}{t-1}\right)\simeq\frac{0}{0}\text{ form}$$

$$=-\left(\lim_{t\to 1}\left(\frac{n\,t^{n-1}}{1}\right)\right)=\boxed{-n}$$

Q. $\lim_{x \to 5} \left[\frac{\log_e x - \log_e 5}{x-5}\right] = ? = \frac{0}{0}$

$= \lim_{x \to 5} \left[\frac{1/x - 0}{1 - 0}\right]$

$= \frac{1}{5}$ ✓

Note: $x \to 0 \Rightarrow x$ is very small

i.e. $\sin x$ is also very small

i.e. $\sin x \to 0$

Q. $\lim_{x \to 0} \left[\frac{e^{\sin x} - 1}{x}\right] = ? = \frac{0}{0}$ form

M-I $= \lim_{x \to 0} \left(\frac{e^{\sin x}(\cos x)}{1}\right) = e^{0} \cdot \cos 0 = 1 \times 1 = 1$

M-II

$= \lim_{x \to 0} \left(\frac{e^{\sin x} - 1}{\sin x}\right) \times \left(\frac{\sin x}{x}\right)$

$= \lim_{\sin x \to 0} \left(\frac{e^{\sin x} - 1}{\sin x}\right) \times \lim_{x \to 0} \left(\frac{\sin x}{x}\right)$

$= 1 \times 1 = 1$

Q. $\lim_{x\to 0}(\tan x \cdot \log x) = ?$ $= 0\times(-\infty)$ form

(a) 0 (b) 1 (c) −1 (d) DNE

$= \lim_{x\to 0}\left(\frac{\log x}{\cot x}\right) = \frac{-\infty}{\infty}$ form

$= \lim_{x\to 0}\left[\frac{\left(\frac{1}{x}\right)}{-\operatorname{cosec}^2 x}\right] = -\lim_{x\to 0}\left(\frac{\sin^2 x}{x}\right)$

$= -\lim_{x\to 0}\left(\frac{\sin x}{x}\right)\times\lim_{x\to 0}(\sin x)$

$= -1\times 0 = 0$

Q. $\lim_{x\to 0}\left[x\cdot\sin\frac{1}{x}\right] = ?$

(a) 1 (b) 0 (c) −1 (d) ∞

$\lim_{x\to 0}\left[x\sin\frac{1}{x}\right] = 0\times\sin\infty$

$= 0\times(\text{Any No b/n } -1 \text{ \& } 1) = 0$

M-II $\lim_{x\to 0}\left[\frac{\sin\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)}\right] = \lim_{y\to\infty}\left(\frac{\sin y}{y}\right)$

Put $y = \frac{1}{x}$

when $x\to 0$, $y\to\infty$

$= \frac{\sin\infty}{\infty}$

$= \frac{\text{No b/n } -1 \text{ \& } 1}{\infty} = 0$

Q. $\lim_{x\to 0} (\sin x)^{\tan x} = ? = 0^0$ form.

Let $K = \lim_{x\to 0} (\sin x)^{\tan x}$

$\log_e K = \lim_{x\to 0} \log (\sin x)^{\tan x}$

$= \lim_{x\to 0} \left[\tan x (\log \sin x)\right] = 0 \times (-\infty)$

$= \lim_{x\to 0} \left[\frac{\log \sin x}{\cot x}\right] = \frac{-\infty}{\infty}$ form.

$= \lim_{x\to 0} \left(\frac{\frac{1}{\sin x}\cdot \cos x}{-\operatorname{cosec}^2 x}\right) = \lim_{x\to 0} -(\cos x \sin x) = -1\times 0 = 0 \Rightarrow K = e^0 = 1$ Ans

Q. $\lim_{x\to 0} \left(\frac{1}{x}\right)^x = ?$ Ans $= 1$

HWQ $= (\infty^0)$ form

Q $\lim_{x\to 0}(\cos 2x)^{\frac{1}{x^2}} = ? = 1^{\infty}$

(a) e (b) e^2 (c) $\sqrt{e}$ (d) e^{-2}

Sol: Let $K = \lim_{x\to 0}(\cos 2x)^{\frac{1}{x^2}}$

$$\log K = \lim_{x\to 0}\log(\cos 2x)^{\frac{1}{x^2}}$$

$$= \lim_{x\to 0}\left[\frac{\log(\cos 2x)}{x^2}\right] = \frac{0}{0} \text{ form.}$$

$$= \lim_{x\to 0}\left[\frac{\frac{1}{\cos 2x}(-\sin 2x(2))}{2x}\right] = -2\lim_{2x\to 0}\left(\frac{\sin 2x}{2x}\right)\times\lim_{x\to 0}\left(\frac{1}{\cos 2x}\right) = -2(1)\times\frac{1}{1} = -2$$

$$\log_e K = -2 \Rightarrow K = e^{-2} \text{ Ans}$$

HW Q

Q $\lim_{x\to 1}\left[\log_3(3x)\right]^{\log_x 3} = ? = 1^{\infty}$ form.

(a) 0 (b) 1 (c) e (d) e^2

Q. $\lim_{x\to 0}\left(\frac{\log(1+x^3)}{\sin^3 x}\right) = ? \Leftarrow \frac{0}{0}$

(a) 0 (b) 1 (c) e (d) ∞

M-I Using L-Hosp Rule :→ Irritating

M-II Using standard Results :

$$\lim_{x\to 0}\frac{\log(1+x^3)}{x^3}\times\lim_{x\to 0}\left(\frac{x^3}{\sin^3 x}\right)$$

$$\lim_{x^3\to 0}\left[\frac{\log(1+x^3)}{x^3}\right]\times\lim_{x\to 0}\left(\frac{x}{\sin x}\right)^3$$

$$= 1\times 1^3 = 1$$

Q. $\lim_{x\to 0}\frac{\sin[\pi\cos^2 x]}{x^2} = ?$

(a) 1 (b) e (c) π (d) 0

$$= \lim_{x\to 0}\frac{\sin[\pi(1-\sin^2 x)]}{x^2} = \lim_{x\to 0}\frac{\sin(\pi-\pi\sin^2 x)}{x^2}$$

$$= \lim_{x\to 0}\left[\frac{\sin(\pi\sin^2 x)}{x^2}\right]\times\frac{\pi\sin^2 x}{\pi\sin^2 x}$$

$$= \lim_{x\to 0}\left[\frac{\sin(\pi\sin^2 x)}{\pi\sin^2 x}\right]\times\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2\times\pi$$

$$= 1\times 1^2\times\pi = \pi$$

Q. $\lim_{x\to 0}\left(\frac{\tan x - \sin x}{x^3}\right) = ?$

(a) 1 (b) 0.5 (c) −0.5 (d) −1

(M-I) → Lengthy

(M-II) $\lim_{x\to 0}\left(\frac{\frac{\sin x}{\cos x} - \sin x}{x^3}\right) = \lim_{x\to 0}\frac{\sin x(1-\cos x)}{\cos x . x^3}$

$= \lim_{x\to 0}\left(\frac{\sin x}{x}\right) \times \lim_{x\to 0}\left(\frac{1-\cos x}{x^2}\right) \times \lim_{x\to 0}\left(\frac{1}{\cos x}\right)$

$= 1 \times \frac{1}{2} \times 1 = \frac{1}{2}$

Q. $\lim_{x\to 0}\left[\frac{e^x + e^{-x} - 2}{x^2}\right] = ?$

(a) 1 (b) 0 (c) 1/2 (d) e

(M-I) → Imitating

(M-II) $\lim_{x\to 0}\left(\frac{e^x + \frac{1}{e^x} - 2}{x^2}\right)$

$= \lim_{x\to 0}\left(\frac{e^{2x} + 1 - 2e^x}{e^x . x^2}\right) = \lim_{x\to 0}\frac{(e^x - 1)^2}{e^x . x^2}$

$= \lim_{x\to 0}\left(\frac{e^x - 1}{x}\right)^2 \times \lim_{x\to 0}\left(\frac{1}{e^x}\right) = 1 \times 1 = 1$

Q. if $2-\frac{x^2}{3} < \frac{x\sin x}{1-\cos x} < 2$ then Evaluate $\lim_{x\to 0}\left(\frac{x\sin x}{1-\cos x}\right) = \underline{\ ?\ }$

(M-I) $\lim_{x\to 0}\left(\frac{x\sin x}{1-\cos x}\right) = \lim_{x\to 0}\left(\frac{\sin x}{x}\right)\times\left(\frac{x^2}{1-\cos x}\right) = 1\times 2 = 2$

SANDWITCH THEOREM → if $f(x)\le g(x)\le h(x)$

then $\boxed{\lim_{x\to a} f(x) \le \lim_{x\to a} g(x) \le \lim_{x\to a} h(x)}$

(M-II) By S. Th, $\lim_{x\to 0}\left(\frac{x\sin x}{1-\cos x}\right) = 2$

Continuity → f(x) is said to be cont if,

$$\boxed{\lim_{x \to a} f(x) = f(a)}$$

App. Value = Exact Value

$$\boxed{(LHL = RHL) = f(a)}$$

M-I Either exact Value DNE

M-II or App. Value " "

M-III Both exist But Not equal.

Methods to Check Discontinuity

Q. Check the Continuity of the following functions

(1) $f(x) = \begin{cases} x \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ (Cont)

Sol: At $x=0$, Exact Value $= f(0) = 0$ (given)

App Value $= \lim_{x\to 0} f(x) = \lim_{x\to 0}\left(x \sin\frac{1}{x}\right)$

$= 0 \times \sin\infty = 0 \times (\text{No b/w } -1 \,\&\, 1)$

$= 0$

$\because$ Approx Value = Exact Value

So $f(x)$ is Cont at $x=0$

(2) $f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$ (Cont)

Already Solved YESTERDAY

Q. $f(x)=\begin{cases}1+x^2, & 0\le x\le 1\\ (2-x), & x>1\end{cases}$ (Discont)

At $x=1$, Exact Value $= f(1) = (1+x^2)_{x=1} = 2$

LHL at $(x=1) = f(1^-) = \lim_{x\to 1^-} f(x)$

$= \lim_{x\to 1}(1+x^2) = 1+1^2 = 2$

RHL at $(x=1) = f(1^+) = \lim_{x\to 1^+} f(x)$

$= \lim_{x\to 1}(2-x) = 2-1 = 1$

$\because$ LHL $\neq$ RHL So limit DNE $\Rightarrow f(x)$ is Discont at $x=1$ by M-II

Q. $f(x)=\begin{cases}\frac{|x|}{x}, & x\neq 0\\ 1, & x=0\end{cases}$ (Discont)

$=\begin{cases}-1, & x<0\\ +1, & x>0\\ 1, & x=0\end{cases}$

At $x=0$
- Exact Value $= f(0) = 1$
- LHL $= f(1^-) = -1$
- RHL $= f(1^+) = +1$

$\because$ LHL $\neq$ RHL So Discont at $x=0$

Note → Here $f(x)$ is Not a Signum funcⁿ

Q. HWQ If $f(x)=\begin{cases} \dfrac{ae^{x}-b\cos x+ce^{-x}}{x\sin x}, & x\neq 0 \\ 2, & x=0 \end{cases}$

is continuous everywhere then correct option is?

(a) ✗ $a=2,\ b=1,\ c=1$

(b) ✗ $a=1,\ b=-2,\ c=1$

(c) ✗ $a=1,\ b=1,\ c=2$

(d) ✓ $a=1,\ b=2,\ c=1$

(e) None

$\because f(x)$ is cont at $x=0$ so

App Value = Exact Value

$$\lim_{x\to 0} f(x) = f(0)$$

$$\lim_{x\to 0}\left(\frac{ae^{x}-b\cos x+ce^{-x}}{x\sin x}\right)=2$$

$$\frac{a-b+c}{0}=2 \Rightarrow \boxed{a-b+c=0}$$

Now, if we take $a-b+c=0$ then LHS will be in $\frac{0}{0}$ form.

$$\lim_{x\to 0}\left(\frac{ae^{x}+b\sin x-ce^{-x}}{x\cos x+\sin x}\right)=2$$

$$\frac{a-c}{0}=2 \Rightarrow \boxed{a=c}$$

Now when we take $a=c$, LHS will become again $\frac{0}{0}$ form.
So again applying L-Hosp Rule in LHS.

$$\lim_{x\to 0}\left[\frac{ae^{x}+b\cos x+ce^{-x}}{\underbrace{x(-\sin x)+\cos x}+\cos x}\right]=2$$

$$\frac{a+b+c}{0+1+1}=2 \Rightarrow \boxed{a+b+c=4}$$

Solving these three eqn's, $a=1, b=2, c=1$

Q. $\lim_{x \to 0} \left(\frac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1} \right) = ?$ $= DNE$

HWQ

Q. find K for which $f(x)$ is Continuous at $x=0$

(1) $f(x)=\begin{cases}\dfrac{\log(1+3x)-\log(1-2x)}{x}, & x\neq 0\\ K, & x=0\end{cases}$

(a) 1 (b) 0 (c) -1 (d) 5

Sol: At $x=0$, Exact Value $= f(0)=K$ (given)

Now, App. Value $=\lim_{x\to 0} f(x)=$

$$=\lim_{x\to 0}\left[\frac{\log(1+3x)-\log(1-2x)}{x}\right]=\lim_{x\to 0}\left[\frac{\left(\frac{3}{1+3x}\right)-\left(\frac{-2}{1-2x}\right)}{1}\right]=5$$

for Continuity at $x=0$, Exact Value = App Value $\Rightarrow K=5$

(2) $f(x)=\begin{cases}\sin\frac{1}{x}, & x\neq 0\\ K, & x=0\end{cases}$

(a) 0 (b) 1 (c) $(-\infty,\infty)$

(d) No such value of K

(M-II) $\lim_{x\to 0} f(x) = \lim_{x\to 0}\left[\frac{\log(1+3x)-\log(1-2x)}{x}\right]$

$$= \lim_{3x\to 0}\left(\frac{\log(1+3x)}{3x}\right)\times 3 - \lim_{2x\to 0}\left(\frac{\log(1-2x)}{2x}\right)\times 2$$

$$= 1\times 3-(-1)\,2 = 5$$

$x=1$ & $x=3$ are problem creating points.

PYQ

The values of a and b for which the function

$$f(x) = \begin{cases} 2x+1, \text{ if } x \le 1 \\ ax^2+b \text{ if } 1 < x < 3 \\ 5x+2a \text{ if } x \ge 3 \end{cases}$$

is continuous every where

(a) $a = 2, b = 1$ (b) $a = 1, b = 2$

(c) $a = 3, b = 2$ (d) $a = 2, b = 3$

At $x=1$, LHL = RHL = $f(1)$

$(2x+1)_{x=1} = (ax^2+b)_{x=1} = (2x+1)_{x=1}$

$3 = \boxed{a+b=3}$ — (1)

At $x=3$, LHL = RHL = $f(3)$

$(ax^2+b)_{x=3} = (5x+2a)_{x=3} = 15+2a$

$9a+b = 15+2a \Rightarrow \boxed{7a+b=15}$ — (2)

$a=2, b=1$


Thank
THANK
Keep Hustling!

DATA SCIENCE
&
ARTIFICIAL INTELLIGENCE

& CS / IT

Calculus and Optimization

Lecture No. 05

By- Dr. Puneet Sharma Sir

Recap of previous lecture

Topic

Limit & Continuity

Topics to be Covered

Topic

1. Differentiability
2. Taylor & Maclaurin Series

Q. $\lim_{x\to 0}\left\{\dfrac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}\right\} = ?$ (a) 0 (b) 1 (c) -1 (d) DNE

Sol: $\simeq \dfrac{e^{\infty}-1}{e^{\infty}+1} = \dfrac{\infty-1}{\infty+1} = \dfrac{\infty}{\infty}$ form

But we can't apply L-Hospital's Rule because $f(x)$ is not diff at $x=0$

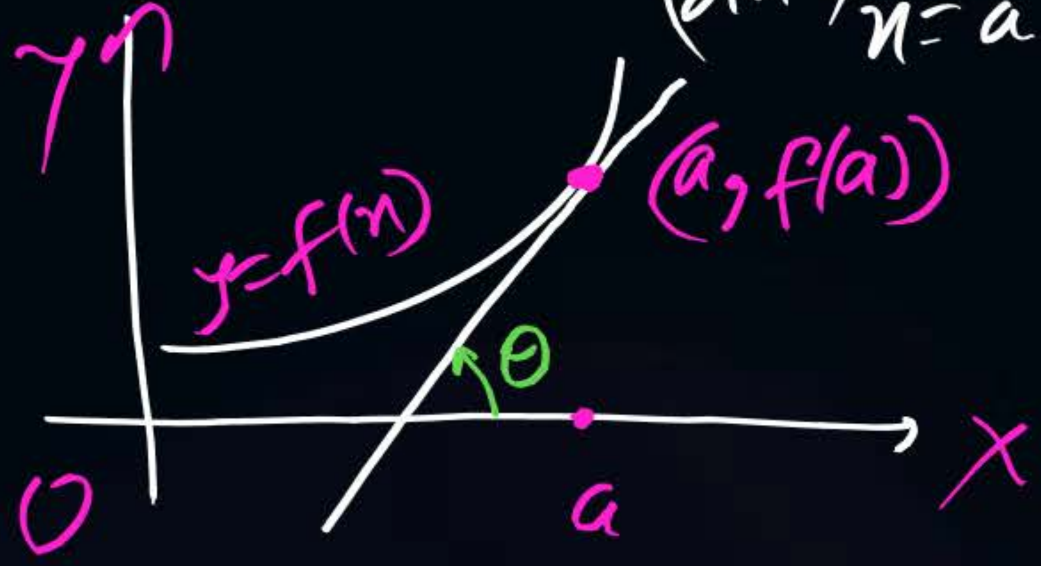
Now $LHL = f(0^-) = f(0-h) = \lim_{h\to 0}\left(\dfrac{e^{\frac{1}{0-h}}-1}{e^{\frac{1}{0-h}}+1}\right) = \dfrac{e^{-\infty}-1}{e^{-\infty}+1} = \dfrac{0-1}{0+1} = -1$

$RHL = f(0^+) = f(0+h) = \lim_{h\to 0}\left(\dfrac{e^{\frac{1}{0+h}}-1}{e^{\frac{1}{0+h}}+1}\right) = \lim_{h\to 0}\left(\dfrac{1-e^{-\frac{1}{h}}}{1+e^{-\frac{1}{h}}}\right) = \dfrac{1-e^{-\infty}}{1+e^{-\infty}} = \dfrac{1-0}{1+0} = 1$

$\because$ LHL $\neq$ RHL so limit DNE.

G. Meaning of Derivative

for $y = f(x)$, $\left(\frac{dy}{dx}\right)_{x=a} = \boxed{f'(a) = \tan\theta}$ = slope of tangent at $(x=a)$



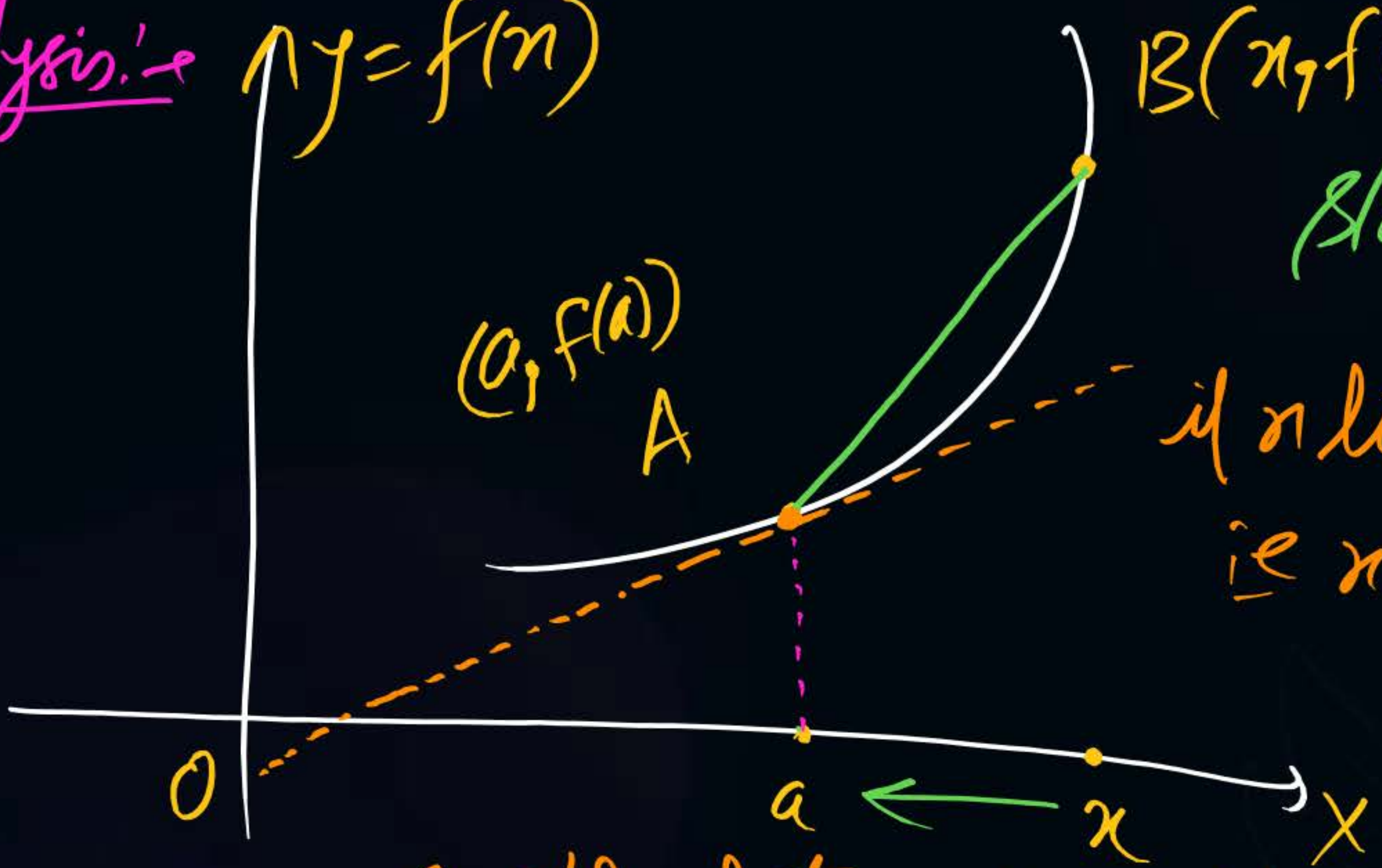
$\because$ $x=a$ is any Random point in the Domain of $f(x)$

So In general,

$\boxed{f'(x) = \tan\theta}$ at any Random Point x

Mathematical Defn of Differentiability → $f(x)$ is said to be differentiable at $x=a$ if $\lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\right)$ exist & Value in this limit is called Derivative of $f(x)$ at $x=a$ & it is denoted as $f'(a)$ i.e. we have $\boxed{f'(a) = \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\right)}$

Analysis:-



Slope of line AB $= \dfrac{f(x)-f(a)}{x-a}$

if x lies in the Nbd of a

i.e x lies very-very close to a

i.e $x \to a$ then

Chord will convert into tangent

& Slope of tangent = Slope of chord (AB)

$$f'(a) = \lim_{x \to a}\left(\frac{f(x)-f(a)}{x-a}\right)$$

Mathematical Def^n → $f(x)$ is said to be differentiable at $x=a$, if it is possible to find the value of $\lim_{x \to a}\left(\frac{f(x)-f(a)}{x-a}\right)$ & value of this limit is called Derivative of $f(x)$ at $x=a$ & it is denoted by $f'(a)$ i.e

$$\boxed{f'(a) = \lim_{x \to a}\left(\frac{f(x)-f(a)}{x-a}\right)}$$

$$LHD = \lim_{h \to 0}\left(\frac{f(a-h)-f(a)}{-h}\right)$$

$$RHD = \lim_{h \to 0}\left(\frac{f(a+h)-f(a)}{h}\right)$$

When LHD = RHD then $f(x)$ is said to be Differentiable at $x=a$

Note (1) Continuity is the Necessary Condition for Differentiability.

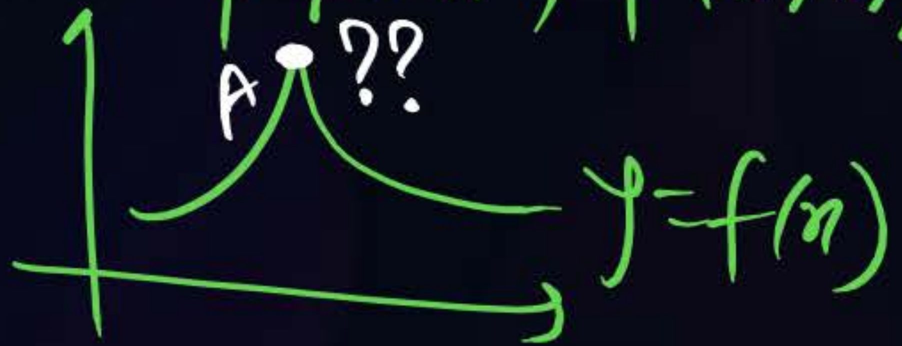
But it is not Sufficient Condition.

(2) N Condⁿ is $\longrightarrow$ Continuity

Sufficient Condⁿ is $\longrightarrow \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\right)$ must exist.

Q: $\underbrace{\text{Differentiability} \Longrightarrow \text{Continuity}}_{\text{True}}$ or $\underbrace{\text{Continuity} \Longrightarrow \text{Differentiablity}}_{\text{False}}$.

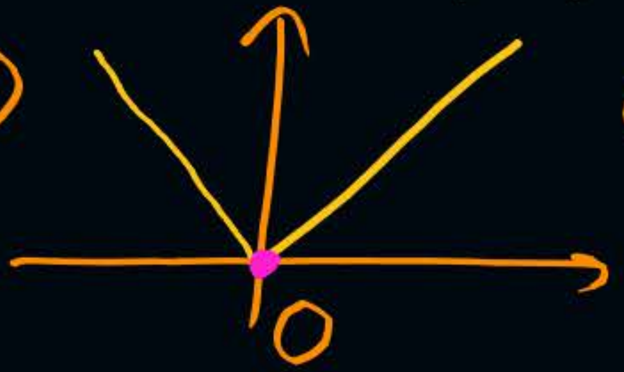
(3) At Sharp Point, $f(x)$ is said to be Non Differentiable.

eg 

at A, $f(x)$ is Non Diff, $\because$ it is not possible to find unique tangent at A.

eg At $x=0$, $f(x)=|x|$ is Continuous but Not Diff

(M I)



(M II) $f(x)=|x|=\begin{cases}-x, & x<0\\ x, & x\geq 0\end{cases}$; $f'(x)=\begin{cases}-1, & x<0\\ 1, & x>0\end{cases}$

LHL $=0$, RHL $=0$, $f(0)=0$ i.e Cont at $x=0$

LHD $=-1$, RHD $=1$, Not Diff

(V) Shortcut Method of finding LHD and RHD of $f(x)$ $\rightarrow$

W.K. that LHL $=\lim\limits_{x\to a^-} f(x)$ & RHL $=\lim\limits_{x\to a^+} f(x)$

Similarly, LHD of $f(x)=$ LHL of $f'(x)$ & RHD of $f(x)=$ RHL of $f'(x)$

$=\lim\limits_{x\to a^-} f'(x)$ $\qquad$ $=\lim\limits_{x\to a^+} f'(x)$

Qs find the Derivative of $y=|x|$?

Sol: $\frac{dy}{dx}=\frac{d}{dx}|x|=\frac{d}{dx}\left(\sqrt{x^2}\right)=\frac{1}{2\sqrt{x^2}}\frac{d}{dx}(x^2)=\frac{2x}{2\sqrt{x^2}}=\frac{x}{|x|}$

$\begin{cases}\frac{-x}{x}, & x<0\\ \frac{+x}{x}, & x>0\end{cases}$

ie $\frac{d}{dx}|x|=\frac{x}{|x|}=\frac{x|x|}{|x|^2}=\frac{x|x|}{x^2}=\frac{|x|}{x}, \quad x\neq 0 = \begin{cases}-1, & x<0\\ \text{DNE}, & x=0\\ +1, & x>0\end{cases}$

$\underbrace{\left(\frac{d}{dx}|x|\right)_{x=0}=\text{DNE}}_{\substack{LHD=-1\\ RHD=+1}}, \quad \underbrace{\left(\frac{d}{dx}|x|\right)_{x=5}=+1}_{\substack{LHD=1\\ RHD=1}}, \quad \underbrace{\left(\frac{d}{dx}|x|\right)_{x=-3}=-1}_{\substack{LHD=-1\\ RHD=-1}}, \quad \underbrace{\left(\frac{d}{dx}|x|\right)_{x=-5}=-1}_{\substack{LHD=-1\\ RHD=-1}}$

ie $f(x)=|x|$ is diff everywhere except at $x=0$

Q: if $y = \sin|x|$ then evaluate $\frac{dy}{dx}$ at $x=0$ & $x = -\frac{\pi}{4}$

Sol: $y = f(x) = \sin|x| = \begin{cases} \sin(-x), & x<0 \\ \sin(+x), & x>0 \end{cases}$

$= \begin{cases} -\sin x, & x<0 \\ \sin x, & x>0 \\ 0, & x=0 \end{cases}$

At $x=0$: LHL $= -0$, RHL $= +0$, $f(0) = 0$ ie Continuous

$f'(x) = \begin{cases} -\cos x, & x<0 \\ +\cos x, & x>0 \end{cases}$

At $x=0$: LHD $= -\cos 0 = -1$, RHD $= +\cos 0 = +1$

ie $\left[\frac{d}{dx}(\sin|x|)\right]_{x=0} = \text{DNE}$

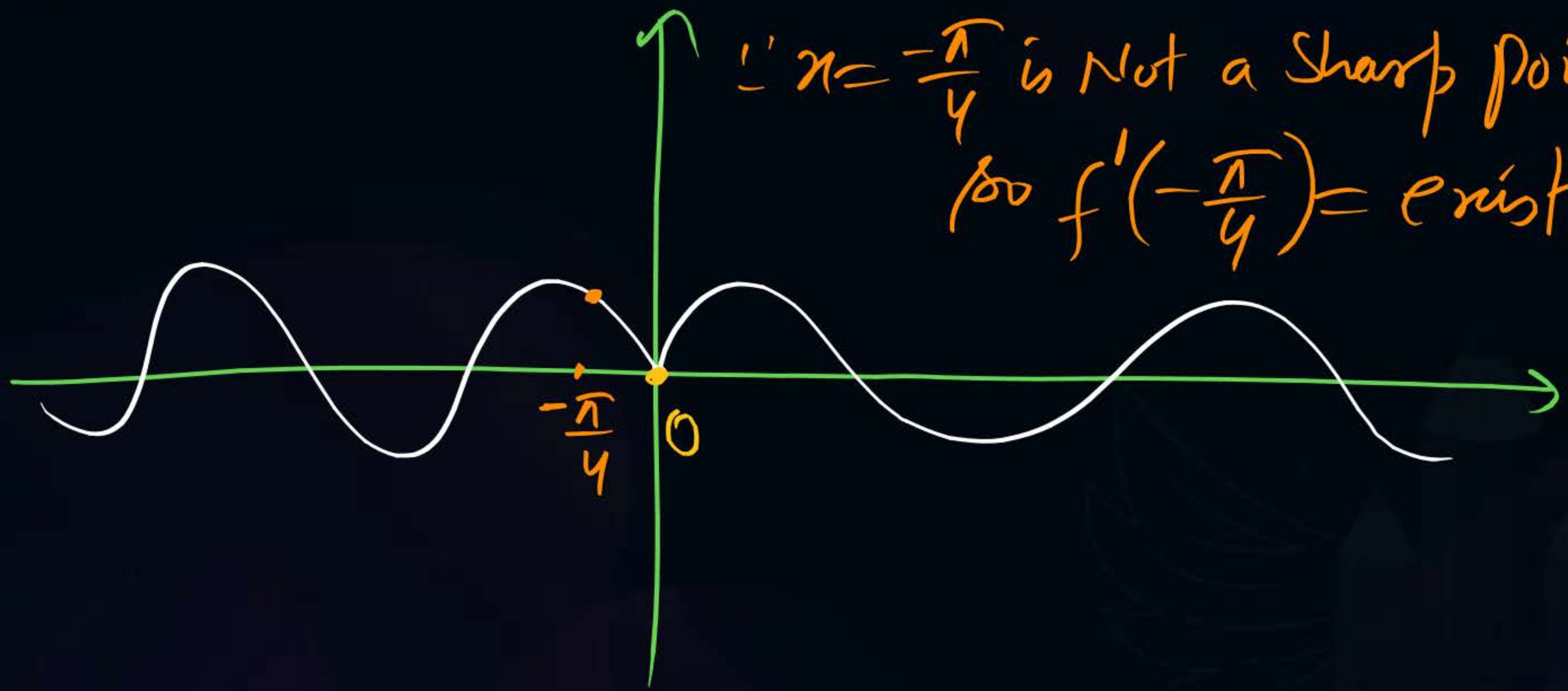
$\because x = -\frac{\pi}{4}$ is not a problem creating Point

so $f'\left(-\frac{\pi}{4}\right) = \left(-\cos x\right)_{x=-\frac{\pi}{4}} = -\cos\left(-\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

Analysis $y = \sin|x|$, $\because x = 0$ is Sharp Point so $f'(0) = \text{DNE}$

$\because x = -\frac{\pi}{4}$ is Not a Sharp point

so $f'\left(-\frac{\pi}{4}\right) = $ exist $= \ldots\ldots = \frac{-1}{\sqrt{2}}$



Dom = $(0,\infty)$

Q At $x=1$, $f(x)=|\log_e x|$ is

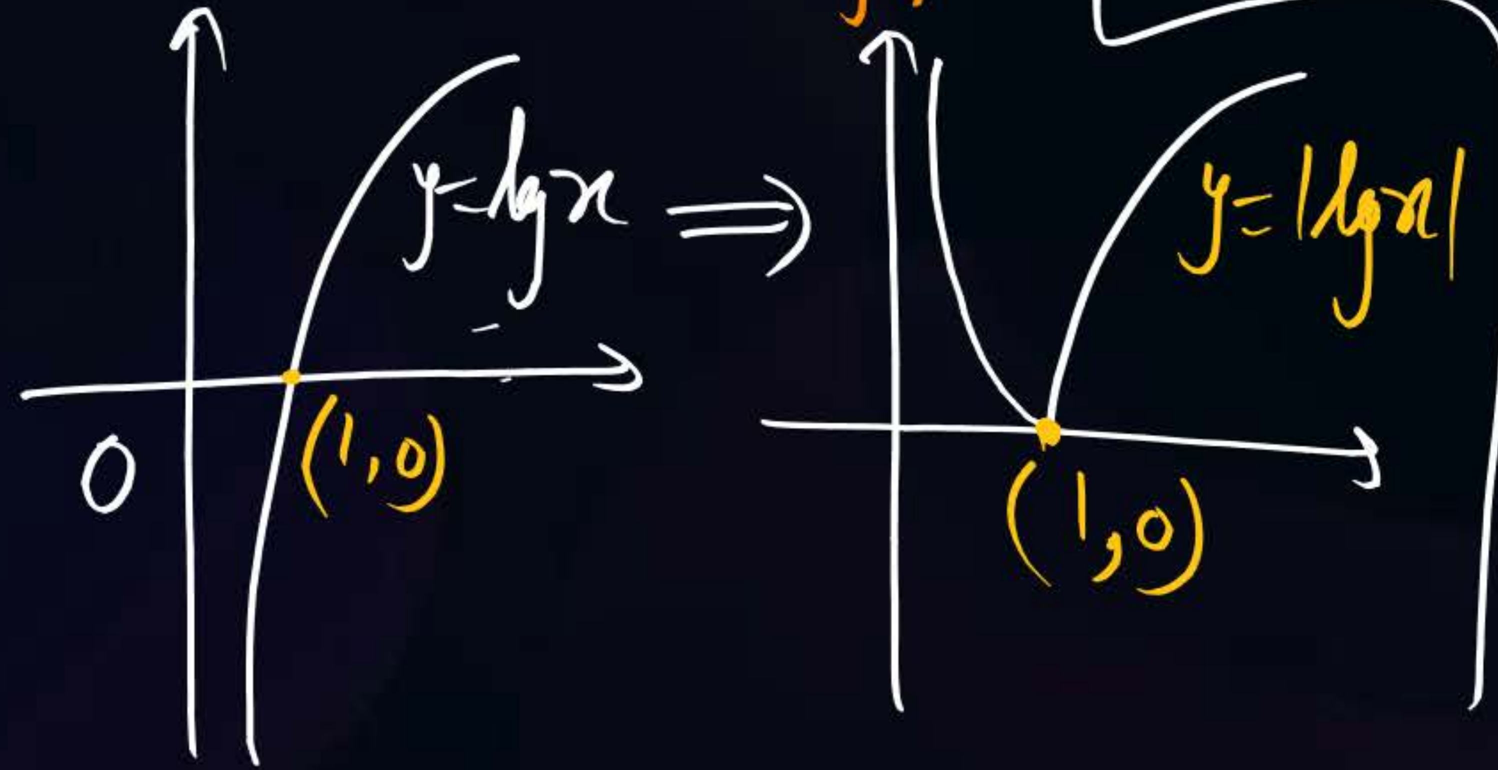
(a) Cont but Not Diff

(b) Diff but Not Cont (PAAP)

(c) Neither Cont Nor Diff

(d) Both Cont as well as Diff

(M-I)



(M-II) $f(x)=|\log x| = \begin{cases} -\log x, & 0<x<1 \\ +\log x, & x>1 \end{cases}$

$f'(x)=\begin{cases} -1/x, & 0<x<1 \\ 1/x, & x>1 \end{cases}$

At $x=1$ $\begin{cases} \text{LHD} = f'(1^-) = -1 \\ \text{RHD} = f'(1^+) = +1 \end{cases}$

i.e. LHD $\neq$ RHD

So $f(x)$ is Not Diff at $x=1$

Q. find points of Discontinuity and Non Diff points of $f(x)=\frac{x-|x-1|}{x}$

HW Q

[Ans: Discont at $x=0$ only & Non Diff at $x=0$ & 1]

Which of the following function is differentiable at $x = 0$?

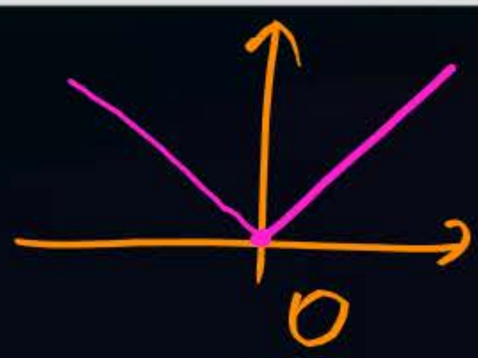
(a) $f(x) = |x|$ ✗

(b) $f(x) = |x| + |x-1|$

(c) $f(x) = x|x|$ ✓

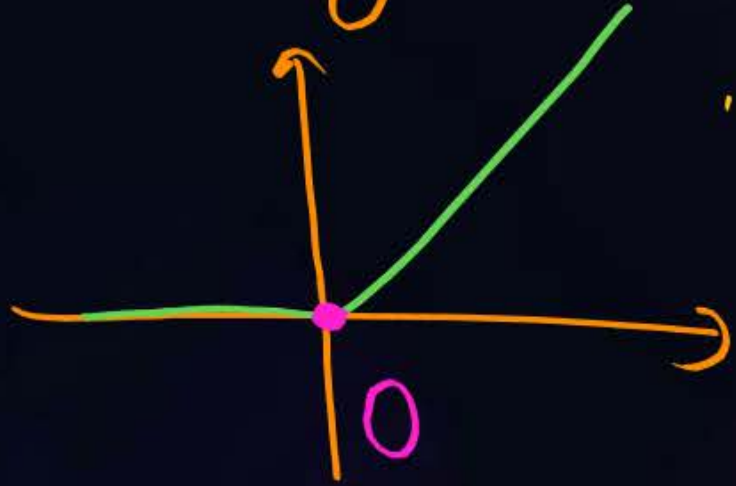
(d) $f(x) = \begin{cases} 0 \text{ if, } & x \le 0 \\ x \text{ if, } & x > 0 \end{cases}$ ✗

(a)



Not Diff at $x = 0$

(d)



" " " "

(c) $y = f(x) = x|x| = \begin{cases} -x^2 & , x < 0 \\ x^2 & , x > 0 \end{cases}$

(M-I)

$\because x = 0$ is not a sharp point so $f(x)$ is Diff at $x = 0$

(M-II) $f'(x) = \begin{cases} -2x & , x < 0 \\ 2x & , x > 0 \end{cases}$

LHD of $f(x) = f'(0^-) = 0$

RHD of $f(x) = f'(0^+) = 0$

$\because$ LHD $=$ RHD so $f(x)$ is diff at $x = 0$

Taking (b): $\boxed{f(x)=|x|+|x-1|}$

Analysis

$$= \begin{cases} -x+(1-x), & x<0 \\ +x+(1-x), & 0\le x<1 \\ +x+(x-1), & x\geq 1 \end{cases}$$

$$f(x) = \begin{cases} 1-2x, & x<0 \\ 1, & 0\le x<1 \\ 2x-1, & x\geq 1 \end{cases}$$

At $x=0$, LHL $=1$, RHL $=1$, $f(0)=1$ Hence Cont at $x=0$

At $x=1$, LHL $=1$, RHL $=1$, $f(1)=1$,, ,, ,, $x=1$

$$f'(x) = \begin{cases} -2, & x<0 \\ 0, & 0<x<1 \\ 2, & x>1 \end{cases}$$

At $x=0$ — LHD $=-2$, RHD $=0$ is Not Diff at $x=0$

At $x=1$ — LHD $=0$, RHD $=2$ is Not Diff at $x=1$

A real function

PYQ.

$$f(x)=\begin{cases}\alpha x^2+\beta x, & \text{for } x<0\\ \alpha x^3+\beta x^2+5\sin x, & x\geq 0\end{cases}$$

If $f(x)$ is twice differentiable then

(a) $\alpha = 1, \beta = 0$ (b) $\alpha = 1, \beta = 5$

(c) $\alpha = 5, \beta = -10$ (d) $\alpha = 5, \beta = 5$

$$f'(x)=\begin{cases}2\alpha x+\beta, & x<0\\ 3\alpha x^2+2\beta x+5\cos x, & x>0\end{cases}$$

At $x=0$, LHD = RHD

$\beta = 5$

$$f''(x)=\begin{cases}2\alpha, & x<0\\ 6\alpha x+2\beta-5\sin x, & x>0\end{cases}$$

LHD = RHD

$2\alpha = 2\beta \Rightarrow \alpha=\beta=5$

Q $\lim_{x\to 0}\left(\frac{\sin x}{x}\right) = ? = \lim_{x\to 0}\left(\frac{x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\cdots}{x}\right) = \lim_{x\to 0}\left[1-\frac{x^2}{3!}+\frac{x^4}{5!}-\cdots\right]$

$= 1-0+0-0+\cdots = 1$

$\because \sin x = x-\frac{x^3}{3!}-\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\cdots$

Note: $\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots\cdots$

$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\cdots$

TAYLOR & MACLAURIN SERIES
(POWER SERIES)

MACLAURIN SERIES → Let $f(x)$ is continuous func^n defined in a Domain D s.t it's all derivative exist then $f(x)$ can be expanded in the Nbd of 0 ($x \to 0$) as follows;

$$\boxed{f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots\cdots}$$

Note 1. this series provides Approx value of $f(x)$ in the Nbd of 0

2. Coeff of x^n in the M.S. Exp $= \dfrac{f^n(0)}{n!}$

3. In this series we are getting Increasing powers of x

Some Standard Results of Maclaurin Series

→ All these Results are valid only when $x \to 0$

① $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$

$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \cdots$

② $a^x = 1 + x(\ln a) + \frac{x^2}{2!}(\ln a)^2 + \frac{x^3}{3!}(\ln a)^3 + \cdots$

Gate ✓ $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

③ $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$

Gate

$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots = \frac{e^x - e^{-x}}{2}$

(4) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots\cdots\infty$

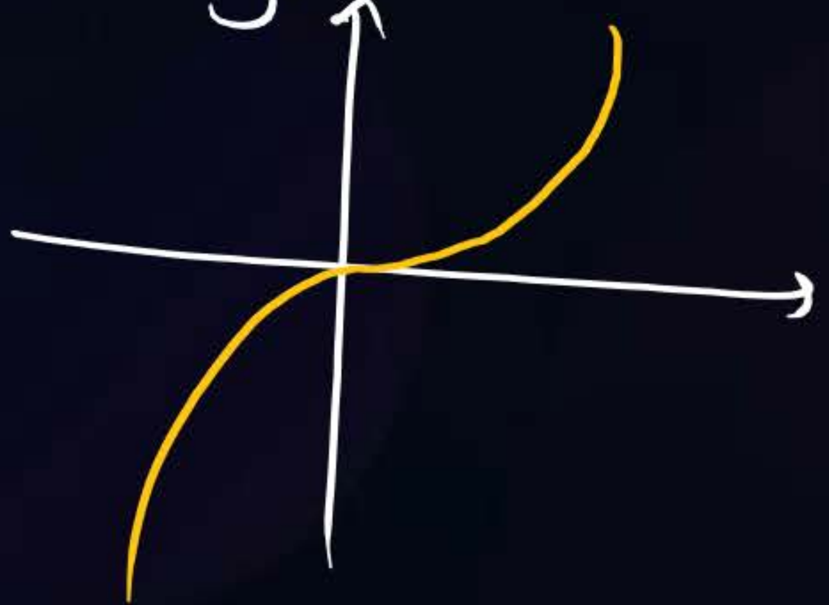
$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots\cdots = \frac{e^x + e^{-x}}{2}$

(5) $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \cdots\cdots$

<u>Note</u>: $\sinh x = \frac{e^x - e^{-x}}{2}$, $\cosh x = \frac{e^x + e^{-x}}{2}$

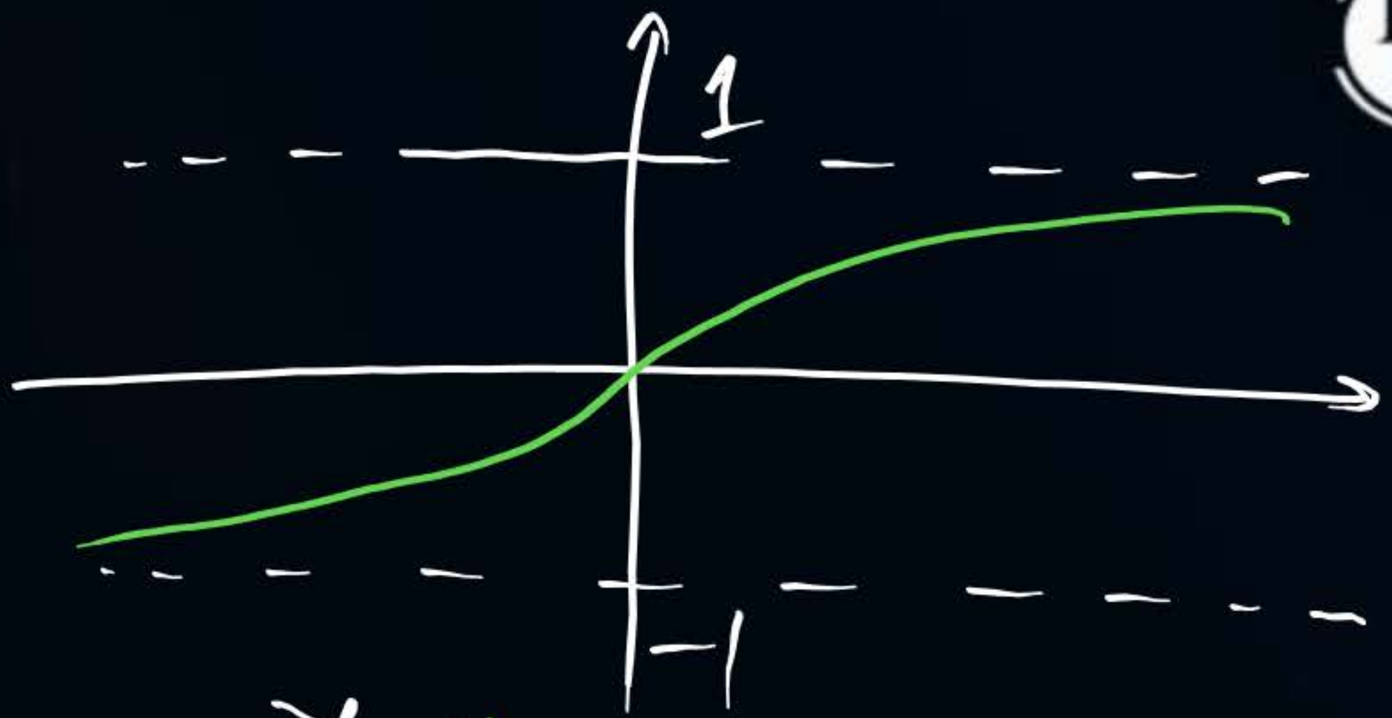
$\frac{d}{dx}(\sinh x) = \int \sinh x \, dx = \cosh x$

$\frac{d}{dx}(\cosh x) = \int \cosh x \, dx = \sinh x$

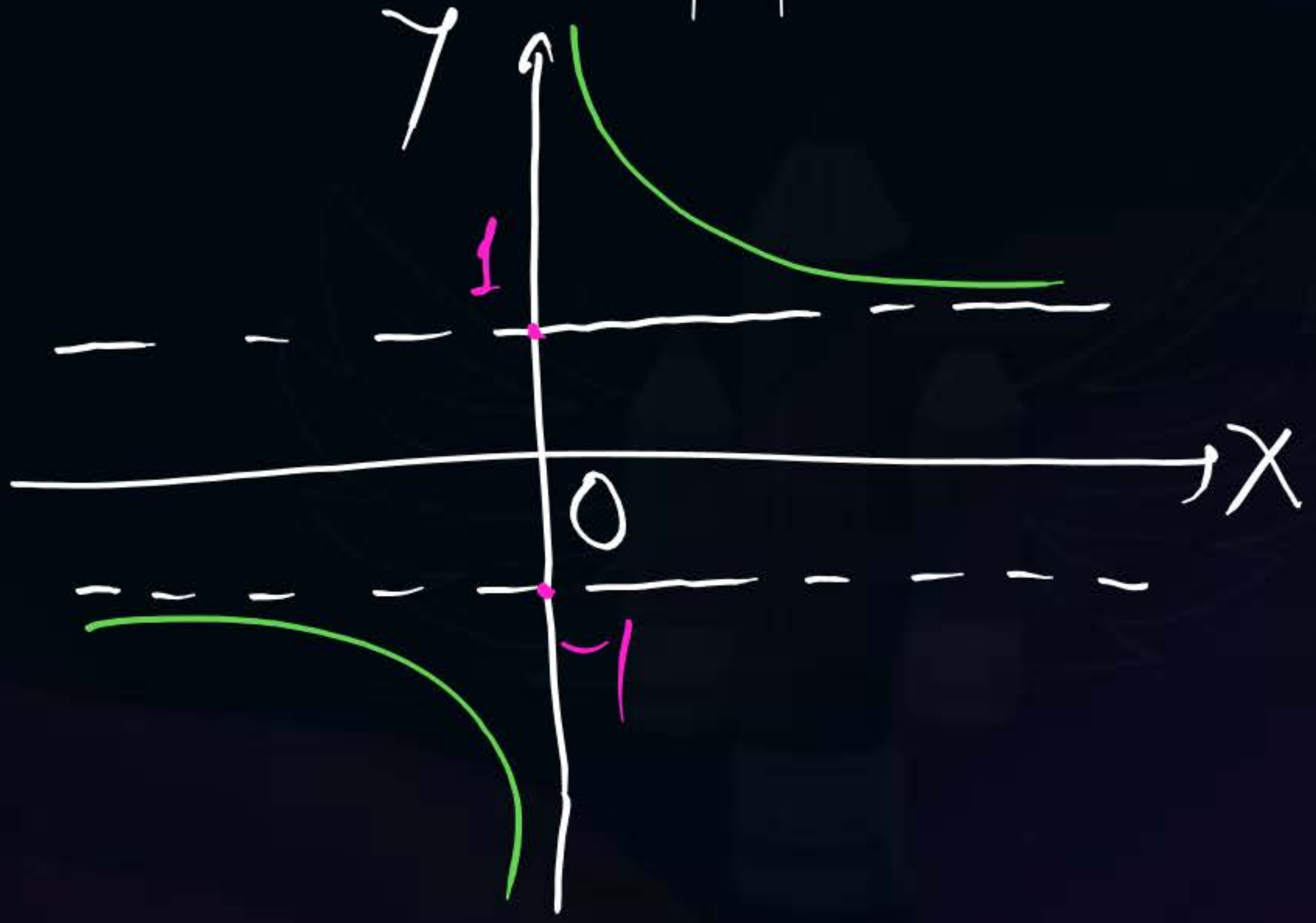


Note:

① $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$



② $\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^{x} + e^{-x}}{e^{x} - e^{-x}}$



Q. Expand $\tan x$ in the nbd of $x=0$

Sol: $f(x)=\tan x$, $f'(x)=\sec^2 x$, $f''(x)=2\sec x(\sec x\tan x)=2\sec^2 x\tan x$

$f(0)=0$, $f'(0)=1$, $f''(0)=0$

$$f'''(x)=2\left[2\sec^2 x\tan x\right]\tan x+2\sec^2 x(\sec^2 x)$$
$$=4\sec^2 x\tan^2 x+2\sec^4 x \implies f'''(0)=0+2=2$$

$f'''(0)=0+2=2$

Mac. series is given as, $f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\ldots\ldots$

$$\tan x=(0)+x(1)+\frac{x^2}{2!}(0)+\frac{x^3}{3!}(2)+\ldots$$

$$\tan x=x+\frac{x^3}{3}+\frac{2}{15}x^5+\ldots\ldots$$

Q. for $x <<< 1$, $\coth x$ can be approximated as ?

(a) x

(b) $1/x$

(c) $1/x^2$

(d) e^x

$\Downarrow$

x is very very very less than 1

so we can take x is very small $\Rightarrow$ x lies in the Nbd of 0

or $x \to 0$

& Mac. Series expansion of $\coth x$ is given as,

$$f(x) = \coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots}{x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots}$$

$$= \frac{1 + \text{Neglect}}{x + \text{Neglect}} \simeq \frac{1}{x}$$

Q. The Taylor Series Expansion of $(3\sin x + 2\cos x)$ is ?

PYQ

(a) $2+3x-x^2-\frac{x^3}{2}+\ldots\ldots$ ✓

(b) $2-3x+x^2-\frac{x^3}{2}+\ldots\ldots$ ✗

(c) $2+3x+x^2+\frac{x^3}{2}+\ldots\ldots$ ✗

(d) $2-3x-x^2+\frac{x^3}{2}+\ldots\ldots$ ✗

we will expansion $f(x)$ in the Nbd of $x=0$

i.e. $f(x) = 3\sin x + 2\cos x$

$$= 3\left[x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots\ldots\right]$$

$$+2\left[1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots\ldots\right]$$

$$f(x) = 2+3x-x^2-\frac{x^3}{2}+\ldots\ldots \quad \text{(a)}$$

Q: Consider the Differential Equation $\boxed{\frac{dy}{dx} = x^2y - 1}$, $y(0)=1$ then value of y at $x = 0.1$ is

(a) 1.0000

(b) 0.90000

(c) 0.90033 ✓

(d) 0.8133

$\because$ 0.1 lies in the nbd of '0' so we will follow Mac. Series expansion for $\boxed{y = f(x)}$ in the nbd of $x = 0$

$y(0) = f(0) = 1$, $f'(0) = \left(x^2y - 1\right)_{x=0} = 0\times1 - 1 = -1$

$f''(x) = \frac{d^2y}{dx^2} = x^2\frac{dy}{dx} + y\cdot(2x) = x^2\left(x^2y - 1\right) + 2xy$

$\boxed{f''(x) = x^4y - x^2 + 2xy} \Longrightarrow f''(0) = 0$

$f'''(x) = x^4\frac{dy}{dx} + y(4x^3) - 2x + 2x\frac{dy}{dx} + y(1)$

$f'''(x) = x^4(x^2y - 1) + 4x^3y - 2x + 2x(x^2y - 1) + y \Longrightarrow f'''(0) = 1$

Now the Maclaurin Series Exp of $f(x)$ in the Nbd of $x=0$ is as follows.

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\text{-------}$$

$$f(x)=(1)+x(-1)+\frac{x^2}{2!}(0)+\frac{x^3}{3!}(1)+\text{-----}$$

$$f(x)=1-x+\frac{x^3}{3!}+\text{-------}$$

$$\text{So } f(0.1)=\underbrace{1-(0.1)+\frac{(0.1)^3}{3!}}_{\approx 0.90033}+\text{Neglect}$$

(C)

Note: x lies in the Nbd of 'a' $\Longrightarrow x \in (a-h, a+h)$

OR

$a-h < x < a+h$ where $h \to 0$

OR

$|x-a| < h$

OR

$x \to a$

OR

x is about a

Note: $|x-a| < h$

$\pm(x-a) < h$

$-(x-a) < h$ & $+(x-a) < h$

$x-a > -h$ & $x < h+a$

$x > -h+a$ & $x < a+h$

$$\underbrace{x > a-h \quad \& \quad x < a+h}$$

$\Rightarrow \quad a-h < x < a+h$

x is at 'a'

$\Rightarrow$ we are considering Exact value

Taylor Series: If we want to Expand our function f(x) in the Nbd of 'a' then expansion is called T.S. Expansion & it is defined as follows;

$$f(x) = \underbrace{f(a) + (x-a)f'(a)}_{\text{Linear Approximation}} + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots$$

Note ① This series will provide Approx value of f(x) in the Nbd of a

② Coeff of $(x-a)^n = \dfrac{f^n(a)}{n!}$

③ Here we are getting Increasing powers of $(x-a)$

④ To find Approx Value of f(x) in LINEAR Form, Neglect 2nd & higher degree term

⑤ " " " " " " " Quadratic form, Neglect 3rd " " " "

Linear Interpolation → it is nothing but the L. Approximation of T-Series

$$f(x)=\underbrace{f(a)+(x-a)f'(a)}+\frac{(x-a)^2}{2!}f''(a)+\cdots$$

for L. App^n: $\boxed{f(x)=f(a)+(x-a)f'(a)}$ + Neglect

$$y=y_1+(x-x_1)\left(\frac{dy}{dx}\right)_{(x_1,y_1)}$$

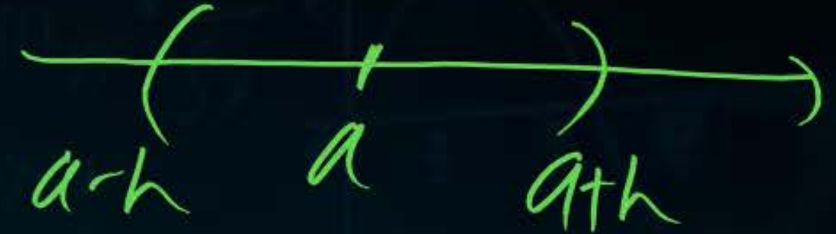
$\simeq (a, f(a))$

$$\boxed{y=y_1+(x-x_1)\left(\frac{y_2-y_1}{x_2-x_1}\right)}$$

equ^n of straight line passing through (x_1,y_1) & (x_2,y_2)

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$$

Various forms of Taylor Series →



$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\cdots$$

Here $x=a+h$ so we have

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\frac{h^3}{3!}f'''(a)+\cdots$$ (Another form)

$\because$ a is any Random point in the Domain of $f(x)$ so we can write

$a \approx x$

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\cdots$$

(Another form)

Q. Expand $\log_e x$ in the Nbd of 1 & Hence Evaluate $\log_e(1.1) = ?$

Sol: $f(x) = \log_e x$, $f'(x) = \frac{1}{x}$, $f''(x) = \frac{-1}{x^2}$, $f'''(x) = \frac{2}{x^3}$ ....

$f(1) = 0$, $f'(1) = 1$, $f''(1) = -1$, $f'''(1) = 2$ ......

So T.S. Exp of $f(x)$ in the Nbd of $x = 1$ is given as,

$$f(x) = f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2!}f''(1) + \frac{(x-1)^3}{3!}f'''(1) + \cdots$$

$$\log x = (0) + (x-1)(1) + \frac{(x-1)^2}{2!}(-1) + \frac{(x-1)^3}{3!}(2) + \cdots$$

$$\log_e x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \cdots \quad \text{Ans}$$

(ii) $\log_e x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \ldots\ldots$

Put $x = 1.1$ in above expression

$$\log_e(1.1) = (1.1-1) - \frac{(1.1-1)^2}{2} + \frac{(1.1-1)^3}{3} - \ldots\ldots$$

$$= (0.1) - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} + \text{Neglect}$$

$$\simeq 0.0953$$

Shortcut: $\log x = \log(1+(x-1)) = \log(1+X) = X - \frac{X^2}{2} + \frac{X^3}{3} - \frac{X^4}{4} + \ldots\ldots$

$x \to 1$

$(x-1) \to 0$

$X \to 0$

$$\log_e x = (x-1) + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} \ldots\ldots$$

Q. In the power Series Expansion of $f(x) = \frac{x-1}{x+1}$ about $x=1$, 3^{rd} term will be ?

(a) $(x-1)^2/2$

(b) $(x-1)^2/4$

(c) $(x-1)^3/8$

(d) $(x-1)^3/4$

Q. if $\boxed{f(x) = x^3 + 8x^2 + 15x - 24}$ then $f\left(\frac{11}{10}\right) = ?$ using T.S. Exp Method.

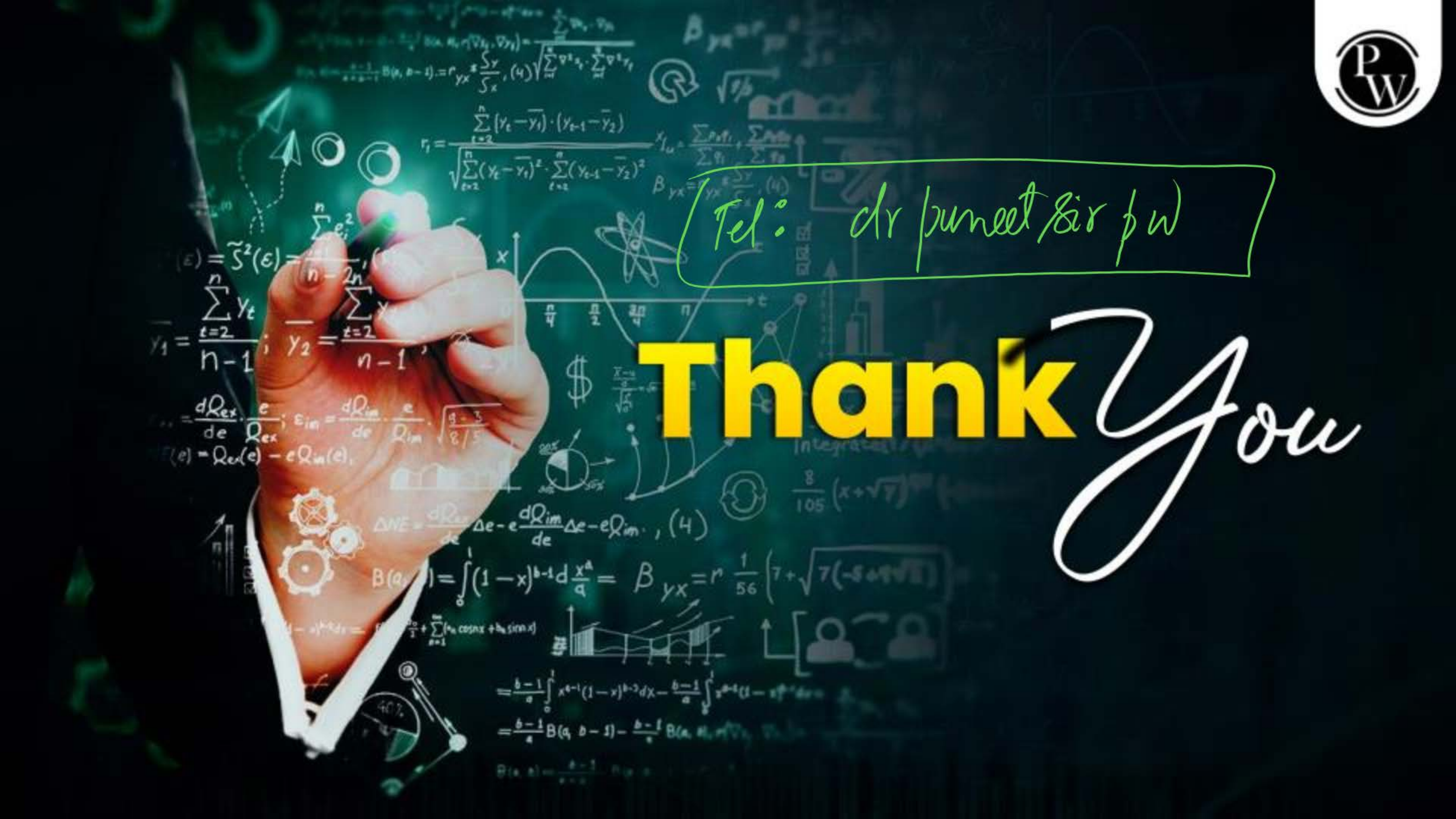
(a) 0

(b) 3.5111

(c) 5.312

(d) 2.179

Tel: dr puneet sir pw



DATA SCIENCE
&
ARTIFICIAL INTELLIGENCE
& CS/IT

Calculus and Optimization

Lecture No. 06

By- Dr. Puneet Sharma Sir

Recap of previous lecture

Topic

TAYLOR & MACLAURIN SERIES

Topics to be Covered

Topic

MEAN VALUE THEOREMS

1. Lagrange's M.V.Th. of Differentials
2. Rolle's M.V.Th.
3. Cauchy's M.V.Th.
4. L-M.V.Th for Integrals.

Q. Find points of Discontinuity and Non Diff points of $f(x)=\frac{x-|x-1|}{x}$

HW Q. [Ans: Discont at $x=0$ only & Non Diff at $x=0$ & 1]

Sol: $f(x)=\frac{x-|x-1|}{x}$, $D_f=R-\{0\}$ ⇒ At $x=0$ $f(x)$ is Discont & Non Diff

Now $f(x)=\begin{cases}\frac{x-(1-x)}{x}, & x<1\\ 1, & x=1\\ \frac{x-(x-1)}{x}, & x>1\end{cases} = \begin{cases}2-\frac{1}{x}, & x<1\\ 1, & x=1\\ 1/x, & x>1\end{cases} \Rightarrow f'(x)=\begin{cases}1/x^2, & x<1\\ -1/x^2, & x>1\end{cases}$

LHL $=1$, RHL $=1$, $f(1)=1$

Hence Cont at $x=1$

LHD $=1$, RHD $=-1$

So Not Diff at $x=1$

Q. In the power series Expansion of $f(x) = \frac{x-1}{x+1}$ about $x=1$, 3^{rd} term will be?

(a) $(x-1)^2/2$

(b) $(x-1)^2/4$

(c) $(x-1)^3/8$ ✓

(d) $(x-1)^3/4$

(M-I) using Conventional Approach of Taylor Series in the Nbd of 1. But it is not feasible during exam time bcoz calculation of Derivatives at $x=1$ will become tedious

(M-II) Put $x-1=t$ or $x=t+1$ then

$$f(x) = \frac{x-1}{x+1} = \frac{t}{t+1+1} = \frac{t+2-2}{t+2} = 1 - \frac{2}{t+2} = 1 - \frac{1}{\left(\frac{t}{2}+1\right)}$$

$$= 1 - \left(1+\frac{t}{2}\right)^{-1} = 1 - \left\{1 - \left(\frac{t}{2}\right) + \left(\frac{t}{2}\right)^2 - \left(\frac{t}{2}\right)^3 + \cdots\right\}$$

$$= \frac{t}{2} - \frac{t^2}{4} + \frac{t^3}{8} - \cdots$$

$$f(x) = \frac{(x-1)}{2} - \frac{(x-1)^2}{4} + \frac{(x-1)^3}{8} - \cdots$$

$\because$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \cdots$$

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$$

Q. Power series Expansion of $f(x) = \frac{\sin x}{x-\pi}$ when $\underbrace{|x-\pi| < \epsilon}$ will be ?

$x \to \pi$ or x lies in the Nbd of π.

Sol (M-I) using fundamental formula of Taylor series $\longrightarrow$ Not very easy.

(M-II) Let $f(x) = \frac{g(x)}{x-\pi} = \frac{\sin x}{x-\pi}$ where $g(x) = \sin x$

Now will will try to find T.S. Exp of $g(x)$ in the Nbd of π.

$$g(x) = g(\pi) + (x-\pi) g'(\pi) + \frac{(x-\pi)^2}{2!} g''(\pi) + \frac{(x-\pi)^3}{3!} g'''(\pi) + \cdots$$

$$\sin x = 0 + (x-\pi)(\cos x)_{x=\pi} + \frac{(x-\pi)^2}{2!}(-\sin x)_{x=\pi} + \frac{(x-\pi)^3}{3!}(-\cos x)_{x=\pi} + \cdots$$

$$= 0 + (x-\pi)(-1) + 0 + \frac{(x-\pi)^3}{3!}(+1) + \cdots$$

$$f(x) = \frac{\sin x}{x-\pi} = -1 + \frac{(x-\pi)^2}{3!} + \cdots$$

(iii) $f(x) = \dfrac{\sin x}{x-\pi}$ about $x = \pi$, Put $x - \pi = t$ when $x \to \pi$, $t \to 0$

$$f(x) = \frac{\sin(\pi + t)}{t} = \frac{-\sin t}{t} = \frac{-\left[t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots\right]}{t}$$

$$f(x) = -1 + \frac{t^2}{3!} - \frac{t^4}{5!} + \frac{t^6}{7!} - \cdots$$

$$\frac{\sin x}{x-\pi} = -1 + \frac{(x-\pi)^2}{3!} - \frac{(x-\pi)^4}{5!} + \frac{(x-\pi)^6}{7!} - \cdots \quad \text{Ans.}$$

Q. if $\boxed{f(x) = x^3 + 8x^2 + 15x - 24}$ then $f\left(\frac{11}{10}\right) = ?$ using T.S. Exp Method.

(a) 0

(b) 3.5111 ✓

(c) 5.312

(d) 2.179

$\frac{11}{10} = 1 + \frac{1}{10}$ i.e $\frac{11}{10}$ lies in the Nbd of 1.

$x = a + h$ where $a = 1$ & $h \to 0$

i.e we will find T.S. Exp of $f(x)$ in the Nbd of $x = 1$

$f(1) = 0$, $f'(x) = 3x^2 + 16x + 15$, $f''(x) = 6x + 16$, $f'''(x) = 6$

$f'(1) = 34$, $f''(1) = 22$, $f'''(1) = 6$

$$f(x) = f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2!}f''(1) + \frac{(x-1)^3}{3!}f'''(1) + 0 + 0 + \cdots$$

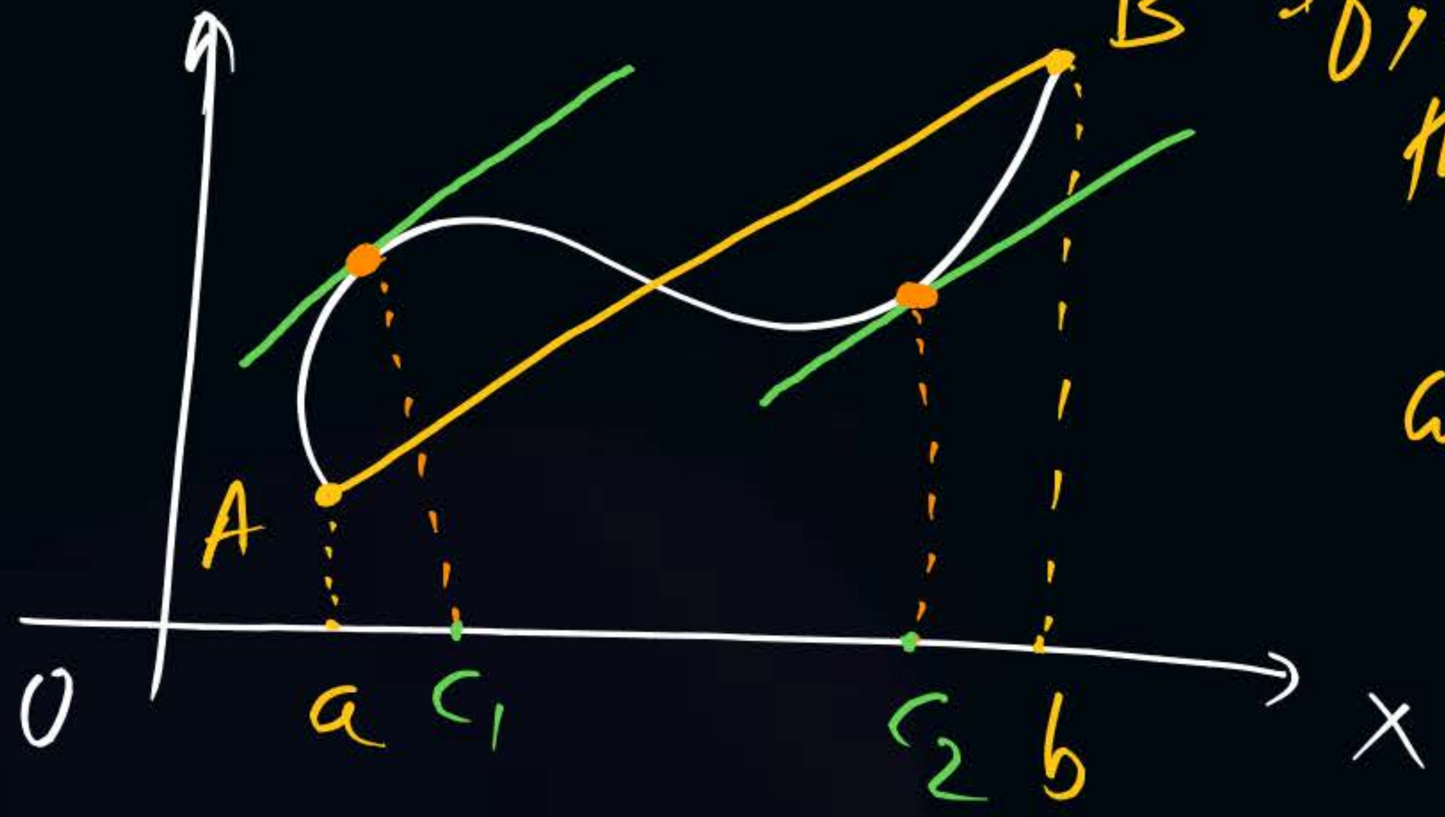
$$f(x) = 0 + (x-1)(34) + \frac{(x-1)^2}{2!}(22) + \frac{(x-1)^3}{3!}(6) + 0 + 0$$

So $f(1.1) = 0.1 \times 34 + \frac{(0.1)^2}{2}(22) + \frac{(0.1)^3}{6}(6) \simeq 3.511$

Mean Value Theorems (New chapter.)

Note:
1. Slope of line passing through (x_1, y_1) & (x_2, y_2) is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$
2. In general, Slope of tangent at any Random point is
$$m = \tan\theta = f'(x)$$
3. Slope of Horizontal line is $m = \boxed{f'(x) = \tan 0 = 0}$
4. Elementary func^n - All polynomial func^n, Exp func^n, log func^n, Trig. Function & Inverse Trig func^n are called E-Functions.
5. All E-func^n are Continuous as well as Differentiable in their Domain.

L.M.V.T $A(a, f(a)), B(b, f(b))$



If, $[a, b]$ Cont as well as Diff

then $\exists$ at least one point $c \in (a, b)$

where tangent is $||^r$ to chord AB

i.e slope of tangent = slope of chord AB

$$\boxed{f'(c) = \frac{f(b) - f(a)}{b - a}}$$

(1) Lagrange's M.V.Th — Let $f(x)$ is defined in $[a,b]$ s.t (i) $f(x)$ is Cont in $[a,b]$, (ii) $f(x)$ is Diff in (a,b)

then $\exists$ at least one c in b/w a & b for which

Tangent at c is $\parallel^r$ to chord AB

i.e Slope of Tangent = Slope of AB

$$\boxed{f'(c) = \frac{f(b)-f(a)}{b-a}}$$

$A(a, f(a))$, $B(b, f(b))$

Note:- Converse of LMVTh is not necessarily True.

Q. Verify L.M.V.Th for $f(x) = x^{1/3}$ in $[-1,1]$ & hence evaluate $c = ?$

Sol: $f'(x) = \frac{1}{3} x^{\frac{1}{3}-1} = \frac{1}{3x^{2/3}}$ i.e At $x = 0$, $f'(0) =$ DNE

So we can't say that $f(x)$ is differentiable throughout in $(-1,1)$

i.e 2^{nd} condition of L.M.V.T is not satisfied

Hence L.M.V.T is not applicable Ans.

(ii) Senseless Question

Analysis : (PODCAST) $\rightarrow f(x)=x^{1/3},\ [-1,1]$

$f(-1)=(-1)^{1/3}=-1$

$f(1)=(1)^{1/3}=1$

$$f'(x)=\frac{1}{3x^{2/3}}$$

By L.M.V.T, $\dfrac{f(1)-f(-1)}{1-(-1)}=f'(c)$

$$\frac{1-(-1)}{1-(-1)}=\frac{1}{3c^{2/3}}$$

$$1=\frac{1}{3c^{2/3}}$$

$$c^{2/3}=\frac{1}{3}\Rightarrow c=\left(\frac{1}{3}\right)^{3/2}=\sqrt{\frac{1}{27}}=\frac{\pm 1}{3\sqrt{3}}$$

i.e Both values of c lies in b/w −1 & 1

Still our func^n is not Diff.

i.e Converse of L.M.V.T is not Necessarily True.

S/p Note:

LMVTh.

if [f(x) is cont & Diff] $\Longrightarrow$ [$\exists\, c \in (a,b)$ s.t $\dfrac{f(b)-f(a)}{b-a} = f'(c)$]

$\not\Longleftarrow$

In previous PODCAST, it is possible to find c

But it does not imply that f(x) is cont & differentiable.

Q. Find the point on the curve $y = \sqrt{x-2}$ in $[2, 3]$ where tangent is $\parallel$ to the chord joining end points of the curve?

(a) $\left(\frac{10}{4}, \frac{1}{2}\right)$

(b) $\left(\frac{9}{4}, 1\right)$

(c) $\left(\frac{9}{4}, \frac{1}{2}\right)$ ✓

(d) $\left(\frac{5}{2}, \frac{5}{2}\right)$

$f(x)$ is cont in $[2, 3]$ ∵ we have no problem creating point.

$f'(x) = \frac{1}{2\sqrt{x-2}}$ = exist everywhere in $(2, 3)$

So $f(x)$ is Diff in $(2, 3)$ so we can apply L.M.V.Th.

$a = 2,\ f(2) = 0 \Rightarrow A(2, 0)\ \&\ B(3, 1)$

$b = 3,\ f(3) = 1$ & slope of $AB = \frac{1-0}{3-2} = 1$

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$\left(\frac{1}{2\sqrt{x-2}}\right)_{x=c} = \frac{f(3)-f(2)}{3-2}$$

$$\frac{1}{2\sqrt{c-2}} = \frac{1-0}{3-2}$$

$$\sqrt{c-2} = \frac{1}{2}$$

$$c-2 = \frac{1}{4}$$

$$c = \frac{9}{4} = 2.25$$

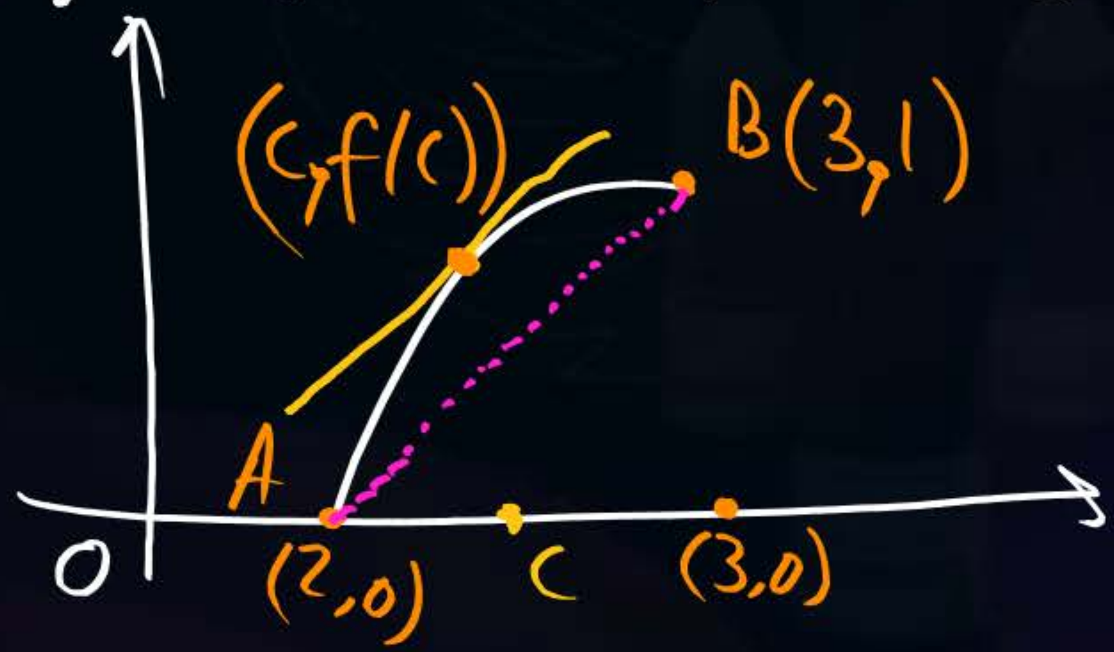
is c lies in b/w 2 & 3 ☺

$$f(c) = \sqrt{c-2}$$

$$f\left(\frac{9}{4}\right) = \sqrt{\frac{9}{4}-2} = \frac{1}{2}$$

So Req Point $= (c, f(c)) = \left(\frac{9}{4}, \frac{1}{2}\right)$

Analysis: $y = \sqrt{x-2}$, $[2, 3]$



Q. Consider the function $f(x)=\sqrt{x-2}$ is defined in $(2,3)$ then at least, at one point in this interval $\frac{dy}{dx}$ equal to ___?___ $= 1$.

By L.M.V.Th, we have proved that,

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

$$f'(c)=\frac{1-0}{3-2}$$

$$\left(\frac{dy}{dx}\right)_{x=c}=1$$

& c lies in the interval $(2,3)$

So our answer is 1

Q. for the function $f(x)=|x|$, Lagrange's Mean Value is not applicable in ?

(a) $1 \le x \le 3$

(b) $x<-1$ or $x>1$

(c) $0<x<1$

(d) $-2<x<2$ ✓

w.k. that at $x=0$, $f(x)=|x|$ is not Diff

i.e. 2^{nd} cond^n of L.M.V.Th is not satisfied at $x=0$

Q. For the function $f(x) = \sin\left(\frac{1}{x}\right)$ Lagrange's Mean Value is applicable in ?

× (a) $[-3, 3]$

(b) × $[-2, 5]$

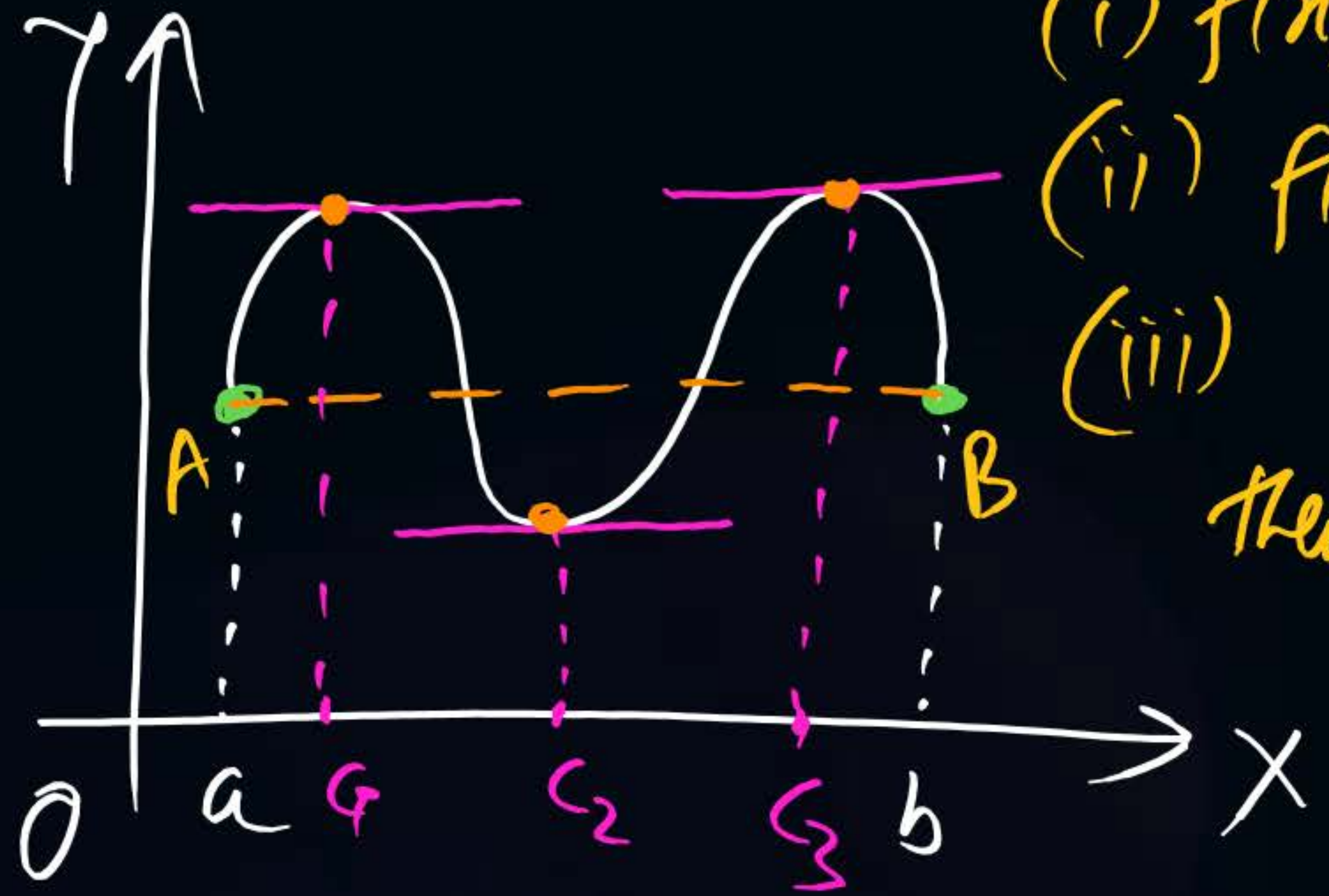
(c) ✓ $[2, 3]$

(d) × $[-1, 4]$

$\text{Dom} = \mathbb{R} - \{0\}$

i.e. At $x = 0$ $f(x)$ is Not Continuous.

(2) <u>Rolle's M.V.Th</u> → Let $f(x)$ is defined in $[a,b]$ s.t



(i) $f(x)$ is Continuous in $[a,b]$

(ii) $f(x)$ is Diff in (a,b)

(iii) $f(a)=f(b)$

then $\exists$ at least one point c in b/w a & b for which

$\boxed{\text{tangent is Horizontal}}$

or tangent is $\parallel^r$ to X axis

or $\boxed{f'(c)=0}$

Q. The ordinate of point on the curve $\boxed{f(x) = \cos x - 1}$; $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ where tangent is $\parallel^r$ to X axis?

(a) π

(b) -2

(c) 2π

(d) 0

KHELA HOO GAYA

$\because f(x)$ is an Elementary func^n so it is Cont as well as Diff

Now $f\left(\frac{\pi}{2}\right) = -1 = f\left(\frac{3\pi}{2}\right)$ i.e. 3rd condition also satisfied.

So we can use Rolle's Th,

$$f'(c) = 0$$

$$(-\sin x)_{x=c} = 0$$

$$\sin c = 0$$

$$\sin c = \sin n\pi$$

$c = n\pi, \; n \in I$

$c = \ldots\ldots -2\pi, -\pi, 0, \pi, 2\pi, 3\pi \ldots\ldots$

$\because$ only $c = \pi$ lies $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

So $f(c) = \cos c - 1 = \cos \pi - 1 = -1 - 1 = -2$

Q. If for the function $\boxed{f(x) = x^3 + bx^2 + ax}$, where $x \in [1, 3]$, Rolle's Theorem holds with $c = 2 + \frac{1}{\sqrt{3}}$ then $a =$ ______ & $b =$ ______ or $a + b = ?$

Sol: $\because f(x)$ is polynomial func^n (i.e an Elementary function) so cont and diff both.

& 3rd cond^n of R.Th is

$$f(a) = f(b)$$

$$f(1) = f(3)$$

$$1^3 + b(1)^2 + a(1) = (3)^3 + b(3)^2 + a(3)$$

$$1 + a + b = 27 + 9b + 3a$$

$$-8b - 2a = 26 \Rightarrow \boxed{a + 4b = -13}$$

By R.Th, $f'(c) = 0$

$$\left(3x^2 + 2bx + a\right)_{x=c} = 0$$

$$3c^2 + 2bc + a = 0$$

$$3\left(2 + \frac{1}{\sqrt{3}}\right)^2 + 2b\left(2 + \frac{1}{\sqrt{3}}\right) + a = 0$$

$$3\left(2+\frac{1}{\sqrt{3}}\right)^2+2b\left(2+\frac{1}{\sqrt{3}}\right)+a=0$$

$$3\left[4+\frac{1}{3}+\frac{4}{\sqrt{3}}\right]+4b+\frac{2b}{\sqrt{3}}+a=0$$

$$a+\left(\frac{4\sqrt{3}+2}{\sqrt{3}}\right)b=-\left(\frac{13\sqrt{3}+12}{\sqrt{3}}\right) \quad 2$$

$$a+4b=-13 \quad 1$$

- - - - - - - - -

- - - - - -

$a=?$, $b=?$

(3) Cauchy's M.V.Th. → Let $f(x)$ & $g(x)$ are two func^n defined in $[a,b]$ s.t

(i) Both $f(x)$ & $g(x)$ are continuous in $[a,b]$

(ii) " " " " differentiable in (a,b)

(iii) $g'(x) \neq 0 \ \forall x \in (a,b)$

Then $\exists$ at least one c in b/w a & b for which

$$\boxed{\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}}$$

(Ratio of Difference of $f(x)$ & $g(x)$ at a & b) = (Ratio of slope of tangent of $f(x)$ & $g(x)$ at c)

Qs if $f(x)=\log_e x$ & $g(x)=\log_e\left(\frac{1}{x}\right)$ defined in $[1,2]$ then By C.M.V.Th,

$c=$?

(a) 1.0

(b) 1.25

(c) 1.5

(d) Any Value b/w 1 & 2

$f(x)$ & $g(x)$ are Cont as well as Diff in $(1,2)$
$\because$ these are Elementary func^n.

$g(x)=\log\frac{1}{x}=-\log x \Rightarrow g'(x)=-\frac{1}{x}$

$\because g'(x)\neq 0 \ \forall x\in(1,2)$

So all the Conditions of C.M.V.Th are satisfied.

$$\frac{f(2)-f(1)}{g(2)-g(1)}=\frac{f'(c)}{g'(c)} \Rightarrow \frac{\ln 2-\ln 1}{-\ln 2+\ln 1}=\frac{1/c}{-1/c}$$

$\Rightarrow -1=-1$ i.e identity.

Q. If $f(x)=\frac{1}{x}$ & $g(x)=\frac{1}{x^2}$ are defined in $[4,6]$ then value of c using Cauchy's M.V.Th ?

(a) 4.5
(b) 5.2
(c) 4.8
(d) 5

At $x=0$, $f(x)$ & $g(x)$ are not Cont and Not Diff.
But $x=0$ is not in the given domain $\because 0 \notin [4,6]$
i.e Both $f(x)$ and $g(x)$ are Cont as well as Diff in $(4,6)$

Now $g'(x)=\frac{-2}{x^3}$, i.e $g'(x)\neq 0 \ \forall x \in (4,6)$

So all the Conditions of C.M.V.Th are Satisfied.

$a=4$, $f(x)=\frac{1}{x}$, $f(4)=1/4$, $f(6)=1/6$, $f'(c)=-1/c^2$

$b=6$, $g(x)=\frac{1}{x^2}$, $g(4)=\frac{1}{16}$, $g(6)=\frac{1}{36}$, $g'(c)=-2/c^3$

So By C.M.V.Th:

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

$$\frac{f(6)-f(4)}{g(6)-g(4)} = \frac{-1/c^2}{-2/c^3}$$

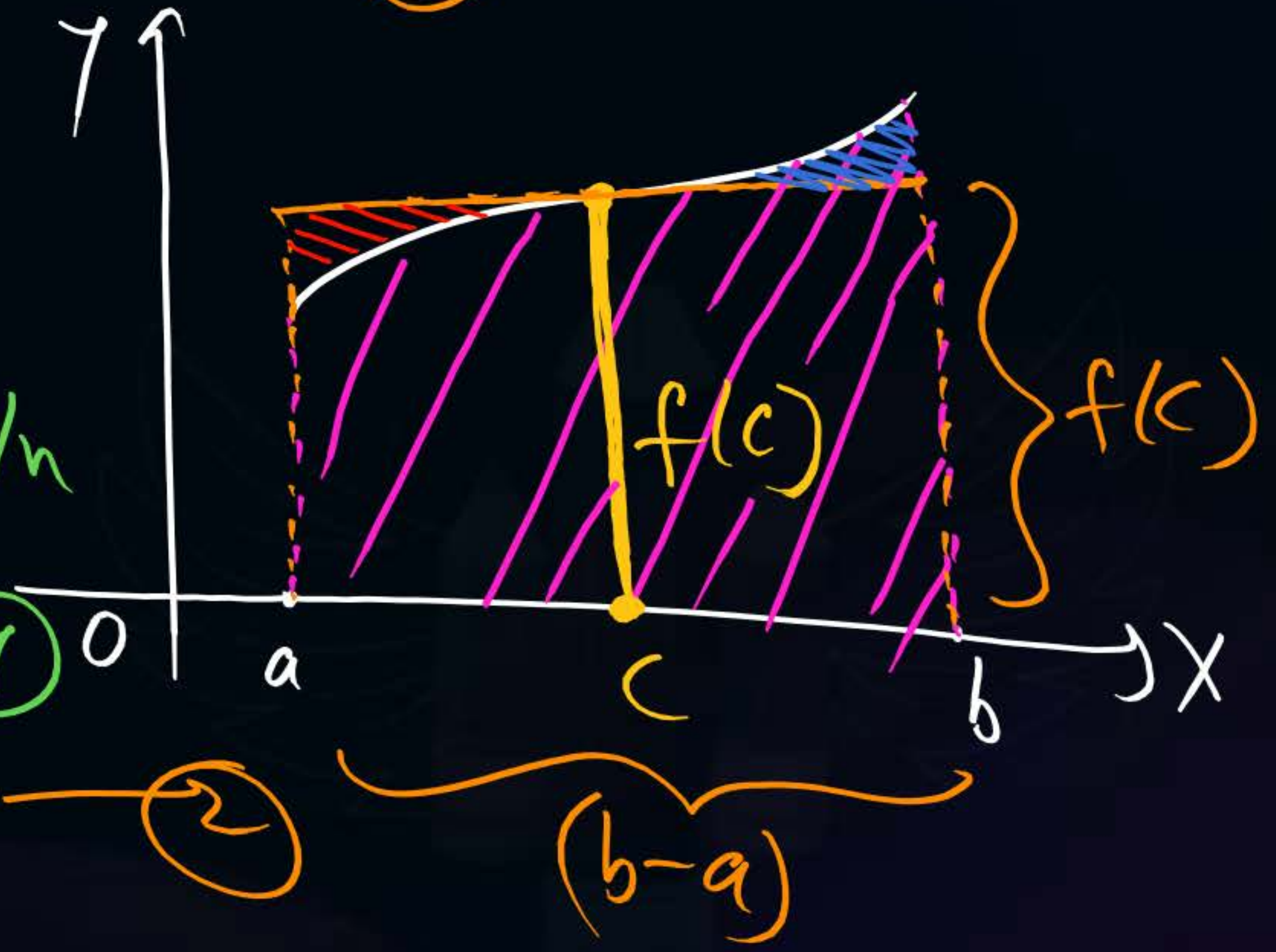
$$\frac{\frac{1}{6}-\frac{1}{4}}{\frac{1}{36}-\frac{1}{16}} = \frac{c}{2} \Rightarrow \frac{1}{\frac{1}{6}+\frac{1}{4}} = \frac{c}{2} \Rightarrow c = 4.8$$

Lagrange's Mean Value Theorem for Integrals →

Average height

if $f(x)$ is continuous func^n in $[a,b]$ then $\exists$ c in b/w a & b s.t

$$\boxed{f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx}$$



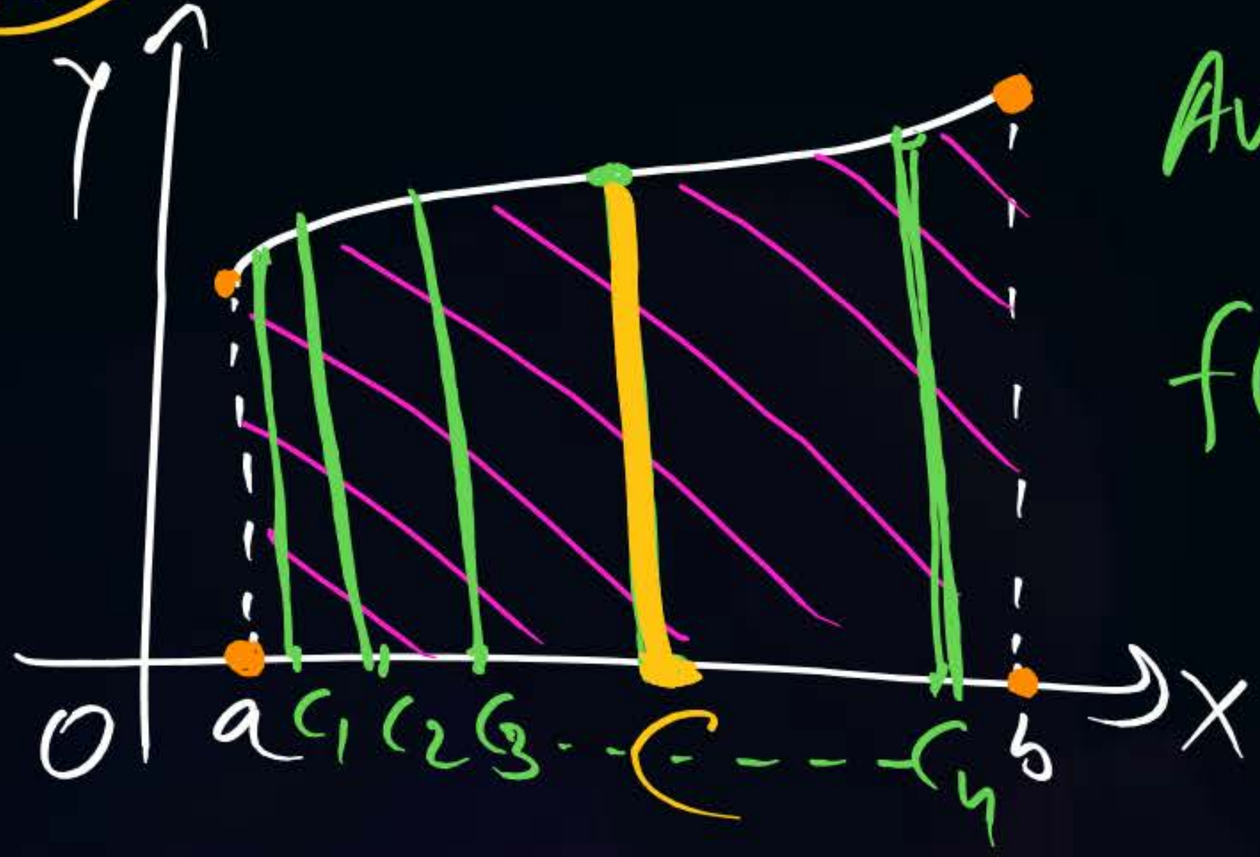
Proof: w.k.that $\int_a^b f(x)\,dx$ = Area under $f(x)$ b/w a & b and x axis —— (1)

Area of this Rectangle = length × height

$= (b-a) \times f(c)$ —— (2)

By ① & ②, $(b-a)\times f(c)=\int_a^b f(x)\,dx \Rightarrow \boxed{f(c)=\frac{1}{b-a}\int_a^b f(x)\,dx}$

M-II :→



Av. Height of $f(x)$ is given as,

$$f(c)=\frac{f(c_1)+f(c_2)+f(c_3)+\dots+f(c_n)}{n}$$

$$f(c)=\frac{\int_a^b f(x)\,dx}{\text{length of interval}}=\frac{\int_a^b f(x)\,dx}{b-a}$$

i.e. Av. Height of Curve $=\frac{1}{b-a}\int_a^b f(x)\,dx$

Note: ie Average Height of Curve b/n a & b is $= f(c)$

& it occurs at $x = c$ where $c \in (a,b)$

Verification: find Av. Height of $y = 2x$ b/n 1 & 6

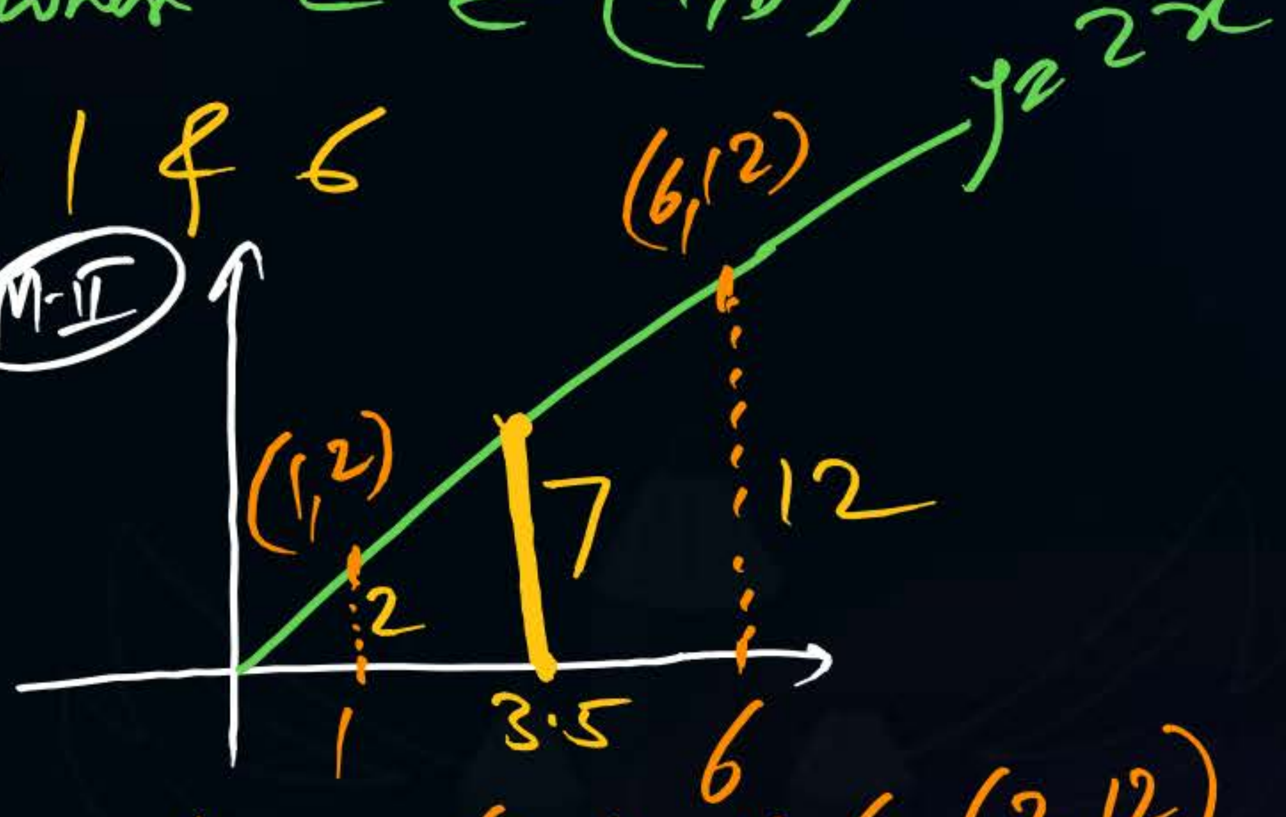
M-I Av Height $= \frac{1}{b-a}\int_a^b f(x)\,dx$

$$f(c) = \frac{1}{6-1}\int_1^6 (2x)\,dx = \frac{2}{5}\left(\frac{x^2}{2}\right)_1^6$$

$$f(c) = \frac{36-1}{5} = 7 \text{ Ans}$$

$$\Rightarrow 2c = 7 \Rightarrow c = 3.5 \text{ Ans}$$

M-II



ie when $x \in (1,6)$, $y \in (2,12)$

Hence Average $y = \frac{2+12}{2} = 7 = f(c)$

& it occurs at $c = 3.5$

eg $X = 3, 4, 5, 6, 7, 8, 9 \Rightarrow \bar{x} = ?$

M-I $\bar{x} = \frac{\sum x}{N} = \frac{3+4+5+6+7+8+9}{7} = \frac{42}{7} = 6$

M-II $\bar{X} = \frac{a+b}{2} = \frac{1^{st}\text{ Point} + \text{Last Point}}{2}$

$= \frac{3+9}{2} = 6$

Q: Find the Average Value of $f(x) = x^2$ b/n 1 & 4 ?

Sol: $f(c) = \frac{1}{4-1}\int_1^4 (x^2)\,dx = \frac{1}{3}\left(\frac{x^3}{3}\right)_1^4 = \frac{64-1}{9} = 7$

(ii) Also find the coordinates of that point ?

(a) $\pm\sqrt{7}$

(b) $(\sqrt{7}, 7)$

(c) $(-\sqrt{7}, 7)$

(d) $(2.5, 7)$

Sol: $\because f(c) = 7$

$c^2 = 7$

$c = \pm\sqrt{7}$

$\because c = -\sqrt{7} \notin (1, 4)$

So possible value of $c = \sqrt{7}$ ✓

So Req Point $= (c, f(c))$

$= (\sqrt{7}, 7)$

Q. The Mean Value of the function $f(x) = 5x^4 + 2$ b/w -1 & 2 is ?

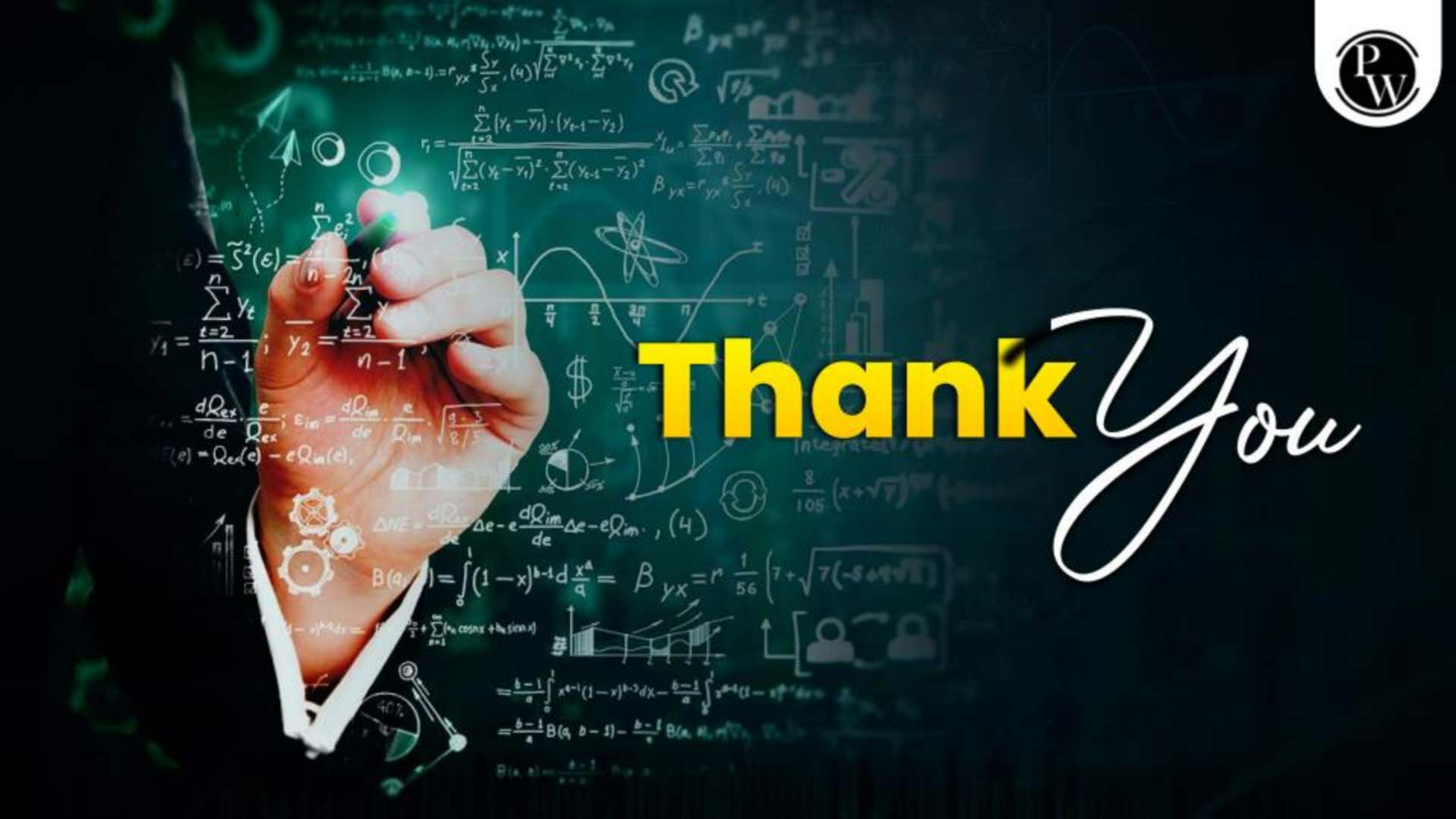
Sol: Av Height of $func^n = \frac{1}{b-a}\int_a^b f(x)\,dx$

$$f(c) = \frac{1}{2-(-1)}\int_{-1}^{2}(5x^4+2)\,dx$$

$$= \frac{1}{3}\left[x^5 + 2x\right]_{-1}^{2}$$

$$= \frac{1}{3}\left[(32+4) - (-1-2)\right]$$

$f(c) = 13$ Ans


Thank You

DATA SCIENCE & ARTIFICIAL INTELLIGENCE

& CS/IT

Calculus and Optimization

Lecture No. 07

By- Dr. Puneet Sharma Sir

Recap of previous lecture

Topic MEAN VALUE THEOREMS

Topics to be Covered

Topic

DERIVATIVES & their Types

(Part-1)

→ ordinary Derivative
→ Partial Derivative

→ Curve & Surface

$y=f(x)$ $z=f(x,y)$

eg $x^2+y^2+z^2=9$

→ Explicit func^n & Implicit func^n

$y=f(x)$ curve $f(x,y)=c$ curve

$z=f(x,y)$ surface $f(x,y,z)=c$ surface

Explicit func^n: If it is possible to separate Dependent and Independent Variables then func^n is called Explicit func^n for eg. $y=f(x)$ E. Curve

$z=f(x,y)$ E. Surface

eg $x^3+y^3+4xy^3=5$

$(1+4x)y^3=5-x^3 \implies y=\left(\frac{5-x^3}{1+4x}\right)^{1/3}$ ie $y=f(x)$

Implicit func^n → If it is not possible to separate Dep and Ind Variables then func^n is called Implicit func^n. eg $f(x,y)=c$, I Curve

$f(x,y,z)=c$, I. Surface

$x^3+y^3+3xy=1$ ie $f(x,y)=c$

Types of Questions

(1) Based on ordinary Derivative exist in case of curve $y = f(x)$

(2) " " Partial Derivative " " " of surface $z = f(x, y)$

(3) " " Total Derivative if $z = f(x, y)$, $x = x(t)$, $y = y(t)$
i.e $z \longrightarrow (x, y) \longrightarrow$ 't' alone

(4) " " Chain Rule of Partial Derivatives, if $z = f(x, y)$, $x = x(r, s)$, $y = y(r, s)$
i.e $z \longrightarrow (x, y) \longrightarrow (r, s)$

(5) " " Jacobian if $(u, v) \longrightarrow (x, y)$

(6) " " Euler Theorem: if $f(x, y)$ is Homogeneous func^n then we can use E. Th.

Ordinary Derivatives $\rightarrow$ exist in case of curve $y=f(x)$ $\rightarrow$

Power formula:$\rightarrow$ $\boxed{\frac{d}{dx}(x^a)=ax^{a-1}}$

$$\frac{d}{dx}(K)=K\frac{d}{dx}(x^0)=K\{0.x^{0-1}\}=0$$

$$\frac{d}{dx}(x)=\frac{d}{dx}(x^1)=1.x^{1-1}=1.x^0=1$$

$$\frac{d}{dx}(x^2)=2x^{2-1}=2x$$

$$\frac{d}{dx}(x^3)=3.x^{3-1}=3x^2$$

Similarly $\frac{d}{dx}(x^4)=4x^3$, $\frac{d}{dx}(x^5)=5x^4$!

$$\frac{d}{dx}\left(\frac{1}{x}\right)=\frac{d}{dx}(x^{-1})=-1.x^{-1-1}=\frac{-1}{x^2}$$

$$\frac{d}{dx}\left(\frac{1}{x^2}\right)=\frac{d}{dx}(x^{-2})=-2x^{-2-1}=\frac{-2}{x^3}$$

$$\frac{d}{dx}(\sqrt{x})=\frac{d}{dx}(x^{1/2})=\frac{1}{2}x^{\frac{1}{2}-1}=\frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right)=\frac{d}{dx}(x^{-1/2})=-\frac{1}{2}x^{-\frac{1}{2}-1}=-\frac{1}{2}x^{-3/2}$$

Similarly $\frac{d^n}{dx^n}(x^n)=n!$ i.e $\boxed{D^n(x^n)=n!}$

$\frac{d^{n+1}}{dx^{n+1}}(x^n)=0$ i.e $\boxed{D^{n+1}(x^n)=0}$

$$\frac{d}{dx}(x^3) = 3x^2$$

$$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

$$\frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right) = -\frac{1}{2}x^{-3/2}$$

② Exponential formula: $\boxed{\frac{d}{dx} a^x = a^x \log_e a}$ for eg $\frac{d}{dx}(e^x) = e^x$

Q. if $y = x^a + a^x + a^a + x^x$ then $\frac{dy}{dx} = ?$

Sol: $\frac{dy}{dx} = ax^{a-1} + a^x \log_e a + 0 + x^x(1+\log x)$

Q. $y = \log_e x + \log_a x + \log_x a + \log_x x + \log_a a$ then $\frac{dy}{dx} = ?$

Sol: $y = \log_e x + \frac{\log_e x}{\log_e a} + \frac{\log_e a}{\log_e x} + 1 + 1$

$$\frac{dy}{dx} = \frac{1}{x} + \frac{1}{\log_e a}\left(\frac{1}{x}\right) + \log_e a\left[\frac{-1}{x(\log x)^2}\right] + 0 + 0$$

③ $\boxed{\frac{d}{dx}(x^x) = x^x(1+\log_e x)}$

④ $\boxed{\frac{d}{dx}(\log_e x) = \frac{1}{x}}$

eg $\frac{d^2}{dx^2}(\log_e x) = \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$

eg $\frac{d}{dx}\left(\frac{1}{\log_e x}\right) = \frac{-1}{(\log x)^2}\left(\frac{1}{x}\right)$

Q. $y = \log_{10} 10 + \log_x x + \log_{10} x + \log_x 10$ then $\frac{dy}{dx} = ?$

$$y = 1 + 1 + \frac{\log_e x}{\log_e 10} + \frac{\log_e 10}{\log_e x}$$

$$\frac{dy}{dx} = 0 + 0 + \frac{1}{\log_e 10}\left[\frac{1}{x}\right] + \log_e 10\left[\frac{-1}{(\log x)^2}\cdot\frac{1}{x}\right]$$

2 Let $f(x) = e^{-|x|}$, where x is real. The value of $\frac{df}{dx} =$ at $x = -1$ is PYQ

(a) $-e$ (b) e

(c) $\frac{1}{e}$ (d) $-\frac{1}{e}$

$$f(x) = e^{-|x|} = \begin{cases} e^{x}, & x<0 \\ e^{-x}, & x>0 \end{cases}$$

$$f'(x) = \begin{cases} e^{x}, & x<0 \\ -e^{-x}, & x>0 \end{cases}$$

$$f'(-1) = e^{-1} = \frac{1}{e}$$

(5) Chain Rule : $\boxed{\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)}$

eg $\frac{d}{dx}\left(\sin\sqrt{\tan x^3}\right) = ?$

$= \cos\left(\sqrt{\tan x^3}\right) \cdot \frac{d}{dx}\left(\sqrt{\tan x^3}\right)$

$= \cos\left(\sqrt{\tan x^3}\right) \cdot \frac{1}{2\sqrt{\tan x^3}} \cdot \frac{d}{dx}\left(\tan x^3\right)$

$= \cos\left(\sqrt{\tan x^3}\right) \cdot \frac{1}{2\sqrt{\tan x^3}} \cdot \sec^2(x^3) \cdot \frac{d}{dx}(x^3)$

$= \cos\left(\sqrt{\tan x^3}\right) \cdot \frac{1}{2\sqrt{\tan x^3}} \cdot \sec^2(x^3) \cdot (3x^2)$

Note :

$\frac{d}{dx}(xy) = x\frac{d}{dx}(y) + y\frac{d}{dx}(x)$

$= x\frac{dy}{dx} + y(1)$

$\frac{d}{dy}(xy) = x\frac{d}{dy}(y) + y\frac{d}{dy}(x)$

$= x(1) + y\frac{dx}{dy}$

$d(xy) = x\,d(y) + y\,d(x)$

$= x\,dy + y\,dx$

eg $\frac{d}{dx}(y^3) = \frac{d}{dy}(y^3) \cdot \frac{dy}{dx}$

$= (3y^2)\, dy/dx$

Some More Standard Results :-

(6) $\frac{d}{dx}(\sin x) = \cos x$

(7) $\frac{d}{dx}(\cos x) = -\sin x$

(8) $\frac{d}{dx}(\tan x) = \sec^2 x$

(9) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

(10) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

(11) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(12) $\frac{d}{dx}(\sinh x) = \cosh x$ (13) $\frac{d}{dx}(\cosh x) = \sinh x$

(14) **Product formula :-**

$$\frac{d}{dx}(fg) = fg' + gf'$$

$$\frac{d}{dx}(fgh) = f'gh + fg'h + fgh'$$

(15) **Quotient formula :-**

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{gf' - fg'}{(g)^2}$$

or $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$

(16) Log Diff :→ Q: if $y = x^x$ then $\frac{dy}{dx} = ? = \ldots$ H.W $\ldots = x^x(1+\log x)$

Q: if $x^y = e^{x-y}$ then $\frac{dy}{dx} = ?$

Sol: $\log(x^y) = \log e^{x-y}$

$y \log x = (x-y)\log_e e$

$y \log x = x - y$

$y(1+\log x) = x$

$y = \frac{x}{1+\log x}$

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{x}{1+\log x}\right]$$

$$= \frac{(1+\log x)(1) - x\left(\frac{1}{x}\right)}{(1+\log x)^2}$$

$$\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$$

$x^{\sin y} = y^{\sin x}$, then $\dfrac{dy}{dx}$ is equal to

(a) $\dfrac{x^2 \cos x \log x - y \sin y}{x^2 \cos x \log x - x \sin x}$

(b) $\dfrac{y^2 \cos y \log y - x \sin x}{y^2 \cos y \log y - y \sin y}$

(c) $\dfrac{xy \cos x \cos y - y \sin y}{xy \cos x \cos y - x \sin x}$

(d) $\dfrac{xy \cos x \log y - y \sin y}{xy \log x \cos y - x \sin x}$

$$x^{\sin y} = y^{\sin x}$$

$$\sin y(\log x) = \sin x(\log y)$$

$$\frac{d}{dx}\left[\sin y(\log x)\right] = \frac{d}{dx}\left[\sin x(\log y)\right]$$

$$\sin y \frac{d}{dx}(\log x) + \log x \frac{d}{dx}(\sin y) = \sin x \frac{d}{dx}(\log y) + \log y \frac{d}{dx}(\sin x)$$

$$\sin y\left(\frac{1}{x}\right) + \log x\left[\cos y \frac{dy}{dx}\right] = \sin x\left(\frac{1}{y}\frac{dy}{dx}\right) + \log y(\cos x)$$

$$\left[\cos y \log x - \frac{\sin x}{y}\right]\frac{dy}{dx} = \cos x \log y - \frac{\sin y}{x}$$

$$\frac{dy}{dx} = \left[\frac{x \cos x \log y - \sin y}{y \cos y \log x - \sin x}\right] \cdot \frac{y}{x}$$

(d)

(17) Differentiation of Infinite Series →

Q. $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \ldots\ldots \infty}}}$

then $\frac{dy}{dx} = ?$ Ans: $\frac{\cos x}{2y-1}$

Sol: $y = \sqrt{\sin x + y}$

$$y^2 = \sin x + y$$

$$y^2 - y = \sin x$$

$$\frac{d}{dx}(y^2) - \frac{d}{dx}(y) = \frac{d}{dx}(\sin x)$$

$$2y\frac{dy}{dx} - \frac{dy}{dx} = \cos x$$

$$(2y-1)\frac{dy}{dx} = \cos x$$

$$\frac{dy}{dx} = \frac{\cos x}{2y-1} \quad \underline{\text{Ans}}$$

Q.

$$y = x^{x^{x^{x\cdots\infty}}} \text{ then } \frac{dy}{dx} = ? \quad \left(= \frac{y^2}{x(1-y\log x)}\right)$$

Sol:

$$y = x^y$$

$$\log y = \log(x^y)$$

$$\log y = y \log x$$

$$\frac{d}{dx}(\log y) = \frac{d}{dx}[y \cdot \log x]$$

$$\frac{1}{y}\left(\frac{dy}{dx}\right) = y\left(\frac{1}{x}\right) + \log x\left(\frac{dy}{dx}\right)$$

$$\left(\frac{1}{y} - \log x\right)\frac{dy}{dx} = \frac{y}{x}$$

$$\frac{(1-y\log x)}{y}\frac{dy}{dx} = \frac{y}{x}$$

$$\frac{dy}{dx} = \frac{y^2}{x(1-y\log x)}$$ //

(18) Differentiation of parametric func^n →

Curve	Parametric Coordinates	Equation
(1) Circle	$(a\cos\theta, a\sin\theta)$	$x^2+y^2=a^2$
(2) Parabola	$(at^2, 2at)$	$y^2=4ax$
(3) Ellipse	$(a\cos\theta, b\sin\theta)$	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$
(4) Hyperbola	$(a\sec\theta, b\tan\theta)$	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

(*) $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\cos 2\theta = 2\cos^2\theta - 1 \rightarrow \frac{1+\cos 2\theta}{2} = \cos^2\theta$

$\cos 2\theta = 1 - 2\sin^2\theta \rightarrow \frac{1-\cos 2\theta}{2} = \sin^2\theta$

(*) if $x=f(t)$, $y=g(t)$ then $\boxed{\frac{dy}{dx}=?=\frac{dy}{dt}\cdot\frac{dt}{dx}}=g'(t)\left(\frac{1}{f'(t)}\right)$

$\frac{dx}{dt}=f'(t)$, $\frac{dy}{dt}=g'(t)$

Q. if $x=at^2$, $y=2at$ then $\frac{dy}{dx}=?$

(a) t

(b) $\frac{1}{t}$ ✓

(c) t^2

(d) $-t$

$\frac{dx}{dt}=2at$, $\frac{dy}{dt}=2a$

$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}$

$=(2a)\left(\frac{1}{2at}\right)$

$=\frac{1}{t}$

Q. if $x=a(\theta-\sin\theta)$, $y=a(1-\cos\theta)$, $\frac{dy}{dx}=?$

(a) $\cot\frac{\theta}{2}$ ✓ (b) $\tan\frac{\theta}{2}$ (c) $\frac{\theta}{2}$ (d) $\cot\theta$

Sol: $\frac{dx}{d\theta}=a[1-\cos\theta]$, $\frac{dy}{d\theta}=a[0+\sin\theta]$

$\frac{dy}{dx}=\frac{dy}{d\theta}\cdot\frac{d\theta}{dx}=(a\sin\theta)\left[\frac{1}{a(1-\cos\theta)}\right]$

$=\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}=\cot\frac{\theta}{2}$

(19) Differentiation of func^n w.r. to another func^n →

if $u = f(x)$, $v = g(x)$ then $\frac{du}{dv} = ? = \frac{du/dx}{dv/dx} = \frac{f'(x)}{g'(x)}$

Q. Differentiate x^x w.r. to $x \log x$?

(a) x^x

(b) $x^x(1+\log x)$

(c) $1+\log x$

(d) $x \log x$

Let $u = x^x \Rightarrow \frac{du}{dx} = f'(x) = x^x(1+\log x)$

& $v = x \log x \Rightarrow \frac{dv}{dx} = g'(x) = x\left(\frac{1}{x}\right) + \log x(1) = 1 + \log x$

$$\frac{du}{dv} = \frac{du}{dx} \cdot \frac{dx}{dv} = x^x(1+\log x)\left[\frac{1}{1+\log x}\right] = x^x$$

M-II $\frac{du}{dv} = \frac{f'(x)}{g'(x)} = \frac{x^x(1+\log x)}{(1+\log x)} = x^x$

Q. Differentiate $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ w.r. to $\tan^{-1}x$? , $-1 < x < 1$

(a) $\tan\theta$ (b) θ (c) 2 (d) 2θ where $\theta = f(x)$

Note: $\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

$\frac{d}{dx}(\cot^{-1}x) = \frac{-1}{1+x^2}$

$\sin^{-1}x + \cos^{-1}x = \pi/2$

$\tan^{-1}x + \cot^{-1}x = \pi/2$

Sol: let $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$

let $u = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \sin^{-1}(\sin 2\theta) = 2\theta$

& let $v = \tan^{-1}x = \theta$

$$\frac{du}{dv} = \frac{du/d\theta}{dv/d\theta} = \frac{2}{1} = 2$$

PARTIAL DIFF. [for $z = f(x,y)$.]

$$\frac{\partial z}{\partial x} = \lim_{h\to 0}\left[\frac{f(x+h,y)-f(x,y)}{h}\right]_{y=\text{Const.}}, \quad \frac{\partial z}{\partial y} = \lim_{k\to 0}\left[\frac{f(x,y+k)-f(x,y)}{k}\right]_{x=\text{Const.}}$$

Note:-

In 2-D.

Equⁿ of X axis : $y = 0$

Equⁿ of line ∥ʳ to x axis : $y = K$

Equⁿ of Y axis : $x = 0$

Equⁿ of line ∥ʳ to y axis ; $x = h$

In 3D.

Equⁿ of XY plane ; $z = 0$

" " YZ plane ; $x = 0$

" ZX plane ; $y = 0$

Equⁿ of plane ∥ʳ to XZ plane $\boxed{y = K}$

" ∥ʳ to YZ plane ; $x = h$

Significance of $\frac{\partial z}{\partial x} \rightarrow$ $z=f(x,y)$

If we cut our surface by the plane $\parallel^r$ to xz plane (i.e. form $y=$ const) then we will get a curve of the type $z=f(x)$ & now

$\frac{\partial z}{\partial x}=$ slope of tangent at any Random point on this curve.

Similarly we can define $\frac{\partial z}{\partial y}=$?

$z=f(x)$ (curve) $\Rightarrow$ slope of tangent on this curve.

$z=f(y)$ (curve) $\Rightarrow$ slope of tangent on this curve.

Note: $z = f(x,y)$ then $z_x = \frac{\partial z}{\partial x} = f_x = \frac{\partial f}{\partial x}$ all are same

(1) & $z_{xx} = f_{xx} = \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 f}{\partial x^2}$

& $z_{xy} = f_{xy} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = ?$

(2) $$\boxed{\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}}$$

(3) All the Results that are applicable in case of ordinary derivatives are also valid in case of partial derivatives keeping other variable constant

(4) Don't assume Dependent Variable as Constant, if we are solving Questions Based on Partial Derivatives.

Q. if $r^2 = x^2+y^2+z^2$ then evaluate $\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial r}{\partial z}$

$$r = f(\underbrace{x, y, z}_{\text{Ind. Variables}})$$

r ↓ Dep. Variable

$$\frac{\partial}{\partial x}(r^2) = \frac{\partial}{\partial x}(x^2+y^2+z^2)$$

$$\frac{\partial}{\partial r}(r^2)\cdot\frac{\partial r}{\partial x} = 2x+0+0$$

$$(2r)\frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$ & $\frac{\partial r}{\partial z} = \frac{z}{r}$

Q. PYQ

if $\boxed{r = x^2 + y - z}$ & $\boxed{y^3 + z^3 + yz - xy = 1}$

where x & y are Independent Variables then at $(2,-1,1)$

evaluate $\frac{\partial r}{\partial x} = ?$

(a) 4.0

(b) 2.0

(c) 5.5

(d) 4.5 ✓

$r = f(x, y, z)$

i.e. $r = x^2 + y - z$

$$\frac{\partial r}{\partial x} = 2x + 0 - \frac{\partial z}{\partial x}$$

$$\frac{\partial r}{\partial x} = 2x - \left[\frac{y}{3z^2 + y}\right]$$

$$\left(\frac{\partial r}{\partial x}\right)_{(2,-1,1)} = 2(2) - \left[\frac{-1}{3(1)^2 - 1}\right] = 4.5$$

& $f(x, y, z) = c$ i.e. $z = f(x, y)$

or we can say that z is also Dependent Variable

$$\frac{\partial}{\partial x}(y^3) + \frac{\partial}{\partial x}(z^3) + \frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial x}(xy) = \frac{\partial}{\partial x}(1)$$

$$0 + 3z^2 \cdot \frac{\partial z}{\partial x} + y\left(\frac{\partial z}{\partial x}\right) - y(1) = 0$$

$$\frac{\partial z}{\partial x} = \frac{y}{3z^2 + y}$$

Q. if $u = \log_e(x^2+y^2)$ then $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = ?$

(a) 2

(b) 0

(c) -1

(d) $\frac{2(x+y)}{(x^2+y^2)}$

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[\log(x^2+y^2)\right] = \frac{1}{x^2+y^2}\frac{\partial}{\partial x}(x^2+y^2) = \frac{1}{x^2+y^2}(2x+0) = \frac{2x}{x^2+y^2}$$

$$u_{xx} = \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{2x}{x^2+y^2}\right)$$

$$= \left[\frac{(x^2+y^2).(2) - 2x(2x+0)}{(x^2+y^2)^2}\right] = \frac{2(y^2-x^2)}{(x^2+y^2)^2}$$

Similarly $u_{yy} = \frac{\partial^2 u}{\partial y^2} = \frac{2(x^2-y^2)}{(x^2+y^2)^2}$

So $u_{xx} + u_{yy} = \frac{2(y^2-x^2)}{(x^2+y^2)^2} + \left[\frac{-2(y^2-x^2)}{(x^2+y^2)^2}\right] = 0$

Q. if $u = e^{xyz}$ then evaluate $\frac{\partial^3 u}{\partial x \partial y \partial z} = ?$ At $(2,-1,0)$

Sol: $\frac{\partial u}{\partial z} = e^{xyz} \cdot \frac{\partial}{\partial z}(xyz) = xy e^{xyz}(1)$

$$\frac{\partial^2 u}{\partial y \partial z} = \frac{\partial}{\partial y}\left[\frac{\partial u}{\partial z}\right] = \frac{\partial}{\partial y}\left[\underset{\mathrm{I}}{xy}\, \underset{\mathrm{II}}{e^{xyz}}\right]$$

$$= x\left[y \frac{\partial}{\partial y}(e^{xyz}) + e^{xyz}\frac{\partial}{\partial y}(y)\right]$$

$$= x\left[y\, e^{xyz} \cdot \frac{\partial}{\partial y}(xyz) + e^{xyz}(1)\right]$$

$$= x\left[y \cdot e^{xyz} \cdot (xz) + e^{xyz}\right]$$

$$\frac{\partial^2 u}{\partial y \partial z} = e^{xyz}\left[x^2yz + x\right]$$

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = \frac{\partial}{\partial x}\left[\frac{\partial^2 u}{\partial y \partial z}\right]$$

$$= \frac{\partial}{\partial x}\left[\underset{\mathrm{I}}{e^{xyz}}\underset{\mathrm{II}}{(x^2yz + x)}\right]$$

$$= e^{xyz}\left[2xyz + 1\right] + (x^2yz + x)\left[e^{xyz} \cdot yz\right]$$

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = e^{xyz}\left[2xyz + 1 + x^2y^2z^2 + xyz\right]$$

At $(2,-1,0) \Rightarrow$ Ans $= 1\left[0+1+0+0\right] = 1$

Q. If $z = f(x-by) + \phi(x+by)$ then Evaluate $b^2 \frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = \underline{? = 0}$

(a) 0

(b) 1

(c) -1

(d) $(a^2-b^2)(f'' + \phi'')$

$$z_x = \frac{\partial z}{\partial x} = f'(x-by)(1-0) + \phi'(x+by)(1+0)$$

$$= f'(x-by) + \phi'(x+by)$$

$$z_{xx} = \frac{\partial^2 z}{\partial x^2} = f''(x-by)(1-0) + \phi''(x+by)(1+0)$$

$$= \boxed{f''(x-by) + \phi''(x+by)} \text{ --- } (1)$$

$$z_y = \frac{\partial z}{\partial y} = f'(x-by)(0-b) + \phi'(x+by)(0+b)$$

$$= -b f'(x-by) + b\phi'(x+by)$$

$$z_{yy} = \frac{\partial^2 z}{\partial y^2} = -b f''(x-by)(-b) + b\phi''(x+by)(b)$$

$$= \boxed{b^2[f''(x-by) + \phi''(x+by)]} \text{ --- } (2)$$

Analysis $z = f(x-by)$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} f(x-by) = \frac{\partial}{\partial f} f(x-by) \cdot \frac{\partial f}{\partial x} = f' \cdot (1-0)$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y} f(x-by) = \frac{\partial}{\partial f} f(x-by) \frac{\partial f}{\partial y} = f'(-b)$$

Let $f(x, y) = \dfrac{ax^2 + by^2}{xy}$, where a and b are constants. If $\boxed{\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial y}}$ at $x = 1$ and $y = 2$, then the relation between a and b is

(a) $a = \dfrac{b}{4}$ (b) $a = \dfrac{b}{2}$

(c) $a = 2b$ (d) $a = 4b$

$$f(x,y) = \frac{ax^2+by^2}{xy} = a\left(\frac{x}{y}\right) + b\left(\frac{y}{x}\right) \quad (1)$$

$$\frac{\partial f}{\partial x} = a\left[\frac{1}{y}\right] + b\left[\frac{-y}{x^2}\right]$$

$$\left(\frac{\partial f}{\partial x}\right)_{\substack{x=1\\y=2}} = \frac{a}{2} - 2b = \frac{a-4b}{2}$$

$$\frac{\partial f}{\partial y} = a\left(\frac{-x}{y^2}\right) + b\left(\frac{1}{x}\right)$$

$$\left(\frac{\partial f}{\partial y}\right)_{\substack{x=1\\y=2}} = -\frac{a}{4} + b = \frac{-a+4b}{4}$$

ATQ, $f_x = f_y$

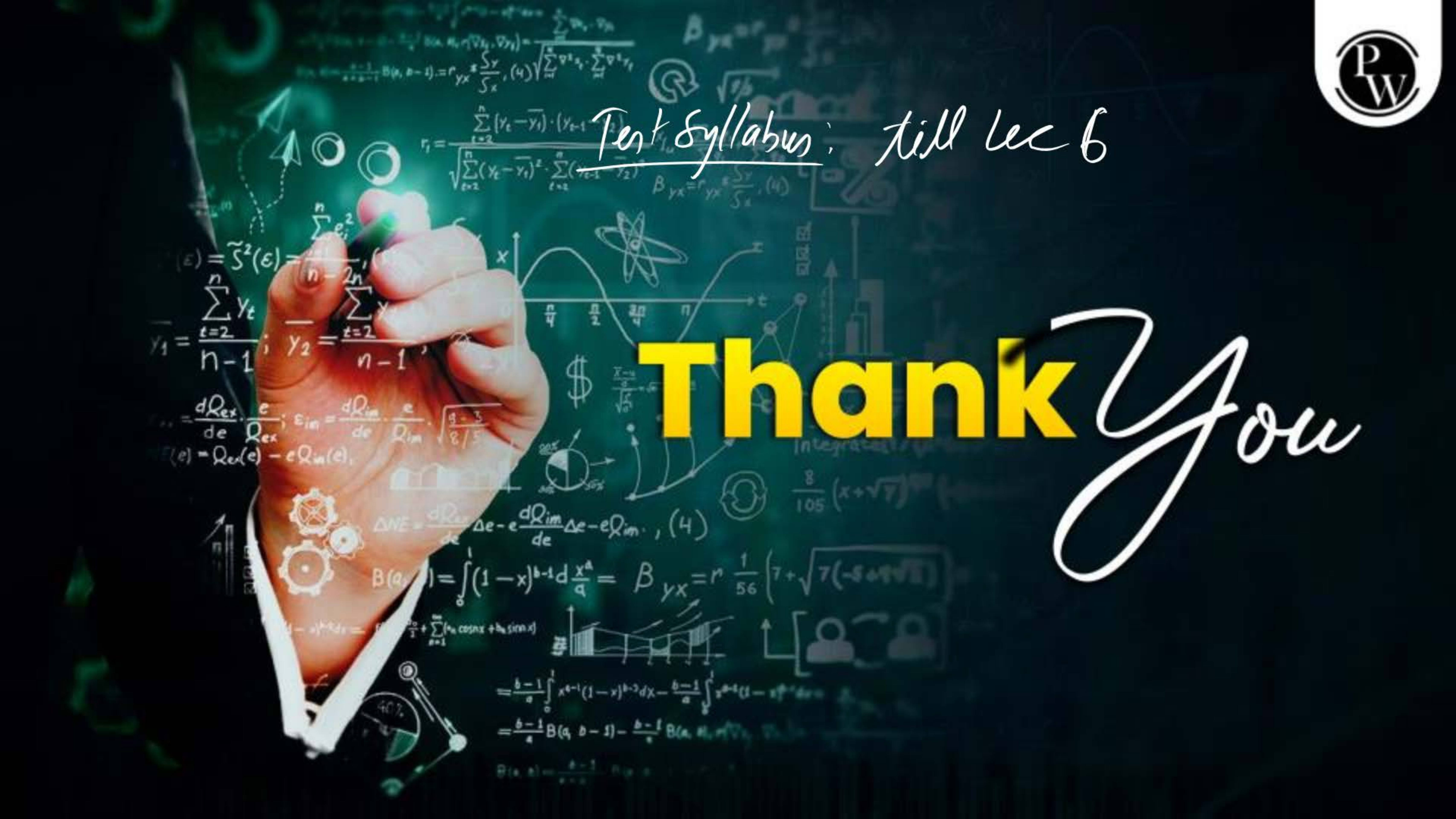
$$\frac{a-4b}{2} = \frac{-a+4b}{4}$$

$$2a - 8b = -a + 4b$$

$$3a = 12b$$

$$a = 4b$$

Test Syllabus: till lec 6



DATA SCIENCE
&
ARTIFICIAL INTELLIGENCE

& CS / IT

Calculus and Optimization

Lecture No. 08

By- Dr. Puneet Sharma Sir

Recap of previous lecture

Topic

DERIVATIVES & THEIR TYPES

(PART-1)

Topics to be Covered

Topic

DERIVATIVES & their Types

(Part 2)

Types of Questions

(1) Based on ordinary Derivative exist in case of curve $y = f(x)$

(2) " " Partial Derivative " " " of surface $z = f(x, y)$

(3) " " Total Derivative if $z = f(x, y)$, $x = x(t)$, $y = y(t)$
i.e $z \longrightarrow (x, y) \longrightarrow$ 't' alone

(4) " " Chain Rule of Partial Derivatives, if $z = f(x, y)$, $x = x(r, s)$, $y = y(r, s)$
i.e $z \longrightarrow (x, y) \longrightarrow (r, s)$

(5) " " Jacobian if $(u, v) \longrightarrow (x, y)$

(6) " " Euler Theorem: if $f(x, y)$ is Homogeneous func^n then we can use E. Th.

Homogeneous func^n :-

eg: $f(x,y) = x^5 + 2x^3y^2 + 4y^5 + xy^4$

$f(\lambda x, \lambda y) = (\lambda x)^5 + 2(\lambda x)^3(\lambda y)^2 + 4(\lambda y)^5 + (\lambda x)(\lambda y)^4$

$= \lambda^5\left[x^5 + 2x^3y^2 + 4y^5 + xy^4\right]$

$= \lambda^5 \cdot f(x,y)$

So $f(x,y)$ is H. func^n of degree 5.

eg $f(x,y) = \dfrac{x^4+y^4}{\sqrt{x}-\sqrt{y}}$

$f(\lambda x, \lambda y) = \dfrac{(\lambda x)^4 + (\lambda y)^4}{\sqrt{\lambda x} - \sqrt{\lambda y}} = \dfrac{\lambda^4}{\sqrt{\lambda}}\left[\dfrac{x^4+y^4}{\sqrt{x}-\sqrt{y}}\right]$

$f(\lambda x, \lambda y) = \lambda^{\frac{7}{2}} \cdot f(x,y)$ is H. func^n of degree 3.5

eg $f(x,y) = x^3 + xy^2 + \sin(x^3)$

$f(\lambda x, \lambda y) = \lambda^3x^3 + (\lambda x)(\lambda^2y^2) + \sin(\lambda^3x^3)$

$= \lambda^3\left[x^3 + xy^2 + \dfrac{1}{\lambda^3}\sin(\lambda^3x^3)\right]$

$\neq \lambda^n \cdot f(x,y)$ So Not Homog func^n!

Homogeneous func^n: Consider a func^n $\boxed{u = u(x, y)}$ then this func^n is called Homog. func^n of degree n, if each term is of same degree (n).

Trick: if $\boxed{u(\lambda x, \lambda y) = \lambda^n \cdot u(x, y)}$ then u is called Homog func^n of degree 'n'.

where $n \in R$.

eg $u = x^3 + 4xy^2 + 5x^2y$

H. func^n of degree $= 3$

eg $u = \dfrac{x^2 + y^2}{\sqrt{x} + \sqrt{y}}$

Homog func^n of degree $= 1.5$

Euler theorem for Homog. func^n →

if u is Homog. func^n of degree n then $\boxed{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n.u}$

Note: (1) $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u$

(2) Sp. Note: if $\boxed{Z=u+v}$ where u & v are Homog. func^n of degree n_1 & n_2
But Z is Non Homog. func^n then.

(i) $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=\boxed{n_1u+n_2v}$

(ii) $x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=\boxed{n_1(n_1-1)u+n_2(n_2-1)v}$

Proof ① $\boxed{z = u + v}$ —— ①

$\frac{\partial z}{\partial x} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}$ & $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n_1 u$ —— ②

$\frac{\partial z}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$ & $x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = n_2 v$ —— ③

Now, $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\right) + y\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right)$

$= \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) + \left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right)$

$= \boxed{n_1 u + n_2 v}$ Hence Proved

Proof ② Similarly we can show that,

$x^2 z_{xx} + 2xy z_{xy} + y^2 z_{yy} = n_1(n_1-1)u + n_2(n_2-1)v$

eg $u = x^5 + 4x^2y^3 + \log(x^5)$.

Non homog. func^n : $f(\lambda x, \lambda y) = \lambda^n f(x, y)$

$$f(\lambda x, \lambda y) = (\lambda x)^5 + 4(\lambda x)^2(\lambda y)^3 + \log(\lambda^5 x^5)$$

$$= \lambda^5 \underbrace{\left[x^5 + 4x^2y^3 + \frac{1}{\lambda^5}\log(\lambda^5 x^5)\right]}$$

$\neq \lambda^n f(x,y)$ so Non Homog.

Also evaluate, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = ??$

$\because$ u is Non Homog func^n so we can't apply E. Th.

Hence Do yourself by Conventional approach.

eg $u = \log x - \log y$

$$u(x,y) = \log\left(\frac{x}{y}\right)$$

$$u(\lambda x, \lambda y) = \log\left(\frac{\lambda x}{\lambda y}\right)$$

$$= \lambda^0 \log\left(\frac{x}{y}\right)$$

is Homog func^n of degree $n=0$

(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = ? = 0 \times u = 0$

(ii) $x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = ? = 0(0-1)u$

$= 0$

Q. if $z = \dfrac{x^{\frac14}+y^{\frac14}}{x^{\frac13}+y^{\frac13}}$ then

① $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y} = ? = nz = -\dfrac{1}{12}z$

② $x^2 z_{xx} + 2xy\, z_{xy} + y^2 z_{yy} = ?$

$\because z = z(x,y)$ so

$$z(\lambda x, \lambda y) = \frac{\lambda^{\frac14}}{\lambda^{\frac13}}\left[\frac{x^{\frac14}+y^{\frac14}}{x^{\frac13}+y^{\frac13}}\right]$$

$$= \lambda^{-\frac{1}{12}}\, z(x,y)$$

So Homog func^n of degree $n = -\dfrac{1}{12}$

$$= n(n-1)z$$

$$= -\frac{1}{12}\left(-\frac{1}{12}-1\right)z$$

$$= \frac{13}{144}z$$

Q. If $u = \log_e\left(\frac{x^2+y^2}{x+y}\right)$ then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = ?$ ~~(a) 1~~ (b) e^u (c) 0 (d) $\log u$

Sol: $\because u(\lambda x, \lambda y) \neq \lambda^n u(x,y)$ i.e. u is Non Homog func^n & we can't apply E. Theorem.

Let, $\boxed{e^u = v} = \frac{x^2+y^2}{x+y}$ & v is Homog func^n of degree $n=1$

Hence Applying E. Th for v,

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = n.v$$

$$x\frac{\partial}{\partial x}(e^u) + y\frac{\partial}{\partial y}(e^u) = 1\cdot e^u$$

$$x(e^u)\frac{\partial u}{\partial x} + y(e^u)\frac{\partial u}{\partial y} = e^u$$

i.e. $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{e^u}{e^u} = 1$ Ans

Q. if $u = \tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)$ then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = ?$

(a) $\sin 2u$

(b) $\tan u$

(c) $\cot u$

(d) $\cos 2u$

$\because u(\lambda x, \lambda y) \neq \lambda^n u(x, y)$, i.e. u is Non Homog so Can't Apply E. Th

Let $\boxed{\tan u = v} = \frac{x^3+y^3}{x+y}$, Here v is Homog func^n of $n = 2$

So $x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = n \cdot v$

$x\frac{\partial}{\partial x}(\tan u) + y\frac{\partial}{\partial y}(\tan u) = 2 \cdot \tan u$

$x(\sec^2 u)\frac{\partial u}{\partial x} + y(\sec^2 u)\frac{\partial u}{\partial y} = 2 \tan u$

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\tan u}{\sec^2 u} = \cdots\cdots = \sin 2u$

Q if $Z = x^2y^4\sin^{-1}\left(\frac{x}{y}\right)+\log x-\log y$ then $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y} = ?$

(a) $6Z$

(b) 0

(c) $6u$

(d) $6(u+v)$

$Z = \underbrace{x^2y^4\sin^{-1}\left(\frac{x}{y}\right)}_{u}+\underbrace{\log\left(\frac{x}{y}\right)}_{v}$, Z is Non Homog func^n but u & v are Homog func^n of degree $n_1=6$, $n_2=0$ resp.

ie $\boxed{Z = u+v}$ So Applying Standard Result,

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y} = n_1u+n_2v$$

$$= 6u+0.v = 6u \text{ Ans}$$

(ii) Also evaluate, $x^2Z_{xx}+2xyZ_{xy}+y^2Z_{yy} = ?$

$$= n_1(n_1-1)u+n_2(n_2-1)v$$

$$= 6(6-1)u+0(0-1)v$$

$$= 30u \text{ Ans}$$

Q. If $\boxed{u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)}$ then $x\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} = \underline{?} = nu = 0 \times u = 0$

Sol: $\because u(\lambda x, \lambda y) = \lambda^{0} \cdot u(x, y)$

So u is homog func^n of $n = 0$

Blunder

(ii) Also evaluate $x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = ? = 0(0-1)u = -u$?

$= 0$ Ans

Q. If $\boxed{u = x^3 y^2 \sin^{-1}\left(\frac{y}{x}\right)}$ then Evaluate?

(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = ?$ (ii) $x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = ?$

(a) u

(b) 5

(c) $5u$

(d) $20u$

$\because u(\lambda x, \lambda y) = \lambda^5 u(x, y)$ i.e. u is Homog func^n of $n = 5$

(i) $x u_x + y u_y = n u = 5u$

(ii) $x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = 5(5-1)u = 20u$

Derivative of an Implicit func^n :-

Consider $f(x,y) = C$ ⟶ (1) then $\boxed{\dfrac{dy}{dx} = -\dfrac{f_x}{f_y}}$

Proof: Diff both sides

$$df = d(C)$$

$$df = 0$$

Now using the S. Result of T.D in LHS

$$\left(\frac{\partial f}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y}\right)dy = 0 \Rightarrow \frac{dy}{dx} = -\left(\frac{\partial f/\partial x}{\partial f/\partial y}\right) = -\frac{f_x}{f_y}$$

Q. If $x^3+y^3+3xy=1$ then evaluate $\frac{dy}{dx}=?$

M-I

$$\frac{d}{dx}[x^3]+\frac{d}{dx}(y^3)+3\frac{d}{dx}(xy)=\frac{d}{dx}(1)$$

$$3x^2+\frac{d}{dy}(y^3)\frac{dy}{dx}+3\left[x\frac{dy}{dx}+y(1)\right]=0$$

$$3y^2\frac{dy}{dx}+3x\frac{dy}{dx}=-3x^2-3y$$

$$(y^2+x)\frac{dy}{dx}=-(x^2+y)$$

$$\frac{dy}{dx}=-\left(\frac{x^2+y}{y^2+x}\right)$$

M-II Let $f\equiv x^3+y^3+3xy-1=0$

$$\frac{\partial f}{\partial x}=3x^2+3y \quad \& \quad \frac{\partial f}{\partial y}=3y^2+3x$$

Now, $\boxed{\frac{dy}{dx}=-\left(\frac{\partial f/\partial x}{\partial f/\partial y}\right)}$

MAG

$$=-\left[\frac{3x^2+3y}{3y^2+3x}\right]=-\left(\frac{x^2+y}{y^2+x}\right)$$

Q. If $x^2+xy+y^2=3$ then $\frac{dy}{dx}=$??

(M-I) Let $\boxed{f \equiv x^2+xy+y^2-3}$

$$\frac{\partial f}{\partial x}=2x+y \quad \& \quad \frac{\partial f}{\partial y}=x+2y$$

$$\boxed{\frac{dy}{dx}=-\left(\frac{f_x}{f_y}\right)}=-\left(\frac{2x+y}{2y+x}\right)$$

(M-II) Using Conventional Approach

→ Do yourself.

Total Derivative →

→ if $u = u(x,y,z)$ then $\boxed{du = \left(\frac{\partial u}{\partial x}\right)dx + \left(\frac{\partial u}{\partial y}\right)dy + \left(\frac{\partial u}{\partial z}\right)dz}$

→ if $u = u(x,y,z)$, where $x = x(t)$, $y = y(t)$, $z = z(t)$

i.e. $u \longrightarrow (x,y,z) \longrightarrow t$ alone.

T.D is $du = \left(\frac{\partial u}{\partial x}\right)dx + \left(\frac{\partial u}{\partial y}\right)dy + \left(\frac{\partial u}{\partial z}\right)dz$

$$\frac{du}{dt} = \left(\frac{\partial u}{\partial x}\right)\frac{dx}{dt} + \left(\frac{\partial u}{\partial y}\right)\frac{dy}{dt} + \left(\frac{\partial u}{\partial z}\right)\frac{dz}{dt}$$

Total Derivative :→ if $w = f(x, y, z)$ where $x = x(t)$, $y = y(t)$, $z = z(t)$

i.e $w \longrightarrow (x, y, z) \longrightarrow t$ ' alone.

Standard Result is ;
$$\boxed{dw = \left(\frac{\partial w}{\partial x}\right) dx + \left(\frac{\partial w}{\partial y}\right) \cdot dy + \left(\frac{\partial w}{\partial z}\right) \cdot dz}$$
→ Learn by ❤

& Total Derivative of w with resp to t is given as.

$$\frac{dw}{dt} = \left(\frac{\partial w}{\partial x}\right) \frac{dx}{dt} + \left(\frac{\partial w}{\partial y}\right) \frac{dy}{dt} + \left(\frac{\partial w}{\partial z}\right) \frac{dz}{dt}$$

Note : if $w \longrightarrow (x, y, z) \longrightarrow x$ alone then $\dfrac{dw}{dx} = \left(\dfrac{\partial w}{\partial x}\right) + \left(\dfrac{\partial w}{\partial y}\right) \cdot \dfrac{dy}{dx} + \left(\dfrac{\partial w}{\partial z}\right) \cdot \dfrac{dz}{dx}$ //

Halka Question:

Q: $d(x.y) = ?$

(a) $dx\,dy$

(b) $dx + dy$

(c) $x\,dx + y\,dy$

(d) $x\,dy + y\,dx$ ✓

M-I using product formula:

$d(xy) = x\,d(y) + y\,d(x)$

$= x\,dy + y\,dx$

M-II using T.D concept,

let $u = xy$

$$\boxed{du = \left(\frac{\partial u}{\partial x}\right)dx + \left(\frac{\partial u}{\partial y}\right)dy}$$

$= (y)\,dx + (x)\,dy$

$=$ (d) Ans

Note:

$\frac{d}{dx}(xy) = x\frac{d}{dx}(y) + y\frac{d}{dx}(x)$

$= x\frac{dy}{dx} + y.(1)$

$\frac{d}{dy}(xy) = x\frac{d}{dy}(y) + y\frac{d}{dy}(x)$

$= x(1) + y\frac{dx}{dy}$

$d(xy) = x\,d(y) + y\,d(x)$

$= x\,dy + y\,dx$

Q: if $u = x^2 - y^2 + 2\sin yz$ then

$$\frac{\partial u}{\partial x} = ? = 2x - 0 + 0 = 2x$$

$$\frac{\partial u}{\partial y} = ? = 0 - 2y + 2\cos yz \frac{\partial}{\partial y}(yz)$$

$$= -2y + 2z\cos(yz)$$

$$\frac{\partial u}{\partial z} = ? = 0 - 0 + 2\cos(yz)\frac{\partial}{\partial z}(yz)$$

$$= 2y\cos(yz)$$

Q: if $y = e^x$ then

$$\frac{dy}{dx} = ? = e^x$$

Q: if $z = \ln x$ then

$$\frac{dz}{dx} = ? = \frac{1}{x}$$

Q. if $u = \underbrace{x^2 - y^2 + z\sin yz}$ where $y = e^x$, $z = \ln x$ then $\frac{du}{dx} = ?$

Explicit func^n

Explicit func^n.

Sol: $u \longrightarrow (x, y, z) \longrightarrow x$ alone.

ie Quest is Based on T.D Concept

So $\boxed{du = \left(\frac{\partial u}{\partial x}\right)dx + \left(\frac{\partial u}{\partial y}\right)dy + \left(\frac{\partial u}{\partial z}\right)dz}$

$$\frac{du}{dx} = \left(\frac{\partial u}{\partial x}\right) + \left(\frac{\partial u}{\partial y}\right)\frac{dy}{dx} + \left(\frac{\partial u}{\partial z}\right)\frac{dz}{dx}$$

$$= (2x) + (-2y + z^2\cos yz)(e^x) + (2yz\cos yz)\left(\frac{1}{x}\right)$$

Q: find the Derivative of x^2y w.r.to x where x & y are connected by the relation $x^2+xy+y^2=1$

Sol: Let $U=x^2y$ —— (1) & $\underbrace{x^2+xy+y^2=1}_{f(x,y)=c \text{ Implicit func}^n}$ —— (2)

T.D of u is as follows,

$$\boxed{du=\left(\frac{\partial u}{\partial x}\right)dx+\left(\frac{\partial u}{\partial y}\right)dy}$$

$$\Rightarrow \frac{du}{dx}=\left(\frac{\partial u}{\partial x}\right)+\left(\frac{\partial u}{\partial y}\right)\frac{dy}{dx}$$

$$=(2xy)+(x^2)\left\{-\left(\frac{2x+y}{x+2y}\right)\right\}$$

$$\Rightarrow \frac{dy}{dx}=-\frac{f_x}{f_y}=-\left(\frac{2x+y}{x+2y}\right)$$

Analysis:

(1) if $y=f(x)$ then $\frac{dy}{dx}=\boxed{\frac{d}{dx}f(x)}=f'(x)=$ ordinary Derivative

(2) if $z=f(x,y)$ $\to$

$\frac{\partial z}{\partial x}=$ P.D

$\frac{\partial z}{\partial y}=$ P.D

$\frac{dz}{dx}=\boxed{\frac{d}{dx}f(x,y)}=$ T.D.

Q $z=x^2y \Rightarrow \frac{\partial z}{\partial x}=2xy,\ \frac{\partial z}{\partial y}=x^2,\ \frac{dz}{dx}=\frac{d(x^2y)}{dx}=$ use T.D. Concept

& $dz=?=d(x^2y)=x^2dy+y\,d(x^2)=x^2dy+y(2x\,dx)$

Given, $z(x, y) = e^{x-2y}$, where $x(t) = e^t$ and $y(t) = e^{-t}$. All the variables are real. The total differential $\frac{dz}{dt}$ is ?

(a) $-z(x + 2y)$
(b) $-z(x - 2y)$
(c) $z(x + 2y)$
(d) $z(x - 2y)$

$z = e^{x-2y}$, $x = e^t$, $y = e^{-t}$

$\because z \longrightarrow (x, y) \longrightarrow t$ alone

i.e. Questn Based on T.D Concept.

$\frac{\partial z}{\partial x} = e^x \cdot e^{-2y}$ & $\frac{\partial z}{\partial y} = e^x \cdot e^{-2y}(-2)$

$$\boxed{dz = \left(\frac{\partial z}{\partial x}\right) dx + \left(\frac{\partial z}{\partial y}\right) dy}$$

$$\frac{dz}{dt} = \left(\frac{\partial z}{\partial x}\right)\frac{dx}{dt} + \left(\frac{\partial z}{\partial y}\right)\frac{dy}{dt}$$

$$= \left(e^{x-2y}\right)e^t + \left(-2e^{x-2y}\right)\left(-e^{-t}\right)$$

$$= x \cdot z + 2yz$$

$$= z(x + 2y)$$

Type IV: Chain Rule of Partial Derivative (change of Variable Concept) →

if $w = f(x, y, z)$ where $x = x(r, s, t)$, $y = y(r, s, t)$, $z = z(r, s, t)$

i.e. $w \longrightarrow (x, y, z) \longrightarrow (r, s, t)$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\left(\frac{\partial x}{\partial r}\right) + \frac{\partial w}{\partial y}\left(\frac{\partial y}{\partial r}\right) + \frac{\partial w}{\partial z}\left(\frac{\partial z}{\partial r}\right)$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\left(\frac{\partial x}{\partial s}\right) + \frac{\partial w}{\partial y}\left(\frac{\partial y}{\partial s}\right) + \frac{\partial w}{\partial z}\left(\frac{\partial z}{\partial s}\right)$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\left(\frac{\partial x}{\partial t}\right) + \frac{\partial w}{\partial y}\left(\frac{\partial y}{\partial t}\right) + \frac{\partial w}{\partial z}\left(\frac{\partial z}{\partial t}\right)$$

Shortcut:

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\quad}\,\frac{}{\partial r} + \frac{\partial w}{\quad}\,\frac{}{\partial r} + \frac{\partial w}{\quad}\,\frac{}{\partial r}$$

$$= \frac{\partial w}{\partial x}\left(\frac{\partial x}{\partial r}\right) + \frac{\partial w}{\partial y}\left(\frac{\partial y}{\partial r}\right) + \frac{\partial w}{\partial z}\left(\frac{\partial z}{\partial r}\right)$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\quad}\,\frac{}{\partial s} + \frac{\partial w}{\quad}\,\frac{}{\partial s} + \frac{\partial w}{\quad}\,\frac{}{\partial s}$$

$$= \frac{\partial w}{\partial x}\left(\frac{\partial x}{\partial s}\right) + \frac{\partial w}{\partial y}\left(\frac{\partial y}{\partial s}\right) + \frac{\partial w}{\partial z}\left(\frac{\partial z}{\partial s}\right)$$

Similarly, $\frac{\partial w}{\partial t} = ?$

Shortcut to learn above Result: -

$$u \longrightarrow (r, s, t) \longrightarrow (x, y)$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\quad}\frac{\quad}{\partial x} + \frac{\partial u}{\quad}\frac{\quad}{\partial x} + \frac{\partial u}{\quad}\frac{\quad}{\partial x} = \frac{\partial u}{\partial r}\left(\frac{\partial r}{\partial x}\right) + \frac{\partial u}{\partial s}\left(\frac{\partial s}{\partial x}\right) + \frac{\partial u}{\partial t}\left(\frac{\partial t}{\partial x}\right)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\quad}\frac{\quad}{\partial y} + \frac{\partial u}{\quad}\frac{\quad}{\partial y} + \frac{\partial u}{\quad}\frac{\quad}{\partial y} = \frac{\partial u}{\partial r}\left(\frac{\partial r}{\partial y}\right) + \frac{\partial u}{\partial s}\left(\frac{\partial s}{\partial y}\right) + \frac{\partial u}{\partial t}\left(\frac{\partial t}{\partial y}\right)$$

If $u = f(r, s)$ where $r = x + y$, $s = x - y$ then then $u_x + u_y =$

(a) $2u_r$ (b) $2u_s$

(c) $-2u_r$ (d) $-2u_s$

$u \longrightarrow (r, s) \longrightarrow (x, y)$

ie Quest is Based on Chain Rule Concept.

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\left(\frac{\partial r}{\partial x}\right) + \frac{\partial u}{\partial s}\left(\frac{\partial s}{\partial x}\right) = u_r(1) + u_s(1)$$

$$u_y = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\left(\frac{\partial r}{\partial y}\right) + \frac{\partial u}{\partial s}\left(\frac{\partial s}{\partial y}\right) = u_r(1) + u_s(-1)$$

$$u_x + u_y = 2u_r \quad (a)$$

Q. If $v = f(2x-3y, 3y-4z, 4z-2x)$ then $\boxed{6V_x + 4V_y + 3V_z = ?}$

(a) 1

(b) −2

(c) 13

(d) 0

Let $r = 2x-3y$, $s = 3y-4z$, $t = 4z-2x$

$\rightarrow V = f(r,s,t)$ where $(r,s,t) = g(x,y,z)$

i.e. $V \longrightarrow (r,s,t) \longrightarrow (x,y,z)$

$$V_x = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\left(\frac{\partial r}{\partial x}\right) + \frac{\partial v}{\partial s}\left(\frac{\partial s}{\partial x}\right) + \frac{\partial v}{\partial t}\left(\frac{\partial t}{\partial x}\right) = V_r(2) + V_s(0) + V_t(-2)$$

$$V_y = \frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\left(\frac{\partial r}{\partial y}\right) + \frac{\partial v}{\partial s}\left(\frac{\partial s}{\partial y}\right) + \frac{\partial v}{\partial t}\left(\frac{\partial t}{\partial y}\right) = V_r(-3) + V_s(3) + V_t(0)$$

$$V_z = \frac{\partial v}{\partial z} = \frac{\partial v}{\partial r}\left(\frac{\partial r}{\partial z}\right) + \frac{\partial v}{\partial s}\left(\frac{\partial s}{\partial z}\right) + \frac{\partial v}{\partial t}\left(\frac{\partial t}{\partial z}\right) = V_r(0) + V_s(-4) + V_t(4)$$

$$V_x = 2V_r - 2V_t$$

$$V_y = -3V_r + 3V_s$$

$$V_z = -4V_s + 4V_t$$

$$6V_x + 4V_y + 3V_z = (12V_r - 12V_t) + (-12V_r + 12V_s) + (-12V_s + 12V_t)$$

$$= 0$$

<u>Ans</u>

Q. if $V=V(x,y)$ where $x+y=2e^{\theta}\cos\phi$, $x-y=2ie^{\theta}\sin\phi$ then evaluate $\frac{\partial V}{\partial \theta}$ and $\frac{\partial V}{\partial \phi}=?$

HW

$$V \longrightarrow (x,y) \longrightarrow (\theta,\phi)$$

$\therefore$ Question is Based on chain Rule of Partial Derivative.

$$\frac{\partial V}{\partial \theta}=?$$

$$\frac{\partial V}{\partial \phi}=?$$

JACOBIAN: if $u \to (x,y)$ & $v \to v(x,y)$

i.e. $(u,v) \to (x,y)$ then

Derivative of (u,v) w.r.to (x,y) is called Jacobian

& it is defined as $$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

* if $(u,v,w) \to (x,y,z)$

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

* if $(u,v) \to (x,y)$ then $\frac{\partial(x,y)}{\partial(u,v)} = ? = J^{-1} = \frac{1}{J}$

w.k.that $\boxed{JJ^{-1} = 1}$

i.e J^{-1} = Reciprocal of J

JACOBIAN →

if $u = u(x,y,z)$, $v = v(x,y,z)$, $w = w(x,y,z)$

i.e. $(u,v,w) \to (x,y,z)$

then Derivative of (u,v,w) with respect to (x,y,z) is called Jacobian & it is defined as

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Note: ① if $(u,v) \to (x,y)$ then

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

② if $(x,y) \to (u,v)$ then $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = J'$

③ $JJ' = 1$

Q. If $u = 3x+5y$ & $v = 4x-3y$ then $\frac{\partial(u,v)}{\partial(x,y)} = ?$

(a) 29

(b) -29

(c) 11

(d) -11

$\because (u,v) \longrightarrow (x,y)$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 3 & 5 \\ 4 & -3 \end{vmatrix}$$

$$= (3)(-3)-(4)(5)$$

$$= -9-20 = -29$$

Q. if $x = u$, $y = u\tan v$, $z = w$ then Derivative of (u, v, w) w.r.to (x, y, z) will be ?

(a) $\frac{\cos^2 v}{u}$ ✓

(b) $u\cos^2 v$

(c) $u\sec^2 v$

(d) 0

Here $(x, y, z) \longrightarrow (u, v, w)$

then $J = \frac{\partial(x,y,z)}{\partial(u,v,w)}$, $J' = \frac{\partial(u,v,w)}{\partial(x,y,z)}$ & $\boxed{JJ' = 1}$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ \tan u & u\sec^2 v & 0 \\ 0 & 0 & 1 \end{vmatrix} = u\sec^2 v$$

So $J' = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{1}{J} = \frac{1}{u\sec^2 v} = \frac{\cos^2 v}{u}$

eg $\boxed{f(x) = x\sqrt{x}} = x^{3/2}$

$f'(x) = \frac{3}{2}x^{\frac{3}{2}-1} = \frac{3}{2}\sqrt{x}$

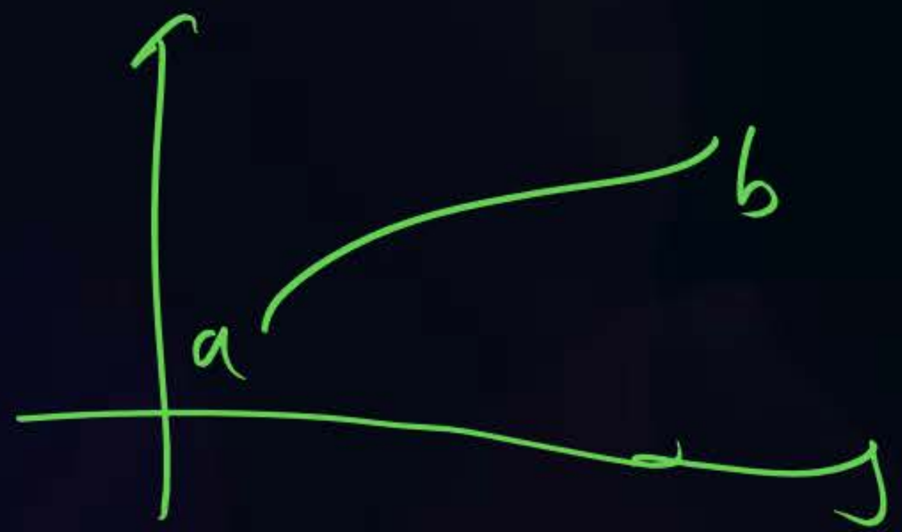
$f'(0) =$ exist.

eg $f(x) = \sqrt{x}$

$f'(x) = \frac{1}{2\sqrt{x}}$

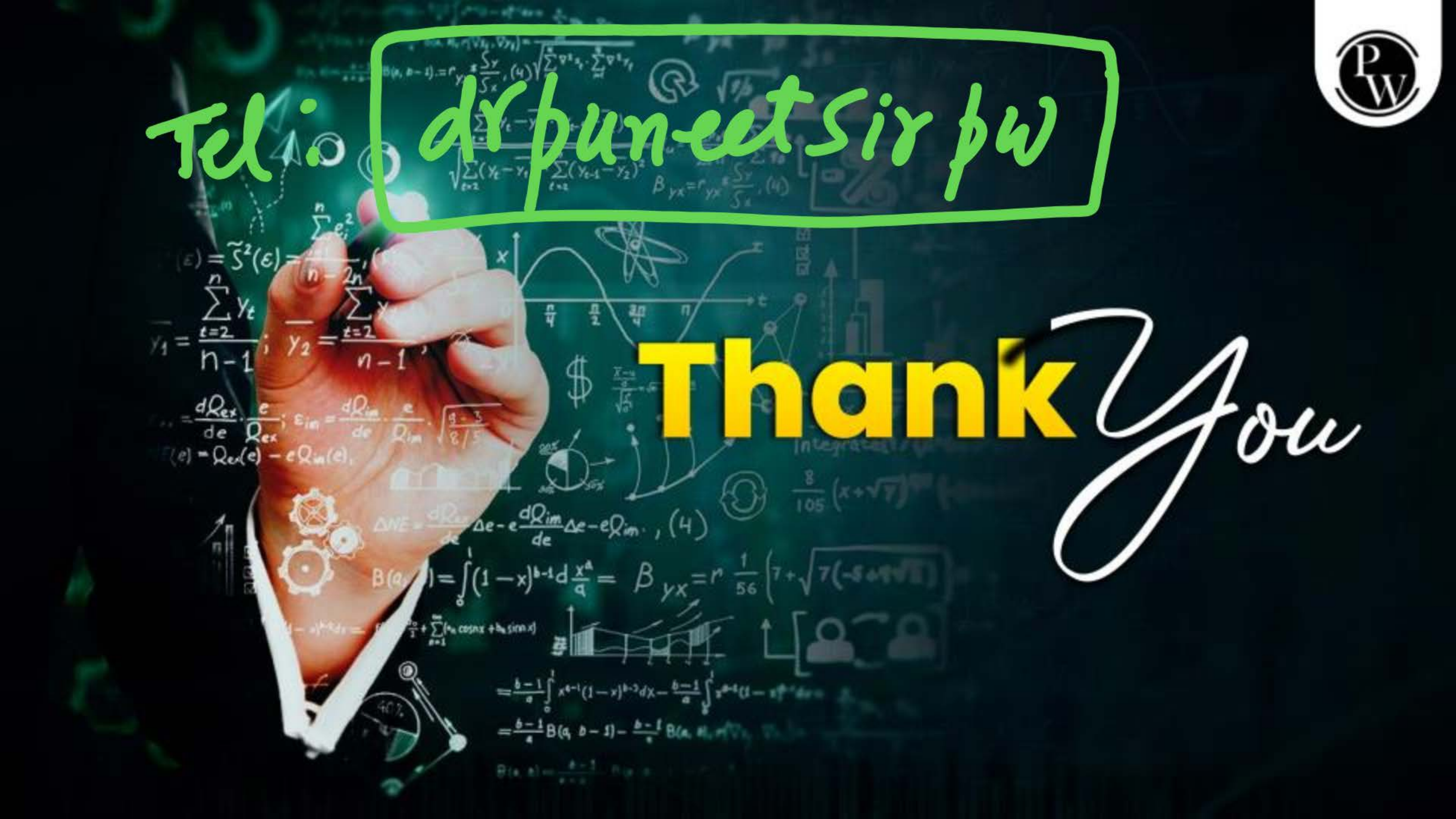
$f'(0) =$ DNE

eg.



$f'(a^+) =$ Check

$f'(b^-) =$ "


Tel : drpuneetsirpw
Thank You

DATA SCIENCE
&
ARTIFICIAL INTELLIGENCE

& CS / IT

Calculus and Optimization

Lecture No. 09

By- Dr. Puneet Sharma Sir

Recap of previous lecture

Topic

DERIVATIVES & THEIR TYPES

(PART-2)

Topics to be Covered

Topic

MAXIMA & MINIMA

(Part 1)

Types of Questions — RECAP

1. Based on ordinary Derivative — exist in case of curve $y = f(x)$
2. " " Partial Derivative — " " " of surface $z = f(x, y)$
3. " " Total Derivative — if $z = f(x, y)$, $x = x(t)$, $y = y(t)$
   i.e. $z \longrightarrow (x, y) \longrightarrow$ 't' alone
4. " " Chain Rule of Partial Derivatives, if $z = f(x, y)$, $x = x(r, s)$, $y = y(r, s)$
   i.e. $z \longrightarrow (x, y) \longrightarrow (r, s)$
5. " " Jacobian if $(u, v) \longrightarrow (x, y)$
6. " " Euler Theorem: if $f(x, y)$ is Homogeneous func^n then we can use E. Th.

Q. HW Q. If $v = v(x, y)$ where $x+y = 2e^{\theta}\cos\phi$, $x-y = 2ie^{\theta}\sin\phi$ then evaluate $\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial \phi} = ?$

Sol:

$$\left.\begin{array}{l} x+y = 2e^{\theta}\cos\phi \\ x-y = 2ie^{\theta}\sin\phi \end{array}\right\} \begin{array}{l} 2x = 2e^{\theta}(\cos\phi + i\sin\phi) \\ 2y = 2e^{\theta}(\cos\phi - i\sin\phi) \end{array} \Rightarrow \begin{array}{l} x = e^{\theta}(e^{i\phi}) = e^{\theta+i\phi} \\ y = e^{\theta}(e^{-i\phi}) = e^{\theta-i\phi} \end{array}$$

$\because v \longrightarrow (x, y) \longrightarrow (\theta, \phi)$ so Quest is Based on Chain Rule.

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\left(\frac{\partial x}{\partial \theta}\right) + \frac{\partial v}{\partial y}\left(\frac{\partial y}{\partial \theta}\right) = V_x\left(e^{\theta+i\phi}\right) + V_y\left(e^{\theta-i\phi}\right)$$ Ans

$$\frac{\partial v}{\partial \phi} = \frac{\partial v}{\partial x}\left(\frac{\partial x}{\partial \phi}\right) + \frac{\partial v}{\partial y}\left(\frac{\partial y}{\partial \phi}\right) = V_x\left(ie^{\theta+i\phi}\right) + V_y\left(-ie^{\theta-i\phi}\right)$$ Ans

JACOBIAN → if $u = u(x,y,z)$, $v = v(x,y,z)$, $w = w(x,y,z)$

RECAP

is $(u, v, w) \to (x, y, z)$

then Derivative of (u, v, w) with respect to (x, y, z) is called Jacobian & it is defined as

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Note ① if $(u, v) \to (x, y)$ then

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

② if $(x, y) \to (u, v)$ then $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = J'$

③ $JJ' = 1 \Rightarrow J' = \frac{1}{J}$ = Reciprocal of J

Application of Jacobian → let $u = u(x,y)$ & $v = v(x,y)$ are two functions

Functionally Dependent funcⁿ → u & v are functionally dependent if there exist Mathematical Relation b/n them & it's condition is $J = 0$

Functionally Independent funcⁿ → u & v are called functionally independent if there DNE any Relation b/n them. & it's condition is $J \neq 0$

Q ⇒ If $u = \sin^{-1}x + \sin^{-1}y$ & $v = x\sqrt{1-y^2} + y\sqrt{1-x^2}$ then $\frac{\partial(u,v)}{\partial(x,y)} = ?$

(a) 1

(b) 4

(c) 0 ✓

(d) $\sin u$

M-I $\because (u,v) \longrightarrow (x,y)$ so

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} \frac{1}{\sqrt{1-x^2}} & \frac{1}{\sqrt{1-y^2}} \\ ? & ? \end{vmatrix} = \text{--------} = 0$$

M-II By observation, we can write,

$u = \sin^{-1}x + \sin^{-1}y$

$u = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$

$u = \sin^{-1}(v) \Rightarrow v = \sin u$

i.e. ∃ Relationship b/w u & v or u & v are Dependent

$\Rightarrow J = 0$

MAXIMA MINIMA

T-1 → Increasing Dec func^n

T-2 → Max-Min of Curve $y=f(x)$

T-3 → Max Min of Surface $z=f(x,y)$

* I^{st} Derivation
  - → Geometrical Significance → To know the Slope of Tangent
  - → Application → To know the Inc or Dec Nature of func^n.

* 2^{nd} Derivatives
  - → Geometrical Significance → To know Concavity of func^n
  - → Application → To know Inflexion Point, Point of Max & Point of minima

INCREASING & DECREASING FUNCTION

w.k.that for $y = f(x)$, $\frac{dy}{dx} = \boxed{f'(x) = \tan\theta} =$ slope of tangent at any Random point x on $f(x)$

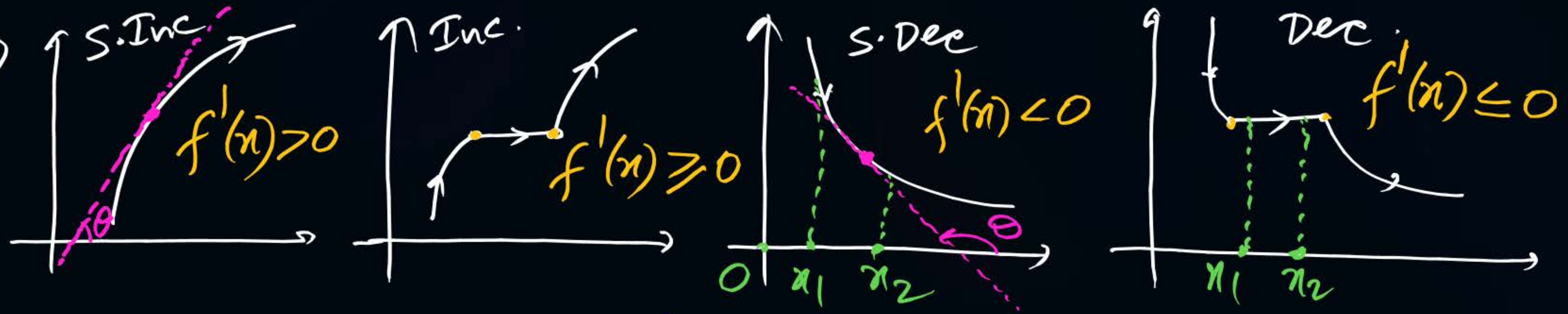
(1) for Acute angle tangent, $f'(x) > 0$

(2) " obtuse " " , $f'(x) < 0$

(3) for Horizontal tangent, $f'(x) = 0$

(4) for Vertical Tangent, $f'(x) =$ D.N.E

(*)



eg: if $f(x_1)>f(x_2)\ \forall\ x_1<x_2$; where $x_1, x_2 \in D_f$ then $f(x)$ is ___S.Dec.___

eg: if $f(x_1)\geq f(x_2)$ " " " " " " " " " ___Decreasing___

(*) Monotonic func's → Those functions which are either S.I or S.D are called Monotonic functions

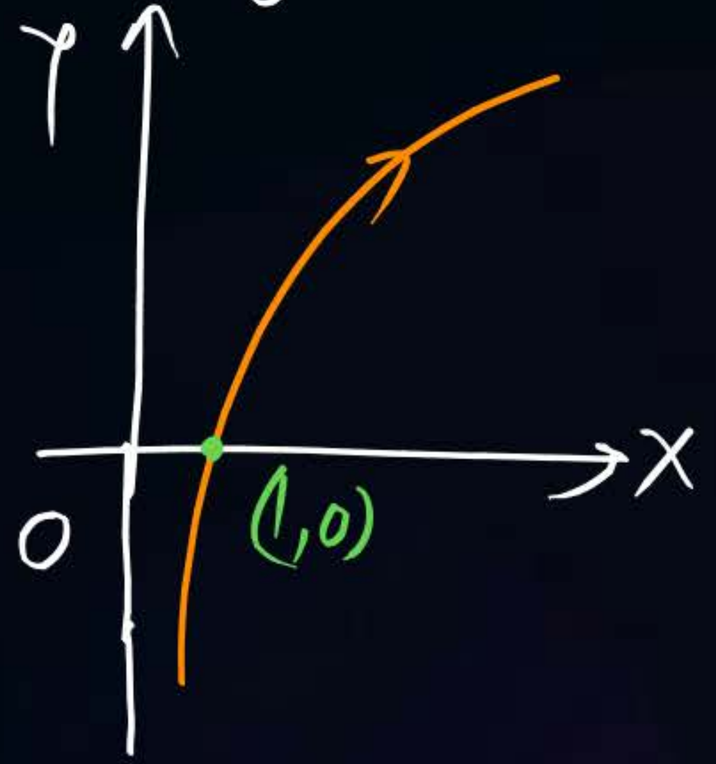
(*) Advice → It is advisable to first write the Domain of function while Solving Questions Based on Inc / Dec functions.

eg: $f(x) = \log_a x$ is S.Inc

where $(a > 1)$

$D_f = (0, \infty)$

M-I (Using Graph) →



M-II

$$f(x) = \log_a x = \frac{\log_e x}{\log_e a}$$

$$\boxed{f'(x) = \frac{1}{x \log_e a}}$$

$$= \frac{1}{(+ve)(+ve)} > 0$$

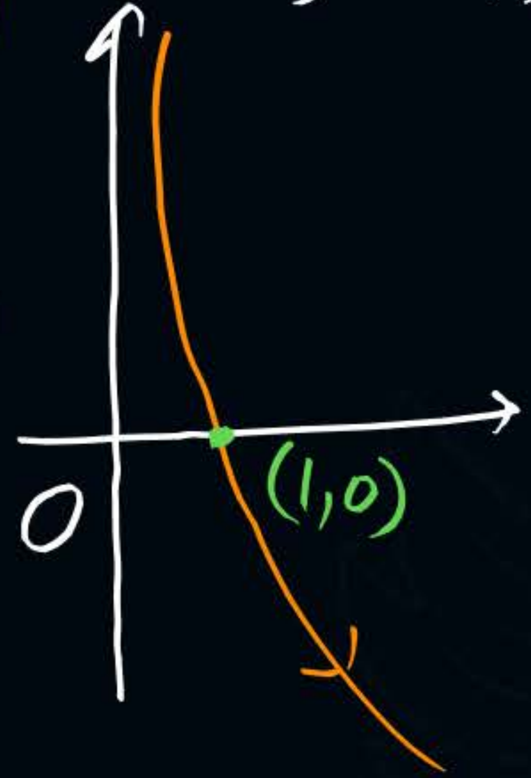
So $f(x)$ is S.Inc.

eg $f(x) = \log_a x$ is S.Dec

where $0 < a < 1$

$D_f = (0, \infty)$

M-I (Using Graph) →



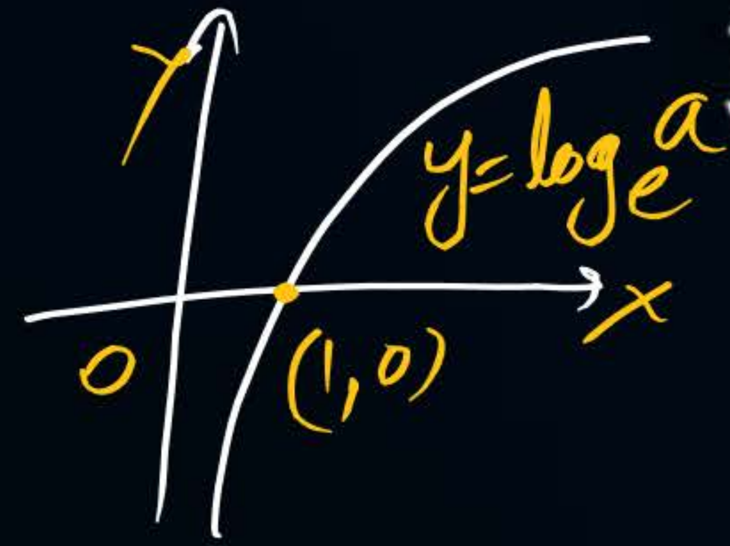
M-II $f(x) = \log_a x = \frac{\log_e x}{\log_e a}$

$$\boxed{f'(x) = \frac{1}{x \log_e a}}$$

$$= \frac{1}{(+ve)(-ve)} < 0$$

So $f(x)$ is S.Dec.

Note: ① $a > 1$ then $\boxed{\log_e a = ? = +ve}$ Learn



$\because$ let $a = 2$, $\log_e 2 = 0.693 = +ve$.

② $0 < a < 1$, $\boxed{\log_e a = ? = -ve}$ Learn.

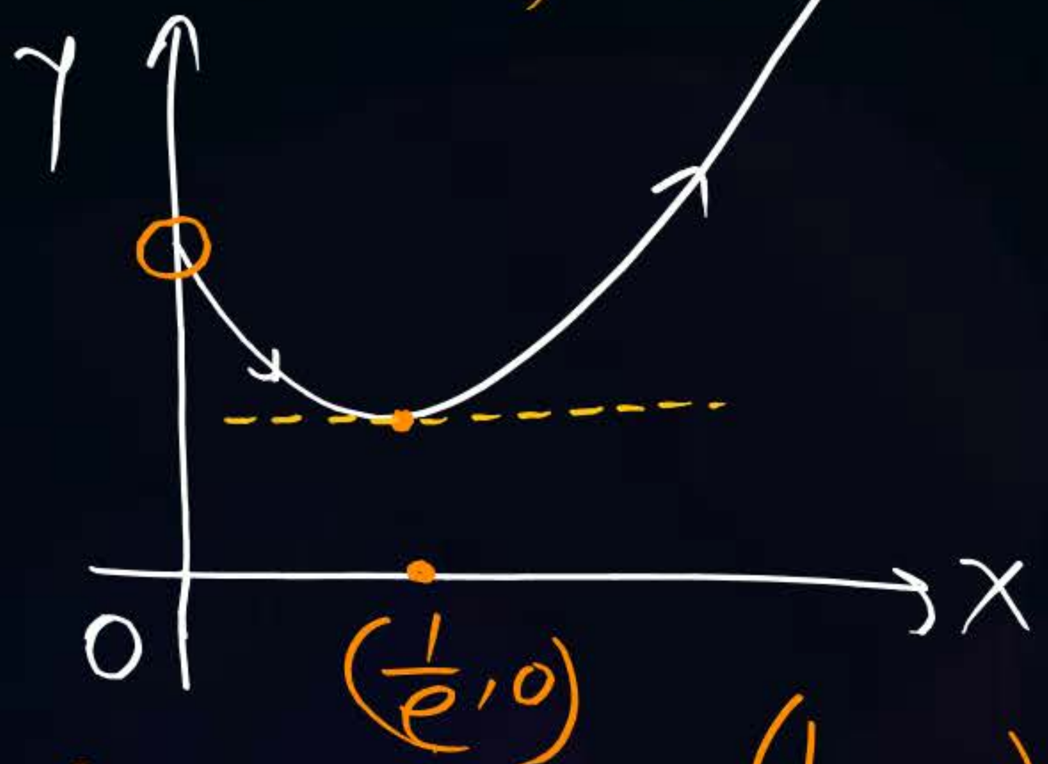
$\because$ let $a = \frac{1}{2}$, $\log_e\left(\frac{1}{2}\right) = \log_e(2)^{-1} = -1 \log_e 2 = -0.6932$

$= -ve$.

Q. find the Interval in which $\boxed{f(x) = x^x}$ Increases & Decreases.

M-I (Using Graph) $\rightarrow$

$y = x^x$; $D_f = (0, \infty)$



$f(x)$ is S.Inc in $\left(\frac{1}{e}, \infty\right)$

$f(x)$ is S.Dec in $\left(0, \frac{1}{e}\right)$

M-II (w/o Graph) $\rightarrow$

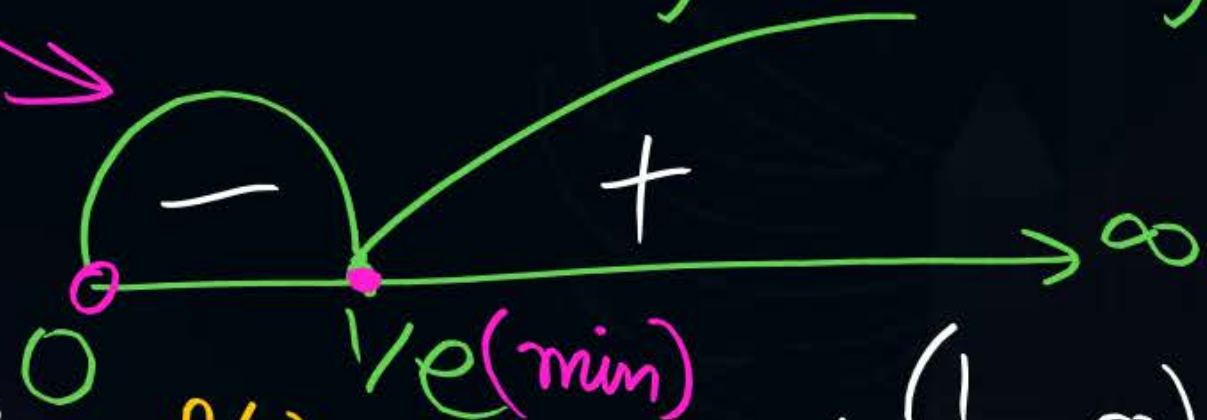
$f'(x) = x^x(1 + \log_e x)$

$f'\left(\frac{1}{e^2}\right) = -ve$

$f'(e) = +ve$

Turning points are; Put $f'(x) = 0$

$x^x(1 + \log x) = 0 \Rightarrow \log_e x = -1 \Rightarrow x = e^{-1}$



So $f(x)$ is S.Inc in $\left(\frac{1}{e}, \infty\right)$

& $f(x)$ is S.Dec in $\left(0, \frac{1}{e}\right)$

MSQ

Q Find the Interval of Increasing & Decreasing for $y = \frac{\log_e x}{x}$, $D = (0, \infty)$

(a) Increases in $(0, e]$

(b) Decreases in $(0, e]$

(c) Increases in $[e, \infty)$

(d) Decreases in $[e, \infty)$

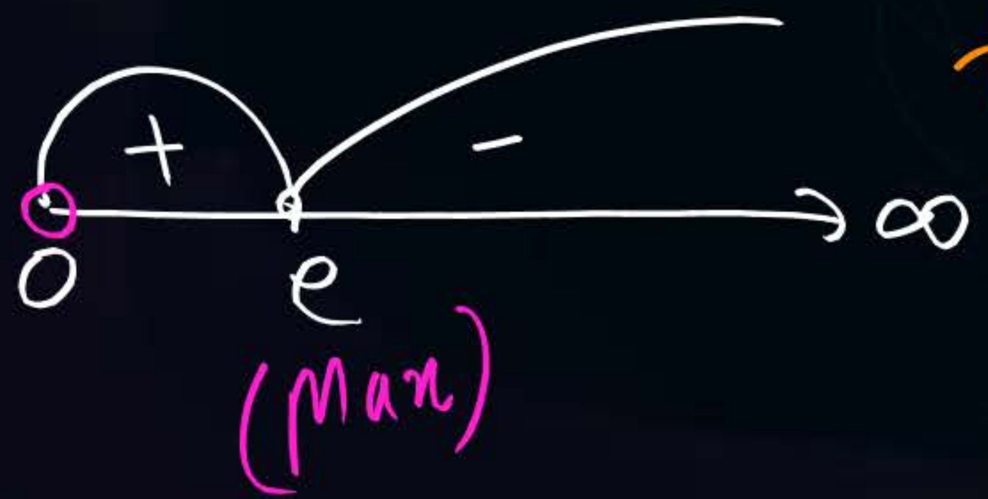
$$f'(x) = \frac{x\left(\frac{1}{x}\right) - \log x (1)}{(x)^2} = \frac{1 - \log_e x}{x^2}$$

Turning Point: $f'(x) = 0 \Rightarrow x = e$

$$\boxed{f'(x) = \frac{1 - \log_e x}{x^2}}$$

$f'(1) = +ve$

$f'(e^2) = -ve$



$f(x)$ S.Inc in $(0, e)$

$f(x)$ Inc in $(0, e]$

$f(x)$ S.Dec in (e, ∞)

$f(x)$ Dec in $[e, \infty)$

Analysis : (1) $f'(x) = \frac{1-\log x}{x^2}$, f

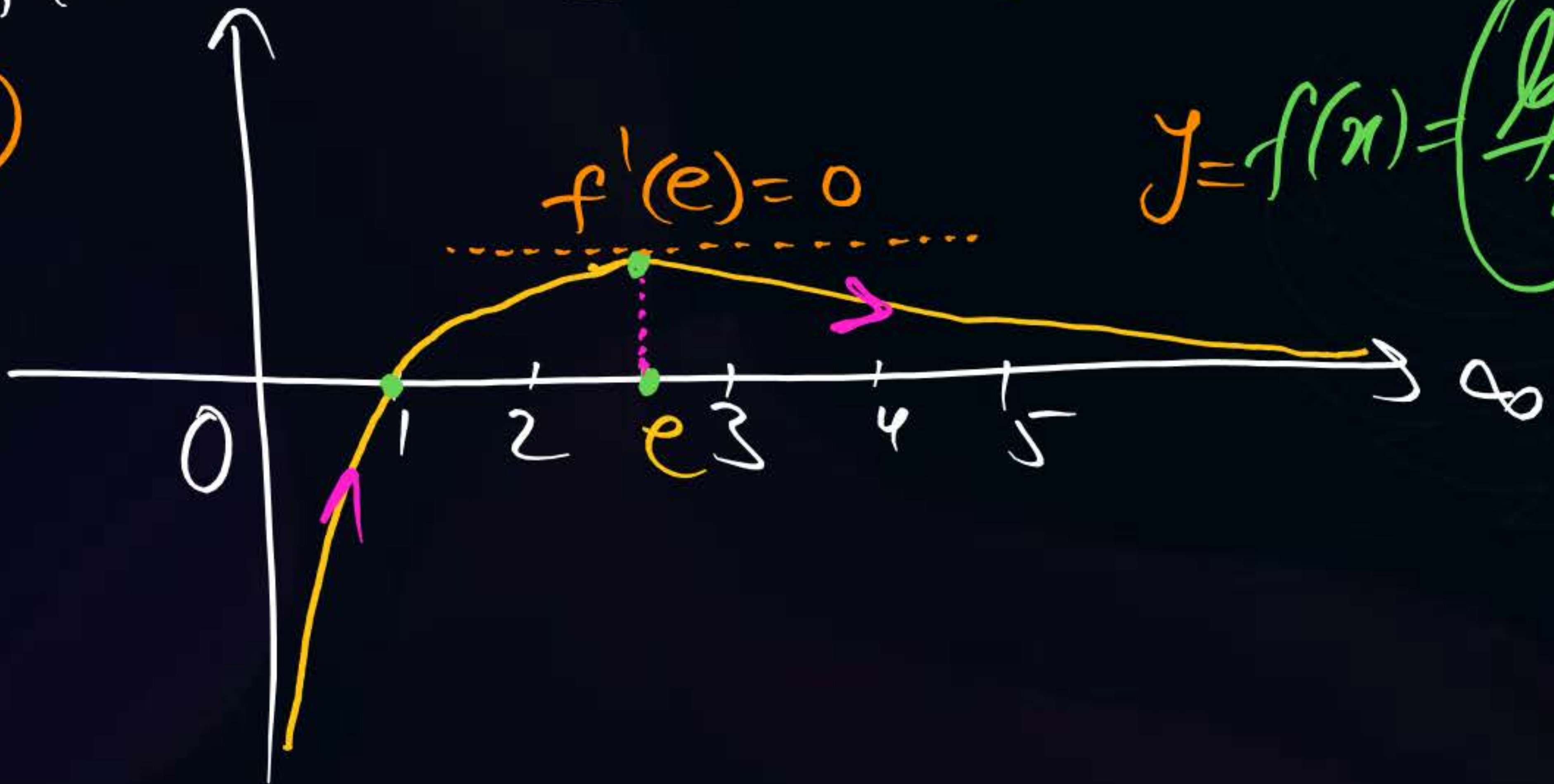
$f(x)$ is S.Inc in $(0, e)$

$f(x)$ is Inc in $(0, e]$ $\because f'(e) = 0$

$f(x)$ is S.Dec in (e, ∞)

$f(x)$ is Dec in $[e, \infty)$ $\because f'(e) = 0$

(2)



$y = f(x) = \frac{\log x}{x}$, $D_f = (0, \infty)$

$f(1) = \frac{0}{1} = 0$

Dom $= (-\infty, \infty)$

Q $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 21$, then find the interval in which $f(x)$

MSQ Increases and Decreases?

(a) $f(x)$ Inc in $(1,2) \cup (3,\infty)$

(b) $f(x)$ Dec in $(-\infty,1) \cup (2,3)$

(c) $f(x)$ Inc in $x < -1$ or $2 < x < 3$

(d) $f(x)$ Dec in $1 < x < 2 \cup x > 3$

$f(x) = x^4 - 8x^3 + 22x^2 - 24x + 21$

$f'(x) = 4x^3 - 24x^2 + 44x - 24$

$= 4(x^3 - 6x^2 + 11x - 6)$

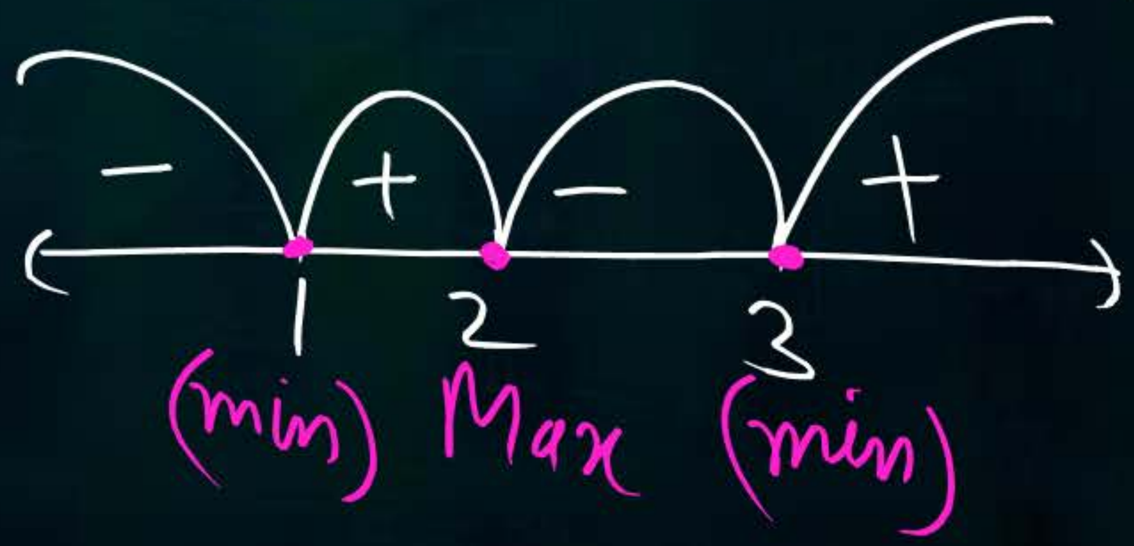
$$\boxed{f'(x) = 4(x-1)(x-2)(x-3)}$$

$f'(0) = -ve$

$f'(1.5) = +ve$

$f'(2.5) = -ve$

$f'(4) = +ve$



$f(x)$ Inc in $[1,2] \cup [3,\infty)$

$f(x)$ Dec in $(-\infty,1] \cup [2,3]$

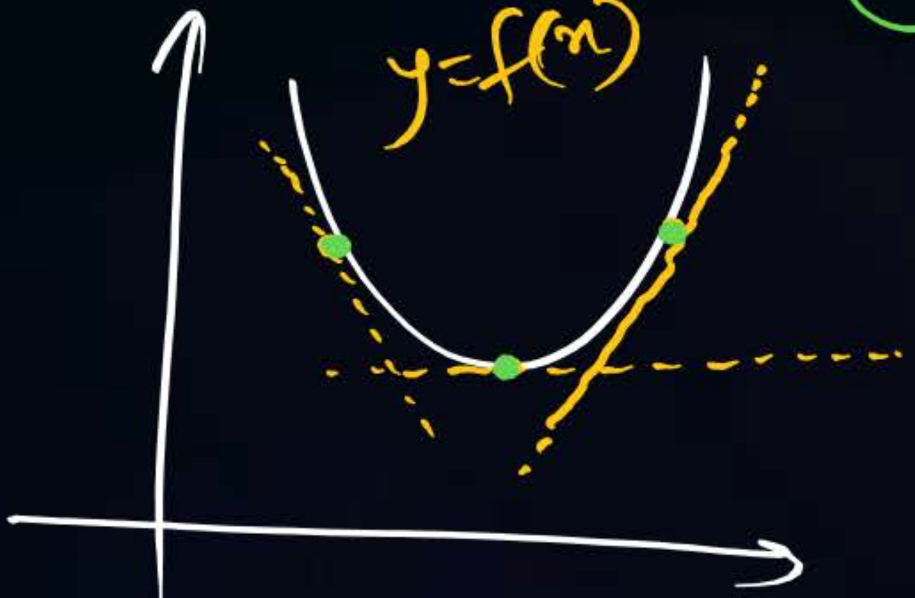
Information :

Concave upward Curve :

if curve lies above the tangent always then Curve is called $\underbrace{\text{concave upward Curve}}$

$\Downarrow$

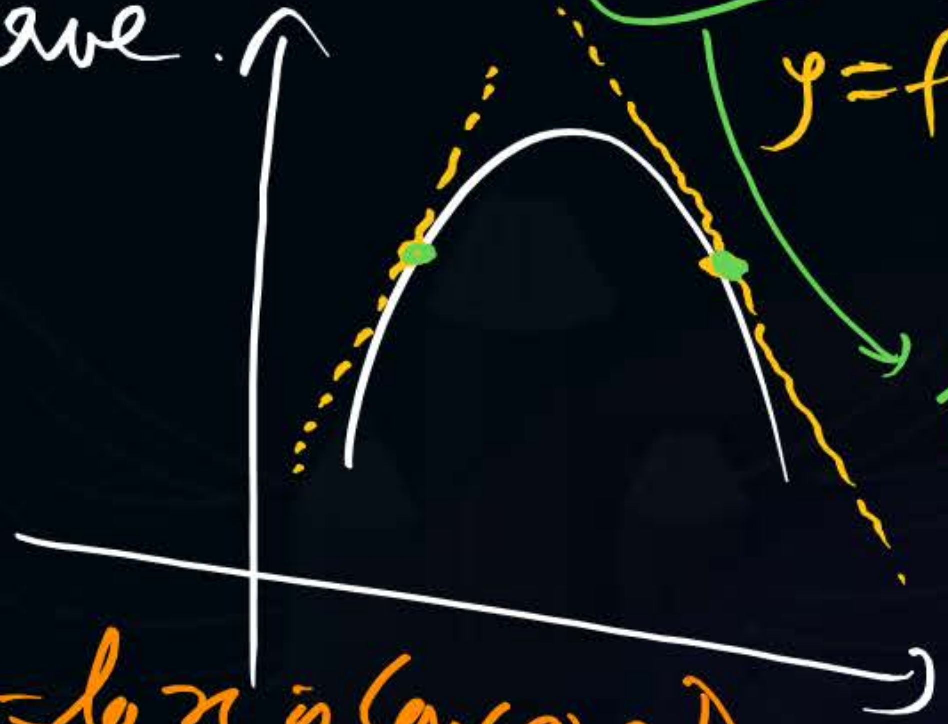
$f''(x) > 0$



eg $f(x) = e^x$ is Concave upward Curve

Concave Downward Curve

if Curve lies below the tangent always then Curve is called $\underbrace{\text{Concave Downward}}$ Curve.



$f''(x) < 0$

eg $f(x) = \log_e x$ is Concave Downward Curve

$f(x) = \boxed{y = e^x}, (-\infty, \infty)$

eg

$f''(x) > 0$

Concave up. Curve

M-II

$f'(x) = e^x$

$f''(x) = e^x$

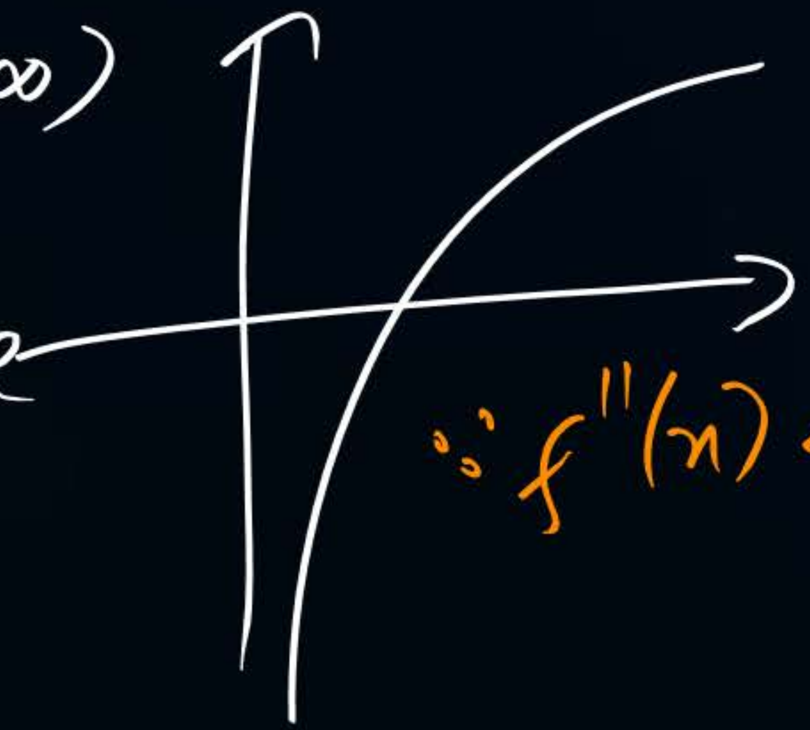
i.e $f''(x) > 0$ always in Domain

$\Longrightarrow$ $\underline{f(x)}$ is Concave upward Curve

eg $f(x) = e^{-x}$ is also $\underline{\text{Concave upward}}$

eg $f(x) = \boxed{y = \log_e x}, (0, \infty)$

Concave Down Curve



$\because f''(x) < 0$

$f'(x) = \frac{1}{x}$

$f''(x) = -\frac{1}{x^2}$

i.e $f''(x) < 0$ always

$\Rightarrow$ $\underline{f(x)}$ is Concave Downward Curve

eg $f(x) = \log_a x$, $0 < a < 1$ is also $\underline{\text{Concave up}}$

Q. The function $f(x) = x^4$ is ?
2014 (MSQ)

(a) strictly Increases

(b) strictly Decreases

(c) Concave Downward

(d) Concave upward

$\boxed{f(x) = x^4}$, $(-\infty, \infty)$

$f'(x) = 4x^3 \Rightarrow$

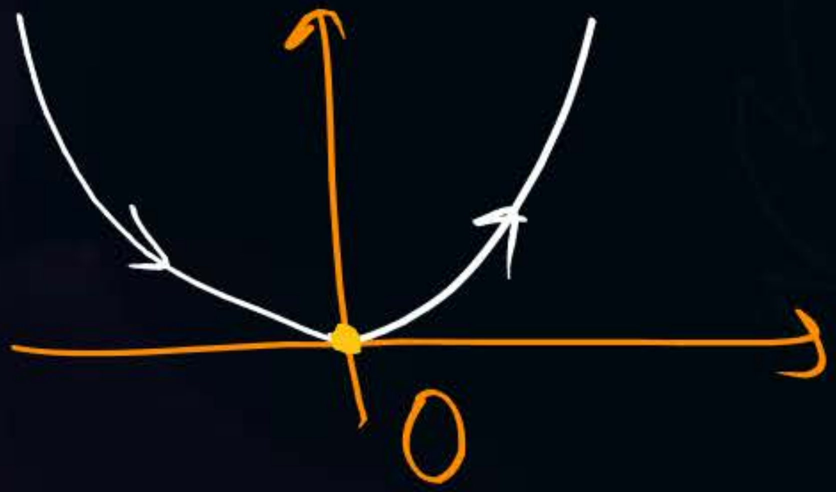
$f(x)$ is S. Inc $(0, \infty)$

$f(x)$ is S. Dec in $(-\infty, 0)$

$f''(x) = 12x^2 > 0$ always $\because$

$\Rightarrow f(x)$ is Concave upward. in $(-\infty, \infty)$

(M-II) $y = x^4$



<u>Shortcuts</u>: Put $\boxed{f'(x)=0}$ & Try to find T. Points (say $x = c_1, c_2$)

(1) then check sign of $f'(x)$ as follows;

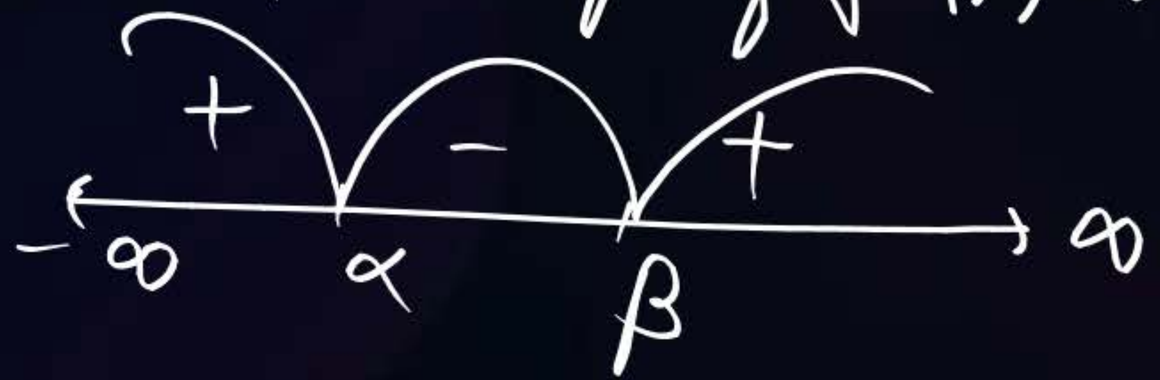
$\Rightarrow f(x)$ Inc in $(-\infty, c_1] \cup [c_2, \infty)$

$f(x)$ Dec in $[c_1, c_2]$

(2) <u>To find the Interval where $f(x)$ is Concave up or Concave Downward</u> $\longrightarrow$

Put $\boxed{f''(x)=0}$ & Try to find x (say $x = \alpha, \beta$)

then check the sign of $f''(x)$ as follows;



$\Rightarrow f(x)$ is Concave upward in $(-\infty, \alpha] \cup [\beta, \infty)$

$f(x)$ is Concave Downward in $[\alpha, \beta]$

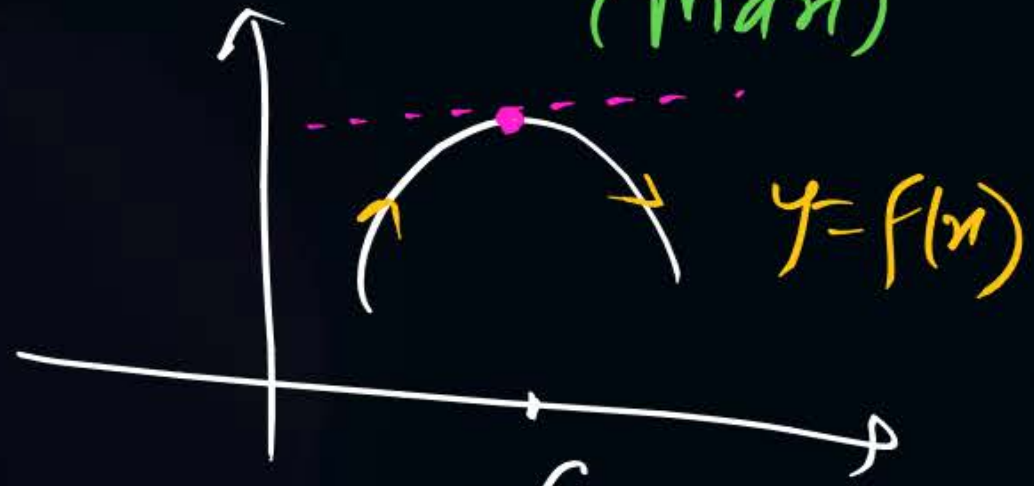
(*) Consider $y = f(x)$ is given func^n & it's T. points is $x = c$

then sign of $f'(x) \Longrightarrow$

c
(Max)

c
(min)

eg



c
(Max)

c
(min)

c
(NMNM)

,

c (NMNM)

Saddle point / point of Inflexion →

"Those points where curve changes it's concavity are called points of Inflexion"

Note: (1) those points where we are getting Neither Maxima, Nor minima are called Points of Inflexion (F)

(2) if ($x=\alpha$ is the point of Inflexion) $\implies$ (At $x=\alpha$ we will get N.M.N.M) $\not\Longleftarrow$

Shortcut: Put $f''(x)=0$ & Try to get x (Say it is $x=\alpha$)

if $f'''(\alpha)\neq 0 \Rightarrow x=\alpha$ is point of Inflexion

& if $f'''(\alpha)=0 \Rightarrow$ we can't say anything about α.

Q. Find the interval in which $f(x) = x^4 - 24x^2 + 11$ is S.Inc, S.Dec, Concave upward & Concave Downward ?

MSQ

(a) S.Dec in $(-\infty, -2\sqrt{3}) \cup (0, 2\sqrt{3})$

(b) S.Inc in $(-2\sqrt{3}, 0) \cup (2\sqrt{3}, \infty)$

(c) Concave upward in $(-\infty, -2) \cup (2, \infty)$

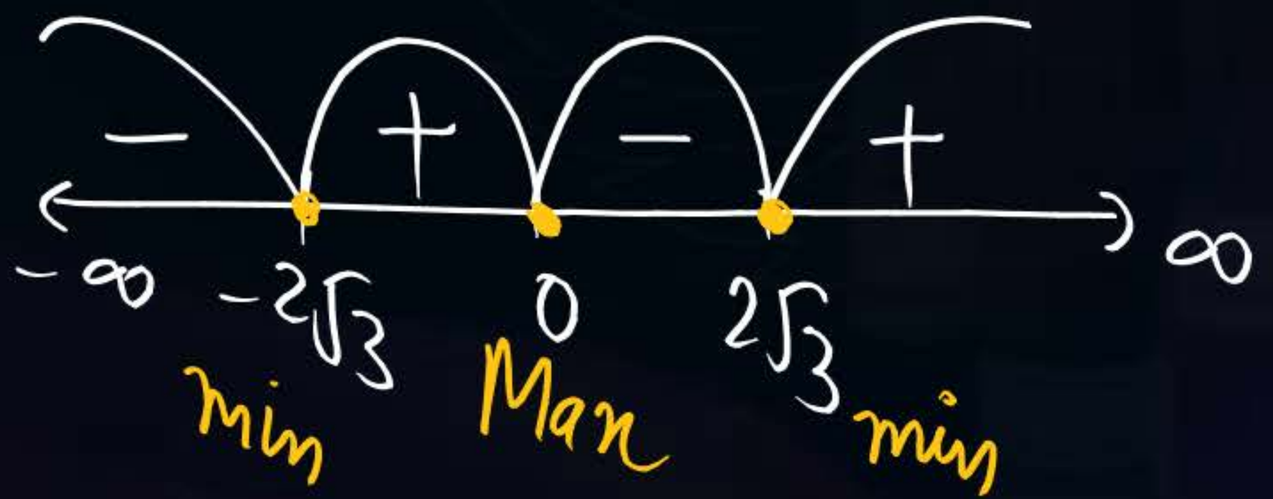
(d) Concave Downward in $(-2, 2)$

$f(x) = x^4 - 24x^2 + 11 \quad , (-\infty, \infty)$

$f'(x) = 4x^3 - 48x = 4x(x^2 - 12)$

$$\boxed{f'(x) = 4x(x - 2\sqrt{3})(x + 2\sqrt{3})}$$

T.Points are $x = -2\sqrt{3}, 0, 2\sqrt{3}$



Again, $f(x) = x^4 - 24x^2 + 11$

$$f'(x) = 4x^3 - 48x$$

$$f''(x) = 12x^2 - 48$$

$$= 12(x^2 - 4)$$

$$\boxed{f''(x) = 12(x-2)(x+2)}$$

(P.O.I) (P.O.I)

$f(x)$ is concave up in $(-\infty, -2) \cup (2, \infty)$

" " Concave Down in $(-2, 2)$

Note: $\because$ Concavity changes at $x = -2$ & 2

So these two points are saddle points or Points of Inflexion

& At $x = \pm 2$ we will get NMNM

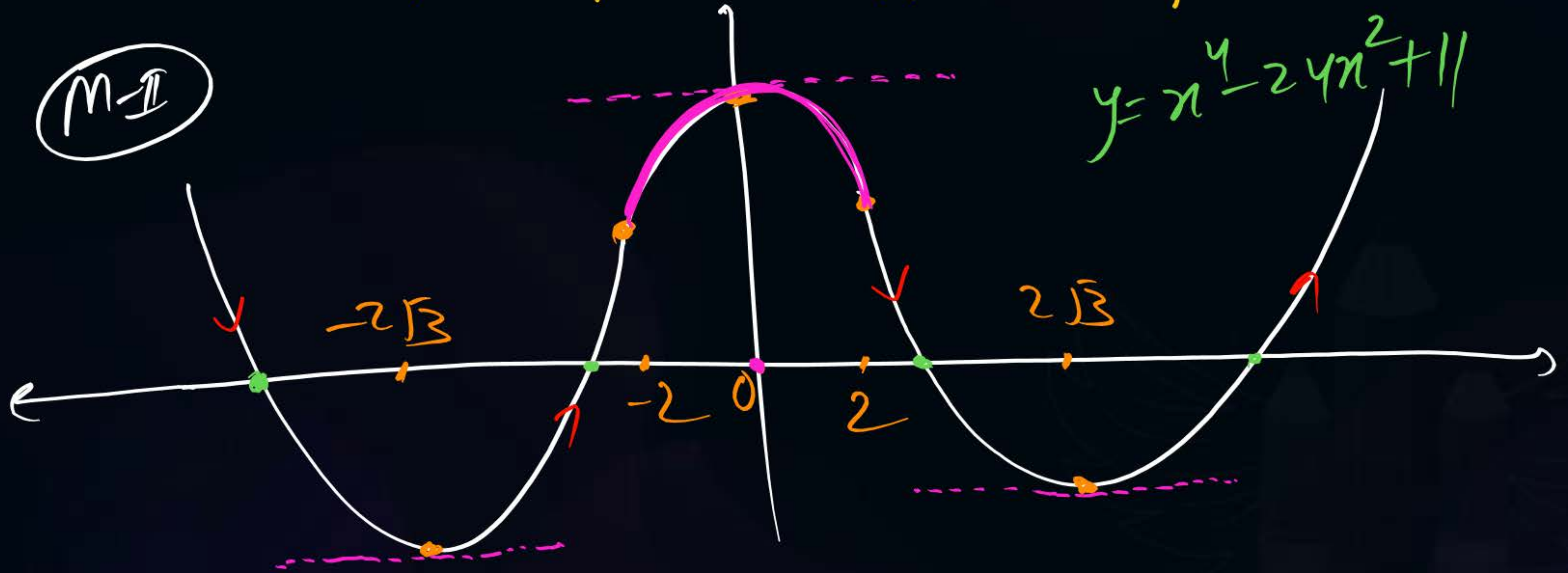
(*) if ($x = \alpha$ is P.O.I) $\Longrightarrow$ (we will get NMNM at α)

$\not\Longleftarrow$

In previous Ex., T-points are $x = -2\sqrt{3}, 0, 2\sqrt{3}$

& Saddle points are $x = -2$ & 2

M-II



Q. The Number of Inflexion Points of $f(x)=x+12x^4$ is/are?

(a) 0
(b) 1
(c) 2
(d) 3

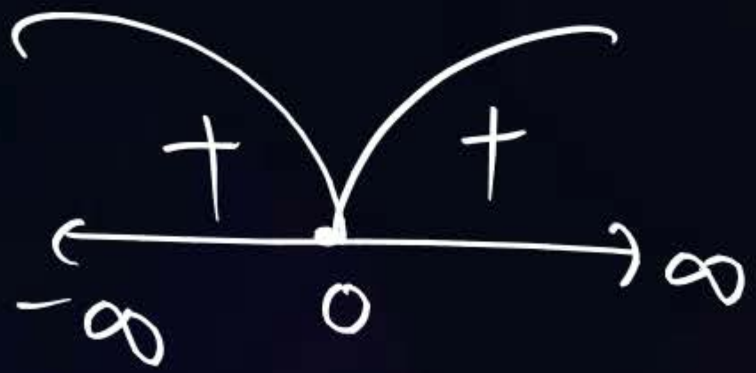
ZERO

M-I $f(x)=x+12x^4$

$f'(x)=1+48x^3$

$\boxed{f''(x)=144x^2}$

Put $f''(x)=0 \Rightarrow x=0$



$f(x)$ is concave upward always.

M-II $f'(x)=1+48x^3$

$f''(x)=144x^2$, $f'''(x)=288x$

Putting $f''(x)=0 \Rightarrow x=0$

Now $f'''(0)=(288x)_{x=0}=0$

i.e. $x=0$ is not a Point of Inflexion.

Hence given $f(x)$ has <u>NO</u> POI

Q. Number of Inflexion points of $\boxed{f(x) = x^4 - 18x^2 + 9}$ is/are ? Two.

(a) 0
(b) 1
(c) 2 ✓
(d) 3

M-I

$f'(x) = 4x^3 - 36x$

$f''(x) = 12x^2 - 36$
$= 12(x^2 - 3)$

$$\boxed{f''(x) = 12(x - \sqrt{3})(x + \sqrt{3})}$$

$f(x)$ is concave upward in $(-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$
" " " Down in $(-\sqrt{3}, \sqrt{3})$

M-II

$f'(x) = 4x^3 - 36x$

$f''(x) = 12x^2 - 36$, $f'''(x) = 24x$

Putting $f''(x) = 0 \Rightarrow x = \pm\sqrt{3}$

Now $f'''(\sqrt{3}) = (24x)_{x=\sqrt{3}} \neq 0$

& $f'''(-\sqrt{3}) = (24x)_{x=-\sqrt{3}} \neq 0$

ie Both the points are points of Inflexion

(ii) Also find Max/min of $f(x) = x^4 - 18x^2 + 9$

$$f'(x) = 4x^3 - 36x = 4x(x^2 - 9)$$

$$\boxed{f'(x) = 4x(x+3)(x-3)}$$

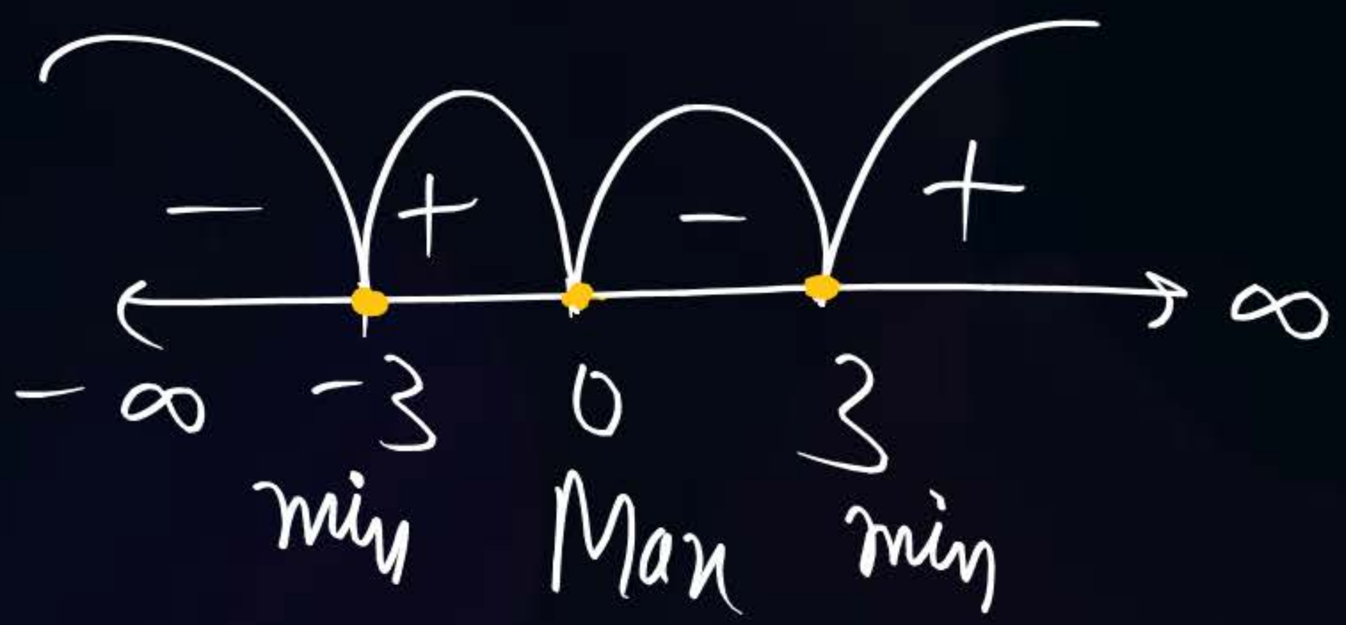
T-Points are $x = -3, 0, 3$

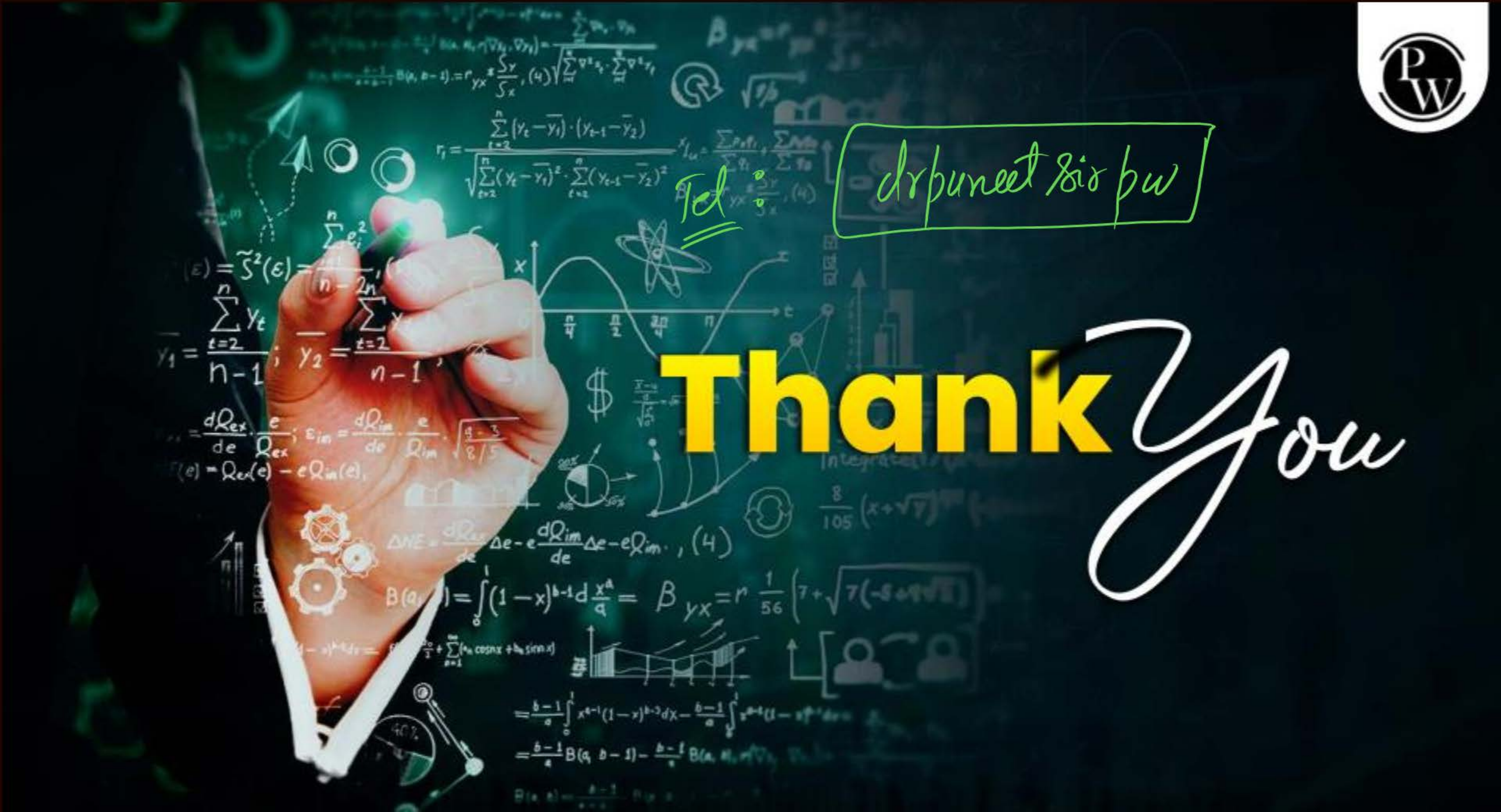
$f'(-5) = -ve$

$f'(-1) = +ve$

$f'(1) = -ve$

$f'(4) = +ve$




Tel : drpuneet sir pw
Thank You

DATA SCIENCE
&
ARTIFICIAL INTELLIGENCE

& CS / IT

Calculus and Optimization

Lecture No. 10

By- Dr. Puneet Sharma Sir

Recap of previous lecture

Topic Maxima - Minima (Part 1)

Topics to be Covered

Topic

MAXIMA & MINIMA
(Part 2)

MAXIMA MINIMA

RECAP

T-1 → Increasing Dec. func^n

T-2 → Max-Min of Curve $y=f(x)$

T-3 → Max Min of Surface $z=f(x,y)$

* I^{st} Derivation
  - → Geometrical Significance → To know the Slope of Tangent
  - → Application → To know the Inc or Dec Nature of func^n.

* 2^{nd} Derivatives
  - → Geometrical Significance → To know Concavity of func^n
  - → Application → To know Inflexion Point, Point of Max & Point of minima

RECAP

INCREASING & DECREASING FUNCTION

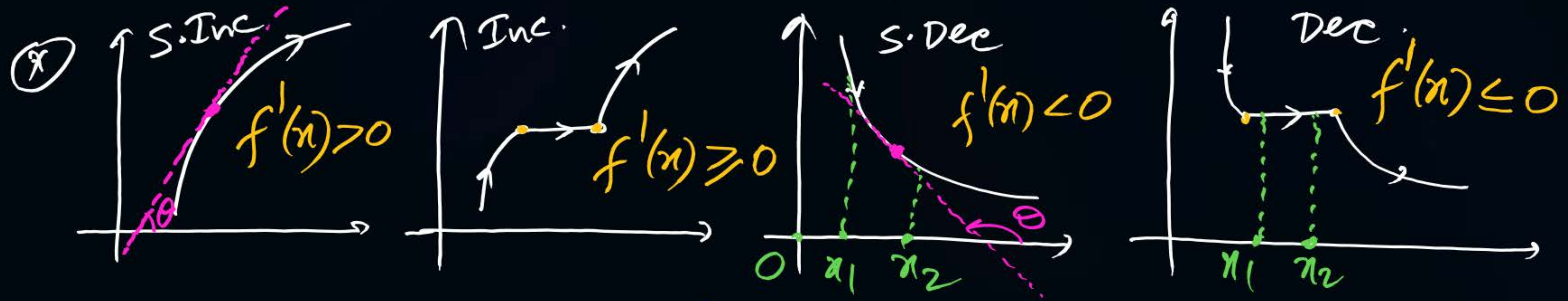
w.k.that for $y = f(x)$, $\frac{dy}{dx} = \boxed{f'(x) = \tan\theta} =$ slope of tangent at any Random point x on $f(x)$

(1) for Acute angle tangent, $f'(x) > 0$

(2) " obtuse " " , $f'(x) < 0$

(3) for Horizontal tangent, $f'(x) = 0$

(4) for Vertical Tangent, $f'(x) =$ D.N.E



eg: if $f(x_1) > f(x_2)\ \forall\ x_1 < x_2$; where $x_1, x_2 \in D_f$ then $f(x)$ is S.Dec.

eg: if $f(x_1) \geq f(x_2)$ " " " " " " " " " Decreasing

* **Monotonic func's** → Those functions which are either S.I or S.D are called Monotonic functions

* **Advice** → It is advisable to first write the Domain of function while Solving Questions Based on Inc / Dec functions.

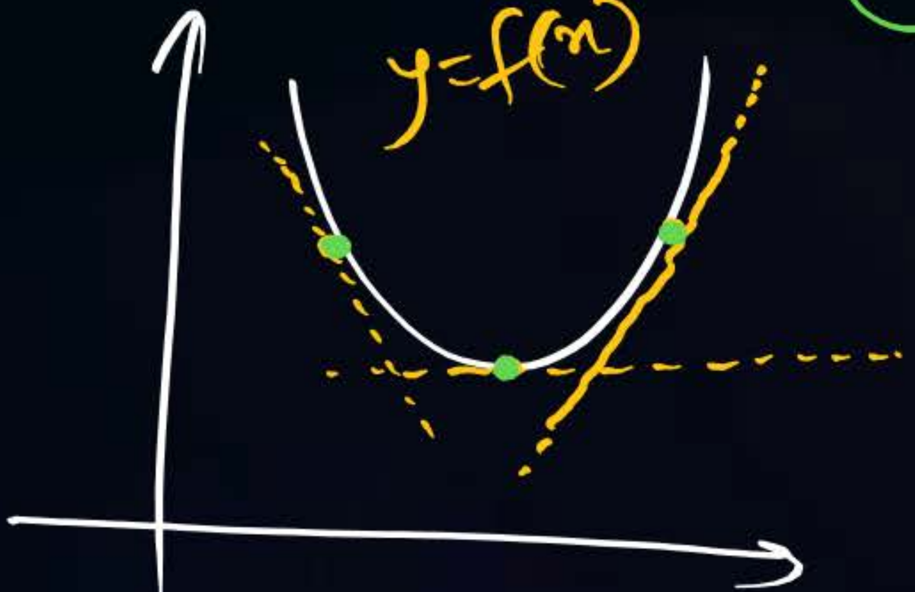
RECAP

Information: RECAP

Concave upward Curve:

if curve lies above the tangent always then curve is called concave upward curve

$$f''(x) > 0$$

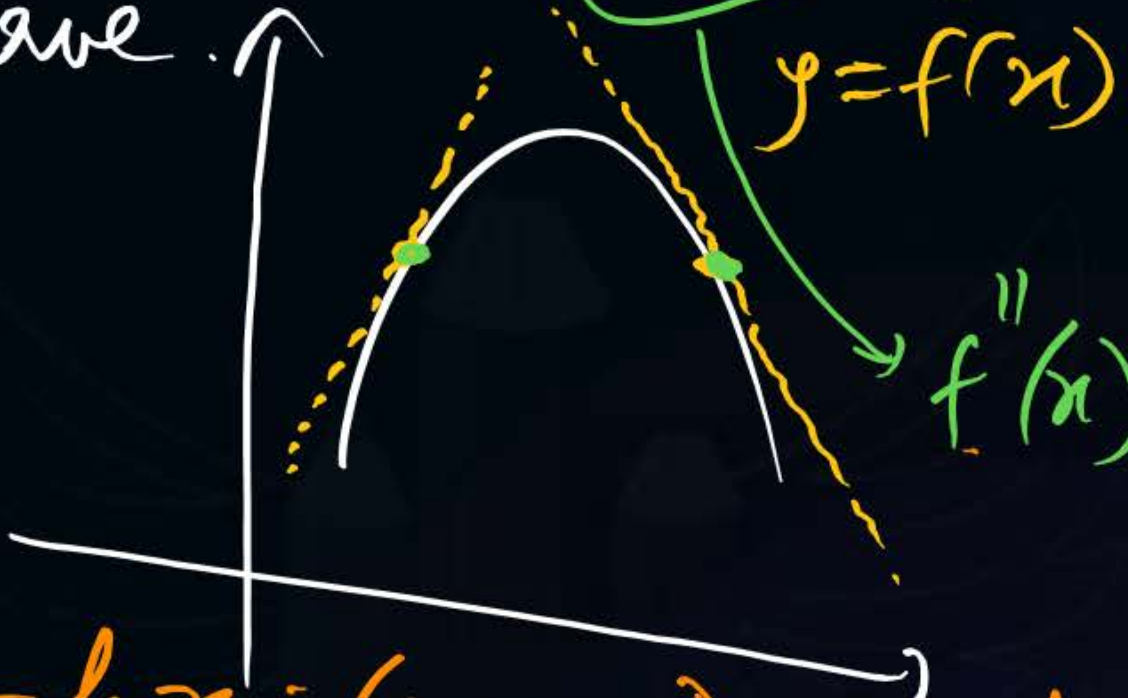


eg $f(x)=e^x$ is Concave upward Curve

Concave Downward Curve

if curve lies below the tangent always then curve is called Concave Downward curve.

$$f''(x) < 0$$



eg $f(x)=\log_e x$ is Concave Downward Curve

Shortcuts: Put $\boxed{f'(x)=0}$ & Try to find T. Points (Say $x = c_1, c_2$)

① then check sign of $f'(x)$ as follows;

RECAP

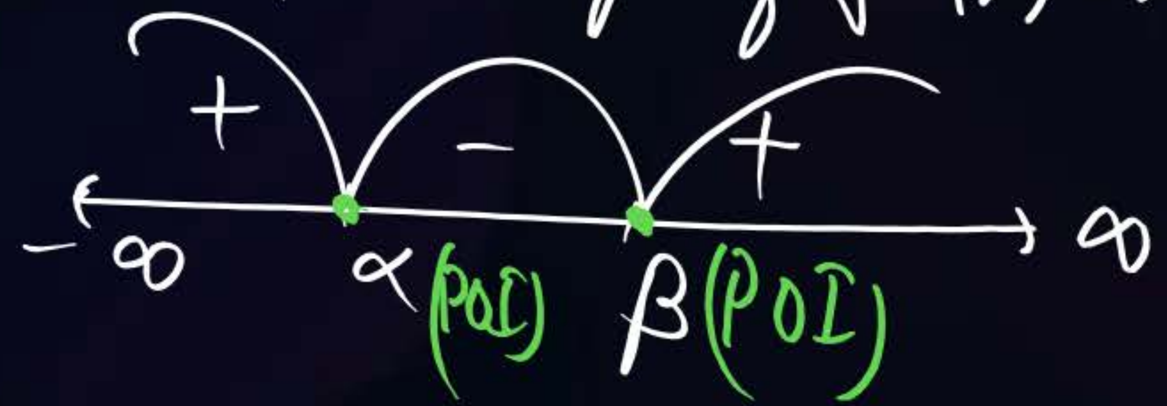
$\Rightarrow f(x)$ Inc in $(-\infty, c_1] \cup [c_2, \infty)$

$f(x)$ Dec in $[c_1, c_2]$

② To find the Interval where $f(x)$ is concave up or concave Downward $\longrightarrow$

Put $\boxed{f''(x)=0}$ & Try to find x (Say $x = \alpha, \beta$)

then check the sign of $f''(x)$ as follows;



$\Rightarrow f(x)$ is concave upward in $(-\infty, \alpha] \cup [\beta, \infty)$

$f(x)$ is concave Downward in $[\alpha, \beta]$

Saddle point / point of Inflexion →

"Those points where curve changes it's concavity are called Points of Inflexion"

Note: ① those points where we are getting Neither Maxima, Nor minima are called Points of Inflexion (F)

RECAP

② if (x = α is the point of Inflexion) $\Longrightarrow$ (At x = α we will get N.M.N.M)
$\not\Longleftarrow$

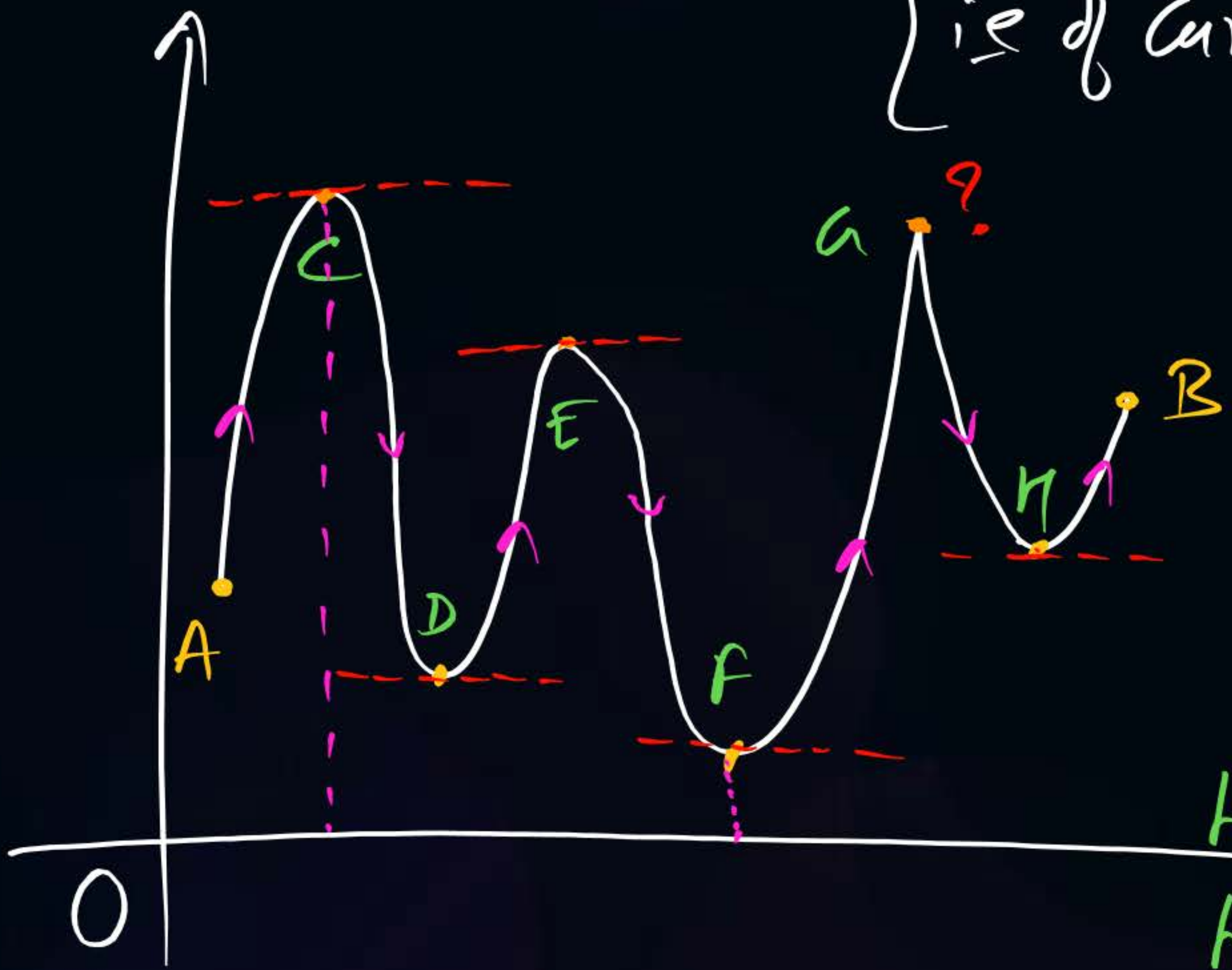
Shortcut: Put $f''(x) = 0$ & Try to get x (Say it is $x = \alpha$)

if $f'''(\alpha) \neq 0 \Rightarrow x = \alpha$ is point of Inflexion

& if $f'''(\alpha) = 0 \Rightarrow$ we can't say anything about α.

MAXIMA-MINIMA of func^n of Single Variable

[i.e. of curve $y = f(x)$.] → 12^{th} class.



Point ≃ x

Value ≃ y

Local Max points → C, E, G, B

Local Max Values → $f(C), f(E), f(G), f(B)$

Absolute Maxima / Global → $f(C)$

Local min Points → A, D, F, H

Local Min Values → $f(A), f(D), f(F), f(H)$

Absolute Minima / Global: $f(F)$

① Necessary Condition for Maxima-Minima → $\boxed{f'(x) = 0 \text{ or Not Defined}}$ ①

② Stationary Points: eg C, D, E, F, H

those points where tangent is horizontal are called stationary points.

i.e at stationary points, $f'(x) = 0$

③ Turning Points/critical Points → eg C, D, E, F, H, G

those points where tangent is either horizontal or Not defined are called T. Points. i.e at T. Points, $f'(x) = 0$ or undefined

→ All the stationary points are Turning points But converse is not necessarily True

→ By solving eqn ① we can find Turning Points. (T)

(4) Extreme Points / Optimal Points → eg $A, \underbrace{C, D, E, F, G, H}_{\text{T. Points}}, B$

those points where we get either Maxima or Minima are called E-Points

→ Corner Points are also Extreme points (True)

But take Care; Corner points are not critical Points.

(5) Points of Inflexion / Saddle points →

those points where Curve Changes it's Concavity are Called Saddle points

(6) Maxima & Minima occurs alternately (True)

(7) Sufficient Cond^n for Maxima-Minima → $f(x)$ must satisfy either 1^{st} Derivative test or 2^{nd} Derivative test.

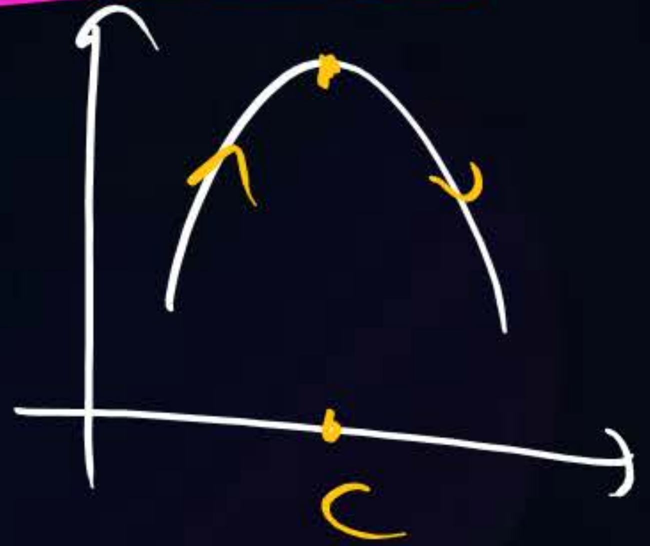
(8) 1^{st} Derivative test :→ Let $x = c$ is the Turning point.

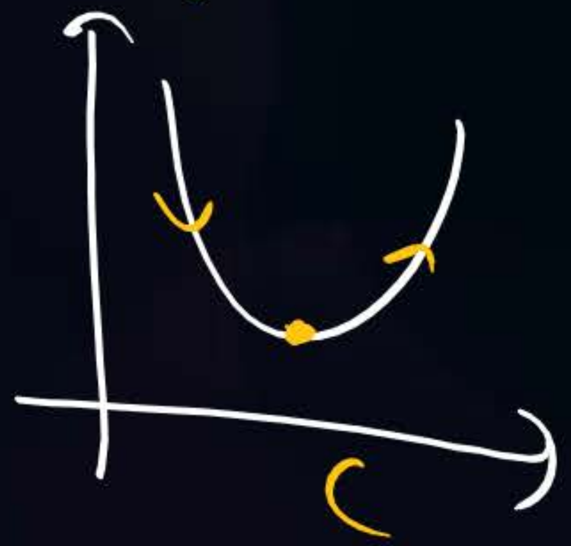
(it is obtained by solving equ^n (1)) i.e $f'(c) = 0$ or undefined.

Now we will check the sign of $f'(c)$ as follows, M.Imp.

(Maxima) (minima) (NNNM) (NMNM)

Explanation :-





DEVELOPING NEURONS →

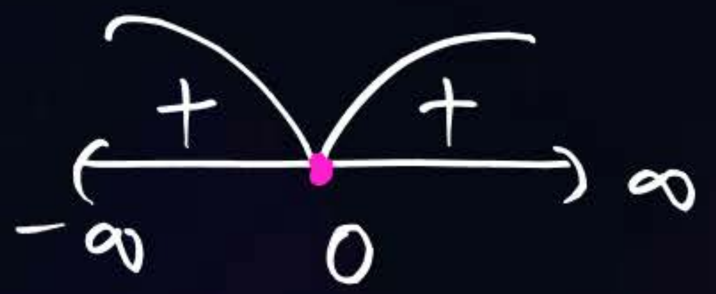
eg $y=f(x)=x^3$, $\boxed{f'(x)=3x^2}$, $f''(x)=6x$, $f'''(x)=6$

Putting $f'(x)=0$

$3x^2=0$

$x=0$

So $\boxed{\text{T-Point is } x=0}$



NMNM.

Putting $f''(x)=0$

$6x=0$

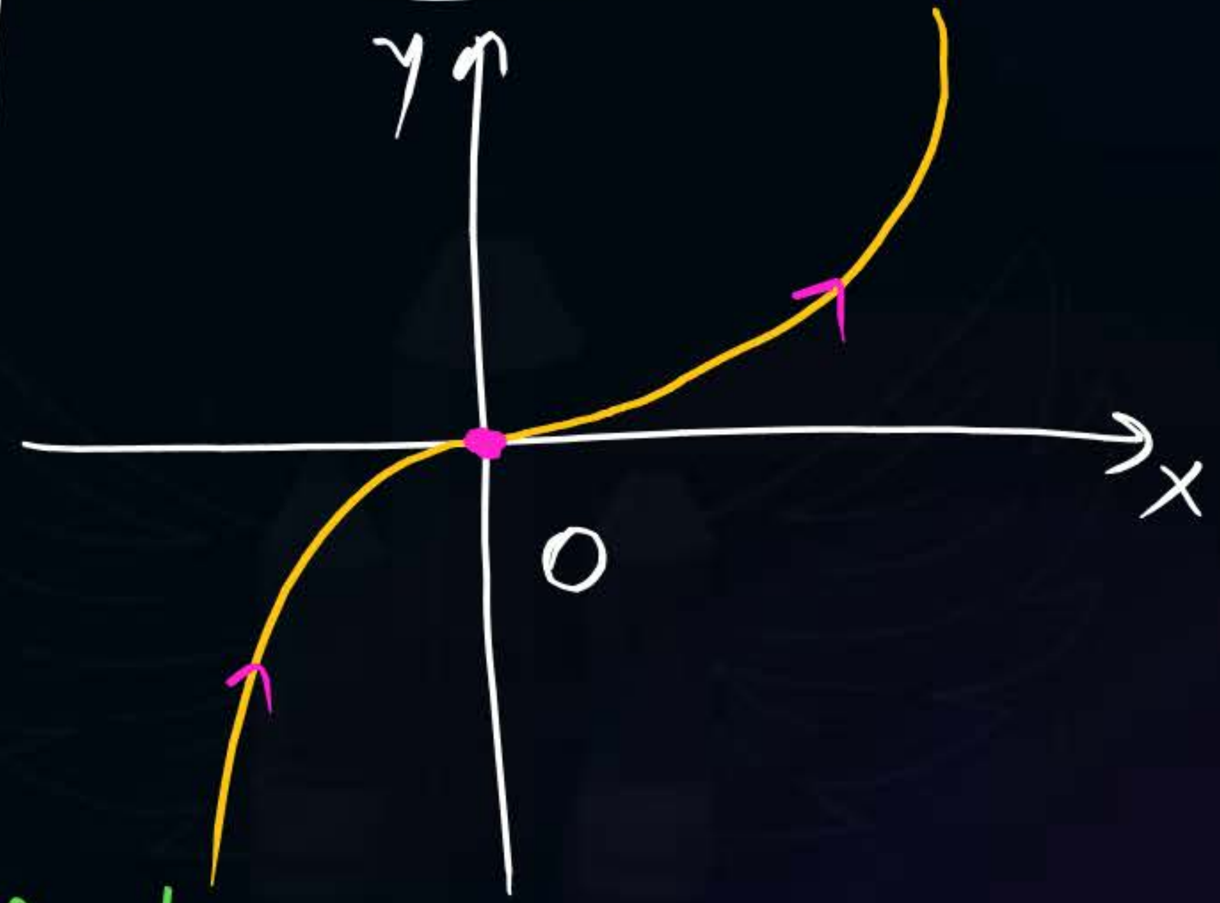
$x=0$

So $f'''(0)=[6]_{x=0}=6$

i.e. $f'''(0)\neq 0$

∴ $\boxed{x=0 \text{ is P.O.I}}$

M-II $\quad y=x^3$



(*) Inflexion Points may or may not be a Turning Points.

(9) 2nd Derivative test → Let $x = c$ is critical point then

$$\boxed{f'(c) = 0}$$

Note: if $f^{n}(c) = 0$ & $f^{n+1}(c) \neq 0$;
$n =$ even then
$x = c$ is point of Inflexion

$f''(c) > 0$

($x = c$ is Point of Minima)

$\Downarrow$

$f(x)$ is minimum at $x = c$

$f''(c) = 0$

$\downarrow$

$f'''(c) \neq 0$

($x = c$ is saddle point)

$f''(c) < 0$

($x = c$ is Point of Maxima)

$\Downarrow$

$f(x)$ is Maximum at $x = c$

(10) If we want to check the Nature of Corner points, then we have No Shortcut Method 🙁

it's Nature can be Calculated by using following theorem only

"$\boxed{\text{Maxima \& Minima occurs Alternately}}$"

for eg let $f(x)$ is defined in $[a,b]$ & it's T.Points are $x = c_1, c_2, c_3$

Case I:

min. Max min Max min

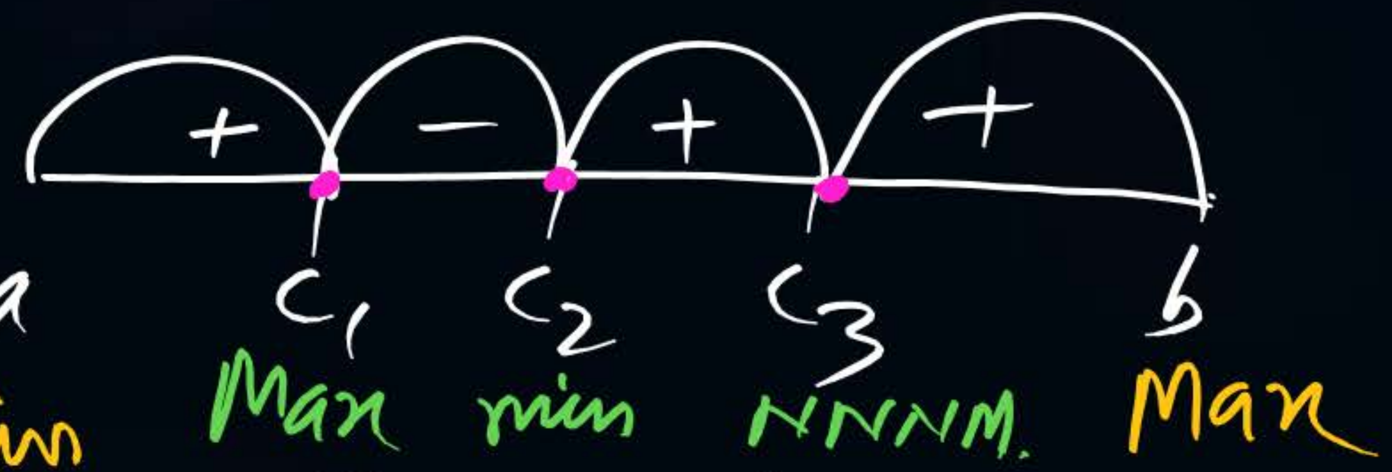
Sign of $f'(x)$

Case II:

Max. (min) Max min Max

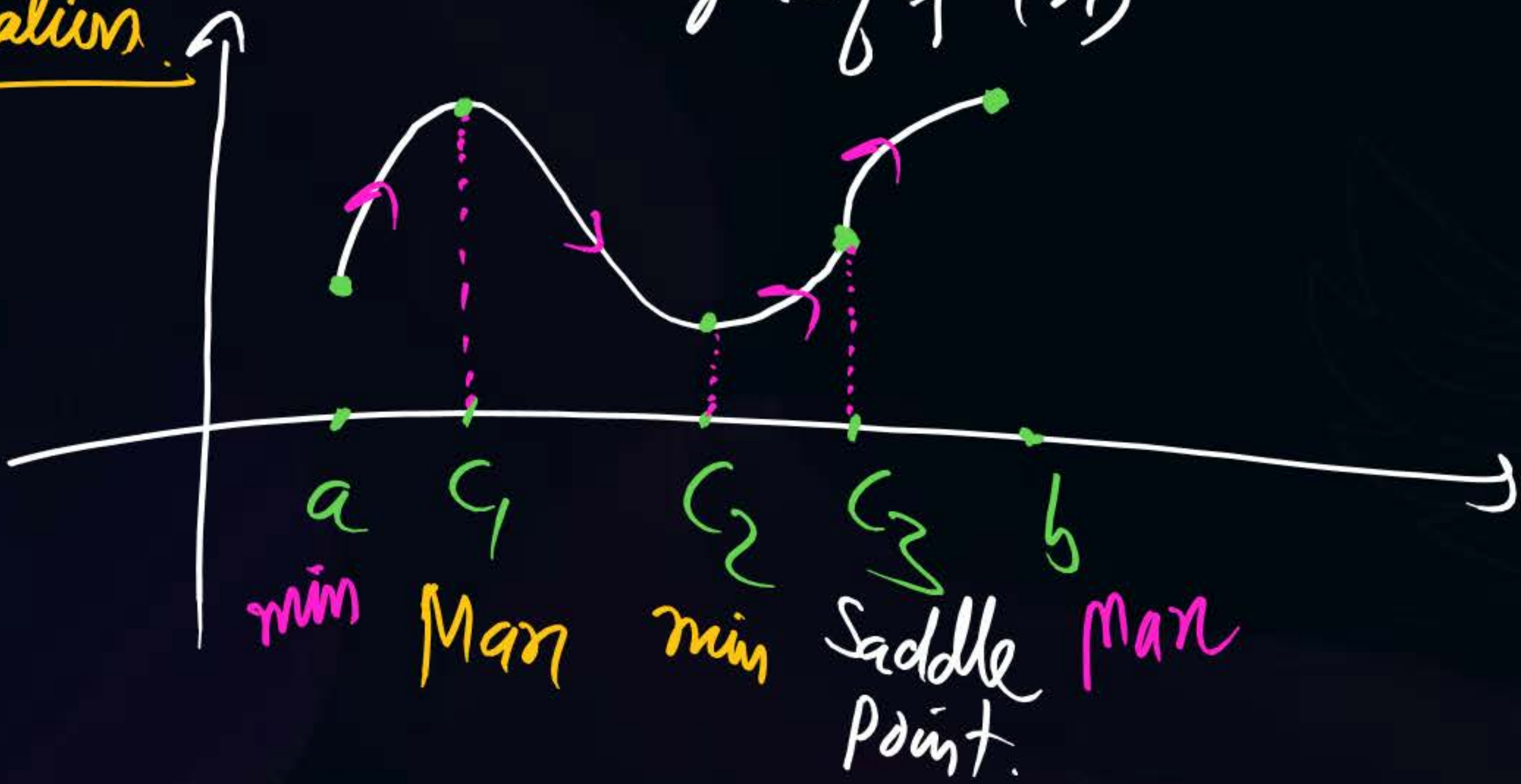
sign of $f'(x)$

Case III :



Sign of $f'(x)$

Explanation :



(1) An n^{th} degree polynomial Bends at Most $(n-1)$ times

So it has at Most $(n-1)$ extremas (Maxima & Minima both)

eg $y=(x^2-4)^2$ (it is polynomial of degree 4)

$\Rightarrow$ it has three extremas (one Local Maxima & two L. Minima)

eg: A cubic polynomial with Real Coefficients has

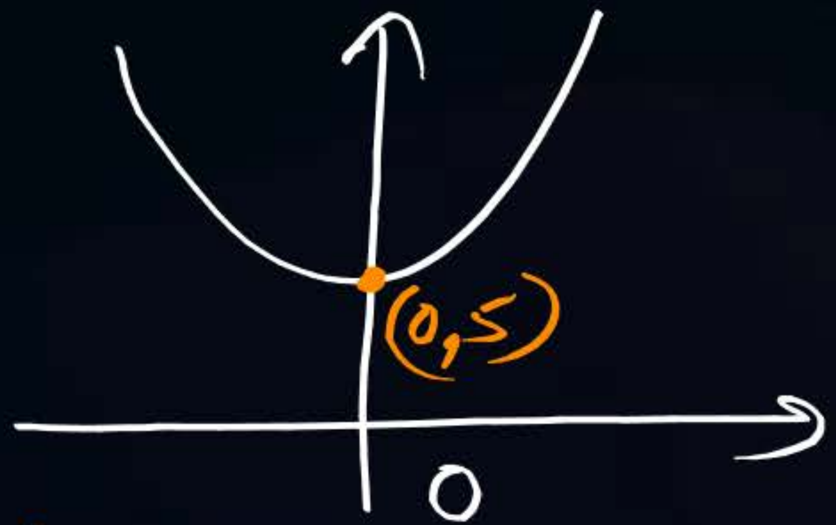
Gate at Most two extremas & three zero crossings.

(Roots)

Questions Based on type I :- (Common sense Based Questions)

Q1:- find Maxima & Minima of (1) $y = x^2 + 5$ (2) $y = \sin 3x + 4$, (3) $y = |x+3|$

Sol (1) $y = x^2 + 5$, $(-\infty, \infty)$



Point of Minima is $x = 0$
Minimum Value is $y = 5$
Max. Value = DNE

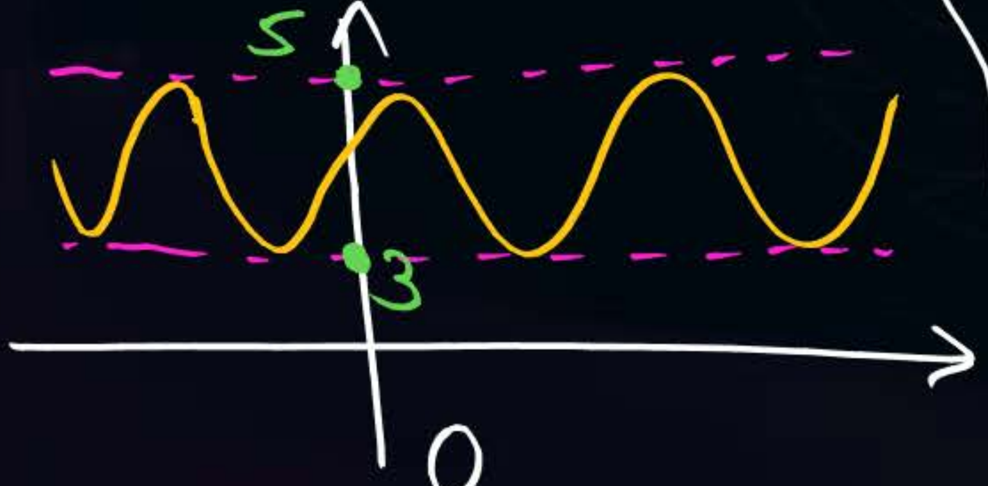
(2) $y = \sin 3x + 4$

$-1 \le \sin 3x \le 1$

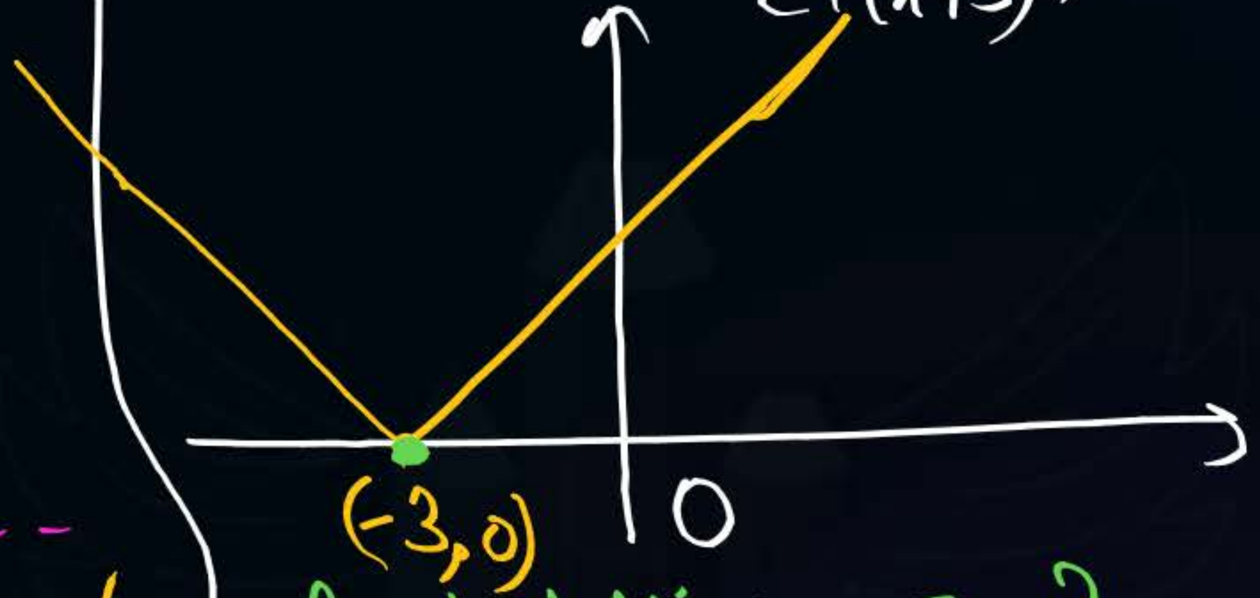
$3 \le (\sin 3x + 4) \le 5$

$3 \le y \le 5$

i.e. Min. Value = 3
Max Value = 5



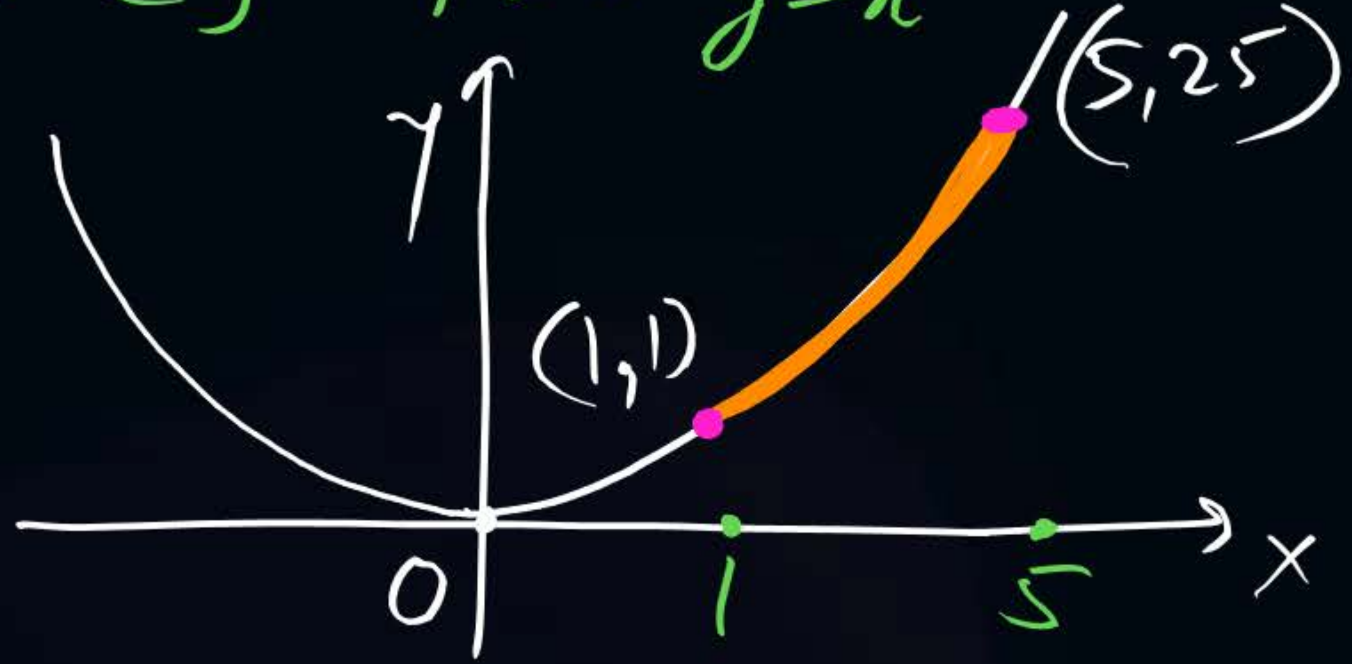
(3) $y = |x+3| = \begin{cases} -(x+3), & x < -3 \\ +(x+3), & x > -3 \end{cases}$



Point of Minima = -3
Min. Value = 0
Max Value = DNE

GATE Max Value of $f(x)=x^2$ in $[1,5]$ will be $\underline{?} = 25$

M-I (Using Graph) $\rightarrow$ $y=x^2$



Min Value $= f(1) = 1$ & it occurs at $x=1$

Max Value $= f(5) = 25$ & " " at $x=5$

M-II (W/O Graph) $\rightarrow$

$f(x)=x^2$, $[1,5]$

$\boxed{f'(x)=2x}$ $\rightarrow$ $f'(0^-)=-ve$, $f(0^+)=+ve$

T-Point is $f'(x)=0$

$2x=0$

$x=0$

But $x=0$ is not in given Domain So it is Total WASTE of TIME

Now, $f(x)=x^2$, $[1,5]$

$f'(x)=2x$

$\because f'(x)>0 \ \forall x\in[1,5]$

So $f(x)$ is S-Inc in $[1,5]$

Hence Min & Max Value will occur at $x=1$ & $x=5$ resp.

Min Value $= f(1)=1$

Max Value $= f(5)=25$

Q. Find Min & Max Values of $f(x)=x^2$ in $(1,5)$

(a) 1, 5 resp (b) 1, 25 resp

(c) 0, 25 resp (d) DNE, DNE resp.

$\because$ Corner points are not there in the given Domain.

So Min Value occurs in the Right Nbd of 1 which is not defined.

& Max Value will occur in the left Nbd of 5 which is also not defined.

For real values of x, the minimum value of th
function f(x) = exp(x) + exp(– x) is

(a) 2 (b) 1

(c) 0.5 (d) 0

we know that if $x \in R^+$ then

$$\boxed{x+\frac{1}{x} \geq 2}$$ learn.

eg $1+\frac{1}{1}=2$

eg $2+\frac{1}{2}>2$

eg $3+\frac{1}{3}>2$

eg $0.2+\frac{1}{0.2}>2$

eg $0.5+\frac{1}{0.5}>2$

eg $1.5+\frac{1}{1.5}>2$

....

$f(x)=e^{x}+e^{-x}$

$f(x)=e^{x}+\frac{1}{e^{x}}$ $(\because e^{x} \in R^{+})$

= No + it's Reciprocal

$f(x) \geq 2$

so Min $f(x)=2$

& Max $f(x)$ = DNE

& Point of Minima is $x=0$

Doubt : Also find Point of Minima? for $f(x) = e^x + e^{-x}$

Sol: $f(x) = e^x + e^{-x}$

$f'(x) = e^x - e^{-x}$

Put $f'(x) = 0$

$e^x - e^{-x} = 0$

$e^x = e^{-x}$

$e^x = \frac{1}{e^x}$

$e^{2x} = 1$

$\log_e(e^{2x}) = \log_e 1$

$2x(1) = 0$

$x = 0$

Q. $f(x)=x^3-9x^2+24x+5$, defined in $[1,6]$ then minimum and maximum values are respectively

TYPE II

(a) 21, 41

(b) 4, 6

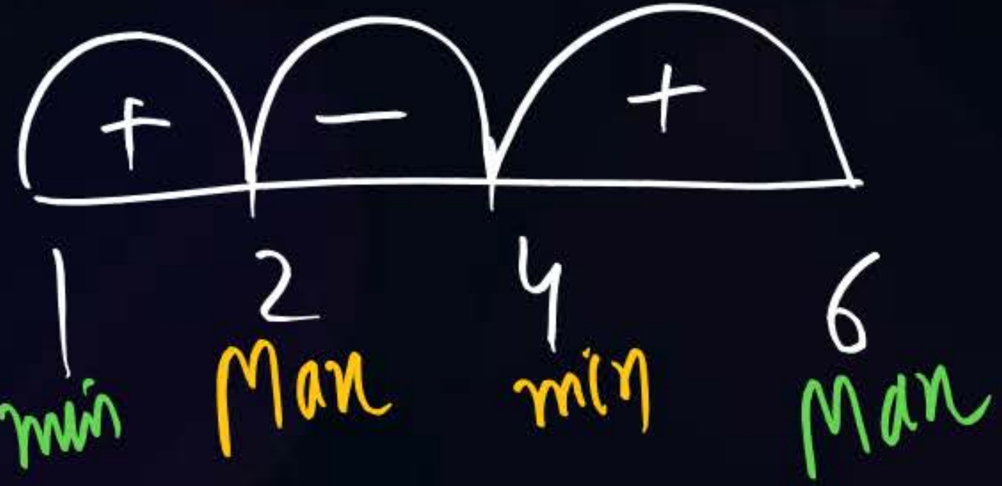
(c) 21, 25

(d) DNE, DNE

$$\boxed{f(x)=x^3-9x^2+24x+5}$$

$$f'(x)=3x^2-18x+24$$
$$=3(x^2-6x+8)$$
$$\boxed{f'(x)=3(x-4)(x-2)}$$

So T. Points are $x=2$ & 4



L. Min Points are $x=1$ & 4

L. Min Values $\begin{cases} f(1)=21 \\ f(4)=21 \end{cases}$

So Absolute Min. Value $=21$

L. Max Points are $x=2$ & 6

L. Max Values $\begin{cases} f(2)=25 \\ f(6)=41 \end{cases}$

So Absolute Max Value $=41$ & it occurs at $x=6$

Q. $f(x) = x^3 - 9x^2 + 24x + 5$ then min & Max Values are resp.

(a) 21, 41

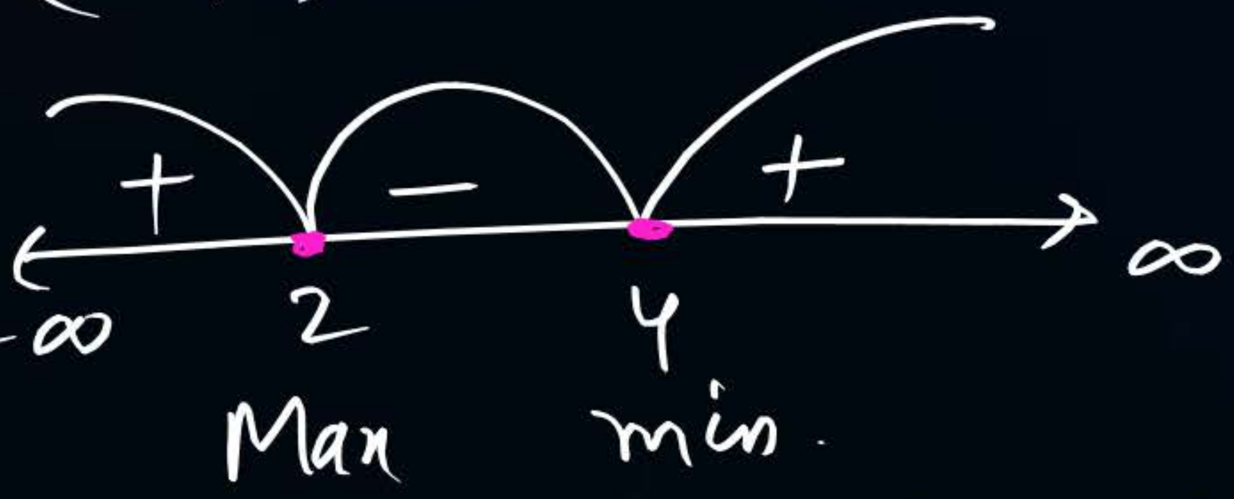
(b) 4, 6

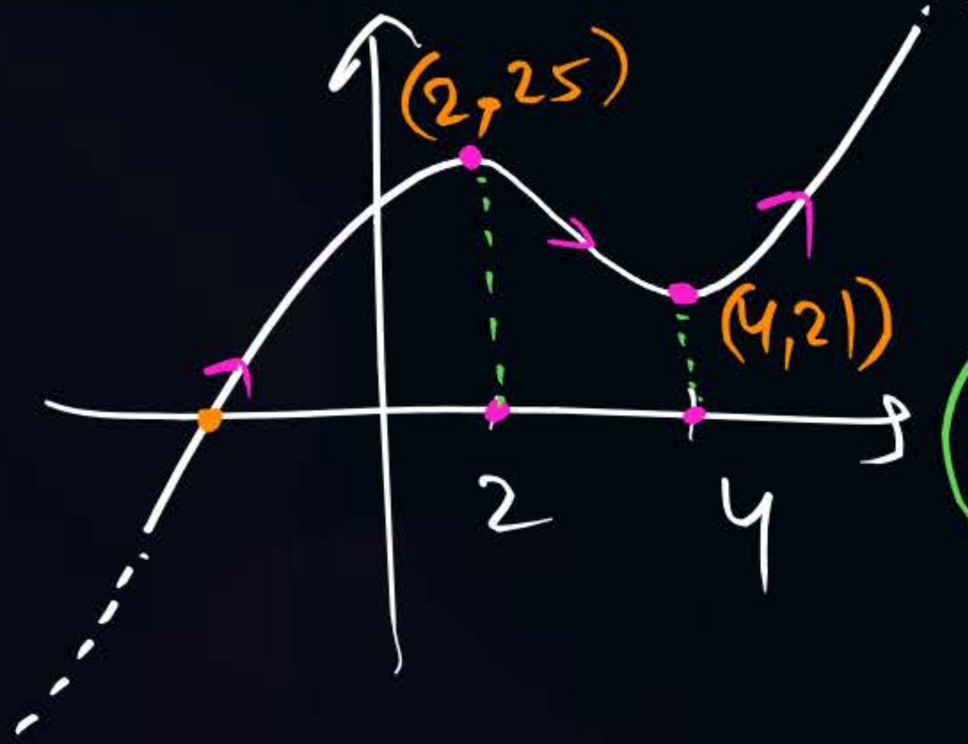
(c) 21, 25

(d) DNE, DNE ✓

$f(x) = x^3 - 9x^2 + 24x + 5$; $(-\infty, \infty)$

$$\boxed{f'(x) = 3(x-2)(x-4)}$$





Local Max Value = $f(2) = 25$

Local Min Value = $f(4) = 21$

Absolute Min Value = DNE

Absolute Max Value = DNE

Q. The Maximum slope of $y = x^3 - 9x^2 + 24x + 5$ will be? ______

HW

(a) 2

(b) 3

(c) 4

(d) DNE

Q. $f(x) = \dfrac{4x^2+1}{x}$ then find the Interval in which $f(x)$ Inc & Dec. Also find minimum & Man Value of $f(x)$ if exist?

Sol: $f(x) = \dfrac{4x^2+1}{x}$, $D_f = R - \{0\}$

$$f(x) = 4x + \frac{1}{x}$$

$$f'(x) = 4 - \frac{1}{x^2} = \frac{4x^2-1}{x^2}$$

$$f'(x) = \frac{(2x-1)(2x+1)}{x^2}$$

$f'(x) = 0$ or undefined

$\Rightarrow x = \pm\frac{1}{2}$ or $x = 0$

But $x = 0 \notin D_f$ so T. Points are $x = \pm\frac{1}{2}$

$f(x)$ Inc in $\left(-\infty, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, \infty\right)$

$f(x)$ Dec in $\left[-\frac{1}{2}, 0\right) \cup \left(0, \frac{1}{2}\right]$

Local Max Point is $x = -\frac{1}{2}$

" " Value $= f\left(-\frac{1}{2}\right) = \frac{4\left(-\frac{1}{2}\right)^2+1}{-1/2} = \frac{1+1}{-1/2} = -4$

Global Maxima = DNE

Local Min Point is $x = \frac{1}{2}$

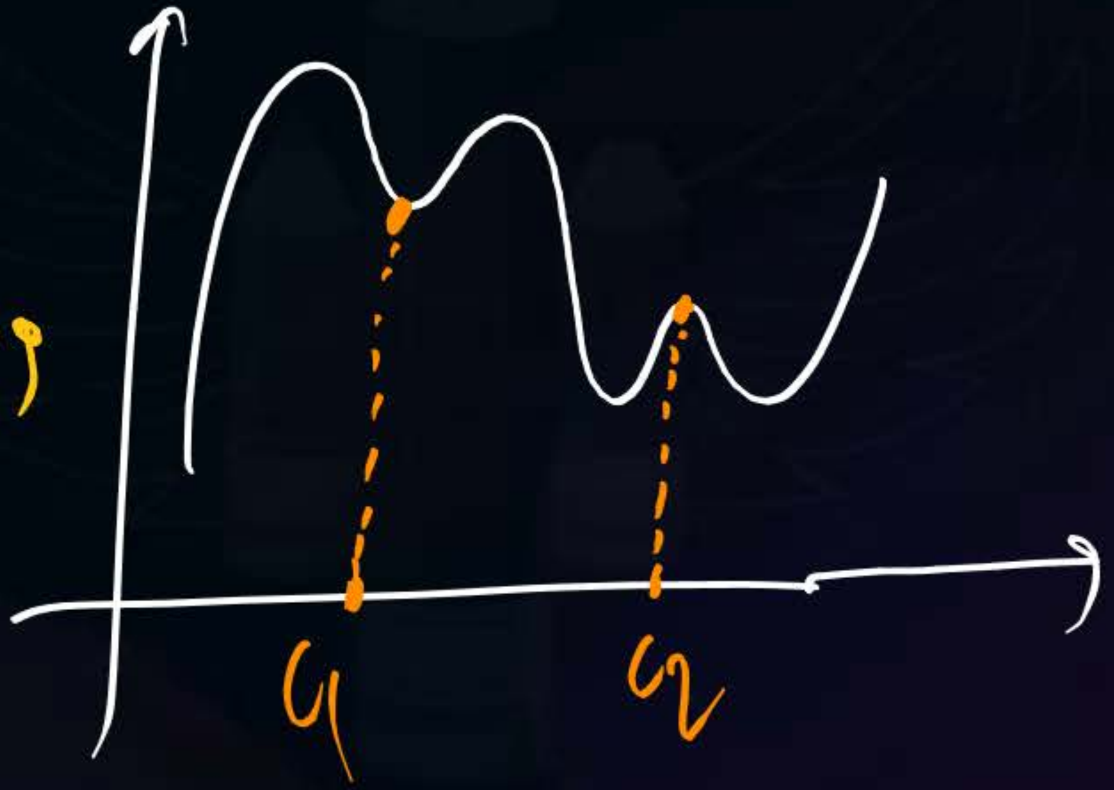
" " Value $= f\left(\frac{1}{2}\right) = \frac{4\left(\frac{1}{2}\right)^2+1}{1/2} = \frac{1+1}{1/2} = 4$

Global Minima is = DNE

GAZAB KA Conclusion →

"Local Minimum Value may be gretar that Local Max Value"

Here $f(c_1) > f(c_2)$ Hence Verifeed



Note: $(1)^{1/3} = 1, \omega, \omega^2$ & $(-1)^{1/3} = -1, -\omega, -\omega^2$

Q. The Minimum Value of $y = (x-1)^{2/3}$ occurs at $x = 1$ & Min Value is $f(1) = 0$

Sol: $\boxed{f(x) = (x-1)^{2/3}}$, $D_f = (-\infty, \infty)$, $f(0) = (0-1)^{2/3} = \left[(-1)^2\right]^{\frac{1}{3}} = (1)^{1/3} = 1$ exist

$f'(x) = \frac{2}{3}(x-1)^{\frac{2}{3}-1} = \boxed{\frac{2}{3(x-1)^{1/3}}}$

$f'(0) = \frac{2}{3(-1)^{1/3}} = \frac{2}{3(-1)} = -ve$

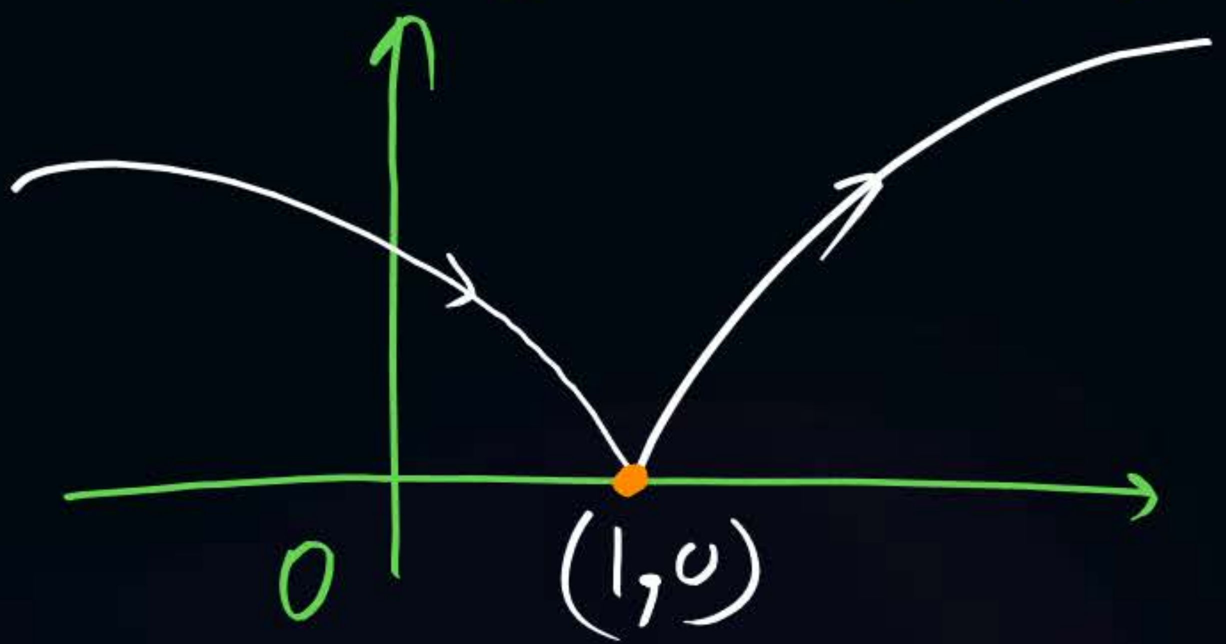
$f'(2) = \frac{2}{3(1)^{1/3}} = \frac{2}{3(1)} = +ve$

T-Point: $f'(x) = 0$ or $f'(x) = N.D$

$\Downarrow$ No Value of x exist

$\Downarrow$ $x = 1$ $\because f'(1) = DNE$

Graph: $y=(x-1)^{2/3}$, $(-\infty,\infty)$



PODCAST: $f(x)=(x-1)^{2/3}$

(1) $f'(x)=\dfrac{2}{3(x-1)^{1/3}}$, $f''(x)=\dfrac{-2}{9(x-1)^{4/3}}$

$\because f''(x)<0$ always

$\Rightarrow$ $f(x)$ is Concave Downward Curve at every point in the Domain of $f(x)$

(2) Solve above Question by using 2nd Derivative test.

Not possible to solve because $f''(1)$ = DNE.

Q. $f(x)=\dfrac{e^{\sin x}}{e^{\cos x}}$, $x \in R$ then Max Value of $f(x)$ is ______

HW

(a) $\frac{3\pi}{4}$

(b) $e^{-\sqrt{2}}$

(c) $e^{\sqrt{2}}$ ✓

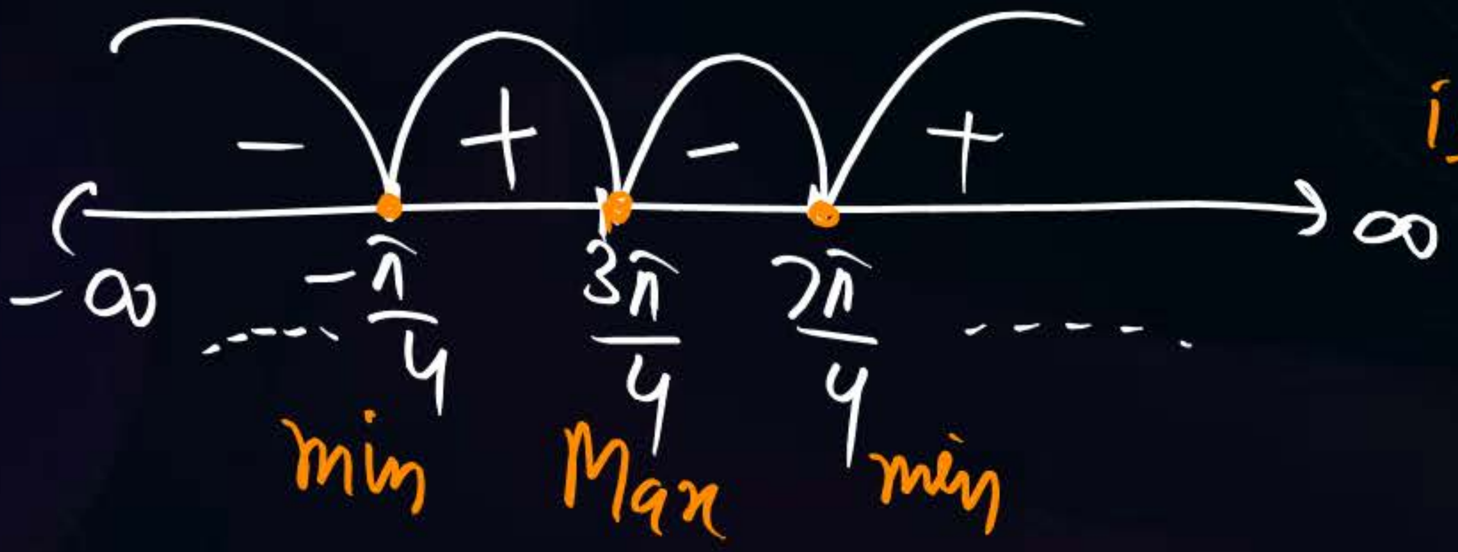
(d) -1

$$f(x)=e^{\sin x-\cos x} \Rightarrow \boxed{f'(x)=e^{\sin x-\cos x}(\cos x+\sin x)}$$

T-Points; $f'(x)=0 \Rightarrow e^{\sin x-\cos x}(\cos x+\sin x)=0$

$\because e^{\sin x-\cos x} \neq 0$ so $\cos x+\sin x=0 \Rightarrow \tan x=-1$

$x = \ldots\ldots -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}, \ldots\ldots$



is $x=\frac{3\pi}{4}$ is Point of Maxima

& Max Value $= f\left(\frac{3\pi}{4}\right) = \ldots\ldots = e^{\sqrt{2}}$

M-II T. Points are $x = \dots\dots -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}, \dots\dots$

$f''\left(-\frac{\pi}{4}\right) = \dots\dots = +ve$ & $f''\left(\frac{3\pi}{4}\right) = \dots\dots = -ve$

$x = -\frac{\pi}{4}$ is Point of minima & $x = \frac{3\pi}{4}$ is Point of Maxima

Min Value $= f\left(-\frac{\pi}{4}\right) = \left(e^{\sin x - \cos x}\right)_{x=-\frac{\pi}{4}}$

$= e^{-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}}$

$= e^{-\frac{2}{\sqrt{2}}} = e^{-\sqrt{2}}$

Max Value $= f\left(\frac{3\pi}{4}\right) = \left(e^{\sin x - \cos x}\right)_{x=\frac{3\pi}{4}}$

$= e^{\frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right)}$

$= e^{\frac{2}{\sqrt{2}}} = e^{\sqrt{2}}$

Q: $f(x)=(x-1)^3(x+1)^2$ then Number of Extremas will be ?

(a) 1

(b) 2

(c) 3

(d) 5

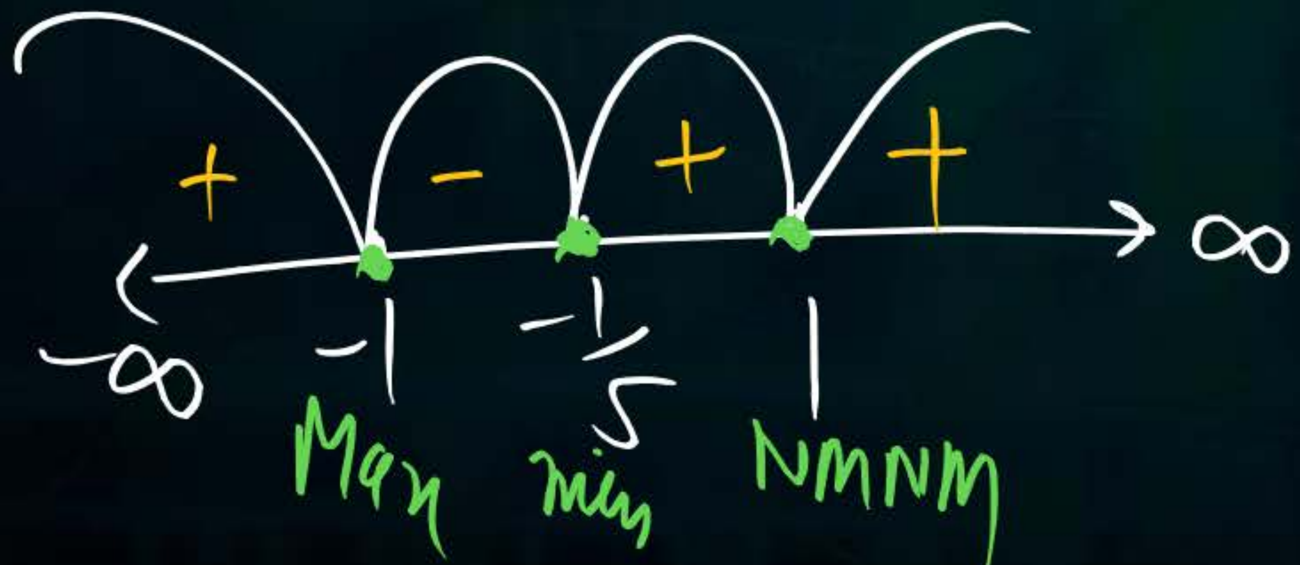
(e) 4

$$f'(x)=(x-1)^3\{2(x+1)\}+(x+1)^2\{3(x-1)^2\}$$

$$=(x+1)(x-1)^2\left[2(x-1)+3(x+1)\right]$$

$$\boxed{f'(x)=(x+1)(x-1)^2\left[5x+1\right]}$$

T-Points are $x=-1,-\frac{1}{5},1$



$f(-2)=+ve$

$f(-2/5)=-ve$

$f(0)=+ve$

$f(2)=+ve$

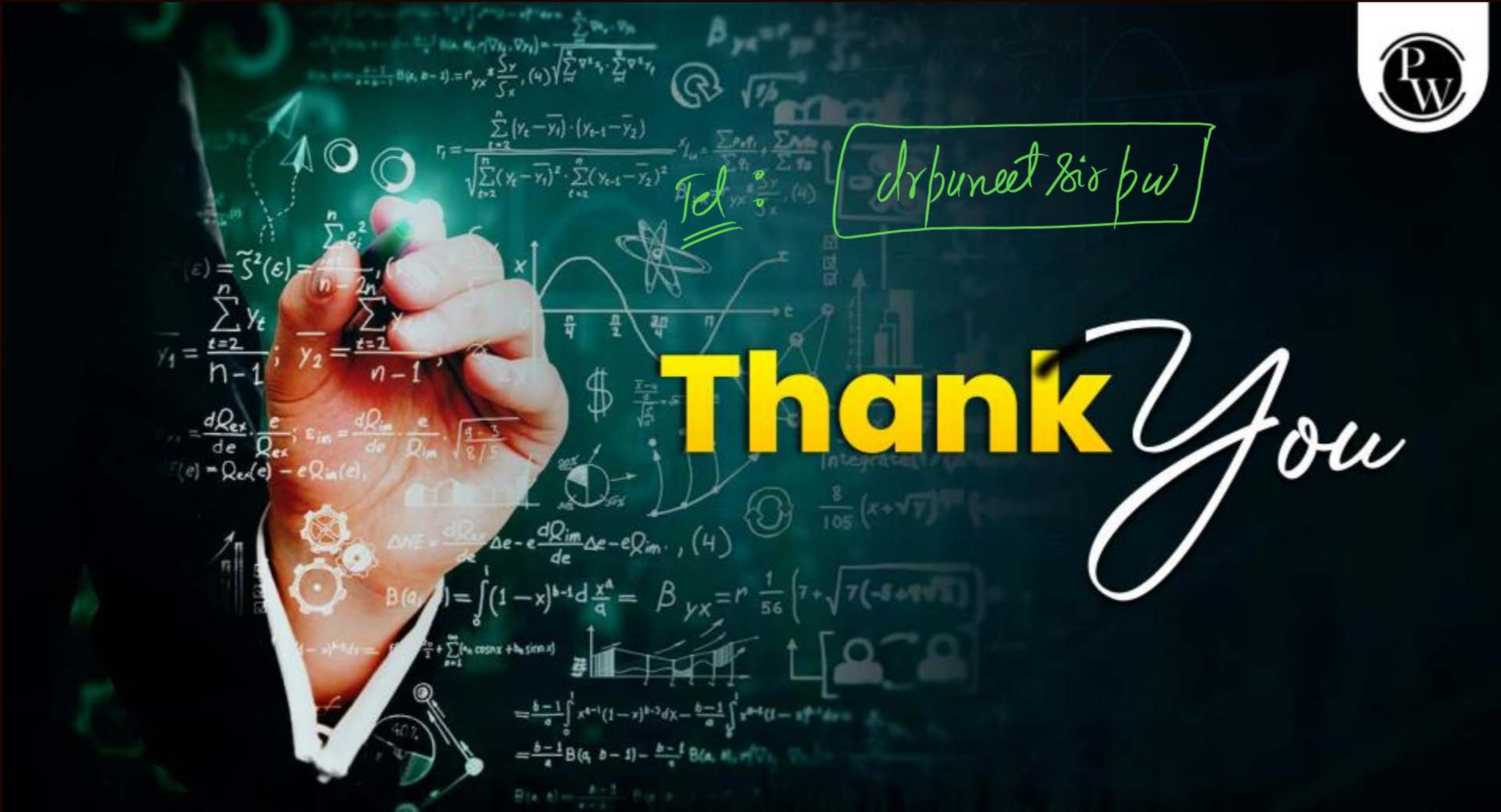
i.e. Number of Extremas = two

Qs A Right Circular Cone of Maximum Volume is to be inscribed in a sphere of Radius 1 mtr then find the height of such cone.

Type II

(Application Based Questions)

Ans $h = \frac{4}{3}$


Tel : drpuneet sir pw
Thank You

DATA SCIENCE
&
ARTIFICIAL INTELLIGENCE

& CS/IT

Calculus and Optimization

Lecture No. 11

By- Dr. Puneet Sharma Sir

Recap of previous lecture

Topic Maxima - Minima (Part 2)

Topics to be Covered

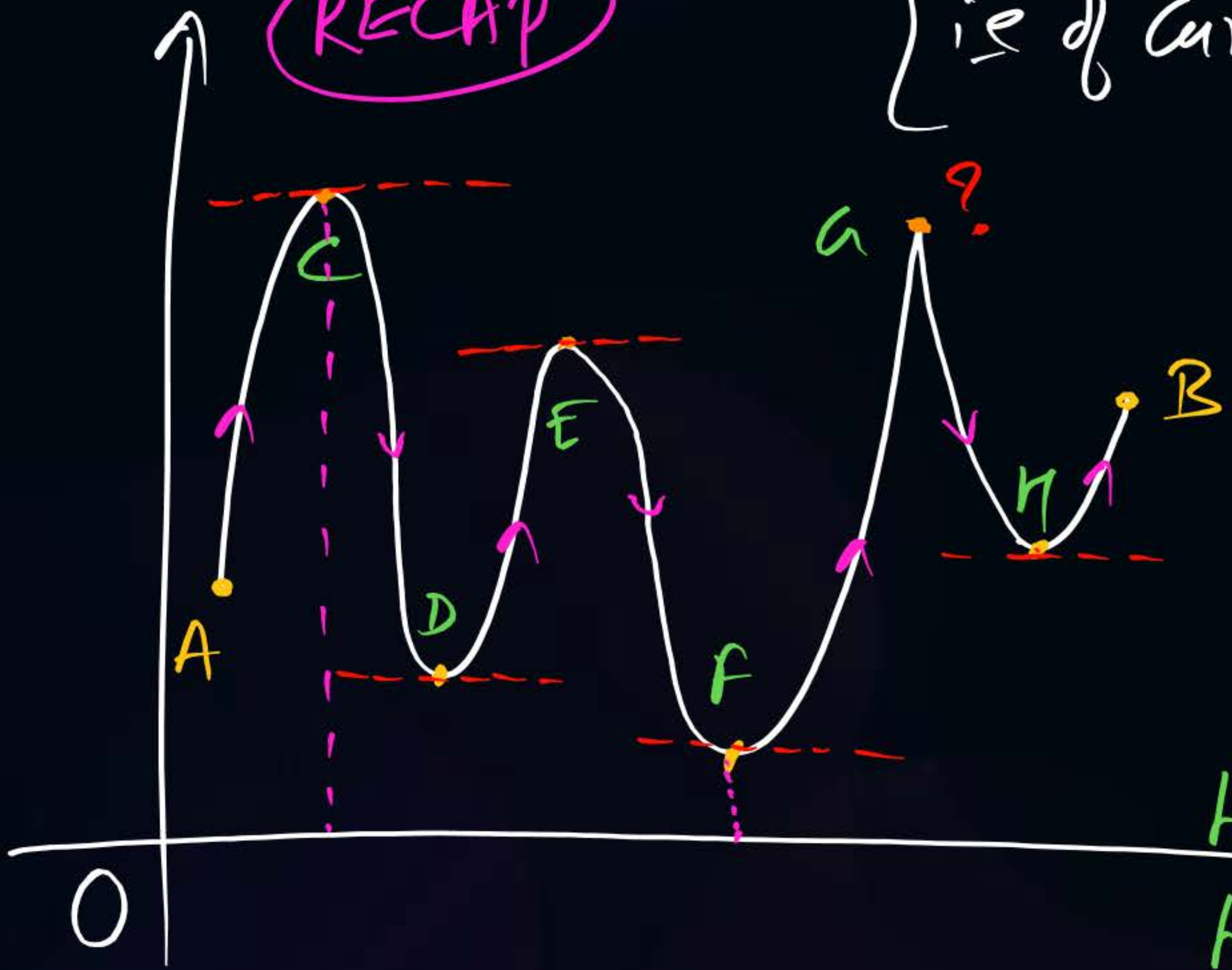
Topic

① Maxima-Minima of Surface
② Application Based Questions of Max-Minima
③ Integration (Part 1)

MAXIMA - MINIMA of func^n of Single Variable

[i.e of curve $y = f(x)$.] → 12^{th} class.

RECAP



Point ≃ x

Value ≃ y

Local Max points → C, E, G, B

Local Max Values → $f(C), f(E), f(G), f(B)$

Absolute Maxima /Global → $f(C)$

Local min Points → A, D, F, H

Local Min Values → $f(A), f(D), f(F), f(H)$

Absolute Minima /Global: $f(F)$

Q: The Maximum slope of $y = x^3 - 9x^2 + 24x + 5$ will be ?

HW

(a) 2

(b) -3

(c) 4

(d) DNE

Sol: $f(x) = x^3 - 9x^2 + 24x + 5$ — (1)

$f'(x) = 3x^2 - 18x + 24$ — (2)

so we will try to find the Max value of $f'(x)$

let $f'(x) = g(x) = 3x^2 - 18x + 24$

$g'(x) = 6x - 18 = \boxed{6(x-3)}$

T. Point of $g(x)$ is $x = 3$

i.e. At $x = 3$, we will get minima

& Min Value $= g(3) = (3x^2 - 18x + 24)_{x=3}$

i.e. Min slope of $f(x) = -3$ Ans

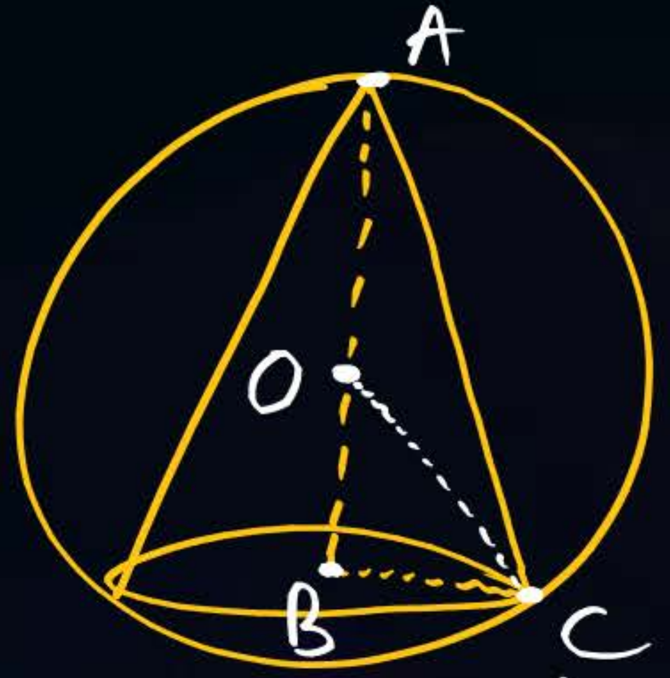
& Max Value of $g(x) =$ DNE

i.e. Max slope of $f(x) =$ DNE

Ans

Qs (Type II) A Right Circular Cone of Maximum Volume is to be inscribed in a sphere of Radius 1 mtr then find the Height of such Cone.

(Application Based Questions)



given, $OC = OA = 1$ mtr

let $AB = h$

$BC = r$

In ΔOBC,

$OC^2 = OB^2 + BC^2$

$1^2 = (h-1)^2 + r^2$

$V = \frac{1}{3}\pi r^2 h$ $\{$i.e. $V = f(r, h)$ surface$\}$

$V = \frac{\pi}{3}\left[1-(h-1)^2\right]\cdot h$

$= \frac{\pi}{3}\left[1-(h^2+1-2h)\right]h$

$V = \frac{\pi}{3}\left[2h^2 - h^3\right]$ $\{$Now $V = f(h)$ curve$\}$

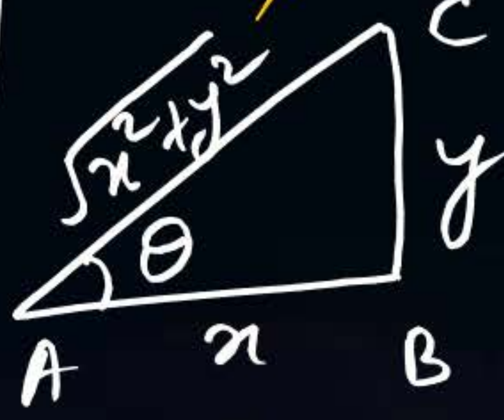
for Max. Volume, $\frac{dV}{dh} = 0 \; \cdots\cdots \Rightarrow h = 0, \frac{4}{3}$

Neglect $h=0$ so Ans is $h = 4/3$.

Q. In a Right angle Δ, if sum of the Hypotenuse & one side is kept constant, in order to find Maximum area then find the angle b/n them ?

2015

(a) 30°
(b) 60°
(c) 45°
(d) 90°



$\theta = ?$

ATQ: $x+\sqrt{x^2+y^2}=K$

$x^2+y^2=(K-x)^2$

$x^2+y^2=K^2+x^2-2Kx$

$y^2=K^2-2Kx$ —— (1)

$A=\frac{1}{2}xy$ $\{A=f(x,y)$ surface$\}$

$A^2=\frac{1}{4}x^2y^2=\frac{1}{4}x^2\left[K^2-2Kx\right]$

Let $A^2=U=\frac{K^2}{4}x^2-\frac{K}{2}x^3$ $\{$Now $U=f(x)$ curve$\}$

where $A=\sqrt{U} \Rightarrow A_{max}=\sqrt{U_{max}}=?$

Now for Max U, $\frac{dU}{dx}=0$

$$U = \frac{K^2}{4}x^2 - \frac{K}{2}x^3 \Rightarrow \frac{du}{dx} = \frac{K^2}{2}x - \frac{3K}{2}x^2$$

for Max U, $\frac{du}{dx} = 0 \Rightarrow \frac{K^2}{2}x - \frac{3K}{2}x^2 = 0$

$$\frac{Kx}{2}(K - 3x) = 0 \Rightarrow x = \frac{K}{3}$$

Now $y^2 = K^2 - 2Kx = K^2 - 2K\left(\frac{K}{3}\right) = \frac{3K^2 - 2K^2}{3} = \frac{K^2}{3} \Rightarrow y = \frac{K}{\sqrt{3}}$

In $\triangle ABC$, $\tan\theta = \frac{y}{x} = \frac{K/\sqrt{3}}{K/3} = \sqrt{3} \Rightarrow \theta = 60^\circ$

Note: $(\sin\theta, \cos\theta, \tan\theta) = \frac{P.\ B.\ P}{H\ H\ B}$

A political party orders an arch for the entrance to the ground in which the annual convention is being held. The profile of the arch follows the equation y $y = 2x - 0.1x^2$ where y is the height of the arch in meters. The maximum possible height of the arch is

$y_{max} = ?$

(a) 8 meters
~~(b)~~ 10 meters
(c) 12 meters
(d) 14 meters

$y = f(x) = 2x - 0.1x^2$ —— (1)

$f'(x) = 2 - 0.2x = 0.2[10 - x]$

T. Point is $x = 10$

so Max Value $= f(10)$

$= [2x - 0.1x^2]_{x=10}$

$= 20 - 10 = 10$ mtrs

Max-Min of func^n of two Variables (i.e. of $z = f(x,y)$).

Point $\simeq (x, y)$

Value $\simeq z$

N. Cond^n for Max-Min: $\rightarrow \dfrac{\partial z}{\partial x} = 0 \quad \rightarrow (1)$, $\dfrac{\partial z}{\partial y} = 0 \quad \rightarrow (2)$

Stationary Point / Critical Point $\rightarrow$ Points obtained by solving (1) & (2).

for eg after solving we get $x = a$ & $y = b$ then $P(a, b)$ is critical Point.

Sufficient Cond^n for Max-Min $\rightarrow$ function must satisfy Lagrange's cond^n.

Note :- Let $P(a,b)$ is critical point then we can use following symbols.

$$\left(\frac{\partial^2 z}{\partial x^2}\right)_{P(a,b)} = r \quad , \left(\frac{\partial^2 z}{\partial y^2}\right)_{P(a,b)} = t \quad , \left(\frac{\partial^2 z}{\partial x \partial y}\right)_{P(a,b)} = s$$

Lagrange's Conditions :-

(1) if $rt - s^2 > 0$ & $r > 0$ then $P(a,b)$ is point of minima

(2) if $rt - s^2 > 0$ & $r < 0$ " " " " " Maxima

(3) if $rt - s^2 < 0$ then " " " " Inflexion

(4) if $rt - s^2 = 0$ then case fails i.e L-conditions are unable to give an idea about Maxima or minima & we need further investigation.

Q. $\boxed{z=f(x,y)}$ s.t $f_x(a,b)=0$, $f_y(a,b)=0$ N. Condⁿ

and also we have

$$f_{xy}^2(a,b)-f_{xx}(a,b)f_{yy}(a,b)<0 \ \& \ f_{xx}(a,b)<0$$

S. Condⁿ

then $P(a,b)$ is Point of ________

(a) Minima

(b) Maxima

(c) Inflexion

(d) Data Insufficient

$$s^2-rt<0 \ \& \ r<0$$

or $rt-s^2>0$ & $r<0$

So $P(a,b)$ is Point of Maxima

Given a function $f(x, y) = 4x^2 + 6y^2 - 8x - 4y + 8$. The optimum value of $f(x, y)$ is

(a) a minimum equal to 10/3

(b) a maximum equal to 10/3

(c) a minimum equal to 8/3

(d) a maximum equai to 8/3

$$Z = 4x^2 + 6y^2 - 8x - 4y + 8$$

$$Z_x = 8x - 8, \quad Z_{xx} = 8$$

$$Z_y = 12y - 4, \quad Z_{yy} = 12$$

$$Z_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial Z}{\partial y}\right) = \frac{\partial}{\partial x}(12y - 4) = 0$$

Critical Points: Put $Z_x = 0$ & $Z_y = 0$

$8x - 8 = 0$ & $12y - 4 = 0$

$x = 1$ & $y = 1/3$

So $P(1, \frac{1}{3})$ is C.Point.

Here $r = (Z_{xx})_P = 8$, $t = (Z_{yy})_P = 12$,

$s = (Z_{xy})_P = 0$

$\therefore rt - s^2 = (8)(12) - (0)^2 = 96 > 0$ & $r > 0$

So $P(1, \frac{1}{3})$ is Point of Minima.

$$\text{Min. Value} = (f(x,y))_P = (4x^2 - 6y^2 - 8x - 4y + 8)_{P(1, \frac{1}{3})}$$

$$= 10/3$$

Find the absolute maxima and minima of the function respectively

$f(x, y) = x^2 - xy - y^2 - 6x + 2$ on the rectangular plate $0 \le x \le 5$, $-3 \le y \le 0$

(a) 7, −3

(b) 3, −7

(c) −3, 7

(d) DNE, DNE

$$z = x^2 - xy - y^2 - 6x + 2 \quad \text{---} (1)$$

$$z_x = 2x - y - 6, \quad z_{xx} = 2$$

$$z_y = -x - 2y, \quad z_{yy} = -2$$

$$z_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}(-x-2y) = -1$$

for critical points, $z_x = 0$ & $z_y = 0$

$2x - y = 6$ & $x + 2y = 0$

After solving we get $x = \frac{+12}{5}$ & $y = \frac{-6}{5}$

So stationary point is $P\left(\frac{+12}{5}, \frac{-6}{5}\right)$.

$r = 2, \ t = -2, \ s = -1$

$\because rt - s^2 = (2)(-2) - (-1)^2 = -5 < 0$

So $P\left(\frac{12}{5}, \frac{-6}{5}\right)$ is P.O.I ie At P, NMNM occurs

How we will check Max & Minima at corner points.

$0 \le x \le 5$ & $-3 \le y \le 0$ so $P(0,-3)$, $Q(0,0)$, $R(5,-3)$, $S(5,0)$ are corner points.

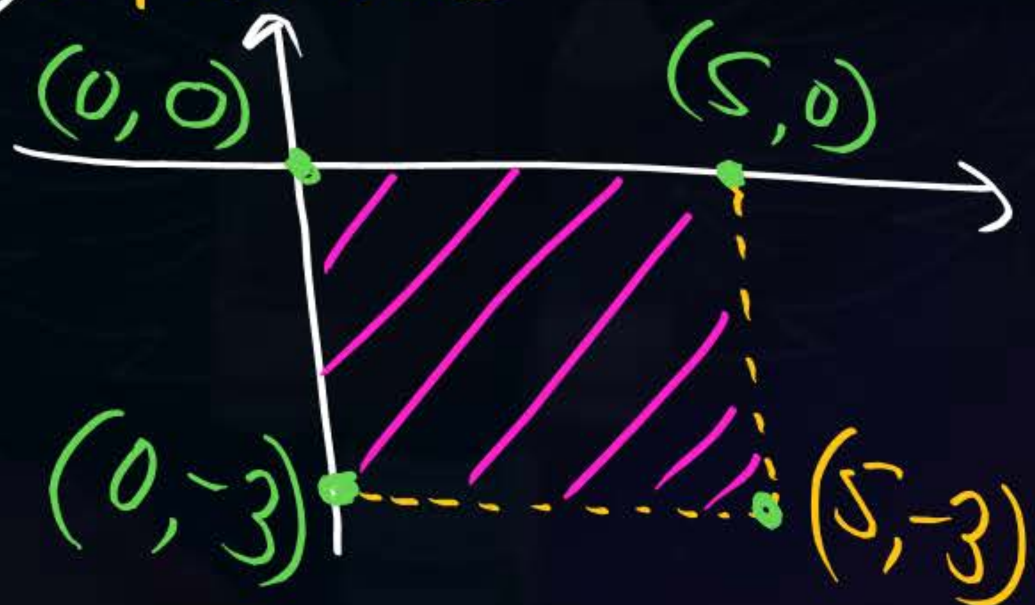
Given, $\boxed{f(x,y) = x^2 - xy - y^2 - 6x + 2}$

At P, $f(0,-3) = 0 - 0 - (-3)^2 - 0 + 2 = -7$ = min. Value

At Q, $f(0,0) = 2$

At R, $f(5,-3) = 25 - (-15) - 9 - 30 + 2 = 3$ = Max Value

At S, $f(5,0) = 25 - 0 - 0 - 30 + 2 = -3$



2025 EE

$\boxed{f(x,y) = (x+y-1)^2 + (x+y)^2}$ & $S = \{(x,y) : x \in R, y \in R\}$

where S contains set of points which minimizes $f(x,y)$ then

(a) S contains exactly one point

(b) S is empty

(c) S contains Infinitely many points

(d) S Contains finite Number of Points.

At $P(\frac{1}{2},0)$; $z = (\frac{1}{2})^2 + (\frac{1}{2})^2 = 1/2$

At $P(\frac{1}{4},\frac{1}{4})$; $z = (\frac{1}{2})^2 + (\frac{1}{2})^2 = 1/2$ & so on.....

$z_x = 2(x+y-1) + 2(x+y)$

$z_{xx} = 2+2 = 4 = r$

$z_y = 2(x+y-1) + 2(x+y)$

$z_{yy} = 2+2 = 4 = t$

$z_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = 2+2 = 4 = s$

$\because rt - s^2 = (4)(4) - (4)^2 = 0$ Case fails.

For Stationary Points:

$$\frac{\partial z}{\partial x}=0 \Rightarrow 2(x+y-1)+2(x+y)=0 \Rightarrow 4x+4y=2$$

$$\frac{\partial z}{\partial y}=0 \Rightarrow 2(x+y-1)+2(x+y)=0 \Rightarrow 4x+4y=2$$

i.e. By both the eqn, we have only one conclusion, $\boxed{x+y=\frac{1}{2}}$ — (1)

So $(x,y)=\left(\frac{1}{4},\frac{1}{4}\right),\left(\frac{3}{2},-1\right),\left(\frac{1}{8},\frac{3}{8}\right),\left(\frac{1}{2},0\right)$ ....

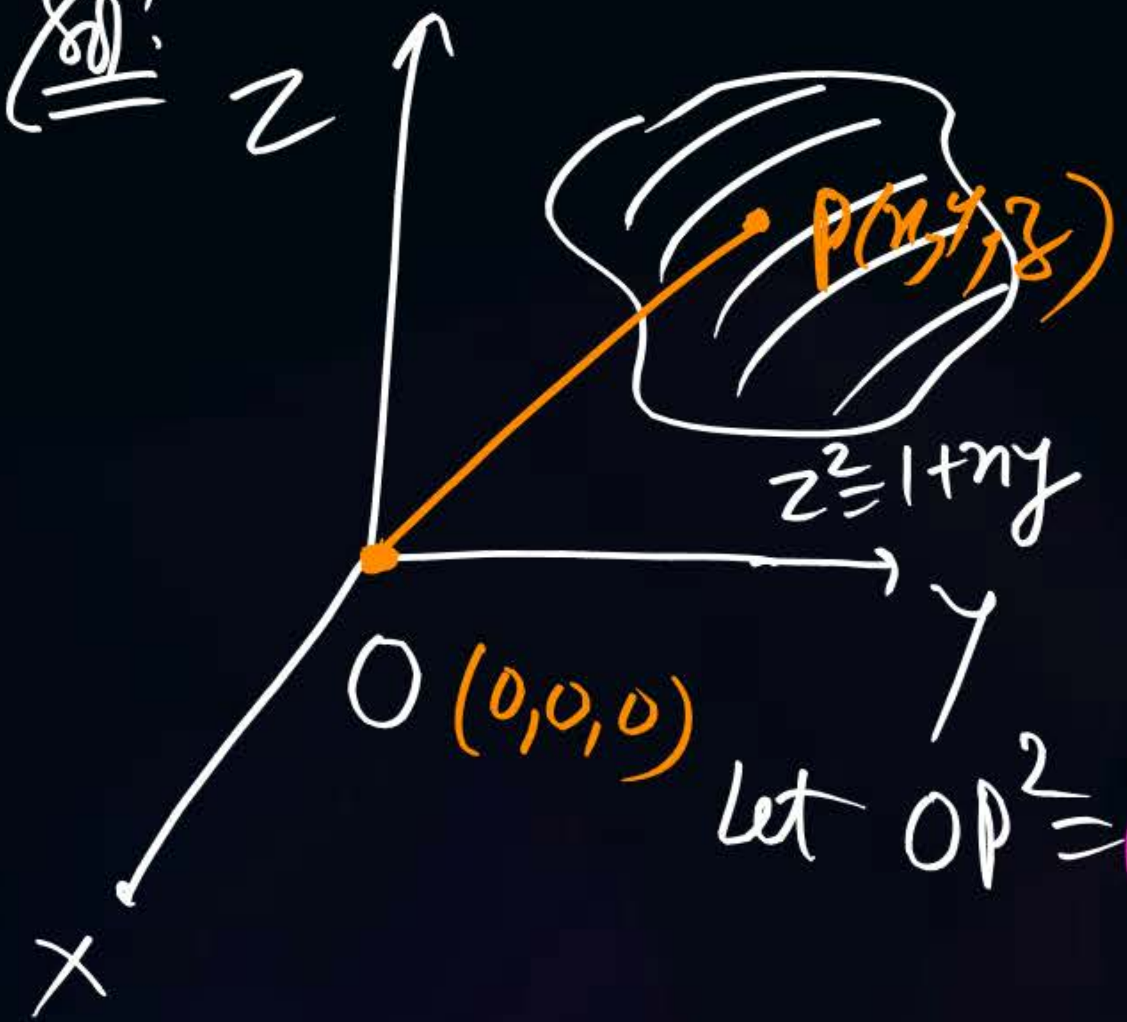
are the possible solutions of (1)

$\Rightarrow f(x,y)$ has $\underline{\infty}$ Number of Stationary Points.

S: Contains ∞ Points

Q: The minimum distance of any point on the surface $\boxed{z^2 = 1+xy}$ from origin will be ? ______

Sol:



Let $P(x,y,z)$ be any random point on this surface

$OP = \sqrt{x^2+y^2+z^2}$ $\quad \{ \because OP = f(x,y,z) \}$

$OP = \sqrt{x^2+y^2+1+xy}$ $\quad \{ \text{Now } OP = f(x,y) \}$

Let $OP^2 = \boxed{u = x^2+y^2+1+xy}$ —— (1) where $OP = \sqrt{u}$

& $OP_{min} = \sqrt{u_{min}} = ?$

$$u = x^2 + y^2 + 1 + xy \quad \text{--- (1)}$$

For S. Points: $\frac{\partial u}{\partial x} = 0$, $\frac{\partial u}{\partial y} = 0$ $\Rightarrow$ $x = 0$, $y = 0$.

i.e. $P(0,0)$ is Stationary Point.

Now $r = 2,\ t = 2,\ s = 1$

$\because rt - s^2 > 0$ & $r > 0$

So $P(0,0)$ is Point of Minima

So $u_{min} = \left(x^2 + y^2 + 1 + xy\right)_{P(0,0)}$

$= (0 + 0 + 1 + 0) = 1$

So $OP_{min} = \sqrt{u_{min}} = \sqrt{1} = 1$

INTEGRATION

$y = \sin(e^{x^2})$ → $\dfrac{dy}{dx} = \cos(e^{x^2}) \cdot e^{x^2} \cdot (2x)$

→ $\int \sin(e^{x^2})\,dx = \; ? \; =$ Not possible.

i.e Diff is easy just because of the presence of Chain Rule.

& Int is tough bcoz Chain Rule DNE.

$\boxed{\text{Area} = \iint (1)\,dx\,dy} \simeq \int y\,dx = \int_{x=a}^{b} f(x)\,dx =$ Area under $f(x)$ b/n $x=a$ & $x=b$

Volume $= \iiint (1)\,dx\,dy\,dz \simeq \iint z\,dx\,dy = \iint f(x,y)\,dx\,dy$

Mass $= \iiint \rho \cdot dx\,dy\,dz \quad \because D = \dfrac{M}{V}$

(3) Linear Algebra, Prob & Stats, Calculus → Diff, → Integration

→ Single Int, Multiple Int, $\int_0^\infty \frac{\sin x}{x}\,dx = ?$ (Laplace Transf), $\int_0^\infty e^{-x^2}\,dx = ?$ (β-γ func^n), $\int(\text{Complex func}^n)\,dz = ?$ (C-I Th / C-R Th)

$\int(\text{vector})\,d\vec{r}$ (G-Div / G-Th / S Th), C.R.V., $P(a<x<b) = \int_a^b f(x)\,dx$ (Prob & Stats), $\frac{d^2y}{dx^2} + y = 0$ (D Eq) → (twice Integrate)

$\int_1^5 (?)\,dx = ?$

N.Tech (T Rule / Simpson's)

For CS & DA :-

(1) L. Algebra
(2) Calculus
(3) Prob & Stats

INDEFINITE INT → Collection of all the Antiderivatives in Called Indefinite Int.

Primitives

if $\frac{d}{dx}\left(f(x)+c\right) = \varphi(x) \implies \boxed{\int \varphi(x)\,dx = f(x)+c}$

Antiderivative

Derivative of $f(x)$

eg $\frac{d}{dx}(x^2) = 2x$

$\frac{d}{dx}(x^2-5) = 2x$

$\frac{d}{dx}(x^2+7) = 2x$

$\frac{d}{dx}(x^2+11) = 2x$ ......

$\implies \int 2x\,dx = x^2+c$

Methods of Solving Indefinite Integration →

Methods

1. using Standard Result.
2. using substitution (M. Imp).
3. using Integration by part
4. using partial Fraction

Standard Results :

(1) Power formula: $\boxed{\int x^a dx = \frac{x^{a+1}}{a+1}, a \neq -1}$

(2) $\int \frac{1}{x} dx = \log x + C$

(3) $\int a^x dx = \frac{a^x}{\log_e a} + C$

(4) $\int e^x dx = e^x + C$

(5) $\int \sin x \, dx = -\cos x + C$

(6) $\int \cos x \, dx = \sin x + C$

(7) $\int \tan x \, dx = \log \sec x + C$

(8) $\int \cot x \, dx = \log \sin x + C$

(9) $\int \sec x \, dx = \log(\sec x + \tan x) + C$

(10) $\int (\operatorname{cosec} x) dx = \log(\operatorname{cosec} x - \cot x) + C$

(11) $\int \sec^2 x \, dx = \tan x + C$

(12) $\int \operatorname{cosec}^2 x \, dx = -\cot x + C$

(13) $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx)$

(14) $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx)$

(15) $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

(16) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right) + C$

(17) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right) + C$

(18) $\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left(x+\sqrt{x^2+a^2}\right) + C$

(19) $\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left(x+\sqrt{x^2-a^2}\right) + C$

(20) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$

(21) $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left(x+\sqrt{x^2+a^2}\right) + C$

(22) $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left(x+\sqrt{x^2-a^2}\right) + C$

(23) $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$

(24) $\int \sec x \tan x\,dx = \sec x + C$, (25) $\int (\operatorname{cosec} x \cot x)\,dx = -\operatorname{cosec} x + C$

(26) $\boxed{\int \frac{f'(x)}{f(x)}dx = \log f(x) + C}$; (27) $\boxed{\int e^x\{f(x)+f'(x)\}dx = e^x f(x) + C}$

eg $\int \frac{dx}{x} = ? = \ln x + C$

eg $\int \tan x \, dx = ?$

$= \int \frac{\sin x}{\cos x}dx = -\int \frac{(-\sin x)}{\cos x}dx$

$= -\log(\cos x) + C$

$= \log(\sec x) + C$

eg $\int e^x(\sin x + \cos x)dx = ? = e^x \cdot \sin x + C$

eg: $\int [\sin(\log x) + \cos(\log x)]dx = ?$ Ans: $x\sin(\log x) + C$

Put $\log_e x = t \Rightarrow x = e^t$

$dx = e^t dt$

$= \int (\sin t + \cos t)e^t dt$

$= \int e^t(\sin t + \cos t)dt = e^t \cdot \sin t$

$= x \cdot \sin(\log_e x)$

(*) $\boxed{\int x^a\,dx = \frac{x^{a+1}}{a+1}}$, $a \neq -1$; eg $\int k\,dx = ? = k\int x^0\,dx = k\left(\frac{x^{0+1}}{0+1}\right) = kx + c$

eg $\int (x^2)\,dx = ? = \frac{x^{2+1}}{2+1} = \frac{x^3}{3} + c$

eg $\int \left(\frac{1}{x^2}\right)dx = ?$ $= \int x^{-2}\,dx = \frac{x^{-2+1}}{-2+1} = \frac{-1}{x}$

eg $\int (\sqrt{x})\,dx = ? = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} = \frac{x^{3/2}}{3/2} = \frac{2}{3}x^{3/2}$

eg $\int \left(\frac{1}{\sqrt{x}}\right)dx = ? = \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = 2\sqrt{x}$

eg $\int (x)\,dx = ? = x^2/2$

eg $\int \left(\frac{1}{x}\right)dx = ? = \log_e x$

Q. $\int\left(\frac{e^{5\log x}-e^{4\log x}}{e^{3\log x}-e^{2\log x}}\right)dx=?$

$$=\int\left(\frac{e^{\log_e x^5}-e^{\log_e x^4}}{e^{\log_e x^3}-e^{\log_e x^2}}\right)dx$$

$$=\int\left(\frac{x^5-x^4}{x^3-x^2}\right)dx=\int\frac{x^4(x-1)}{x^2(x-1)}dx$$

$$=\int x^2dx=\frac{x^3}{3}+C$$

Q. $I=\int\frac{x^4}{x^2+1}dx=?$

$$=\int\frac{x^4-1+1}{x^2+1}dx=\int\frac{x^4-1}{x^2+1}dx+\int\frac{1}{x^2+1}dx$$

$$=\int\frac{(x^2-1)(x^2+1)}{x^2+1}dx+\tan^{-1}x$$

$$=\int(x^2-1)dx+\tan^{-1}x$$

$$=\frac{x^3}{3}-x+\tan^{-1}x+C$$

(PQ) $I=\int \frac{x^2+x+1}{(x-1)^3}dx = ?$

(M-I) $I=\int\left\{\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{(x-1)^3}\right\}dx$

= Lengthy Approach ....

Sol: Put $x-1=t \Rightarrow x=t+1$

$dx=dt$

$I=\int\left(\frac{(t+1)^2+(t+1)+1}{t^3}\right)dt$

$=\int\left(\frac{t^2+1+2t+t+1+1}{t^3}\right)dt$

$=\int\left(\frac{t^2+3t+3}{t^3}\right)dt$

$I=\int\left(\frac{1}{t}+\frac{3}{t^2}+\frac{3}{t^3}\right)dt$

$=\log t+3\left(\frac{-1}{t}\right)+3\left(\frac{t^{-3+1}}{-3+1}\right)$

$=\log(x-1)-\frac{3}{x-1}-\frac{3}{2(x-1)^2}+C$

Questions Based on (M-II) →

Q. $\int x^2 \cdot \sin(x^3)\, dx = ?$

Put $x^3 = t$

$3x^2\, dx = dt$

$x^2\, dx = dt/3$

$I = \int \sin(t) \cdot \frac{dt}{3}$

$= -\frac{1}{3} \cos t + C$

$= -\frac{1}{3} \cos(x^3) + C$

Q. $I = \int \sec^3 x \cdot \tan x\, dx = ?$

~~Put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$~~

~~So $I = \int (\sec x) \cdot t \cdot dt$~~ = Not good.

→ Put $\sec x = t \Rightarrow \sec x \tan x\, dx = dt$

$I = \int \sec^2 x \cdot \sec x \tan x\, dx$

$= \int t^2 \cdot dt = \frac{t^3}{3} = \frac{\sec^3 x}{3} + C$

Q. $I = \int \sin(\log x)\,dx = ?$

Put $\log x = t \Rightarrow x = e^t$

$dx = e^t\,dt$

$I = \int \sin t \cdot e^t\,dt$

$= \int e^t \cdot \sin t\,dt$

(By Formula 13), $a = 1$, $b = 1$

$= \frac{e^t}{1^2+1^2}\left[1 \cdot \sin t - 1 \cdot \cos t\right]$

$= \frac{x}{2}\left[\sin(\log x) - \cos(\log x)\right]$

Q. $I = \int \frac{e^x}{\sqrt{4 - e^{2x}}}\,dx = ?$

Put $e^x = t \Rightarrow e^x\,dx = dt$

$I = \int \frac{dt}{\sqrt{4 - t^2}} = \int \frac{1}{\sqrt{2^2 - t^2}}\,dt$

$= \sin^{-1}\left(\frac{t}{2}\right) + c$ (By Formula 20)

$= \sin^{-1}\left(\frac{e^x}{2}\right) + c$

(2) Integration by parts $\rightarrow$ $\boxed{\int \underset{I}{u} \cdot \underset{II}{v}\, dx = u\int v\,dx - \int\left\{\frac{du}{dx}\cdot\int v\,dx\right\}dx + c}$

Log func^n — Trig func^n

$\boxed{I \quad L \quad A \quad T \quad E}$

Inverse Trig func^n — Algebraic func^n — Exp. func^n

the letter which is coming 1st in the word ILATE should be assumed as 1st func^n 'u'

Shortcut – $\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots\cdots$

where Dash denotes diff & suffix denotes integration.

Q. $\int x\sin x\,dx = ?$

(M-I) $= x(-\cos x) - \int (1)(-\cos x)\,dx$

$= -x\cos x + \sin x + C$

(M-II) $\int x.\sin x\,dx = x(-\cos x) - 1(-\sin x) + 0$

(M-II)

Q. $\int x^3 \sin x\,dx = ?$

(M-I) use u-v formula 3 times (very lengthy)

(M-II) $\int x^3 \sin x\,dx = x^3(-\cos x) - 3x^2(-\sin x) + 6x(\cos x) - 6(\sin x) + 0$

Q $\int x^2 e^x\,dx = ? = x^2(e^x) - 2x(e^x) + 2(e^x) - 0$

Take Care: this method is applicable only when u is polynomial

Q. $I=\int (x \log x)\,dx = ? = \int (\log x)\cdot x\,dx$

$u = \log x$, $v = x$

$= \log x\left(\frac{x^2}{2}\right) - \int\left\{\frac{1}{x}\left(\frac{x^2}{2}\right)\right\}dx$

$= \frac{x^2}{2}\log x - \frac{1}{2}\left(\frac{x^2}{2}\right) + C$

Q. $I=\int (\log x)\,dx = ? = \int (\log x)\cdot 1\,dx$

$u = \log x$, $v = 1$

$= \log x\,(x) - \int\left\{\frac{1}{x}(x)\right\}dx$

$= \boxed{x\log x - x}$ Learn as Standard Result.

Q. $I = \int (\log x)^2\,dx = ?$

$= \int (\log x)^2\cdot 1\,dx$

$u = (\log x)^2$, $v = 1$

$= (\log x)^2\cdot(x) - \int\left(\frac{2\log x}{x}\right)\cdot x\,dx$

$= x(\log x)^2 - 2\left[x\log x - x\right]$

Note: $\int x^3 dx = \frac{x^4}{4} \Rightarrow \int (2x+5)^3 dx = ? = \frac{(2x+5)^4}{4} \cdot \left(\frac{1}{2}\right)$ Put $2x+5 = t$

$\int \sin x = -\cos x \Rightarrow \int \sin(5x) dx = ? = -\frac{\cos(5x)}{5}$ Put $5x = t$

Quest Based on M-IV →

Q. $I=\int \frac{\cos\theta}{(2+\sin\theta)(3+4\sin\theta)}\,d\theta=?$ $=\int \frac{1}{(2+t)(3+4t)}\,dt=\int\left(\frac{A}{2+t}+\frac{B}{3+4t}\right)dt$

$\left(\text{P.F}: A=-\frac{1}{5},\ B=\frac{4}{5}\right)$

Put $\sin\theta=t$

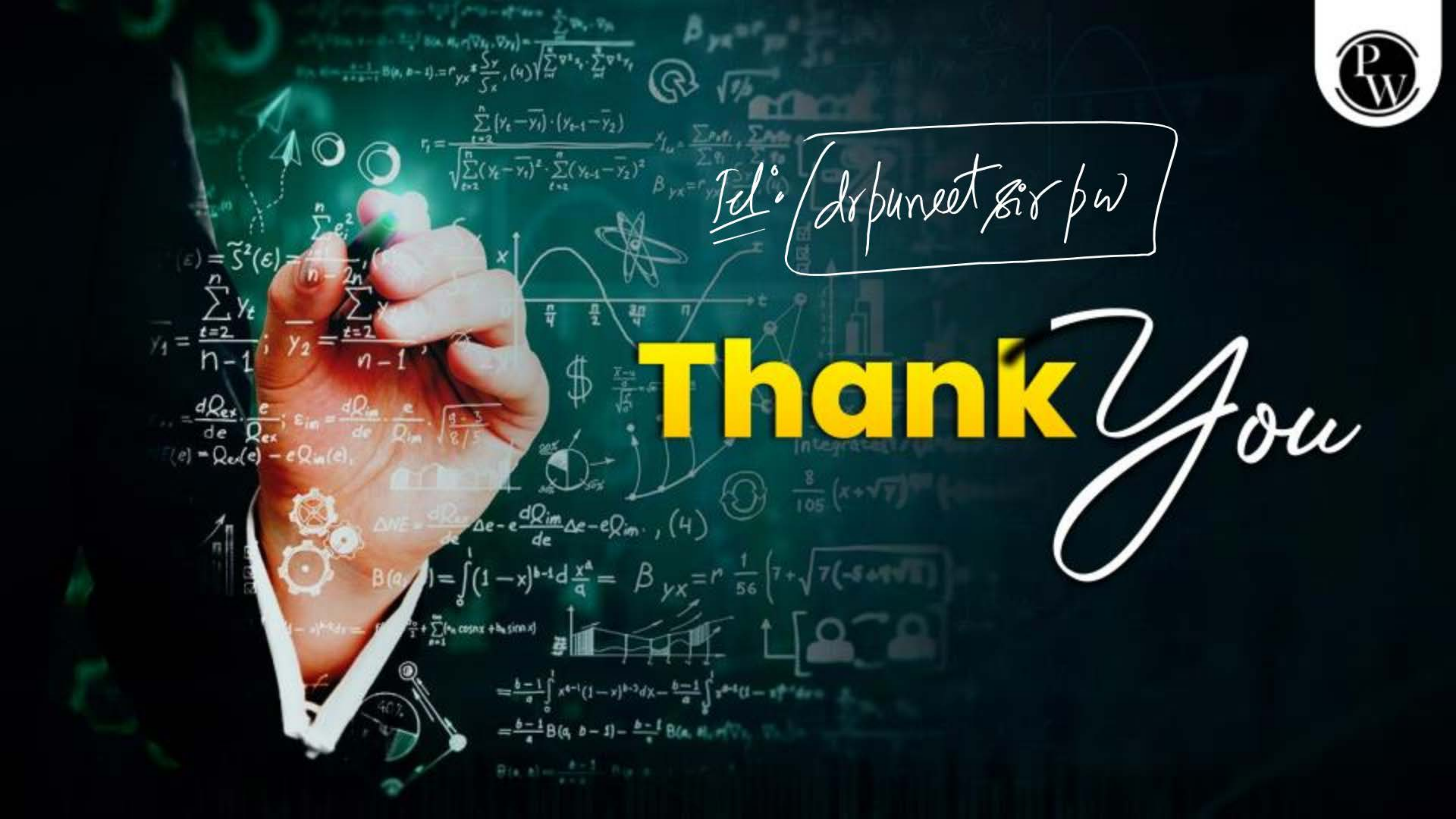
$\cos\theta\, d\theta=dt$

$=\int \frac{-1/5}{2+t}\,dt+\int\frac{+4/5}{3+4t}\,dt$

$=-\frac{1}{5}\log(2+t)+\frac{4}{5}\frac{\log(3+4t)}{4}$

$=-\frac{1}{5}\log(2+\sin\theta)+\frac{1}{5}\log(3+4\sin\theta)$

$=\frac{1}{5}\log\left(\frac{3+4\sin\theta}{2+\sin\theta}\right)$



Id: drpuneetsir pw

DATA SCIENCE
&
ARTIFICIAL INTELLIGENCE
& CS/IT

Calculus and Optimization

Lecture No. 12

By- Dr. Puneet Sharma Sir

Recap of previous lecture

Topic INTEGRATION (Part-1)

(Indefinite Integration)

Topics to be Covered

Topic

INTEGRATION (Part 2)

- → Definite Integration
- → Leibnitz Rule of Diff under Integration
- → Application of Definite Integration (Finding Length of Curve & Volume of Revolution)

Standard Results :

(1) Power formula: $\boxed{\int x^a\,dx = \frac{x^{a+1}}{a+1},\ a \neq -1}$

(2) $\int \frac{1}{x}\,dx = \log x + C$

(3) $\int a^x\,dx = \frac{a^x}{\log_e a} + C$

(4) $\int e^x\,dx = e^x + C$

(5) $\int \sin x\,dx = -\cos x + C$

(6) $\int \cos x\,dx = \sin x + C$

(7) $\int \tan x\,dx = \log \sec x + C$

(8) $\int \cot x\,dx = \log \sin x + C$

(9) $\int \sec x\,dx = \log(\sec x + \tan x) + C$

(10) $\int (\operatorname{cosec} x)\,dx = \log(\operatorname{cosec} x - \cot x) + C$

(11) $\int \sec^2 x\,dx = \tan x + C$

(12) $\int \operatorname{cosec}^2 x\,dx = -\cot x + C$

(13) $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$

(14) $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

(15) $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

(16) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right) + C$

(17) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right) + C$

(18) $\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left(x+\sqrt{x^2+a^2}\right) + C$

(19) $\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left(x+\sqrt{x^2-a^2}\right) + C$

(20) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$

(21) $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left(x+\sqrt{x^2+a^2}\right) + C$

(22) $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left(x+\sqrt{x^2-a^2}\right) + C$

(23) $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$

(24) $\int \sec x \tan x\,dx = \sec x + C$, (25) $\int (\operatorname{cosec} x \cot x)\,dx = -\operatorname{cosec} x + C$

DEFINITE INTEGRAL

Fundamental Theorem of Integral Calculus →

if $\frac{d}{dx}(f(x)+c) = \phi(x)$

then $\int \phi(x)\,dx = f(x) + c$

F.Th: $\int_a^b \phi(x)\,dx = \Big[f(x)+c\Big]_{x=a}^{x=b}$

$= (f(b)+c) - (f(a)+c)$

$= \boxed{f(b) - f(a)}$

Q. $I = \int_0^{\pi/4} \tan^2(x)\,dx = ?$

$= \int_0^{\pi/4} (\sec^2 x - 1)\,dx$

$= (\tan x - x)_0^{\pi/4}$

$= (1-0) - \left(\frac{\pi}{4} - 0\right) = 1 - \frac{\pi}{4}$

Q. $I = \int_0^{\pi/2} \sqrt{\cos\theta}\,\sin^3\theta\, d\theta = ?$ (a) 21 (b) 8/3 (c) $\frac{8}{21}$ (d) 21/8

Put $\cos\theta = t$ $\to$ At $\theta = 0,\ t = 1$; At $\theta = \frac{\pi}{2},\ t = 0$

$\sin\theta\, d\theta = -dt$

$$I = \int_0^{\pi/2} \sqrt{\cos\theta}\cdot \sin^2\theta \cdot \sin\theta\, d\theta$$

$$= \int_1^0 \sqrt{t}\cdot(1-t^2)\,(-dt)$$

$$= \int_0^1 \left(t^{1/2} - t^{5/2}\right) dt = \left(\frac{t^{3/2}}{3/2} - \frac{t^{7/2}}{7/2}\right)_0^1 = \frac{8}{21} \quad (c)$$

Properties of Definite Integral →

① $$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$$

② $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

③ $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

where $a < c < b$

④ $$\int_a^b f(x)\,dx = \int_a^b f(\underbrace{a+b-x})\,dx$$

⑤ $$\int_0^a f(x)\,dx = \int_0^a f(\underbrace{a-x})\,dx$$

Property (6) →

$$\int_0^{2a} f(x)\,dx = \boxed{\int_0^{a}\{f(x)+f(2a-x)\}\,dx} = \begin{cases} 2\int_0^{a} f(x)\,dx, & \text{if } f(2a-x)=f(x) \\ 0 \;; & \text{if } f(2a-x)=-f(x) \end{cases}$$

Property (7) →

$$\int_{-a}^{a} f(x)\,dx = \boxed{\int_0^{a}\{f(x)+f(-x)\}\,dx} = \begin{cases} 2\int_0^{a} f(x)\,dx, & \text{if } f(-x)=f(x) \text{ i.e. } f(x) \text{ is Even} \\ 0 & \text{, if } f(-x)=-f(x) \text{ i.e. } f(x) \text{ is odd} \end{cases}$$

Prop (1) $I=\int_1^3 x\,dx = ? = \left(\frac{x^2}{2}\right)_1^3 = \frac{9}{2}-\frac{1}{2} = 4$

$I=\int_1^3 t\,dt = ? = \left(\frac{t^2}{2}\right)_1^3 = \frac{9}{2}-\frac{1}{2} = 4$

Prop (2) $I=\int_3^1 x\,dx = ? = \left(\frac{x^2}{2}\right)_3^1$

$= \frac{1}{2}-\frac{9}{2} = -4$

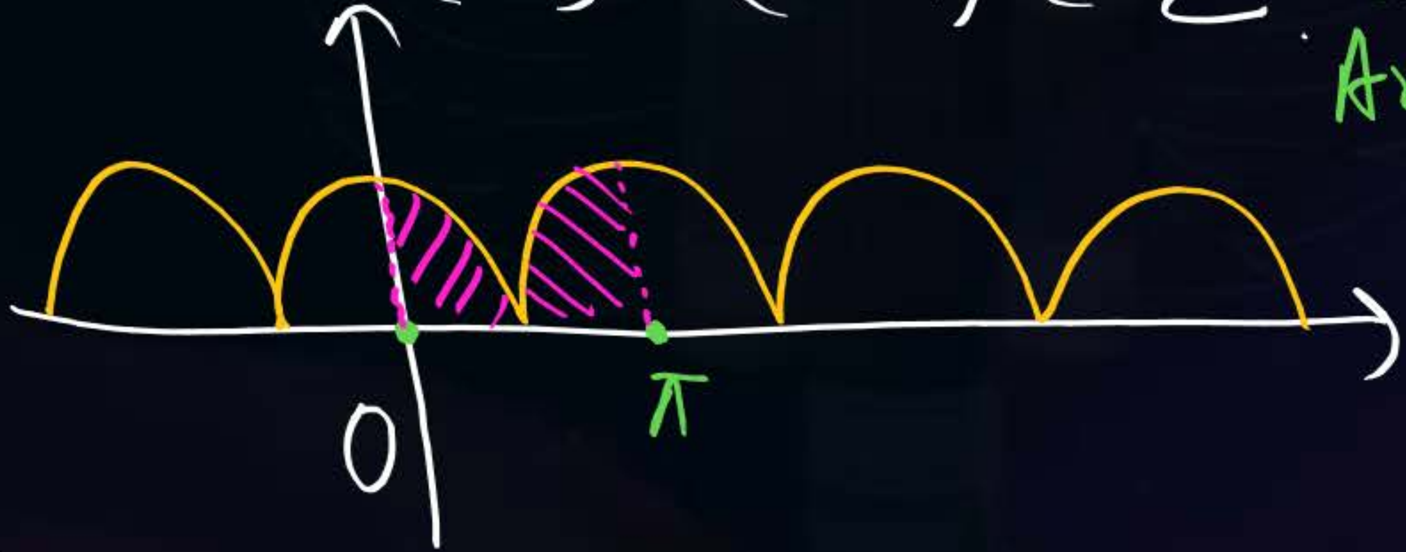
Prop (3) : $I=\int_0^{\pi} |\cos x|\,dx = ?$ (Ans = 2)

$\because |\cos x| = \begin{cases} +\cos x, & 0<x<\frac{\pi}{2} \\ -\cos x, & \frac{\pi}{2}<x<\pi \end{cases}$

$I=\int_0^{\pi} |\cos x|\,dx = \int_0^{\pi/2}(+\cos x)\,dx + \int_{\pi/2}^{\pi}(-\cos x)\,dx$

$= (\sin x)_0^{\pi/2} - (\sin x)_{\pi/2}^{\pi}$

$= (1-0)-(0-1) = 2$ = Shaded Area



Q. $I = \int_{-1}^{2} (1+|x|)\,dx = ?$ (a) 5.5 (b) 4.5 (c) $\frac{2}{11}$ (d) 3

$$I = \int_{-1}^{0} (1+|x|)\,dx + \int_{0}^{2} (1+|x|)\,dx$$

$$= \int_{-1}^{0} (1-x)\,dx + \int_{0}^{2} (1+x)\,dx$$

$$= \left(x - \frac{x^2}{2}\right)_{-1}^{0} + \left(x + \frac{x^2}{2}\right)_{0}^{2}$$

$$= \left[(0-0) - \left(-1-\frac{1}{2}\right)\right] + \left[(2+2) - (0+0)\right] = 5.5$$

Property 4 →

$$\int_a^b f(x)\,dx = \int_a^b f(\underbrace{a+b-x})\,dx$$

Property 5 →

$$\int_0^a f(x)\,dx = \int_0^a f(\underbrace{a-x})\,dx$$

Q. $I = \int_1^2 \frac{\sqrt{x}}{\sqrt{3-x}+\sqrt{x}}\,dx = ?$ —— (1)

$a=1, b=2$, so $a+b-x = 3-x$

i.e. By Prop(4); $x \longrightarrow (3-x)$

$$I = \int_1^2 \frac{\sqrt{3-x}}{\sqrt{x}+\sqrt{3-x}}\,dx \text{ —— (2)}$$

$$2I = \int_1^2 \left(\frac{\sqrt{x}+\sqrt{3-x}}{\sqrt{x}+\sqrt{3-x}}\right) dx$$

$$2I = (x)_1^2 = 2-1 = 1$$

$$I = \frac{1}{2}$$

Q. $I = \int_0^{\pi/2} \frac{\sin x}{\sin x+\cos x}\,dx = ?$ —— (1)

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$, (c) π (d) 0

$a=0, b=\frac{\pi}{2}$ so $a+b-x = \frac{\pi}{2}-x$

so by Prop(5); $x \longrightarrow \frac{\pi}{2}-x$

$$I = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}\,dx$$

$$= \int_0^{\pi/2} \left(\frac{\cos x}{\cos x+\sin x}\right) dx \text{ —— (2)}$$

$$2I = \int_0^{\pi/2} (1)\,dx = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$$

Q. $I=\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx=?$ (a) $\frac{\pi}{4}$ (b) $\frac{\pi^2}{4}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi^2}{2}$

By Prop (5); $x\longrightarrow(\pi-x)$

$$I=\int_0^{\pi}\frac{(\pi-x)\cdot\sin(\pi-x)}{1+\cos^2(\pi-x)}\cdot dx$$

$$=\int_0^{\pi}\frac{(\pi-x)\cdot\sin x}{1+(-\cos x)^2}dx$$

$$I=\pi\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}dx-\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx$$

$$I=\downarrow\quad -I$$

$$2I=\pi\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}dx$$

Put $\cos x=t$ — At, $x=0$, $t=1$; At, $x=\pi$, $t=-1$

$-\sin x\,dx=dt$

$$=\pi\int_{+1}^{-1}\frac{(-dt)}{1+t^2}=+\pi\int_{-1}^{1}\left(\frac{1}{1+t^2}\right)dt$$

$$=+2\pi\int_0^1\left(\frac{1}{1+t^2}\right)dt=+2\pi\left(\tan^{-1}t\right)_0^1$$

$$2I=+2\pi\left(\frac{\pi}{4}-0\right)\Rightarrow I=+\pi^2/4$$

Property (6) $\rightarrow$

$$\int_0^{2a} f(x)\,dx = \boxed{\int_0^{a}\{f(x)+f(2a-x)\}\,dx} = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(2a-x)=f(x) \\ 0, & \text{if } f(2a-x)=-f(x) \end{cases}$$

Property (7) $\rightarrow$

$$\int_{-a}^{a} f(x)\,dx = \boxed{\int_0^{a}\{f(x)+f(-x)\}\,dx} = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(-x)=f(x) \text{ i.e. } f(x) \text{ is Even} \\ 0, & \text{if } f(-x)=-f(x) \text{ i.e. } f(x) \text{ is odd} \end{cases}$$

$$\int_0^{\alpha} f(x)\,dx = \begin{cases} 2\int_0^{\alpha/2} f(x)\,dx, & \text{if } f(\alpha-x)=f(x) \\ 0, & \text{if } f(\alpha-x)=-f(x) \end{cases}$$

Q. $I = \int_{-\pi/2}^{\pi/2} \underbrace{|\sin x|}_{\text{even func}^n} dx = ?$ $= 2\int_0^{\pi/2} |\sin x|\, dx$

Prop 7

(a) 0

(b) 1

(c) 2

(d) 4

$= 2\int_0^{\pi/2} (+\sin x)\, dx$

$= 2(-\cos x)_0^{\pi/2}$

$= -2[0-1] = 2$

M-II $I = \int_{-\pi/2}^{0} (-\sin x)dx + \int_0^{\pi/2} (+\sin x) dx$

$= (+\cos x)_{-\pi/2}^{0} - (\cos x)_0^{\pi/2}$

$= (1-0) - (0-1) = 2$

Q. $\int_{-\pi/3}^{\pi/3} \underbrace{x^5 \cdot \sin^4 x}_{\text{odd func}^n} dx = ? = 0$ (By Prop 7)

$f(x) = x^5 \sin^4 x$

$f(-x) = (-x)^5 \sin^4(-x)$

$= -(x^5) \cdot (-\sin x)^4$

$= - x^5 \cdot \sin^4 x$

$= -f(x)$ so $f(x)$ is odd func^n

Q. $I = \int_{-\pi/2}^{\pi/2} (\underbrace{\sin|x| + \cos|x|}_{\text{even func}^n}) \cdot dx = ?$

(a) 2 (b) 0 (c) 4 (d) 1

By prop (7);

$I = 2\int_0^{\pi/2} (\sin|x| + \cos|x|) dx$

$= 2\int_0^{\pi/2} (\sin x + \cos x) dx$

$= 2\left[-\cos x + \sin x\right]_0^{\pi/2}$

$= 2\left[\{-0+1\} - \{-1+0\}\right] = 4$

Q. $I = \int_{-\pi/2}^{\pi/2} \log\left(\frac{2-\sin x}{2+\sin x}\right) dx = ?$

$f(x) = \log\left(\frac{2-\sin x}{2+\sin x}\right)$

$f(-x) = \log\left[\frac{2+\sin x}{2-\sin x}\right] = \log\left(\frac{2-\sin x}{2+\sin x}\right)^{-1}$

$= -\log\left(\frac{2-\sin x}{2+\sin x}\right) = -f(x)$

i.e. $f(x)$ is an odd func^n so $I = 0$

By Prop (7)

Q: $I = \int_0^{2\pi} \cos^5 x \, dx = ?$ $= 2\int_0^{\pi} \cos^5 x \, dx = 2[0] = 0$

Prop 6

(a) 0

(b) 1

(c) 2

(d) 8

$f(x) = \cos^5 x$

$f(2\pi - x) = \cos^5(2\pi - x)$

$= (\cos x)^5$

$= \cos^5 x$

$= f(x)$

Let $f(x) = \cos^5 x$

$f(\pi - x) = [\cos(\pi - x)]^5$

$= (-\cos x)^5 = -\cos^5 x$

$= -f(x)$

Q. $I = \int_0^{2\pi} \sin x \, dx = ?$ (a) 0 (b) 1 (c) 2 (d) 4

$$= \int_0^{\pi} (\sin x)\, dx + \int_{\pi}^{2\pi} (\sin x)\, dx = 2 + (-2) = 0$$

Q. if $f(x) = \sin x$ then find area bounded by $f(x)$ b/w $x = 0$ & $x = 2\pi$

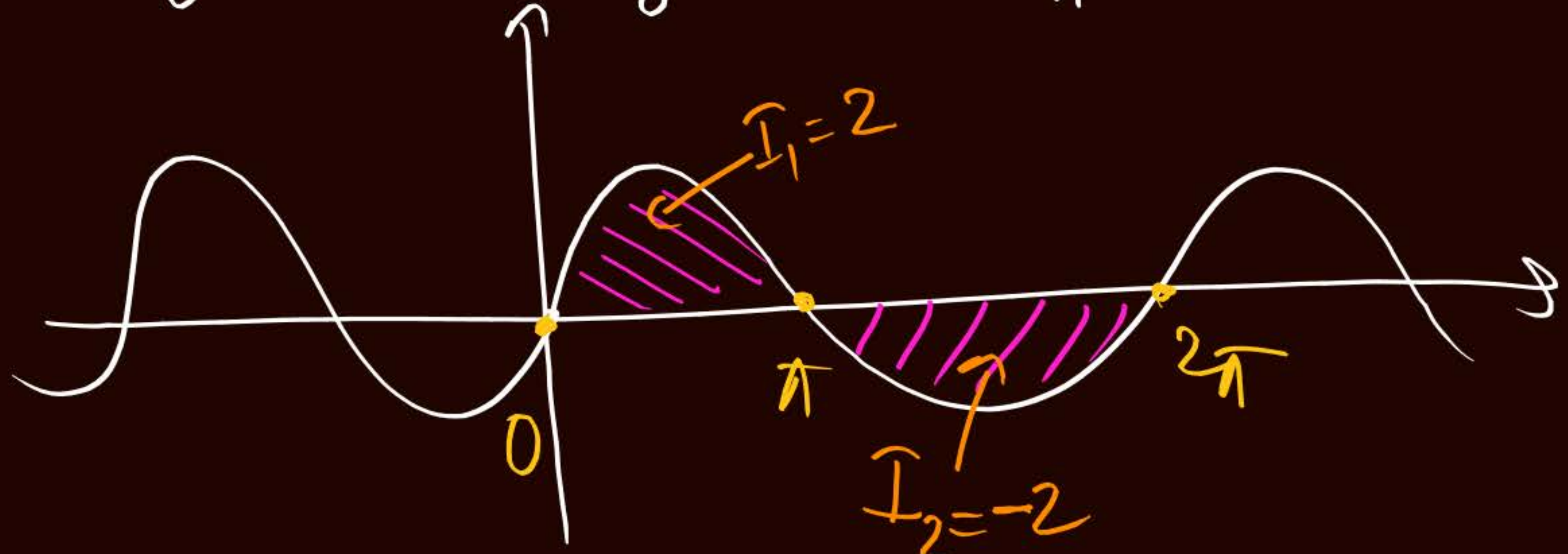
(a) 0

(b) 1

(c) 2

(d) 4

$$\text{Area} = \int_0^{2\pi} |\sin x|\, dx = \int_0^{\pi} (+\sin x)\, dx + \int_{\pi}^{2\pi} (-\sin x)\, dx = 2 - (-2) = 4$$



Q. $I = \int_0^{\pi/2} \log(\sin x)\,dx = ?$ —— (1)

By Prop (5), $x \to \frac{\pi}{2} - x$

$$I = \int_0^{\pi/2} \log \sin\left(\frac{\pi}{2} - x\right) dx = \int_0^{\pi/2} \log \cos x \, dx \quad \text{—— (2)}$$

$$2I = \int_0^{\pi/2} (\log \sin x + \log \cos x)\,dx = \int_0^{\pi/2} \log(\sin x \cos x)\,dx$$

$$= \int_0^{\pi/2} \log\left(\frac{2\sin x \cos x}{2}\right) dx = \int_0^{\pi/2} (\log \sin 2x - \log 2)\,dx$$

$$= \int_0^{\pi/2} \log \sin 2x \, dx - \log 2 \int_0^{\pi/2} (1)\,dx$$

$$2I = I_1 - \log 2\left(\frac{\pi}{2} - 0\right) \quad \text{—— (3)}$$

Now $I_1 = \int_0^{\pi/2} \log \sin(2x)\,dx$

Put $2x = t$ → At $x = 0$, $t = 0$; At $x = \frac{\pi}{2}$, $t = \pi$

$dx = \frac{dt}{2}$

$$I_1 = \int_0^{\pi} \log \sin t \cdot \frac{dt}{2} = \int_0^{\pi/2} \log \sin t \, dt = I$$

So By (3), $2I = I_1 - \frac{\pi}{2}\log 2$

$$2I = I - \frac{\pi}{2}\log 2$$

$$I = -\frac{\pi}{2}\log 2$$

(*) $\boxed{\int_0^{\pi/2} \log \sin x \, dx = \int_0^{\pi/2} \log \cos x \, dx = -\frac{\pi}{2} \log 2}$ Learn

Q. $I = \int_0^{\pi/2} \log(\tan x + \cot x)\, dx$

(a) $\frac{\pi}{2} \ln 2$

(b) $\pi \ln 2$ ✓

(c) $-\frac{\pi}{2} \ln 2$

(d) 0

$$= \int_0^{\pi/2} \log\left\{\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\right\} dx$$

$$= \int_0^{\pi/2} \log\left\{\frac{1}{\sin x \cos x}\right\} dx$$

$$= \int_0^{\pi/2} \left[\log 1 - \underbrace{\log(\sin x \cos x)}\right] dx$$

$$I = \int_0^{\pi/2} \left\{0 - \underbrace{(\log \sin x + \log \cos x)}\right\} dx$$

$$= -\left[\int_0^{\pi/2} \log \sin x \, dx + \int_0^{\pi/2} \log \cos x \, dx\right]$$

$$= -\left[-\frac{\pi}{2} \log 2 - \frac{\pi}{2} \log 2\right]$$

$$= \pi \log 2$$

Leibnitz Rule of Diff under the Sign of Integration →

$$\frac{d}{dx}\int_{\phi(x)}^{\psi(x)} f(t)\,dt = \boxed{\frac{d}{dx}(\psi)\cdot f(\psi) - \frac{d}{dx}(\phi)\, f(\phi)}$$

eg $I = \dfrac{d}{dx}\displaystyle\int_{x^2}^{x^3}\left(\frac{1}{\log t}\right)dt = ?$

$$= \frac{d}{dx}(x^3)\frac{1}{\log(x^3)} - \frac{d}{dx}(x^2)\frac{1}{\log(x^2)}$$

$$= 3x^2\left(\frac{1}{3\log x}\right) - 2x\left(\frac{1}{2\log x}\right)$$

$$= \frac{x^2 - x}{\log x}$$

Q. Consider the func^n $\boxed{f(x)=\int_0^x e^{-\left(\frac{x^2}{2}\right)}dx}$ & let it's T.S. Exp about $x=0$ is given as $f(x)=a_0+a_1x+\underline{a_2}x^2+a_3x^3+\cdots\cdots$ then $a_2=\underline{?}$

(a) 0

(b) 0.5

(c) 1

(d) 2

In Maclaurin series Exp, Coeff of $x^2=a_2=\dfrac{f''(0)}{2!}$

$$\frac{d}{dx}f(x)=\frac{d}{dx}\int_0^x e^{-\left(\frac{x^2}{2}\right)}dx=\boxed{\frac{d}{dx}(x)\cdot e^{-\frac{x^2}{2}}-\frac{d}{dx}(0)e^{\frac{0^2}{2}}}=e^{-\frac{x^2}{2}}-0$$

$$f''(x)=\frac{d}{dx}(f'(x))=\frac{d}{dx}\left(e^{-\frac{x^2}{2}}\right)=e^{-\frac{x^2}{2}}\left(\frac{-2x}{2}\right)=-xe^{-x^2/2}$$

$$a_2=\frac{f''(0)}{2!}=0$$

Let $f(x)=\int_0^x e^t(t-1)(t-2)dt$. Then $f(x)$ decreases in the interval.

(a) $x \in (1, 2)$ (b) $x \in (2, 3)$

(c) $x \in (0, 1)$ (d) $x \in (0.5, 1)$

$$f(x)=\int_0^x e^t(t-1)(t-2)dt$$

$$f'(x)=\frac{d}{dx}\int_0^x e^t(t-1)(t-2)dt$$

$$=\frac{d}{dx}(x)\left\{e^x(x-1)(x-2)\right\}-\frac{d}{dx}(0)\left\{e^0(0-1)(0-2)\right\}$$

$$= e^x(x-1)(x-2)-0$$

$$\boxed{f'(x)=e^x(x-1)(x-2)}$$

T. Points are $x=1$ & 2

Max min

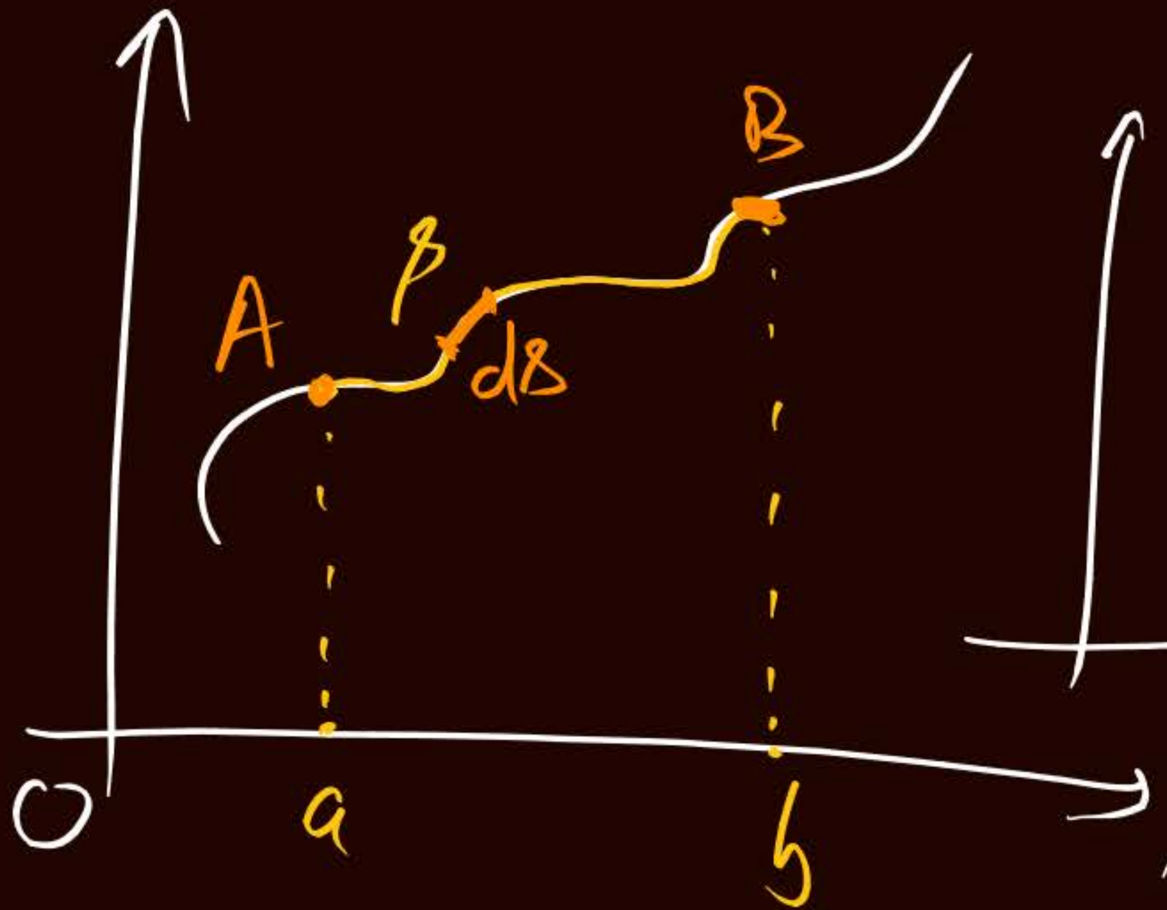
$f(x)$ Dec in $(1,2)$

Application of Definite Integration

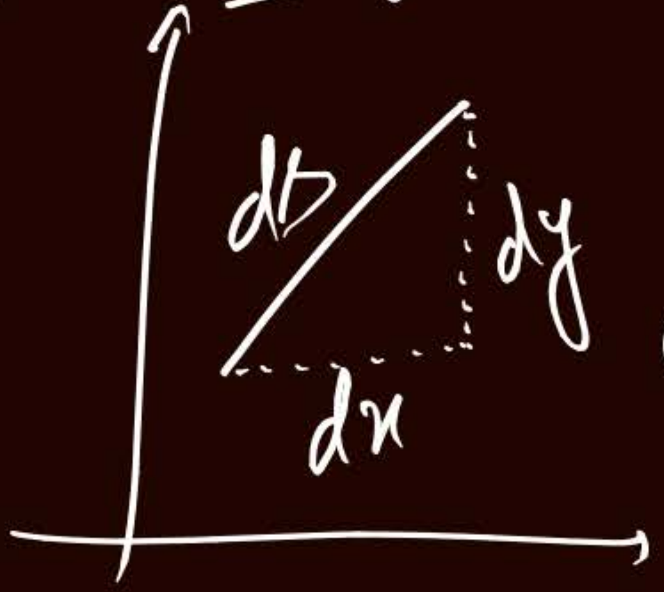
- To find Length of Curve
- To find Volume of Solid formed by Revolution.

Length of Curve: $y=f(x)$ b/w $x=a$ & $x=b$ is

$$\boxed{s=\int_{x=a}^{b}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\cdot dx}$$



Proof:



$$(ds)^2=(dx)^2+(dy)^2$$

$$(ds)^2=\left[1+\left(\frac{dy}{dx}\right)^2\right](dx)^2$$

$$\int_0^s ds=\int_{x=a}^{b}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\cdot dx$$

$$s=$$ Hence Prooved.

Case I: length of curve $y = f(x)$ b/w $x = a$ to b is $s = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

Case II: " " $x = f(y)$ b/w $y = c$ to d is $s = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2} \cdot dy$

w/o proof

Case III: if $x = x(t)$, $y = y(t)$, $z = z(t)$ then

w/o proof

length of curve b/w t_1 & t_2 is $s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2} \cdot dt$

Case IV: if $r = f(\theta)$ then length of curve b/w θ_1 & θ_2 is $s = \int_{\theta_1}^{\theta_2} \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2} \cdot d\theta$

w/o proof

(Not in a syllabus of CS & DSAI)

Q. The length of curve $y = \frac{2}{3} x^{3/2}$ b/n $x = 0$ & 1 is ____?

(a) 1.732

(b) 1.414

(c) 1.22

(d) 3.14

$$\frac{dy}{dx} = \frac{2}{3}\left[\frac{3}{2} \cdot x^{\frac{3}{2}-1}\right] = x^{1/2} = \sqrt{x}$$

$$S = \int_{x=0}^{1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_{x=0}^{1} \sqrt{1+x}\, dx = \left(\frac{(1+x)^{3/2}}{3/2}\right)_{x=0}^{1}$$

$$= \frac{2}{3}\left[2^{3/2} - 1^{3/2}\right] = \frac{2}{3}\left[\sqrt{2^3} - 1\right] = \frac{2}{3}\left[2\sqrt{2} - 1\right]$$

$$= \frac{2}{3}\left[2 \times 1.414 - 1\right] = 0.66\left[2.828 - 1\right] = 0.66 \times 1.828$$

$$= 1.22$$

Consider a spatial curve in three-dimensional space given in parametric form by

$$x(t) = \cos t,\ y(t) = \sin t,\ z(t) = \frac{2}{\pi}t,\ 0 \le t \le \frac{\pi}{2}$$

The length of the curve is ________.

Use Case III

(a) 1

(b) 2

(c) 1.86

(d) 3.14

A parabolic cable is held between two supports at the same level. The horizontal span between the supports is L. The sag at the mid-span is h. The equation of the parabola is $y = 4h\frac{x^2}{L^2}$, where x is the horizontal coordinate and y is the vertical coordinate with the origin at the centre of the cable. The expression for the total length of the cable is

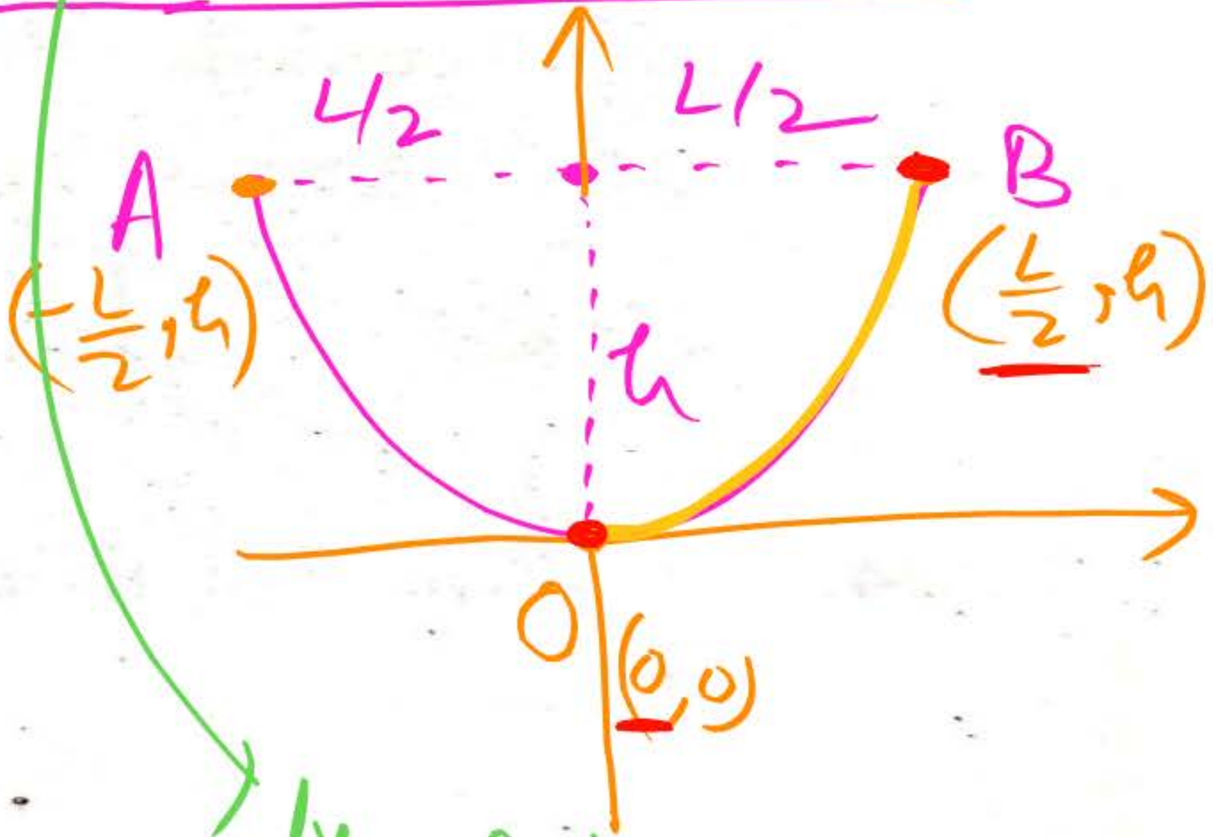
depth

(a) $\int_0^{L} \sqrt{1 + 64\frac{h^2x^2}{L^4}}\,dx$

(b) $2\int_0^{L/2} \sqrt{1 + 64\frac{h^3x^2}{L^4}}\,dx$

(c) $\int_0^{L/2} \sqrt{1 + 64\frac{h^2x^2}{L^4}}\,dx$

(d) $2\int_0^{L/2} \sqrt{1 + 64\frac{h^2x^2}{L^4}}\,dx$



$\frac{dy}{dx} = \frac{8xh}{L^2}$

Total Length AOB

$= 2$ length of OB

$= 2\int_{x=0}^{L/2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\cdot dx$

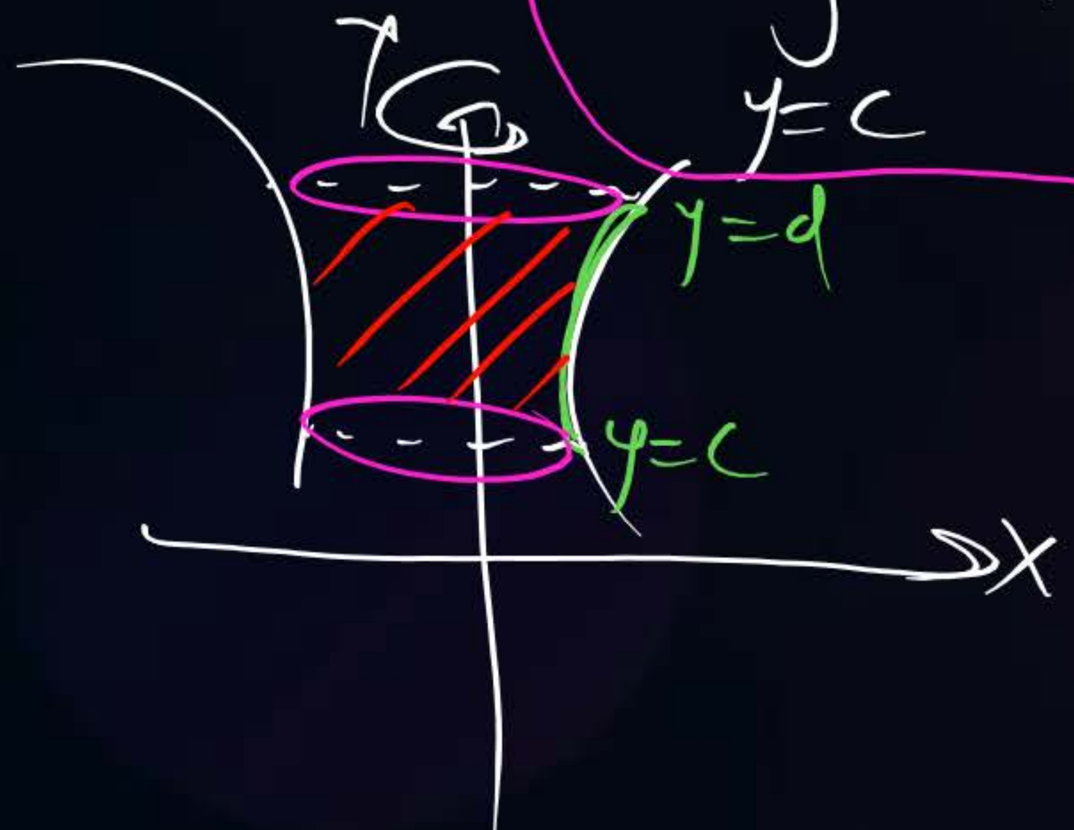
$= 2\int_0^{L/2} \sqrt{1 + \frac{64x^2h^2}{L^4}}\cdot dx$

(2) Volume of Solid formed by Revolution →

Case I: (Revolution about Y axis) →

if the curve $x = f(y)$ is to be Revolved about Y axis b/n $y = c$ to d then volume of Solid formed is

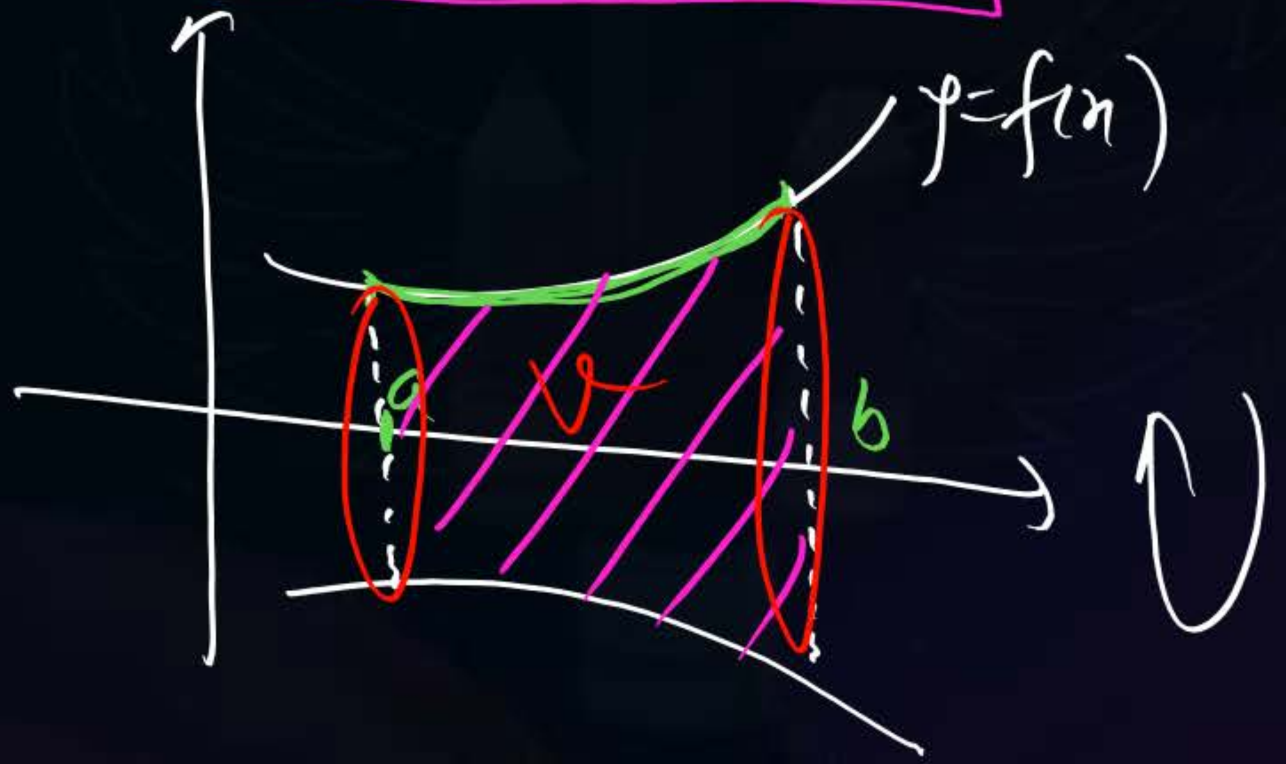
$$\boxed{V = \int_{y=c}^{d} \pi x^2 \, dy}$$



Case II: (Revolution about X axis) →

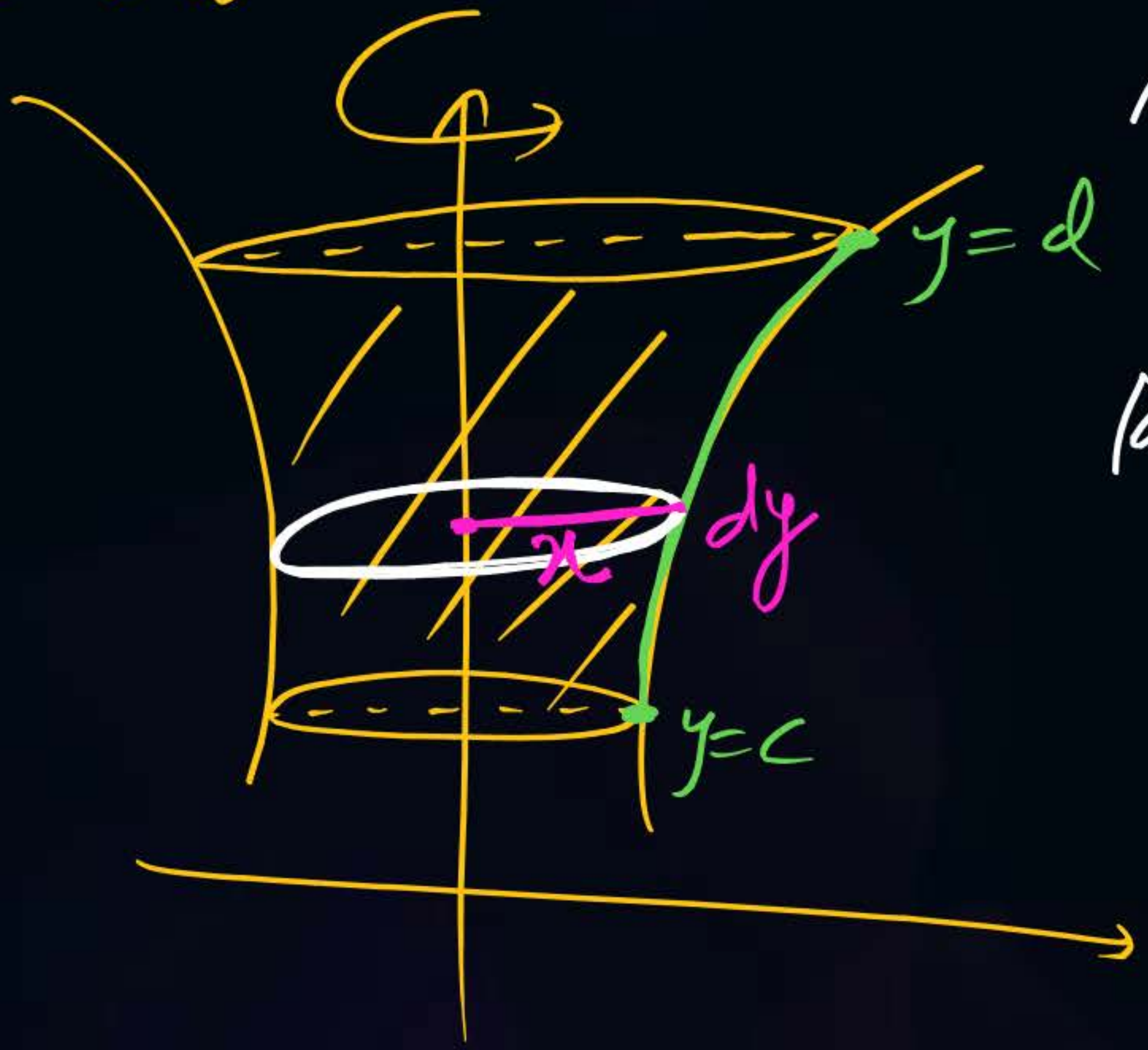
if the curve $y = f(x)$ is to be Revolved about X axis b/n $x = a$ to b then volume of Solid formed is

$$\boxed{V = \int_{x=a}^{b} \pi y^2 \, dx}$$



Proof of Case I:

Consider the small disc as shown by white marker.



So we can assume this disc as a cylinder of Radius x & height dy

So volume of this disc is

$$dV = \pi x^2 dy$$

$$\int_0^V (1)\,dV = \int_{y=c}^{d} \pi x^2 dy$$

$$V = \int_c^d \pi x^2 dy$$ Hence Proved

Q. Find the Volume of Solid formed by Revolution of the Arc $y=\sqrt{x}$ b/w $x=1$ & 2 about x axis ?

(a) $3\pi/2$

(b) $\pi/2$

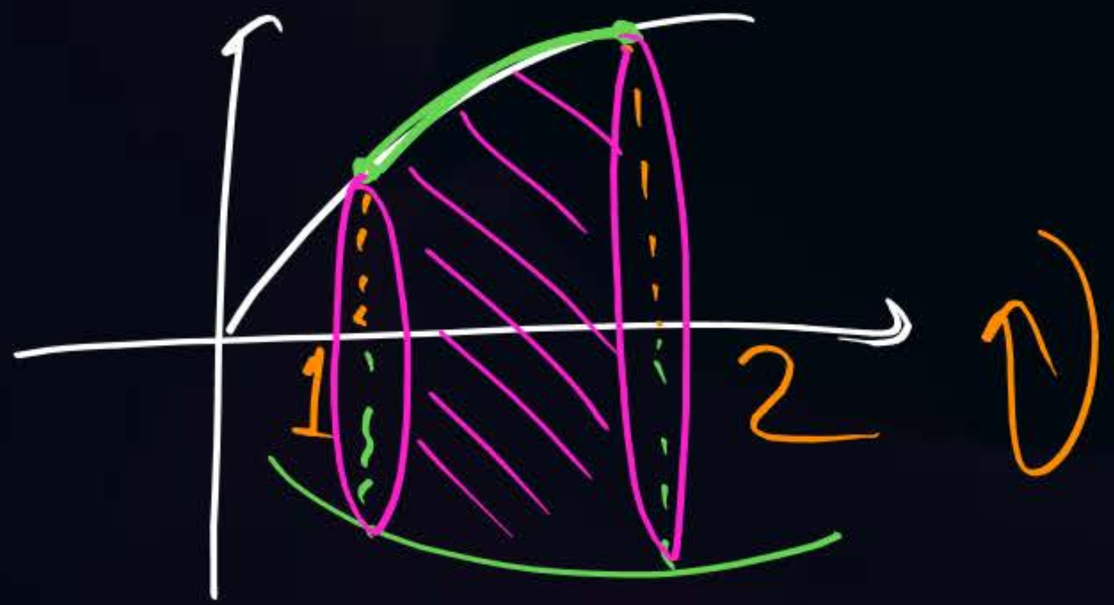
(c) $3/2$

(d) 3π

$$\text{Req Volume} = \int_{x=a}^{b} \pi y^2\,dx = \int_{x=1}^{2} \pi(\sqrt{x})^2\,dx = \pi\int_1^2 x\,dx$$

$$= \frac{3\pi}{2}$$

M-II $y=\sqrt{x}$



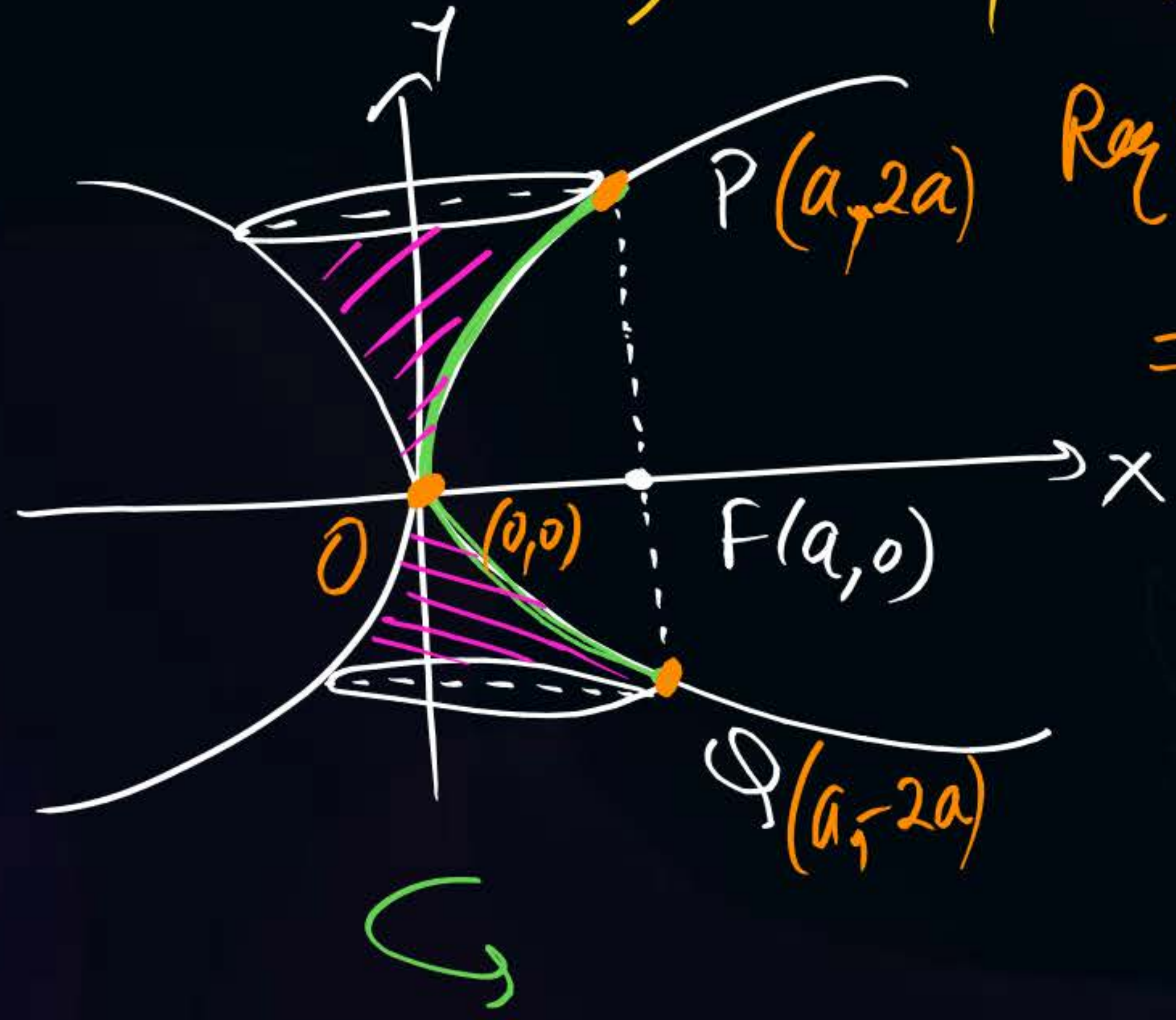
Q. Find the Volume of Reel shaped Solid formed by Revolution of arc of the $y^2=4ax$ (cut off by it's Latus Rectum) about Y axis?

(a) $32\pi a^3$

(b) $\frac{16\pi a^3}{5}$

(c) $\frac{32}{5}\pi a^3$

(d) $\frac{4}{5}\pi a^3$



Req Volume = 2 Volume of above portion.

$$= 2\int_{y=0}^{2a} \pi x^2\, dy = 2\pi \int_{y=0}^{2a} \left(\frac{y^2}{4a}\right)^2 dy$$

$$= \frac{2\pi}{16a^2}\left(\frac{y^5}{5}\right)_0^{2a} = \frac{\pi}{8\times 5a^2}\left(2^5 a^5 - 0\right)$$

$$= \frac{4\pi a^3}{5}$$

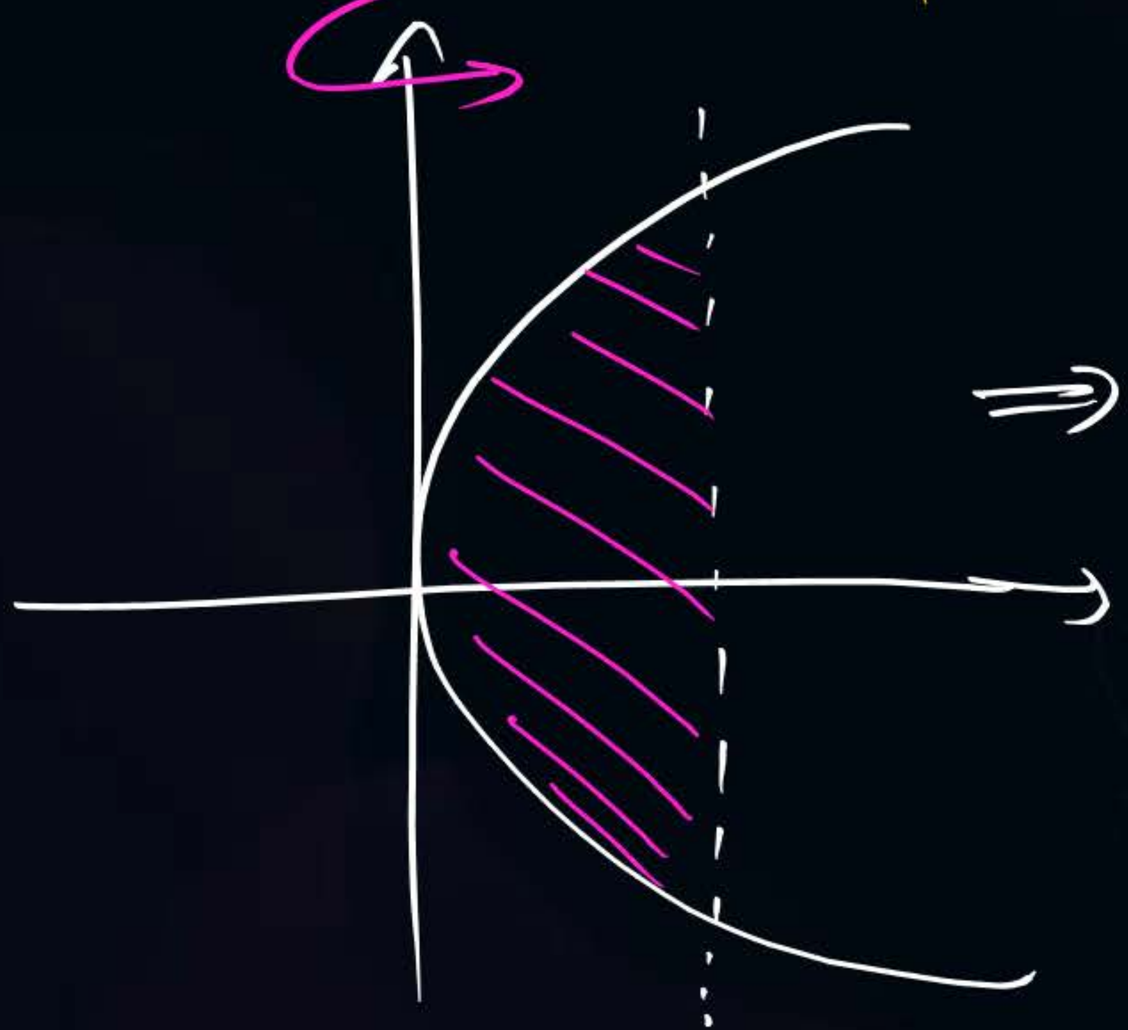
Q. Find the volume of solid formed by revolution of area of the $y^2 = 4ax$ (cut off by it's latus rectum) about Y axis?

(a) $\frac{4\pi}{5} a^3$

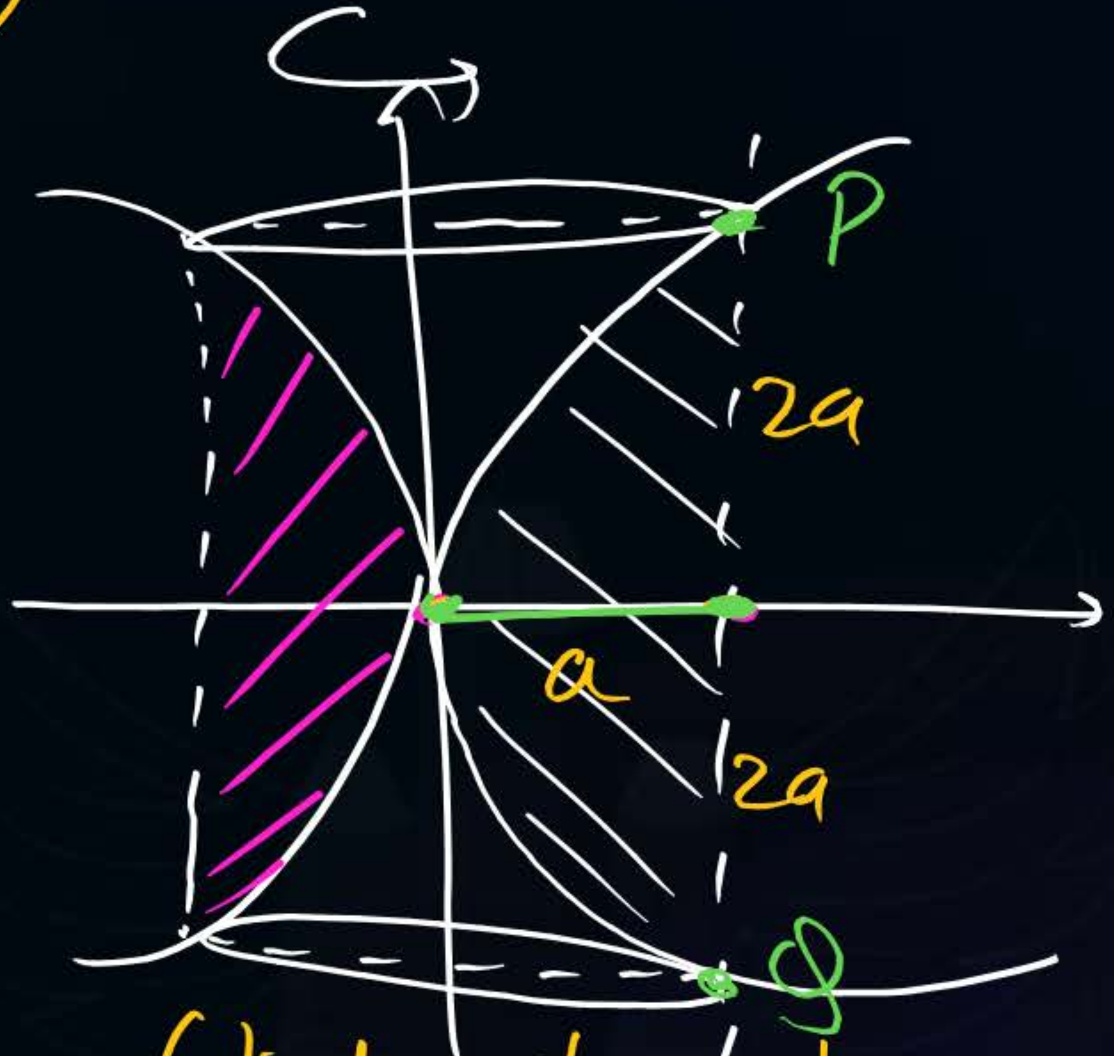
(b) $\frac{2\pi}{5} a^3$

(c) $\frac{128\pi}{5} a^3$

(d) $\frac{16}{5}\pi a^3$



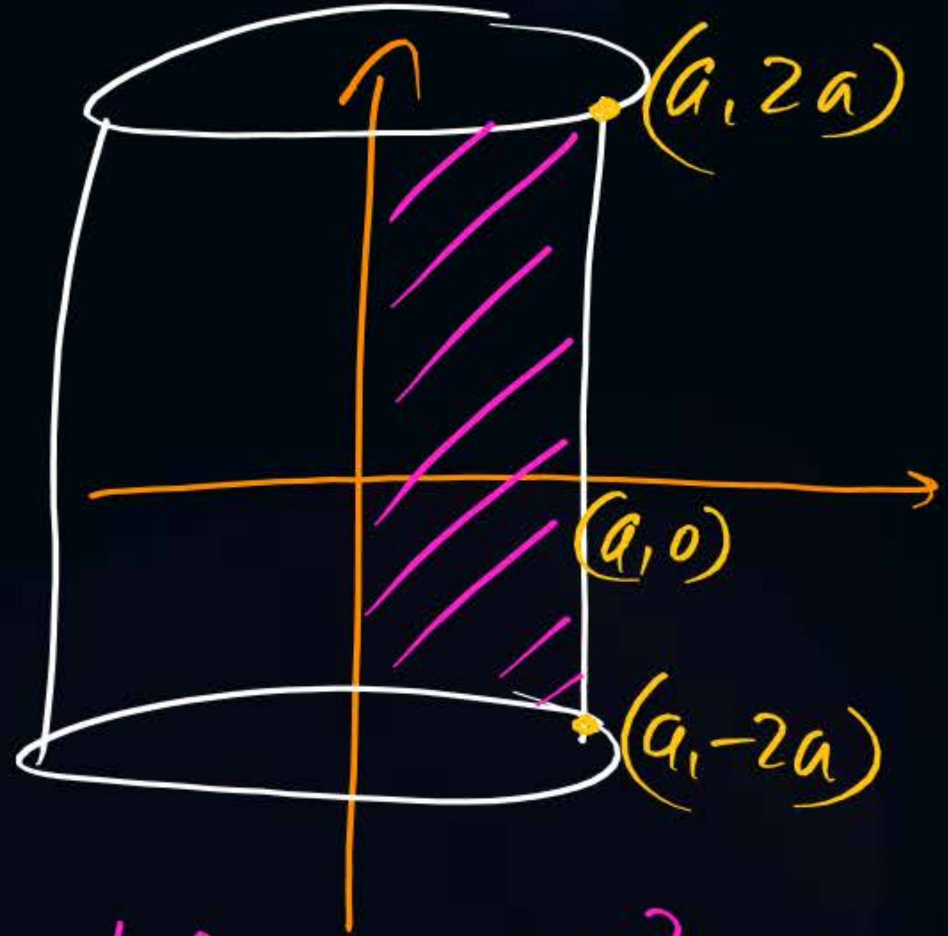
$\Longrightarrow$



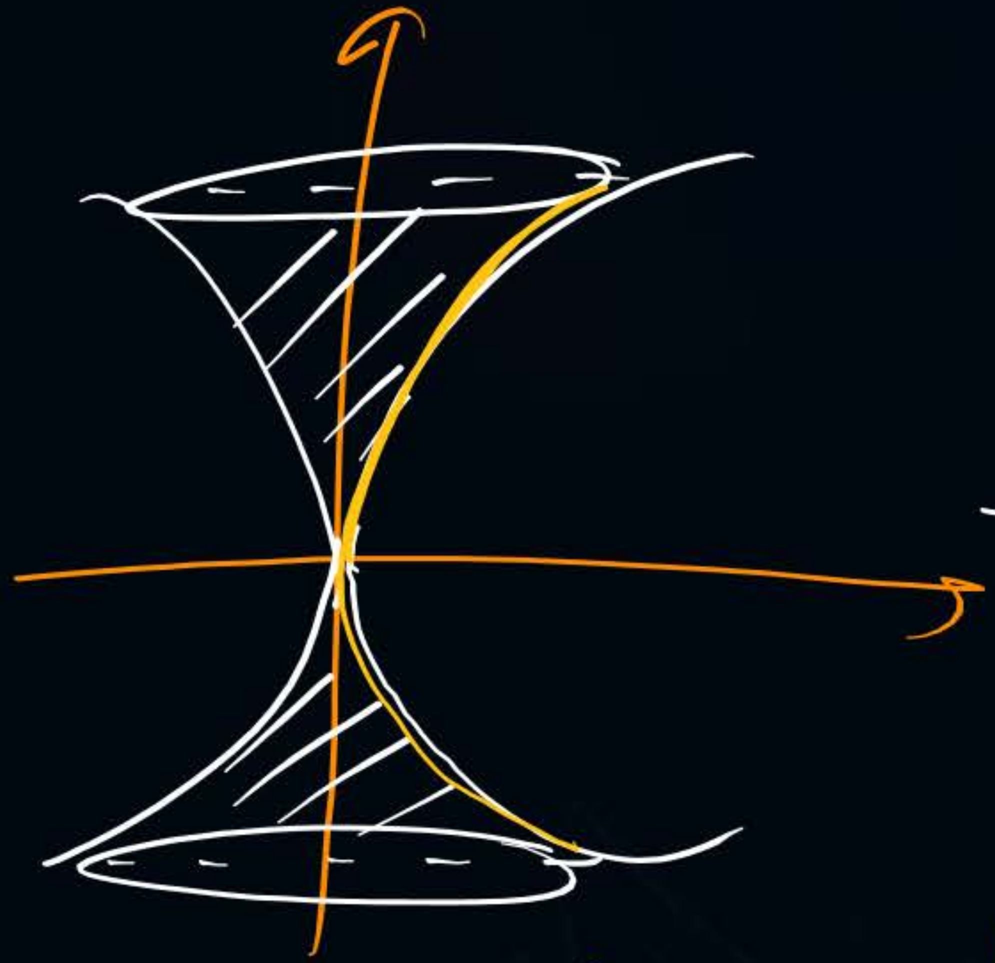
Cylinder with cavity

$r = a$ & $h = 2a$

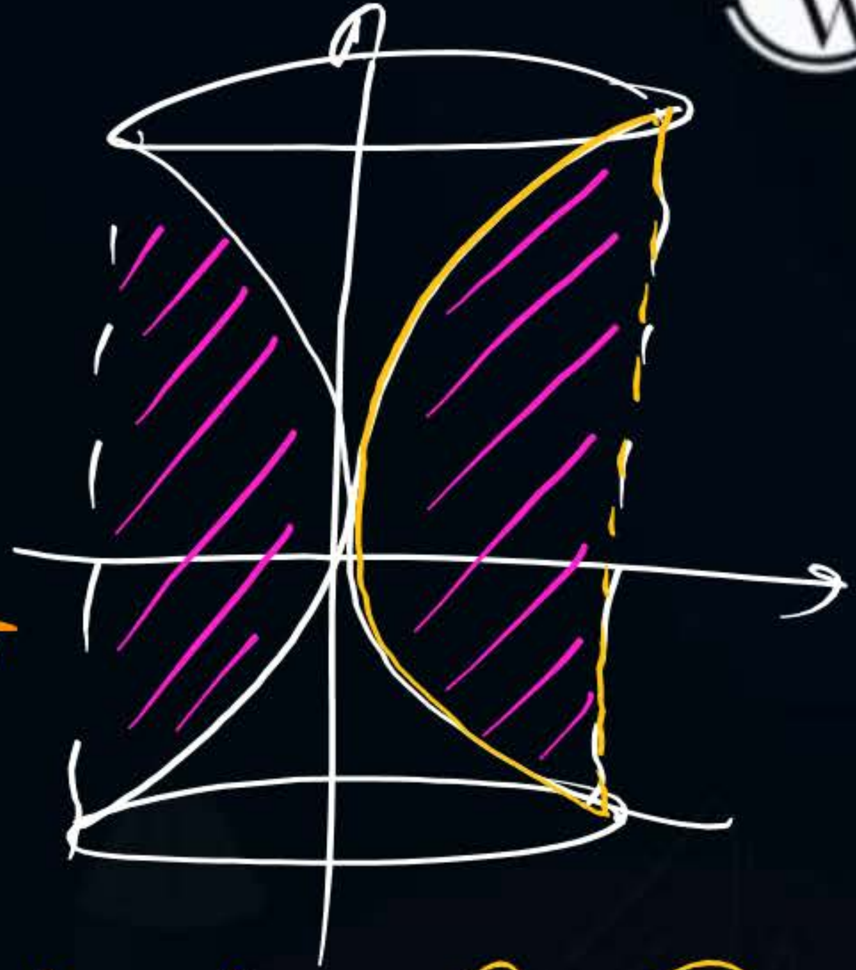
Explanation :-



$$\text{Volume} = \pi (a)^2 (2a) \quad \text{—— } (1)$$



$$\text{Volume} = \frac{4\pi a^3}{5} \quad \text{—— } (2)$$

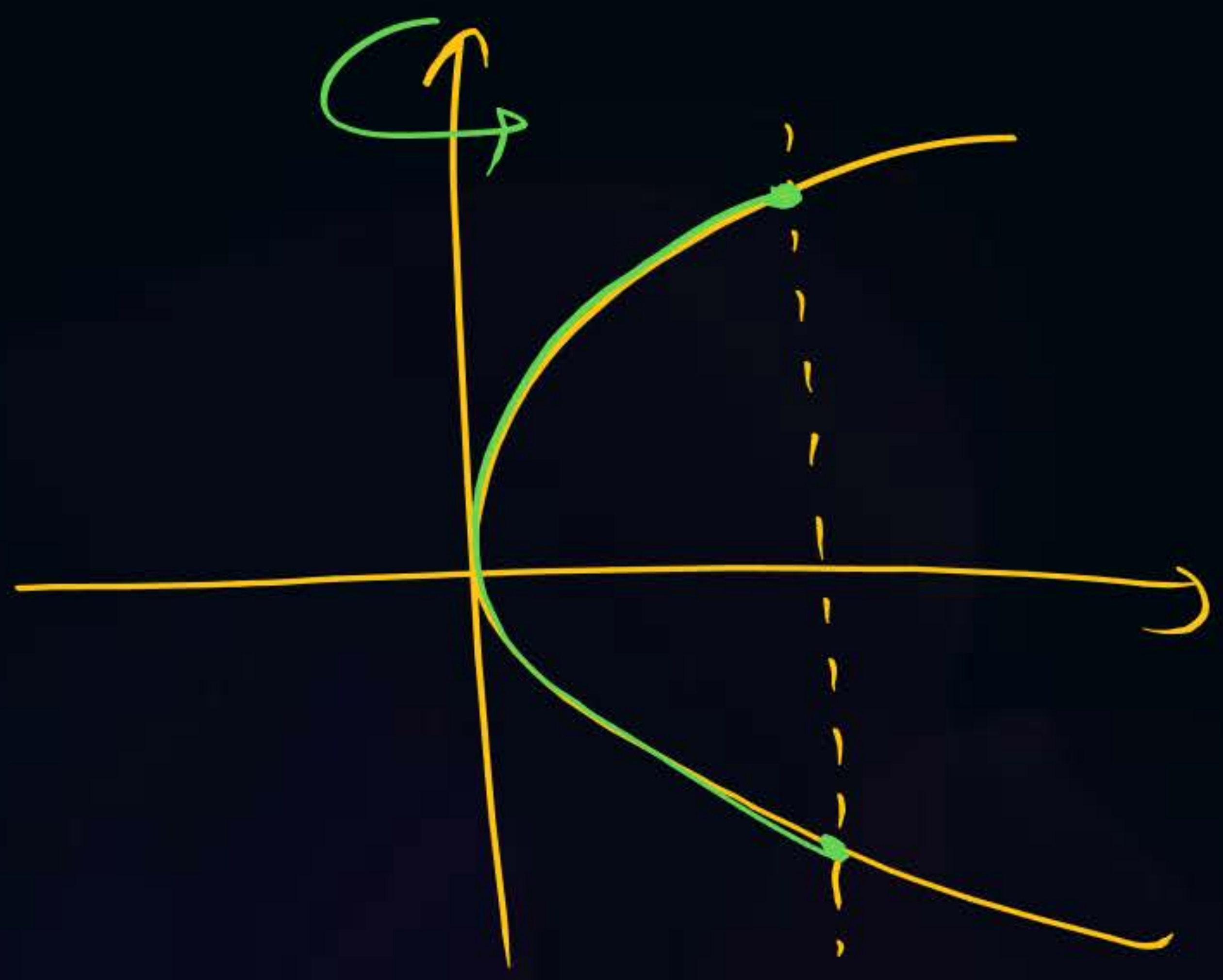


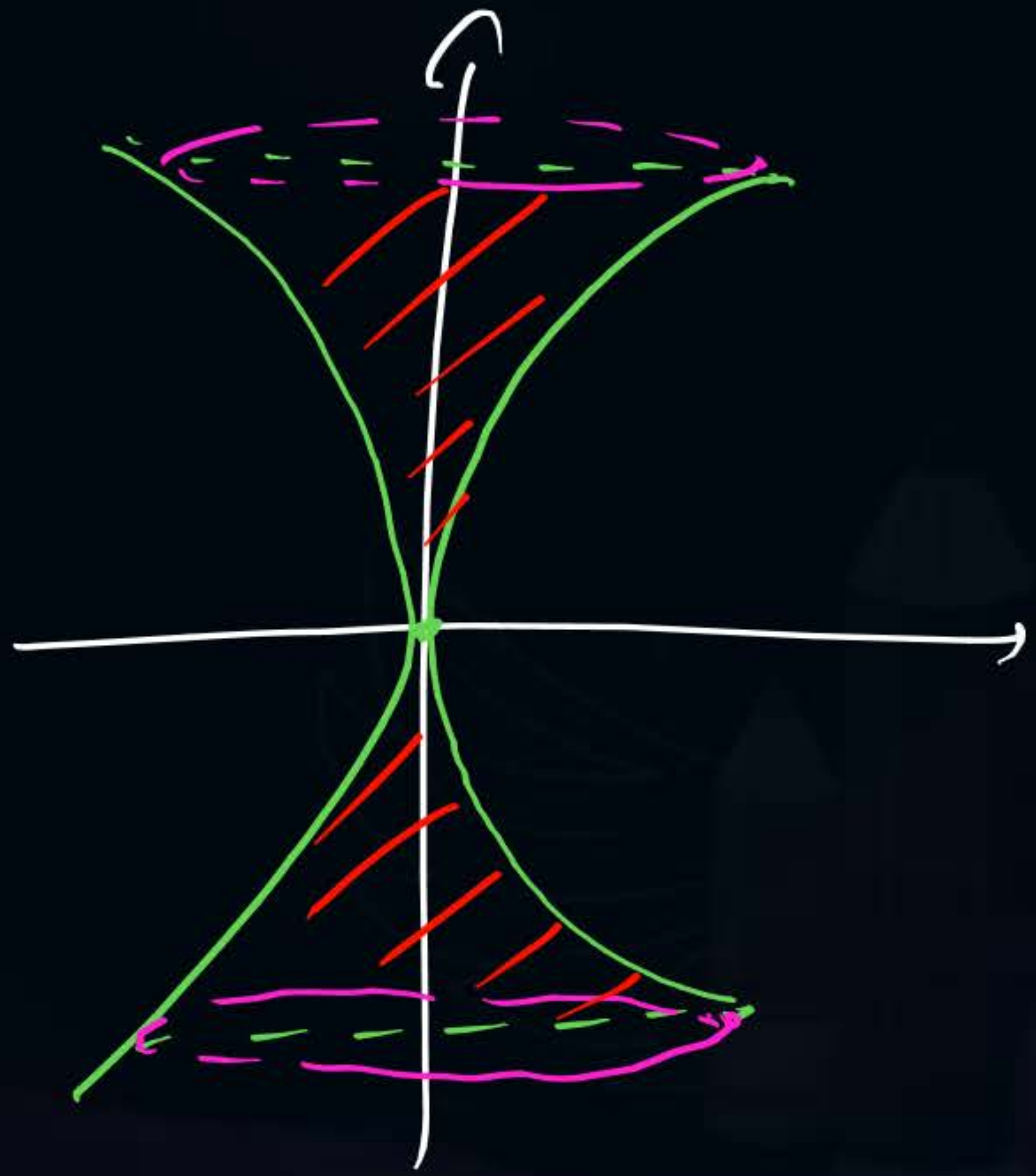
$$\text{Req Volume} = (1) - (2)$$

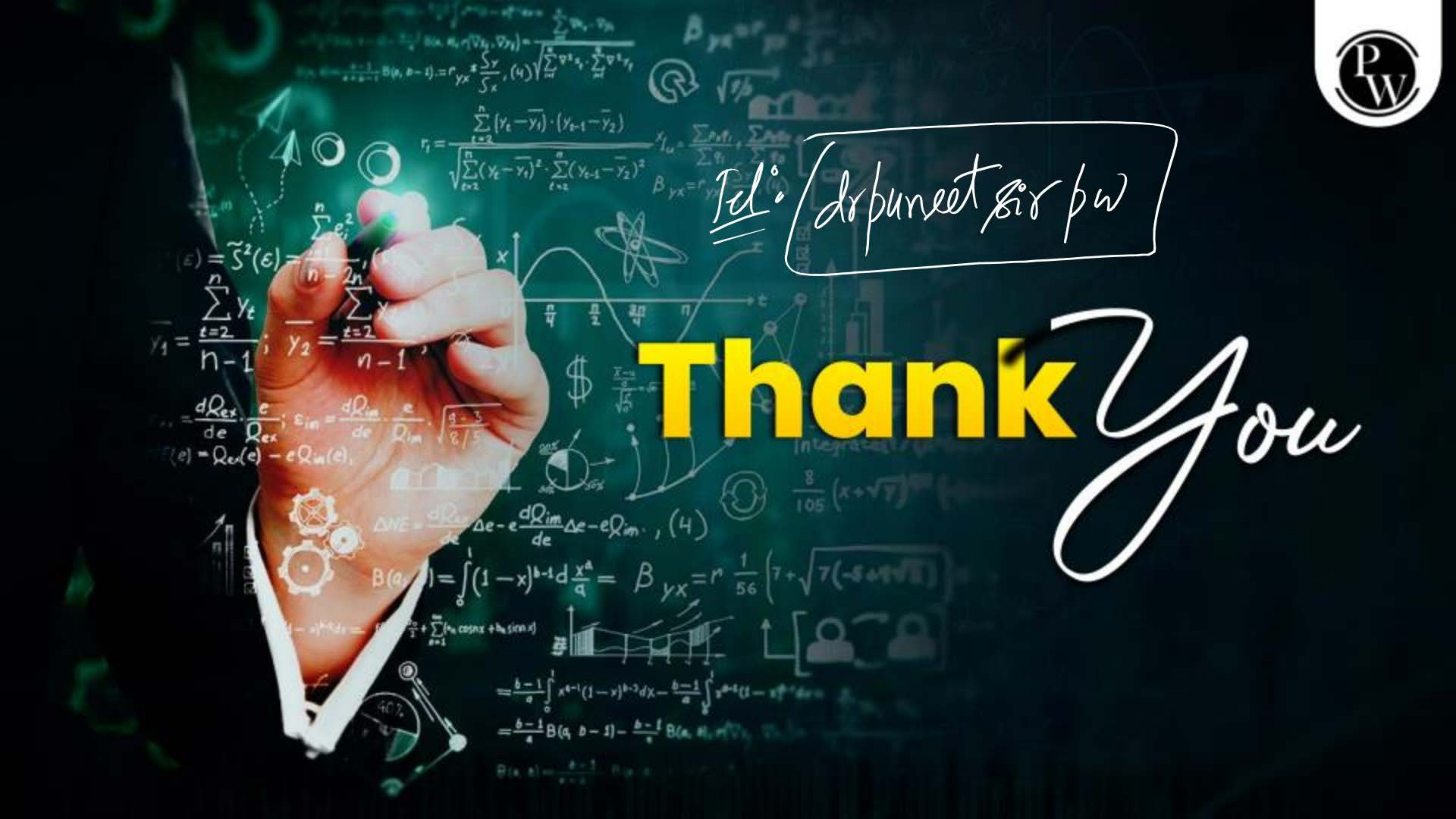
$$= \frac{16}{5}\pi a^3$$

$y^2 = 4ax$, LR $x = a$

length of LR $= 4a$






Id: drpuneetsirpw
Thank You

DATA SCIENCE
&
ARTIFICIAL INTELLIGENCE

& CS/IT

Calculus and Optimization

Lecture No. 13

By- Dr. Puneet Sharma Sir

Recap of previous lecture

Topic

INTEGRATION

Topics to be Covered

Topic

→ Beta & Gamma functions

→ Nature of Roots

→ PRACTICE QUESTIONS

BETA & GAMMA func^n

Properties of Gamma func^n →

① $\Gamma(n+1) = \begin{cases} n! & , n \in +\text{Integer} \\ n\Gamma n & , n \in +\text{ve Rational.} \end{cases}$

eg $\Gamma 5 = \Gamma 4+1 = 4!$, $\Gamma 4 = 3!$, $\Gamma 3 = 2!$

$\Gamma 2 = 1!$, $\Gamma 1 = 0!$, $\boxed{\Gamma 0 = \text{N.D}}$

Note: $\boxed{\Gamma \frac{1}{2} = \sqrt{\pi}}$, $\boxed{\Gamma \text{-ve int} = \text{N.D}}$

$\Gamma\text{-ve Rational}$ = Defined by formula ②

eg $\Gamma \frac{3}{2} = \Gamma \frac{1}{2}+1 = \frac{1}{2}\Gamma\frac{1}{2} = \frac{\sqrt{\pi}}{2}$

eg $\Gamma \frac{5}{2} = \Gamma \frac{3}{2}+1 = \frac{3}{2}\cdot\Gamma\frac{3}{2} = \frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\frac{1}{2} = \frac{3\sqrt{\pi}}{4}$

eg $\Gamma \frac{7}{2} = ? = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}$

eg $\Gamma \frac{9}{2} = ? = \frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\frac{1}{2}$

eg $\Gamma \frac{17}{4} = ?$

$\cancel{= \frac{15}{4}\cdot\frac{13}{4}\cdot\frac{11}{4}\cdot\frac{9}{4}\cdot\frac{7}{4}\cdot\frac{5}{4}\cdot\frac{3}{4}\cdot\frac{1}{4}}\,\Gamma\frac{1}{4}$

$= \frac{13}{4}\cdot\frac{9}{4}\cdot\frac{5}{4}\cdot\frac{1}{4}\cdot\Gamma\frac{1}{4}$

② $\boxed{\Gamma n \cdot \Gamma 1-n = \dfrac{\pi}{\sin n\pi}}$, $n \notin I$ — ①

eg Evaluate $\Gamma\frac{1}{2}$, $\Gamma-\frac{1}{2}$, $\Gamma-\frac{3}{2}$, $\Gamma\frac{1}{4} \cdot \Gamma\frac{3}{4} = ?$

Sol ① Put $n = \frac{1}{2}$ in ①

$$\Gamma\frac{1}{2} \cdot \Gamma 1-\frac{1}{2} = \frac{\pi}{\sin\left(\frac{\pi}{2}\right)}$$

$$\Gamma\frac{1}{2} \cdot \Gamma\frac{1}{2} = \frac{\pi}{(1)}$$

$$\left(\Gamma\frac{1}{2}\right)^2 = \pi \Rightarrow \Gamma\frac{1}{2} = \sqrt{\pi}$$ Ans

Neglecting -ve sign

Sol ② Put $n = \frac{3}{2}$ in ①.

$$\Gamma\frac{3}{2} \cdot \Gamma 1-\frac{3}{2} = \frac{\pi}{\sin\left(\frac{3\pi}{2}\right)}$$

$$\frac{1}{2}\Gamma\frac{1}{2} \cdot \Gamma-\frac{1}{2} = \frac{\pi}{(-1)}$$

$$\frac{\sqrt{\pi}}{2} \cdot \Gamma-\frac{1}{2} = -\pi \Rightarrow \Gamma-\frac{1}{2} = -2\sqrt{\pi}$$

Sol 3 : Put $n = \frac{5}{2}$ in ①.

$$\Gamma\frac{5}{2} \cdot \Gamma 1-\frac{5}{2} = \frac{\pi}{\sin\left(\frac{5\pi}{2}\right)}$$

$$\frac{3}{2} \cdot \frac{1}{2} \cdot \Gamma\frac{1}{2} \cdot \Gamma-3/2 = \frac{\pi}{(1)}$$

$$\frac{3}{4}\sqrt{\pi}\,\Gamma-3/2 = \pi \Rightarrow \Gamma-\frac{3}{2} = \frac{4}{3}\sqrt{\pi}$$

Sol: (4) Put $x = \frac{1}{4}$ in (1) $\Rightarrow \Gamma\frac{1}{4} \cdot \Gamma\left(1-\frac{1}{4}\right) = \frac{\pi}{\sin\left(\frac{\pi}{4}\right)} \Rightarrow \Gamma\frac{1}{4} \cdot \Gamma\frac{3}{4} = \pi\sqrt{2}$ Ans

Property (3) $\displaystyle\int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \frac{\Gamma\frac{m+1}{2} \cdot \Gamma\frac{n+1}{2}}{2\,\Gamma\frac{m+n+2}{2}}$

Q. $\displaystyle I = \int_0^{\pi/2} \sin^3\theta \cos^4\theta \, d\theta = ?$

Here, $m = 3, n = 4$

$$= \frac{\Gamma\frac{3+1}{2} \cdot \Gamma\frac{4+1}{2}}{2\,\Gamma\frac{3+4+2}{2}} = \frac{\Gamma 2 \cdot \Gamma\frac{5}{2}}{2\,\Gamma\frac{9}{2}} = \frac{\underline{1} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}}{2 \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}}$$

$$= \frac{2}{35}$$

Q. $I = \int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta = ? = \int_0^{\pi/2} \sqrt{\frac{\cos\theta}{\sin\theta}}\, d\theta = \int_{\theta=0}^{\pi/2} \sin^{-\frac{1}{2}}\theta \cos^{\frac{1}{2}}\theta\, d\theta$

Here $m = -\frac{1}{2}$, $n = \frac{1}{2}$

(a) $\pi/\sqrt{2}$

(b) $\pi\sqrt{2}$

(c) 2π

(d) $\pi/2$

$$= \frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right) \cdot \Gamma\left(\frac{\frac{1}{2}+1}{2}\right)}{2\,\Gamma\left(\frac{-\frac{1}{2}+\frac{1}{2}+2}{2}\right)} = \frac{\Gamma\frac{1}{4} \cdot \Gamma\frac{3}{4}}{2\,\Gamma 1} = \frac{\pi\sqrt{2}}{2(1)} = \frac{\pi}{\sqrt{2}}$$

PYQ
2M

$$I=\int_0^{\pi/6}\cos^4(3\theta)\cdot\sin^3(6\theta)\,d\theta=?=\int_0^{\pi/2}\cos^4(t)\cdot\sin^3(2t)\cdot\frac{dt}{3}$$

(a) 3

(b) 1/90

(c) 1/15 ✓

(d) 0

Put $3\theta=t$

$d\theta=\frac{dt}{3}$

At $\theta=0,\ t=0$

At $\theta=\frac{\pi}{6},\ t=\frac{\pi}{2}$

$$=\frac{1}{3}\int_0^{\pi/2}\cos^4(t)\cdot\left[2\sin t\cos t\right]^3\cdot dt$$

$$=\frac{8}{3}\int_0^{\pi/2}\sin^3 t\cos^7 t\,dt=\frac{8}{3}\cdot\frac{\Gamma\frac{3+1}{2}\cdot\Gamma\frac{7+1}{2}}{2\Gamma\frac{3+7+2}{2}}$$

$$=\frac{4}{3}\cdot\frac{\Gamma 2\cdot\Gamma 4}{\Gamma 6}=\frac{4}{3}\cdot\frac{\lfloor 1\lfloor 3}{\lfloor 5}=\frac{1}{15}\ \text{(c)}$$

$$I=\int_0^{\pi} x\cos^2 x\,dx = ?$$

use property 5 of Definite Integration

ie $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$$I=\int_0^{\pi} x\cdot\cos^2 x\,dx = \int_0^{\pi}(\pi-x)\cos^2 x\,dx$$

$$I = \pi\int_0^{\pi}\cos^2 x\,dx - \int_0^{\pi} x\cos^2 x\,dx$$

The value of the integral $\int_0^{\pi} x\cos^2 x\,dx$ is

(a) $\frac{\pi^2}{8}$ (b) $\frac{\pi^2}{4}$

(c) $\frac{\pi^2}{2}$ (d) π^2

$$I = \pi\int_0^{\pi}\cos^2 x\,dx - I$$

$$2I = 2\pi\int_0^{\pi/2}\cos^2 x\,dx$$

$\because \int_0^{\alpha} f(x)dx = 2\int_0^{\alpha/2} f(x)dx$ if $f(\alpha-x)=f(x)$

$$I = \pi\int_0^{\pi/2}\sin^0 x\cdot\cos^2 x\,dx = \pi\cdot\frac{\Gamma\frac{0+1}{2}\,\Gamma\frac{2+1}{2}}{2\,\Gamma\frac{0+2+2}{2}}$$

$(m=0, n=2)$

$$=\frac{\pi}{2}\cdot\frac{\sqrt{\pi}\cdot\frac{1}{2}\sqrt{\pi}}{\Gamma 2} = \frac{\pi^2}{4}$$

(4) Def^n of Gamma func^n → $\boxed{\int_0^{\infty} e^{-x} \cdot x^{n-1}\, dx = \Gamma n}$

(2nd Eulerian Integral)

eg $I = \int_0^{\infty} \frac{x^3}{e^x} dx = ? = \int_0^{\infty} e^{-x} \cdot x^3 dx = \int_0^{\infty} e^{-x} \cdot x^{4-1} dx = \Gamma 4 = 3! = 6$

eg $I = \int_{x=0}^{\infty} e^{-\frac{x^2}{8}} dx = ? = \int_0^{\infty} e^{-y} \cdot \frac{\sqrt{2}\, dy}{\sqrt{y}} = \sqrt{2} \int_0^{\infty} e^{-y} \cdot y^{-\frac{1}{2}} dy$

Put $\frac{x^2}{8} = y \Rightarrow x = \sqrt{8} \cdot \sqrt{y}$

$dx = \sqrt{8} \cdot \frac{1}{2\sqrt{y}} dy = \sqrt{2} \cdot \frac{dy}{\sqrt{y}}$

$= \sqrt{2} \int_0^{\infty} e^{-\infty} \cdot y^{\frac{1}{2}-1} dy$

$= \sqrt{2} \cdot \Gamma \frac{1}{2} = \sqrt{2}\sqrt{\pi} = \sqrt{2\pi}$

Q. $\int_{-\infty}^{\infty} e^{-(x^2/2)}\,dx = ?$ (Even func^n) $= 2\int_0^{\infty} e^{-\left(\frac{x^2}{2}\right)}dx = 2\int_0^{\infty} e^{-y}\cdot\frac{1}{\sqrt{2}\cdot\sqrt{y}}\,dy = \sqrt{2}\int_0^{\infty} e^{-y}\cdot y^{-\frac{1}{2}}\,dy$

(a) $\frac{1}{2}$ (b) $\sqrt{2\pi}$ ✓

(c) 1 (d) ∞

Put $\frac{x^2}{2} = y \Rightarrow x = \sqrt{2}\cdot\sqrt{y}$

$dx = \sqrt{2}\cdot\frac{1}{2\sqrt{y}}\,dy$

$= \sqrt{2}\cdot\int_0^{\infty} e^{-y}\cdot y^{\frac{1}{2}-1}\,dy = \sqrt{2}\cdot\Gamma\frac{1}{2} = \sqrt{2\pi}$

Question of Country :-

$I = \int_0^{\infty} e^{-x^2}\,dx = ? = \int_0^{\infty} e^{-y}\cdot\frac{dy}{2\sqrt{y}} = \frac{1}{2}\int_0^{\infty} e^{-y}\cdot y^{-\frac{1}{2}}\,dy = \frac{1}{2}\int_0^{\infty} e^{-y}\cdot y^{\frac{1}{2}-1}\,dy$

Put $x^2 = y \Rightarrow x = \sqrt{y}$

$dx = \frac{dy}{2\sqrt{y}}$

$= \frac{1}{2}\Gamma\frac{1}{2} = \frac{\sqrt{\pi}}{2}$ i.e. $\boxed{\int_0^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}}$

(5) Def^n of Beta func^n
(1st Eulerian Integral)

$$\boxed{\int_0^1 x^{m-1}\cdot(1-x)^{n-1}\,dx = B(m,n)} = \frac{\Gamma m\cdot\Gamma n}{\Gamma m+n}$$

Note (1) Relationship b/n Beta & Gamma func^n is $B(m,n) = \dfrac{\Gamma m\cdot\Gamma n}{\Gamma m+n}$

(2) Beta func^n is symmetrical about m & n i.e $B(m,n) = B(n,m)$

(3) $$\boxed{\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = B(m,n)} = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}$$

Q. $I = \int_0^2 x(8-x^3)^{1/3}\,dx = ?$

ⓐ 16

ⓑ 16π

ⓒ $\dfrac{16\pi}{3\sqrt{3}}$

ⓓ $\dfrac{16\pi}{9\sqrt{3}}$

Put $x^3 = 8y$

$x = (8y)^{\frac{1}{3}} = 2y^{\frac{1}{3}}$

$dx = 2\cdot\frac{1}{3}y^{-2/3}dy$

At $x=0,\ y=0$

At $x=2,\ y=1$

$$= \int_0^1 2\cdot y^{\frac{1}{3}}\cdot(8-8y)^{\frac{1}{3}}\cdot\frac{2}{3}y^{-\frac{2}{3}}dy$$

$$= \frac{8}{3}\int_0^1 y^{-\frac{1}{3}}(1-y)^{\frac{1}{3}}dy = \frac{8}{3}\int_0^1 y^{\frac{2}{3}-1}\cdot(1-y)^{\frac{4}{3}-1}\cdot dy$$

$$= \frac{8}{3}\cdot B\left(\frac{2}{3},\frac{4}{3}\right) = \frac{8}{3}\cdot\frac{\Gamma\frac{2}{3}\,\Gamma\frac{4}{3}}{\Gamma\left(\frac{2}{3}+\frac{4}{3}\right)}$$

$$= \frac{8}{3}\cdot\frac{\Gamma\frac{2}{3}\cdot\frac{1}{3}\Gamma\frac{1}{3}}{\Gamma 2} = \frac{8}{9}\cdot\Gamma\frac{1}{3}\cdot\Gamma\left(1-\frac{1}{3}\right) = \frac{8}{9}\cdot\frac{\pi}{\sin\left(\frac{\pi}{3}\right)}$$

$$= \frac{8}{9}\,\frac{\pi}{(\sqrt{3}/2)} = \frac{16\pi}{9\sqrt{3}}$$

Q: $I=\int_0^\infty \frac{dx}{1+x^4}=?$ $=\int_0^\infty \frac{\frac{1}{4}y^{-\frac{3}{4}}dy}{(1+y)^1}=\frac{1}{4}\int_0^\infty \frac{y^{\frac{1}{4}-1}}{(1+y)^{\frac{1}{4}+\frac{3}{4}}}\cdot dy$

(a) $\pi/2$

(b) 0

(c) $\pi/2\sqrt{2}$

(d) $\frac{\pi}{4}$

Put $x^4=y \Rightarrow x=y^{\frac{1}{4}}$

$dx=\frac{1}{4}y^{-3/4}dy$

$=\frac{1}{4}B\left(\frac{1}{4},\frac{3}{4}\right)=\frac{1}{4}\frac{\Gamma\frac{1}{4}\cdot\Gamma\frac{3}{4}}{\Gamma\left(\frac{1}{4}+\frac{3}{4}\right)}$

$=\frac{1}{4}\left[\frac{\frac{\pi}{\sin\left(\frac{\pi}{4}\right)}}{\Gamma 1}\right]=\frac{1}{4}(\pi\sqrt{2})=\frac{\pi}{2\sqrt{2}}$

W.K.That $\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx=B(m,n)$

Question of Country: ① $\int_0^{\pi/2} \log(\sin x)\,dx = -\frac{\pi}{2}\log 2$, ② $\int_0^{\infty} e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$

Learn by ♡

③ $\int_0^{\infty}\left(\frac{\sin ax}{x}\right)dx = \frac{\pi}{2}$ $(a>0)$, ④ $\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\,dx\,dy = \frac{\pi}{4}$

(using Laplace Transform) (Using Double Integral)

eg $\boxed{\int_0^{\infty}\frac{\sin x}{x}dx = \frac{\pi}{2}}$, $\int_0^{\infty}\left(\frac{\sin 2x}{x}\right)dx = \frac{\pi}{2}$, $\int_0^{\infty}\left(\frac{\sin 3x}{x}\right)dx = \frac{\pi}{2}$ .....

Proof of ④: $I = \int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\,dx\,dy = \int_0^{\infty} e^{-x^2}dx \times \int_0^{\infty} e^{-y^2}dy = \frac{\sqrt{\pi}}{2}\times\frac{\sqrt{\pi}}{2} = \frac{\pi}{4}$

(18) $I=\int_0^{\infty}\frac{12\cos\pi t\,\sin 2\pi t}{\pi t}\,dt=\,?=\frac{6}{\pi}\int_0^{\infty}\frac{2.\sin(2\pi t)\cos(\pi t)}{t}\,dt$

(a) 0

(b) $6/\pi$

(c) 6

(d) $\pi/2$

$2\sin A\cos B$

$=\sin(A+B)+\sin(A-B)$

$=\frac{6}{\pi}\int_0^{\infty}\left[\frac{\sin(3\pi t)+\sin(\pi t)}{t}\right]dt$

$=\frac{6}{\pi}\left[\int_0^{\infty}\left(\frac{\sin 3\pi t}{t}\right)dt+\int_0^{\infty}\left(\frac{\sin\pi t}{t}\right)dt\right]$

$=\frac{6}{\pi}\left[\frac{\pi}{2}+\frac{\pi}{2}\right]=6$

$$\int_0^1 x^6\sqrt{1-x^2}\,dx =$$

(a) $\frac{5\pi}{256}$ ✓ (b) $\frac{5\pi}{128}$

(c) $\frac{5\pi}{512}$ (d) $\frac{3\pi}{512}$

$$I=\int_0^1 x^6\sqrt{1-x^2}\,dx$$

$$=\int_0^1 x^6(1-\underline{x^2})^{\frac{1}{2}}dx$$

Put $x^2=y \Rightarrow x=\sqrt{y}$, $dx=\frac{1}{2\sqrt{y}}dy$

$$I=\int_0^1(\sqrt{y})^6(1-y)^{\frac{1}{2}}\cdot\frac{1}{2\sqrt{y}}dy$$

$$=\frac{1}{2}\int_0^1 y^3\cdot y^{-\frac{1}{2}}\cdot(1-y)^{\frac{1}{2}}\cdot dy$$

$$I=\frac{1}{2}\int_0^1 y^{\frac{5}{2}}\cdot(1-y)^{\frac{1}{2}}dy$$

$$=\frac{1}{2}\int_0^1 y^{\frac{7}{2}-1}\cdot(1-y)^{\frac{3}{2}-1}dy$$

$$=\frac{1}{2}B\left(\frac{7}{2},\frac{3}{2}\right)=\frac{1}{2}\cdot\frac{\Gamma\frac{7}{2}\cdot\Gamma\frac{3}{2}}{\Gamma\left(\frac{7}{2}+\frac{3}{2}\right)}$$

Ans.

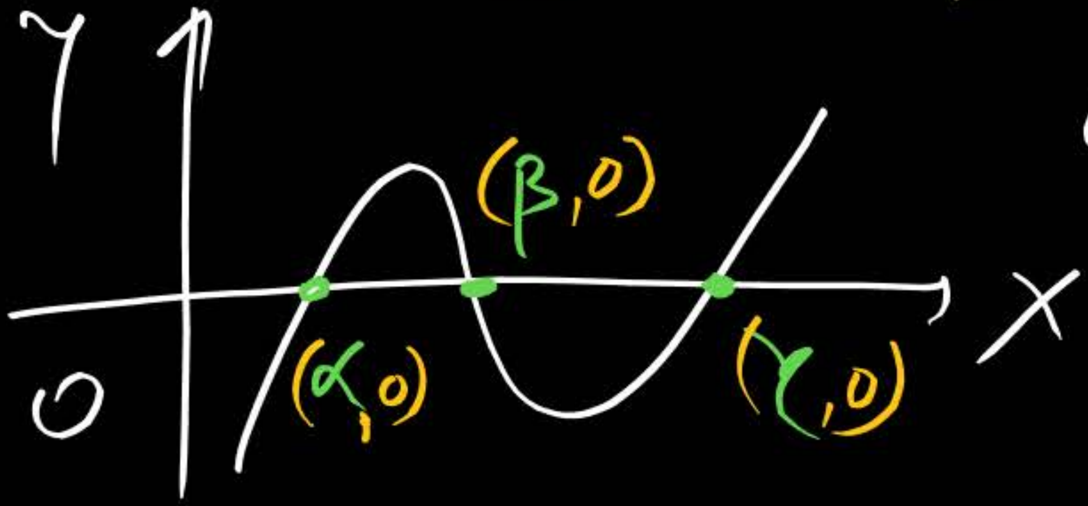
$$=\frac{1}{2}\cdot\frac{\left(\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}\right)\cdot\left(\frac{1}{2}\cdot\sqrt{\pi}\right)}{\Gamma 5}$$

$$=\frac{5\pi}{256}$$ Ans.

NATURE of ROOTS

(*) Roots / Solutions / ZEROCROSSING / ZEROS $\longrightarrow$ (points touching X axis)

(Real Roots) (Complex Roots)



Consider $y = f(x)$ s.t α, β, γ are the Roots

then $y = f(\alpha) = f(\beta) = f(\gamma) = 0$

(*) Complex Roots occurs in pair only when coefficients are Real.

eg Consider $x^2 - (i+1)x + i = 0$

$$x^2 - ix - x + i = 0$$

$$x(x-i) - 1(x-i) = 0 \Rightarrow (x-i)(x-1) = 0 \Rightarrow x = 1 \,\&\, i$$

(*) n^{th} degree poly has exactly $\underline{n}$ roots whether Real or Complex.

(*) n^{th} degree poly (with Real Coeff) has exactly $\underline{n}$ roots in which at least $\underline{\text{one}}$ will be Real & at Most $\underline{n-1}$ will complex. (where n = odd)

bcoz in that situation Complex Roots will be in pair.

(*) <u>Descartes Rule of Sign</u> →

(i) No. of +ve Real Roots of $f(x) \leq$ No. of times sign changes in $f(x)$

(ii) No. of -ve Real Roots of $f(x) \leq$ No. of times sign changes in $f(-x)$

Q→ choose the possible correct options for $f(x) = x^9 + 5x^3 - x^2 + 7x + 2$

MSQ

(a) $f(x)$ has at Most 2 +ve Roots ($\because$ No. of times sign changes in $f(x) = 2$)

(b) $f(x)$ has at most 3 -ve Roots

$f(-x) = (-x)^9 + 5(-x)^3 - (-x)^2 + 7(-x) + 2$

$= -x^9 - 5x^3 - x^2 - 7x + 2$

(c) $f(x)$ has at least 4 Complex Roots

So No. of times sign changes in $f(-x)$ = one

(d) $f(x)$ has at least one Real Root ($\because$ $f(x)$ is an odd degree poly with Real Coeffs)

The polynomial $p(x) = x^5 + x + 2$ has

(a) all real roots

(b) 3 real and 2 complex roots

(c) 1 real and 4 complex roots

(d) all complex roots

$p(x) = x^5 + x + 2$

No sign change in $p(x)$

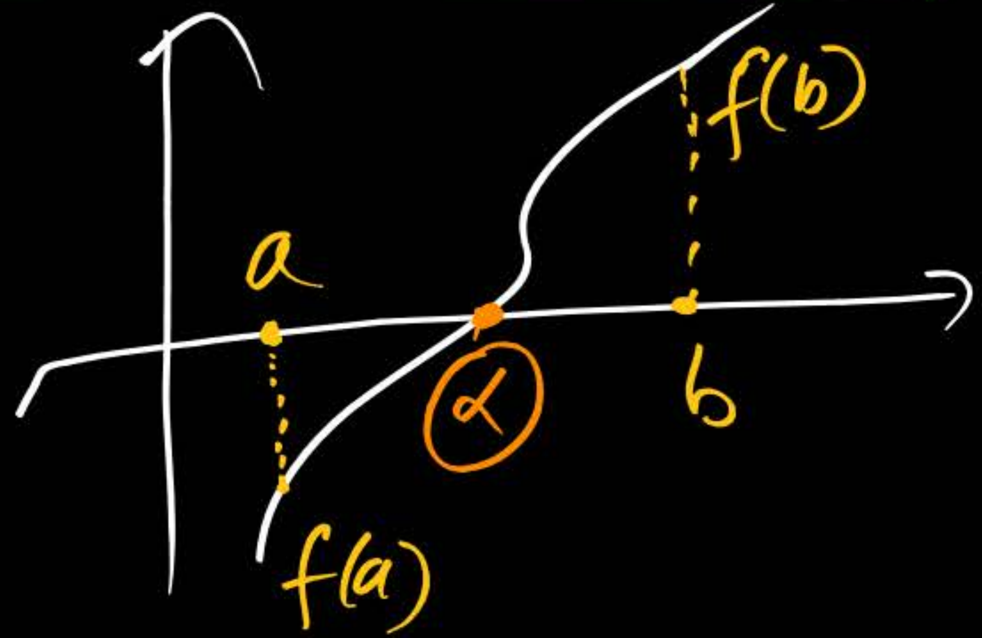
$\Rightarrow$ No +ve Root.

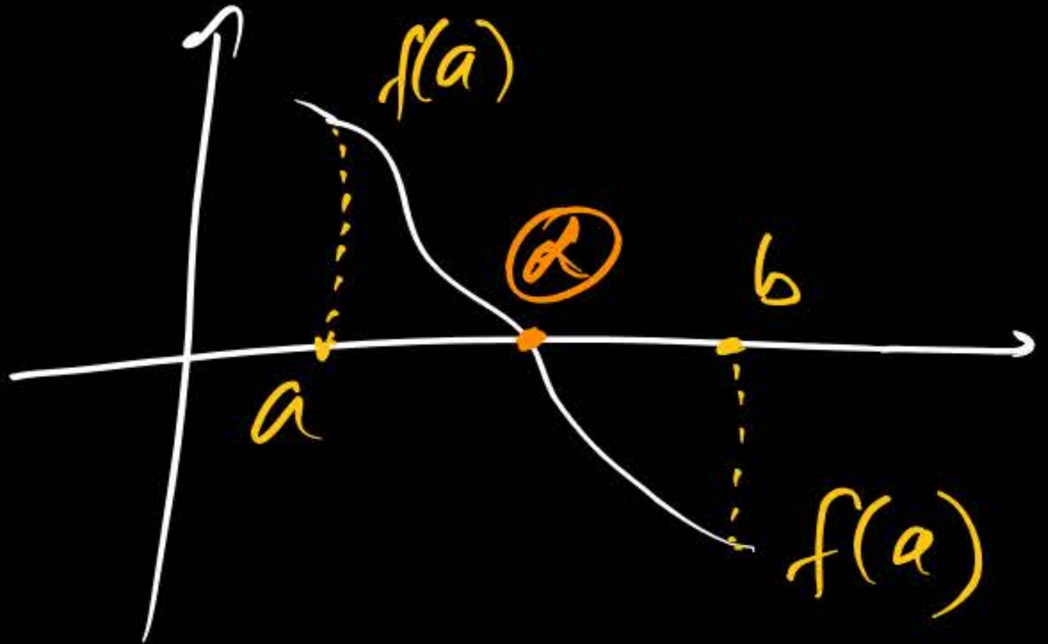
$p(-x) = -x^5 - x + 2$

No of -ve Real Roots ≤ 1

So No of Complex Roots ≤ 4

BOLZANO THEOREM →





$\Rightarrow \quad f(a)\cdot f(b) < 0$

$\Rightarrow \exists\, \alpha \in (a,b)$

$f(a)\cdot f(b) < 0$

$\exists\, \alpha \in (a,b)$

"if at $x=a$ & at $x=b$, $f(a)$ & $f(b)$ are of opposite sign then $\exists$ at least one Root α of $f(x)$ b/w a & b"

or "if $f(a)\cdot f(b)<0$ then $\exists$ at least one $\alpha \in (a,b)$ s.t $f(\alpha)=0$."

Q: Consider $\boxed{f(x) = x^4 - x^3 - x^2 - 4}$, $[1, 9]$ then $\alpha = ?$ where $\alpha \in (1, 9)$

Sol: $\because f(1) = -ve$

$f(9) = +ve$ So By B. Th $\alpha \in (1, 9)$

Again $f(5) = +ve$ So By B. Th $\alpha \in (1, 5)$

Again $f(3) = +ve$ " " " $\alpha \in (1, 3)$

Again $\boxed{f(2) = 0}$ So $\alpha = 2$ Ans

Q. if $xe^x = \cos x$ then one of the sol of this eqn lies in b/w

ⓐ $(2, 3)$

ⓑ $(-1, 0)$

ⓒ $(0.56, 0.60)$

ⓓ $(1, 2)$

Let $\boxed{f(x) = xe^x - \cos x}$

ⓐ $f(2) = +ve$, $f(3) = +ve \Rightarrow \alpha \notin (2, 3)$

ⓑ $f(-1) = -1 \cdot e^{-1} - \cos(-1) \approx -\frac{1}{e} - \cos\left(\frac{\pi}{3}\right) = -ve$

$f(0) = 0 - \cos(0) = -1 = -ve \Rightarrow \alpha \notin (-1, 0)$

ⓓ $f(1) = e - \cos(1) = 2.71 - (\text{value b/w } -1 \,\&\, 1) = +ve$

$f(2) = 2e^2 - \cos(2) = +ve \Rightarrow \alpha \notin (1, 2)$

ⓒ $f(0) = -\cos 0 = -1$, $f(1) = e - \cos(1) = +ve \Rightarrow \alpha \in (0, 1)$ ✓

A non-zero polynomial f(x) of degree 3 has roots at $x = 1$, $x = 2$ and $x = 3$. Which one of the following must be TRUE?

(a) $f(0)\,f(4) < 0$ (b) $f(0)\,f(4) > 0$

(c) $f(0) + f(4) > 0$ (d) $f(0) + f(4) < 0$

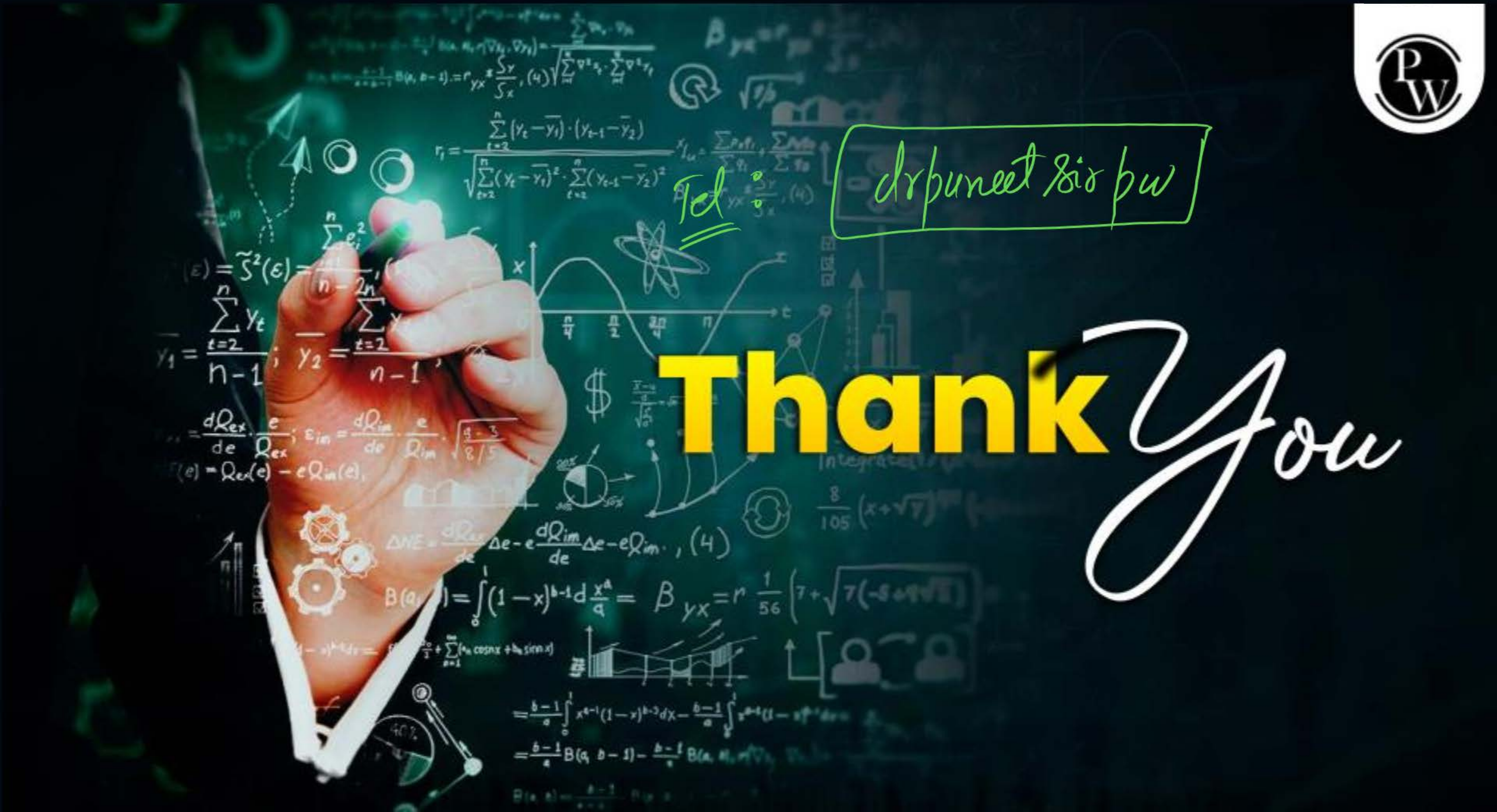
$f(x)$: $\alpha = 1$, $\beta = 2$, $\gamma = 3$

if $f(0) \cdot f(4) < 0$

i.e. $f(0)$ & $f(4)$ are of opposite sign

So By B. Th, $\exists$ at least one root α of $f(x)$ b/w 0 & 4

& Here $\alpha = 1, 2, 3 \in (0, 4)$ So (a)


Tel : drpuneet sir pw
Thank You