PORTFOLIO OPTIMIZATION

A Project Report Submitted
in Partial Fulfilment of the Requirements
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in

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by

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to the

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CERTIFICATE

This is to certify that the work contained in this report entitled **"Portfolio Optimization"** submitted by **Priyanshu Gupta** (**Roll No: 202123034**) to Department of Mathematics, Indian Institute of Technology Guwahati towards the requirement of the course **MA699 Project** has been carried out by him under my supervision.

Guwahati - 781 039 | (Prof. Chandan Pal)

April, 2022 | Project Supervisor

ABSTRACT

The main aim of the project is to optimize the portfolio consisting of risky as well as non risky assets. I started with defining several securities or contracts namely, bonds, stocks, forward, futures, swaps and options. I also described the time value of money - how simple/periodic compounding /continuous compounding interest works and how can we compare any two scheme , be it of any type of interest rate. In chapter 3, initially I explained the expected returns and risks associated with any (single) risky asset followed by two and then for several securities using few theorems and results. At last, I implemented the theory described in third chapter. I used an excel file for real market returns. The excel file contains the weekly returns of 87 stocks for 5 years (i.e., 261 weeks). I used an open source programming language - R, for statistical analysis of this data and concluded with the weights in both cases - MVP and MVL. I also gave the returns and risk value with this weighted portfolio.

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Chapter 1

Prerequisites

1.1 Classification of Securities/Contracts

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   (b) Risky (e.g. - Stocks).
2. Derivatives and Contracts
   (a) Forward or Futures
   (b) Swaps
   (c) Options

1.2 Bonds

Bonds are fixed income securities which give the owner the right to a fixed, pre-determined payment (also called **nominal/face/par value or princi-**

pal), at a future determined date (also called the **maturity date**). The party that promises to pay is the **debtor** and the party that will get paid is the **creditor**. Both of these parties are called the **counter-parties**. The difference between the bond price and the nominal value is called **interest**. The interest as the percentage of the total value is called **interest rate**. Bonds are risk free securities in principal as you know the nominal value in advanced. If a person has some purchasing power that (s)he would prefer to delay in order to earn interest, then (s)he could buy a bond with long maturity.

1.2.1 Types of Bonds

- Depending on Maturity :
    1. Short Term Bonds (Less than or equal to 1 year maturity usually)
    2. Long Term Bonds (More than 1 year maturity usually)
- Depending on Discounts :
    1. Pure discount bonds (Involves only an initial and final payment)
    2. Coupon bonds (Debtor makes various periodic payments called coupons, a predetermined percentage of face value, to the creditor)

Price at which the bond is sold can be higher , exactly same and lower than the nominal value, than it's called **above par**, **at par** and **below par** respectively. Pure discount bonds are always below par while coupon bonds can be anything - above/at/below par.

1.2.2 Risk associated with bonds

1. **Credit or Default Risk** - If the debtor fails to meet the payment obligations of the bond (i.e. the debtor defaults).

2. **Inflation Risk** - Since the future prices are uncertain due to inflation, the actual face value (inflation adjusted) may be lesser than the (current) invested value.

3. **Liquidity Risk** - If the creditor sells the bonds prior to maturity, (s)he may get a lower price than expected.

1.3 Stocks

Stock is a security that gives its owner (stock-holder) the right to a proportion of any profit (dividends) that might be distributed rather than reinvested by the (stock-issuing) firm. The dividends are not known in advance and depends on the firm's profit and policy. Hence, there is no guarantee of any nominal value unlike in bonds. Stock can be sold to another person and that person will be the new owner of that stock from that time. Note that, there can be negative return (if the selling price is lower than the buying price), zero return (if the selling price is equal to the buying price) and positive return (if the selling price is greater than the buying price) in case of stock. There are two sources of returns in case of stocks -

- **Dividend Gains** - Dividends received while in ownership of the stocks.

- **Capital Gains** - Difference between selling price and buying price of the stocks.

1.3.1 Short and Long Positions

- **Short Position** - Consists of borrowing the stock from someone who owns it and then selling it (here the short seller is in short position), with the hope of dropping of stock price to buy the stock at a lower price and return it to the owner (here the short seller is covering the short position). In case of bonds, the debtor is in short position.

- **Long Position** - The act of buying the stock is said to be the long position. In case of bonds, the creditor is in long position.

1.4 Forward or Futures

These are the contracts where one party agrees to buy the underlying asset at a future predetermined date (**Maturity date**) at a predetermined price (**Forward/Future Price**). The current market price of that asset is called **Spot Price**. The terms of a forward contract are negotiated between buyer and seller. Hence it is customizable. Also forward contracts are settled on a maturity date. Conversely, a futures contract is a standardized one where the conditions relating to quantity, date, and delivery are standardized. The future contract is marked to market on a daily basis, i.e. the profit or losses are settled daily. This is a zero sum game i.e. the profit of one party is the loss of other party.

1.5 Swaps

These are the contracts by which two parties exchange cash flows. In case of interest rate swaps, suppose you are paying back a loan whose interest rate is a variable and you give a fixed payment to a person (**swap seller**) and he takes care of all the variable payments up to a certain cap. The principal amount is called as **notional principal**.

1.6 Options

An option is a financial derivative or security that gives its owner the right to buy/sell another (underlying) security, on or before a predetermined date (**Maturity/Expiration Date** T) for a predetermined price (**Strike/Exercise Price** K). This is different than forward or futures in a context that the owner of a option may or may not buy/sell underlying asset. The act of buying/selling the underlying asset is called **exercising the option**. Since the owner has some additional right, he has to pay some upfront fee (**Premium/Option Price**) to the seller of the option right at the beginning. Options work as the hedge of the risk of the market.

1.6.1 Classifications of options

- **Call Option** - The owner has the right to buy the underlying asset.
- **Put Option** - The owner has the right to sell the underlying asset.
- **European Option** - The owner has the right to buy/sell the underlying asset only on a fixed future date.

- **American Option** - The owner has the right to buy/sell the underlying asset on or before (including) a fixed future date. (Early exercise is also possible).

So there can be basically four different combinations, namely -

1. **European Call**

   Let us take a scenario where the owner of the option has the right to buy some stock on maturity date T and strike price K. Let $S(t)$ be the spot price of that stock at time t and $S(T)$ be the price of the stock at time $T(> t)$. If $S(T) > K$, the owner will exercise the call option and get a profit of $S(T) - K$ otherwise the owner will not exercise the call option. Hence,

   $$Payoff = max\{S(T) - K, 0\}$$

2. **European Put**

   $$Payoff = max\{K - S(T), 0\}$$

3. **American Call**

   Suppose the option is exercised at time $T1 \leq T$

   $$Payoff = max\{S(T1) - K, 0\}$$

4. **American Put**

   $$Payoff = max\{K - S(T1), 0\}$$

Chapter 2

Risk-Free Assets

2.1 Time Value of Money

$N\$$ after some time is way less than $N\$$ today, simply because of two main points that they are locked till future point and prices may increase due to inflation which will decrease the purchasing power. So one should be compensated for this postponed time. Usually, this compensation is said to be the interest.

2.1.1 Simple Interest

Suppose we deposit some amount into a bank to earn some interest. The future value is determined by initial deposit (Principal P), interest rate $r(> 0)$ and time t. In case of simple interest, where our interest is not earning interest, the value of our investment at time $t \geq 0$ will be

$$V(t) = (1 + rt)P \tag{2.1}$$

the number $1 + rt$ is called the growth factor. If the Principal is invested at time s, then the value at time $t(\geq s)$ will be

$$V(t) = (1 + r(t - s))P \tag{2.2}$$

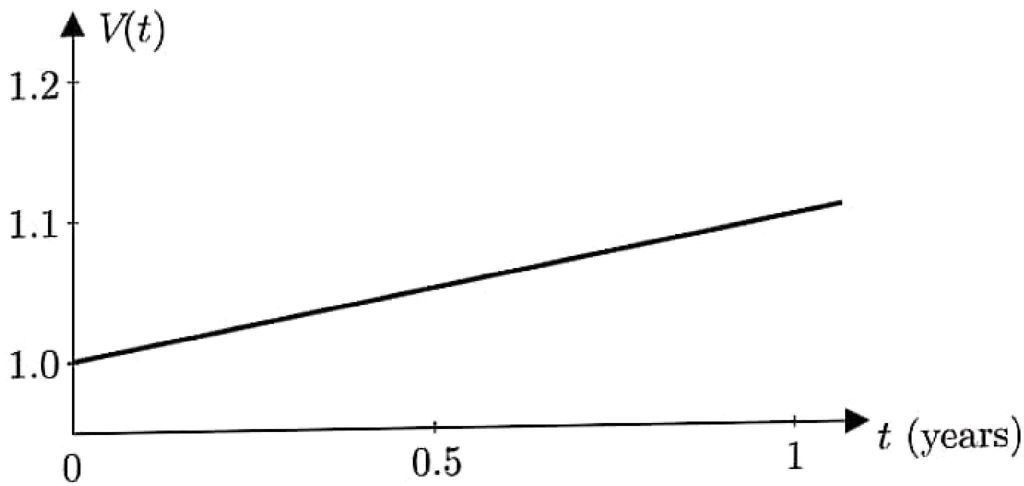

Figure 2.1: Principal attracting simple interest at 10% ($r = 0.1, P = 1$)

The return of this investment will be

$$K(s,t) = \frac{V(t) - V(s)}{V(s)} \tag{2.3}$$

In case of simple interest,

$$K(s,t) = r(t - s) \tag{2.4}$$

It is clear that interest rate is equal to return over one year (put $t = s + 1$ in equation 2.4).

Present/discounted value of $V(t)$ is given by

$$V(0) = V(t)(1 + rt)^{-1}. \tag{2.5}$$

where $(1 + rt)^{-1}$ is called the discount factor.

A perpetuity is a sequence of payment of fixed amount to be made at equal time intervals and continuing indefinitely into the future (e.g. - Rent on real estate). For example, an initial deposit of $\frac{P}{r}$ will get you P amount payable every year (in case of simple interest).

2.1.2 Periodic Compounding

Generally simple interest is used only for short term investments or some specific loans. In principal, your earned interest also earns interest periodically, which is known as **compounding**. If m interest payments are made per annum and the interest rate r remains unchanged, after t years the future value of an initial principal P will become

$$V(t) = (1 + \frac{r}{m})^{tm} P \tag{2.6}$$

where t must be a whole multiple of $\frac{1}{m}$. The number $(1 + \frac{r}{m})^{tm}$ is the growth factor.

The return of this investment will be

$$K(s,t) = \frac{V(t) - V(s)}{V(s)} = (1 + \frac{r}{m})^{(t-s)m} - 1 \tag{2.7}$$

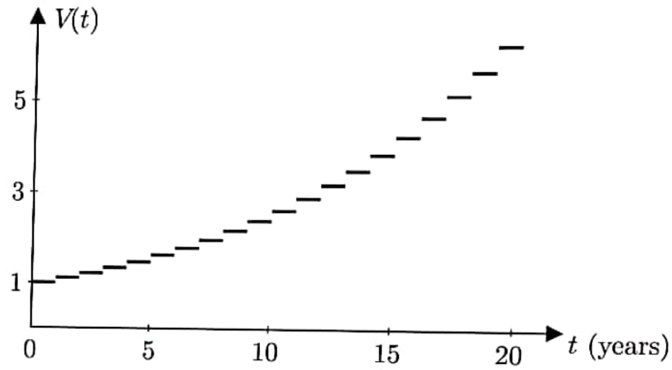

Figure 2.2: Annual compounding at 10% ($m = 1, r = 0.1, P = 1$)

It is clear that interest rate is equal to return over one year provided $m = 1$ (put $t = s + 1$ in equation 2.7).

Clearly, the return $K(s,t)$ is not additive $\{\because K(0,1) + K(1,2) \neq K(0,2)\}$

In case of compound interest, the present/discounted value of $V(t)$ is

$$V(0) = V(t)(1 + \frac{r}{m})^{-tm}, \tag{2.8}$$

where the number $(1 + \frac{r}{m})^{-tm}$ is the discount factor.

It can be proven easily (using binomial formula) that the future value $V(t)$ increases if m, t, r or P increases. Also if $V(t)$ is kept fixed, present value increases if r, t or m decreases.

2.1.3 Streams of Payments

An annuity is a sequence of finitely many payments of a fixed amount due at equal time intervals (e.g. - Regular Deposit to

a savings account). For example, regular deposit of P (with annual compound interest r) made once a year for n years will have the present value

$$PV = \frac{P}{1+r} + \frac{P}{(1+r)^2} + \cdots + \frac{P}{(1+r)^n} = P \times \frac{1-(1+r)^{-n}}{r} = P \times PA(r,n)$$

where $PA(r,n)$ is called present value factor for an annuity.
Formula for present value of a perpetuity can be obtained from above equation by letting $n \to \infty$

$$\lim_{n\to\infty} P \times PA(r,n) = \frac{P}{r}$$

We can observe that this is the same value as we studied in the case of simple interest. The reason is that the amount remaining to earn interest in the following year is always $\frac{P}{r}$.

2.1.4 Continuous Compounding

Suppose the invested money is compounded annually and one withdraws the money after 11 months, the person gets no interest. If $m \to \infty$ in equation (2.6), we get

$$V(t) = e^{tr}P \tag{2.9}$$

which is called continuous compounding. The growth factor is e^{tr}. The rate of growth is directly proportional to the current wealth. It gives higher returns compared to any frequency m. In case of continuous compounding, the present/discounted value of $V(t)$ is

$$V(0) = V(t)e^{-tr}, \tag{2.10}$$

where the number e^{-tr} is the discount factor.

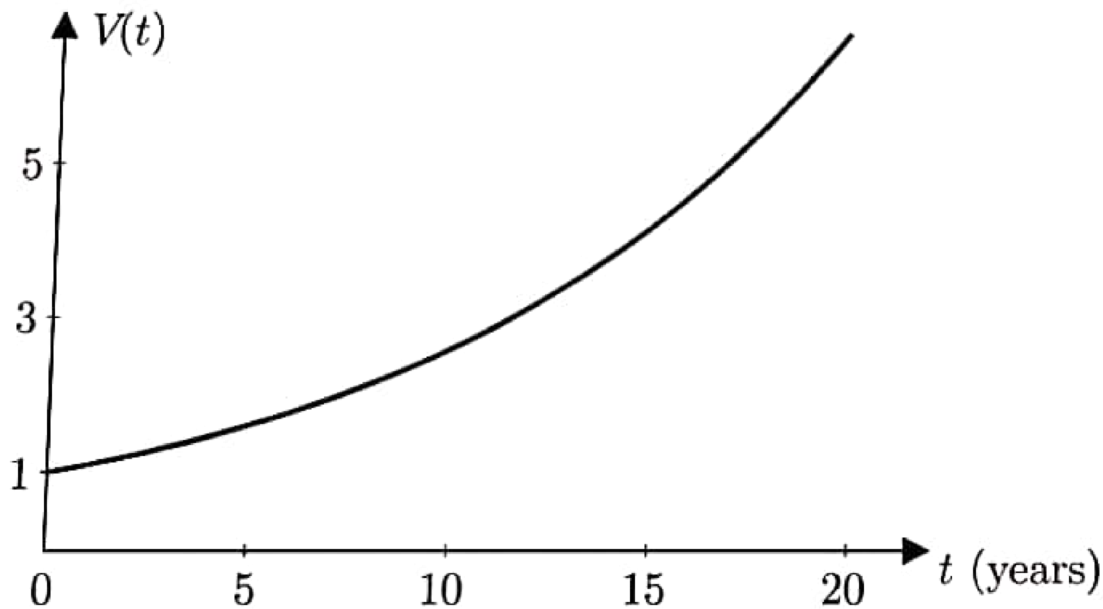

Figure 2.3: Continuous compounding at 10% ($r = 0.1, P = 1$)

Clearly, the usual return $K(s,t)$ is not additive $\{\because K(0,1) + K(1,2) \neq K(0,2)\}$, so we introduce the logarithmic returns

$$k(s,t) = \ln \frac{V(t)}{V(s)} = (t-r)s. \tag{2.11}$$

Note that, logarithmic returns are additive. $\{k(s,t) + k(t,u) = k(s,u)\}$

2.1.5 How to Compare Compounding Methods

We say that two compounding methods are equivalent if the corresponding growth factors over a period of one year are the same. If first growth factor exceeds the second, we say that the first method is preferable. For example, 10% semi-annual compounding (growth factor = $(1+\frac{0.1}{2})^2 = 1.1025$) is preferable to 9% monthly compounding (growth factor = $(1+\frac{0.9}{12})^{12} = 1.0938$).
For a given compounding method with interest rate r the effective rate r_e is one that gives the same growth factor over the one year period under annual compounding, i.e. -

$$1 + r_e = \begin{cases} (1+\frac{r}{m})^m & \text{, Periodic compounding} \\ e^r & \text{, Continuous compounding} \end{cases}$$

Hence two compounding methods are equivalent if and only if the corresponding effective rates are equal. If first effective rate exceeds the second, we say that the first method is preferable.
In terms of the effective rate r_e the future value can be written as

$$V(t) = (1+r_e)^t P, \qquad \forall t \geq 0. \tag{2.12}$$

Cash flows of several payments

Consider a time interval $[0, T]$ divided into m equal sub-intervals, say (t_i, t_{i+1}), with r interest rate per sub-interval, $t_0 = 0$ and $t_m = T$. We invest $P_i (\in \mathbb{R})$ after the end of i^{th} sub-interval ($\forall i \geq 0$). Negative P_i shows the amount invested and positive P_i shows the amount withdrawn. The future value of these cash flows

$$V(T) = \begin{cases} \sum_{i=0}^{m} P_i(1+r)^{(m-i)} & \text{, Periodic compounding} \\ \sum_{i=0}^{m} P_i e^{r(T-t_i)} & \text{, Continuous compounding} \end{cases}$$

Suppose we need to compare two project with complex investments and pick out the better one. The investment with better present value will be the better one.The present value of these cash flows

$$V(0) = \begin{cases} \sum_{i=0}^{m} \frac{P_i}{(1+r)^i} & \text{, Periodic compounding} \\ \sum_{i=0}^{m} \frac{P_i}{e^{rt_i}} & \text{, Continuous compounding} \end{cases}$$

Chapter 3

Portfolio Management

Portfolio is basically a collection of (risky or risk free) assets. The general goal of a rational investor is to minimize the risk and maximize the profit. But high returns are generally associated with high risks.

3.1 Risk and Return

Warren Edward Buffett is an American investor and CEO of Berkshire Hathaway. He says -

- Don't put all your eggs in one basket.
- Diversification may preserve wealth, but concentration builds wealth.

These two statements at the first glance seem to be contradictory but the first one says to minimize risk and the second one says to maximize returns.

3.1.1 Expected Return

Suppose we invest $S(0)$ amount in some stock at time $t = 0$, at time $t = T$ the price $S(T)$ is unknown and hence assumed to be a random variable. $S(T) : \Omega \to [0, +\infty]$ on a probability space Ω. The return will be

$$K = \frac{S(T) - S(0)}{S(0)} \tag{3.1}$$

This K is a random variable with expected value $\mathbb{E}(K) = \mu_K = \mu$. By the linearity of mathematical expectation

$$\mathbb{E}(K) = \frac{\mathbb{E}(S(T)) - S(0)}{S(0)} \tag{3.2}$$

Hence

$$S(T) = S(0)(1 + K)$$

$$\mathbb{E}(S(T)) = S(0)(1 + \mu)$$

3.1.2 Standard Deviation as Risk Measure

One should not always look at the expected returns whereas risk should also be considered. The spread of expected returns combined with the probabilities can be considered as a component to compute risk. Hence, the obvious choice is the standard deviation as the risk measure. The standard deviation of the return K

$$\sigma_K = \sqrt{Var(K)}.$$

where $Var(K)$ is the variance of the return K.

$$Var(K) = \mathbb{E}(K - \mu)^2 = \mathbb{E}(K^2) - \mu^2$$

It is clear that $\sigma_{aK} = |a|\sigma_K$ and $Var(aK) = a^2 Var(K)$ where $a \in \mathbb{R}$

Example

Suppose there are two stocks $S_1 \& S_2$. The returns of S_1 are 11% with probability 0.25 and 13% with probability 0.75, whereas the returns of S_2 are 2% with probability 0.25 and 22% with probability 0.75.

Expected return for stock j $(\mu_j) = \displaystyle\sum_{i=1}^{n} K_i p_i$.

Risk associated with stock j $(\sigma_j) = \sqrt{\displaystyle\sum_{i=1}^{n} (K_i - \mu_j)^2 p_i}$.

where n is the number of cases.

Hence $\mu_1 = 12.5\%, \mu_2 = 17\%, \sigma_1 = 0.866 \,\&\, \sigma_2 = 8.66$

$\Rightarrow S_2$ is giving higher returns but also taking higher risk.

3.2 Two Securities

We invest in two stocks here to mitigate some risk. letś introduce weights to identify the proportional allocation of funds to each security instead of specifying the number of shares of each security. For two securities, weights are

$$w_1 = \frac{x_1 S_1(0)}{V(0)}, w_2 = \frac{x_2 S_2(0)}{V(0)}$$

where $V(0)$ is the initial investment and $S_k(0)$ is the price of k^{th} stock at time t=0. Also x_k is the number of shares of stock k (for k = 1,2).

Observations

1. Clearly, $w_1 + w_2 = \frac{x_1 S_1(0) + x_2 S_2(0)}{V(0)} = 1$

2. If short selling is allowed, one of the weights will be negative and other will be greater than 1 but their sum remains 1.

3. Even though the actual number of shares remain unchanged, the weights change as the stocks prices change.

If we allow short selling, we can get more benefit in some cases. For example, suppose a portfolio with $V(0) = 1000\$, S_1(0) = 30\$, S_2(0) = 40\$, w_1 = 1.20$ and $w_2 = -0.2$, then $x_1 = w_1 * \frac{V(0)}{S_1(0)} = 40$ and $x_2 = -5$.
Suppose at time $t = T, S_1(T) = 35\$$ and $S_2(T) = 39\$$, the worth of the portfolio is

$$V(T) = x_1 S_1(T) + x_2 S_2(T) = V(0)\left(w_1 \frac{S_1(T)}{S_1(0)} + w_2 \frac{S_2(T)}{S_2(0)}\right) = 1205\$$$

If we choose weights to be 0.6 and 0.4 respectively, we get $V(T) = 1090\$$ which is lower than before.

Theorem 3.2.1. *The return K_V on a portfolio consisting of two securities is the weighted average*

$$K_V = w_1 K_1 + w_2 K_2 \tag{3.3}$$

where w_1 and w_2 are the weights and K_1 and K_2 are the returns on the two components.

Proof. From previous example,

$$\begin{aligned}
V(T) &= V(0)\left(w_1 \frac{S_1(T)}{S_1(0)} + w_2 \frac{S_2(T)}{S_2(0)}\right) \\
&= V(0)\left(w_1(1+K_1) + w_2(1+K_2)\right) \\
K_V &= \frac{V(T) - V(0)}{V(0)} \\
&= w_1(1+K_1) + w_2(1+K_2) - 1 \\
&= w_1 K_1 + w_2 K_2
\end{aligned}$$

□

3.2.1 Risk and Expected Return of a Portfolio

Theorem 3.2.2. *The expected return $\mathbb{E}(K_V)$ on a portfolio of two securities is given by*

$$\mathbb{E}(K_V) = w_1 \mathbb{E}(K_1) + w_2 \mathbb{E}(K_2) \tag{3.4}$$

Proof. This follows directly from (3.3) and the additivity of mathematical expectation. □

Theorem 3.2.3. *The variance $Var(K_V)$ for a portfolio of two securities is given by*

$$Var(K_V) = w_1^2 Var(K_1) + w_2^2 Var(K_2) + 2w_1w_2Cov(K_1, K_2). \qquad (3.5)$$

Proof.

$$\begin{aligned} Var(K_V) &= \mathbb{E}(K_V^2) - \mathbb{E}(K_V)^2 \\ &= w_1^2[\mathbb{E}(K_1^2) - \mathbb{E}(K_1)^2] + w_2^2[\mathbb{E}(K_2^2) - \mathbb{E}(K_2)^2] + 2w_1w_2[\mathbb{E}(K_1K_2) - \mathbb{E}(K_1)\mathbb{E}(K_2)] \\ &= w_1^2 Var(K_1) + w_2^2 Var(K_2) + 2w_1w_2Cov(K_1, K_2) \end{aligned}$$

$\square$

Let us introduce the following notations -

$$\mu_V = \mathbb{E}(K_V), \qquad \mu_1 = \mathbb{E}(K_1), \qquad \mu_2 = \mathbb{E}(K_2)$$

$$\sigma_V = \sqrt{Var(K_V)}, \qquad \sigma_1 = \sqrt{Var(K_1)}, \qquad \sigma_2 = \sqrt{Var(K_2)}$$

$$c_{12} = Cov(K_1, K_2), \qquad \rho_{12} = \frac{c_{12}}{\sigma_1\sigma_2}$$

where $\rho_{12}(\in [-1, 1])$ is correlation coefficient. It is undefined when $\sigma_1\sigma_2 = 0$. It tells the strength of the linear relationship between two securities. $\rho_{12} = +1$ indicates that the two move in same direction. $\rho_{12} = -1$ indicates that the two move in opposite direction. $\rho_{12} = 0$ indicates that there is no relationship between them.

Theorem 3.2.4. *The variance σ_V^2 of a portfolio can not be more than $max\{\sigma_1^2, \sigma_2^2\}$*

provided no short selling is allowed.

Proof. Suppose $\sigma_1^2 \le \sigma_2^2$.
Short selling is not allowed means, $w_1, w_2 \ge 0$.

$$\begin{aligned}
\sigma_V^2 &= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{12}\sigma_1\sigma_2 \\
&\le w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2 \\
&= (w_1\sigma_1 + w_2\sigma_2)^2 \\
&\le (w_1\sigma_2 + w_2\sigma_2)^2 \\
&= \sigma_2^2 = max\{\sigma_1^2, \sigma_2^2\}
\end{aligned}$$

The other case has the similar proof. □

Remark 3.2.5. If short selling is allowed, σ_V^2 may be strictly greater than $max\{\sigma_1^2, \sigma_2^2\}$.

3.2.2 Feasible Set

The collection of all portfolios that can be manufactured by investing in two given assets is called the feasible or attainable set. Each such portfolio can be represented by a point (σ_V, μ_V) in the σ, μ plane. i.e.,

$$\mu_V = w_1\mu_1 + w_2\mu_2 \tag{3.6}$$

$$\sigma_V^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{12}\sigma_1\sigma_2 \tag{3.7}$$

where $w_1, w_2 \in \mathbb{R}$ and $w_1 + w_2 = 1$

Putting $w_1 = s$ and $w_2 = 1 - s$ in above equations, we get

$$\mu_V = s\mu_1 + (1-s)\mu_2 \tag{3.8}$$

$$\sigma_V^2 = s^2\sigma_1^2 + (1-s)^2\sigma_2^2 + 2s(1-s)\rho_{12}\sigma_1\sigma_2 \tag{3.9}$$

where $s \in \mathbb{R}$.

Theorem 3.2.6. *If $\rho_{12} < 1$ or $\sigma_1 \neq \sigma_2$, then σ_V^2 as a function of s attains its minimum value at*

$$s_0 = \frac{\sigma_2^2 - c_{12}}{\sigma_1^2 + \sigma_2^2 - 2c_{12}}. \tag{3.10}$$

The corresponding values of the expected return μ_V and variance σ_V^2 are -

$$\mu_0 = \frac{\mu_1\sigma_2^2 + \mu_2\sigma_1^2 - (\mu_1 + \mu_2)c_{12}}{\sigma_1^2 + \sigma_2^2 - 2c_{12}}, \sigma_0^2 = \frac{\sigma_1^2\sigma_1^2 - c_{12}^2}{\sigma_1^2 + \sigma_2^2 - 2c_{12}} \tag{3.11}$$

If $\rho_{12} = 1$ or $\sigma_1 = \sigma_2$, then all feasible portfolios have the same variance equal to $\sigma_1^2 = \sigma_2^2$.

Proof. **CASE 1** if $\rho_{12} < 1$,

$\sigma_1^2 + \sigma_2^2 - 2c_{12} > \sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma 2 = (\sigma_1 - \sigma_2)^2 \geq 0$

if $\sigma_1 \neq \sigma_2$,

$\sigma_1^2 + \sigma_2^2 - 2c_{12} \geq \sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma 2 = (\sigma_1 - \sigma_2)^2 > 0$

By first derivative test,

$\frac{d(\sigma_V^2)}{ds} = 0$

$2s(\sigma_1^2 + \sigma_2^2 - 2c_{12}) - 2(\sigma_2^2 - c_{12}) = 0$

$s = \frac{\sigma_2^2 - c_{12}}{\sigma_1^2 + \sigma_2^2 - 2c_{12}} (= s_0, say)$

By second derivative test,

$\frac{d^2(\sigma_V^2)}{ds^2} = 2(\sigma_1^2 + \sigma_2^2 - 2c_{12}) > 0$

$\Rightarrow \sigma_V^2$ attains its maximum at s_0.

Putting the value of $s = s_0$ in equation (3.8) and (3.9), we get μ_0 and σ_0^2 as per equation (3.11).

CASE 2 if $\rho_{12} = 1$ and $\sigma_1 = \sigma_2$,

$\sigma_V^2 = s^2\sigma_1^2 + (1-s)^2\sigma_2^2 + 2s(1-s)\rho_{12}\sigma_1\sigma_2 = (s\sigma_1 + (1-s)\sigma_2)^2 = \sigma_1^2 = \sigma_2^2$

$\square$

Theorem 3.2.7. *Let $-1 < \rho_{12} < 1$ and $\mu_1 \neq \mu_2$. Then for each portfolio V in the feasible set, $x = \sigma_V$ and $y = \mu_V$ satisfy the equation of a hyperbola*

$$x^2 - A^2(y - \mu_0)^2 = \sigma_0^2 \tag{3.12}$$

with μ_0 and $\sigma_0^2 > 0$ given as above, and with

$$A^2 = \frac{\sigma_1^2 + \sigma_2^2 - 2c_{12}}{(\mu_1 - \mu_2)^2} > 0.$$

The two asymptotes of the hyperbola are $y = \mu_0 \pm \frac{1}{A}x$.

Proof. Since $-1 < \rho_{12} < 1, \sigma_1^2 + \sigma_2^2 - 2c_{12} > \sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma 2 = (\sigma_1 - \sigma_2)^2 \geq 0$ and $\mu_1 \neq \mu_2, (\mu_1 - \mu_2)^2 > 0$

Hence $A^2 > 0$

Again $\rho_{12} < 1$ implies that $c_{12} < \sigma_1\sigma_2$

Hence $\sigma_0^2 > 0$ From (3.8), $s = \frac{(\mu_V - \mu_2)}{(\mu_1 - \mu_2)}$

Substituting this in (3.9), we get

$$\sigma_V^2 - A^2(\mu_V - \mu_0)^2 = \sigma_0^2$$

The transformation $x = \sigma_V$ and $y = \mu_V$, gives us

$x^2 - A^2(y-\mu_0)^2 = \sigma_0^2 \Rightarrow \frac{(x-0)^2}{\sigma_0^2} - \frac{(y-\mu_0)^2}{(\frac{\sigma_0}{A})^2} = 1$,

which is indeed a hyperbola with asymptotes $y = \mu_0 \pm \frac{1}{A}x$.

{ The hyperbola $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ has asymptotes $y = k \pm \frac{b}{a}(x-h)$.} $\square$

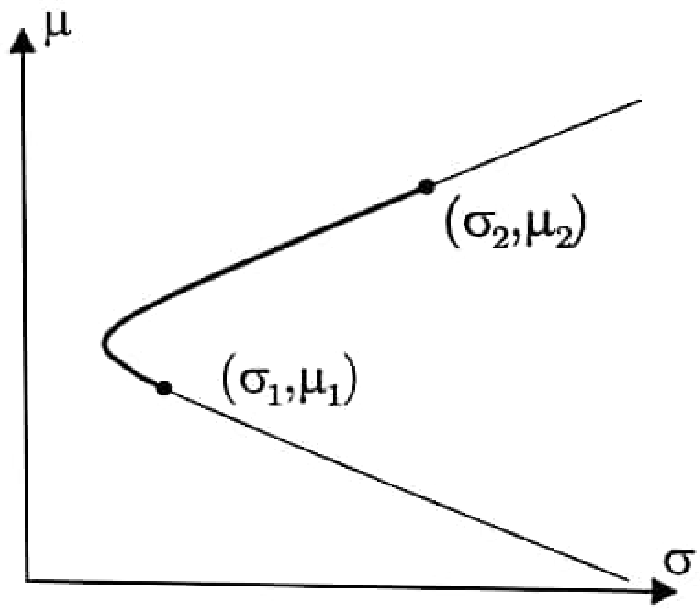

Figure 3.1: Hyperbola representing feasible portfolios with $-1 < \rho_{12} < 1$

Theorem 3.2.8. *Suppose $\rho_{12} = 1$ and $\sigma_1 \neq \sigma_2$, then $\sigma_V = 0$ if and only if $w_1 = -\frac{\sigma_2}{\sigma_1-\sigma_2}, w_2 = \frac{\sigma_1}{\sigma_1-\sigma_2}$. This involves short selling as $w_1 < 0$ or $w_2 < 0$. Suppose $\rho_{12} = -1$, then $\sigma_V = 0$ if and only if $w_1 = \frac{\sigma_2}{\sigma_1+\sigma_2}, w_2 = \frac{\sigma_1}{\sigma_1+\sigma_2}$. This requires no short selling as $w_1 > 0$ or $w_2 > 0$.*

Proof. Let $\rho_{12} = 1$ and $\sigma_1 \neq \sigma_2$.

$$\begin{aligned}
\sigma_V = 0 &\iff \sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2} = 0 \\
&\iff w_1\sigma_1 + w_2\sigma_2 = 0 \\
&\iff w_1 = -\frac{\sigma_2}{\sigma_1 - \sigma_2}, w_2 = \frac{\sigma_1}{\sigma_1 - \sigma_2} \qquad \{\text{Since } w_1 + w_2 = 1\}
\end{aligned}$$

Let $\rho_{12} = -1$.

$$\begin{aligned}
\sigma_V = 0 &\iff sqrt w_1^2\sigma_1^2 + w_2^2\sigma_2^2 - 2w_1w_2\sigma_1\sigma_2 = 0 \\
&\iff w_1\sigma_1 - w_2\sigma_2 = 0 \\
&\iff w_1 = \frac{\sigma_2}{\sigma_1 + \sigma_2}, w_2 = \frac{\sigma_1}{\sigma_1 + \sigma_2} \qquad \{\text{Since } w_1 + w_2 = 1\}
\end{aligned}$$

□

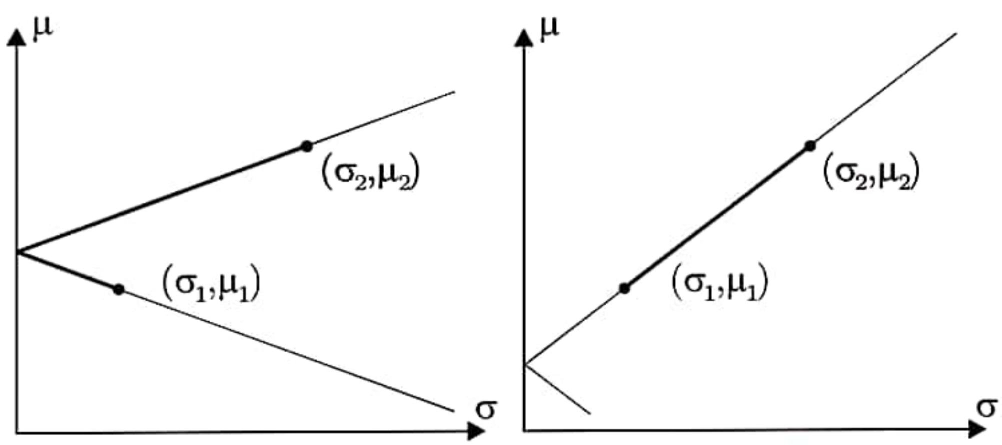

Figure 3.2: Typical portfolio lines with ρ_{12} = -1 and 1

Theorem 3.2.9. *Suppose that $\sigma_1 \leq \sigma_2$. The following five cases are possible.*

1. *If* $\rho_{12} = 1$*, then there is a feasible portfolio* V *with short selling such that* $\sigma_V = 0$ *whenever* $\sigma_1 < \sigma_2$*. Each portfolio* V *in the feasible set has the same* σ_V *whenever* $\sigma_1 = \sigma_2$*.*

2. *If* $\frac{\sigma_1}{\sigma_2} < \rho_{12} < 1$*, then there is a feasible portfolio* V *with short selling such that* $\sigma_V = \sigma_1$*, but for each portfolio without short selling* $\sigma_V \geq \sigma_1$*.*

3. *If* $\rho_{12} = \frac{\sigma_1}{\sigma_2}$*, then* $\sigma_V \geq \sigma_1$ *for each feasible portfolio* V*.*

4. *If* $-1 < \rho_{12} < \frac{\sigma_1}{\sigma_2}$*, then there is a feasible portfolio* V *without short selling such that* $\sigma_V < \sigma_1$*.*

5. *If* $\rho_{12} = -1$*, then there is a feasible portfolio* V *without short selling such that* $\sigma_V = 0$*.*

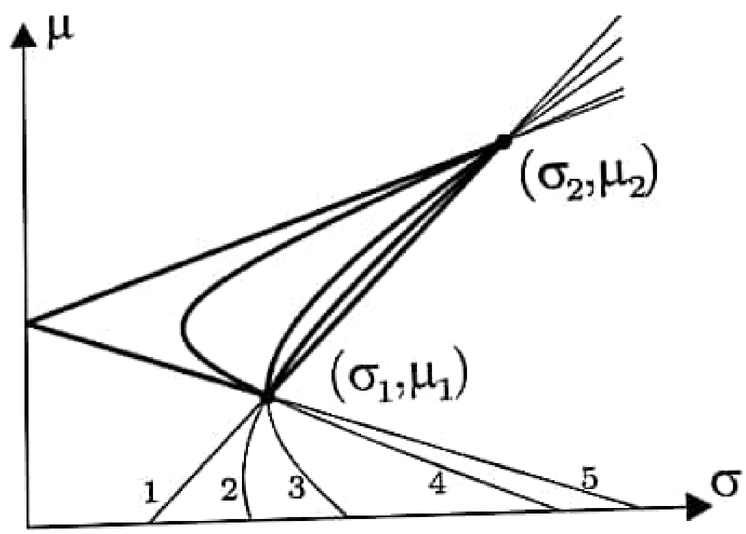

Figure 3.3: portfolio lines for various values of ρ_{12}

3.3 Several Securities

3.3.1 Risk and Expected Return of a Portfolio

Let us consider a portfolio which is constructed from n different securities with the weight of each asset being

$$w_i = \frac{x_i S_i(0)}{V(0)}, \qquad i = 1, \cdots, n,$$

where x_i is the number of shares of type i in the portfolio, $S_i(0)$ is the initial price of security i and $V(0)$ is the initial amount invested in portfolio. We introduce following notions

$$\text{Weight vector } \mathbf{w} = [\ w_1\ w_2\ \cdots\ w_n\ ]$$

$$\text{Unit vector } \mathbf{u} = [\ 1\ 1\ \cdots\ 1\ ]$$

$$\text{Expectation vector } \mathbf{m} = [\ \mu_1\ \mu_2\ \cdots\ \mu_n\ ] \qquad \text{where } \mu_i = \mathbb{E}(K_i)\ \forall i = 1, \cdots, n$$

$$\text{Covariance matrix } \mathbf{C} = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{bmatrix}$$

$$\text{where } c_{ij} = Cov(K_i, K_j) \text{ and } c_{ii} = \sigma_i^2 = Var(K_i)\ \forall i, j = 1, \cdots, n$$

The condition of the sum of all weights being 1, can be written as

$$1 = \mathbf{w}\mathbf{u}^T \tag{3.13}$$

Assume that $det\,\mathbf{C} \neq 0 \Rightarrow \mathbf{C}^{-1}$ exists.

Return on portfolio V

$$K_V = w_1K_1 + w_2K_2 + \cdots + w_nK_n \tag{3.14}$$

Expected returns

$$\mu_V = \mathbb{E}(K_V) = \mathbf{w}\mathbf{m}^T \tag{3.15}$$

Varianvce

$$\sigma_V^2 = Var(K_V) = \mathbf{w}\mathbf{C}\mathbf{w}^T \tag{3.16}$$

$$\{\because Var(X) = Cov(X,X)\,\&Cov(\sum_{i=1}^{n} w_iK_i, \sum_{j=1}^{m} w_jK_j) = \sum_{i=1}^{n}\sum_{j=1}^{m} w_iw_jCov(K_i,K_j)\}$$

3.3.2 Minimum Variance Portfolio

The portfolio with the smallest variance among all feasible portfolios will be called the minimum variance portfolios or MVP. To find this, we need to solve,

$$\text{Min } \sigma_V^2 = \text{Min } \mathbf{w}\mathbf{C}\mathbf{w}^T \qquad \text{subject to } \mathbf{w}\mathbf{u}^T = 1 \qquad \text{for } \mathbf{w} \in \mathbb{R}^n. \tag{3.17}$$

Theorem 3.3.1. *Let $det\,\mathbf{C} \neq 0$, then the MVP has weights $w_{MVP} = \frac{\mathbf{u}\mathbf{C}^{-1}}{\mathbf{u}\mathbf{C}^{-1}\mathbf{u}^T}$.*

Proof. Using Lagrange Multiplier Method, ($\lambda \Rightarrow$ Lagrange Multiplier)

$$F(\mathbf{w},\lambda) = \mathbf{w}\mathbf{C}\mathbf{w}^T - \lambda(\mathbf{w}\mathbf{u}^T - 1)$$

First order necessary condition gives, $\mathbf{2wC} - \lambda\mathbf{u} = \mathbf{0}$ or $\mathbf{w} = \frac{\lambda}{\mathbf{2}}\mathbf{uC}^{-1}$.

$$\Rightarrow \frac{\lambda}{2}\mathbf{uC}^{-1}\mathbf{u}^T = 1$$

$$\Rightarrow \frac{\lambda}{2} = \frac{1}{\mathbf{uC}^{-1}\mathbf{u}^T}$$

$\{\because c_{ij} \geq 0 \Rightarrow C$ is SPD $\Rightarrow C^{-1}$ is also SPD $\Rightarrow \mathbf{uC}^{-1}\mathbf{u}^T > 0\}$

$\{\because \sigma_V^2$ is bounded below by zero , so it must have a minimum $\}$

$$\Rightarrow w_{MVP} = \frac{\mathbf{uC}^{-1}}{\mathbf{uC}^{-1}\mathbf{u}^T}$$

$\square$

3.3.3 Efficient Frontier

Dominating Security

A security with expected return μ_1 and standard deviation σ_1 is said to dominate another security with expected return μ_2 and standard deviation σ_2 provided $\mu_1 \geq \mu_2$ and $\sigma_1 \leq \sigma_2$.

Efficient Frontier

A portfolio is called efficient if there is no other portfolio except itself, that dominates it. The subset of efficient portfolios among all feasible portfolios is called the efficient frontier. In order to determine the efficient frontier, we need to identify and eliminate the dominated portfolios.

Minimum Variance Line

The family of portfolio V, parameterised by $\mu \in \mathbb{R}$ such that $\mu_V = \mu$ and $\sigma_V^2 \leq \sigma_{V'}^2$ for each portfolio V' with $\mu_{V'} = \mu$ is called the minimum variance line or MVL.

Min $\sigma_V^2 = \mathbf{wCw}^T \qquad$ subject to $\mathbf{wu}^T = 1$ and $\mathbf{wm}^T = \mu \qquad$ for $\mathbf{w} \in \mathbb{R}^n, \mu \in \mathbb{R}$.

Using Lagrange Multiplier Method, ($\lambda_1, \lambda_2 \Rightarrow$ Lagrange Multipliers)

$$G(\mathbf{w}, \lambda_1, \lambda_2) = \mathbf{wCw}^T - \lambda_1(\mathbf{wm}^T - \mu) - \lambda_2(\mathbf{wu}^T - 1)$$

First order necessary condition gives,

$$\mathbf{2wC} - \boldsymbol{\lambda_1}\mathbf{m} - \boldsymbol{\lambda_2}\mathbf{u} = \mathbf{0} \text{ or } \mathbf{w} = \frac{\mathbf{1}}{\mathbf{2}}(\boldsymbol{\lambda_1}\mathbf{m} + \boldsymbol{\lambda_2}\mathbf{u})\mathbf{C}^{-1}.$$

$$\Rightarrow \frac{\mathbf{1}}{\mathbf{2}}(\boldsymbol{\lambda_1}\mathbf{m} + \boldsymbol{\lambda_2}\mathbf{u})\mathbf{C}^{-1}\mathbf{m}^T = \mu \text{ and } \frac{\mathbf{1}}{\mathbf{2}}(\boldsymbol{\lambda_1}\mathbf{m} + \boldsymbol{\lambda_2}\mathbf{u})\mathbf{C}^{-1}\mathbf{u}^T = 1$$

$$Define M = \begin{bmatrix} \mathbf{mC^{-1}m^T} & \mathbf{uC^{-1}m^T} \\ \mathbf{mC^{-1}u^T} & \mathbf{uC^{-1}u^T} \end{bmatrix} = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \qquad \text{(say)} \qquad (3.18)$$

$$\Rightarrow \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = 2M^{-1} \begin{bmatrix} \mu \\ 1 \end{bmatrix}$$

$$\Rightarrow w_{MVL} = \mu\frac{(\mathbf{smC^{-1}} - \mathbf{ruC^{-1}})}{(\mathbf{ps} - \mathbf{qr})} + \frac{(\mathbf{puC^{-1}} - \mathbf{qmC^{-1}})}{(\mathbf{ps} - \mathbf{qr})} = \mu\mathbf{a} + \mathbf{b} \text{ where } \mathbf{a}, \mathbf{b} \in \mathbb{R}^n$$

This can be summarised in the following theorem.

Theorem 3.3.2. *Let det* $\mathbf{C} \neq 0$ *and* $\mathbf{m}, \mathbf{u}$ *are linearly independent vectors, then the weight vector* $\mathbf{w}$ *represents a portfolio* V *on the minimum variance line if and only if*

$$\Rightarrow w_{MVL} = \mu \frac{(\mathbf{smC^{-1}} - \mathbf{ruC^{-1}})}{(\mathbf{ps} - \mathbf{qr})} + \frac{(\mathbf{puC^{-1}} - \mathbf{qmC^{-1}})}{(\mathbf{ps} - \mathbf{qr})}$$

with $\mu = \mu_V$ *and notations of (3.18).*

Theorem 3.3.3 (Two Fund Theorem)**.** *Under the assumptions of the previous theorem, let* $\mathbf{w_1}$ *and* $\mathbf{w_2}$ *be the weights of any two portfolios* V_1, V_2 *on the minimum variance line with different expected returns* $\mu_{V_1} \neq \mu_{V_2}$*. Then each portfolio* V *on the minimum variance line can be obtained as a linear combination of these two, i.e.,*

$$\mathbf{w} = \alpha \mathbf{w_1} + (1 - \alpha)\mathbf{w_2} \qquad \textit{for some } \alpha \in \mathbb{R}$$

Proof. Define $\alpha = \frac{\mu_V - \mu_{V_2}}{\mu_{V_1} - \mu_{V_2}}$

$$\begin{aligned} \alpha \mathbf{w_1} + (1 - \alpha)\mathbf{w_2} &= \alpha(\mu_{V_1}\mathbf{a} + \mathbf{b}) + (1 - \alpha)(\mu_{V_2}\mathbf{a} + \mathbf{b}) \\ &= (\alpha\mu_{V_1} + (1 - \alpha)\mu_{V_2})\mathbf{a} + \mathbf{b} \\ &= \mu_V \mathbf{a} + \mathbf{b} = \mathbf{w} \end{aligned}$$

□

Following pdf contains the practical implementation of the theory described in this chapter.

```
> # Markowitz Model : Minimum Variance Portfolio and Minimum Variance
Line
> #install.packages("readxl")
> #install.packages("matlib")
> #Select CRAN - UK(London)

> #Libraries used
> library(readxl)
> library(matlib)


> #Reading Excel files and converting into matrix
> A1 <-
read_excel("C:/Users/priya/Desktop/Coding/Project/Portfolio/stockdata.xls
x")
> A<-as.matrix(A1)
> T = nrow(A) #Number of weeks
> N = ncol(A) #Number of assets


> #Creating Unit Vector u
> u = matrix(1,1,N)


> #Creating Expectation Vector m
> m = matrix(0,1,N)
> for(j in 1:N){
+ sum=0
+ for(i in 1:T){
+ sum = sum+A[i,j]
+ }
+ m[1,j] = sum/T
+ }


> #Creating Covariance Matrix c
> c = matrix(0,N,N)
> for(i in 1:N){
+ for(j in 1:N){
+ c[i,j] = cov(A[,i],A[,j])
+ }
+ }


> #precalculating for faster runtime
> ic = inv(c)
> p = m%*%ic%*%t(m)
> q = u%*%ic%*%t(m)
> r = m%*%ic%*%t(u)
> s = u%*%ic%*%t(u)
> M = matrix(c(p,q,r,s),2,2)
> d = det(M)
> iM = inv(M)


> #Calcultaing Weight Vector w
> w = u%*%ic/(s)[1,1]
> t(w) #Showing weight vector in column form
               [,1]
 [1,]  0.0243566620
```

```
 [2,] -0.0044270332
 [3,] -0.0335702868
 [4,] -0.0303228009
 [5,] -0.0441272608
 [6,]  0.0065446941
 [7,]  0.0105525793
 [8,]  0.0012463628
 [9,]  0.0318452979
[10,] -0.0121803050
[11,] -0.0286437661
[12,] -0.0332489669
[13,]  0.0320382760
[14,] -0.0264416111
[15,]  0.0120607412
[16,]  0.0083295225
[17,]  0.0179366697
[18,] -0.0652738850
[19,] -0.0645395856
[20,] -0.0091513382
[21,]  0.1306528883
[22,]  0.0585871614
[23,]  0.1148090302
[24,] -0.0032177814
[25,]  0.0534614238
[26,] -0.0142151370
[27,]  0.0242424399
[28,]  0.0187299613
[29,] -0.0054426368
[30,] -0.0926940834
[31,]  0.0061514431
[32,]  0.0265009623
[33,]  0.1862419081
[34,]  0.0027285102
[35,]  0.0138284847
[36,] -0.0888200544
[37,]  0.0108172157
[38,] -0.0067138674
[39,]  0.0379099595
[40,] -0.0055923978
[41,]  0.0611113954
[42,] -0.0140887970
[43,]  0.0117090110
[44,]  0.0502736106
[45,] -0.1033725415
[46,] -0.0233200002
[47,]  0.0911662336
[48,]  0.0520427161
[49,] -0.0207778495
[50,]  0.0511867102
[51,]  0.0231554688
[52,] -0.0169294702
[53,]  0.0629559571
[54,] -0.0310305925
[55,]  0.0777064520
[56,]  0.0012725254
[57,] -0.0056713789
[58,] -0.0017550681
[59,]  0.0003563647
[60,]  0.0200071184
[61,]  0.0651279670
```

```
[62,]  0.0072060278
[63,]  0.0514505271
[64,]  0.0490919485
[65,] -0.0123118731
[66,]  0.1305779988
[67,]  0.1781652578
[68,]  0.0351650349
[69,] -0.0253761158
[70,]  0.0161198800
[71,] -0.0364316859
[72,] -0.0494487846
[73,] -0.0070731233
[74,]  0.0379258958
[75,] -0.0327905165
[76,]  0.0159593142
[77,]  0.0065318997
[78,]  0.0217801318
[79,]  0.0114202592
[80,] -0.0166455624
[81,] -0.0137823807
[82,]  0.0024704668
[83,] -0.0088650553
[84,]  0.0093935111
[85,] -0.0174558021
[86,]  0.0132988001
[87,]  0.0215487176
> rowSums(w) #checking if weight sums are 1
[1] 1


> #Associated Risk and Returns
> returns = w%*%t(m)
> risk = w%*%c%*%t(w)
> returns
             [,1]
[1,] 0.0009568303
> risk
             [,1]
[1,] 8.921528e-05


> #For a given level of returns mu, calculating weights on MVL
> mu = 0.10
> wmvl = mu*((s[1,1]*m%*%ic-r[1,1]*u%*%ic)/d) + (p[1,1]*u%*%ic-
q[1,1]*m%*%ic)/d
> t(wmvl)#Showing weight vector in column form
               [,1]
 [1,] -1.4470984228
 [2,]  1.0899476018
 [3,]  0.1287137968
 [4,]  0.0295020476
 [5,] -0.3115786633
 [6,] -0.2456921191
 [7,]  0.1453006846
 [8,] -0.4841253669
 [9,] -0.4959702664
[10,] -0.5200209230
[11,]  0.7979398892
[12,]  0.6031095945
[13,] -0.2791914258
```

```
[14,]  0.5508494311
[15,]  0.2021002318
[16,] -0.0480791213
[17,]  1.6482077055
[18,] -1.1787429552
[19,]  0.4198838832
[20,]  0.2980894401
[21,] -0.8476883347
[22,] -1.2141961158
[23,] -1.4375228825
[24,] -0.0004151066
[25,]  0.6068040534
[26,] -0.0709488144
[27,]  0.3022893279
[28,] -0.4581018133
[29,]  0.1890578156
[30,]  1.0596000006
[31,]  0.0764900626
[32,]  0.0197454842
[33,]  1.9985944848
[34,] -0.1014329089
[35,] -0.0709344584
[36,]  0.5942171829
[37,] -0.2907703225
[38,]  1.3385628303
[39,] -0.9820779911
[40,]  0.4794740485
[41,] -0.4536910420
[42,] -0.5856285819
[43,]  0.9479929665
[44,] -0.7831984949
[45,] -1.3035712720
[46,]  0.6895881308
[47,] -0.2234463506
[48,]  1.3432032161
[49,]  0.2925246967
[50,] -0.5178500986
[51,] -1.1767590502
[52,] -0.2315874917
[53,] -0.1076124191
[54,] -0.3312724824
[55,] -0.2184513327
[56,]  0.9093253940
[57,] -0.3283227595
[58,]  1.1157241203
[59,]  0.0951282264
[60,] -0.4656106794
[61,] -0.5787740729
[62,] -0.7503952723
[63,]  1.3196828423
[64,]  0.2493405000
[65,]  0.1000762381
[66,]  0.1416668013
[67,] -0.2273525755
[68,] -0.3864029207
[69,]  1.0809322072
[70,]  0.4936652002
[71,] -0.8819471880
[72,]  0.8020897000
[73,] -0.7940234766
```

```
[74,]  0.1670302962
[75,] -0.2388392568
[76,] -0.4646610968
[77,] -0.1349846151
[78,] -0.5223214898
[79,]  0.2893920593
[80,] -0.0934290538
[81,]  0.1986619871
[82,]  1.2737439138
[83,]  0.2915144522
[84,] -0.3798397502
[85,] -0.3393291956
[86,] -0.4760503780
[87,]  0.1001778635
> rowSums(wmvl) #checking if weight sums are 1
[1] 1


> #Associated Risk and Returns
> returnsmvl = wmvl%*%t(m)
> riskmvl = wmvl%*%c%*%t(w)
> returnsmvl
     [,1]
[1,]  0.1
> riskmvl
             [,1]
[1,] 8.921544e-05


> #END OF FILE
```

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