The Augmented Black-Litterman Model: A Ranking-Free Approach to Factor-Based Portfolio Construction and Beyond*

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Abstract

The Fama and French (1992 and 1993 etc.) factor-ranking approach has been extensively applied in quantitative fund management. However, this approach suffers from hidden factor view, information inefficiency etc. issues. Based on the Black-Litterman model (Black and Litterman, 1992; as explained in Cheung, 2010), we develop a technique that endogenises the ranking process and elegantly resolves these issues. This model explicitly seeks forward-looking factor views and smoothly blends them to deliver robust allocation to securities. Our numerical experiments show this is an intuitive and practical framework for factor-based portfolio construction, and beyond. This article is featured by: -

- ◇ A new and unified framework for strategy combination, factor mimicking and security-specific bets
- ◇ An elegant and ranking-free approach to factor style construction
- ◇ Worked examples based on the FTSE EUROTOP 100 universe
- ◇ Insight into the classic issue of confidence parameter setting
- ◇ An implementation guidance in the appendix

JEL Classification: C10, C11, C61, G11, G14

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1 Introduction

Active portfolio management is about leveraging information and information is often embodied in investment views. The Black-Litterman (BL) Model (Black and Litterman, 1992; as explained in Satchell and Scowcroft, 2000; Idzorek, 2004; and Cheung, 2010 etc.) represents an elegant view processor. We highlight the following advantages of the BL model over the traditional optimiser:

- **Taking view uncertainty into account**. The underlying Bayes' Rule admits probabilistic views, and so view uncertainties as well as mean estimates are naturally taken into consideration.
- **Smoothness in view processing**. The BL-based allocation framework employs the Bayes' Rule, a mathematical law for view updating, before portfolio optimisation.
- **Robust allocation due to shrinkage**. The allocation framework not only blends portfolio manager (PM) views but also shrinks them towards the prior view[1]. This mechanism is ideal for portfolio managers to exercise discretion.
- **Freedom in views**. While the traditional optimiser requires one absolute view (as opposite to relative view) for each security, the BL model is fully flexible: it allows relative, strategy (or portfolio) views as well as absolute views; but if there is no view, the prior holding will be maintained.

However, the BL model only admits views on the securities or their combinations. This is inconvenient to quant PMs who consider economic and financial factors in their portfolio construction process. In equity portfolio construction, a common belief is that there are some fundamental (e.g., market capitalisation, book-to-price, dividend yield etc.), technical (e.g., price momentum etc.), industrial (e.g., banks, automobiles etc.), and macroeconomic (e.g., energy price, interest rate, unemployment rate, FX rate, credit spread, etc.) factors driving stock returns. These factors should be utilised as information sources in forecasts, risk management and portfolio construction. The question is how to utilise such information. We observe the following existing practices:

(1) Where cross-sectional information is observed, e.g., the fundamental stock characteristics, a ranking-based factor-mimicking (FM) portfolio construction technique is extensively applied as inspired by Fama and French (FF) (1992, 1993 and 1996). In its original fashion, a sorted universe is divided into some buckets (e.g., quintiles, deciles) of securities and the equal-weighted long-top-bucket and short-bottom-bucket rule is applied to mimic the factor return. To combine factors, joint ranking is employed which essentially takes intersections of these ranking buckets.

(2) Where factor return time series are directly observable, e.g., the macro economic factors, it is common to fit a factor model. With this, several approaches to portfolio construction are possible: (i)

[1] As explained in Cheung (2010), the BL model involves a prior view (i.e., the market view) towards which, PM investment views are shrunken according to the Bayes' Rule. Since ***shrinkage*** is implicit in Bayesian inference, it is used in this paper. It is however worth highlighting that we use shrinkage in a subtly different sense from that of ***shrinkage estimator*** commonly understood in statistics. While *shrinkage estimation* commonly attempts to 'scientifically' determine an estimator, we leave the balance of view *shrinkage* to the 'art' of investor discretion in our framework.

rank the universe by the factor exposures, and follow the FF method; (ii) directly use the factor model to predict stock returns based on factor views; and (iii) construct optimisation-based factor mimicking portfolios which are only exposed to their respective target factors.

The characteristics-ranking process (1) throws away the distance information between characteristics upfront. The 'long-top, short-bottom' portfolio construction rule that follows also suffers from arbitrariness and information inefficiency in stock selection. More subtly, the ranking-based process implicitly bears directional assumption on factor return, which is dangerous and should be made explicit in practice. In process (2), due to estimation errors, it is desirable to incorporate some robust technique.

In this article, the second of a series, we develop, based on the BL technique, a model which enables a factor-based yet ranking-free approach to portfolio construction. This approach seeks explicitly and utilises PM's factor views with a shrinkage feature. In our forthcoming articles, we will unveil that what we are establishing here is indeed a unified Bayesian allocation framework which not only solves PM's FM problems with elegance and efficiency, but also addresses a breadth of other practical challenges, e.g., in strategy combination, hedging, 'portable alpha' generation, strategy implementation with trading restrictions, etc., to the highest standard.

In the remainder of this article, we first brief the BL model in its original form. Taking into account a linear factor model, we then formulate the *Augmented Black-Litterman* (ABL) model so as to admit factor and security-specific views. Finally, we illustrate its applications to factor-based equity portfolio construction and stock-specific betting with worked examples, followed by a summary of the model's new features. An implementation guidance is included in the Appendix.

2 A Brief Review of the Black-Litterman Model

Black and Litterman (1992) espouse that the lack-of-robustness problem with traditional portfolio optimisation should not be attributed to the optimiser itself, but to the quality of inputs, i.e., the security return forecasts and the error matrix. To improve these optimisation inputs, they recommend a Bayesian framework.

Suppose there are n securities in the universe and we are after a probabilistic view on the security return vector $\vec{\widetilde{r}}_{[n\times 1]}$[2]. In line with the notion of the *semi-strong form of market efficiency* (Fama, 1970), if all we have are publicly available information and techniques, $\mathcal{G}$, the market view $\vec{\widetilde{\mu}}_{M[n\times 1]}$ should be the best available view, superior to our estimation $\vec{\widetilde{\mu}}_{[n\times 1]}$[3]. What is the market view then? Supposing the market is already in equilibrium, the equilibrium view can be assessed using the *Capital Asset Pricing Model* (CAPM), i.e., $\vec{\widetilde{\pi}}_{[n\times 1]} = \vec{\beta}_{[n\times 1]}[\mathbb{E}(\widetilde{R}_M|\mathcal{G}) - r_f]$[4], from the market portfolio. However, the market is not necessarily in

[2]Unless otherwise specified, in this article, all returns refer to those excess of the risk-free return.

[3]See Cheung (2010) for an elaboration. In terms of notation, we use "$\widetilde{X}$" to represent a random variable; "$\widehat{X}$" to represent an estimate of a random variable; "$\vec{X}_{[n\times 1]}$" to represent a vector with n entries; "$\mathbf{X}_{[n\times m]}$" to represent a matrix of dimensions n by m; and "$\widetilde{X}_{|\mathcal{I}}$" to represent a random variable conditional on information $\mathcal{I}$. Dimensions are given at the first appearance of a vector or matrix and wherever considered helpful.

[4]Here, $\vec{\beta}$ is the vector of security exposures to the market; $\mathbb{E}(\widetilde{R}_M|\mathcal{G})$ is the expected gross market return; and r_f is the risk-free

equilibrium. The following normality assumption is therefore made:

$$\vec{\tilde{r}}_{|\mathcal{G}[n\times 1]} \overset{\text{delegate}}{\rightarrow} \vec{\tilde{\mu}}_M \sim \mathbb{N}(\vec{\tilde{\pi}}, \tau\mathbf{\Sigma}_{[n\times n]}) \tag{2.1}$$

where $\vec{\tilde{r}}_{|\mathcal{G}}$ is our *prior* view given public information $\mathcal{G}$; $\mathbf{\Sigma}$ is the security covariance matrix; and τ is a scalar representing the PM's uncertainty about the estimation. This is to say, in the BL model, we defer our prior view $\vec{\tilde{r}}_{|\mathcal{G}}$ to the market view $\vec{\tilde{\mu}}_M$ when all we have is only the public information; and the market view $\vec{\tilde{\mu}}_M$ is still believed to be best approximated by the equilibrium view $\vec{\tilde{\pi}}$, but is subject to some (normally distributed) error proportional to the security covariance matrix $\mathbf{\Sigma}$.

With private information $\mathcal{H}$, suppose the PM forms k views in the following format:

$$\mathbf{P}_{[k\times n]}\vec{\tilde{r}}_{|\mathcal{H},\mathcal{G}[n\times 1]} = \vec{\tilde{y}}_{|\mathcal{H},\mathcal{G}[k\times 1]} + \vec{\tilde{\varepsilon}}_{[k\times 1]} \tag{2.2}$$

where $\mathbf{P}$ is the view structure matrix; $\vec{\tilde{r}}_{|\mathcal{H},\mathcal{G}}$ is the required *posterior* vector of return estimates; $\vec{\tilde{y}}_{|\mathcal{H},\mathcal{G}}$ is the vector of updated view estimates; and $\vec{\tilde{\varepsilon}}$ is the vector of view estimation errors. This view formulation allows a view to be expressed as linear combination of security returns. Suppose the PM believes her best estimation of $\vec{\tilde{y}}_{|\mathcal{H},\mathcal{G}}$ is $\vec{\tilde{q}}_{[k\times 1]}$, subject to normally distributed errors:

$$\vec{\tilde{\varepsilon}} \sim \mathbb{N}(\vec{0}_{[k\times 1]}, \mathbf{\Omega}_{[k\times k]}) \tag{2.3}$$

where $\vec{0}$ is a vector of zeros; and $\mathbf{\Omega}$ is a diagonal variance matrix of view-estimation errors, which are considered, for simplicity, as independent across views[5], then these views contain valuable information that can be combined to update our view on $\vec{\tilde{r}}$. Based on the *Bayes' Rule*, the following analytical result for the posterior return vector $\vec{\tilde{r}}_{|\mathcal{H},\mathcal{G}}$ is derived:

Theorem 2.1 (Posterior Return Estimates) *The posterior return estimates are normally distributed, i.e., $\vec{\tilde{r}}_{|\vec{\tilde{q}},\mathcal{H},\mathcal{G}} \sim \mathbb{N}(\vec{\tilde{m}}_{[n\times 1]}, \widehat{\mathbf{V}}_{[n\times n]})$, where the updated mean estimates are:*

$$\vec{\tilde{m}} = \left[(\tau\mathbf{\Sigma})^{-1} + \mathbf{P}^{\mathrm{T}}\mathbf{\Omega}^{-1}\mathbf{P}\right]^{-1}\left[(\tau\mathbf{\Sigma})^{-1}\vec{\tilde{\pi}} + \mathbf{P}^{\mathrm{T}}\mathbf{\Omega}^{-1}\vec{\tilde{q}}\right] \tag{2.4}$$

and the updated variance-covariance matrix is:

$$\widehat{\mathbf{V}} = \left[(\tau\mathbf{\Sigma})^{-1} + \mathbf{P}^{\mathrm{T}}\mathbf{\Omega}^{-1}\mathbf{P}\right]^{-1} \tag{2.5}$$

Proof. See Appendix A to Cheung (2010).□

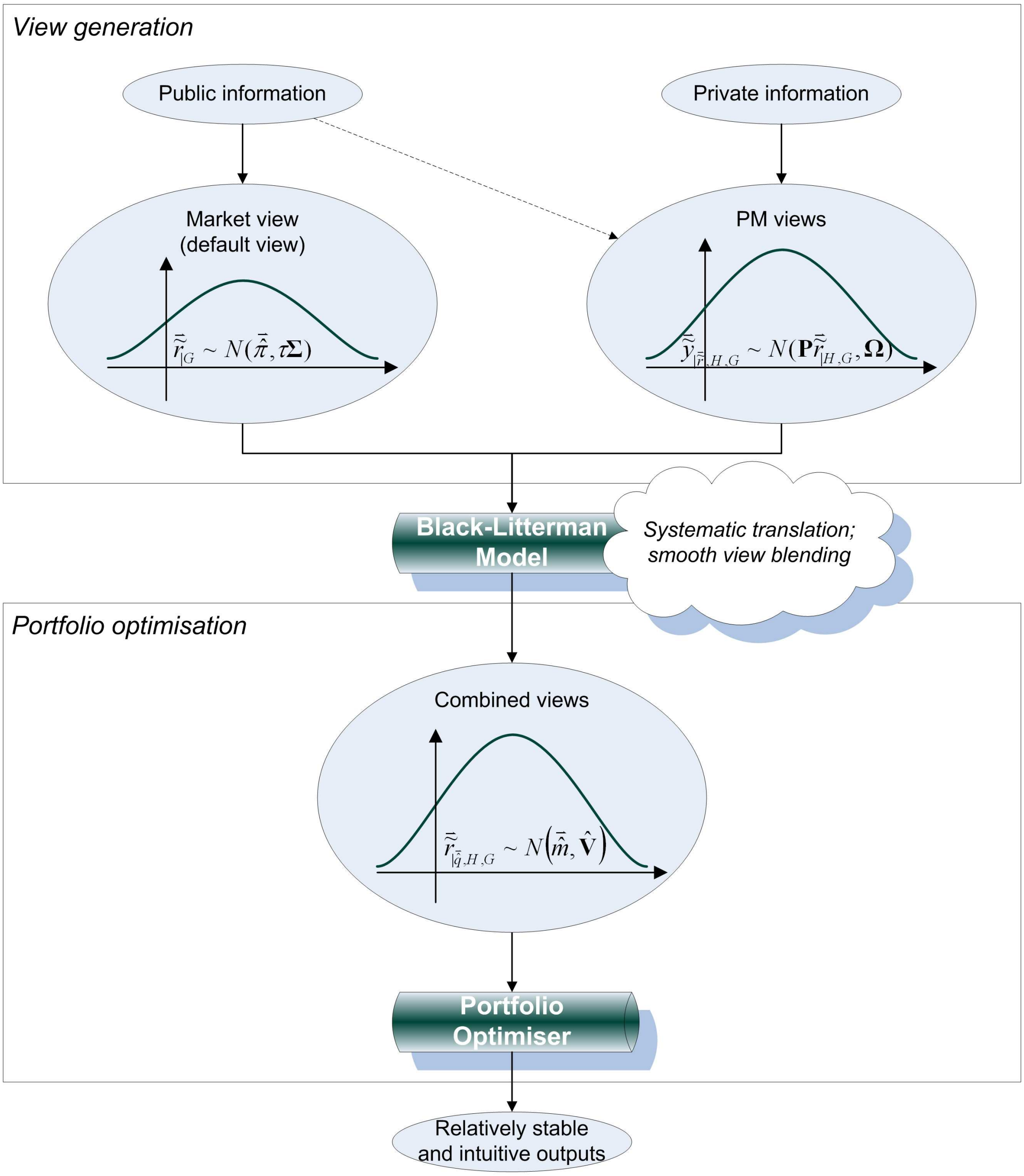
Theorem 2.1 gives the distribution of the posterior return vector $\vec{\tilde{r}}_{|\mathcal{H},\mathcal{G}}$ through updating its first and second moments. PM can therefore feed these into a mean-variance optimiser for portfolio construction. Figure 1 gives the whole picture of the BL-based portfolio optimisation framework.

To take factor views into account in this framework, let us consider linear factor model first.

interest rate.

[5]This simplifying assumption follows the classical linear regression model.

Figure 1 The BL-Based Portfolio Optimisation Framework



3 Linear Factor Models

In the equity world, linear factor models based on economic and financial factors have been applied to risk management as well as alpha generation processes (see Conner *et al* (2010) for general factor modelling theory and portfolio risk management). A linear factor model takes the following form:

$$\vec{\tilde{r}}_{[n\times 1]} = \vec{a}_{[n\times 1]} + \mathbf{B}_{[n\times f]}\vec{\tilde{r}}_{F[f\times 1]} + \vec{\tilde{\xi}}_{[n\times 1]} \tag{3.1}$$

where $\vec{\tilde{r}}$ is the vector of security returns; $\vec{a}$ is the intercept vector; $\mathbf{B}$ is the matrix of factor loadings; $\vec{\tilde{r}}_F$ represents the vector of factor returns; and $\vec{\tilde{\xi}}$ stands for the vector of security-specific returns which are considered independent of factor returns and of each other.

An immediate result from the above set-up is the following factor risk relationship:

$$\mathbf{\Sigma}_{[n\times n]} = \mathbf{B}\mathbf{\Sigma}_{F[f\times f]}\mathbf{B}^{\mathrm{T}} + \mathbf{\Sigma}_{\xi[n\times n]} \tag{3.2}$$

where $\mathbf{\Sigma}$ is the covariance matrix of the security returns; $\mathbf{\Sigma}_F$ represents the covariance of factor returns; and $\mathbf{\Sigma}_\xi$ stands for the covariance of the security-specific returns which are considered orthogonal.

In equity portfolio analytics, fitting factor risk model (3.2) has become a common effort, e.g., the Lehman Equity Risk Analyzer (ERA) Model, the Nomura Fundamental Risk Model (FRM), and the BARRA multi-factor risk model etc. Now, let us utilise such model information for factor-based portfolio construction.

4 The Augmented Black-Litterman Model

To admit PM factor views in the Bayesian framework, we augment the universe from n securities to also include all the f ($\leqslant n$) relevant factors and the n idiosyncratic components. In terms of return, the *view universe*[6] is represented by $\vec{\tilde{\gamma}}^a_{[(2n+f)\times 1]} = \left\{\vec{\tilde{r}}^{\mathrm{T}}_{[n\times 1]}, \vec{\tilde{r}}^{\mathrm{T}}_{F[f\times 1]}, \vec{\tilde{\xi}}^{\mathrm{T}}_{[n\times 1]}\right\}^{\mathrm{T}}$. By so doing, views can be expressed on all the three parts as such:

$$\mathbf{Q}^a_{[K\times(2n+f)]}\vec{\tilde{\gamma}}^a_{|\mathcal{H},\mathcal{G}} = \vec{\tilde{Y}}^a_{|\mathcal{H},\mathcal{G}[K\times 1]} + \vec{\tilde{\varepsilon}}^a_{[K\times 1]} \tag{4.1}$$

where

$$\mathbf{Q}^a_{[K\times(2n+f)]} = \begin{pmatrix} \mathbf{P}_{[k_1\times n]} & & \mathbf{0} \\ & \mathbf{P}_{F[k_2\times f]} & \\ \mathbf{0} & & \mathbf{P}_{\xi[k_3\times n]} \end{pmatrix} \tag{4.2}$$

is the augmented view structure matrix[7], containing k_1 ($\leqslant n$) security (or portfolio) return view structures $\mathbf{P}_{[k_1\times n]}$, k_2 ($\leqslant f$) factor return (or their linear combinations) view structures $\mathbf{P}_{F[k_2\times f]}$, k_3 ($\leqslant n$) security

[6] In developing ABL, we consider three universes: we form views freely on the *view universe* $\vec{\tilde{\gamma}}^a_{[(2n+f)\times 1]} = \left\{\vec{\tilde{r}}^{\mathrm{T}}_{[n\times 1]}, \vec{\tilde{r}}^{\mathrm{T}}_{F[f\times 1]}, \vec{\tilde{\xi}}^{\mathrm{T}}_{[n\times 1]}\right\}^{\mathrm{T}}$; later, analyse based on the *analytical universe* $\vec{\tilde{r}}^a_{[(n+f)\times 1]} = \left\{\vec{\tilde{r}}^{\mathrm{T}}_{[n\times 1]}, \vec{\tilde{r}}^{\mathrm{T}}_{F[f\times 1]}\right\}^{\mathrm{T}}$, which is required to be full-rank; and eventually, allocate capital to the *investment universe* $\vec{\tilde{r}}_{[n\times 1]}$, in which only tradable securities are included.

[7] Note the block-diagonal specification does not admit cross-type views, i.e., those combining security, factor and idiosyncratic returns, such as $\mathbf{P}\vec{\tilde{r}}_{|\mathcal{H},\mathcal{G}} + \mathbf{P}_{\sim F}\vec{\tilde{r}}_{F|\mathcal{H},\mathcal{G}} + \mathbf{P}_{\sim\xi}\vec{\tilde{r}}_{\xi|\mathcal{H},\mathcal{G}} = \vec{\tilde{Y}}_{|\mathcal{H},\mathcal{G}} + \vec{\tilde{\varepsilon}}$. This is purely for simplicity. Extension is however possible at the cost of slight complexity.

idiosyncratic return (or their linear combinations) view structures $\mathbf{P}_{\xi[k_3 \times n]}$, and $K = k_1 + k_2 + k_3$;

$$\vec{\vec{\gamma}}^{\,a}_{|\mathcal{H},\mathcal{G}[(2n+f)\times 1]} = \begin{pmatrix} \vec{\vec{r}}_{|\mathcal{H},\mathcal{G}[n\times 1]} \\ \vec{\vec{r}}_{F|\mathcal{H},\mathcal{G}[f\times 1]} \\ \vec{\vec{\xi}}_{|\mathcal{H},\mathcal{G}[n\times 1]} \end{pmatrix} \tag{4.3}$$

is the unknown but required augmented *posterior* vector of returns, containing n posterior security returns $\vec{\vec{r}}_{|\mathcal{H},\mathcal{G}[n\times 1]}$, f posterior factor returns $\vec{\vec{r}}_{F|\mathcal{H},\mathcal{G}[f\times 1]}$, and n posterior security idiosyncratic returns $\vec{\vec{\xi}}_{|\mathcal{H},\mathcal{G}[n\times 1]}$;

$$\vec{\vec{Y}}^{\,a}_{|\mathcal{H},\mathcal{G}[K\times 1]} = \begin{pmatrix} \vec{\vec{y}}_{|\mathcal{H},\mathcal{G}[k_1\times 1]} \\ \vec{\vec{y}}_{F|\mathcal{H},\mathcal{G}[k_2\times 1]} \\ \vec{\vec{y}}_{\xi|\mathcal{H},\mathcal{G}[k_3\times 1]} \end{pmatrix} \tag{4.4}$$

is the augmented vector of updated views, containing k_1 views $\vec{\vec{y}}_{|\mathcal{H},\mathcal{G}[k_1\times 1]}$ on security returns, k_2 views $\vec{\vec{y}}_{F|\mathcal{H},\mathcal{G}[k_2\times 1]}$ on factor returns, k_3 views $\vec{\vec{y}}_{\xi|\mathcal{H},\mathcal{G}[k_3\times 1]}$ on security idiosyncratic returns; and

$$\vec{\vec{\varepsilon}}^{\,a}_{[K\times 1]} = \begin{pmatrix} \vec{\vec{\varepsilon}}_{[k_1\times 1]} \\ \vec{\vec{\varepsilon}}_{F[k_2\times 1]} \\ \vec{\vec{\varepsilon}}_{\xi[k_3\times 1]} \end{pmatrix} \tag{4.5}$$

is the augmented vector of view estimation errors, containing k_1 view errors $\vec{\vec{\varepsilon}}_{[k_1\times 1]}$ on security returns, k_2 view errors $\vec{\vec{\varepsilon}}_{F[k_2\times 1]}$ on factor returns, k_3 view errors $\vec{\vec{\varepsilon}}_{\xi[k_3\times 1]}$ on security idiosyncratic returns.

Factor model (3.1) dictates that $\vec{\vec{r}}$, $\vec{\vec{r}}_F$ and $\vec{\vec{\xi}}$ are linearly dependent. To ensure full-rank representation, we define our *analytical universe* $\vec{\vec{r}}^{\,a}_{[N\times 1]} = \left\{\vec{\vec{r}}^{\,\mathrm{T}}_{[n\times 1]}, \vec{\vec{r}}^{\,\mathrm{T}}_{F[f\times 1]}\right\}^{\mathrm{T}}$, where $N = n + f$. The view formulation (4.1) is then equivalently expressed in this universe as follows:

$$\mathbf{P}^a_{[K\times N]}\vec{\vec{r}}^{\,a}_{|\mathcal{H},\mathcal{G}[N\times 1]} = \vec{\vec{y}}^{\,a}_{|\mathcal{H},\mathcal{G}[K\times 1]} + \vec{\vec{\varepsilon}}^{\,a}_{[K\times 1]} \tag{4.6}$$

where

$$\mathbf{P}^a_{[K\times N]} = \begin{pmatrix} \mathbf{P}_{[k_1\times n]} & \mathbf{0}_{[k_1\times f]} \\ \mathbf{0}_{[k_2\times n]} & \mathbf{P}_{F[k_2\times f]} \\ \mathbf{P}_{\xi[k_3\times n]} & -\mathbf{P}_{\xi[k_3\times n]}\mathbf{B}_{[n\times f]} \end{pmatrix} \tag{4.7}$$

is the (analytical) augmented view structure matrix;

$$\vec{\vec{r}}^{\,a}_{|\mathcal{H},\mathcal{G}[N\times 1]} = \begin{pmatrix} \vec{\vec{r}}_{|\mathcal{H},\mathcal{G}[n\times 1]} \\ \vec{\vec{r}}_{F|\mathcal{H},\mathcal{G}[f\times 1]} \end{pmatrix} \tag{4.8}$$

is the unknown but required (analytical) augmented *posterior* vector of returns, containing n posterior security returns $\vec{\vec{r}}_{|\mathcal{H},\mathcal{G}}$, and f posterior factor returns $\vec{\vec{r}}_{F|\mathcal{H},\mathcal{G}}$; and $\vec{\vec{y}}^{\,a}_{|\mathcal{H},\mathcal{G}} = \vec{\vec{Y}}^{\,a}_{|\mathcal{H},\mathcal{G}} + \vec{c}_{[K\times 1]}$ where

$$\vec{c}_{[K\times 1]} = \begin{pmatrix} \vec{0}_{[k_1\times 1]} \\ \vec{0}_{[k_2\times 1]} \\ \mathbf{P}_{\xi[k_3\times n]}\vec{a}_{[n\times 1]} \end{pmatrix} \tag{4.9}$$

is a constant vector that collects the extra returns due to the deterministic 'drift' term in the linear factor model not captured by the factor set.

As in the BL specification, we make the following assumption regarding the prior view, given only public information $\mathcal{G}$:

Assumption 4.1 (Augmented Prior Return Forecasts) *The prior return forecasts (based on the public information) are normally distributed as follows:*

$$\vec{\tilde{r}}^{a}_{|\mathcal{G}[N\times 1]} \overset{\text{delegate}}{\rightarrow} \vec{\tilde{\mu}}^{a}_{M[N\times 1]} \sim \mathbb{N}(\vec{\tilde{\pi}}^{a}_{[N\times 1]}, \tau \mathbf{\Sigma}^{a}_{[N\times N]}) \tag{4.10}$$

where

$$\vec{\tilde{\mu}}^{a}_{M[N\times 1]} = \begin{pmatrix} \vec{\tilde{\mu}}_{M[n\times 1]} \\ \vec{\tilde{\mu}}_{MF[f\times 1]} \end{pmatrix} \tag{4.11}$$

is the augmented vector of market *return forecasts, containing n market estimates on security returns $\vec{\tilde{\mu}}_M$, and f market estimates on factor returns $\vec{\tilde{\mu}}_{MF}$;*

$$\vec{\tilde{\pi}}^{a}_{[N\times 1]} = \begin{pmatrix} \vec{\tilde{\pi}}_{[n\times 1]} \\ \vec{\tilde{\pi}}_{F[f\times 1]} \end{pmatrix} \tag{4.12}$$

is the augmented vector of CAPM-assessed equilibrium returns, containing n equilibrium security returns $\vec{\tilde{\pi}}$, and f equilibrium factor returns $\vec{\tilde{\pi}}_F$;

$$\mathbf{\Sigma}^{a}_{[N\times N]} = \mathbb{C}\mathrm{ov}\left(\vec{\tilde{r}}^{a}, (\vec{\tilde{r}}^{a})^{\mathrm{T}}\right) \tag{4.13}$$

is the covariance matrix of the augmented returns $\vec{\tilde{r}}^{a}$; and τ is the BL scalar representing the PM's uncertainty about the estimation.

With private information $\mathcal{H}$, view (4.1), or equivalently, (4.6), is formulated. In terms of the view estimation errors $\vec{\tilde{\varepsilon}}^{a}_{[K\times 1]}$, we make the following normality assumption:

Assumption 4.2 (Augmented View Errors) *The view errors are normally distributed as follows:*

$$\vec{\tilde{\varepsilon}}^{a}_{[K\times 1]} \sim \mathbb{N}(\vec{0}_{[K\times 1]}, \mathbf{\Omega}^{a}_{[K\times K]}) \tag{4.14}$$

where $\vec{0}$ is a vector of zeros; and

$$\mathbf{\Omega}^{a}_{[K\times K]} = \begin{pmatrix} \mathbf{\Omega}_{[k_1\times k_1]} & & \mathbf{0} \\ & \mathbf{\Omega}_{F[k_2\times k_2]} & \\ \mathbf{0} & & \mathbf{\Omega}_{\xi[k_3\times k_3]} \end{pmatrix} \tag{4.15}$$

is the augmented block-diagonal variance matrix of view-estimation errors provided by the PM[8]*, containing the view uncertainty matrices $\mathbf{\Omega}_{[k_1\times k_1]}$ for security (or portfolio) returns, $\mathbf{\Omega}_{F[k_2\times k_2]}$ for factor returns, and*

[8]Note again, the block-diagonal specification here assumes that cross-'asset class' views are independent, i.e., those views covering security, factor and idiosyncratic returns are treated as independent.

$\mathbf{\Omega}_{\xi[k_3 \times k_3]}$ *for security idiosyncratic returns, which can be individually considered, for simplicity, as diagonal matrices*[9].

Therefore, we have the following perception conditional on a realisation of $\vec{\tilde{r}}^a_{|\mathcal{H},\mathcal{G}}$ or $\vec{\tilde{\gamma}}^a_{|\mathcal{H},\mathcal{G}}$: -

$$\vec{\tilde{y}}^a_{|\vec{\tilde{r}}^a,\mathcal{H},\mathcal{G}} \sim \mathbb{N}\left(\mathbf{P}^a \vec{\tilde{r}}^a_{|\mathcal{H},\mathcal{G}}, \mathbf{\Omega}^a\right) = \mathbb{N}\left(\mathbf{Q}^a \vec{\tilde{\gamma}}^a_{|\mathcal{H},\mathcal{G}} + \vec{c}, \mathbf{\Omega}^a\right) \tag{4.16}$$

Applying Theorem 2.1 to the prior knowledge $\vec{\tilde{r}}^a_{|\mathcal{G}} \sim \mathbb{N}(\vec{\tilde{\pi}}^a, \tau \mathbf{\Sigma}^a)$ and the PM views (4.16) with the final conviction $\vec{\tilde{Y}}^a_{|\vec{\tilde{\gamma}}^a,\mathcal{H},\mathcal{G}} \overset{\text{belief}}{\rightarrow} \vec{\tilde{q}}^a_{[K\times 1]}$ (the 'ultimate' view mean estimation), or equivalently

$$\vec{\tilde{y}}^a_{|\vec{\tilde{r}}^a,\mathcal{H},\mathcal{G}} \overset{\text{belief}}{\rightarrow} \vec{\tilde{q}}^a + \vec{c} \tag{4.17}$$

where

$$\vec{\tilde{q}}^a_{[K\times 1]} = \begin{pmatrix} \vec{\tilde{q}}_{[k_1\times 1]} \\ \vec{\tilde{q}}_{F[k_2\times 1]} \\ \vec{\tilde{q}}_{\xi[k_3\times 1]} \end{pmatrix} \tag{4.18}$$

is the augmented vector of best view estimates provided by the PM, containing k_1 view estimates $\vec{\tilde{q}}_{[k_1\times 1]}$ on security (or portfolio) returns, k_2 view estimates $\vec{\tilde{q}}_{F[k_2\times 1]}$ on factor (or combination of factors) returns, and k_3 view estimates $\vec{\tilde{q}}_{\xi[k_3\times 1]}$ on security idiosyncratic returns, we obtain the closed-form result as follows:

Theorem 4.1 (Augmented Posterior Return Estimates) *The posterior return estimates are normally distributed, i.e.,* $\vec{\tilde{r}}^a_{|\vec{\tilde{q}}^a,\mathcal{H},\mathcal{G}} \sim \mathbb{N}(\vec{\tilde{m}}^a_{[N\times 1]}, \widehat{\mathbf{V}}^a_{[N\times N]})$*, where the updated mean estimates are:*

$$\vec{\tilde{m}}^a = \left[(\tau\mathbf{\Sigma}^a)^{-1} + (\mathbf{P}^a)^{\mathrm{T}}(\mathbf{\Omega}^a)^{-1}\mathbf{P}^a\right]^{-1}\left[(\tau\mathbf{\Sigma}^a)^{-1}\vec{\tilde{\pi}}^a + (\mathbf{P}^a)^{\mathrm{T}}(\mathbf{\Omega}^a)^{-1}(\vec{\tilde{q}}^a + \vec{c})\right] \tag{4.19}$$

and the updated variance-covariance matrix is:

$$\widehat{\mathbf{V}}^a = \left[(\tau\mathbf{\Sigma}^a)^{-1} + (\mathbf{P}^a)^{\mathrm{T}}(\mathbf{\Omega}^a)^{-1}\mathbf{P}^a\right]^{-1} \tag{4.20}$$

where

$$\mathbf{\Sigma}^a_{[N\times N]} = \begin{bmatrix} \mathbf{\Sigma}_{[n\times n]} & \mathbf{B}_{[n\times f]}\mathbf{\Sigma}_{F[f\times f]} \\ \mathbf{\Sigma}_{F[f\times f]}\mathbf{B}^{\mathrm{T}}_{[f\times n]} & \mathbf{\Sigma}_{F[f\times f]} \end{bmatrix} \tag{4.21}$$

$$\vec{\tilde{\pi}}^a_{[N\times 1]} = \begin{pmatrix} \vec{\tilde{\pi}}_{[n\times 1]} \\ \vec{\tilde{\pi}}_{F[f\times 1]} \end{pmatrix} = \kappa \begin{pmatrix} \mathbf{\Sigma} \\ \mathbf{\Sigma}_F \mathbf{B}^{\mathrm{T}} \end{pmatrix} \vec{w}_{M[n\times 1]} \tag{4.22}$$

$$\kappa = \frac{\mathbb{E}(\widetilde{R}_M|\mathcal{G}) - r_f}{\sigma_M^2} \tag{4.23}$$

$\widetilde{R}_M$ *is the long-term gross return of the market;* σ_M *is the long-term volatility of the market return;* $\vec{w}_M$ *is the market portfolio weight vector; and other variables have been explained above.*

[9] Although there is no technical difficulty in relaxing the view independence assumption made in the original BL model, we continue assuming this for the convenience of user. With our recommended confidence parameter setting process discussed in our forthcoming article, it can be shown that this assumption is not restrictive.

Proof. See Appendix A.□

Please refer to Figure 2 for the whole picture of the ABL-based portfolio optimisation process.

5 Worked Examples

In this section, we illustrate the application of the ABL model to factor-based but ranking-free equity portfolio construction through worked examples. We take FTSE EUROTOP 100 INDEX as the universe (including 104 stocks at the time of analysis) as well as the benchmark. We aim to illustrate how to construct portfolios with factor or stock-idiosyncratic views[10] in the ABL framework and how the model behaves.

We choose the Lehman ERA (Europe Weekly) as our factor model input. View inputs are therefore expressed with regard to the ERA factors. However, it should be noted that while the methodology requires risk model feeds, it is not restricted to any specific factor model. In case the PM prefers a certain vendor's factor model or even a customised factor model, she has the freedom.

Our data source is the Lehman proprietary equity database. The optimisation is carried out based on the following variance-minimisation specification:

$$\underset{\vec{w}}{\operatorname{argmin}}\{\vec{w}^{\mathrm{T}}\widehat{\mathbf{V}}\vec{w}\} \tag{5.1}$$

$$\text{s.t.} \begin{cases} \vec{w}^{\mathrm{T}}\vec{\widetilde{m}} \geqslant TR \\ \vec{w}^{\mathrm{T}}\vec{1} = 1 \\ \vec{w}^{\mathrm{T}}\vec{\beta} = 1 \end{cases} \tag{5.2}$$

where $\vec{\widetilde{w}}_{[n\times 1]}$ is the vector of portfolio weights; $\vec{\widetilde{m}}_{[n\times 1]}$ is the stock part of ABL the posterior mean $\vec{\widetilde{m}}^{a}_{[N\times 1]}$ with the factor part truncated; $\widehat{\mathbf{V}}_{[n\times n]}$ is the stock part of the ABL posterior variance $\widehat{\mathbf{V}}^{a}_{[N\times N]}$ with the factor part truncated; and TR is the portfolio target return.

There are 45 factors relevant to our universe. These are listed in Appendix C. Suppose we have the following 2 factor views and a composite stock-specific view:

$$\begin{cases} \mathbb{E}(\widetilde{r}_{\text{Auto}}|\mathcal{H},\mathcal{G}) = 0.20 \\ \mathbb{E}(\widetilde{r}_{\text{Bank}}|\mathcal{H},\mathcal{G}) = 0.15 \\ -\mathbb{E}(\widetilde{\xi}_{\text{TIT}}|\mathcal{H},\mathcal{G}) - \mathbb{E}(\widetilde{\xi}_{\text{ROG}}|\mathcal{H},\mathcal{G}) + 3\mathbb{E}(\widetilde{\xi}_{\text{RWE}}|\mathcal{H},\mathcal{G}) - \mathbb{E}(\widetilde{\xi}_{\text{DWP}}|\mathcal{H},\mathcal{G}) = 0.20 \end{cases}. \tag{5.3}$$

The first view basically says, within the prediction horizon, the Automobiles factor return is expected to be 20%. The second view forecasts the Banks factor return at 15%. The third view is expressed on a set of stock-specific returns, with an expected relative performance of 20%. This view (on a combination of stock-specific returns) resembles 'portable alpha' since it excludes the factor common part.

The corresponding view structure $\mathbf{P}_{[3\times N]}$ and view estimates $\vec{\widetilde{q}}_{[3\times 1]}$ are given in Table 1.

[10]This model contains the traditional BL as a special case and handles stock/portfolio views in precisely the same way. The reader is referred to Cheung (2009) for tests of such traditional BL views.

Figure 2 The ABL-Based Portfolio Optimisation Framework

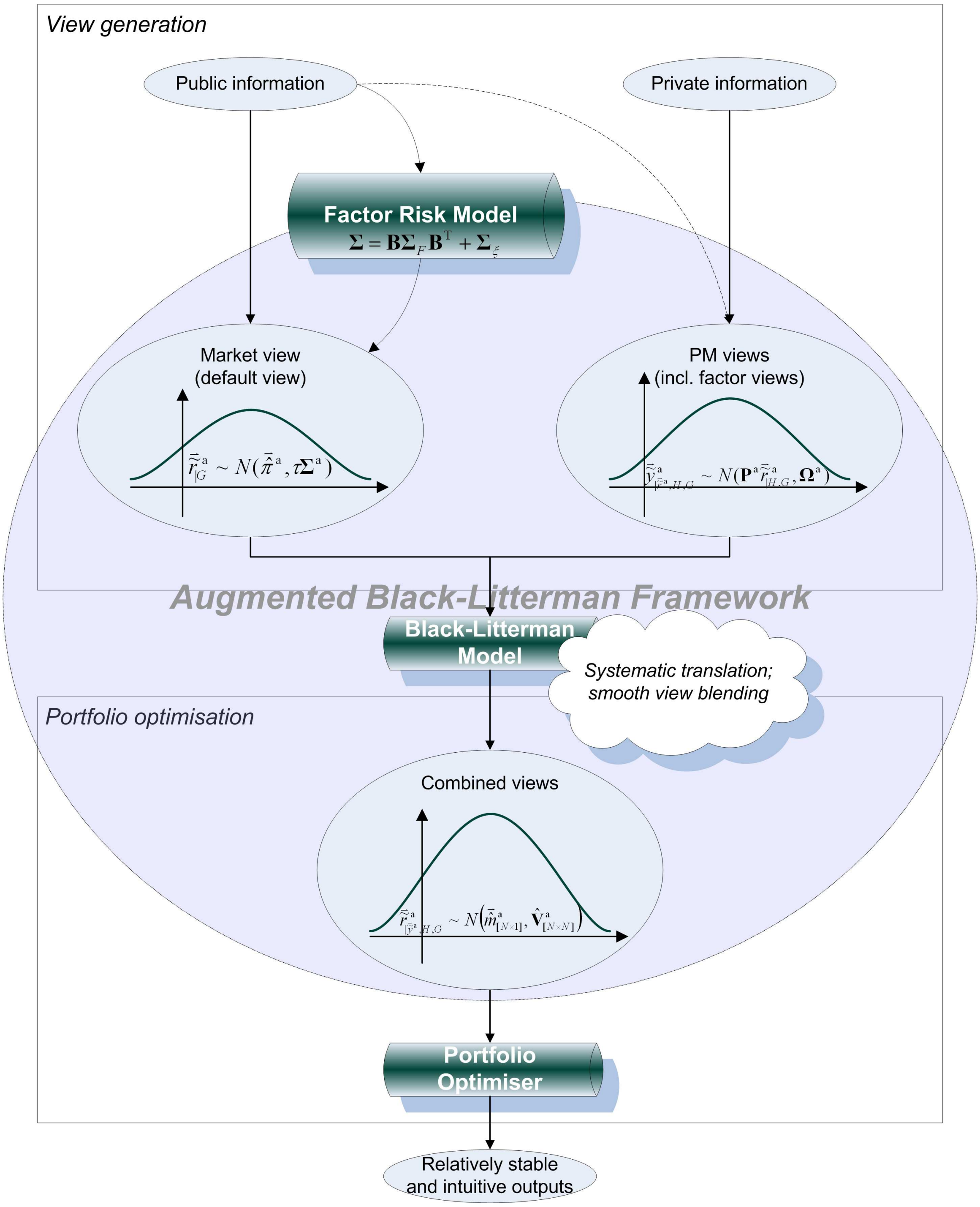


Table 1: View Mean Forecast Inputs

View / Securities / forecast	View Structure $\underset{[N\times 3]}{\mathbf{P}^{\mathrm{T}}}$ View 1 (factor view)	View 2 (factor view)	View 3 (stock-specific view)
Automobiles	1	0	0
Banks	0	1	0
TIT IM	0	0	-1
BATS LN	0	0	0
ROG VX	0	0	-1
NESN VX	0	0	0
ABBN VX	0	0	0
RWE GY	0	0	3
AV/ LN	0	0	0
DPW GY	0	0	-1
EDF FP	0	0	0
BT/A LN	0	0	0
…	…	…	…
GLE FP	0	0	0
BNP FP	0	0	0
BLT LN	0	0	0
SIE GY	0	0	0
View forecast $\underset{[1\times 3]}{\vec{q}^{\mathrm{T}}}$	**0.20**	**0.15**	**0.20**

Table 2: Four View Scenarios

		Scenario 1 Strong Auto view			Scenario 2 Strong Bank view			Scenario 3 Factor combined			Scenario 4 Stock-specific view		
	Mean estimate	Var	95% confidence Interval		Var	95% confidence Interval		Var	95% confidence Interval		Var	95% confidence Interval	
View 1	20.00%	0.01	0.00%	40.00%	100	-1980.00%	2020.00%	0.01	0.00%	40.00%	100	-1980.00%	2020.00%
View 2	15.00%	100	-1985.00%	2015.00%	0.01	-5.00%	35.00%	0.01	-5.00%	35.00%	100	-1985.00%	2015.00%
View 3	20.00%	100	-1980.00%	2020.00%	100	-1980.00%	2020.00%	100	-1980.00%	2020.00%	0.01	0.00%	40.00%

In the traditional optimisation framework, views are signified only by mean estimates. In the BL model, *view significance* is captured by the view uncertainties as well. For example, with the above three views, the four scenarios in Table 2 should be treated differently. In Scenario 1, View 2 and View 3 are by far dominated by View 1 which gives a relatively clear confidence interval $[0\%, 40\%]$. Thus Scenario 1 can be considered as a 'strong Auto view' with the other two views effectively ignored. Likewise, Scenario 2 represents a 'strong Bank view', and Scenario 4, a 'strong stock-specific view'. In Scenario 3, since both factor views are assigned low variances (i.e., high confidence), we consider it a 'factor combination' scenario. Now let us see how consistent our numerical outputs are with these assertions[11].

5.1 Scenario 1: Optimisitic View on the Automobiles Factor

The computed posterior forecasts are given in Table 3[12]. We observe that,

[11] Note in our analysis, we set our uncertainty multiplier for the market view τ to 1. We believe this is a good choice in normal applications, e.g., benchmarked active portfolio management, enhanced indexation etc. We will discuss the classic issue of view confidence parameter setting with greater detail in a forthcoming article.

[12] Since the list is too long, for illustration purpose, we just partially list the results.

Table 3: Black-Litterman Forecasts (Strong Autos View)

Assets	Market Weight	Prior Returns	Posterior Returns	Difference
TIT IM	0.61%	3.21%	4.60%	1.39%
BATS LN	0.78%	2.64%	4.57%	1.93%
ROG VX	2.00%	2.68%	4.33%	1.65%
NESN VX	2.37%	1.78%	2.02%	0.23%
ABBN VX	0.60%	10.64%	18.71%	8.07%
RWE GY	0.65%	2.83%	4.51%	1.68%
AV/ LN	0.60%	10.04%	16.49%	6.44%
DPW GY	0.44%	7.70%	11.73%	4.02%
EDF FP	0.34%	6.59%	10.35%	3.76%
BT/A LN	0.79%	3.51%	6.98%	3.47%
...	...	...	...	...
GLE FP	1.27%	7.46%	13.53%	6.07%
BNP FP	1.53%	7.06%	12.52%	5.46%
BLT LN	0.83%	11.84%	20.06%	8.22%
SIE GY	1.52%	9.39%	16.37%	6.97%

Table 4: Optimisation Outputs (Strong Autos View)

Assets	Beta	Market Weight	Posterior Weight	Difference
DAI GY	1.63	1.33%	11.25%	9.92%
VOW GY	1.61	0.52%	9.40%	8.88%
BMW GY	1.07	0.28%	7.51%	7.23%
VOW3 GY	1.60	0.17%	7.16%	6.99%
SAP GY	0.19	0.68%	7.09%	6.41%
RNO FP	1.37	0.39%	5.79%	5.40%
PHIA NA	1.15	0.72%	4.19%	3.48%
BT/A LN	0.50	0.79%	4.12%	3.33%
AI FP	1.10	0.47%	3.67%	3.20%
BAS GY	1.05	0.90%	4.02%	3.12%
...	...	...	...	...
ZURN VX	1.12	0.66%	-2.70%	-3.36%
HSBA LN	0.72	3.22%	-1.75%	-4.97%
DGE LN	0.68	0.86%	-4.63%	-5.49%
OR FP	0.75	0.54%	-8.61%	-9.15%

- most of the stock returns look more optimistic, and
- the posterior stock ranking is adjusted but still similar to the prior ranking.

Since the dominating view is about the Automobiles factor, rather than any particular stock, its one-to-many factor relationship with stocks gives rise to the difficulty in finding an intuitive pattern from the posterior view adjustments. The broad amplification effect is due to the high confidence on the positive factor view which is treated by the ABL model as a universally 'good' news.

Table 4 presents the recommended weigh changes (or active weights or tilts) sorted from high to low. Note the automobile manufacturers Daimler AG (DAI GY), Volkswagen AG (VOW GY), Bayerische Motoren Werke AG (BMW GY), and Volkswagen AG (VOW3 GY) top the league, followed by software firm SAP AG (SAP GY) and another automobile manufacturer Renault SA (RNO FP).

To see the influences of this view on different sectors, we aggregate these recommended active weights according to MSCI Sector 2[13] as shown in Figure 3. The optimistic Automobiles factor view leads to a significant tilt of 38.4% to the Automobiles sector. This spiky sectoral overweight indicates that our factor tilt is indeed a FM portfolio, similar to what the FF factor-ranking approach attempts to achieve[14]. It can be shown that, the ABL model offers the same functionality as the FF method by elegantly endogenising it. Some other 'winners' are Software, Capital Goods, Financials etc. Due to correlations, an Auto industry boom may trigger optimism in these industries as well.

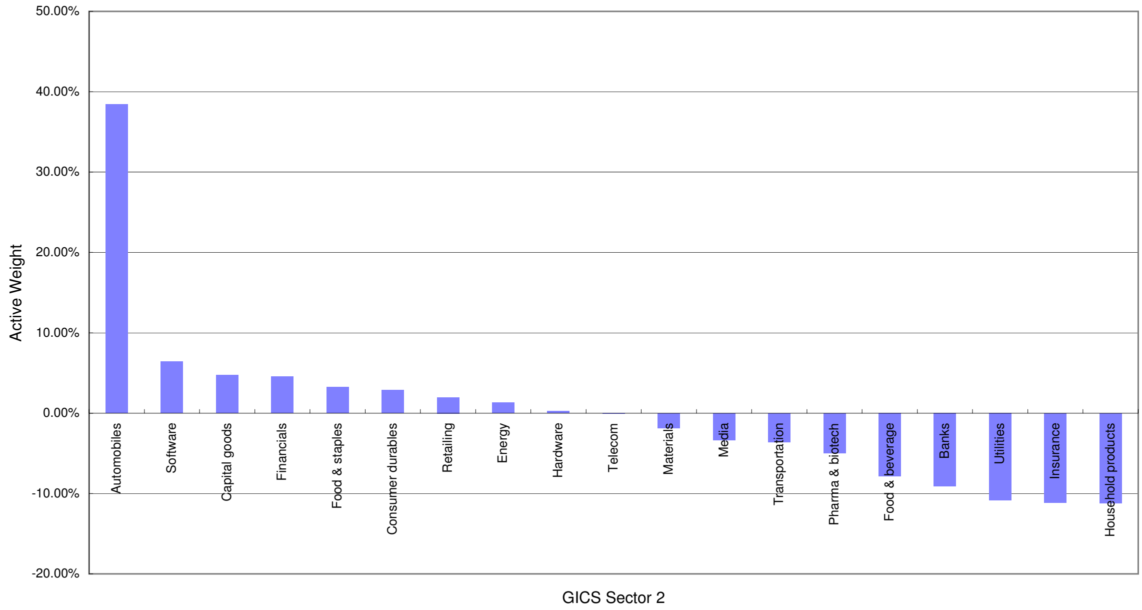
An interesting observation is the separation of the Financials and Banks. The most underweighted sectors are Insurance and Household Products. This is mainly due to (a) a relatively low correlation between Automobiles and these sectors; and (b) the *leveraging effect* of the weight constraints (i.e., active weights totalling 0) and thus, some sectors have to be use to leverage the sectors that are positively influenced by the factor view.

In this scenario, the overall turnover is 199.8%. The magnitude of the turnover is determined by the PM's view significance. Should the PM feel the turnover excessively high, she may weaken her views. For

[13] This classification has the closest proximity to the ERA *Industries* factors definition.

[14] In a forthcoming article, we will explore the theoretical linkage between the FF ranking approach and the ABL framework. In due course, the FM property of the factor tilt will be explained further.

Figure 3: The Optimised Active Weights (Strong Automobiles View)



example, should View 1 be less certain, e.g.,

$$\widetilde{r}_{\text{Auto}|\mathcal{H},\mathcal{G}} \sim \mathbb{N}(0.20, 0.05), \tag{5.4}$$

the turnover would have been 58.6% but the sectoral active allocation pattern remains exactly the same (see Figure 4 compared with Figure 3). Alternatively, the PM can also weaken her view by lowering the mean estimate to attain similar effects.

5.2 Scenario 2: Optimistic View on the Banks Factor

Our observation from the positive Banks view scenario (see Figure 5) is very similar to those from the Automobiles. We highlight the following:

(1) The posterior stock ranking is adjusted but still similar to the prior ranking.

(2) The ABL-optimised weights are intuitive - most banks are picked up for aggressive allocation. The picture becomes clearer when we compare sector aggregate allocations.

(3) The Financials sector is also overweighed; but Insurance is underweighted.

(4) The Automobile sector is underweighted. This is consistent with the Scenario 1 observation where Banks were used to leverage the view on Automobiles; and here is the opposite.

Through the above tests, we believe the ABL-based optimisation framework makes intuitive and reliable sense in terms of tilt pattern.

Figure 4: The Optimised Active Weights (Weaker Automobiles View)

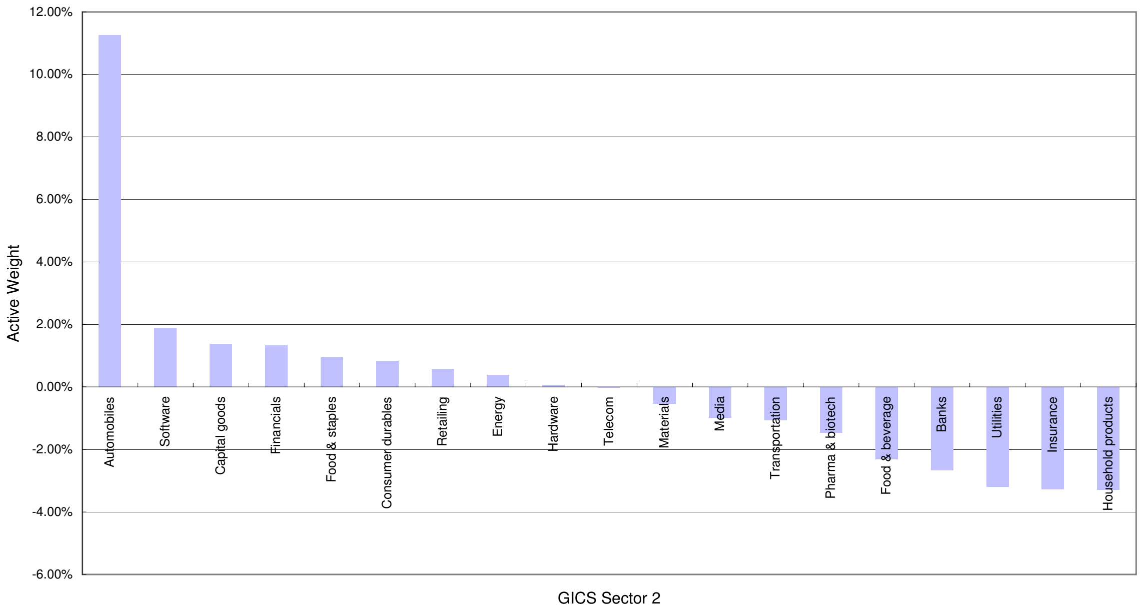


Figure 5: The Optimised Active Weights (Strong Banks View)

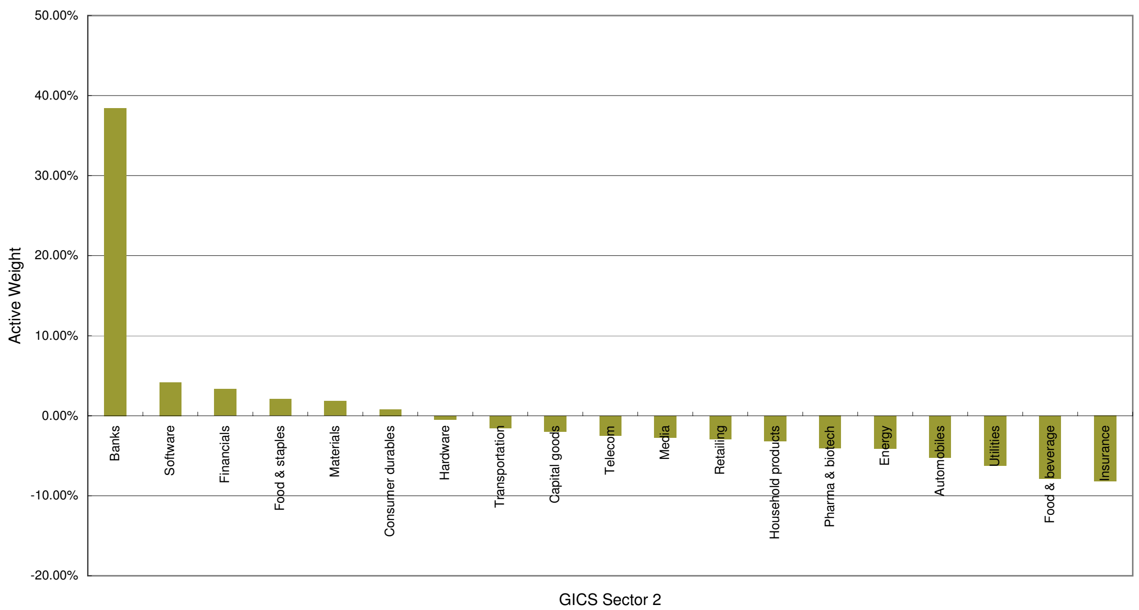


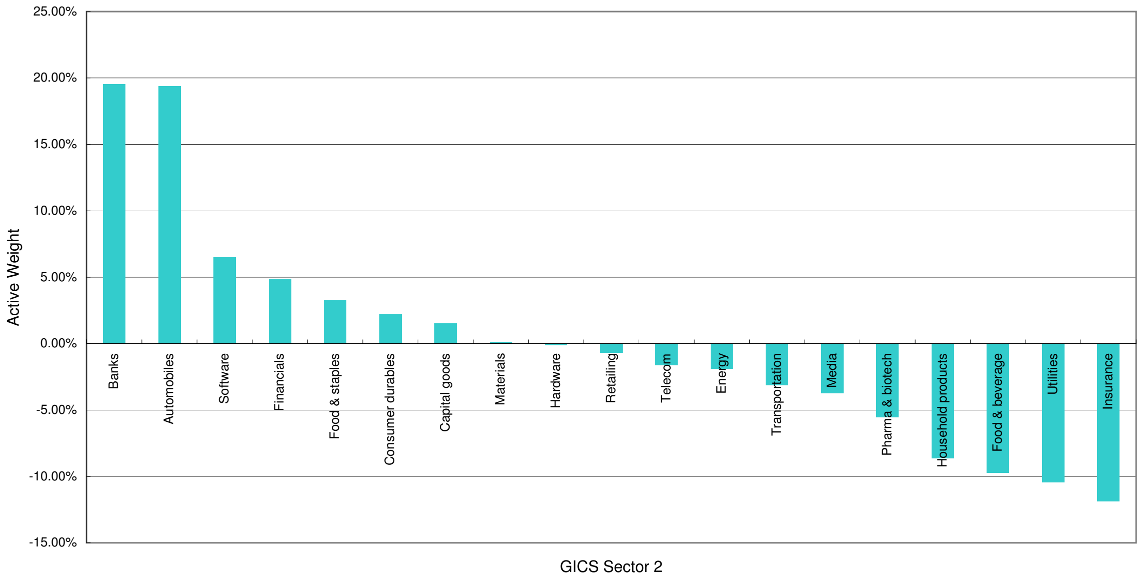
Table 5: Black-Litterman Forecasts (Combined Factor View)

Assets	Market Weight	Prior Returns	Posterior Returns	Difference
TIT IM	0.61%	3.21%	5.20%	1.99%
BATS LN	0.78%	2.64%	5.03%	2.40%
ROG VX	2.00%	2.68%	4.96%	2.29%
NESN VX	2.37%	1.78%	2.17%	0.39%
ABBN VX	0.60%	10.64%	20.33%	9.69%
RWE GY	0.65%	2.83%	5.17%	2.34%
AV/ LN	0.60%	10.04%	18.16%	8.12%
DPW GY	0.44%	7.70%	12.53%	4.83%
EDF FP	0.34%	6.59%	11.13%	4.54%
BT/A LN	0.79%	3.51%	7.54%	4.03%
…	…	…	…	…
GLE FP	1.27%	7.46%	15.71%	8.25%
BNP FP	1.53%	7.06%	14.50%	7.44%
BLT LN	0.83%	11.84%	22.03%	10.19%
SIE GY	1.52%	9.39%	17.69%	8.30%

Table 6: Optimisation Outputs (Combined Factor View)

Assets	Beta	Market Weight	Posterior Weight	Difference
SAP GY	0.19	0.68%	7.17%	6.49%
VOW GY	1.61	0.52%	6.90%	6.39%
BBVA SQ	1.15	1.39%	6.57%	5.17%
VOW3 GY	1.60	0.17%	5.25%	5.07%
GLE FP	1.07	1.27%	6.25%	4.98%
NDA SS	1.04	0.66%	4.85%	4.19%
SAN SQ	1.01	1.78%	5.70%	3.92%
DAI GY	1.63	1.33%	5.19%	3.86%
BNP FP	1.01	1.53%	5.11%	3.58%
FORA NA	1.34	0.47%	3.92%	3.45%
…	…	…	…	…
VIV FP	0.88	0.74%	-3.01%	-3.75%
NESN VX	0.25	2.37%	-1.40%	-3.77%
DGE LN	0.68	0.86%	-4.72%	-5.58%
OR FP	0.75	0.54%	-6.63%	-7.17%

Figure 6: The Optimised Active Weights (Banks and Autos Combined) - Sector 2 Aggregation



5.3 Scenario 3: Combining Factor Views on Automobiles and Banks

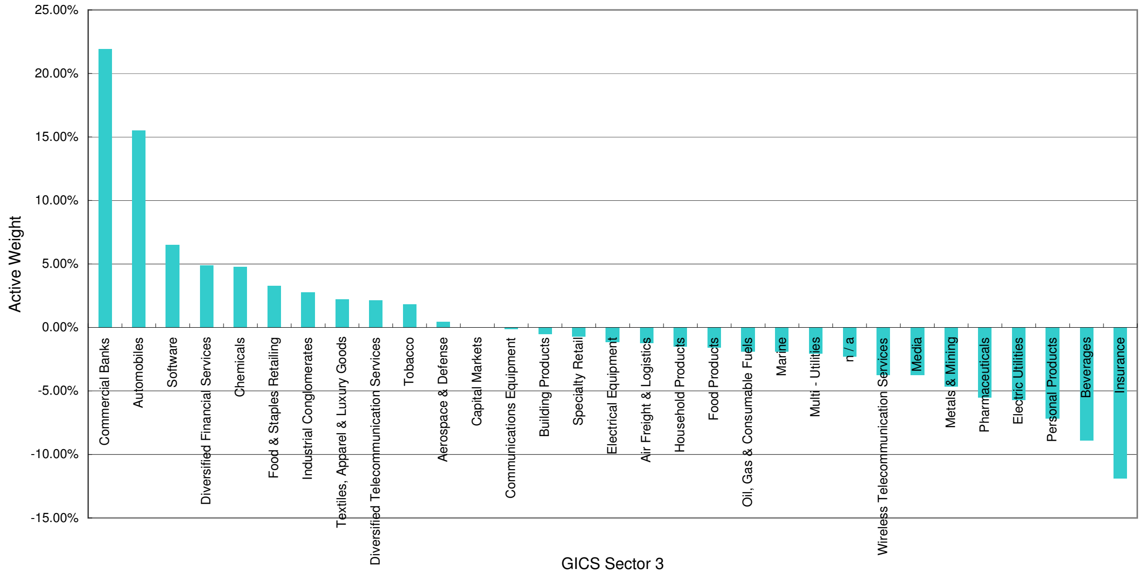
The computed posterior forecasts for Scenario 3 (combined factor views) are given in Table 5. Again,

- most of the stock returns appear higher (i.e., more optimistic), and
- the posterior stock ranking is adjusted but still similar to the prior ranking.

Table 6 presents the recommended active weights sorted from high to low. From this, it is clear that, among the top 10 stocks selected for overweighing, all (except one Software stock) belong to the Automobiles, Banks, and Financials sectors. Figure 6 gives the sector 2 aggregation picture. We observe that the optimistic factor views combined give rise to significant tilts to the Banks (19.5%) and Automobiles (19.4%) sectors.

Figure 7 gives the picture of Sector 3 aggregation. Not surprisingly, Commercial Banks, Automobiles, Software and Diversified Financial Services are the major 'winners'; and Personal Products, Beverages, and

Figure 7: The Optimised Active Weights (Banks and Autos Combined) - Sector 3 Aggregation



Insurance are the major 'losers'. The underweighting of these sectors can be attributed to their relatively low correlations with the optimistic factors. With the two strong factor views combined, the optimiser recommends a 192.2% turnover, which is similar to Scenarios 1 and 2. Again, the turnover can be controlled by adjusting the view uncertainty parameters without imposing efficiency-undermining weight constraints that destroy the intuitive pattern as in traditional optimisation.

5.4 Scenario 4: Stock-Specific View

The ABL framework also admits stock-specific views. Table 7 shows that, with view Scenario 4, forecasts are adjusted significantly for those stocks covered directly by the view (i.e., Telecom Italia SPA (TIT IM), Roche Holding AG (ROG VX), RWE AG (RWE GY) and DreamNex (DWP GY)), and trivially for others. This is expected since the view is 'stock-specific' in nature. The linear combination of the view does not affect the fact that the view is not about systematic returns and therefore has minimal implication for other stocks.

Table 8 and Figure 8 present the optimised active weights. As expected, the main active allocation adjustments are made to the four view-covered stocks, with the directions intuitive. Aside from these major tilts, active weights to other stocks are non-negligible. These additional tilts create a 'synthetic basket' that hedges the common risk of the main tilts, leaving the net effect reflecting precisely the 'idiosyncratic' nature of the view, often referred to as 'portable alpha' view.

5.5 Summary

Through the worked examples, we observe the following features of the ABL model:

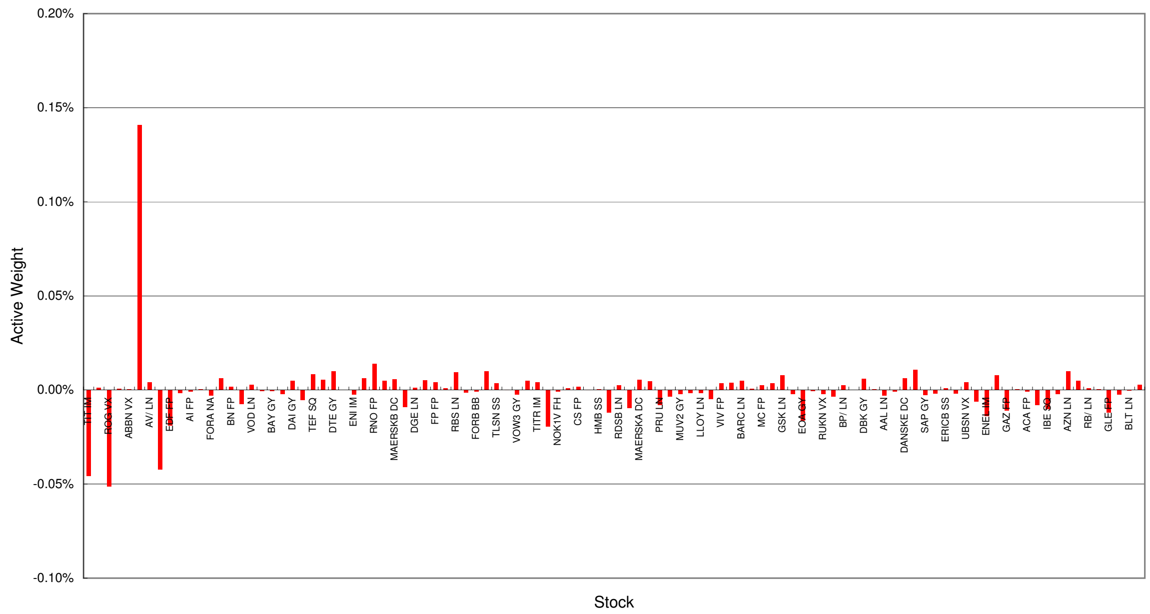
Table 7: Black-Litterman Forecasts (Stock-Specific View)

Assets	Market Weight	Prior Returns	Posterior Returns	Difference
TIT IM	0.61%	3.21%	2.16%	1.05%
BATS LN	0.78%	2.64%	2.67%	-0.03%
ROG VX	2.00%	2.68%	1.28%	1.40%
NESN VX	2.37%	1.78%	1.79%	-0.01%
ABBN VX	0.60%	10.64%	10.65%	-0.01%
RWE GY	0.65%	2.83%	7.85%	-5.02%
AV/ LN	0.60%	10.04%	10.00%	0.04%
DPW GY	0.44%	7.70%	5.87%	1.83%
EDF FP	0.34%	6.59%	6.59%	0.00%
BT/A LN	0.79%	3.51%	3.58%	-0.07%
...	...	...	...	...
GLE FP	1.27%	7.46%	7.48%	-0.01%
BNP FP	1.53%	7.06%	7.07%	-0.01%
BLT LN	0.83%	11.84%	11.85%	-0.01%
SIE GY	1.52%	9.39%	9.40%	-0.01%

Table 8: Optimisation Outputs (Stock-Specific View)

Assets	Beta	Market Weight	Posterior Weight	Difference
TIT IM	0.46	0.61%	0.57%	-0.05%
BATS LN	0.38	0.78%	0.78%	0.00%
ROG VX	0.38	2.00%	1.95%	-0.05%
NESN VX	0.25	2.37%	2.37%	0.00%
ABBN VX	1.52	0.60%	0.60%	0.00%
RWE GY	0.40	0.65%	0.80%	0.14%
AV/ LN	1.43	0.60%	0.60%	0.00%
DPW GY	1.10	0.44%	0.40%	-0.04%
EDF FP	0.94	0.34%	0.32%	-0.02%
BT/A LN	0.50	0.79%	0.79%	0.00%
...	...	...	...	...
GLE FP	1.07	1.27%	1.26%	-0.01%
BNP FP	1.01	1.53%	1.53%	0.00%
BLT LN	1.69	0.83%	0.83%	0.00%
SIE GY	1.34	1.52%	1.52%	0.00%

Figure 8: The Optimised Active Weights (Stock-Specific View)



(1) **Probablistic views**. Unlike in the traditional optimisation framework, views in the ABL framework are signified by both their mean ('signal') and variance ('noise'). By adjusting these parameters, we can express our view strength. Stronger views dominate weaker views in active allocation.

(2) **Intuitiveness**. Our examples show that a positive sector view causes spiky active allocation to the stocks in the sector; and a stock-specific view leads to significant overweight to the targeted stocks and certain allocation to the rest of the universe that hedges their common risk. The results are expected and intuitive at both the factor and stock-specific levels.

(3) **Robustness**. Not only are the ABL active allocation patterns intuitive, but they are also invariant up to view strength changes. In contrast to the erratic behaviour of the traditional optimiser in face of small input adjustments or estimation errors, this robustness feature of the ABL model is practically appealing.

(4) **Posterior estimates less informative**. Unlike the active weights that are highly explainable as the result of our view inputs, the posterior return estimates alone appear less informative. The posterior mean vector needs the companion of the posterior variance matrix to make economic sense.

(5) **Ranking-free factor portfolio construction**. The ABL model generates FM portfolios as the FF technique but without factor ranking.

(6) **Unified approach**. By admitting factor views and security-specific views, as well as traditional BL views at the security level, it can not only help stock pickers optimise their portfolios as in the traditional BL model, but can also facilitate factor-based quant strategies and portable alpha strategies, following consistent allocation principles.

6 Conclusion

In quantitative portfolio management, the Fama-French (FF) factor-ranking process is broadly applied. This approach however suffers from hidden factor views, information inefficiency etc. issues. In this article, we augment the Black-Litterman (BL) model to provide a more elegant factor-mimicking (FM) functionality. In this framework, instead of performing factor ranking, we simply need to form our factor views. Our Augmented Black-Litterman (ABL) model then blends them based on the Bayes' Rule, and generates the composite FM portfolio drawing on the security-factor covariance structure contained in the user-chosen factor risk model. Our worked examples demonstrate that the ABL model is an intuitive and robust new approach to factor-based portfolio construction.

Superficially, our approach avoids the factor-ranking process; whereas in fact, it replaces the latter by a mathematical process involving the covariance structure. Since the ranking process essentially sorts security's factor exposures that are part of the covariance structure, the two processes are meant to do fundamentally similar things but the ABL approach is more precise and elegant. Provided the popularity of the ranking process in the quantitative investment community, the potential practical value of the ABL model in helping factor-based portfolio construction alone is considered significant.

Moreover, the model also enables a neat and structural approach to security-specific betting or portable alpha generation. Existing methods for security-specific view implementation typically involve common risk hedging either through an index wrapper or a synthetic basket. The ABL technique provides an efficient one-shot solution which endogenises the hedge and avoids unnecessary cross trades. With this technique, fund managers can focus on forming their security-specific views and rely on the model to construct the portfolio with a built-in hedge. Given these additional important applications, the ABL model significantly boosts the practical value of the BL approach.

Finally, our approach paves the way for a separation of the *art* of alpha generation from the *science* of portfolio construction. Traditionally, different types of views are utilised by different groups of portfolio managers (PM) who design and follow different allocation processes within which, alpha generation is mixed with portfolio construction. Whereas while the market needs a colourful world of alpha generation tastes and ideas to prosper, there is no justification for a segmented world of *ad hoc* processes to implement them. Fundamentally, the ABL technique rightly offers a unified framework within which, different types of views can be implemented in a technically consistent fashion. In this framework, while investors enjoy flexibility in view generation (the art), the methodologies for view implementation into portfolios (the science) are unified. This unification potential is expected to have far-reaching impact on the industry's investment processes.

Recall our practical issue checklist in Cheung (2010), the following have been addressed: -

✓ *Curse of dimensionality*. The BL model spits out a fully populated numerical posterior matrix. In case of a large universe, this poses significant burden to the optimiser. We resolve this issue by obtaining a sparse representation of the matrix drawing on the principal component approximation. (Cheung, 2010)

✓ *Factor-based portfolio construction*. The ABL model does the job, as well as security-specific view-based portfolio construction, by design.

✓ *Confidence parameter setting*. This issue is partially addressed. As observed in the worked examples, portfolio turnover increases with view confidence. More generally, in the BL/ABL framework, portfolio tilts magnitude is monotonically increasing in view confidence. In a forthcoming article, we will explore this property for view confidence parameter setting and beyond.

In our forthcoming articles, we will address the remaining issues: -

☐ *View correlations*. For simplicity, the BL model treats views as orthogonal. In reality, view correlations may exist but are hard to quantify. Is this a problem?

☐ *Prior setting*. In practice, there are a variety of investment styles. Equilibrium is an abstract concept;

and the market portfolio does not seem to be the starting point for all strategies. How should we choose an appropriate prior?

- *Optimiser issues*. The BL framework calls an optimiser for allocation purpose. Whereas the optimiser behaves as a 'black box'. There are risk-aversion as well as transparency issues. Can we make the framework more open and intuitive?

- *Risk model quality*. Risk model has errors itself. How can we economically use it, or take multiple models into account, in allocation?

- *Linearity and normality assumptions*. In the BL model, security returns are considered normal; and the factor model, linear. How to apply this model to non-normal, non-linear markets?

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Appendix A: Proof

Theorem 4.1 (Augmented Posterior Return Estimates) *The posterior return estimates are also normally distributed, i.e.,* $\vec{\tilde{r}}^a_{|\vec{\tilde{q}}^a,\mathcal{H},\mathcal{G}} \sim \mathbb{N}(\vec{\hat{m}}^a_{[N\times 1]}, \widehat{\mathbf{V}}^a_{[N\times N]})$*, where the updated mean estimates are:*

$$\vec{\hat{m}}^a = \left[(\tau\mathbf{\Sigma}^a)^{-1} + (\mathbf{P}^a)^{\mathrm{T}}(\mathbf{\Omega}^a)^{-1}\mathbf{P}^a\right]^{-1}\left[(\tau\mathbf{\Sigma}^a)^{-1}\vec{\hat{\pi}}^a + (\mathbf{P}^a)^{\mathrm{T}}(\mathbf{\Omega}^a)^{-1}(\vec{\hat{q}}^a + \vec{c})\right] \tag{6.1}$$

and the updated variance-covariance matrix is:

$$\widehat{\mathbf{V}}^a = \left[(\tau\mathbf{\Sigma}^a)^{-1} + (\mathbf{P}^a)^{\mathrm{T}}(\mathbf{\Omega}^a)^{-1}\mathbf{P}^a\right]^{-1} \tag{6.2}$$

where

$$\mathbf{\Sigma}^a_{[N\times N]} = \begin{bmatrix} \mathbf{\Sigma}_{[n\times n]} & \mathbf{B}_{[n\times f]}\mathbf{\Sigma}_{F[f\times f]} \\ \mathbf{\Sigma}_{F[f\times f]}\mathbf{B}^{\mathrm{T}}_{[f\times n]} & \mathbf{\Sigma}_{F[f\times f]} \end{bmatrix} \tag{6.3}$$

$$\vec{\hat{\pi}}^a_{[N\times 1]} = \begin{pmatrix} \vec{\hat{\pi}}_{[n\times 1]} \\ \vec{\hat{\pi}}_{F[f\times 1]} \end{pmatrix} = \kappa \begin{pmatrix} \mathbf{\Sigma} \\ \mathbf{\Sigma}_F\mathbf{B}^{\mathrm{T}} \end{pmatrix} \vec{w}_{M[n\times 1]} \tag{6.4}$$

$$\kappa = \frac{\mathbb{E}(\widetilde{R}_M|\mathcal{G}) - r_f}{\sigma_M^2} \tag{6.5}$$

$\widetilde{R}_M$ *is the long-term gross return of the market;* σ_M *is the long-term volatility of the market return;* $\vec{w}_M$ *is the market portfolio holding weights; and other variables have been explained in the main text.*

Proof. Note (4.10), (4.6) and (4.14) are just the augmented version of (2.1), (2.2) and (2.3), respectively. With Theorem 2.1, we reach (6.1) and (6.2). Therefore, it suffices to prove (6.3)-(6.4) only. We start with (6.4). The CAPM states that the expected market equilibrium return of security i should be the return to the market portfolio adjusted by its risk exposure:

$$\hat{\pi}_i = \frac{\mathbb{C}\mathrm{ov}\left(\tilde{r}_i, \tilde{r}_M|\mathcal{G}\right)}{\mathbb{C}\mathrm{ov}\left(\tilde{r}_M, \tilde{r}_M|\mathcal{G}\right)}\left[\mathbb{E}(\widetilde{R}_M|\mathcal{G}) - r_f\right] \tag{6.6}$$

where $\tilde{r}_i$ represents the return of security i; and $\tilde{r}_M$ is the excess returns of the market.

For the whole universe of n securities, the following holds in the spirit of (6.6):

$$\vec{\hat{\pi}} = \frac{\mathbb{C}\mathrm{ov}\left(\vec{\tilde{r}}_{[n\times 1]}, \tilde{r}_M|\mathcal{G}\right)}{\mathbb{C}\mathrm{ov}\left(\tilde{r}_M, \tilde{r}_M|\mathcal{G}\right)}\left[\mathbb{E}(\widetilde{R}_M|\mathcal{G}) - r_f\right]. \tag{6.7}$$

Noting:

$$\tilde{r}_M = \vec{\tilde{r}}^{\mathrm{T}}\vec{w}_{M[n\times 1]}, \tag{6.8}$$

we have:

$$\begin{aligned} \vec{\hat{\pi}} &= \frac{\mathbb{C}\mathrm{ov}\left(\vec{\tilde{r}}, \vec{\tilde{r}}^{\mathrm{T}}\vec{w}_M|\mathcal{G}\right)}{\mathbb{C}\mathrm{ov}\left(\vec{w}_M^{\mathrm{T}}\vec{\tilde{r}}, \vec{\tilde{r}}^{\mathrm{T}}\vec{w}_M|\mathcal{G}\right)}\left[\mathbb{E}(\widetilde{R}_M|\mathcal{G}) - r_f\right] \\ &= \frac{\tau\mathbf{\Sigma}\vec{w}_M}{\vec{w}_M^{\mathrm{T}}\left(\tau\mathbf{\Sigma}\right)\vec{w}_M}\left[\mathbb{E}(\widetilde{R}_M|\mathcal{G}) - r_f\right] \\ &= \kappa\mathbf{\Sigma}\vec{w}_M. \end{aligned} \tag{6.9}$$

Similarly,

$$\begin{aligned}
\vec{\tilde{\pi}}_F &\overset{\text{CAPM}}{=} \frac{\mathbb{C}\text{ov}\left(\vec{\tilde{r}}_F, \vec{\tilde{r}}^{\mathrm{T}}\vec{w}_M|\mathcal{G}\right)}{\mathbb{C}\text{ov}\left(\vec{w}_M^{\mathrm{T}}\vec{\tilde{r}}, \vec{\tilde{r}}^{\mathrm{T}}\vec{w}_M|\mathcal{G}\right)}\left[\mathbb{E}(\widetilde{R}_M|\mathcal{G}) - r_f\right] \\
&= \frac{\mathbb{C}\text{ov}\left(\vec{\tilde{r}}_F, (\mathbf{B}\vec{\tilde{r}}_F + \vec{\tilde{\xi}})^{\mathrm{T}}\vec{w}_M|\mathcal{G}\right)}{\vec{w}_M^{\mathrm{T}}\left(\tau\mathbf{\Sigma}\right)\vec{w}_M}\left[\mathbb{E}(\widetilde{R}_M|\mathcal{G}) - r_f\right] \\
&= \frac{\tau\mathbf{\Sigma}_F\mathbf{B}^{\mathrm{T}}\vec{w}_M}{\tau\sigma_M^2}\left[\mathbb{E}(\widetilde{R}_M|\mathcal{G}) - r_f\right] \\
&= \kappa\mathbf{\Sigma}_F\mathbf{B}^{\mathrm{T}}\vec{w}_M. \qquad (6.10)
\end{aligned}$$

Combining (6.9) and (6.10) gives (6.4).

To prove (6.3), recall (4.13):

$$\begin{aligned}
\mathbf{\Sigma}^a &= \mathbb{C}\text{ov}\left(\vec{\tilde{r}}^a, (\vec{\tilde{r}}^a)^{\mathrm{T}}\right) \\
&= \mathbb{C}\text{ov}\left[\begin{pmatrix} \vec{\tilde{r}}_{[n\times 1]} \\ \vec{\tilde{r}}_{F[f\times 1]} \end{pmatrix}, \begin{pmatrix} \vec{\tilde{r}}^{\mathrm{T}}_{[1\times n]} & \vec{\tilde{r}}^{\mathrm{T}}_{F[1\times f]} \end{pmatrix}\right] \\
&= \begin{bmatrix} \mathbb{C}\text{ov}\left(\vec{\tilde{r}}_{[n\times 1]}, \vec{\tilde{r}}^{\mathrm{T}}_{[1\times n]}\right) & \mathbb{C}\text{ov}\left(\vec{\tilde{r}}_{[n\times 1]}, \vec{\tilde{r}}^{\mathrm{T}}_{F[1\times f]}\right) \\ \mathbb{C}\text{ov}\left(\vec{\tilde{r}}_{F[f\times 1]}, \vec{\tilde{r}}^{\mathrm{T}}_{[1\times n]}\right) & \mathbb{C}\text{ov}\left(\vec{\tilde{r}}_{F[f\times 1]}, \vec{\tilde{r}}^{\mathrm{T}}_{F[1\times f]}\right) \end{bmatrix} \qquad (6.11)
\end{aligned}$$

(6.3) is easily obtained by noting the factor model

$$\vec{\tilde{r}} = \vec{a} + \mathbf{B}\vec{\tilde{r}}_F + \vec{\tilde{\xi}}. \qquad (6.12)$$

This completes the proof. □

Appendix B: Implementation of the ABL Model

The ABL framework in practice involves 4 steps:

Step 1: *Data collection for the market-wide variables, portfolio-specific variables, and the PM views.*

The following inputs are needed:

General economy-related input:

r_f: the risk-free interest rate

Benchmark portfolio-related inputs: need to pick an index type of portfolio representing the universe and the market portfolio, of which, the following are obtained:

$\vec{w}_M$: the market portfolio weight vector

$\mathbb{E}(\widetilde{R}_M|\mathcal{G})$: the estimated long-term gross market portfolio return

Portfolio-related inputs:

τ: the risk multiplier. Normally, we recommend this parameter to be set to 1. Also, the value choice for this parameter should be considered together with the view-uncertainty matrix $\mathbf{\Omega}^a$ to achieve a desired balance for shrinkage

$\mathbf{\Sigma}^a_{[N\times N]}$: the augmented variance-covariance matrix, involving the covariance matrix of the security returns $\mathbf{\Sigma}$; the covariance of factor returns $\mathbf{\Sigma}_F$; the factor loading matrix $\mathbf{B}$; and the diagonal covariance of the security-specific returns $\mathbf{\Sigma}_\xi$. The latter two components should be obtained from a linear factor risk model, e.g., the ERA model or a customised factor regression model that the PM chooses to use. $\mathbf{\Sigma}$ can be obtained drawing on (3.2)

$\vec{a}$: the factor model intercept (in case of standard factor model, set $\vec{a} = \vec{0}$)

View-related inputs:

$\mathbf{P}^a_{[K\times N]}$: the (analytical) augmented view/strategy structure, which contains k_1 view structures on security (or portfolio) returns $\mathbf{P}_{[k_1\times n]}$, k_2 view structures on factor returns (or their linear combinations) $\mathbf{P}_{F[k_2\times f]}$, k_3 view structures on the security idiosyncratic returns (or their linear combinations) $\mathbf{P}_{\xi[k_3\times n]}$ and the factor loading matrix $\mathbf{B}$

$\vec{\hat{q}}^a_{[K\times 1]}$: the augmented view/strategy estimates, containing k_1 view estimates on security (or portfolio) returns $\vec{\hat{q}}_{[k_1\times 1]}$, k_2 view estimates on factor returns $\vec{\hat{q}}_{F[k_2\times 1]}$, k_3 view estimates on security idiosyncratic returns $\vec{\hat{q}}_{\xi[k_3\times 1]}$

$\mathbf{\Omega}^a_{[K\times K]}$: the augmented block-diagonal matrix containing the independent view covariance blocks on its diagonal, i.e., respectively the view uncertainty matrices for security (or portfolio) returns $\mathbf{\Omega}_{[k_1\times k_1]}$, for factor returns $\mathbf{\Omega}_{F[k_2\times k_2]}$, and for security idiosyncratic returns $\mathbf{\Omega}_{\xi[k_3\times k_3]}$

Step 2: *Back-out the market view from the market portfolio based on the CAPM.*

This requires the CAPM-assessed equilibrium means $\vec{\bar{\pi}} = \kappa \boldsymbol{\Sigma} \vec{w}_M$ and $\vec{\bar{\pi}}_F = \kappa \boldsymbol{\Sigma}_F \mathbf{B}^{\mathrm{T}} \vec{w}_M$, and then assemble them into $\vec{\bar{\pi}}^a_{[N\times 1]}$ according to (4.22).

Step 3: *Update according to Theorem 4.1*.

This involves substituting the results from Step 2, together with other inputs as specified in Step 1, into Theorem 4.1 to evaluate the posterior mean $\vec{\bar{m}}^a$ and error matrix $\widehat{\mathbf{V}}^a$.

Step 4: *Optimise the allocation based on the truncated posterior estimation to decide how to tilt the market portfolio*.

The general mean-variance optimisation problem is:

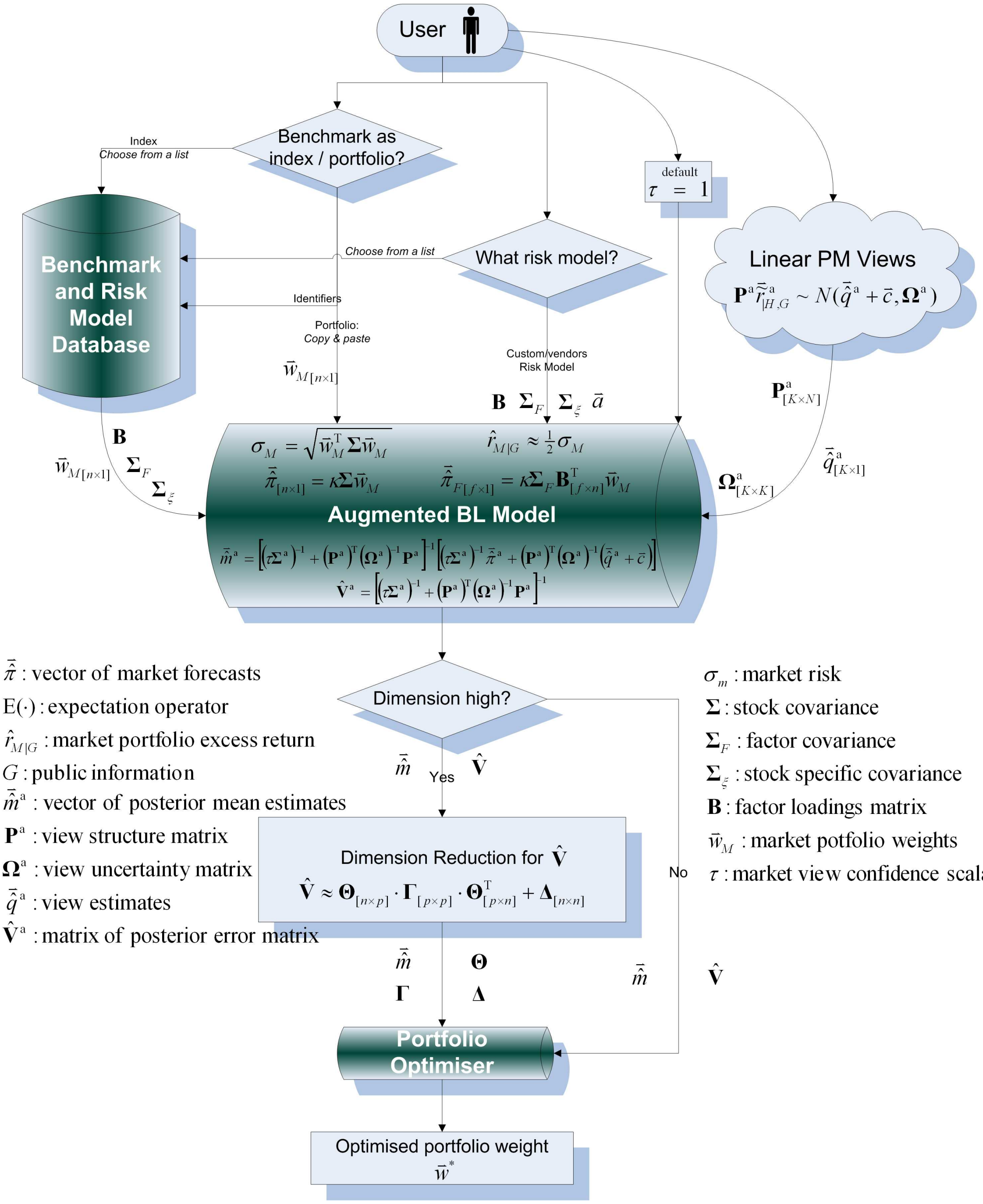
$$\underset{\vec{w}}{\operatorname{argmax}}\{\vec{w}^{\mathrm{T}}\vec{\bar{m}} - \lambda \vec{w}^{\mathrm{T}}\widehat{\mathbf{V}}\vec{w}\} \tag{6.13}$$

where $\vec{\bar{m}}$ and $\widehat{\mathbf{V}}$ are the security mean and variance 'carved out' respectively from the ABL posterior mean $\vec{\bar{m}}^a$ and error matrix $\widehat{\mathbf{V}}^a$.

Since most of the data items can be obtained from a market database, the PM just needs to concentrate on the view-related inputs: τ, $\mathbf{P}^a$, $\vec{\bar{q}}^a$ and $\boldsymbol{\Omega}^a$.

Figure B1 illustrates how the user should prepare inputs and how the model interacts with various components in a typical equity analytical framework.

Figure B1 Interactions between Different Components in the ABL Framework



$\bar{\hat{\pi}}$: vector of market forecasts
$\mathrm{E}(\cdot)$: expectation operator
$\hat{r}_{M|G}$: market portfolio excess return
G : public information
$\bar{\hat{m}}^a$: vector of posterior mean estimates
$\mathbf{P}^a$: view structure matrix
$\mathbf{\Omega}^a$: view uncertainty matrix
$\bar{\hat{q}}^a$: view estimates
$\hat{\mathbf{V}}^a$: matrix of posterior error matrix

σ_m : market risk
$\mathbf{\Sigma}$: stock covariance
$\mathbf{\Sigma}_F$: factor covariance
$\mathbf{\Sigma}_\xi$: stock specific covariance
$\mathbf{B}$: factor loadings matrix
$\bar{w}_M$: market potfolio weights
τ : market view confidence scalar

Appendix C The Relevant ERA Factors

1	Automobiles
2	Banks
3	Capital goods
4	Consumer durables
5	Energy
6	Financials
7	Food & beverage
8	Food & staples
9	Hardware
10	Household products
11	Insurance
12	Materials
13	Media
14	Pharma & biotech
15	Retailing
16	Software
17	Telecom
18	Transportation
19	Utilities
20	Book value/price
21	Dividend yield
22	Earnings growth (5yr consensus)
23	Earnings/price (consensus)
24	Historic earnings growth (past 5yrs)
25	Return-on-equity
26	Size factor
27	Short-term momentum (past 6M)
28	Long-term momentum (past 24M)
29	Refined momentum (1M minus 2M)
30	Commodity price (EUR base)
31	Commodity price (GBP base)
32	Energy price (EUR base)
33	Energy price (GBP base)
34	EUR interest rate spread (10yr-3M)
35	GBP interest rate spread (10yr-3M)
36	Europe credit spread (BAA-AAA)
37	Fundamental FX factor (USD per EUR)
38	Fundamental FX factor (USD per GBP)
39	Fundamental FX factor (USD per JPY)
40	Investment FX factor (USD per CHF)
41	Investment FX factor (USD per DKK)
42	Investment FX factor (USD per EUR)
43	Investment FX factor (USD per GBP)
44	Investment FX factor (USD per NOK)
45	Investment FX factor (USD per SEK)