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Master's thesis

Evaluating performance of an image compression scheme based on non-negative matrix factorization

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THANKS (remove entirely in case you do not with to thank anyone)

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Abstrakt

V několika větách shrňte obsah a přínos této práce v českém jazyce.

Klíčová slova Replace with comma-separated list of keywords in Czech.

Abstract

Summarize the contents and contribution of your work in a few sentences in English language.

Keywords Replace with comma-separated list of keywords in English.

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Introduction

Data encoding and consequently data compression are both problems which lie at the heart of many modern technologies - digital television, videogames, mobile communications, security cameras and all other kinds of multimedia. As the amount of data only grows in the current world, the quality of compression ends up becoming a very serious problem, since good compression can very significantly reduce the costs of data storage as well as the costs and speed of data transfer.

To put the problem into perspective, in order to store images in the resolutions currently considered as high resolution (1920x1080 pixels, the second most common resolution used on desktop devices [4]. When storing an image with these dimensions using an uncompressed standard encoding, the filesize would be almost 6 megabytes. Such a high size impacts many areas, such as speed of transfer or storage costs. Fortunately, modern image compression formats such as *PNG* or *JPEG* are able to reduce this size remarkably.

Currently, images contribute to the amount of data on the internet significantly - not only are images a common form of media for professional purposes but some of the currently largest platofrms on the internet are based on image sharing and image hosting. One modern social media platform centered around image sharing had over 67 million new posts each day [26]. Being able to obtain these images fast and in good quality is therefore highly important for both users, just as it is important for the owners of these services to be able to store this data.

In the recent years with growht related to machine learning and similar areas, such as artifical intelligence, new applications of mathematical concepts were discovered. Some of these concepts which are enjoying high success are algorithms related to *dimensionality reduction*. These algorithms aim to reduce the number of random variables under consideration [23].

While these algorithms have been enjoying success mostly in areas such as data mining or machine learning, their nature of reducing the amount of random variables under inspection means that the algorithms are essentially also compression algorithms. One of the algorithms used for dimensionality reduction and the one which this thesis focuses on is *non-negative matrix factorization* (abbreviated as *NMF*). The *NMF* algorithm is currently used in areas such as facial recognition or astronomy. Heart of the algorithm is the factorization of a matrix consisting of non-negative values into matrices.

The research in image compression methods using dimensionality reduction algorithms is currently being performed - another dimensionality reduction algorithm and its potential usage for image compression which has been well researched is the *singular value decomposition* algorithm, for example in [17]. This algorithm, just like the *NMF*, factorizes a matrix - however, without restricting the values to be non-negative. While certain similar research to see whether *NMF* can be used for image compression exists, the works are related to very specific use-case scenarios.

Thus, this thesis aims to analyze the potential of the *NMF* algorithm as a tool for image compression. In order to achieve this, the following points will be explored in the thesis:

- *NMF* and its current applications will be studied (Chapter 1).
- The theory and practice related to digital image encoding and image compression will be explored and described together with modern image compression . (Chapters 2 and 3)

By analyzing these concepts, a proof of concept image compression algorithm using *NMF* will be designed and implemented. By doing so, these issues will be addressed:

- Whether a certain way of representing an uncompressed image is better suited for non-negative matrix factorization.
- Utilizing both subjective as well as objective metrics commonly used for evaluating quality of image compression, the performance of the proof of concept compression scheme will be evaluated.
- How well suited *NMF* is for usage as an algorithm for image compression.

At the end of the thesis, the proof of concept algorithm will be compared to the state of the art image compression algorithms and possible points related to further analysis or will be explored.

The first three chapters are related to the theoretical part of the problem, studying *NMF*, image encoding and image compression. The following chapters are related to the practical part of this thesis - design and implementation of the image compression algorithm and its evaluation.

Chapter 1

Non-negative matrix factorization

This chapter discusses the *non-negative matrix factorization* - defines the problem, describes some of the existing solutions to the problem and offers some observations. By doing so, the basics for the rest of the thesis are provided.

Non-negative matrix factorization as a problem was first formulated by Paatero and Tapper in [21], but the works which have given this problem far more popularity are the works of Lee and Seung [15], where *NMF* was applied to areas of machine learning and artificial intelligence - more specifically to facial recognition and discovering semantic features in encyclopedic articles.

1.1 Problem definition

Let V be a $n \times p$ non-negative matrix, (i.e. with $x_{ij} \geq 0$, denoted $X \geq 0$), and $r > 0$ an integer. Non-negative matrix factorization consists in finding an approximation

$$V \approx WH \tag{1.1}$$

where W, H are $n \times r$ and $r \times p$ non-negative matrices, respectively - meaning all the elements of the matrices are non-negative. In practice, the rank r is often chosen such that $r \ll min(n, p)$. This is due to the reason than in many common applications of non-negative matrix factorization, the information contained in the matrix V is summarized and split into r factors as the columns of W [7].

It should be noted here that the name of the problem might be misleading, as the term *"factorization"* is usually understood more as an exact decomposition, whereas *NMF* is in reality an approximation. Thus, the problem is called *non-negative matrix approximation* in certain other works, such as [29].

1.2 Problem solution

In this section, two of the commonly used algorithms for solving the non-negative matrix factorization problem will be described. The first of these two algorithms will be the algorithm based on *multiplicative updates* as used in [15], where the attention to part-based analysis, simplicity of the *multiplicative updates* and interpretability of the results helped to spread the influence into many other research fields, such as image processing or text processing [10]. Due to these reasons, this will be one of the algorithms described. The second algorithm which will be described is the *alternating least squares* algorithm, which is the earliest algorithm proposed for solving the non-negative matrix factorization problem (positive matrix factorization in the original work) [21].

The reason for choosing these two algorithms is that both of them are very commonly used in practice. Other algorithms exist and are often researched, example being the *projected gradient* method [12]. However, they will not be explored within this thesis.

1.2.1 Multiplicative updates

The algorithm described here is described more thoroughly in [16], a work by Lee and Seung where the algorithms are described in detail together with proving correctness of the algorithms.

In order to find an approximate factorization $V \approx WH$, a way how to quantify the quality of the approximation needs to be defined. Such a metric (or a cost function) can be constructed by measuring the distance between two non-negative matrices A and B. One of the measures provided in [16] is the square of the Euclidean distance between A and B.

$$||A - B||^2 = \sum_{ij}(A_{ij} - B_{ij})^2 \tag{1.2}$$

This distance is lower bounded by zero and vanishes if and only if $A = B$.

By using this cost function, the non-negative matrix factorization can be formulated as an optimization problem:

Problem 1 *Minimize* $||V - WH||^2$ *with respect to* W *and* H*, subject to the constraints* $W, H \geq 0$.

It is shown in [16] that an algorithm cannot realistically solve this problem by finding a global minimum. However, it is possible using various techniques from numerical optimization which make it possible to find local minimum.

Thus, the multiplicative update rules are a compromise between speed and ease of implementation offered in [16] for solving this problem. The multiplicative updates can be described as an algorithm below:

Input: Non-negative matrix V
Output: Non-negative factors W and H

- Initialize W and H as non-negative matrices
- Until $||V - WH||^2$ is minimized, update W and H by computing the following, with n as an index of the iteration:

$$H^{n+1}_{[i,j]} = H^{n}_{[i,j]} \frac{((W^n)^T V)_{[i,j]}}{((W^n)^T W^n H^n)_{[i,j]}} \tag{1.3}$$

and

$$W^{n+1}_{[i,j]} = W^{n}_{[i,j]} \frac{(V(H^{n+1})^T_{[i,j]}}{W^n H^{n+1}(H^{n+1})^T_{[i,j]}} \tag{1.4}$$

Algorithm 1: Multiplicative update algorithm for NMF

It is shown in [16] that the Euclidean distance $||V - WH||$ (and consequently the cost function $||V - WH||^2$) is nonincreasing under these rules. The work by Lee and Seung also considers another possible cost function and proves the convergence of these rules.

1.2.2 Alternating least squares method

Alternating least squares method is the first algorithm proposed for solving the non-negative matrix factorization problem, which was proposed in the work by Paatero. [21] Fixing either of the factors W or H, the problem essentially becomes a least squares problem, which is commonly used in *regression analysis.*

The alternating least squares problem then solves NMF using the algorithm shown in 2.

As the least squares algorithm does not enforce the constraint of non-negativity, all the negative elements in matrices are set to 0 after each evaluation of the least squares problem.

The name of the algorithm reflects its nature where it keeps alternating between solving the least squares problem for one fixed matrix and then the other.

Input: Non-negative matrix V
Output: Non-negative factors W and H

- Initialize W as random dense matrix
- Until a stopping condition:
  (LS) Solve $min_{H\geq 0}||V - WH||^2$
  (NONNEG) Set all the negative elements of H to 0.
  (LS) Solve $min_{W\geq 0}||V^T - H^T W^T||^2$
  (NONNEG) Set all the negative elements of W to 0.

Algorithm 2: Basic Alternating least squares algorithm for NMF. [22]

1.3 Common NMF applications

In this section, a short example of a common application of non-negative matrix factorizations will be provided, for the purpose of showing concrete examples so that the reader may become more adjusted to the problem as well as showing certain observations about the properties of non-negative matrix factorization.

1.3.1 Part-based analysis

What has inspired the work of Lee and Seung [15] was the human activity of recognizing objects from basic parts - especially when using human vision which is shown to be designed to detect the presence or absence of features (parts) of physical objects in order to recognize them [6].

Thus, assuming that features of an object would be independent and it would be possible to compile all the features together, an object could be described formally as:

$$Object_i = Part_1(b_{i1}) \text{ with } Part_2(b_{i2}) \text{ with...}, \tag{1.5}$$

where b_{ij} either has the value *present* if part i is present in object j or the value *absent* if part i is absent in object j.

If the possible states *present* and *absent* in the model are replaced by non-negative values, it is possible to signify not only the presence or absence of a feature but also its quantity or significance ($b_{ij} \geq 0$). Thus, mathematically, a description of an object could look like the following:

$$Object_i = b_{i1} \times Part_1 + b_{i2} \times Part_2 + ... \tag{1.6}$$

[10]

When utilizing non-negative matrix factorization, the matrix W can be considered the set of features present in the data and matrix H the set of hidden variables. Non-negative matrix factorization can also be implemented as:

$$v_i \approx W h_i \tag{1.7}$$

Represented this way, the concept of non-negative matrix factorization can be understood intuitively - each column in the original matrix V is a data point. Each column in the matrix W is a basis element and columns of the matrix H give the coordinates of a data point in the basis W. The product matrix WH (the approximation of V) is a linear combination of the column vectors - the features extracted from the data points and its significance in the data point.

This way, features and parts can be extracted from the original data and understood, as shown in the next subsection on the image learning example.

1.3.2 Image learning

Digital image processing is a field which is currently enjoying high popularity when it comes to research. Image processing is the field thanks to which it is possible to extract features from images or recognize various patterns. Principal component analysis *(PCA)* is a dimensionality reduction technique similar to *NMF* commonly used for recognizing faces in images [5] - however, without the non-negativity constraint. It has been shown in [15] that *NMF* can be used in a similar fashion while potentially classifying the data in a way which is easier to be understood.

The Figure 1.1 has been taken from [15], where a database of 2,429 facial images has been taken and used to create a matrix V. By applying both *NMF* as well as *PCA* to the matrix, feature and coeffiecient matrices were created and particular instance of a face was approximately represented by a linear superposition of basis images. The coefficients used in the linear superposition are shown in the montages, where black pixels indicate positive values and red pixels indicate negative values. It can be seen on these montages that while *NMF* can be used for the same purposes as *PCA*, the main strength of the algorithm is that certain parts can be recognized meaningfully - for example parts of a nose and such, whereas in the case of *PCA*, discovering meaning in the matrices is difficult.

It is for these reasons that non-negativity is a powerful and meaningful constraint, as parts are never subtracted from the particular instance (as it can happen in the case of *PCA*), but are only added together [10].

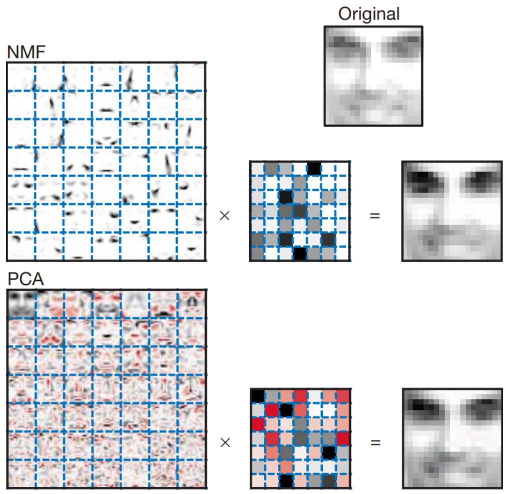


Figure 1.1: Comparison between dimensionality reduction algorithms - *NMF* and *PCA*, as shown in [15].

1.3.3 NMF properties

Previous sections in this chapter have described non-negative matrix factorization, more specifically the definition of the problem, common solutions and an example usage of *NMF*. In this section, certain properties of non-negative matrix factorization will be pointed out. When the proof of concept image compression scheme is designed, the usefulness of these properties for compression will be discussed.

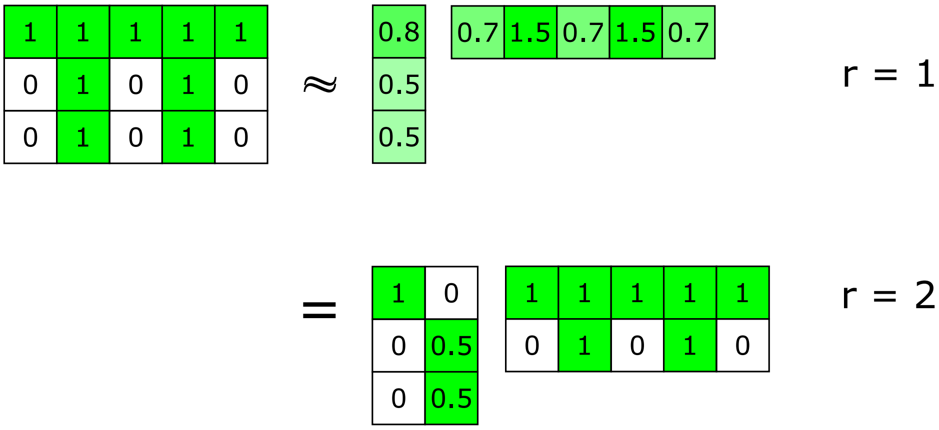
The first property which will be shown is that the solution to non-negative matrix factorization is **not unique**. A matrix and its inverse can transform the two factorization matrices, for example as:

$$V \approx WH = WBB^{-1}H \tag{1.8}$$

If the matrices $\bar{W} = WB$ and $\bar{H} = B^{-1}H$ are non-negative then they form another solution to the *NMF* problem.[31]

Another important property is that the non-negative matrix factorization is not hierarchical, meaning that the factor matrices using rank r can be completely different to those of rank $r+1$, as shown in 1.2, where choice of rank $r = 1$ provides only approximation of the target matrix V yet rank $r = 2$ makes it possible to calculate $V = WH$ instead of the approximation. Due to these reasons, the results provided by using *NMF* highly depend on choice of the rank parameter.

Figure 1.2: Visual display of non-hierarchical properties of NMF. [20]



However, one of the most important properties to note is that non-negative matrix factorization is very difficult to solve and the existing general algorithms solve *NMF* as an optimization problem. Not only that but it has even been proven that non-negative matrix factorization is an *NP-hard* problem.[27] With certain constraints, it is however possible to solve the *NMF* problem in polynomial time - for example it is shown in [13] that in case the matrix V is symmetric and contains a diagonal principal submatrix of rank r, it is possible to solve the problem in polynomial time $O(rm^2)$.

The last property of non-negative matrix factorization to be noted has been shown in the figures above. The factors extracted are often *sparse* - it is precisely for this reason that interpreting results of non-negative matrix factorization is easy in fields such as image processing or text mining (but many others as well).[8]

1.3.4 Choice of rank r

The choice of rank r is one of the most important problems when utilizing non-negative matrix factorization - as when used for the examples above, choice of rank r changes the amount of features to be extracted in order to approximate the target matrix V.

There are several common methods used for choosing the rank r:

- Trial and error - try different values of r and choose the one performing the best for application at hand.
- Estimate the rank r using various statistical approaches (such as by using *SVD*).

- The use of expert insights.

[8]

1.4 NMF and compression

Most common use-case scenarios of non-negative matrix factorization are related to machine learning, as shown in the previous sections. However, *NMF* can also be looked upon as a lossy compression tool (more on lossy compression in chapter 3), as an original matrix of size $n \times p$ is approximated by the product of two smaller matrices. Assuming the target matrix V is represented in the same way as the factor matrices W and H, if the amount of elements contained in matrices W and H is lower than the amount of elements in matrix V, then *NMF* was used to perform compression.

Whether the amount of elements in factor matrices W and H is lower than the amount of elements in target matrix V depends on the choice of rank r. More specifically, if the dimensions of matrix V are $n \times p$, the dimensions of matrix W $n \times r$ and the dimensions of matrix H $r \times p$, then this requirement can be formalized as:

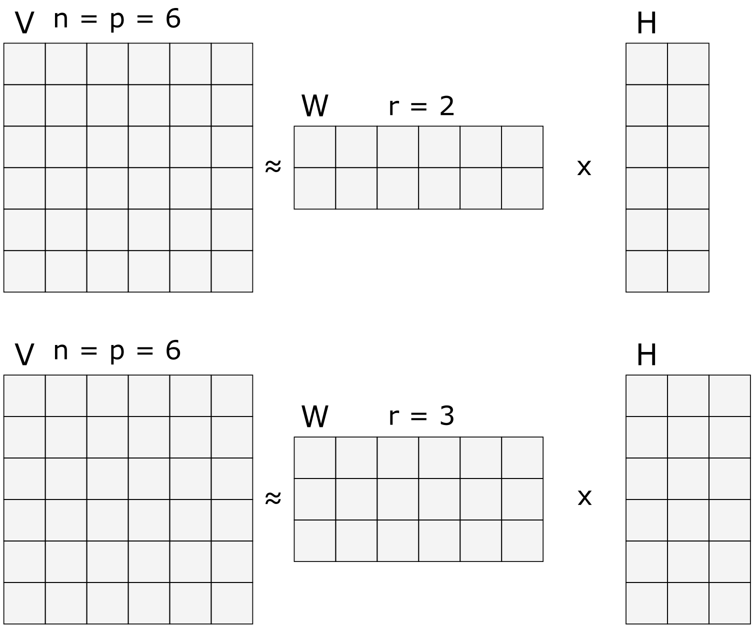
$$\begin{aligned} n \times r + r \times p &< n \times p \\ r(n+p) &< n \times p \\ r &< \frac{n \times p}{n+p} \end{aligned}$$

Considering a square matrix ($n = p$), the following relationship can also be deduced:

$$\begin{aligned} r &< \frac{n \times n}{n+n} \\ r &< \frac{n^2}{2n} \\ r &< \frac{n}{2} \end{aligned}$$

This relationship is also demonstrated in the figure 1.3.

Figure 1.3: Visualization of NMF as a compression tool. When using a rank r lower than $n/2$, the total amount of elements in matrices W and H is less than the amount of elements in matrix V. When the rank r reaches the value $n/2$, the amount of elements is the same.



Chapter **2**

Digital image encoding

The second chapter of this thesis is related to digital images and their representations color models. The importance of this chapter is related to creating the image compression scheme and its empirical testing on various different digital image representations. The possible ways of specifying colors in digital images which will be explored are *RGB*, *grayscale* and $Y'C_BC_R$. As this thesis is related to image compression and non-negative matrix factorization and not image processing, only the most important elements of image encoding and color spaces will be explored.

2.1 Digital images

A digital image I is stored as a matrix of *pixels* (abbreviation for a *picture element*). These matrices described as 2D discrete space are derived from analog images in 2D continuous space through the process called *sampling.* More about sampling can be found for example in [32].

The value assigned to a pixel $I[m, n]$ determines its color. The following sections explore the common color encoding options.

2.2 Color models

A color model is a mathematical model used for describing the way of representing colors, usually as tuples of numbers (although this is not necessary, as in i.e. the *grayscale* model). When associated with a description of how the tuple components are interpreted, colors can be interpreted. The color models described are the *RGB* color model, the $Y'C_BC_R$ color space (the set of colors which can be used) which is a transformation using the *RGB* model and the *grayscale* model.

2.2.1 RGB

The *RGB* color model is closely related to the way the human eye perceives colors with the r (red), g (green) and b (blue) receptors in our retinas.[11] In order to represent a color, components of each color (red, green and blue) are added together. As these components are added together, the *RGB* model is considered to be an additive one.

In order to store image data using the *RGB* color model, the color components need to be quantified. Common way of storing the values of components in a pixel is storing the color intensity value using 8 bits (range [0, 255], where the value 0 indicates no inclusion of the color component and 255 indicates maximum possible inclusion of the component). If all the values of components are equal to 0, the resulting color is black, if all the values of components are equal to the defined maximum value (255 in this case), the resulting color is white. An uncompressed image format which represents images this way is, for example, the Windows BMP.[1]

Thus, encoding an uncompressed image using the *RGB* color scheme with the common 8-bit per component component representation results in each pixel being represented by 24 bits. The size of an image in bytes would then be $width * height * 3$ bytes (not counting the header and other data used by the specific file format).

A colored image together with its decomposition into the R, G and B channels can be seen on the figure 2.1

2.2.2 $Y'C_BC_R$

Y'CbCr, also written as $Y'C_BC_R$ is an encoding system of colors commonly used in digital image systems, which is defined by a mathematical coordinate transformation from an associated RGB color space. The Y' represents the *luma* value (brightness of an image). The C_B and C_R values are considered the *chroma components*, and represent the color information.

2.2.2.1 Luma

The *luma* value is represented in the $Y'C_BC_R$ model by the symbol Y' and representes the brightness of an image. Y itself is considered to be the *relative luminance*. Relative luminance is a metric of light intensity as it appears to the human eye. The prime symbol (Y') denotes that *gamma correction* has been utilized. Gamma correction is an operation related to nonlinearity of light perception - when twice the number of photons hit a camera sensor, twice the signal is received, denoting a linear relationship. However, the human eye does not perceive change of light in a linear way. Gamma correction thus aims to translate the human eye's light sensitivity and that of a camera. [19] As gamma correction is not an important topic for the rest of this thesis, it will not be explored further.

Luma is calculated as the weighted sum of gamma-compressed $R'G'B'$ components. The prime again represents gamma correction. Luma can be calculated in the following way, as described in [2]:

$$Y' = 0.2126R' + 0.7512G' + 0.0722B' \tag{2.1}$$

2.2.2.2 Chrominance

Chrominance is the signal conveying the color information of a picture, separately from the accompanying luma. the C_B and C_R values represent the blue-difference (and red-difference, respectively) when compared to the luma. Multiple ways of calculating C_B and C_R exist, such as the one for HDTVs in [2]. In digital images, other transformations exist, such as the one used in the JPEG image format:

$$\begin{aligned} C_B &= 128 - (0.168736R') - (0.331264G') + (0.5B') \\ C_R &= 128 + (0.5R') - (0.418688G') + (0.081312B') \end{aligned} \tag{2.2}$$

[9]

2.2.2.3 Use of $Y'C_BC_R$

Common usage of $Y'C_BCR$ stems from the human eye being more sensitive to the differences in luminance than color differences. By transforming colors into the $Y'C_BCR$ color space, it is possible to separate the color data from luminance and reduce the amount of color difference signals - thus less resolution can be allocated for chrominance while keeping high resolution for the luma information. This process is called *chroma subsampling* and is widely used in video encoding schemes, as well as in the JPEG image format. [14]

An image decomposed into the Y', C_B and C_R components can be seen on the figure 2.2

2.2.3 Grayscale

Grayscale images are images composed exclusively of shades of gray. When using the grayscale model, each pixel therefore carries only the intensity information. Commonly, grayscale images are stored using 8 bits per pixel. The range of colors represented by these 8 bits goes from black (the value 0) through possible shades of gray to white (255 or another maximum possible value).

It is possible to transform colors from an *RGB* color space into the grayscale model by calculating the luma using the formula 2.1, as luma is a representation of an image's brightness.

Encoding an uncompressed grayscale image which uses the common 8 bit per pixel representation would therefore create an image of a size of $width * height$, not counting the header and other data used by the specific file format.

Figure 2.1: An image decomposed into the R, G and B channels as used in the *RGB* color model.

Figure 2.2: An image decomposed into the Y', C_B and C_R components, as used in the $Y'C_BC_R$ color space. The Y' component also represents the grayscale image.

Chapter **3**

Image compression

As it has been shown in the introduction of this thesis, storing uncompressed images requires a lot of space. The art of reducing the amount of space required for storing data by encoding information with less bits than would otherwise be necessary is called data compression.

This chapter will explore the basics of data compression in order to build a framework for the rest of this thesis. Image compression basics with examples of current algorithms used for image compression will be described. The end of the chapter will explore compression metrics which can be used for determining quality of data compression (as well as image compression).

3.1 Data compression

When the terms *compression algorithm* or *compression technique* are used, they refer to two algorithms. One algorithm which takes an input X and generates a representation Y, which requires fewer bits to store. The second algorithm is the reconstruction algorithm, which operates on the representation Y and generates the reconstruction Z. Depending on whether the reconstruction Z is completely identical to the original data source X, the compression algorithms can be divided into two broad classes, being either *lossless algorithms* or *lossy algorithms*. Lossy algorithms generally provide much higher compression than lossless algorithms do, but at the cost of losing information.[25]

3.1.1 Lossless algorithms

Lossless compression techniques involve no loss of information - by using a lossless algorithm, the original data can be recovered exactly from the compressed data. The applications for lossless algorithms commonly include text data, where small differences could easily result in wrong statements (errors in e.g. bank records are, for obvious reasons, very undesirable).

3.1.2 Lossy algorithms

However, not all usage scenarios require compression to be lossless. In these cases, the requirement of retrieving absolutely identical data can be relaxed. By relaxing this requirement, very often higher compression ratios (more on compression ratio in section 3.2.2) can be obtained, at the cost of having distortions in the reconstruction.

Very common scenarios where lossy compression algorithms become often more desirable than lossless ones are when storing audio content. Modern audio compression algorithms are almost all lossy - the often used *MP3* format used for storing audio stores audio with a lossy compression algorithm. As long as audio is stored without audible artifacts (distortions which are not present in the original data), the quality of sound does not have to be perfect - especially when compressing speech. When compressing other forms of audio, such as music, the compression needs to be more accurate, but still does not have to be absolutely perfect - as long as the listener does not notice the difference.

Other typical use case for a lossy compression algorithm is compressing images or video content. Small distortions are acceptable in images and video as long as they are not easily noticable by the human eye. However, this can also depend on the use case - for example photos commonly taken can be compressed in a lossy way, but medical images often have to be compressed in a lossless way, as artifacts can be very undesirable for medical usage.

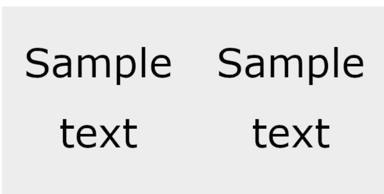


Figure 3.1: A sample image showing how accurate a lossy algorithm can be. The left image has been compressed using a lossless algorithm, the right image has been compressed using a lossy algorithm (JFIF/JPEG). In this case, essentially no artifact can be seen unless zoomed in (as seen in the figure 3.2).

Figure 3.2: A zoomed in image used in the figure 3.1. While almost impossible to notice any artifact in the former example, the artifacts can be seen on the borders between the text and the gray background easily once zoomed in. Color levels were adjusted in order to emphasize the presence of artifacts.

3.1.3 Modeling and coding

One of the important factors when designing a compression scheme which needs to be accounted for are characteristics of data which is to be compressed. An approach which works the best will depend to a large extent on the redundancies (more on redundancies in section 3.2.1) inherent to the data compressed.

In the phase of modeling, information about any redundancy is extracted in a form of a model, which can then be utilized in the compression algorithm. The second phase important in a compression scheme is *coding*. A description of model and how data differs from the model can be encoded (usually using a binary alphabet).[25]

A very simple example of modeling and coding and its importance can be seen on *delta encoding* (sometimes also called *delta compression*). Delta encoding encodes a sequence of messages not by their values but by calculating the difference between two elements.[28] Thus, a sequence such as

$$10000, 10002, 10001, 99999, 10000$$

can be modelled as a sequence where the numbers have very small differences. When shown as a sequence of differences, with the original value, the sequence then becomes:

$$10000, +2, -1, -2, +1$$

After a representation of this data sequence has been modeled, the coding phase creates a way how to encode this sequence using this model. One possible coding is shown in the table 3.1. Utilizing this coding, the differences

+1	00
+2	01
-1	11
-2	11

Table 3.1: Example coding for a sample data sequence using delta encoding

require only 2 bits in order to be stored.

However, delta encoding would not be efficient in a scenario when the data has large differences - especially if the differences would be larger than storing the data itself. Understanding the type of data when creating a model is therefore highly important.

3.2 Image compression

Commonly, data compression algorithms are discussed as universal techniques which are able to compress essentially any data. While these algorithms might still have preferred applications, literature commonly separates techniques from the applications. However, image compression is a specific field where separating techniques from the application is essentially impossible as the techniques are meant specifically for compressing images.[25]

While compressing image data using standard compression techniques, none of them are satisfactory for color or grayscale images (which were discussed in chapter 2). For example, statistical methods can be very good for compressing data where each value has different probability (such as in standard text, where certain letters appear far less than others). However, in images, certain colors or shades of gray often have the same probabilities.[24]

3.2.1 Redundancies in images

In order to understand how image compression techniques work, certain aspects of images need to be characterized, as compression techniques try to use these aspects for their advantage. A fundamental component of image compression is reducing the amount of redundancies present in the signal source. The redundancies commonly present in images are these ones:

- Coding redundancy
- Interpixel redundancy (also sometimes spatial redundancy)
- Psychovisual redundancy

[30]

Coding redundancy occurs when length of the code words is larger than required. A simple example of coding redundancy can be shown on a grayscale image where only certain shades of gray are used. Instead of requiring 8 bits per shade of gray, such an image might require far less bits to encode, as can be seen on **??**.

Interpixel redundancy refers to the fact that usually, neighbouring pixels tend to have similar colors. Therefore, it can often be possible to predict the color of neighbouring pixels. As a simple example, black-and-white images might be encoded using *run-length encoding*, where instead of encoding the specific colors per pixel, an information about how many pixels of the same color are stored in a row. This can be particularly efficient when considering bi-level (black-and-white) images, as can be seen on **??**.

The last kind of redundancy which is commonly utilized in image compression algorithms is called *psychovisual redundancy.* The human eye is not able to perceive certain visual information in an image - such information could be discarded without any noticeable artifacts present in the compressed image. A tool commonly used for reducing psychovisual redundancies in an image is *discrete cosine transform*, which is also used in the JFIF/JPEG image format.

3.2.2 Compression metrics

When evaluating the quality of a compression technique, utilizing certain metrics is necessary - especially when attempting to compare compression algorithms.

One of the most important metrics is the *compression ratio.* Compression ratio is a very simple metric which quantifies the reduction in data representation needed. It can be easily defined in the following way:

$$Compression\ Ratio = \frac{Uncompressed\ size}{Compressed\ size} \tag{3.1}$$

It is easy to see that compression ratio is an essential metric as it makes it very easy to compare compression algorithms or even quantify their compression power in general. Certain metrics extend compression ratio and use it for scoring purposes, such as the *Weissman score* which can be used for lossless compression algorithms. [**?**].

However, while compression ratio is a very valuable metric and highly valuable for lossless algorithms, it potentially stops being the most important metric when evaluating lossy algorithms. When evaluating lossy algorithms, the important information is not only quantification of how much was the data compressed but also evaluating how much information was actually lost in the compression process. As an example, if a lossy algorithm has better compression ratio than another one, it might still not be the algorithm of choice, in the case where it would introduce too many artifacts into the data.

Thus, it is necessary to introduce other compression metrics, which can be utilized for evaluating the performance of lossy compression algorithms, related to information loss - or in other words, the errors present in data after reconstruction. The two error metrics which are commonly used when measuring the performance of a lossy compression algorithm are the *mean squared error* (MSE) and *peak signal-to-noise ratio* (PSNR).[?]

Mean squared error is a statistical metric measuring the average of the squared of errors (the difference between original values and the values after decompressing data and obtaining the reconstruction). When used for image compression, the metric can be formalized in the following way:

$$MSE = \frac{1}{MN} \sum_{y=1}^{M} \sum_{x=1}^{N} (I(x,y) - I'(x,y))^2 \tag{3.2}$$

where

- M and N are the dimensions of an image
- $I(x,y)$ is the value of a pixel on coordinates x, y in the original image.
- $I'(x,y)$ is the value of a pixel on coordinates x, y in the reconstructed image.

A lower value of MSE is better, as it directly displays less errors present in the decompressed image.

Peak signal-to-noise ratio is a metric derived from MSE describing the ratio between the maximum possible power of a signal and the power of corrupting noise which affects the representation. Due to signals possibly having a very wide dynamic range, PSNR is usually expressed in the logarithmic scale.

With MSE defined as above, PSNR can be formalized as:

$$PSNR = 20 \cdot log10 \left(\frac{MAX_I^2}{MSE} \right) \tag{3.3}$$

where MAX_I is the maximum possible value of a pixel in an image. In the case of grayscale images stored with 8 bits per pixel, this value would be 255. Unlike MSE, higher PSNR values are better, as they are a sign of less errors in data - signal being the original colors and noise being errors.

When evaluating MSE (and consecutively PSNR) for color images, the calculation becomes slightly more difficult, as a pixel value of I[x, y] holds multiple values for multiple components, which are all perceived differently by the human eye. Some of possible approaches can therefore be the following:

- Evaluate MSE and PSNR for each component. Resulting MSE (or PSNR respectively) is the average of MSE values of each component (or PSNR values of each component, respectively).

- Due to brightness being the most important element for human eye, rather than color components, evaluate MSE and PSNR values only of luma (or simply the grayscale image). The definition of luma as a weighted sum has been explored in chapter 2.

[18][3]

However, it needs to be noted that PSNR is not a perfect metric in the sense that it would provide absolute definite conclusions. When utilizing PSNR, it is still important to also consider data subjectively, as it is only conclusively valid in scenarios where the compared results come from the same codec (or codec type) and the same content. Not only that, but it is a metric which is not performing strongly as a quality metric when it comes to human perception of image data. [**?**] [**?**] Still, it is a metric often utilized to evaluate lossy compression of images.

Last but not least, one of the important metrics, just like when evaluating performance of any other algorithm, is the required time to execute the compression algorithm (as well as decompression time).

3.2.3 JPEG image compression

As noted in chapter 1, non-negative matrix factorization can be understood as a lossy compression algorithm - approximating the data in an original matrix by two smaller matrices which have less elements in total than the original matrix. As this process necessarily involves data loss (as non-negative matrix factorization solves the problem of approximating the original matrix, not finding an exact representation), a compression algorithm based on non-negative matrix factorization has to be a lossy one. Thus, the image compression algorithms which are going to be discussed in this thesis are lossy ones. Also, as this thesis is related to creating a proof of concept image compression scheme based on non-negative matrix factorization and not describing all the existing lossy image compression schemes, only the JPEG compression scheme will be explored, as the results of the *NMF* compression scheme will be compared to

JPEG.

The JPEG compression method is commonly used for compressing digital photographies and makes it possible to adjust the degree of compression - making it possible to decide the tradeoff between image quality and compression. Commonly, JPEG is able to achieve 10:1 compression ratio without significant perceptible loss in image quality.[**?**]

The JPEG compression algorithm can use various modes of operation, but the most popular one works in the following way:

- An image is converted from RGB to $Y'C_BC_R$.
- The resolution of chroma is reduced - this is done as the human eye is less sensitive to color details than brightness.
- The image is split into 8×8 pixel blocks. For each component of $Y'C_BC_R$ the 8x8 block is transformed using the discrete cosine transform. By doing so, the image data is represented in the frequency domain.
- The amplitudes of the frequencies are quantized, meaning that components with high frequencies are stored with less accuracy than the ones with lower frequencies.
- The resulting data of 8×8 blocks is further compressed using lossless encoding.

These steps are all able to reduce all three redundancies presented in the previous sections - psychovisual redundancies are removed by storing higher frequencies with lower accuracy and interpixel redundancy by working no 8×8 pixel blocks. Coding redundancy is reduced using a lossless compression algorithm at the end.

Chapter **4**

NMF compression scheme

The previous three chapters all explored the essentials required to construct an image compression scheme based on non-negative matrix factorization. Utilizing these concepts, proof of concept compression schemes will be designed in this chapter. The following implementation is described in the following chapter 5.

4.1 Compression scheme design

As it has been shown in chapter 1, non-negative matrix factorization can be understood as a lossy compression scheme for matrices, as the original matrix can be approximated by factor matrices which possibly require less space to store. However, as there are multiple ways of representing and encoding an image with different tradeoffs, two compression schemes will be constructed and compared with each other - a naive one and a scheme based on compressing chroma values.

4.1.1 NMF properties

However, before the compression schemes are designed, there are some features of non-negative matrix factorization which should be noted here, as understanding the effects and limitations of non-negative matrix factorization is important for constructing these schemes. These properties are the following:

- Optimization problem - takes pretty long

4.1.2 Naive compression scheme

fgs

4.1.3 $Y'C_BC_R$ compression scheme

fds

Chapter 5

Implementation

fgsfds

Conclusion

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Appendix A

Acronyms

NMF Non-negative matrix factorization

PNG Portable Network Graphics

JPEG

PCA Principal Component Analysis

SVD Singular Value Decomposition

Appendix **B**

Contents of enclosed CD

```
readme.txt ....................... the file with CD contents description
exe ..................................... the directory with executables
src ..................................... the directory of source codes
    wbdcm ...................................... implementation sources
    thesis .............. the directory of LaTeX source codes of the thesis
text ........................................ the thesis text directory
    thesis.pdf .......................... the thesis text in PDF format
    thesis.ps ............................ the thesis text in PS format
```