Lec 11 - Sets and Relations

Relations

A relation from set A to B is a subset of $A \times B$ $(R \subseteq A \times B)$

Ex $R = \{(a,b) : \text{student } a \text{ is taking class } b\}$

$(a,b) \in R$: $a\,R\,b$, $a \sim_R b$ } a is related with b

A relation on A is a subset $R \subseteq A \times A$

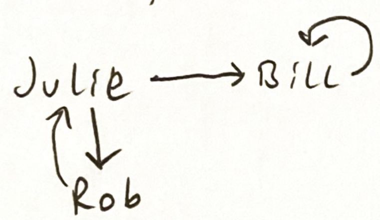
Ex $A = \mathbb{Z}$: $x R y$ iff $x \equiv y \pmod 5$

$A = \mathbb{N}$: $x R y$ iff $x \mid y$

$A = \mathbb{N}$: $x R y$ iff $x \leq y$

** set A together with R is a directed graph.

$G = (V, E)$ with $V = A$, $E = R$



Properties

A relation R on A is;

* Reflexive: If xRx for all $x \in A$
* Symmetric: If $xRy \Rightarrow yRx$ for all $x, y \in A$
* Anti-symmetric: If $xRy \wedge yRx \Rightarrow x = y$
* Transitivity: If $xRy \wedge yRz \Rightarrow xRz$

Ex

	Reflective	Symmetric	Anti-sym	Transitivity
$x \equiv y$	Y	Y	N	Y
$x \mid y$	Y	N	Y	Y
$x \leq y$	Y	N	Y	Y

equivalence relations

} Partial orders

Equivalence relation: An eq. rel. is reflexive, symmetric and transitive

Ex) Equality $(=)$'s itself

$x \equiv y \pmod{n}$

• The equivalence class of $x \in A$ is the set of all elements in A that are related to x by R: denoted $[x]$

$$[x] : \{y : xRy\}$$

Ex) $x \equiv y \pmod{5}$

$[7] : \{\ldots, -3, 2, 7, 12, 17, 22, \ldots\}$

$[7] = [12] = [17]$

Partition: A partition of A is a collection of this joint non-empty sets $A_1, \ldots, A_n, \subseteq A$, whose union is A

Ex)
$$\left.\begin{array}{l} \{\ldots -5, 0, 5, 10, \ldots\} \\ \{\ldots -4, 1, 6, 11, \ldots\} \\ \{\ldots -3, 2, 7, 12, \ldots\} \\ \{\ldots -2, 3, 8, 13, \ldots\} \\ \{\ldots -1, 4, 9, 14, \ldots\} \end{array}\right\}$$
Each set is actually a different partition of $\mathbb{Z}$

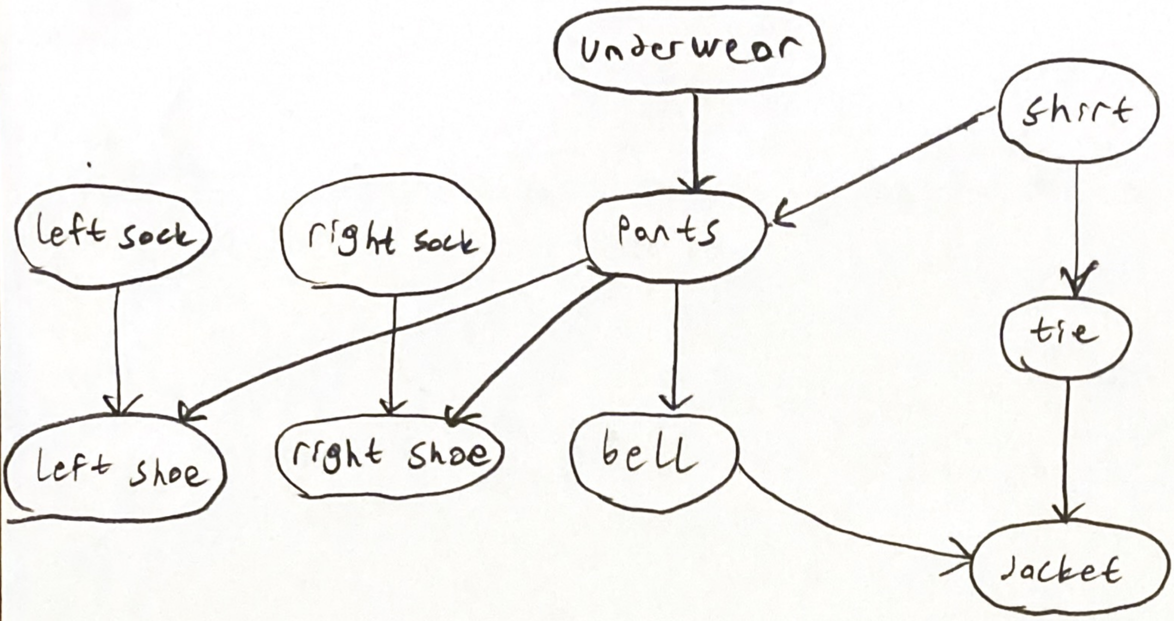
Theorem) The equivalance class of an equivalance relation on a ~~set~~ set A form a partition of A

Partial order: A relation is a (weak) partial order if it is reflexive, antisymmetric and transitive

A partial order relation is denoted with $\leq$ instead of R

$(A, \leq)$ is called a partially order set or poset

A poset is a directed graph such that with vertex set A and edge set $\leq$



Hesse Diagram

Hesse Diagram: A Hesse diagram for a poset $(A, \preceq)$ is a directed graph with vertex set A and edge set $\preceq$ minus

* all self loops
* All the edges implied by transitivity

Theorem A poset has no directed cycles other than self-loops

Proof By contradiction

suppose $\exists n \geq 2$, distinct elements $a_1, \dots, a_n$ such that we

$$\underbrace{\underbrace{a_1 \preceq a_2 \preceq a_3}_{a_1 \preceq a_3} \preceq a_4}_{a_1 \preceq a_4} \dots \preceq a_{n-1} \preceq \underbrace{a_n \preceq a_1}$$

use induction

$$\underbrace{a_1 \preceq a_n}$$

$a_1 \neq a_n \rightarrow$ contradiction ✗

* So, deleting self-loops from a poset makes a directed acyclic graph (DGA)
* a and b are <u>incomparable</u> if neither $a \preceq b$ nor $b \preceq a$

<u>comparable</u> if either $a \preceq b$ or $b \preceq a$

Total order: A total order is a partial order in which every pair of elements is comparable

* A total order is consistent with a partial order then it is called a topological sort.

Topological sort: A top. sort of a poset $(A, \preceq)$ is a total order $(A, \preceq_T)$ such that $\preceq \subseteq \preceq_T$

$\preceq$ This relation; $\subseteq$ is a subset of; $\preceq_T$ that relation

$(x \preceq y \Rightarrow x \preceq_T y)$

Theorem Every finite poset has a top. sort

** $x \in A$ is minimal if $\neg \exists y \in A, y \neq x$ s.t. $y \preceq x$ → This means that there is no element y that is smaller than x

$x \in A$ is maximal .... $y \succeq x$

Lemma Every finite poset has a min. elem.

Def A chain is a sequence of distinct elements such that

$a_1 \preceq a_2 \preceq a_3 \preceq \dots \preceq a_t$ → length

Proof Let $a_1 \preceq a_2 \preceq \dots \preceq a_n$ be a max. length chain (exists!)

Case 1: $a \notin \{a_1, a_2, \dots, a_n\}$ if $a \preceq a_1$ → longer chain ✗

Case 2: $a \in \{a_1, a_2, \dots, a_n\}$ if $a \preceq a_1$ → cycle ✗

$\neg\, a \preceq a_1$

Then a_1 is minimal ∎