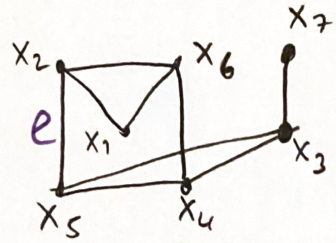
6.042 - Lecture 6 - Graph Theory and Coloring

Claim (U Chicago): on average, men have 74 % more opposite gender partners than women

74 % → 233 %

Claim (ABC News):



e is incident to x_1 and x_2

degree of x_5 is 3

Def: A graph G is a pair of sets (V, E) where V is a nonempty set of items called vertices or nodes. E is a set of 2-item subsets of V called edges.

$V = \{x_1, x_2, \ldots, x_7\}$

$E = \{\{x_1, x_2\}, \{x_1, x_6\}, \{x_4, x_3\}, \ldots \{x_7, x_3\}\}$

* We can have graphs with nodes and no edges

Ex) $\cdot x_1 \quad \cdot x_2 \quad \cdot x_3$

$G = (V, E) \qquad V = \{x_1, x_2, x_3\}$

$E = \{\ \}$

Def) Two nodes x_i & x_j are <u>adjacent</u> if $\{x_i, x_j\} \in E$

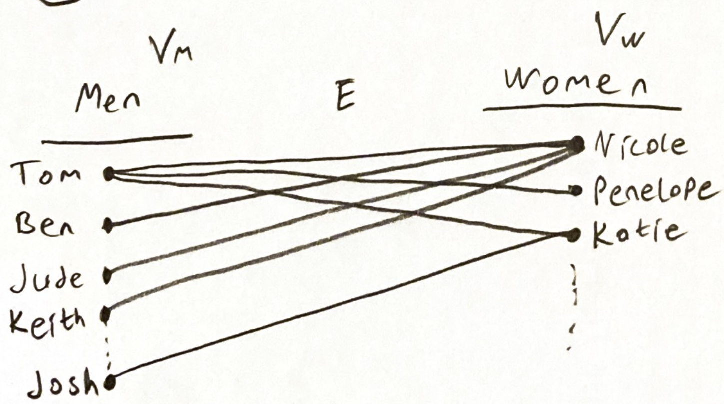
Def) An edge $e = \{x_i, x_j\}$ is <u>incident</u> to x_i & x_j

Def) The number of edges incident to a node is the degree of a node

Def) A graph is simple if it has no loops or multiple edges

Loop

multiple edges



$|V| \approx 300$ million

$|V_M| \approx 147.6$ M

$|V_W| \approx 152.4$ M

$|E| = ??$

Def: A_m = average # of opposite-gender partners for men

A_w = " " " " " " " women

What is $A_m / A_w \overset{?}{=} 1.74, 2.33$

Cardinality ⇒ the number of elements in a given mathematical set

$$A_M = \frac{\sum\limits^{x \in V_M} \deg(x)}{|V_M|} = \frac{|E|}{|V_M|} \quad , \quad A_W = \frac{\sum\limits^{x \in V_W} \deg(x)}{|V_W|} = \frac{|E|}{|V_M|}$$

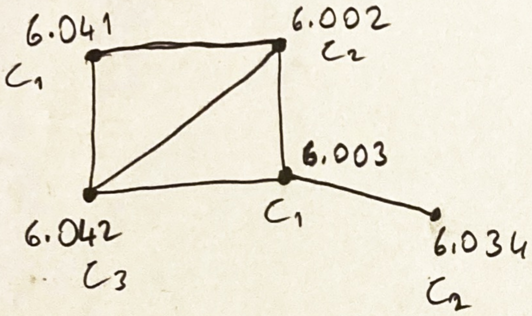
$$A_M / A_W = \frac{|E| / |V_M|}{|E| / |V_W|} = \frac{|V_W|}{|V_M|} = 1.0325$$

Graph Coloring Problem: Given a graph G and K colors, assign a color to each node so adjacent nodes get different colors.

Def, The minimum value of K for which such a coloring exists is the **Chromatic Number** of G

$\chi(G)$

Ex Exam scheduling



	Slots	
C_1	Wed	5 − 7 pm
C_2	"	7 − 9 pm
C_3	"	9 − 11 pm
C_4	"	11 − 1 am
C_5	"	1 − 3 am

Basic Coloring Algorith for $G = (V, E)$

1- Order the nodes $V_1, V_2, \dots, V_n$
2- Order the colors $C_1, C_2, \dots, C_n$
3- For $i = 1, 2, \dots, n$
Assign the lowest legal color to V_i

the right thing to do in a graph proof, is to put n for number of nodes and if P(n) doesn't work, put e for number of edges in graph and try to proof P(e)

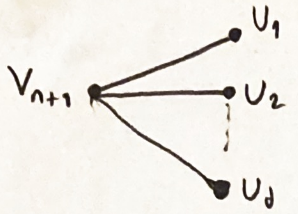
Thm: If every node in a n-node graph G has degree $\leq d$, then the Basic Alg. uses at most $d+1$ colors for G

Proof By induction

Inductive hypothesis: $P(n)$

Base case: $n = 1 \Rightarrow 0$ edges $\rightarrow d = 0$, $\rightarrow$ 1 color $= d + 1$ ✓

Inductive step: Assume $P(n)$ is true for induction
Let $G = (V, E)$ be any $(n+1)$-node graph. Let d = max. degree in G
Order the nodes $V_1, V_2, \dots, V_n, V_{n+1}$
Remove V_{n+1} from G to create $G' = (V', E')$
G' has max. degree $\leq d$ and n nodes so $P(n)$ says Basic Coloring Algorithm
Uses $\leq d+1$ colors for $V_1, V_2, \dots, V_n$

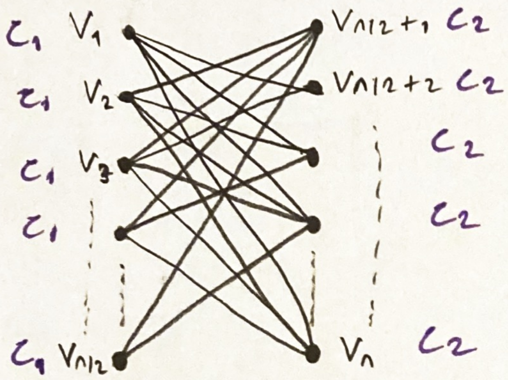


$\leq d$ neighbors

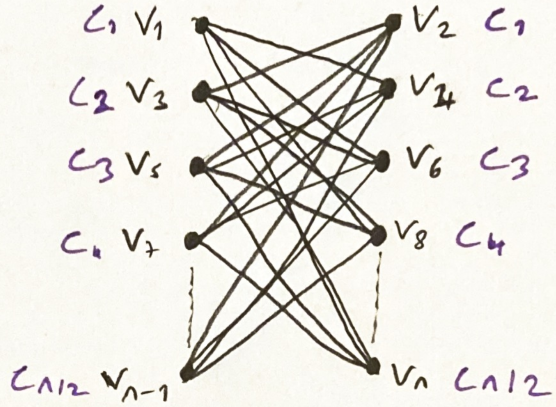
V_{n+1} has $\leq d$ neighbors $\Rightarrow \exists$ color in $\{c_1, c_2, \dots, c_{d+1}\}$ not used by any neighbor. Give V_{n+1} that color.

$\Rightarrow$ Basic Alg. uses $\leq d+1$ colors on $G \Rightarrow P(n+1)$ ∎

ordering matters...



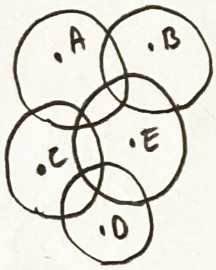
2 colors used

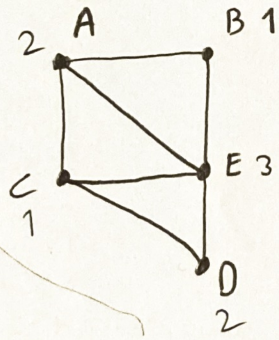


$n/2$ colors used

Def A graph $G = (V, E)$ is <u>bipartite</u> if V can be split into V_L, V_R so that all the edges connect a node in V_L to a node in V_R.

Overlaping radio towers





$\chi(G) = 3$