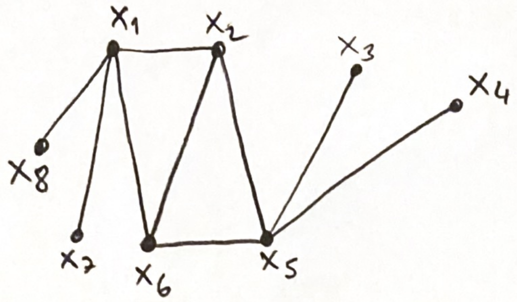
6.042 - Lecture 7 - Matching Problems

Def) Given graph $G = (V, E)$ a <u>matching</u> is a subgraph of G where every node has degree 1.

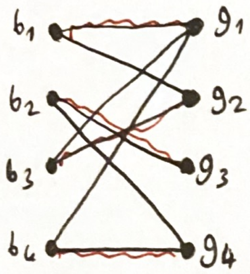
Ex)



$\{x_1 - x_6, x_2 - x_5\}$ is a matching of size 2

Def) A matching is perfect if it has size $\frac{|V|}{2}$

Ex)

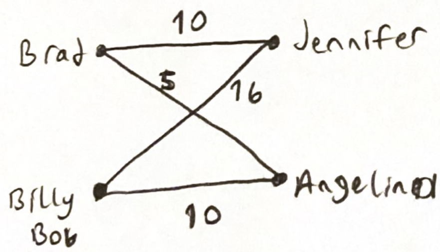


$\{b_1 - g_1, b_2 - g_3, b_3 - g_2, b_4 - g_4\}$ is a perfect matching

Def) The <u>weight</u> of a matching M is the sum of the weights on the edges in M

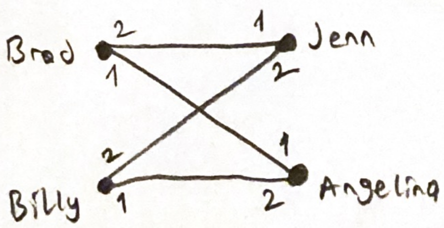
Def) A <u>min-weight matching</u> for G is a perfect matching for G with the min weight.

Ex)



Brad - Jenn
Billy - Angelina $\}$ 20

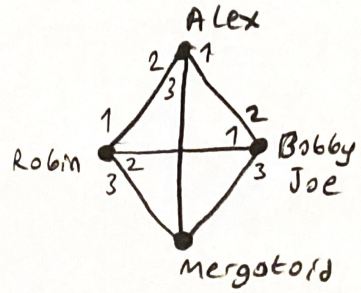
Def) Given a matching M, x & y form a <u>rogue couple</u> if they each other to their mates in M



if we married Brad to Jenn and Billy to Angelina, Brad and Angelina form a rogue couple because they like each other better than who they were hooked up with.

Def A matching is <u>stable</u> if there are no rogue couples.

Goal: Find a perfect matching that is stable.



Thm $\neg\exists$ stable matching for this graph

Pf By contradiction. Assume $\exists$ stable matching M
Then Mergatoid match w/ someone in M.
WLOG (by symmetry), assume Mergatoid matched
to Alex. Alex & Robin form a rogue couple for M.
$\Rightarrow$ M not stable. # $\square$

* Without Loss of Generality (WLOG) means all of the choices will end up with same result, so choose and proof one choice will be enough.

Stable Marriage Problem

- N boys & N girls
- Each boy his own ranked list of all the girls he prefers
- " girl " " " " " " " boys she "

Goal: Find perfect matching w/o rogue couples

Boys		Girls	
♂ 1	C, B, E, A, D	♀ A	3, 5, 2, 1, 4
♂ 2	A, B, E, C, D	♀ B	5, 2, 1, 4, 3
♂ 3	D, C, B, A, E	♀ C	4, 3, 5, 1, 2
♂ 4	A, C, D, B, E	♀ D	1, 2, 3, 4, 5
♂ 5	A, B, D, E, C	♀ E	2, 3, 4, 1, 5

Serenaders

Girls	Day 1	Day 2	Day 3	Day 4
A	~~2~~, ~~4~~, 5	5	5	5
B	—	2	2, ~~1~~	2
C	1	~~1~~, 4	4	4
D	3	3	3	3
E	—	—	—	1

Crossouts

Boys	Day 1	Day 2	Day 3
1		C	B
2	A		
3			
4	A		
5			

Need to show:

- TMA terminates (quickly)
- Everyone get married
- No rogue couples
- Fairness

Thm 1 TMA terminates in $\leq N^2+1$ days.

Pf By contradiction. Suppose TMA does not terminates in N^2+1 days.

Claim: If we don't terminate on a day, then some boy [redacted] crosses a girl off his list that night

N list with N names $\Rightarrow \leq N^2$ crossouts $\neq N^2+1$ crossouts $\square$

Let $P ::=$ "If a girl G ever rejected a boy B, then G has a suitor who she prefers to B."

Lemma 1 P is an invariant for TMA

Pf By induction on # days

Base Case: Day 0. No one is rejected yet $\Rightarrow$ vacuously true

Ind. step: Assume P holds at the end of day d

Case 1: G rejects B on day d+1. Then there was smn. a better boy
$\Rightarrow$ P true at d+1

Case 2: G rejected B before d+1.
P $\Rightarrow$ G had a better suitor on day d
$\Rightarrow$ G has same or better suitor on d+1 $\Rightarrow$ P true on d+1 $\square$

Thm 2 Everyone is married in TMA

Pf By contradiction. Assume that some boy B is not married at the end.

$\Rightarrow$ B rejected by every girl

$\Rightarrow$ Every girl has better suitor (Lemma 1)

$\Rightarrow$ Every girl married

$\Rightarrow$ Every boy married $\longrightarrow$ Contradiction $\square$

Thm 3 TMA produces a stable matching

Pf Let Bob & Gail be any pair that are not married

Case 1: Gail rejected Bob

$\Rightarrow$ Gail marries smn. better than Bob (Lemma 1)

$\Rightarrow$ Gail & Bob are not rogue

Case 2: Gail did not reject Bob

$\Rightarrow$ Bob never serenaded Gail

$\Rightarrow$ Gail is lower on Bob's list than Bob's wife

$\Rightarrow$ Bob & Gail are not rogue

$\Rightarrow$ No rogue couples $\rightarrow$ Contradiction $\square$

Let S = set of all stable matchings $S \neq \emptyset$

For each person P, we define the realm of possibility for P to be

$$\{Q \mid \exists M \in S, \{P, Q\} \in M\}$$

Def A person's optimal mate is his/her favorite from the realm of possibility

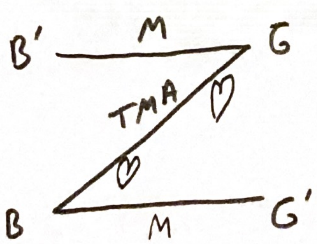
Def " " pessimal " " " least favorite " " " " "

Thm 4 TMA marries every boy w/ his optimal mate

Thm 5 " " " girl " her pessimal "

Pf of Thm 5 By contradiction

Suppose that $\exists$ stable matching M $\exists$ girl G who fares worse than TMA



B & G rogue in M $\rightarrow$ Contradiction

$\Rightarrow$ M is not stable $\square$