Lecture 2 - 6.042 MCS

Proof by Contradiction

To prove P is true, we assume P is False (i.e., $\neg P$ is T) then use that hypothesis to derive a falsehood or contradiction.

If $\underbrace{\neg P}_{\text{False}} \Rightarrow F$ is true

False or P is true

Ex) Thm: $\underbrace{\sqrt{2} \text{ is irrational}}_{P}$

Proof: (by cont.)

Assume for purpose of contradiction that $\underbrace{\sqrt{2} \text{ is rational}}_{\neg P}$.

$\Rightarrow \sqrt{2} = \frac{a}{b}$ (fraction in lowest terms)

$\Rightarrow 2 = \frac{a^2}{b^2}$

$\Rightarrow 2b^2 = a^2$

$\Rightarrow a$ is even $(2 | a)$

$\Rightarrow 4 | a^2$

$\Rightarrow 4 | 2b^2$

$\Rightarrow 2 | b^2$

$\Rightarrow b$ is even

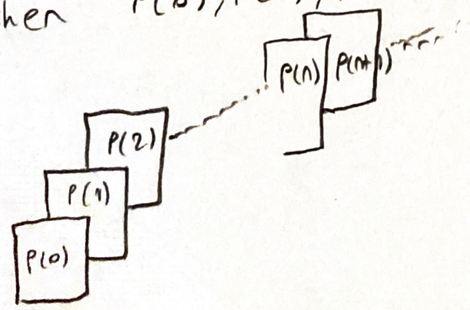
$\Rightarrow a/b$ is not in lowest terms $\Rightarrow$ contradiction

$\Rightarrow \sqrt{2}$ is irrational. ∎

Induction axiom

Let $P(n)$ be a predicate. If $P(0)$ is true and $\forall n \in \mathbb{N} \; (P(n) \Rightarrow P(n+1))$ is true then $\forall n \in \mathbb{N} \; P(n)$ is true.

If $P(0)$, $P(0) \Rightarrow P(1)$, $P(1) \Rightarrow P(2)$, ... are true then $P(0), P(1), P(2)$ ...... are true



Thm: $\forall n \geq 0 \quad 1+2+3+\dots+n = \dfrac{n*(n+1)}{2}$

$$\sum_{i=1}^{n} i = \sum^{1\leq i \leq n} i = \sum_{1 \leq i \leq n} i$$

If $n=1 \quad 1+2+\dots+n = 1$

If $n \leq 0 \quad 1+2+\dots+n = 0$

$n=4 \quad 1+2+3+4 = 10 = \dfrac{4.(5)}{2}$

Pf: by induction

Let $P(n)$ be proposition that $\sum_{i=1}^{n} i = \dfrac{n(n+1)}{2}$

Base case: $P(0)$ is true

$$\sum_{i=1}^{0} i = 0 = \frac{0(0+1)}{2} \checkmark$$

Inductive step: For $n \geq 0$, show $P(n) \Rightarrow P(n+1)$ is true

Assume $P(n)$ is true for purposes of induction.

(i.e, assume $1+2+\dots+n = \dfrac{n(n+1)}{2}$)

need to show $1+2+\dots+(n+1) = \dfrac{(n+1)(n+2)}{2}$

$$\underbrace{1+2+\dots+n}_{\frac{n(n+1)}{2}} + (n+1)$$

$$\frac{n(n+1)}{2} + (n+1) = \frac{n^2+3n+2}{2} = \frac{(n+1)(n+2)}{2} \checkmark$$

$\forall n \geq 0, P(n) \Rightarrow P(n+1)$ $\blacksquare$

Thm: $\forall n \in N$: 3 divides $(n^3 - n)$ Ex) $n=5$ 3 divides $(125-5)$

Proof: by induction

let $P(n) =: 3 \mid (n^3 - n)$

Base case: $n=0 \quad 3 \mid (0-0)$

Inductive step: For $n \geq 0$, show $P(n) \Rightarrow P(n+1)$ is true

Assume $P(n)$ true, i.e. $3 \mid (n^3 - n)$

~~Exam~~ Examine $(n+1)^3 - (n+1) = n^3 + 3n^2 + 3n + 1 - (n+1)$

$$= n^3 + 3n^2 + 2n$$

$$= (n^3 - n) + 3n^2 + 3n$$

$3 \mid 3n^2$

$3 \mid 3n$

$3 \mid (n^3 - n)$ by Inductive hypothesis

$\Rightarrow 3 \mid (n+1)^3 - (n+1)$ ∎

You don't have to use 0 as your base case for example the following will also be valid

Base Case: $P(6)$ is true

Ind. Step: $\forall n \geq 6 \quad P(n) \Rightarrow P(n+1)$

Conclude: $\forall n \geq 6 \quad P(n)$

Thm (NOT!!) ALL horses are the same color.

Proof: by induction

$P(n) ::=$ In any set of $n \geq 1$ horses, the horses in the set are all the same color.

Base case: $P(1)$ it's true since just 1 horse.

Inductive Step: Assume $P(n)$ to prove $P(n+1)$

Consider any set of $n+1$ horses $H_1, H_2, \ldots, H_n, H_{n+1}$

Then $H_1, H_2, \ldots, H_n$ are the same color.

$H_2, H_3, \ldots, H_{n+1}$ " " " " (1) for $n=1$, this is an empty set

Since $\text{color}(H_1) = \text{color}(H_2, \ldots, H_n) = \text{color}(H_{n+1})$

$\Rightarrow$ ALL $n+1$ are same color $\Rightarrow P(n+1)$ ∎

$P(1)$ $\quad P(2) \Rightarrow P(3),\ P(3) \Rightarrow P(4) \ldots$

$\forall n \geq 2 \quad P(n) \Rightarrow P(n+1)$

$P(1) \Rightarrow P(2)$

This is false actually

* The moral here; ALWAYS CHECK THE BASE CASE

Thm: $\forall n$, $\exists$ way to tile a $2^n \times 2^n$ region with any ~~center~~ ~~corner~~ square missing (for Bill).

Proof: by induction

$P(n)$

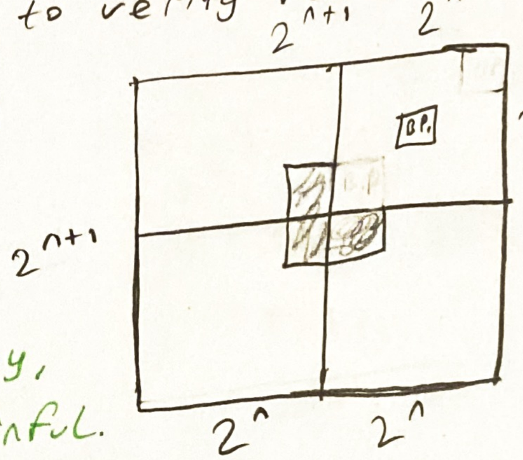
Base case: $P(0)$ ✓

If you are struggling to prove by induction, make your hypothesis much stronger. In this problem we changed problem statement as 'any square missing'

Inductive step: For $n \geq 0$, assume $P(n)$ to verify $P(n+1)$

So we need to show $P(n+1)$ is True

Consider a $2^{n+1} \times 2^{n+1}$ courtyard



It's all the art of what's your induction hypothesis.
Picking a good one, life is easy,
Picking the wrong one, very painful.