6.042 - Lecture 10 - Graph Theory III

Def An Euler Tour is a walk that traverses every edge exactly once and starts and finishes at the same vertex

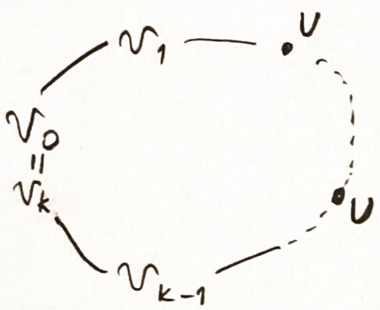
Thm $\underbrace{\text{A connected graph has an Euler Tour}}_{1}$ iff $\underbrace{\text{every vertex of the graph has even degree}}_{2}$

we need to prove in both ways

if (1) then (2)   if (2) then (1)

Pf of if (1) then (2) Assume $G=(V,E)$ has an Euler Tour

→ this means that we have a walk that goes all around the whole graph



since every edge in E is traversed once

Now, consider a vertex v. Even if v repeat itself in this path, degree of it will be 2 times # times it appears in this tour, so; (because every edge is traversed once);

$\deg(v) = \#$ times v appears in tour $v_0 \dots v_k$, times 2

Pf of if (2) then (1) For $G=(V,E)$, assume $\deg(v)$ is even for all $v \in V$

Let $W: v_0 - v_1 - \dots - v_k$ be longest walk that traverses no edge more than once

(1) $v_k - u$ not in W; $v_0 - v_1 - \dots - v_k - u$, longer ✗

→ Suppose that this edge is not in W, so W - u is longer path than the longest path which is a contradiction

All edges that are incident to v_k are used in W

(2) $v_k = v_0$, otherwise v_k has odd degree in W

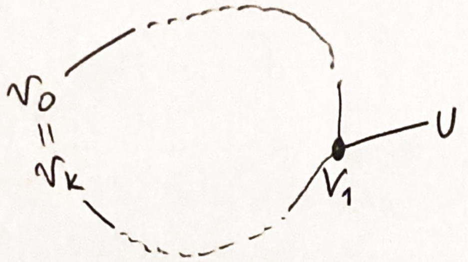
by (1); v_k has odd degree in G ✗

Suppose W is not an Euler Tour

G is connected; so $\exists$ edge is not in W but it is incident to some vertex in W

Let $u - v_i$ be this edge

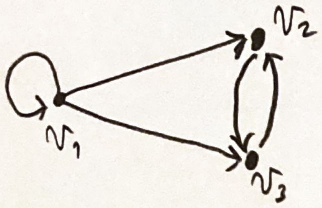
not in W



$u - v_i - v_{i+1} - \cdots - v_k = v_0 - v_1 - \cdots - v_j$

↳ this is a longer walk ✗

Directed Graph (Digraphs)



$v_2 \longrightarrow v_3$

tail head

number of incoming edge's

$\text{indegree}(v_2) = 3$

$\text{Outdegree}(v_2) = 1$

↳ number of outgoing edges

Theorem) Let $G = (V, E)$ be an n-node graph with $V = \{v_1, \ldots, v_n\}$

Let $A = \{a_{ij}\}$ denote adjacency matrix for G. That is

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \to v_j \text{ is an edge} \\ 0, & \text{otherwise} \end{cases}$$

Let p_{ij}^{k} = # directed walks of length k, from v_i to v_j

Then $A^k = \{p_{ij}^{(k)}\}$

Consider the adjacency matrix of the graph above

$$\begin{array}{c|c} & \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \end{array} & \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \end{array} = A \qquad A^2 = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad A^3 = \begin{pmatrix} 1 & 3 & 3 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Pf) $a_{ij}^{(k)}$ denote the $(i,j)^{th}$ entry in A^k

By induction) $P(k)$ = "Theorem is true for k"

$P(k) = $ "$\forall_{i,j} \; a_{ij}^{(k)} = p_{ij}^{(k)}$"

Base case) $k = 1$; Edge $v_i \to v_j$: $p_{ij}^{(1)} = 1 = a_{ij}^{(1)}$ ✓

↳ notice that this is the definition of adjacency matrix

No edge $\quad ; p_{ij}^{(1)} = 0 = a_{ij}^{(1)}$ ✓

Induction Step Assume $P(k)$

$$p_{ij}^{(k+1)} = \sum_{\substack{h: v_h \to v_j \\ \text{is an edge} \\ \text{in } G}} p_{ih}^{(k)} = \sum_{h=1}^{n} p_{ih}^{(k)} \cdot a_{hj}$$

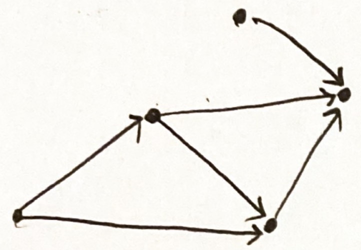
$v_i \xrightarrow{k} v_h \to v_j$

$$= \underbrace{\sum_{h=1}^{n} a_{ih}^{(k)} a_{hj}}_{\text{Ind. step } P(k)} = \underbrace{a_{ij}^{(k+1)}}_{\text{matrix mult.}}$$

$\square$

Def A digraph $G=(V,E)$ is <u>strongly connected</u>

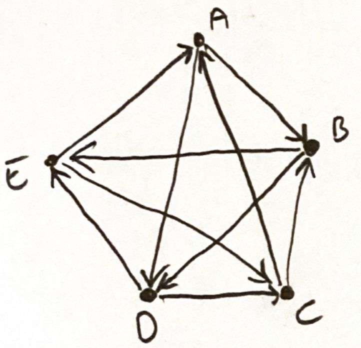
If $\forall u,v \in V$, $\exists$ directed path from u to v in G



Def A directed graph is called a directed acyclic graph (DAG)

If it does not contain any directed cycles

Tournament Graphs



either u beats v ; $u \to v$

or v beats u ; $v \to u$

Can we find a path that covers all of the teams and gives us ~~an~~ ordering on who is the best player like the one at the top?

Ex $A \to B \to D \to E \to C$ (wait a minute! we also have $C \to A$ in this ~~path~~ graph!!!)

Ex $C \to B \to D \to E \to A$

This concept is actually called <u>directed Hamiltonian Path</u> → a walk that goes around the graph and visits every vertex exactly once

Now we are going to prove that a tournament graph has this beautiful property.

Def A <u>directed Hamiltonian Path</u> is a directed walk that visits every vertex exactly once.

Thm Every tournament graph contains a directed Hamiltonian path.

pf By induction on n.

$P(n)$ = "Every tournament graph on n nodes contains a directed hamiltonian path"

Base case $n=1$ ✓
↳ single vertex is a directed hamiltonian path

Inductive step Assume $P(n)$

Consider a tournament graph on $n+1$ nodes.
Take out one node v. This gives a tournament graph on n nodes.
By $P(n)$: $v_1 \to v_2 \to \dots \to v_n$

Case 1 $v - v_1$ ✓

Case 2 $v_1 - v$

Consider smallest i such that $v \to v_i$

$v_1 \to \dots \to v_{i-1} \to v_i \to \dots \to v_n$

$v \nearrow v_i$

since v_i is the smallest number that v beats, than anything smaller than i must have beat v.
$\to v \to v_{i-1}$ ✗

$v \to v_{i-1}$ ✗

$v_{i-1} \to v$

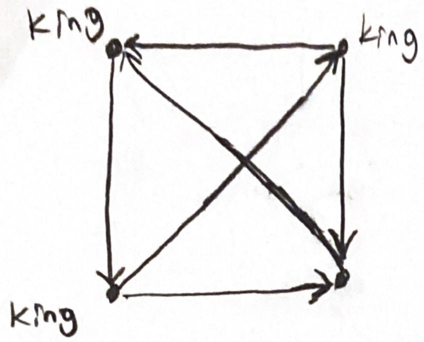
Also notice that, we can use a completely similar and symmetrical argument such that
largest i such that $v_i \to v$

Ex Either chicken u pecks chicken v ; $u \to v$
or 〃 v 〃 〃 u ; $v \to u$

u virtually pecks v if

- $u \to v$, or
- $\exists_w \; u \to w \to v$

A chicken that virtually pecks every other chicken, it's called a <u>king chicken</u>



Thm The chicken with highest outdegree is a king

Pf By contradiction

Let u have highest outdegree

Suppose u is not king.

$\exists v: v \to u$ and

$\forall w: \underbrace{\neg u \to w}_{w \to u}$ OR $\underbrace{\neg w \to v}_{v \to w}$

If $u \to w$ then $v \to w$

$\text{outdegree}(v) \geq \text{outdegree}(u) + 1$ $\times$ $\blacksquare$