Lecture 3: Strong Induction Notes

Good Proofs are

- correct
- complete: All of the key steps have to be there
- clear: So we can understand what's going on.
- brief: You don't want to crush somebody with all the details
- "elegant": Clever
- well-organized: Use lemmas like subroutines in programming. This helps to make your proof clear.
- in order: Steps must be in correct order.

Fermat's Last Thm.

$$\forall n > 2 \quad \neg \exists x, y, z \in \mathbb{N}^+ : x^n + y^n = z^n$$

Problem: Find a sequence of moves to go from

A	B	C
D	E	F
H	G	

1 inversion

to

A	B	C
D	E	F
G	H	

0 inversion

Legal move: Slide a letter into adjacent blank square.

You think that's a easy thing to do??? Actually it's impossible but this is a course about proofs right? So lets proof that this is impossible

Thm: There is no sequence of legal moves to invert G & H and return all other letters to their former position.

Row Move

Ex:

A	B	C
D	G	
E	F	H

$\Rightarrow$

A	B	C
D		G
E	F	H

Actual order

1	2	3
4	5	6
7	8	9

Lemma 1: A row move doesn't change the order of the items

Proof: In a row move, we move an item from cell i into a adjacent cell $i-1$ or $i+1$. Nothing else moves. Hence the order of items preserved. $\square$

Column Moves:

Ex:

A	B	C
D	F	
H	E	G

$\Rightarrow$

A	B	C
D	F	G
H	E	

Ex:

A	B	C
D		G
H	E	F

$\Rightarrow$

A		C
D	B	G
H	E	F

Lemma 2: A column move changes the relative order of precisely 2 pairs of items.

Proof: In a column move, we move an item in cell i to a blank spot in cell $i-3$ or $i+3$. When an item moves 3 positions, it changes order with 2 items. ($i-1, i-2$ or $i+1, i+2$)

Def: A pair of letters L_1 and L_2 is an inversion (a.k.a. inverted pair) if L_1 precedes L_2 in alphabet but L_1 is after L_2 in puzzle.

A	B	C
F	D	G
E	H	

(D, F), (E, F), (E, G) 3 inversions

Lemma 3: During a move, the # of inversions can only increase by 2, decrease by 2 or stay the same.

Proof: Row move: no changes (by Lemma 1)

Column move: 2 pairs change order (by Lemma 2)

Case A: both pairs were inorder $\Rightarrow$ # inversions $\uparrow 2$

Case B: " " " inverted $\Rightarrow$ # inversions $\downarrow 2$

Case C: One pairs inverted $\Rightarrow$ # inversions stay same $\square$

Corollary 1: During a move, the parity (evenness or oddness) of # inversions does not change.

Proof; Adding or subtracting 2 doesn't change parity

Lemma 4: In every state reachable from

A	B	C
D	E	F
H	G	

, the parity of # inversions is odd.

Proof: By induction

$P(n) ::=$ After any sequence of n moves from

A	B	C
D	E	F
H	G	

, the parity of # inversions is odd.

Base case: $n=0$. # inversions $= 1 \Rightarrow$ Parity is odd.

Inductive step: For $n \geq 0$, show $P(n) \Rightarrow P(n+1)$

Consider any sequence of $n+1$ moves $M_1, \ldots, M_{n+1}$

By inductive hypothesis ($P(n)$), we know that parity after n moves $M_1, \ldots, M_n$ is odd.

By corollary 1, we know parity of # inversions does not change during M_{n+1} $\Rightarrow$ the parity after $M_1, M_2, \ldots, M_n, M_{n+1}$ is odd $\Rightarrow P(n+1)$ □

Proof of Theorem: The parity of # inversions in desired state is even (0 inversion). By Lemma 4, the desired state cannot be reached from

A	B	C
D	E	F
H	G	

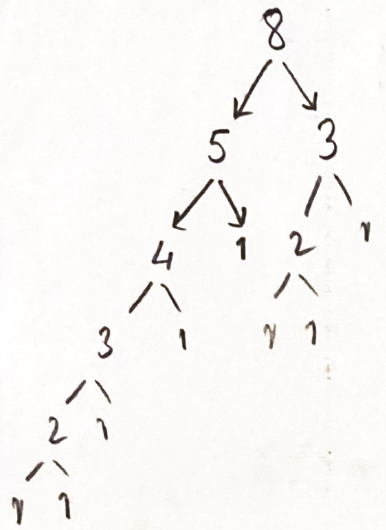
□

** The idea is you're looking for a property that holds at the beginning,
* is preserved by every step, but is not present in the target state.

Strong Induction Axiom

Let $P(n)$ be any predicate. If $P(0)$ is true and $\forall n\ (P(0) \wedge P(1) \wedge \ldots \wedge P(n)) \Rightarrow P(n+1)$ is true, then $\forall n\ P(n)$ is true.

Unstacking Game



pts
15
4
2
1
3
2
+ 1
28

Thm: All strategies for the n-block game produce the same score, $S(n)$

Ex) $S(8) = 28$.

Proof: By strong induction

$P(n)$

Base case: $n=1$ $\quad S(1)=0 \quad \frac{1\cdot(1-1)}{2} = 0$ ✓

Inductive step: Assume $P(1), P(2), \ldots, P(n)$ to prove $P(n+1)$

Look at $n+1$ blocks

$n+1$

$k \qquad n+1-k$

$1 \le k \le n$

$S(n+1)$

Score $= k*(n+1-k) + P(k) + P(n+1-k)$

We're stuck in here. Let's change inductive hypothesis,

$S(n) = \frac{n(n-1)}{2} \longrightarrow S(1) = 0$

$$\text{Score} = K*(n+1-k) + \frac{k(k-1)}{2} + \frac{(n+1-k)(n-k)}{2}$$

$$= \frac{2Kn + 2K - 2k^2 + k^2 - K + (n+1)n - Kn - K - Kn + K}{2}$$

$$= \frac{(n+1)\cdot n}{2} \quad \text{which equals } S(n+1)$$