Lecture 8 - Graph Theory II: Minimum Spanning Trees

Reading: 5.4 - 5.7

- Walks and Paths
- Connectivity
- Cycles and Closed walk
- Spanning Trees
- Minimum Weight spanning Trees (MST)

Walks and Paths

Def A walk is a sequence of vertices connected by edges

$$\underset{\text{start}}{v_0} - v_1 - \cdots\cdots - \underset{\text{end}}{v_k} \Big\} \text{ length } k$$

Def A path is a walk where all v_i's are different.

Lemma 1 If $\exists$ walk from u to v, then $\exists$ path from u to v

Proof $\exists$ walk from u to v

By well ordering principle: walk of minimal length

$$u = v_0 - v_1 - \cdots\cdots - v_k = v$$

Case 1 $k = 0 \rightarrow$ single vertex

Case 2 $k = 1 \rightarrow u - v \rightarrow$ single edge

Case 3 $k \geq 2$; suppose walk is not a path

$\exists\ i \neq j \quad v_i = v_j$

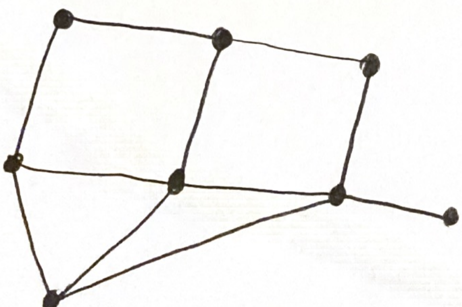
$u = v_0 - \cdots\cdots - v_i = v_j - \cdots\cdots v_k = v$

is a shorter walk $\longrightarrow$ contradiction ∎

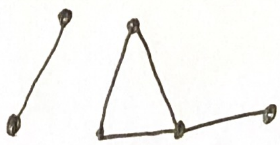
Connectivity

Def u and v are <u>connected</u> if there is a path from u to v

Def A graph is <u>connected</u> when every pair in the graph are connected



A connected graph



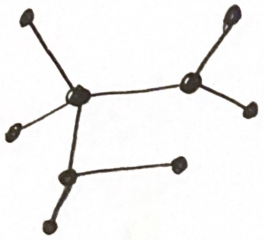
Not connected graph

Cycles & Closed walks

Def A closed walk is a walk which starts and ends at the same vertex: $v_0 - v_1 - \cdots - v_k = v_0$

Def If $k \geq 3$ and $v_0, v_1, \ldots, v_{k-1}$ are all different, then it is called cycle.

Trees



Def A connected and noncyclic graph is called tree

Def A leaf is a node with degree 1 in a tree

Lemma Any connected subgraph of a tree is a tree

Pf By contradiction, suppose the connected subgraph is not a tree:

- Has a cycle $\rightarrow$ whole graph has this cycle
  $\underbrace{}_{\text{Tree}} \rightarrow$ contradiction

Lemma A tree with n vertices has $n-1$ edges

Pf By induction on n

$P(n)$ = "There are $n-1$ edges in any n-vertex tree"

Base case: $P(1) \rightarrow$ One vertex, zero edges ✓

Inductive step: suppose $P(n)$

Let T be a tree with $n+1$ vertices.

Let v be a leaf of the tree

Remove v: this creates a connected subgraph $\rightarrow$ This is also a tree

By $P(n)$: It has $n-1$ edges

Re-attach v: we will get original T, It has $(n-1)+1 = n$ edges

Spanning Trees

Def A spanning tree (ST) of a connected graph is a subgraph that is tree with the same vertices as the graph

Theorem Every connected graph has a spanning Tree

Pf By contradiction

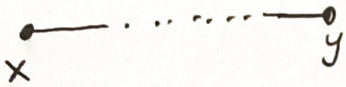
Assume a connected graph G has no spanning tree

Let T be a connected subgraph of G with the same vertices as G and with the smallest number of edges possible

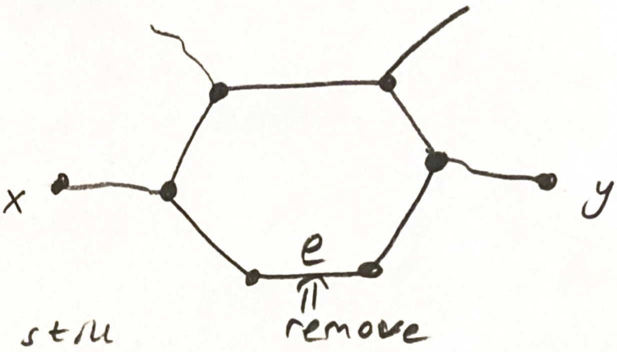
T is not a spanning tree

So, it has a cycle.

Case 1



doesn't contain e



After removing edge e, there is still a path between x and y

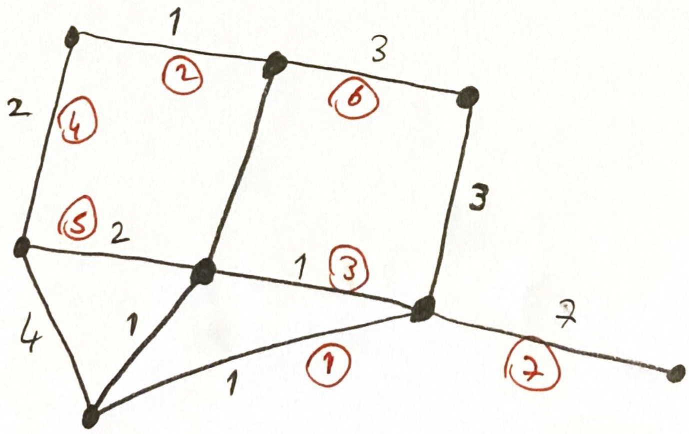
All vertices in G are still connected after removing e from T

contradiction ∎

Minimum Weight Spanning Trees

Def The MST of an edge-weighted graph G is the spanning Tree of G with the smallest possible sum of edge-weights.

Algorithm Grow a subgraph one edge at a time such that at each step:
- Add the minimum weight edge that keeps the subgraph acyclic



Lemma Let S consists of the first m edges selected by the Algorithm.
Then $\exists$ MST $T=(V,E)$ for G such that $S \subseteq E$

Theorem For any connected weighted graph G, the Alg. produces a MST.

Pf of thm

$\#V = n$

1) If $m < n-1$ edges are picked, then $\exists$ edge in $E - S$ that can be added without creating a cycle.

2) Once $m = n-1 \Rightarrow$ we know S is an MST

Pf of Lemma: By induction on m

$P(m) = $ " $\forall G \; \forall S$ consisting of the first m selected edges $\exists_{MST} \; T = (V,E)$ of G such that $S \subseteq E$ "

Base case: $m = 0 \Rightarrow S = \emptyset$, $S \subseteq E$ for any MST $T = (V,E)$ ✓

Inductive step: Assume $P(m)$ holds

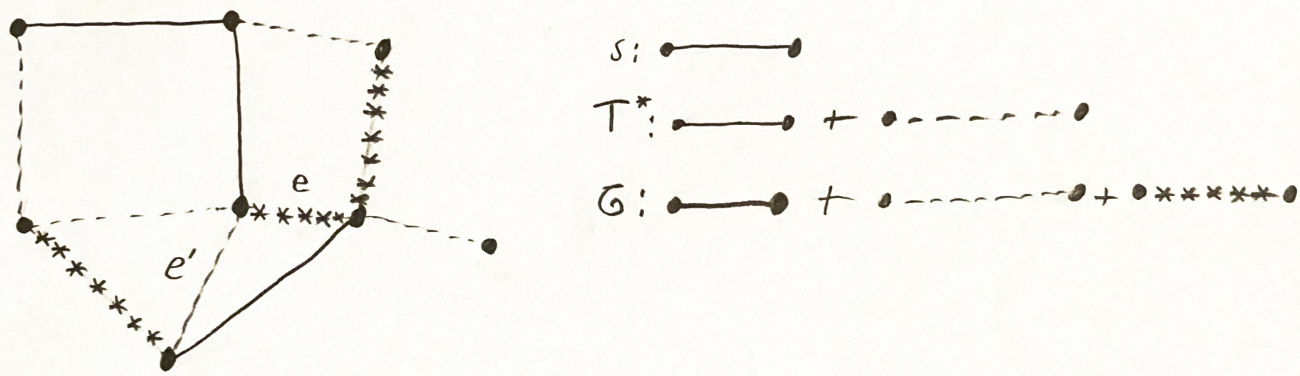
Let e denote the $(m+1)$st selected edge

Let S denote the first m " edges

By $P(m)$: Let $T^* = (V,E)$ a MST such that $S \subseteq E^*$

Case 1: $e \in E^*$: $S \cup \{e\} \subseteq E^* \longrightarrow P(m+1)$ ✓

Case 2: $e \notin E^*$:



(Alg. $\Rightarrow S \cup \{e\}$ has no cycle

(T^* is a tree $\Rightarrow (V, E^* \cup \{e\})$ has a cycle

$\rightarrow$ this cycle has an edge $e' \in E^* - S$

Alg. could have selected e or $e' \Rightarrow$ weight of $e \leq$ weight of e'

Swap e and e' in T:

Let $T^{**} = (V, E^{**})$, $E^{**} = (E^* - \{e'\}) \cup \{e\}$

T^{**} is acyclic because removed e' from the only cycle in $E^* \cup \{e\}$

T^{**} is connected since e' was in a cycle

T^{**} contains all vertices in G

$\rightarrow T^{**}$ is a spanning Tree of $G \Rightarrow$ T^* is a MST, Weight $T^{**} <$ Weight T^* $\Big\}$ T^{**} is a MST