6.042 - Lecture 12: Sums

Def) An n-year \$M-payment annuity pays \$M at the start of each year for n years

Assumption: Fixed interest rate p

$$\$1 \text{ today} = \$(1+p) \text{ in 1 year}$$
$$= \$(1+p)^2 \text{ in 2 years}$$
$$= \$(1+p)^3 \text{ in 3 years}$$

$$\$\frac{1}{1+p} = \$1 \text{ in a year}$$
$$\$\frac{1}{(1+p)^2} = \$1 \text{ in two years}$$

Current-value		Payments
$\$M$	=	\$M now
$\$\frac{M}{1+p}$	=	\$M in 1 yr
$\$\frac{M}{(1+p)^2}$	=	\$M in 2 yrs
$\$\frac{M}{(1+p)^{n-1}}$	=	\$M in $(n-1)$ yrs

Total Current Value $\rightarrow V = \sum_{i=0}^{n-1} \frac{M}{(1+p)^i}$

$$= M * \sum_{i=0}^{n-1} x^i \quad \text{where } x = \frac{1}{1+p}$$

$$= M \frac{1-x^n}{1-x}$$

Thm) $\forall n \geq 1, x \neq 1, \sum_{i=0}^{n-1} x^i = \frac{1-x^n}{1-x}$

Perturbation Method

$$S = 1 + x + x^2 + \dots + x^{n-1}$$
$$-\ xS = x + x^2 + \dots + x^{n-1} + x^n$$
$$(1-x)S = 1 - x^n$$
$$\Rightarrow S = \frac{1-x^n}{1-x}$$

$$V = M\left(\frac{1-x^n}{1-x}\right)$$

$$= M\left(\frac{1-\left(\frac{1}{1+p}\right)^n}{1-\left(\frac{1}{1+p}\right)}\right) = M\left(\frac{1+p-\frac{1}{(1+p)^{n-1}}}{p}\right)$$

Thm) If $|x|<1$, $\displaystyle\sum_{i=1}^{\infty} i x^i = \frac{x - \cancel{(n+1)x^{n+1}}^{\,0} + \cancel{x^{n+2}}^{\,0}}{(1-x)^2}$

$$= \frac{x}{(1-x)^2}$$

Ex) An annuity that pays \$im at the end of year i $(i=1,2,3,\dots,)$ is worth

$m=\$50K$, $p=0.06$

$V=\$14.722,222$

$$m\left(\frac{\frac{1}{1+p}}{\left(1-\frac{1}{1+p}\right)^2}\right) = \frac{m(1+p)}{p^2}$$

$$\downarrow$$

$$\sum_{i=1}^{\infty} i m \frac{1}{(1+p)^i}$$

$$\sum_{i=1}^{\infty} i 2^{-i} = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \cdots$$

$$= \frac{1/2}{(1-1/2^2)} = \frac{1/2}{1/4} = 2$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Guess: $\forall n \ \sum_{i=1}^{n} i^2 = an^3 + bn^2 + cn + d$

Plug in:

$n=0 \Rightarrow 0=d$

$n=1 \Rightarrow 1=a+b+c+d$

$n=2 \Rightarrow 5=8a+4a^2+2c+d$

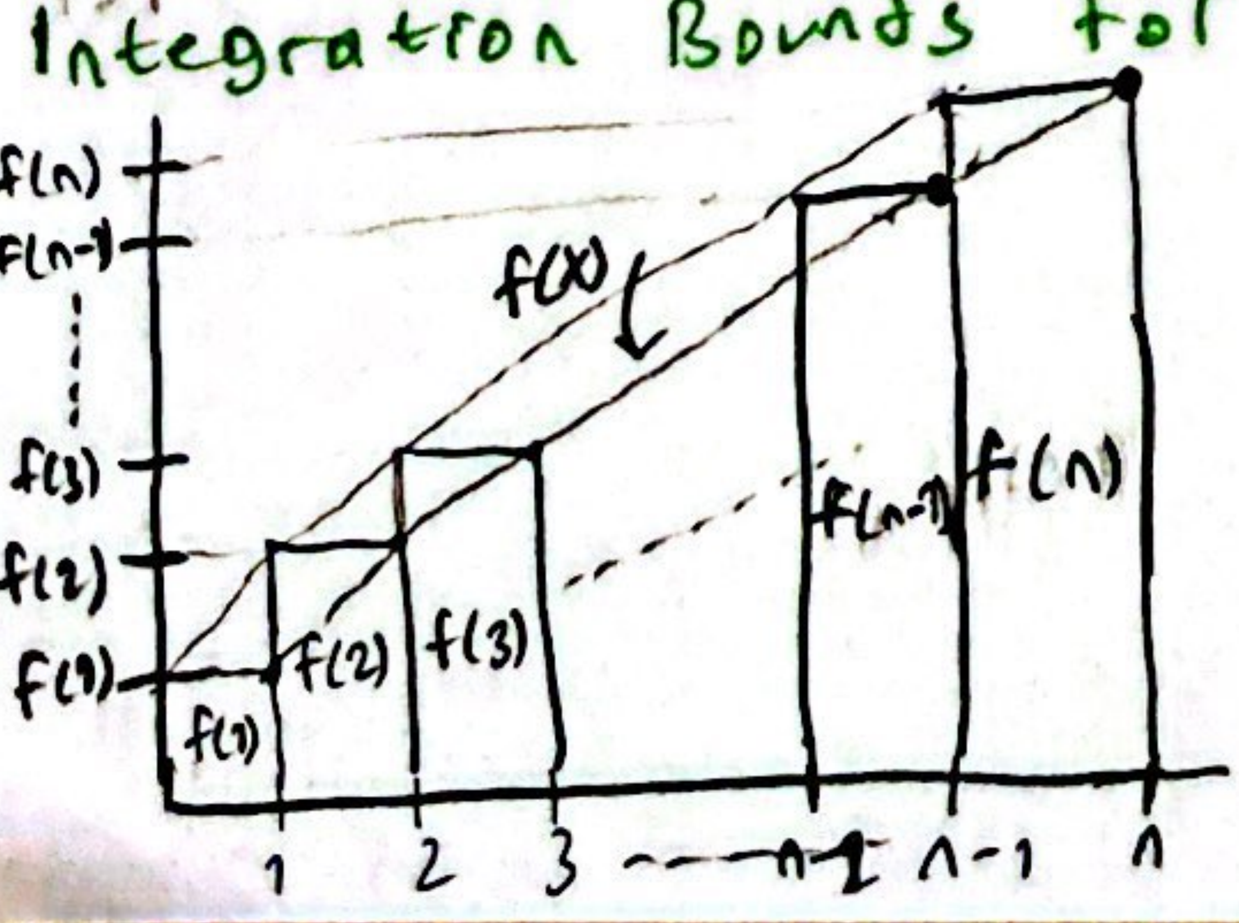
$n=3 \Rightarrow 14=27a+9a^2+3c+d$

$a=1/3,\ b=1/2,\ c=1/6,\ d=0$

* Every time you sum up the powers of numbers, the answer is a polynomial with higher degree for ex, sum of squares equals to polynomial with 3rd order

What about $\sum_{i=1}^{n}\sqrt{i}$

Integration Bounds for $\sum_{i=1}^{n} f(i)$ when f is a positive increasing func.



$$\sum_{i=1}^{n} f(i) \geq f(1) + \int_1^n f(x)\,dx$$

$$\sum_{i=1}^{n} f(i) \leq f(n) + \int_1^n f(x)\,dx$$

For $m = \$50k$, $n = 20$ years, $p = .06$

$V = \$607{,}906$

Claim: If $n = \infty$, then $V = \frac{1+p}{p}$ $\Rightarrow$ For $m = \$50k$, $p = 0.06$

$\Rightarrow V = 883.333\$$

Pf: $\lim_{n\to\infty} \frac{1}{(1+p)^{n-1}} \to 0$

Corollary: If $|x| < 1$, $\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$

Pf: $\lim_{n\to\infty} x^n = 0$ $\square$

Ex) $1 + \frac{1}{2} + \frac{1}{4} - - - - = \frac{1}{1-\frac{1}{2}} = 2$

$1 + \frac{1}{3} + \frac{1}{9} + \cdots = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$

$\sum_{i=1}^{n} i x^i = x + 2x^2 + 3x^3 + \cdots + n x^n$

$$
\begin{aligned}
S &= x + 2x^2 + 3x^3 + \cdots + n x^n \\
-\quad xS &= x^2 + 2x^3 + \cdots + n x^{n+1} \\
\hline
(1-x)S &= \underbrace{x + x^2 + x^3 + \cdots + x^n}_{\frac{1-x^{n+1}}{1-x} - 1} - n x^{n+1}
\end{aligned}
$$

$\Rightarrow (1-x)S = \frac{1-x^{n+1}}{1-x} - 1 - n x^{n+1}$

$\Rightarrow S = \frac{x - (n+1)x^{n+1} + n x^{n+2}}{1-x}$

Derivative method

For $x \neq 1$, $\sum_{i=0}^{n} x^i = \frac{1-x^{n+1}}{1-x}$

$\Rightarrow \sum_{i=0}^{n} i \cdot x^{i-1} = \frac{(1-x)(n+1)(1-x^n) - (-1)(1-x)^{n+1}}{(1-x)^2}$

$\frac{1-(n+1)x^n + n x^{n+1}}{(1-x)^2}$ $\Rightarrow \sum_{i=0}^{n} x^i i = \frac{x - (n+1)x^{n+1} + x^{n+2}}{(1-x)^2}$

Ex) $f(i)=\sqrt{i}$

$$\int_1^n \sqrt{x}\,dx = \frac{x^{3/2}}{3/2}\Bigg|_1^n = \frac{2}{3}\left(n^{3/2}-1\right)$$

$$\Rightarrow \overset{1}{f(1)} + \frac{2}{3}(n^{3/2}-1) \le \sum_{i=1}^{n}\sqrt{i} \le \overset{\sqrt{n}}{f(n)} + \frac{2}{3}\left(n^{3/2}-1\right)$$

$$\Rightarrow \frac{1}{3} + \frac{2}{3}n^{3/2} \le \sum_{i=1}^{n}\sqrt{i} \le \sqrt{n} + \frac{2}{3}n^{3/2} - \frac{2}{3}$$

$$\Rightarrow n=100 \qquad 667 \le \sum_{i=1}^{100}\sqrt{i} \le 676$$

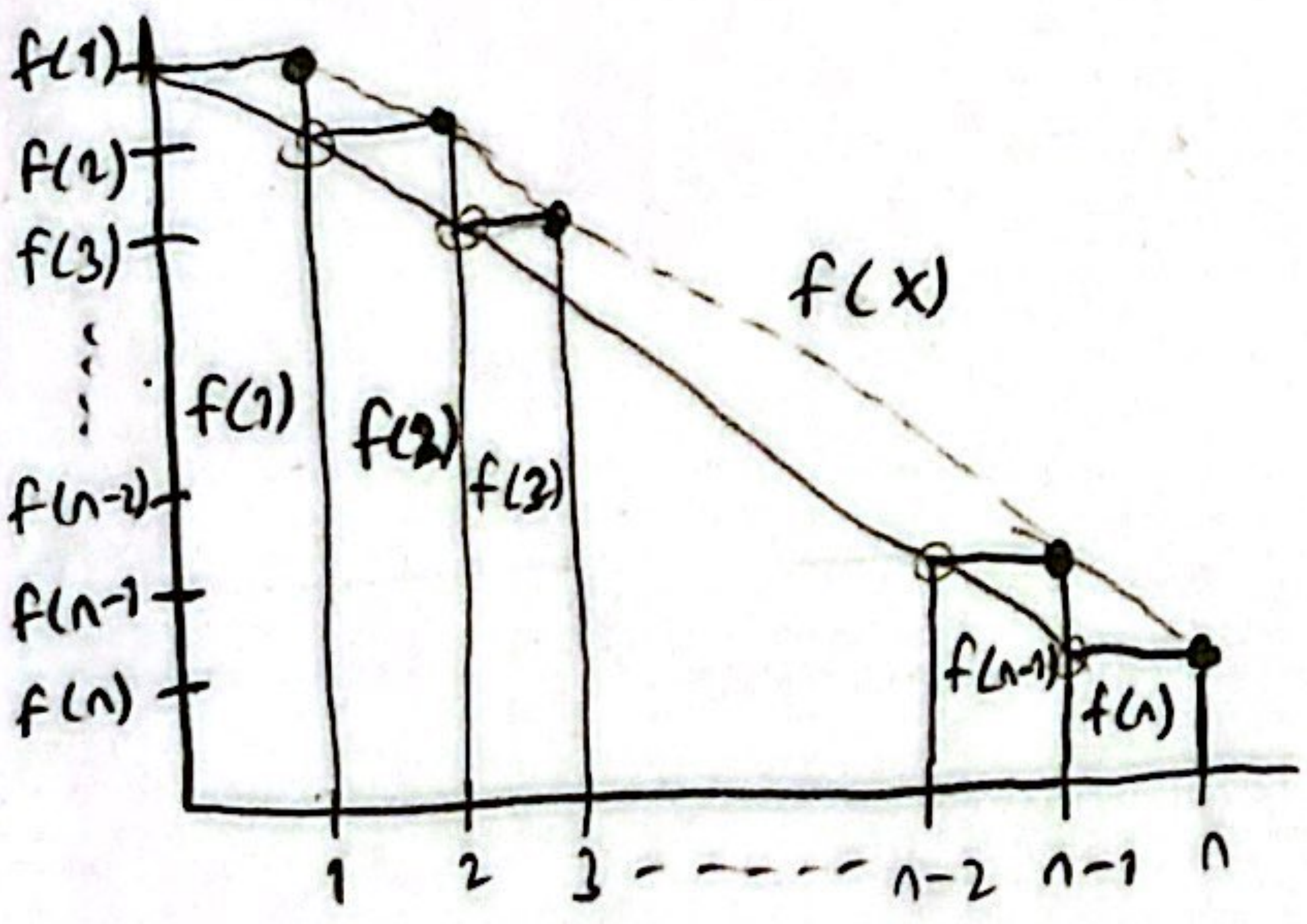
$$\sum_{i=1}^{n}\sqrt{n} = \frac{2}{3}n^{3/2} + \delta(n) \quad \text{where} \quad \delta(n) = \sqrt{n} - \frac{2}{3}$$

$$\sum_{i=1}^{n}\sqrt{i} \sim \frac{2}{3}n^{3/2} \rightarrow \lim_{n\to\infty} \frac{\frac{2}{3}n^{3/2} + \delta(n)}{\frac{2}{3}n^{3/2}} \Rightarrow \lim_{n\to\infty} 1 + \cancelto{0}{\frac{\sqrt{n}-\frac{2}{3}}{\frac{2}{3}n^{3/2}}} = 1$$

Def) $g(x) \sim h(x)$ means $\displaystyle\lim_{x\to\infty}\frac{g(x)}{h(x)} = 1$

Integration bounds when f is decreasing and positive

Ex) $\displaystyle\sum_{i=1}^{n}\frac{1}{\sqrt{i}}$



$$\sum_{i=1}^{n} f(i) \le f(1) + \int_1^n f(x)\,dx$$

$$\sum_{i=1}^{n} f(i) \ge f(n) + \int_1^n f(x)\,dx$$

$f(i) = 1/\sqrt{n}$

$\int^n$

$f(i) = 1/\sqrt{n}$

$$\int_1^n \frac{dx}{\sqrt{x}} = \frac{\sqrt{x}}{1/2}\Bigg|_1^n = 2(\sqrt{n}-1) = 2\sqrt{n}-2$$

$$\Rightarrow f(n) + 2\sqrt{n} - 2 \le \sum_{i=1}^{n}\frac{1}{\sqrt{i}} \le f(1) + 2\sqrt{n} - 2$$

$$\Rightarrow \frac{1}{\sqrt{n}} + 2\sqrt{n} - 2 \le \sum_{i=1}^{n}\frac{1}{\sqrt{i}} \le 2\sqrt{n} - 1$$

$$\Rightarrow \sum_{i=1}^{n} 1/\sqrt{i} = 2\sqrt{n} - \delta(n) \qquad 1 \le \delta(n) \le 2 - \frac{1}{\sqrt{n}}$$

$$\Rightarrow \sum_{i=1}^{n} 1/\sqrt{i} \sim 2\sqrt{n} \Rightarrow \lim_{n\to\infty}\frac{2\sqrt{n}-\delta(n)}{2\sqrt{n}} = 1 - \lim_{n\to\infty}\frac{2}{2\sqrt{n}} = 1$$