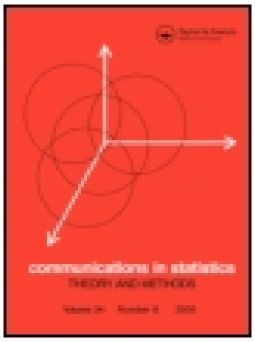

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MULTIVARIATE QUALITY CONTROL

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Key Words and Phrases: multivariate quality control; Hotelling's T; principal components; CUSUM charts; Andrews plots.

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ABSTRACT

This paper includes both the motivation for multivariate quality control, and a discussion of some of the techniques currently available. The emphasis focuses primarily on control charts and includes the T^2-chart, the use of principal components and some recent developments, multivariate analogs of CUSUM charts and the use of the Andrews procedure. Some of the problems associated with multivariate acceptance sampling are presented, and the paper concludes with some recommendations for future research and development.

I. INTRODUCTION

This paper will address the subject of *multivariate quality control*, that is, control procedures where two or more attributes of a product or process

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are being evaluated. Most of the discussion will focus on multivariate control charts, but some attention will also be given to the limited amount of work which has been carried out in the area of multivariate acceptance sampling.

Considerable enthusiasm was generated on the subject of statistical quality control during and after World War II. Much of this was the result of requirements of both the British and American governments for the use of quality control with regard to government contracts during the war and many concerns, having been exposed to these techniques, found them equally applicable with the resumption of non-military production after the war. Unfortunately, the momentum was not enough to carry through the 1960's, when quality seemed to take a back seat to other objectives. The use of multivariate quality control techniques was further hampered at that time by the lack of adequate computational resources. With the renaissance of interest in quality and the development of much more powerful computing capabilities, the stage has now been set for renewed interest in statistical quality control procedures, particularly those dealing with the multivariate case. In fact, most of the techniques discussed in the paper can be administered, if not actually constructed, on a personal computer.

II. CONTROL CHARTS

One of the most powerful tools in the arsenal of quality control practitioners is the statistical control chart. First developed in the 1920's by Walter Shewhart, the control chart found widespread use during

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World War II and has been employed, with various enhancements and modifications, ever since. The standard control charts are designed to detect a significant departure of a process level from its standard. The limits for these charts are usually based on some measure of the inherent variability of this process. Auxilliary charts have been developed to detect changes in this inherent variability. While the bulk of the literature on this subject deals with a single measurement on the process, methods are available which may be employed when two or more characteristics are measured at the same time on a process. Shewhart recognized this problem also, but a great deal of what will be discussed is due to work by Harold Hotelling in the 1930's and 1940's. These techniques include his T^2-procedure and its extensions to multivariate generalizations of control charts for means and standard deviations or ranges. These techniques may be enhanced by the method of *principal components,* a procedure which can result in a more simple and meaningful control situation.

III. WHY MULTIVARIATE CONTROL

To illustrate the need for multivariate control, a fictitious set of data will be introduced. These data come from a process in which an analysis is routinely made on the concentration of one of its chemical constituents. The samples to be tested are split in two and analyzed by two different methods, Method 1 and Method 2. It may be that Method 2 has been proposed as a replacement for Method 1, and that this is the cross-over period, or it may be that neither of these tests is clearly better than the other, and that both are used for control. For whatever reason, each time

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the process is sampled, two related measurements are available to ascertain the condition of the process. (This is not meant to imply that this is a substitute for test comparision procedures. It is not: it is presented merely as an example to introduce this subject.) These data are displayed in Table I, and are also shown in Figure 1. Fifteen observations are not generally considered sufficient information with which to set up control charts, but they will suffice for a numerical example for illustration purpose.

While two control charts may be constructed from this information, the multivariate techniques to be shown possess three important properties: (1) They will produce a single answer to the question: "Is the process in control?"; (2) The specified type I error will be maintained; (3) These techniques will take into account the relationship between these two variables. Later on, another problem will be considered: (4) "If the process *is* out of control, what is the problem?" Property (3) will be discussed in Section IV. It is obvious that any control procedure should contain a single measure to represent the overall condition of the process. This section, then, will be concerned with the problem of error control.

Later, methods for handling groups of observations will be introduced, but for the moment, the control charts will be for individual observations. It will be assumed under the null hypothesis that the process is on standard, both Method 1 and Method 2 will produce results which are normally distributed and that, therefore, control charts may be constructed with guaranteed type I errors. For this example, this will be $\alpha = .05$ for each method.

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The means for both methods are 10.0. The standard deviation for Method 1 is .89, while that of Method 2 is .86. The value of t for 14 degrees of freedom which cuts off .025 in each tail is 2.145. This means that the 95% limits for Method 1 are 10.0 ± (2.145) (.89): 8.09 and 11.91. Similarly, the limits for Method 2 are 8.16 and 11.84. It is implied that if the process were on standard, the analysis for Method 1 would produce values outside these limits 5% of the time as would Method 2 or, conversely, each would produce results inside their respective limits 95% of the time. If the two methods were independent of each other (which they are not in this case), the probability that *both* methods would produce results inside their limits at the same time is (.95) (.95) = .9025, which says that by employing 95% limits for each variable, one is effectively working with a type I error of 10% not 5%. (Taken a bit further, if one had a ten-variable problem instead of two, the type I error would be $1 - (.95)^{10} = .40$.) If, on the other hand, the two variables were perfectly correlated, then the knowledge of one would be all that is necessary to measure the process, and the type I error would remain at .05. A glance at Figure 1 will indicate that the present example is somewhere in between these two extremes (the correlation between Methods 1 and 2 is .887), and hence this should be taken into account in determining the true type I error. The desired solution is to design a control procedure which will guarantee $\alpha = .05$ in the first place.

IV. T^2-Control Charts

The test statistic which is used for the overall control of this process is a generalization of the

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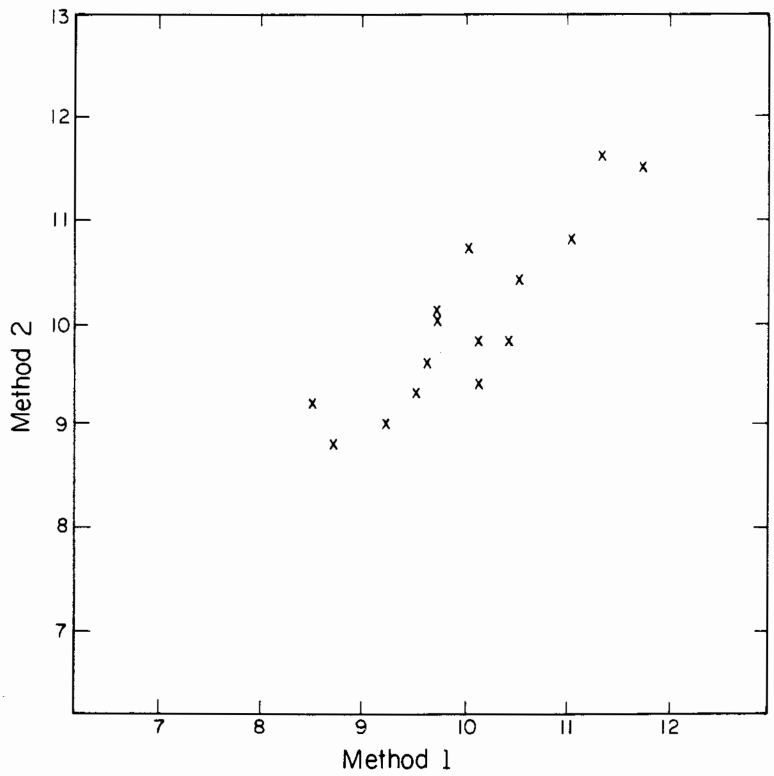

Figure 1. Chemical example: Original data.

t-test, and for single observations, this takes the form:

(IV.1) $$T^2 = (x - \bar{x})' S^{-1} (x - \bar{x})$$

where x is a column vector representing a pair of observations for Methods 1 and 2 , and S is the covariance matrix displayed in Table I. The T^2-distribution was derived by Hotelling (1931), and is a function of the number of variables, p, and the number of observations used in estimating the covariance matrix. T^2 is related to the well-known F-distribution by the relationship:

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$$T^2_{p,n,\alpha} = \frac{np}{(n-p+1)} F_{p,n-p+1,\alpha} \tag{IV.2}$$

In the example from Section II, p = 2 variables, n = 15 observations and α = .05, so $F_{2,14,.05} = 3.74$ which makes $T^2_{2,15,.05} = 8.01$, and this value is the upper control limit for the T^2 control chart. In working with squared quantities, only an upper limit is required. In the case with univariate statistics, if the sample size is large enough, the t-distribution may be replaced with the normal distribution, and in the case of mulitivariate analysis, T^2 may be approximated by the χ^2-distribution with p degrees of freedom.

As an example, consider the first pair of observations in Table I, where Method 1 = 10.0 and Method 2 = 10.7. So

$$T^2=[10.0-10.0,\ 10.7-10.0] \begin{bmatrix} .7986, & .6793 \\ .6793, & .7343 \end{bmatrix}^{-1} \begin{bmatrix} 10.0-10.0 \\ 10.7-10.0 \end{bmatrix}$$

which is less than 8.01, and hence it would be concluded that the process at this point did not differ significantly from the average of the fifteen observed pairs in Table I. Figure 2 shows a T^2-chart for these data plus four additional pairs, included to exhibit some abherent behaviour. These are: A = (12.3, 12.5), B = (7.0, 7.3), C = (11.0, 9.0) and D = (7.3, 9.1). Note that with the exception of C, all of these extra observations would have been out of control using the individual Shewhart charts, but that all four of them are above the T^2 limit.

When there are only two variables as is the case here, the control situation may be represented graphically, and this is done in Figure 3. The concept of an ellipse goes back to Karl Pearson (1901), and has

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TABLE I

CHEMICAL ANALYSES

	METHOD 1	METHOD 2
	10.0	10.7
	10.4	9.8
	9.7	10.0
	9.7	10.1
	11.7	11.5
	11.0	10.8
	8.7	8.8
	9.5	9.3
	10.1	9.4
	9.6	9.6
	10.5	10.4
	9.2	9.0
	11.3	11.6
	10.1	9.8
	8.5	9.2
Mean	10.0	10.0
St.Dev.	.89	.86

$$\text{Covariance Matrix} \begin{bmatrix} .7986 & .6793 \\ .6793 & .7343 \end{bmatrix}$$

been modified here to allow for the small sample properties of the T^2-statistic (Jackson, 1980). Any points which are outside the ellipse will be above the T^2 control limit and vice versa, and therefore for the two-dimensional case, one could use the ellipse itself for a control limit. Note that this ellipse is superimposed on the original univariate limits computed earlier. There is a large region within these combined control limits which is not contained within the ellipse. This is because of the correlation between Methods 1 and 2 implying that a result such as C is highly unlikely even though both methods are inside their respective univariate limits because one is higher than expected and one is lower than expected

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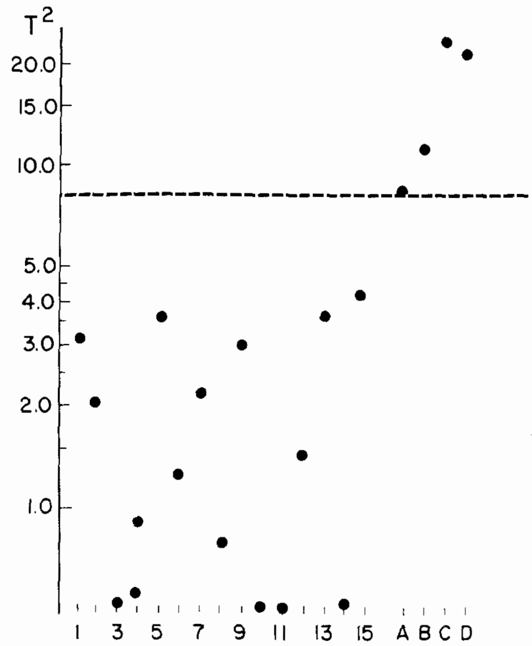

Figure 2. 95% T^2-control chart for chemical example.

when these measurements tend to go up and down together. (There is also a small region inside the ellipse which is *outside* the univariate limits. This also reflects the high positive correlation between the methods.) What has been accomplished, then, is the construction of a single control limit which produces the desired type I error, .05, and takes into account all of the available information, namely the means, variances and correlation between the methods.

V. T^2 CONTROL FOR GROUPS OF OBSERVATIONS

It is generally recognized that one is better off using groups of observations for control rather than a

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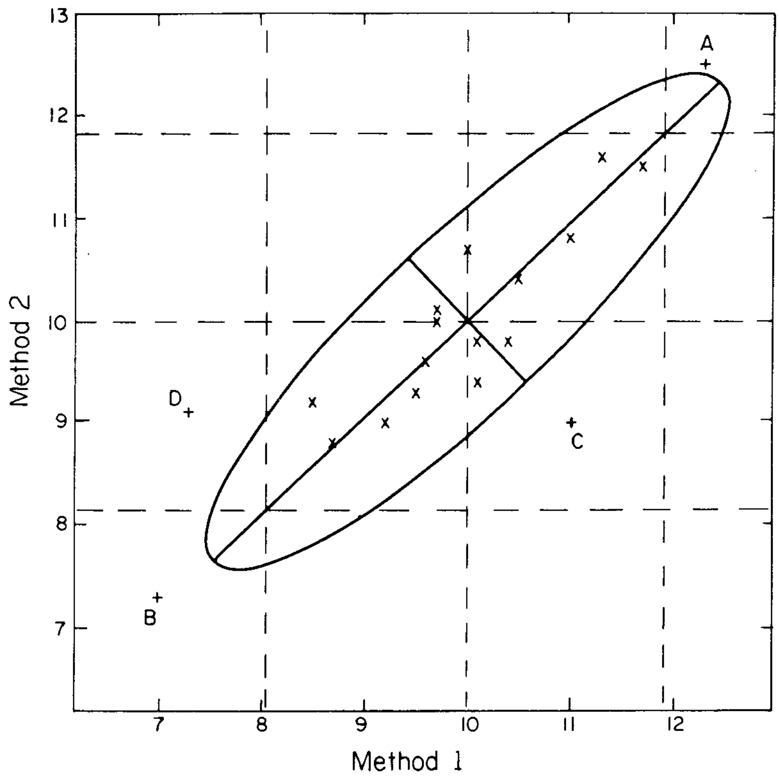

Figure 3. Comparison of 95% control chart limits and 95% control ellipses.

single observation, and the same holds true for multivariate analysis. The obvious advantages are the increased precision of using averages rather than individuals, the fact that averages usually tend towards normality whether or not the underlying distributions do and the ability to control on variability. In addition to the standard Shewhart chart for averages, there would be a companion chart for ranges or standard deviations. There is, fortunately,

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a multivariate analog to this, also due to Hotelling (1947) called *Generalized T^2-Statistics*. This concept can best be illustrated by Figure 4 which shows three observations, their mean and a standard. Three statistics are available: T^2_O, a measure of overall variability, T^2_M, a measure of the distance between the mean of this group of observations and the aim and T^2_D, a generalized measure of the dispersion of the sample around its own mean. The computations are quite simple. Let the size of the sample to be analyzed be m. For purposes of an example, using the extra four observations (A, B, C, and D) from the previous example as a "sample",

1. For each of the four observations in the sample, compute T^2. For this example, they are 8.51, 11.38, 23.14 and 21.55. The sum of these values, 64.58, is denoted by T^2_0 and has an asymptotic χ^2 -distribution with mp degrees of freedom. In this case, this would be (4) (2) = 8, and the 5% limit would be 15.5, so we would conclude that the overall variability of this sample is significantly larger than that of the original 15 observations.

2. Obtain the average of this sample: This is 9.600 for Method 1 and 9.475 for Method 2. These are inserted into the T^2-formula, multiplied by m as follows:

$$T^2_M = (4)\ [9.600 - 10.0,\ 9.475 - 10.0] \begin{bmatrix} .7986 & .6793 \\ .6793 & .7343 \end{bmatrix}^{-1} \begin{bmatrix} 9.600 - 10.0 \\ 9.475 - 10.0 \end{bmatrix} = 1.81.$$

T^2_M has the same distribution as T^2 with the same limits. In this case, with the limit

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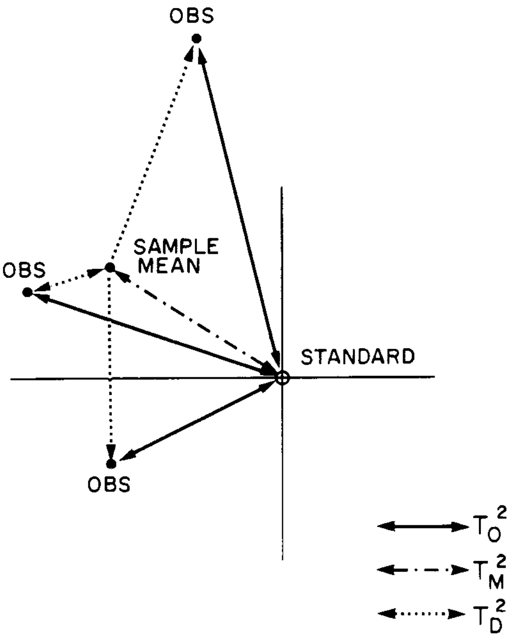

Figure 4. Generalized T^2-statistics.

equal to 8.21, the mean of this sample is not significant from the original sample.

3. The difference $T^2{}_0 - T^2{}_M = T^2{}_D$ has an asymptotic χ^2 -distribution with (m-1)p degrees of freedom, in this case, 6, with a 5% limit of 12.59. $T^2{}_D = 64.58 - 1.81 = 62.77$ which is highly significant.

For this example, then, there is excessive overall variablility made up almost exclusively of the variability of the sample itself. The generalized mean is not significantly different than its "standard", the mean of the original fifteen observations.

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Consider a different set of four observations: (11.5, 11.6), (12.0, 11.8), (11.7, 11.5) and (11.9, 11.6). These produce the *following* T^2-statistics: 3.89, 5.07, 3.64 and 4.52 with T^2_0 = 17.12. The averages of this sample are (11.775, 11.625) with T^2_M = 16.12 and T^2_D = 1.00. In this case, the mean has shifted from the standard, but it is quite consistent about that level.

It has been our experience that T^2_0 is not a particularly useful statistic for multivariate control charts. If it *is* out-of-control, one or both of the other two statistics will usually be out of control as well. On the other hand, T^2_M and T^2_D, the multivariate analogs of the average and range chart, are very useful, and should be used whenever possible. There are cases where, because of cost, time, the nature of the test or whatever, it is not practical or possible to use averages for short-range process control. This is true whether the control procedure is univariate or multivariate. However, in the multivariate case, these generalized T^2-statistics may be used as summary statistics to evaluate the process performance by tricks, days, weeks, machines or batches of material similar to the use of the analysis of variance in the univariate case.

VI. PRINCIPAL COMPONENTS

The axes of the ellipse in Figure 3 have some properties of note. The major axis is the *orthogonal* regression line between Methods A and B. The orthogonal regression line, also due to Pearson, differs from the conventional regression lines by minimizing the sums of squares of residuals *perpendicular* to the

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line itself. This is a useful relationship to employ when neither variable can be considered the cause of variations in the other as is the case here, where both are a function of the process being monitored. In fact, variation along this major axis represents variation in the process. Variation along the minor axis represents a lack of agreement between the two methods. Since these axes are at right angles to each other, they may be considered a principal axis rotation about the intersection of the two means of the original axes. These rotated axes are defined by the method of *principal components* (Hotelling, 1933).

A detailed description of the method of principal components and its relationship to quality control may be found in Jackson (1980, 1981a, 1981b). However, the properties of this technique, as illustrated by the preceding numerical example, are as follows:

1. The principal components are defined as linear transformations of the original two variables x_1 and x_2 into a new set of two variables z_1 and z_2 which are independent of each other. This transformation is:

$$Z = U'(x - \bar{x}) \tag{VI.1}$$

2. The transformation is defined by the following matrix equation:

$$U'SU = L \tag{VI.2}$$

which for the preceding example is:

$$\begin{bmatrix} .7236 & .6902 \\ -.6902 & .7236 \end{bmatrix} \begin{bmatrix} .7986 & .6793 \\ .6793 & .7343 \end{bmatrix} \begin{bmatrix} .7236 & -.6902 \\ .6902 & .7236 \end{bmatrix} =$$

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$$\begin{bmatrix} 1.4465 & 0 \\ 0 & .0864 \end{bmatrix}$$

3. The columns of U are called *characteristic vectors* or *eigenvectors*. The values of the first column, .7236 and .6902, are the direction cosines relating the major axis in Figure 3 to the original coordinate axis. Since variation along this axis represents variation in the process as represented by the two test results, the first principal component may be thought of as a new variable representing process variability. (In a nongeometric setting, the coefficients of the first vector are nearly the same and both positive, making the transformation appear as a weighted average.) The second column contains the direction cosines for the minor axis. These coefficients are also nearly the same, but differ in sign, and hence the second principal may be thought of as representing the difference between the two methods, i.e. testing and measurement variability.

4. The matrix L is the covariance matrix of the principal components. The diagonal elements, 1.4465 and .0864 are called *characteristic roots* or *eigenvalues* and are the variances of z_1 and z_2. The fact that the off-diagonals are zero demonstrates that z_1 and z_2 are uncorrelated.

5. The determinant of L is equal to the determinant of S. This may be considered a genera-

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lized measure of variability as the square root of this determinant is proportional to the area of the ellipse shown in Figure 3.

6. Another measure of variability is the trace of a covariance matrix. The sum of the characteristic roots is equal to the sum of the original variances, 1.5329. This allows one to make statements generally associated with variance components, by taking the ratio of each characteristic root to their sum. For this example, one could conclude that the 1.4465/1.5329 or 94.4% of the total variability of the system is due to the process, and .0864/1.5329 or 5.6% is due to testing and measurement.

VII. USE OF PRINCIPAL COMPONENTS FOR CONTROL CHARTS

Section VIII will extend the methods described above to more than two variables, but while still at the two-variable level, the method of principal components will be applied to control charts. It is customary in this situation to rescale the principal components to have unit variance. This is done by dividing the characteristic vectors by the square roots of their corresponding characteristic roots, viz:

$$W = UL^{-1/2} \tag{VII.1}$$

which for this example is:

$$\begin{bmatrix} .7236/\sqrt{1.4465} & -.6902/\sqrt{.0864} \\ .6902/\sqrt{1.4465} & .7236/\sqrt{.0864} \end{bmatrix} = \begin{bmatrix} .6016 & -2.3481 \\ .5739 & 2.4617 \end{bmatrix}$$

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and Equation (VI.1) is replaced by

$$y = W'(x - \bar{x}) \tag{VII.2}$$

Since y_1 and y_2 both have zero means and unit variances, the 95% limits for both of these variables is ∓ 2.145. Table II shows the original data and the principal components y_1 and y_2. Note in particular the four extra observations A, B, C and D. A and B are clearly out of control due to the level of the process. y_2 is out of control for C because the original methods disagreed. D is in trouble on both counts. These same data are shown in Figure 5. Included in Table II are the corresponding values of T^2 shown in Figure 2. If principal component information is available, an alternative computation for T^2 is available:

$$T^2 = y'y \tag{VII.3}$$

which is just the sum of squares of the principal components - a much easier task than Equation IV.1. The generalized T^2-statistics of Section V also carry over to principal components.

VIII. PRINCIPAL COMPONENTS WITH MORE THAN TWO VARIABLES

The discussion about the properties of principal components given in section VI holds for any number of variables, and as this number gets larger, another property emerges: *one may not need to use a full set of principal components for control.*

Consider the following four-variable problem dealing with the testing of ballistic missiles. A fuller description of this example may be found in

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TABLE II

PRINCIPAL COMPONENTS CALCULATIONS

OBS.NO.	$x_1 - \bar{x}_1$	$x_2 - \bar{x}_2$	y_1	y_2	T^2
1	.0	.7	.40	1.72	3.12
2	.4	-.2	.13	-1.43	2.06
3	-.3	.0	-.18	.70	.52
4	-.3	.1	-.12	.95	.92
5	1.7	1.5	1.88	-.30	3.62
6	1.0	.8	1.06	-.38	1.27
7	-1.3	-1.2	-1.47	.10	2.17
8	-.5	-.7	-.70	-.55	.79
9	.1	-.6	-.28	-1.71	3.00
10	-.4	-.4	-.47	-.05	.22
11	.5	.4	.53	-.19	.32
12	-.8	-1.0	-1.06	-.58	1.46
13	1.3	1.6	1.70	.89	3.68
14	.1	-.2	-.05	-.73	.54
15	-1.5	-.8	-1.36	1.55	4.25
A	2.3	2.5	2.82	.75	8.51
B	-3.0	-2.7	-3.35	.40	11.38
C	1.0	-1.0	.03	-4.81	23.14
D	-2.7	-.9	-2.14	4.12	21.55

Jackson (1959), but for now, it may be summarized by saying that the variables of concern are thrust measurements, and as was the case in the two-variable example, there are two test methods, A and B. There is also some redundancy in the system in that two identical gauges were used, the results of which were fed into both methods, hence the four variables. Again, one would like a single answer to the question: "Does this round (rocket) conform to standard?"

The four variables are: x_1 = Gauge #1, Method A; x_2 = Gauge #1, Method B; x_3 = Gauge #2, Method A and x_4

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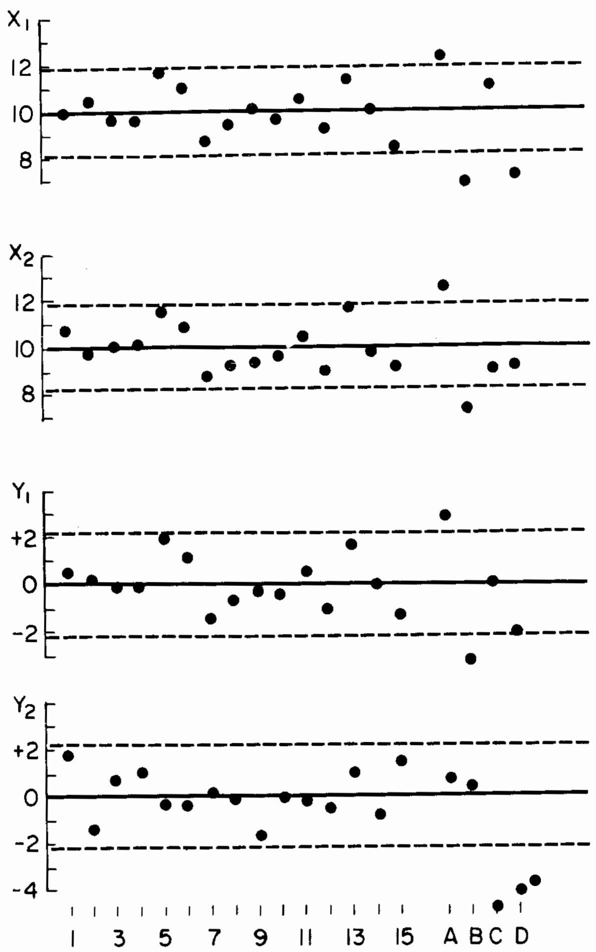

Figure 5. 95% control charts for chemical example: Original observations and principal components.

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= Gauge #2, Method B. The covariance matrix for a sample of forty rounds is shown in Table III. The characteristic roots associated with this matrix are: $l_1 = 335.34$, $l_2 = 48.03$, $l_3 = 29.33$ and $l_4 = 16.41$, and the total of these four roots is 429.11. The third and fourth roots, together, account for about 11% of the total variability. Besides the fact that the third and fourth principal components account for so little of the total variablility, a significance test indicates that these two roots are not significantly different from each other. If two roots are exactly equal, the characteristic vectors associated with them cannot be uniquely defined, so the results of this test would cast doubt on the descriptive validity of these two principal components. The alternative is to carry the first two principal components, T^2 and a measure of the inability of the first two components to predict what the original observations were. From a parsimonious point of view, any system which can adequately predict the original p variables with a smaller number of principal components should be looked on with favor since it should make it easier for the control personnel in diagnosing problems when they arise. In non-quality control operations, one of the main uses of principal components is to reduce the dimensionality of the problem when doing data analysis.

Recalling that Equation VII.2 related the principal components to the original variables, the inverse of this relationship can be used to predict the original variables form the principal components. This is:

(VIII.1) $$\hat{x} = \bar{x} + W'Ly$$

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TABLE III

COVARIANCE MATRIX FOR BALLISTIC MISSILE EXAMPLE

Gauge #1		Gauge #2	
Meth.A	Meth.B	Meth.A	Meth.B
x_1	x_2	x_3	x_4
102.74	88.67	67.04	54.06
88.67	142.74	86.56	80.03
67.04	86.56	84.57	69.42
54.06	80.03	69.42	99.06

If only the first two components are used, W will still have four rows, but only two columns, and L in this case will be a 2x2 matrix containing only the first two characteristic roots. If all four vectors were used, Equation VIII.1 would yield x exactly, but if only two vectors are used, it will only produce an estimate. The difference between the estimate and the original data will represent the residual variability unexplained by the first two principal components. In this case, VIII.1 can be rewritten as:

$$\text{(VIII.2)} \qquad x = \bar{x} + W'Ly + (x - \hat{x})$$

producing an expression not unlike the analysis of variance model, the terms on the right hand side of the equation representing the mean, the amount explained by the first two principal components and the amount unexplained by them.

In this example, the characteristic vectors are:

$$W' = \begin{bmatrix} .0256 & -.0897 \\ .0332 & -.0258 \\ .0251 & .0200 \\ .0245 & .1082 \end{bmatrix}$$

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The coefficients for the first characteristic vector are all positive and roughly the same size. This is similar to the first vector for the two-variable example and again indicates that the first principal component represents variablility in the product (78%). The second vector (11%) has positive coefficients for the second gauge and negative coefficients for the first gauge, hence representing gauge differences. The remaining 11% would represent method differences and method x gauge interactions, and will be assumed to represent the inherent variability. Assume, for convenience , that the means are all equal to zero, and that a sample observation is:

$$X = \begin{bmatrix} 15 \\ 10 \\ 20 \\ -5 \end{bmatrix}$$

Based on a sample size of n = 40, the 95% limits for y_1 and y_2 are +2.20, and for T^2 (using p = 2 because only two principal components are being employed) the limit is 6.67. The principal components for that observation vector are:

$$y = \begin{bmatrix} 1.094 \\ -1.744 \end{bmatrix}$$

The predicted measurements are obtained from Equation VIII.1 viz:

$$\hat{x} = \begin{bmatrix} .0256 & -.0897 \\ .0332 & -.0258 \\ .0251 & .0200 \\ .0245 & .1082 \end{bmatrix} \begin{bmatrix} 335.34 & 0 \\ 0 & 48.03 \end{bmatrix} \begin{bmatrix} 1.094 \\ -1.744 \end{bmatrix} = \begin{bmatrix} 16.9 \\ 14.3 \\ 7.5 \\ -.1 \end{bmatrix}$$

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The residuals are:

$$x - \hat{x} = \begin{bmatrix} -1.9 \\ -4.3 \\ 12.5 \\ -4.9 \end{bmatrix}$$

and the sum of squares of residuals is 202.2. There is a test for this also, and in this case, this quantity is significant. The conclusion is that the two-component model does not fit the data, and the culprit appears to be a mismatch between the two methods on the second gauge.

In practice, the residual sum of squares is tested first because if it is significant, it casts doubt on the other results. It has been our experience that when this result is significant, the reason is usually connected with a testing and measurement discrepancy of some sort, and is quite often a test or recording error. It is a very efficient bad-data-detector. If the residual is not significant, the next statistic to look at is T^2, and if that is also not significant, one may continue operating the process. If T^2 is significant, then one should look at the individual principal components to see why. The procedure for handling residuals is explained more fully in Jackson and Mudholkar (1979) which includes a nine-variable example where only five principal components were retained. There are extensions of this procedure to supplement the generalized T^2-statistics.

It may appear that not very much has been gained by dropping the third and fourth principal components in the missile example. However, if one is able to explain, say, 90% of the variability of a twenty-varia-

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ple problem with only three or four principal components (this is not an unusual occurence), and the principal components are as readily interpretable as the ones shown here (they aren't always - nothing is perfect!) the control personnel will have a lot less charts to monitor, and should be able to better diagnose the problems when they do occur.

There are a number of criteria for choosing the number of principal components to retain. These include (1) Significance tests for equality of roots, (2) Retaining enough principal components to explain a prescribed proportion of the total variance and (3) Stopping when the residual variances are equal to some prescribed amount, usually the inherent variability of the system under study. This last criteria is the most appropriate for quality control operations.

IX. OTHER MULTIVARIATE TECHNIQUES

With the resurgence of interest in quality control, there is beginning to be some more effort applied to statistical quality control, and this is now beginning to spill over into multivariate quality control. One recent paper, (Woodall and Ncube, 1984) considers some multivariate CUSUM procedures. One of these consists of running a CUSUM chart for each variable, and concluding that an out-of-control condition exists if any one of them is out of control. Also considered were simultaneous CUSUM charts on the principal components which appeared to have smaller average run lengths when the original variables were highly correlated and should have other advantages as well, particularly if the principal components can be interpreted.

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Another recent paper (Kulkarni and Paranjape, 1984) transforms the original variables using the Andrews function plot (Andrews, 1972) for use as control variables. An example of this technique is shown in Figures 6 and 7 using the chemical data from Table I. For a two-variable problem, a typical transformation would be:

(IX.1)

$$t = (x_1 - \bar{x}_1)\sin(\theta) + (x_2 - \bar{x}_2)\cos(\theta) \qquad 0 < \theta < \pi$$

Andrews designed this method for use as a multivariate data analysis tool. (This technique was not intended for a two-variable problem which could be handled more efficiently by some of the methods already discussed.) Kulkarni and Paranjape carry this one step farther by constructing control limits for t which are functions of the covariance matrix. These limits for the chemical problem are:

$$\bar{x}_1 \sin(\theta) + \bar{x}_2 \cos(\theta) \pm$$

$$\sqrt{\chi^2_{10,\alpha} \begin{bmatrix}\sin(\theta) & \cos(\theta)\end{bmatrix} \begin{bmatrix} .7986 & .6793 \\ .6793 & .7343 \end{bmatrix} \begin{bmatrix}\sin(\theta) \\ \cos(\theta)\end{bmatrix}}$$

and is a continuous function of θ. (In the example to follow, the limits were set about 0.) The situation for two variables is a very special case because x_1 is displayed at $\theta = \pi/2$, x_2 at $\theta = 0$, $(.707)(x_1 + x_2)$ at $\theta = \pi/4$ and $(.707)(x_1 - x_2)$ at $3\pi/4$. Figure 6 shows the first seven observations from Table I, all of them being within the limits. Figure 7 shows the four aberrant observations A, B, C and D. A is just barely outside of the limits between $0<\theta<\pi/4$ as it was barely outside the control ellipse in Figure 3. B is farther

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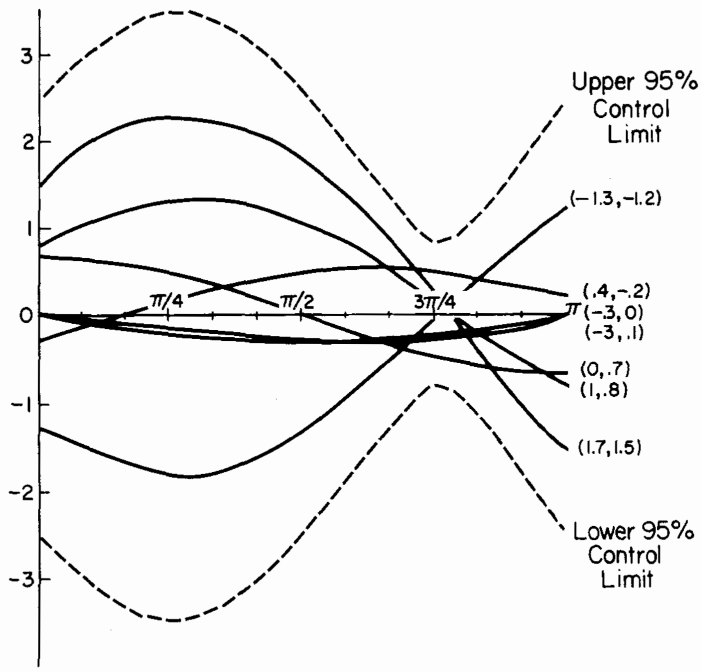

Figure 6. Andrews plots for chemical data: First seven observations.

out in the same region. C, which represented a mismatch between the methods is out of control at $\theta = 3\pi/4$ and D is out of control over a fairly long range indicating its dual problem. If more than two variables are being controlled, the plotted function and their limits become more irregular, and one needs to use the whole range $-\pi<\theta<\pi$. There are, apparently, some type II error problems for the general case. However, this technique, like the CUSUM technique described in the preceeding paragraph, is quite new and both will require more actual use in the field to ascertain their strengths and weaknesses. What is encouraging is the sudden burst of new ideas.

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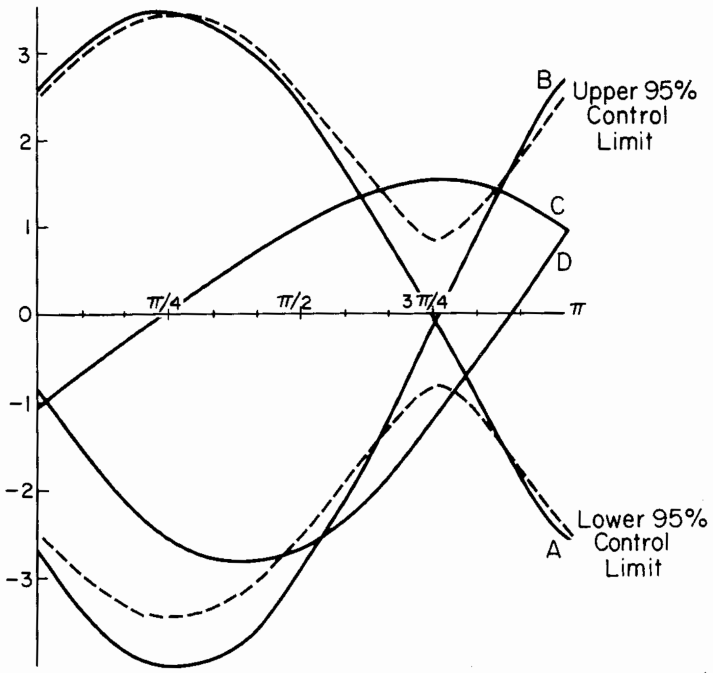

Figure 7. Andrews plots for chemical data: Observations A,B,C and D.

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X. MULTIVARIATE ACCEPTANCE SAMPLING

In contrast to control charts, the technique of *acceptance sampling* consists of sampling a number of items from a lot of finished product to determine whether or not the quality of this lot is such that it should be shipped. ("Finished" in this case refers to any operation at the conclusion of which, the product is shipped somewhere else, be it the ultimate consumer or merely another department in the same organization.)

In the case of sampling for variables, specifications are made on a characteristic of the product, and a sampling scheme is created by establishing a type I error associated with the standard for the product, and a type II error associated with the specification limits. This will determine the sample size required, and the significance test used to make the required decision about the lot. This differs from control charts in that the type II error is rarely set up for them in advance, with the exception of acceptance control charts. (Freund, 1960; Schilling, 1982). As was the case for control charts, great strides were made in acceptance sampling during World War II, with much development work being sponsored by the government from which came the beginning of the various *MIL STD* documents which are in use today.

While there has not been a lot of activity related to multivariate control charts, even less is available for multivariate acceptance sampling. The reason for this is that the specifications for the product will generally consist of a collection of individual specifications for each variable. The intersection of these specifications would form a rectangular solid while the variability would more than likely take the form of an ellipsoid. Jackson and Bradley (1961) considered this problem for some sequential sampling procedures. Their example consisted of a three-variable problem, the variables being related to various characteristics of the performance of a ballistic missile. Jackson and Bradley tried both inscribing and circumscribing ellipsoids proportional to the inherent variability within and about this rectangular solid. The former procedure would be too stringent; the latter, too lenient. Presumably, some optimum might be obtained, but no work has been done along this line to

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date. Shakun (1965) considered specifications which were also in the form of ellipsoids which had the effect of reducing both the specifications and the test statistic to quadratic forms. Jackson and Bradley (1966) later suggested using principal components in which case the specifications could be made on the principal components themselves. This would be possible only if the principal components were easily interpretable and readily understandable by the people responsible for setting the specifications. This is approximately the state of affairs twenty years later, and presents a fertile field of opportunity for research and development.

XI. WHAT THE FUTURE HOLDS

In view of the resurgence of interest in statistical quality control, and the increased facilities available to incorporate multivariate procedures into it, the future should provide an exciting atmosphere for research and development in these activities.

As discussed in the previous section, the area of multivariate acceptance sampling is virtually undeveloped. The main effort here should be directed towards the construction of multivariate specifications which can be more readily translated into sampling plans; once this is done, the actual multivariate tests required will probably entail the use of statistics which have already been developed.

In the area of process control the field is just as wide open. There has been no work done on multivariate acceptance control charts because it involves some of the same problems as acceptance sampling

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itself. Apart from some work by Montgomery and Klatt (1972a and 1972b) and Alt and Deutsch (1978), there seems to have been little done in the area of cost considerations for multivariate control charts. The work of Jackson and Mudholkar on residual analysis related to principal components produced only large sample estimates. While these may be adequate for most QC applications, we do not know for sure, and it would be nice to have the exact distributions anyway. The same holds for T^2_D. These are just a few of the many problems still to be worked on.

As important, for the moment however, is more actual application of the methods already available to real life situations. This should produce more quality control engineers who can handle concepts of multivariate analysis, and at the same time produce more multivariate-minded statisticians who are knowledgeable about the fundamentals of quality control. When this happens, R&D teams made up of both groups of people will be formed to further the efforts of this important topic.

XI. BIBLIOGRAPHY

Alt, F.B. and Deutsch, S.J., (1978). Multivariate Economic Control Charts for the Mean, Proc. Seventh Annual Meeting NE Reg. Con. Amer. Inst. for Dec. Sci., June 1978, 109-112.

Andrews, D.F., (1972). Plots of High-Dimensional Data, Biometrics, 28, 125-136.

Freund, R.A., (1960). A Reconsideration of the Variables Control Chart with Special Reference to the Chemical Industries, Ind. Qual. Control, 16 No. 11, May 1960, 35-41.

Hotelling, H., (1931). The Generalization of Student's Ratio, Ann. Math. Stat., 2, 360-378.

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Hotelling, H., (1933). Analysis of a Complex of Statistical Variables into Principal Components, J. Educ. Psych., 24, 417-441, 489-520.

Hotelling, H., (1947). Multivariate Quality Control, Techniques of Statistical Analysis, C. Eisenhart, M. Hastay and W.A. Wallis, Editors, New York: McGraw-Hill, 111-184.

Jackson, J.E., (1959). Quality Control Methods for Several Related Variables, Technometrics, 1, 359-377.

Jackson, J.E., (1980). Principal Components and Factor Analysis: Part I-Principal Components, J. Qual. Tech., 12, 201-213.

Jackson, J.E., (1981a). Principal Components and Factor Analysis: Part II-Additional Topics Related to Principal Components, J. Qual. Tech., 13, 46-58.

Jackson, J.E., (1981b). Principal Components and Factor Analysis: Part III-What is Factor Analysis?, J. Qual. Tech., 13, 125-130.

Jackson, J.E. and Bradley, R.A., (1961). Sequential χ^2 and T^2 Tests and Their Application to an Acceptance Sampling Problem, Technometrics, 3, 519-534.

Jackson, J.E. and Bradley, R.A., (1966). Sequential Multivariate Procedures with Quality Control Applications, Multivariate Analysis I, P.R. Krishnaiah, Editor, New York: Academic Press, 507-519.

Jackson, J.E. and Mudholkar, G.S., (1979). Control Procedures for Residuals Associated with Principal Components, Technometrics, 21, 341-349.

Kulkarmi, S.R. and Paranjape, S.R., (1984). Use of Andrew's Function Plot Technique to Construct Control Curves for Multivariate Process. Comm. Stat.- Theory and Methods, 13, 2511-2533.

Montgomery, D.C. and Klatt, P.J., (1972a). Economic T Design of T^2 Control Charts to Maintain Current Control of a Process, Man. Sci., 19, 76-89.

Montgomery, D.C. and Klatt, P.J., (1972b). Minimum Cost Multivariate Quality Control Tests, AIIE Transactions, 4, 103-110.

Downloaded by [Tsinghua University] at 19:30 18 May 2016

Pearson, K., (1901). On Lines and Planes of Closest Fit to Systems of Points in Space, Philosophical Magazine, Series 6, 2, 559-572.

Shakun, M.F., (1965). Multivariate Acceptance Sampling Procedures for General Specification Ellipsoids. J. Amer. Stat. Assn., 60, 905-913.

Schilling, E.G., (1982). Acceptance Sampling in Quality Control, New York: Marcel Dekker, 169-173.

Woodall, W.H. and Ncube, M.M., (1984). Given at ASQC-ASA Fall Technical Conference, London, Ontario, Canada.

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