Poincaré gauge theory and Chern-Simons-Witten gravity

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1 Poincaré gauge theory

Poncaré gauge theory (PGT) aims to describe the structure of Einsteinian relativity in close analogy with Yang-Mills theory as a principal G-bundle where the stucture group is that of Poincaré symmetries. Gravity arises from the Lie algebra valued connection 1-form on the principal bundle whose components associated to the translation and Lorentz sub-algebras can be related to the first order quantities in Einstein-Cartan theory: the vielbein and spin connection.

1.1 Bundle structure

Tresguerres *et al.* assert that the appropriate bundle formulation is that of a composite bundle where the total space projects into an intermediate base space whose fibres are the coset of the Poincaré group with its Lorentz subgroup [Tre12; TT05; Jul+94]

$$\begin{tikzcd} P \arrow[d, "\pi_{P\Sigma}"] \arrow[dd, bend right, "\pi_{PM}"'] \\ \Sigma \arrow[d, "\pi_{\Sigma M}"] \\ M \end{tikzcd}$$

Roughly speaking, P is a total space projecting into Σ which is fibred by H=Lorentz. Σ has the form of the base space fibred by the translation group manifold which is isomorphic to Minkowski space-time. Formally

- $P \xrightarrow{\pi_{PM}} M$ is a principal G-bundle with G=Poincaré
- $P \xrightarrow{\pi_{P\Sigma}} \Sigma$ is a principal H-bundle with H=Lorentz
- $\Sigma \xrightarrow{\pi_{\Sigma M}} M$ is the P-associated bundle with typical fibre G/H=translations and structure group G

By choosing a local identity section on the entire bundle, the principal connection 1-form on P can pulled back to the base space to give a Yang-Mills-type field. This section can be decomposed across the two regions of the bundle $\sigma_{MP} = \sigma_{\Sigma P} \circ \sigma_{M\Sigma}$

$$\begin{tikzcd} P \arrow[d, "\pi_{P\Sigma}"'] \arrow[dd, bend right, "\pi_{PM}"'] \\ \Sigma \arrow[u, bend right, "\sigma_{\Sigma P}"'] \arrow[d, "\pi_{\Sigma M}"'] \\ M \arrow[u, bend right, "\sigma_{M\Sigma}"'] \end{tikzcd}$$

The choice of local sections also allows for local trivialisations of the bundle spaces

$$
\begin{tikzcd}
P \arrow[d, "\pi_{P\Sigma}"] & & U \times G \arrow[ll, "s_{MP}"'] \arrow[dd, "\pi^{(1)}_{PM}"'] \\
\Sigma \arrow[u, bend left, "\sigma_{\Sigma P}"] \arrow[d, "\pi_{\Sigma M}"] & U \times G/H \arrow[l, "s_{M\Sigma}"] \arrow[ul, "\sigma_\xi = \sigma_{\Sigma P} \circ s_{M\Sigma}"'] \arrow[d, "\pi^{(1)}_{\Sigma M}"] & \\
U \arrow[u, bend left, "\sigma_{M\Sigma}"] \arrow[uu, bend left=60, "\sigma_{MP}"] & U \arrow[l, "\mathrm{id}_M"] & U \arrow[l, "\mathrm{id}_M"]
\end{tikzcd}
$$

Hence the principal connection 1-form on P which is related to the YM field on M as

$$
\begin{aligned}
\Omega &= \tilde{g}^{-1} \left(d_P + {\pi_{PM}}^* A_M \right) \tilde{g} \\
&= \Lambda^{-1} (d_P + {\pi_{P\Sigma}}^* \Gamma_\Sigma) \Lambda
\end{aligned}
\tag{1}
$$

with

$$
\Gamma_\Sigma = t^{-1} \left(d_\Sigma + {\pi_{\Sigma M}}^* A_M \right) t \tag{2}
$$

We identify the form of the Yang-Mills field on the base manifold as

$$
A_M = {\sigma_{MP}}^* \Omega = -i \left(\overset{(T)}{\Gamma}{}^a_\mu \, \boldsymbol{P}_a + \omega^{ab}_\mu \boldsymbol{J}_{ab} \right) dx^\mu \tag{3}
$$

writing the element $t \in G/H$ as $t = e^{-i\xi^a \boldsymbol{P}_a}$ we write Γ_Σ as

$$
\begin{aligned}
\Gamma_\Sigma &= -i d_\Sigma \xi^a \boldsymbol{P}_a - i {\pi_{\Sigma M}}^* \left[\left(\xi^b {\omega_b}^a + \overset{(T)}{\Gamma}{}^a \right) \boldsymbol{P}_a + \omega^{ab} \boldsymbol{J}_{ab} \right] \\
&= -i \left(\theta^a_\Sigma \boldsymbol{P}_a + {\pi_{\Sigma M}}^* \omega^{ab} \boldsymbol{J}_{ab} \right)
\end{aligned}
\tag{4}
$$

The useful result of this procedure is the definition of an $\mathfrak{iso}(d-1,1)$ valued gauge field of the form

$$
\boldsymbol{A}_\mu = \left(e^a_\mu - D_\mu \xi^a \right) \boldsymbol{P}_a + \omega^{ab}_\mu \boldsymbol{J}_{ab} \tag{5}
$$

and associated curvature form

$$
\begin{aligned}
\boldsymbol{F} &:= d\boldsymbol{A} + \boldsymbol{A} \wedge \boldsymbol{A} \\
&= \left(T^a[e,\omega] - {R^a}_b[\omega] \xi^b \right) \boldsymbol{P}_a + R^{ab}[\omega] \boldsymbol{J}_{ab}
\end{aligned}
\tag{6}
$$

1.2 Relating translational gauge potentials to vielbeins

In many works considering gauged Poincaré treatments of gravity, particularly those concerning Chern-Simons-Witten gravity where the primary focus is on the quantisation of the theory, authors identify the components of the gauge field associated with the translation algebra with the vielbein. However, this identification requires further consideration when examining the gauge transformations of the connection if we wish to be consistent with the first-order variables of Einstein-Cartan gravity. In this framework, the invertibility of the vielbein which allows it to correspond to a valid metric is taken as a postulate, and e^a and ω^{ab} transform only on the Lorentz sub-group of Poincaré. Turning to a theory with the full Poincaré group as a local symmetry, we can see that under a generic transformation $(\Lambda(x), \rho(x)) \in ISO(d-1,1)$, the translational potential will become

$$\overset{(T)}{\Gamma^a} \longrightarrow \overset{(T)}{\Gamma'^a} = \Lambda^a{}_b \overset{(T)}{\Gamma^b} - \Lambda^a{}_b \Lambda_{cd} \omega^{cd} \rho^c + \lambda_b{}^c d\Lambda^a{}_c \rho^b - d\rho^a \tag{7}$$

Clearly, $\overset{(T)}{\Gamma^a}$ has additional transformations when ρ is non-zero which should not affect the vielbein e^a. If we define $\overset{(T)}{\Gamma^a} = e^a - f^a$ where $f^a := D^\omega \xi^a = d\xi^a + \omega^a{}_b \xi^b$ and ξ^a transform as coordinates

$$\xi^a \longrightarrow \xi'^a = \Lambda^a{}_b \xi^b + \rho^a \tag{8}$$

then

$$\begin{aligned} f \rightarrow & d(\Lambda\xi + \rho) + (\Lambda\omega\Lambda^{-1} + \Lambda d\Lambda^{-1})(\Lambda\xi + \rho) \\ = & d\Lambda\xi + \Lambda d\xi + d\rho + \Lambda\omega\xi - d\Lambda\xi + \Lambda\omega\Lambda^{-1}\rho + \Lambda d\Lambda^{-1}\rho \\ = & \Lambda f + \Lambda\omega\Lambda^{-1}\rho + \Lambda d\Lambda^{-1}\rho + d\rho \end{aligned} \tag{9}$$

Now, the transformation of $\overset{(T)}{\Gamma^a}$ will be

$$\overset{(T)}{\Gamma^a} \longrightarrow \overset{(T)}{\Gamma'^a} = e'^a - f'^a = \Lambda(e - f) - \Lambda\omega\Lambda^{-1}\rho - \Lambda d\Lambda^{-1}\rho - d\rho \tag{10}$$

which gives exactly the correct transformation for e^a. This is exactly the non-linear Stückelberg trick. We have introduced an explicit Poincaré gauge invariance at the cost of having to define $d+1$ new coordinate fields. Now the true vielbein will remain invertible and preserve the Lorentzian metric under gauge transformations.

2 Chern-Simons-Witten theory

While in arbitrary dimensions it is not immediately possible to build gravity actions using the Poincaré connection which are analogous to that of general relativity, in 2+1 dimensions a special case emerges. Writing a Chern-Simons type action for $\boldsymbol{A}$, the application of the special invariant inner product on the $\mathfrak{iso}(2,1)$ Lie algebra reveals that the action is identical to the Einstein-Cartan action of the same dimensionality. The action is

$$S_{CS}^{\mathrm{ISO}(2,1)} = \frac{1}{2}\int_M \left\langle \mathbf{A} \wedge d\mathbf{A} + \frac{2}{3}\mathbf{AAA} \right\rangle \tag{11}$$

and the algebra $[\cdot,\cdot]$ and invariant inner product on the algebra $\langle\cdot,\cdot\rangle$ are[1]

$$\left[\boldsymbol{J}^a, \boldsymbol{J}^b\right] = \epsilon^{ab}{}_c \boldsymbol{J}^c \qquad \left[\boldsymbol{J}^a, \boldsymbol{P}^b\right] = \epsilon^{ab}{}_c \boldsymbol{P}^c \qquad \left[\boldsymbol{P}^a, \boldsymbol{P}^b\right] = 0 \tag{13}$$

$$\langle \boldsymbol{J}_a, \boldsymbol{J}_b \rangle = 0 \qquad \langle \boldsymbol{J}_a, \boldsymbol{P}_b \rangle = \eta_{ab} \qquad \langle \boldsymbol{P}_a, \boldsymbol{P}_b \rangle = 0 \tag{14}$$

[1] A convenient curiosity of 2+1 dimensions is that it allows the dualisation of 2-index antisymmetric tensors using the 3-index Levi-Civita tensor. Therefore the following conventional relationships are given:

$$\omega^a = \frac{1}{2}\epsilon^a{}_{bc}\omega^{bc}, \quad \omega^{ab} = -\epsilon^{ab}{}_c \omega^c \tag{12}$$

Varying with respect to $\boldsymbol{A}$, the equations of motion are $\boldsymbol{F} = 0$. Comparing with (6) we see that these equations of motion are Cartan's two structure equations, corresponding to the vanishing of the Riemann and torsion tensors. This is the well-know Chern-Simons-Witten (CSW) theory [Wit89; Wit88; Dun99]. With the inclusion of a negative cosmological constant, the relevant $\mathfrak{so}(2,2)$ AdS$_3$ algebra can be realised by two $SL(2,\mathbb{C})$ gauge fields

$$\boldsymbol{A} = -\left(i\omega^a + \frac{1}{\ell}e^a\right)\boldsymbol{L}_a \qquad \overline{\boldsymbol{A}} = -\left(i\omega^a - \frac{1}{\ell}e^a\right)\overline{\boldsymbol{L}}_a \tag{15}$$

To see the equivalence of the CSW action to that of Einstein-Cartan, we expand $\boldsymbol{A}$ in its generators[2]:

$$\begin{aligned} S_{CS}^{\mathrm{ISO}(2,1)} &= \frac{1}{2}\int_M \mathrm{tr}\Big[\left(e^a\boldsymbol{P}_a + \omega^a\boldsymbol{J}_a\right)\left(de^b\boldsymbol{P}_b + d\omega^b\boldsymbol{J}_b\right) \\ &\qquad + \frac{2}{3}\left(e^a\boldsymbol{P}_a + \omega^a\boldsymbol{J}_a\right)\left(e^b\boldsymbol{P}_b + \omega^b\boldsymbol{J}_b\right)\left(e^c\boldsymbol{P}_c + \omega^c\boldsymbol{J}_c\right)\Big] \\ &= \frac{1}{2}\int_M \eta_{ab}\left(e^a d\omega^b + \omega^a de^b\right) + \epsilon_{abc}e^a\omega^b\omega^c \\ &= \int_M e^a\left(d\omega_a + \frac{1}{2}\epsilon_{abc}\omega^b\omega^c\right) \end{aligned} \tag{16}$$

3 Point particle coupling in Poincaré gravity

3.1 This time...?

In [BBS78; Bal+17], Balachandran *et al.* formulate Lagrangians which yield Wong's equations upon variation by attaching an element of a faithful group representation $\tilde{g} \in \mathscr{R}(G)$ to a point particle worldline as a dynamical degree of freedom. The action is then a Lie algebra inner product of the extended Maurer-Cartan form $\tilde{g}^{-1}D\tilde{g}$ with a constant algebra element K which provides constraints defining the representation of the group via its Casimir invariants.

The Lagrangian coupling a point particle charged under a gauge group to a connection form gauging the symmetry is given by

$$\int d\tau\left\langle K, \tilde{g}^{-1}(\tau)\left(d_\tau + A_\tau\right)\tilde{g}(\tau)\right\rangle = \int_\gamma \left\langle I[\tilde{g}], A - \tilde{g}\dot{\tilde{g}}^{-1}\right\rangle = \int_\gamma \left\langle I[\tilde{g}], A - j[\tilde{g}, \dot{\tilde{g}}]\right\rangle \tag{17}$$

- $\tilde{g} \in G$ is the Lie group valued internal variable which supplies the dynamics to the particle.
- $K \in \mathfrak{g}$ defines the irreducible representation of the particle by fixing the Casimir invariants (is is where Hamiltonian constraints are imposed).
- $I = \tilde{g}K\tilde{g}^{-1} \in \mathfrak{g}$ is the more familiar "colour vector" of Wong's equations
- $j = \tilde{g}d_\tau\tilde{g}^{-1} \in \Omega^1(\gamma) \otimes \mathfrak{g}$ is a Lie algebra valued 1-form corresponding to the chiral current of the non-linear sigma model

Note that $dI = [I, j]$.

Varying w.r.t. $\tilde{g}$ gives the equation of motion for I, the second of Wong's equations

$$[I(\tau), A_\mu(\tau) - j_\mu(\tau)]\dot{x}^\mu = 0 \Leftrightarrow \dot{I}(\tau) + [A_\mu(\tau), I(\tau)]\dot{x}^\mu = 0 \tag{18}$$

This is essentially a parallel transport equation for I along the worldline of the particle.

[2]N.B. here I have left out the extra term $-\int_M D_\omega\phi^a R_a[\omega]$. Assuming that integration by parts with vanishing boundary contributions is valid this should vanish (I hope).

3.1.1 $(d+1)\times(d+1)$-dimensional matrix representation of the Poincaré group

If we consider now specifically the Poincaré group, we can use the $(d+1)\times(d+1)$ dimensional matrix representation to examine this coupling term. Now a group element (Λ, q) is represented as

$$(\Lambda, q) = \begin{pmatrix} \Lambda^i{}_j & q^i \\ 0 & 1 \end{pmatrix} \qquad (\Lambda, q)^{-1} = \begin{pmatrix} (\Lambda^{-1})^i{}_j & -(\Lambda^{-1})^i{}_k q^k \\ 0 & 1 \end{pmatrix} \tag{19}$$

and an algebra element $X = \overset{(L)}{X}{}^{ab}\, \boldsymbol{J}_{ab} + \overset{(T)}{X}{}^{a}\, \boldsymbol{P}_a$ as

$$X = \begin{pmatrix} \overset{(L)}{X}{}^{ab}\, (\boldsymbol{J}^{\text{vec}}_{ab})^i{}_j & \overset{(T)}{X}{}^{a}\, (\boldsymbol{P}^{\text{vec}}_a)^i \\ 0 & 0 \end{pmatrix} \tag{20}$$

Expressing the coupling Lagrangian in this language we have

$$\int_\gamma \left\langle \begin{pmatrix} \overset{(L)}{K}{}^{ab}\, \Lambda^{-1} \boldsymbol{J}^{\text{vec}}_{ab} \Lambda & \overset{(T)}{K}{}^{a}\, \Lambda^{-1} \boldsymbol{P}_a + \overset{(L)}{K}{}^{ab}\, \Lambda^{-1} \boldsymbol{J}^{\text{vec}}_{ab} q \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2}\omega^{ab} \boldsymbol{J}^{\text{vec}}_{ab} & e^a \boldsymbol{P}^{\text{vec}}_a \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} \Lambda^{-1} d\Lambda & \Lambda^{-1} dq \\ 0 & 0 \end{pmatrix} \right\rangle \tag{21}$$

If we focus on scalar particles which transform trivially under Lorentz, we can make Λ non-dynamical and fix a gauge in which it is the identity

$$\int_\gamma \left\langle \begin{pmatrix} \overset{(L)}{K}{}^{ab}\, (\boldsymbol{J}^{\text{vec}}_{ab})^i{}_j & \left(\overset{(T)}{K}{}^{a} + \overset{(L)}{K}{}^{bc}\, (\boldsymbol{J}^{\text{vec}}_{bc})^a{}_k q^k \right) (\boldsymbol{P}^{\text{vec}}_a)^i \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2}\omega^{ab} (\boldsymbol{J}^{\text{vec}}_{ab})^{i'}{}_{j'} & (e^a - dq^a)\, (\boldsymbol{P}^{\text{vec}}_a)^{i'} \\ 0 & 0 \end{pmatrix} \right\rangle \tag{22}$$

In 2+1 dimensions

$$\langle J_{ab}, P_c \rangle = \epsilon_{abc} \tag{23}$$

therefore

$$\begin{aligned} &\int_\gamma \left[\overset{(L)}{K}{}^{ab}\, (e^c - dq^c)\, \epsilon_{abc} + \frac{1}{2}\omega^{ab} \left(\overset{(T)}{K}{}^{c} + \overset{(L)}{K}{}^{fg} (\boldsymbol{J}^{\text{vec}}_{fg})^c{}_k q^k \right) \epsilon_{abc} \right] \\ =& \int_\gamma \left[2\, \overset{(L)}{K}_{a}\, (e^a - dq^a) + \frac{1}{2}\epsilon_{abc}\omega^{ab} \left(\overset{(T)}{K}{}^{c} + 2\epsilon^c{}_{de}\, \overset{(L)}{K}{}^{d}\, q^e \right) \right] \\ =& \int_\gamma \left[2\, \overset{(L)}{K}_{a}\, (e^a - dq^a) + \frac{1}{2}\epsilon_{abc}\omega^{ab}\, \overset{(T)}{K}{}^{c} - 2\, \overset{(L)}{K}_{a}\, \omega^a{}_b q^b \right] \\ =& -2 \int_\gamma \left[\overset{(L)}{K}_{a} \left(\dot{q}^a + \omega^a{}_b q^b - e^a \right) + \frac{1}{4}\omega^{ab}\, \overset{(T)}{K}_{ab} \right] \end{aligned} \tag{24}$$

which can be compared to the Poincaré-gauged worldline action from before. Note that the components of K corresponding to the translation and Lorentz algebras have been dualised. Given its origin, this action in certainly invariant, although we may question the meaning of the extended gauge transformations of $\overset{(L)}{K}_{a}$ and $\overset{(T)}{K}_{ab}$.

In 3+1 dimensions

The invariant bilinear form on the $\mathfrak{iso}(3,1)$ algebra is the mass-squared Casimir

$$\langle \boldsymbol{P}_a, \boldsymbol{P}_b \rangle = \eta_{ab}, \qquad \langle \boldsymbol{J}_{ab}, \boldsymbol{X} \rangle = 0 \quad \forall\, \boldsymbol{X} \in \mathfrak{iso}(3,1) \tag{25}$$

so the action will reduce to

$$\int_\gamma \left[\overset{(T)}{K}_a \left(\Lambda^{-1}\right)^a{}_b \left(dq^b + \frac{1}{2}\omega^{cd} (\boldsymbol{J}^{\text{vec}}_{cd})^b{}_e q^e + e^b \right) \right] \tag{26}$$

The only fully invariant bilinear form on the $\mathfrak{iso}(3,1)$ algebra (as far as I can tell) is:

$$\langle \boldsymbol{J}_{ab}, \boldsymbol{J}_{cd} \rangle = \eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc} \qquad \langle \boldsymbol{P}_a, \boldsymbol{X} \rangle = 0 \tag{27}$$

So the action will reduce to

$$\int_\gamma \left[\overset{(L)}{K}_{ab}\, \omega^{ab} \right] \tag{28}$$

Certainly it is still possible to write an action for a *massless* point particle with a local Poincaré symmetry[3] but I don't know how it could be formulated in this language or how one would maintain gauge invariance while introducing spin degrees of freedom. Possibly by replacing q^a with a Grassmann odd variable and considering the $q_a \omega^a{}_b q^b$ to be a coupling to a spin tensor $S_{ab} = q_a q_b$... drifting into worldline supersymmetry territory here.

3.2 Coupling point particles by gauging the Poincaré symmetry

If we consider the action for a spinless point particle on a worldline in flat Minkowski spacetime

$$S_{pp} = \int d\tau p_a(\tau)\partial_\tau q^a(\tau) - \lambda(\tau) \left(p_a(\tau) p_b(\tau) \eta^{ab} - m^2 \right) \tag{29}$$

we see that it has a global Poincaré symmetry under transformations of the form

$$\begin{aligned} q^a &\to \Lambda^a{}_b q^b + \rho^a \\ p_a &\to \Lambda_a{}^b p_b \end{aligned} \tag{30}$$

making these transformations local, it becomes necessary to introduce connections to maintain the symmetry

$$S_{pp} = \int d\tau \Lambda_a{}^b p_b(\tau)\partial_\tau q^a(\tau) - \lambda(\tau) \left(p_a(\tau) p_b(\tau) \eta^{ab} - m^2 \right) \tag{31}$$

4 Gravitational Wilson lines in the PGT language

The gravitational Wilson loop in a metric formulation of gravity is understood to be a phase related to the pathlength of a massive test particle

$$\Phi_g = \exp im \oint_\gamma d\tau \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} \tag{32}$$

We can also consider the Wilson loops of the Poincaré gauge field

$$\Phi_{\boldsymbol{A}_{\text{grav}}} = \mathrm{tr}_{\mathscr{R}} \mathcal{P} \exp \left[i \oint_\gamma d\tau \left(e^a \boldsymbol{P}_a + \omega^{ab} \boldsymbol{J}_{ab} \right) \right] \tag{33}$$

[3] $S = \int p_a \left(\dot{q}^a + \omega^a{}_b q^b + e^a \right) - \mathscr{H}_0$

We wish to show that $\Phi_{\boldsymbol{A}_{\rm grav}}$ is equal to Φ_g when the appropriate representation is chosen to trace over.

It is claimed by [ACI13; Wit89] and shown for SU(N) by David Tong in his lecture notes on gauge theory that an equivalent way to construct the Wilson loop with a trace in a given representation $\mathcal{R}$ is to exponentiate an action for a test particle with a symmetry gauged by the connection and path integrate out the test particle's degrees of freedom. The path integral corresponds to the trace, and the representation is the Hilbert space of the quantised test particle.

$$W[\gamma] = \int [\mathcal{D}U] e^{iS[U;\boldsymbol{A}]} \tag{34}$$

This technique has been exploited to produce the desired result in the case of AdS_3 CSW gravity by [ACI13; CSZ20]

4.1 Tracing via path integral on a probe particle action

Taking the point particle coupling action in the spinless case and including a Hamiltonian constraint with a Lagrange multiplier λ

$$S_{\rm pp}[q, p, \lambda; A] = \int_\gamma p_a \left(dq^a + \omega^a{}_b q^b + e^a\right) - \lambda \left(p^a p_a - m^2\right) \tag{35}$$

We identify the Wilson loop with[Wit89]

$$W[\gamma] = \int [\mathcal{D}q\, \mathcal{D}p\, \mathcal{D}\lambda] \exp[iS_{\rm pp}(q, p, \lambda; \boldsymbol{A})] \tag{36}$$

and use saddle point approximation to evaluate it in the classical regime, using equation of motion to place the action in the exponential on-shell.

Varying with respect to q^a we have

$$\dot{p}_a - p_b \omega^b_{\tau\, a} = 0 \tag{37}$$

and substituting into the action

$$\begin{aligned} W[\gamma] &= \int [\mathcal{D}p\mathcal{D}\lambda] \exp i \oint_\gamma d\tau \left[p_a e^a_\tau - \lambda \left(p_a p^a - m^2\right)\right] \\ &= \int [\mathcal{D}\lambda] \exp i \oint_\gamma d\tau \left[\frac{1}{4\lambda} e^a_\tau e^b_\tau \eta_{ab} + \lambda m^2\right] \\ &= \exp im \oint_\gamma d\tau \sqrt{\dot{x}^\mu \dot{x}^\nu e^a_\mu e^b_\nu \eta_{ab}} = \exp im \oint_\gamma d\tau \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} \end{aligned} \tag{38}$$

5 Translating metric solutions to Einstein's gravity into the PGT language

Working at the level of solutions to the field equations, we can take existing vacuum metric solutions to the EFEs and determine the form of the PGT gauge field to which they correspond. By translating metric configurations with known single copies and examining the PGT gauge field side by side with the YM single copy gauge field, it may be possible to see further and more geometrically revealing structure within corresponding solutions.

The process for this is as follows:

1. The **vielbein** is computed by finding a solution to the identity

$$\eta(\boldsymbol{e}_\mu, \boldsymbol{e}_\nu) = g_{\mu\nu} \tag{39}$$

   In the special case of fully diagonal metrics, the vielbein can be read off as the square roots of the eigenvalues of $g_{\mu\nu}$.

   Note that the freedom of e^a under Lorentz rotations means that there will in general be a family of equivalent solutions to this equation.

2. The **spin connection** follows as the solution to Cartan's torsion-free structure equation

$$de^a + \omega^a{}_b \wedge e^b = 0 \tag{40}$$

   In many cases, especially in dimensions higher than 2+1 where $\omega^a{}_b$ takes on upwards of 6 components, it is easier to calculate the spin connection using the metric compatible Christoffel connection

$$\omega_\mu{}^a{}_b = e^a_\alpha \Gamma^\alpha_{\mu\nu} (e^{-1})^\nu_b - \partial_\mu e^a_\alpha (e^{-1})^\alpha_b \tag{41}$$

3. Now the **PGT gauge field** is the combination

$$\boldsymbol{A}_\mu = e^a_\mu \boldsymbol{P}_a + \omega^{ab} \boldsymbol{J}_{ab} \tag{42}$$

   with $\xi^a = 0$. In the $2+1$-dimensional case this will automatically satisfy the vacuum field equations $\boldsymbol{F}_{\mu\nu} = 0$[4].

   It may be possible from the form of $\boldsymbol{A}_\mu$ to identify a translational gauge transformation $(\mathbf{1}, \rho)$ which brings the PGT gauge field into a more manageable form. Particularly in moving into a Coulomb gauge which allows for easy comparison with existing YM gauge fields of the same form.

5.1 Translating the conical geometry in $2+1$ dimensions

From the conical metric solution of the Einstein equations we can determine the form of the dreibein and spin connection in the Einstein Cartan formulation, and then construct the relevant ISO(2,1) Chern Simons gauge field. Choosing the metric to be in coordinates which span their entire range ($0 \le r \le \infty$, $0 \le \phi < 2\pi$) we have

$$g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + \frac{dr^2}{(1-\alpha)^2} + r^2 d\phi^2, \quad \alpha = 1 - \sqrt{1 - \frac{\kappa}{\pi} M} \tag{43}$$

The dreibein can be read off from the diagonal entries

$$e^a = \begin{pmatrix} dt & \frac{dr}{1-\alpha} & r d\phi \end{pmatrix}^a \tag{44}$$

[4]It would be interesting to see for higher dimensional solutions what properties the PGT field strength possesses. Immediately I would assume it ought to be zero (curvature and torsion free), although the curvature component corresponds to the Riemann tensor and EFE solutions are Ricci flat but not necessarily Riemann flat. . .

and the spin connection can be determined by solving the torsion free condition.

$$
\begin{aligned}
de^t &= 0 && = -\omega^{tr} \wedge e^r - \omega^{t\phi} \wedge e^\phi && (45)\\
de^r &= 0 && = -\omega^{tr} \wedge e^t - \omega^{r\phi} \wedge e^\phi && (46)\\
de^\phi &= dr \wedge d\phi && = -\omega^{t\phi} \wedge e^t + \omega^{r\phi} \wedge e^r && (47)
\end{aligned}
$$

Evidently, $\omega^{tr} = \omega^{t\phi} = 0$

$$
dr \wedge d\phi = \frac{1}{1-\alpha}\omega^{r\phi} \wedge dr \Rightarrow \omega^{r\phi} = (\alpha - 1)d\phi \tag{48}
$$

Then dualising

$$
\omega^a = \frac{1}{2}\epsilon^a{}_{bc}\omega^{bc} = \begin{pmatrix}(1-\alpha)d\phi & 0 & 0\end{pmatrix}^a \tag{49}
$$

The ISO(2,1) gauge field can now be constructed

$$
\mathbf{A} = \boldsymbol{P}_0 dt + \frac{1}{1-\alpha}\boldsymbol{P}_1 dr + r\boldsymbol{P}_2 d\phi + (1-\alpha)\boldsymbol{J}_0 d\phi \tag{50}
$$

or in coordinates

$$
\begin{aligned}
\mathbf{A}_\mu &= \begin{pmatrix}1\\0\\0\end{pmatrix}_\mu \boldsymbol{P}_0 + \begin{pmatrix}0\\ \frac{1}{1-\alpha}\\ 0\end{pmatrix}_\mu \boldsymbol{P}_1 + \begin{pmatrix}0\\0\\r\end{pmatrix}_\mu \boldsymbol{P}_2 + \begin{pmatrix}0\\0\\1-\alpha\end{pmatrix}_\mu \boldsymbol{J}_0\\
&= \begin{pmatrix}\boldsymbol{P}_0\\ \frac{1}{1-\alpha}\boldsymbol{P}_1\\ r\boldsymbol{P}_2 + (1-\alpha)\boldsymbol{J}_0\end{pmatrix}_\mu
\end{aligned} \tag{51}
$$

whose curvature $\mathbf{F} = D\mathbf{A}$ is also flat.

5.2 Point source and Wu-Yang monopole

Starting from the metric

$$
g = -dt^2 + \frac{dr^2}{(1-\alpha)^2} + r^2 d\phi^2 \tag{52}
$$

and converting to Cartesian coodinates we have

$$
g = -dt^2 + \frac{x^2 + y^2(1-\alpha)^2}{r^2(1-\alpha)^2}dx^2 - \frac{2xy}{r}\left(\frac{1}{(1-\alpha)^2} - 1\right)dxdy + \frac{y^2 + x^2(1-\alpha)^2}{r^2(1-\alpha)^2}dy^2 \tag{53}
$$

The dreibein is as follows I hate it

The CSW gauge field solution is thus

$$
\begin{aligned}
\boldsymbol{A}_0(x) &= \boldsymbol{P}_0\\
\boldsymbol{A}_i(x) &= \frac{1}{1-\alpha}\boldsymbol{P}_i + \frac{\alpha}{r^2}\epsilon_{ij}x^j\boldsymbol{J}_0 + \frac{\alpha}{r^2(1-\alpha)}\epsilon_{ij}x^j x^k \epsilon_k{}^\ell \boldsymbol{P}_m
\end{aligned} \tag{54}
$$

If we consider a gauge transformation of the form $b = \exp \xi^a \boldsymbol{P}_a$ where $\xi^a = \left\{t, \frac{x^i}{1-\alpha}\right\}$ then the gauge field becomes

$$
\begin{aligned}
\boldsymbol{A}_0(x) &= 0\\
\boldsymbol{A}_i(x) &= \frac{\alpha}{r^2}\boldsymbol{J}_a\epsilon^a{}_{ij}x^j = \frac{\alpha}{r^2}\boldsymbol{J}_0\epsilon_{ij}x^j \qquad i = 1, 2
\end{aligned} \tag{55}
$$

Compare this to the known Wu-Yang monopole solution in 3+1 dimensions for the SU(2) gauge potential $\boldsymbol{a}_\mu$ at large distances

$$
\begin{aligned}
\boldsymbol{a}_0(x) &= 0 \\
\boldsymbol{a}_i(x) &= \frac{1}{2er^2}\boldsymbol{\sigma}_a \epsilon_{aij} x^j \qquad\qquad i = 1,2,3
\end{aligned}
\tag{56}
$$

5.3 Multipole solutions

The general multipole solution is

$$
ds^2 = -dt^2 + e^{2\psi(x,y)}(dx^2 + dy^2) = -dt^2 + \Phi(x,y)(dx^2 + dy^2) \tag{57}
$$

$$
\psi = -\frac{\kappa}{2\pi}\sum_n m_n \log|r - r_n| + \frac{1}{2}\log C_0^2 \qquad \Phi(x,y) = C_0^2 \prod_n |r - r_n|^{-\frac{\kappa}{\pi}m_n} \tag{58}
$$

plugging into the vacuum Einstein equation, the function ψ must obey the harmonic equation

$$
e^{-2\psi(x,y)}\nabla^2\psi(x,y) = 0 \tag{59}
$$

Taking the vielbein and spin connection as

$$
e^a = \begin{pmatrix} dt \\ e^{\psi(x,y)}dx \\ e^{\psi(x,y)}dy \end{pmatrix} \qquad\qquad \omega^{12} = \frac{\partial\psi(x,y)}{\partial y}dx - \frac{\partial\psi(x,y)}{\partial x}dy \tag{60}
$$

eliminating the spin connection and trying to eliminate the vielbein via a gauged translation, we have the following set of couple PDEs

$$
\begin{aligned}
\frac{\partial X}{\partial x} &= e^{\psi} - Y\frac{\partial\psi}{\partial y} \qquad\qquad \frac{\partial Y}{\partial x} = X\frac{\partial\psi}{\partial y} \\
\frac{\partial X}{\partial y} &= Y\frac{\partial\psi}{\partial x} \qquad\qquad \frac{\partial Y}{\partial y} = e^{\psi} - X\frac{\partial\psi}{\partial x}
\end{aligned}
\tag{61}
$$

$$
\begin{aligned}
\partial_x(X + iY) &= e^{\psi} + i\partial_y\psi(X + iY) \\
\partial_y(X + iY) &= -ie^{\psi} - i\partial_x\psi(X + iY)
\end{aligned}
\tag{62}
$$

differentiating again and substituting yields the following pair of uncoupled Poissony PDEs

$$
\begin{aligned}
\nabla^2 X(x,y) + |\vec{\nabla}\psi(x,y)|^2 X(x,y) &= 2\frac{\partial\psi(x,y)}{\partial x}e^{\psi(x,y)} \\
\nabla^2 Y(x,y) + |\vec{\nabla}\psi(x,y)|^2 Y(x,y) &= 2\frac{\partial\psi(x,y)}{\partial y}e^{\psi(x,y)}
\end{aligned}
\tag{63}
$$

I don't know how to solve them generally :(

$$
|\vec{\nabla}\psi|^2 = \left(\frac{\kappa}{2\pi}\right)^2 \left[\left(\sum_n \frac{m_n(x - x_n)}{|r - r_n|^2}\right)^2 + \left(\sum_n \frac{m_n(y - y_n)}{|r - r_n|^2}\right)^2\right] \tag{64}
$$

$$
\frac{\partial\psi}{\partial x}e^{\psi} = \partial_x\sqrt{\Phi} = -\frac{\kappa}{2\pi}C_0\left(\sum_n m_n \frac{x - x_n}{|r - r_n|^2}\right)\prod_\ell |r - r_n|^{-\frac{\kappa}{2\pi}m_\ell} \tag{65}
$$

5.4 Multipole solution in complex coordinates

The static vacuum solution to the Einstein equations can be written in complex coordinates as

$$ds^2 = -dt^2 + f(z)g(\bar{z})dzd\bar{z} \tag{66}$$

Choosing the ansatz for the vielbein to be

$$e^a = \begin{pmatrix} dt \\ \frac{1}{2}\left(f(z)dz + g(\bar{z})d\bar{z}\right) \\ -\frac{i}{2}\left(f(z)dz - g(\bar{z})d\bar{z}\right) \end{pmatrix} \tag{67}$$

we can see that purely under coordinate transformations

$$\{t, z, \bar{z}\} \longrightarrow \{t, Z(z), \bar{Z}(\bar{z})\} \qquad Z(z) = \int^z f(\zeta)d\zeta,\ \bar{Z}(\bar{z}) = \int^{\bar{z}} g(\bar{\zeta})d\bar{\zeta} \tag{68}$$

$$e^a_{\{t,Z,\bar{Z}\}} = \begin{pmatrix} dt \\ \frac{1}{2}\left(dZ + d\bar{Z}\right) \\ -\frac{i}{2}\left(dZ - d\bar{Z}\right) \end{pmatrix} \tag{69}$$

and

$$\{t, Z, \bar{Z}\} \longrightarrow \{t, X(Z, \bar{Z}), Y(Z, \bar{Z})\} \qquad X(Z, \bar{Z}) = \frac{1}{2}(Z + \bar{Z}),\ Y(Z, \bar{Z}) = -\frac{i}{2}(Z - \bar{Z}) \tag{70}$$

$$e^a_{\{t,X,Y\}} = \begin{pmatrix} dt \\ dX \\ dY \end{pmatrix} \tag{71}$$

the vielbein is in the form

$$e^a_\mu = \frac{\partial\{t, X, Y\}^a}{\partial\{t, z, \bar{z}\}^\mu} \tag{72}$$

i.e. it is holonomic and the spin connection is zero. However, the sacrifice made to make the vielbein explicitly Minkowski is the obfuscation of topological properties. All this information is now held within X and Y through the restriction of the domain on which they are holomorphic in the complex plane.

Alternatively, we can make a coordinate transform back to real coordinates

$$\{t, z, \bar{z}\} \longrightarrow \{t, x, y\} \tag{73}$$

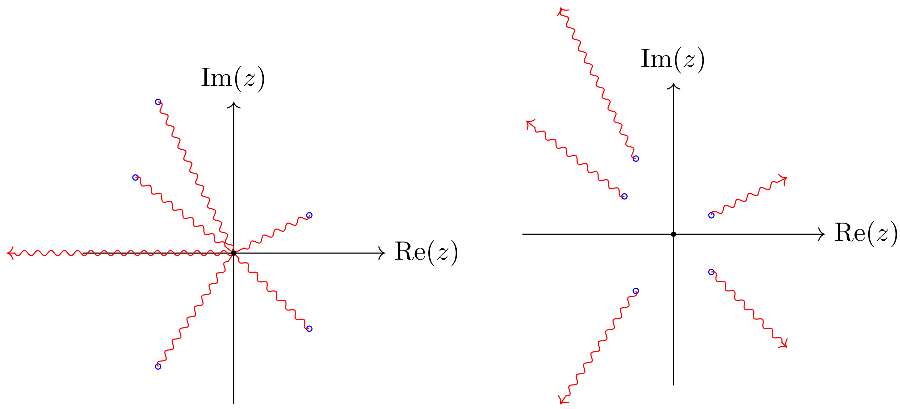
$$e^a_{\{t,x,y\}} = \begin{pmatrix} dt \\ \frac{1}{2}\left[(f + g)dx + i(f - g)dy\right] \\ \frac{1}{2}\left[-i(f - g)dx + (f + g)dy\right] \end{pmatrix} \tag{74}$$

the form of the function $f(z)$ is explicitly

$$\begin{aligned} f(z) &= \mathbb{C}_0 \prod_{n=1}^{N} (z - z_n)^{-\frac{\kappa m_n}{2\pi}} \\ &= \mathbb{C}_0 \prod_{n=1}^{N} |z - z_n|^{-\frac{\kappa m_n}{2\pi}} \exp\left[-i \sum_n \frac{\kappa m_n}{2\pi} \arg_n(z - z_n)\right] \end{aligned} \tag{75}$$

Evidently the first line is multivalued. In the second line, specific branches of the complex logarithm function must be chosen for each $\arg_n(z - z_n)$ to make the function single valued, although it will now be discontinuous. This is done by fixing the domain of $\arg_n$ to be $(\varphi_0 - \pi, \varphi_0 + \pi]$. Giving a specific arrangement of branch cuts is unimportant, but it is this which will encode the topology of the spacetime when coordinates are transformed to be explicitly Minkowski.

Here are two examples of suitable branch cut structures.



With the condition that $(f\bar{(}z)) = g(\bar{z})$, the vielbein is now of the form

$$e^a_{\{t,x,y\}} = \begin{pmatrix} dt \\ \mathbb{C}_0 \prod_{n=1}^N |z - z_n|^{-\frac{\kappa m_n}{2\pi}} \left(\cos\left[\Xi(z)\right] dx + \sin\left[\Xi(z)\right] dy\right) \\ \mathbb{C}_0 \prod_{n=1}^N |z - z_n|^{-\frac{\kappa m_n}{2\pi}} \left(-\sin\left[\Xi(z)\right] dx + \cos\left[\Xi(z)\right] dy\right) \end{pmatrix} \tag{76}$$

$$\Xi(z) = -\sum_n \alpha_n \arg_n z - z_n \tag{77}$$

The discontinuities are all contained in trigonometric functions of $\Xi(z)$, and can be removed by a Lorentz transformation $\Lambda = \exp i\Xi(z)\boldsymbol{J}_0$.

Following this gauged Lorentz transformation, the vielbein is

$$e^a = \begin{pmatrix} dt \\ \mathbb{C}_0 \prod_{n=1}^N |z - z_n|^{-\frac{\kappa m_n}{2\pi}} dx \\ \mathbb{C}_0 \prod_{n=1}^N |z - z_n|^{-\frac{\kappa m_n}{2\pi}} dy \end{pmatrix} \tag{78}$$

which is entirely regular except at the poles z_n. The spin connection is now the pure gauge function

$$(\omega_\mu)^a{}_b = \Lambda^a{}_c \partial_\mu (\Lambda^{-1})^c{}_b = i\partial_\mu \Xi(z)\boldsymbol{J}_0 \tag{79}$$

Meanwhile we can determine the gauged transformation which will eliminate the vielbein by transforming the solution found in holonomic coordinates.

$$\rho^a(t, x, y) = \Lambda^a{}_b \begin{pmatrix} t \\ X(x, y) \\ Y(x, y) \end{pmatrix}^b \tag{80}$$

The final result is that what is left after this manipulation is a pure gauge potential attached to a single Lorentz generator of the form

$$\omega^{ab}_\mu \boldsymbol{J}_{ab} = i\frac{\kappa}{2\pi}\partial_\mu \sum_n m_n \arg(\vec{r} - \vec{r}_n)\boldsymbol{J}_0 \tag{81}$$

Identical to the abelian Chern-Simons gauge field solution (see for example [Dun99]).

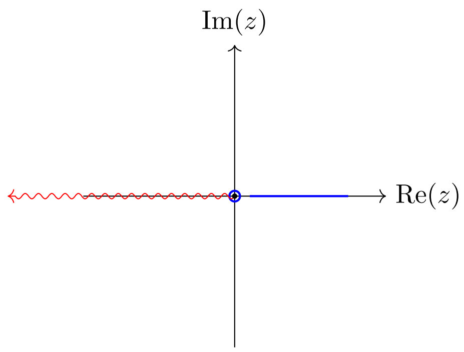
NB that this method assumes *a priori* that the metric is static, all of the point sources follow parallel timelike geodesics wrt one another, there is no winding in time, point sources are spinless.

5.4.1 Monopole solution

Lets try this with a single massive particle at the origin.

$$f(z) = \mathbb{C}_0 |z|^{-\alpha} \exp\left[-i\alpha \arg z\right] \qquad g(\bar{z}) = \mathbb{C}_0 |\bar{z}|^{-\alpha} \exp\left[-i\alpha \arg \bar{z}\right] \tag{82}$$

Choosing the following branch cut arrangement and an integral path as follows



$$\begin{aligned} Z(z) &= \mathbb{C}_0 \int_{\epsilon}^{|z|} dx x^{-\alpha} + i\mathbb{C}_0 |z|^{1-\alpha} \int_0^{\arg z} d\varphi \exp[-i(\alpha-1)\varphi] \\ &= \mathbb{C}_0 \left(\frac{|z|^{1-\alpha} - \epsilon^{1-\alpha}}{1-\alpha} + \frac{|z|^{1-\alpha}}{1-\alpha} \left(e^{i(1-\alpha)\arg z} - 1 \right) \right) \\ &= \mathbb{C}_0 \frac{|z|^{1-\alpha}}{1-\alpha} \left(\cos[(1-\alpha)\arg z] + i \sin[(1-\alpha)\arg z] \right) \end{aligned} \tag{83}$$

Similarly

$$\bar{Z}(z) = \mathbb{C}_0 \frac{|z|^{1-\alpha}}{1-\alpha} \left(\cos[(1-\alpha)\arg z] - i \sin[(1-\alpha)\arg z] \right) \tag{84}$$

Then taking real and imaginary parts

$$X = \mathbb{C}_0 \frac{r^{1-\alpha}}{1-\alpha} \cos[(1-\alpha)\phi] \qquad Y = \mathbb{C}_0 \frac{r^{1-\alpha}}{1-\alpha} \sin[(1-\alpha)\phi] \tag{85}$$

and the presence of the branch cut restricts $-\pi < \phi \leq \pi$

5.4.2 Two pole solutions

Lets set up an example

5.5 Gauging away the vielbein: a general approach

First note the following consistency condition. The gauge transformation purely in the translational sector must satisfy

$$d\xi^a + \omega^a{}_b \xi^b = e^a \tag{86}$$

Taking a second exterior derivative and substituting out $d\xi$

$$de^a + \omega^a{}_b \wedge e^b = R^a{}_b \xi^b \tag{87}$$

Therefore if we expect to have zero torsion, being able to express the vielbein in this form and therefore gauge it away depends on either a) Riemann flatness or b) ξ^a being an eigenvector with zero eigenvalue of all μ, ν components of $\left(R^a{}_b\right)_{\mu\nu}$ simultaneously. N.B. Simultaneous diagonalization of Riemann tensor for Schwarzschild/Taub-NUT is not possible.

To convert the PGT gauge field corresponding to the conical geometry in 2+1 dimensions into a form which was identifiable with the CS monopole it was necessary to identify a gauge transformation which eliminated the components associated with the translation sub-algebra. We can consider this action on a general PGT gauge field to identify if such a transformation generally exists and, if not, what condition must be satisfied to allow it. A suitable transformation ξ^a must satisfy[5]

$$d\xi^a + \omega^a{}_b \xi^b = e^a \tag{89}$$

The solution to the homogeneous equation $D^\omega \xi^a_h = 0$ is given by a holonomy operator

$$\xi^a_h(x) = \hat{\Phi}^a_b(x; x_0)\xi^b_0 \tag{90}$$

where

$$\begin{aligned}
\hat{\Phi}^a_b(x; x_0) &= \hat{\mathcal{P}} \exp\left[-\int_\gamma \omega\right] \\
&= \hat{\mathcal{P}} \exp\left[-\int_{x_0}^{x} \omega_\mu(x') dx'^\mu\right]^a{}_b \\
&= \delta^a_b - \int_{x_0}^{x} dx'^\mu \omega_\mu(x') + \int_{x_0}^{x} dx'^\mu \int_{x_0}^{x'} dx''^\nu \omega_\mu(x')^a{}_c \omega_\nu(x'')^c{}_b \\
&\quad - \int_{x_0}^{x} dx'^\mu \int_{x_0}^{x'} dx''^\nu \int_{x_0}^{x''} dx'''^\rho \omega_\mu(x')^a{}_c \omega_\nu(x'')^c{}_d \omega_\rho(x'')^d{}_b
\end{aligned} \tag{91}$$

The solution to the full inhomogeneous equation can then be written as

$$\xi^a(x) = \Phi^a{}_b(x; x_0)\left[\xi^b_0 + \eta^b(x)\right] \tag{92}$$

substituting into (89)

$$\begin{aligned}
e^a(x) &= d\Phi(x; x_0)\eta(x) + \Phi(x; x_0) d\eta(x) + \omega(x)\Phi(x; x_0)\eta(x) \\
&= \Phi(x; x_0) d\eta(x)
\end{aligned} \tag{93}$$

[5]Note that we could consider a more general transformation including $SO(d-1, 1)$ rotations as well. In this case,

$$e \to \Lambda^{-1}\left(d\xi + \omega\xi + e\right) \stackrel{!}{=} 0 \tag{88}$$

which is an identical constraint, given $\det\Lambda \neq 0$.

$$\left\{\begin{pmatrix} \mathcal{P}_0\sqrt{\frac{r^2-n^2}{n^2+r^2}} \\ \mathcal{P}_1\sqrt{\frac{n^2+r^2}{r^2-n^2}} \\ \mathcal{P}_2\sqrt{n^2+r^2} \\ \mathcal{P}_3\sin(\theta)\sqrt{n^2+r^2}+2n\mathcal{P}_0\cos(\theta)\sqrt{\frac{r^2-n^2}{n^2+r^2}} \end{pmatrix}, \begin{pmatrix} -\frac{n\mathfrak{J}_{23}(n^2-r^2)}{(n^2+r^2)^2} \\ 0 \\ \frac{r\mathfrak{J}_{12}\sqrt{r^2-n^2}}{n^2+r^2} \\ \frac{r\mathfrak{J}_{13}\sin(\theta)\sqrt{r^2-n^2}\,(n^2+r^2)-\mathfrak{J}_{23}\cos(\theta)(n^4-4n^2r^2-r^4)}{(n^2+r^2)^2} \end{pmatrix}\right\}$$

Figure 1: Translational (left) and Lorentz (right) components of the PGT gauge field for Taub-NUT with only NUT charge n

$$\left\{\begin{pmatrix} \mathcal{P}_0\sqrt{1-\frac{rg}{r}} \\ \frac{\mathcal{P}_1}{\sqrt{1-\frac{rg}{r}}} \\ r\,\mathcal{P}_2 \\ r\,\mathcal{P}_3\sin(\theta) \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \mathfrak{J}_{12}\sqrt{1-\frac{rg}{r}} \\ \mathfrak{J}_{13}\sin(\theta)\sqrt{1-\frac{rg}{r}}+\mathfrak{J}_{23}\cos(\theta) \end{pmatrix}\right\}$$

Figure 2: Translational (left) and Lorentz (right) components of the PGT gauge field for Schwarzschild

The remaining equation to be solved is then

$$d\eta^a = (\Phi^{-1})^a{}_b(x;x_0)e^b(x) \tag{94}$$

$$\eta^a(x) = \int \Phi^a{}_b(x_0;x)e^b_\mu dx^\mu \tag{95}$$

The full solution is therefore

$$\xi^a(x) = \Phi^a{}_b(x;x_0)\left[\xi^b_0 + \int_M \Phi^a{}_b(x_0;y)e^b_\mu(y)dy^\mu\right] \tag{96}$$

5.6 PGT gauge fields of 3+1 dimensional Schwarzschild and Taub-NUT BH configurations

Watch tis space...

5.7 Schwarzschild in Kerr-Schild

Starting with the metric in Kerr-Schild coordinates

$$g = \eta_{\mu\nu}(r,\theta) + \kappa\phi(r)k_\mu k_\nu \qquad \phi(r) = \frac{M}{4\pi r},\ k_\mu = \begin{pmatrix}1\\1\\0\\0\end{pmatrix}_\mu \tag{97}$$

The vierbein can immediately be written as

$$e^a_\mu = \tilde{\delta}^a_\mu + \frac{\kappa}{2}\phi(r)k^a k_\mu \qquad \tilde{\delta}^a_\mu = \mathrm{Diag}(1,1,r,r\sin\theta) \tag{98}$$

with k^a raised on the Cartesian Minkowski metric. The spin connection which immediately follows from this vierbein will contain both $\mathcal{O}(1)$ and $\mathcal{O}(\kappa)$ terms, but we can eliminate the $\mathcal{O}(1)$ part from the get-go by first rotating the vierbein

$$\begin{aligned} e' &= R_{23}(\varphi) \cdot R_{12}(\theta) \cdot e \\ R_{23}(\varphi) &= \mathbb{I}_{2\times 2} \oplus \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}, \, R_{12}(\theta) = \mathbb{I}_{1\times 1} \oplus \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \oplus \mathbb{I}_{1\times 1} \end{aligned} \tag{99}$$

Now the vierbein reads

$$e^a = \begin{pmatrix} dt \\ \cos\theta \, dr - r\sin\theta \, d\theta \\ \sin\theta\cos\varphi \, dr + r\cos\theta\cos\varphi \, d\theta - r\sin\theta\sin\varphi \, d\varphi \\ \sin\theta\sin\varphi \, dr + r\cos\theta\sin\varphi \, d\theta - r\sin\theta\cos\varphi \, d\varphi \end{pmatrix} + \frac{\kappa}{2}\phi(r) \begin{pmatrix} -(dt+dr) \\ \cos\theta(dt+dr) \\ \sin\theta\cos\varphi(dt+dr) \\ \sin\theta\sin\varphi(dt+dr) \end{pmatrix} \tag{100}$$

N.B. the very suggestive Cartesian coordinate identities and forward light cone coordinate $du_+ = dt + dr$ The spin connection is

$$\begin{aligned} \omega^a{}_b = \frac{\kappa}{2}\phi(r) &\begin{pmatrix} 0 & \frac{\cos\varphi}{r} & \frac{\sin\theta\cos\varphi}{r} & \frac{\sin\theta\sin\varphi}{r} \\ \frac{\cos\varphi}{r} & 0 & 0 & 0 \\ -\frac{\sin\theta\cos\varphi}{r} & 0 & 0 & 0 \\ -\frac{\sin\theta\sin\varphi}{r} & 0 & 0 & 0 \end{pmatrix} (dt+dr) \\ + \frac{\kappa}{2}\phi(r) &\begin{pmatrix} 0 & \sin\theta \, d\theta & -\cos\theta\cos\varphi \, d\theta + \sin\theta\sin\varphi \, d\varphi & -\cos\theta\sin\varphi \, d\theta - \sin\theta\cos\varphi \, d\varphi \\ \sin\theta \, d\theta & 0 & \cos\varphi \, d\theta - \sin\theta\cos\theta\sin\varphi \, d\varphi & \sin\varphi \, d\theta + \sin\theta\cos\theta\cos\varphi \, d\varphi \\ \cos\theta\cos\varphi \, d\theta - \sin\theta\sin\varphi \, d\varphi & -\cos\varphi \, d\theta + \sin\theta\cos\theta\sin\varphi \, d\varphi & 0 & \sin^2\theta \, d\varphi \\ \cos\theta\sin\varphi \, d\theta + \sin\theta\cos\varphi \, d\varphi & -\sin\varphi \, d\theta - \sin\theta\cos\theta\cos\varphi \, d\varphi & -\sin^2\theta \, d\varphi & 0 \end{pmatrix} \end{aligned} \tag{101}$$

Once again we have an extremely suggestive spin connection, where the $dt + dr$ component looks identical to the field strength tensor of the single-copy gauge field. This is related to the observation made in the appendix of [Ala+21] that due to the null and geodesic properties of k, the spin connection reduces to $(\omega_\mu)_{ab} = \partial_a e_{\mu b} - \partial_b e_{\mu a}$.

The next step we can take is to use translation group gauge transformations to eliminate the $\mathcal{O}(1)$ background components of the vierbein. We identify the following gauge parameters/auxilliary coordinates which will transform the vierbein as $e^a \to e^a - \left(d\xi^a + \omega^a{}_b\xi^b\right)$

$$\xi^a = \begin{pmatrix} t \\ r\cos\theta \\ r\sin\theta\cos\varphi \\ r\sin\theta\sin\varphi \end{pmatrix}^a \tag{102}$$

so that now the vierbein will be

$$e^a = \frac{\kappa}{2}\phi(r) \begin{pmatrix} -2(dt+dr) \\ \frac{r-t}{r}\cos\theta(dt+dr) - (r+t)\sin\theta \, d\theta \\ \frac{r-t}{r}\sin\theta\cos\phi(dt+dr) + (r+t)\left[\cos\theta\cos\phi \, d\theta - \sin\theta\sin\phi \, d\phi\right] \\ \frac{r-t}{r}\sin\theta\sin\phi(dt+dr) + (r+t)\left[\cos\theta\sin\phi \, d\theta + \sin\theta\cos\phi \, d\phi\right] \end{pmatrix} \tag{103}$$

5.8 Kerr-Schild and the kinematical algebra

For a torsion free metric solution, the spin connection can be expressed as

$$(\omega_\mu)_{ab} = \frac{1}{2}\theta^\rho_a\left(\partial_\mu e_{\rho b} - \partial_\rho e_{\mu b}\right) - \frac{1}{2}\theta^\rho_b\left(\partial_\mu e_{\rho a} - \partial_\rho e_{\mu a}\right) - \frac{1}{2}\theta^\rho_a\theta^\sigma_b\left(\partial_\rho e^c_\sigma - \partial_\sigma e^c_\rho\right)e_{\mu c} \tag{104}$$

In a more differential geometry language, this can be written in terms of interior products and Lie derivatives on the vielbein 1-forms along with their inverse vector fields defined by

$$\iota_{\theta_a} e^b = \theta^\mu_a e^b_\mu = \delta^b_a \tag{105}$$

$$(\omega)_{ab} = \frac{1}{2}\left[\mathcal{L}_{\theta_b} e_a - \mathcal{L}_{\theta_a} e_b + \left(\iota_{[\theta_a,\theta_b]} e^c\right) e_c\right] \tag{106}$$

Inserting this into the torsion free equation and noting that in Kerr-Schild the first two terms of (106) are zero[6]

$$de^a + \frac{1}{2}\left(\iota_{[\theta^a,\theta_b]} e^c\right) e_c \wedge e^b \tag{107}$$

The interior product is on the commutator of two vector fields which have a kinematical algebra?

$$[\theta^a, \theta_b] = C^a{}_{bc}\theta^c = \theta^{a\mu}\partial_\mu \theta^\nu_b - \theta^\mu_b \partial_\mu \theta^a \tag{108}$$

Now the torsion free condition becomes

$$\begin{aligned} 0 &= de^a + \frac{1}{2} C^a{}_b{}^c \theta^\alpha_c e^d_\alpha e_d \wedge e^b \\ 0 &= de^a + \frac{1}{2} C^a{}_{bc} e^b \wedge e^c \end{aligned} \tag{109}$$

I think that this can alternatively derived from the Lie derivative of the metric being zero with respect to a Killing field. Choosing a Killing field which is a dual to a vierbein

$$\begin{aligned} \mathcal{L}_{\theta_a} g &= \theta^\alpha_a \partial_\alpha g_{\mu\nu} + \partial_\mu \theta^\alpha_a g_{\alpha\nu} + \partial_\nu \theta^\alpha_a g_{\mu\alpha} \\ &= \theta^\alpha_a \left(\partial_\alpha e^b_\mu e_{\nu b} + \partial_\alpha e^b_\nu e_{\mu b} - \partial_\mu e^b_\alpha e_{\nu b} - \partial_\nu e^b_\alpha e_{\mu b}\right) \\ &= \theta^\alpha_a \left(\partial_{[\alpha} e^b_{\mu]}\, e_{\nu b} + \partial_{[\alpha} e^b_{\nu]}\, e_{\mu b}\right) \end{aligned} \tag{110}$$

in regular degular Kerr-Schild coordinates the commutation relations of the basis fields can be read off as

$$\begin{aligned} &[\boldsymbol{\theta}_0, \boldsymbol{\theta}_1] = -\frac{\kappa\phi}{2r}\left(\boldsymbol{\theta}_0 - \boldsymbol{\theta}_1\right) \\ &[\boldsymbol{\theta}_0, \boldsymbol{\theta}_2] = \frac{\kappa\phi}{2r}\boldsymbol{\theta}_2 \qquad [\boldsymbol{\theta}_1, \boldsymbol{\theta}_2] = -\frac{1}{r}\left(1 - \frac{\kappa}{2}\phi\right)\boldsymbol{\theta}_2 \\ &[\boldsymbol{\theta}_0, \boldsymbol{\theta}_3] = \frac{\kappa\phi}{2r}\boldsymbol{\theta}_3 \qquad [\boldsymbol{\theta}_1, \boldsymbol{\theta}_3] = -\frac{1}{r}\left(1 - \frac{\kappa}{2}\phi\right)\boldsymbol{\theta}_3 \quad [\boldsymbol{\theta}_2, \boldsymbol{\theta}_3] = -\frac{\cos\theta}{r\sin\theta}\boldsymbol{\theta}_3 \end{aligned} \tag{111}$$

The torsion free condition formulated as a field strength implies that e^a is a pure gauge configuration of whatever group it is that $C^a{}_{bc}$ form the structure coefficients of and can be written as $e^a \boldsymbol{T}_a = g^{-1}dg$ [CM21].

By questionably fiddling with the basis[7], I can reduce the algebra to a single commutator giving a the $\mathfrak{g}_{2,1}$ algebra $[\theta^0, \theta^1] = \alpha\theta^1$, meaning that presumably the algebra can be decomposed into $\mathfrak{g}_{\text{kin.}} \simeq \mathfrak{g}_{2,1} \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1)$??? **I don't think this is real or useful**

[6] This is probably obvious as all of the θ^a are Killing vector fields of the metric. Would this then imply that any double copy mechanism stemming from this particular quirk are uniquely confined to maximally symmetric metrics?

[7] I've performed a quasi Gramm-Schmidt basis orthogonalisation with a norm on Minkowksi. This may very well not be real as the vectors $\theta \in \Gamma(TM)$ and to pretend otherwise is Bad.

5.9 Let me try again

Consider the object $\boldsymbol{\theta} = \boldsymbol{\theta}_a dy^a = \theta^\mu \hat{\boldsymbol{\partial}}_\mu = \theta^\mu_a \hat{\boldsymbol{\partial}}_\mu dy^a$ as a connection 1-form on a Minkowski manifold with coordinates $\{y^a\}$ valued in an algebra of diffeomorphisms on a space-time manifold $\mathcal{M}$ with coordinates x^μ. The Lie bracket on the diffeomorphism algebra is given by the Lie derivative

$$[\theta_a, \theta_b] = \mathcal{L}_{\theta_a}\theta_b = (\theta^\nu_a \partial_\nu \theta^\mu_b - \theta^\nu_b \partial_\nu \theta^\mu_a)\, \hat{\boldsymbol{\partial}}_\mu = C_{ab}{}^c \boldsymbol{\theta}_c \tag{112}$$

The curvature of $\boldsymbol{\theta}$ as a connection will be

$$\begin{aligned} \boldsymbol{F}(\boldsymbol{\theta}) &= d\boldsymbol{\theta} + \boldsymbol{\theta} \wedge \boldsymbol{\theta} \\ &= (\partial_a \boldsymbol{\theta}_b - \partial_b \boldsymbol{\theta}_a + \boldsymbol{\theta}_a\boldsymbol{\theta}_b - \boldsymbol{\theta}_b\boldsymbol{\theta}_a)\, dy^a \wedge dy^b \\ &= (\partial_a \theta^\mu_b - \partial_b \theta^\mu_a + [\boldsymbol{\theta}_a, \boldsymbol{\theta}_b]^\mu)\, dy^a \wedge dy^b \otimes \hat{\boldsymbol{\partial}}_\mu \end{aligned} \tag{113}$$

Now including an understanding that θ as a form acts as a map from the tangent space of the Minkowski manifold to that of the spacetime manifold

$$\theta : T\mathbb{M}_4 \longrightarrow T\mathcal{M} \tag{114}$$

I hope it is well defined to identify $\partial_a = \theta^\mu_a \partial_\mu$ which means

$$\begin{aligned} &\boldsymbol{F}[\boldsymbol{\theta}] = 2[\boldsymbol{\theta}_a, \boldsymbol{\theta}_b]^\mu\, dy^a \wedge dy^b \otimes \hat{\boldsymbol{\partial}}_\mu \\ &F^\mu_{ab}[\theta] = 2\theta^\nu_{[a}\partial_\nu \theta^\mu_{b]} \end{aligned} \tag{115}$$

If we now wald general relativity ffffffdffd to the torsion free spin connection, with some rearrangement we can write

$$\begin{aligned} \omega^\mu_{ab} &= \frac{1}{2}[\boldsymbol{\theta}_a, \boldsymbol{\theta}_c]^\nu e_{\nu b}\theta^{\mu c} - \frac{1}{2}[\boldsymbol{\theta}_b, \boldsymbol{\theta}_c]^\nu e_{\nu a}\theta^{\mu c} + \frac{1}{2}[\boldsymbol{\theta}_a, \boldsymbol{\theta}_b]^\mu \\ &= \frac{1}{2}\left(g\left\langle [\boldsymbol{\theta}_a, \boldsymbol{\theta}_c], \boldsymbol{\theta}_b \right\rangle + g\left\langle \boldsymbol{\theta}_a, [\boldsymbol{\theta}_c, \boldsymbol{\theta}_b] \right\rangle\right)\theta^{\mu c} + \frac{1}{2}[\boldsymbol{\theta}_a, \boldsymbol{\theta}_b]^\mu \end{aligned} \tag{116}$$

For Kerr-Schild we find that

$$[\boldsymbol{\theta}_{[a}, \boldsymbol{\theta}_c]^\nu e_{\nu|b]}\theta^{\mu c} = \frac{1}{2}[\boldsymbol{\theta}_b, \boldsymbol{\theta}_c]^\mu \tag{117}$$

5.10 Here is something genuinely interesting

Start from the torsion free equation

$$\partial_\mu e_{\nu c} - \partial_\nu e_{\mu c} + (\omega_\mu)_{cd} e^d_\nu - (\omega_\nu)_{cd} e^d_\mu = 0 \tag{118}$$

If we contract at the end $\theta^\mu_a \theta^\nu_b$ we get to

$$(\partial_a \theta^\rho_b - \partial_b \theta^\rho_a)\, e_{\rho c} = \omega^\rho_{cb} e_{\rho a} + \omega^\rho_{ac} e_{\rho b} \tag{119}$$

Now if we can identify coordinates in which $\tilde{f}_{abc} := \omega^\mu_{ab} e_{\mu c}$ obeys the Jacobi identity, the torsion-free equation imposes that

$$\omega^\mu_{ab} = \partial_a \theta^\mu_b - \partial_b \theta^\mu_a = [\boldsymbol{\theta}_a, \boldsymbol{\theta}_b]^\mu \tag{120}$$

and is equivalent to a “dressed” Abelian field strength tensor for the “connection 1-form” $\boldsymbol{\theta}_a$.

$$\omega^{\mu}_{ab} = e_{a\alpha} g^{\mu\nu} \left(\Gamma^{\alpha}_{\nu\beta} \theta^{\beta}_{b} + \partial_{\nu} \theta^{\alpha}_{\beta} \right) \tag{121}$$

$$\omega^{\mu}_{ab} = g^{\mu\gamma} \theta^{\alpha}_{a} g_{\alpha\sigma} \left(\Gamma^{\sigma}_{\gamma\beta} \theta^{\beta}_{b} + \partial_{\gamma} \theta^{\sigma}_{b} \right) \tag{122}$$

$$\omega^{\mu}_{ab} e_{\mu c} = g_{\alpha\rho} \Gamma^{\rho}_{\beta\gamma} \theta^{\alpha}_{a} \theta^{\beta}_{b} \theta^{\gamma}_{c} + g_{\alpha\beta} \theta^{\alpha}_{a} \theta^{\gamma}_{c} \partial_{\gamma} \theta^{\beta}_{b} \tag{123}$$

$$\begin{aligned} \text{Jacobi}\,[\omega^{\mu}_{ab} e_{\mu c}] = g_{\alpha\rho} \Gamma^{\rho}_{\beta\gamma} &\left(\theta^{\alpha}_{a} \theta^{\beta}_{b} \theta^{\gamma}_{c} + \theta^{\alpha}_{b} \theta^{\beta}_{c} \theta^{\gamma}_{a} + \theta^{\alpha}_{c} \theta^{\beta}_{a} \theta^{\gamma}_{b} \right) \\ &+ g_{\alpha\beta} \left(\theta^{\alpha}_{a} \theta^{\gamma}_{c} \partial_{\gamma} \theta^{\beta}_{b} + \theta^{\alpha}_{c} \theta^{\gamma}_{b} \partial_{\gamma} \theta^{\beta}_{a} + \theta^{\alpha}_{b} \theta^{\gamma}_{a} \partial_{\gamma} \theta^{\beta}_{c} \right) \end{aligned} \tag{124}$$

I've boiled it down to this Jacobi identity needing to be satisfied

$$[\boldsymbol{\theta}_a, \boldsymbol{\theta}_b]^{\nu} e_{\nu c} + [\boldsymbol{\theta}_b, \boldsymbol{\theta}_c]^{\nu} e_{\nu a} + [\boldsymbol{\theta}_c, \boldsymbol{\theta}_a]^{\nu} e_{\nu b} = 0 \tag{125}$$

N.B. that this is true for Schwarzschild and BTZ black holes in their Kerr-Schild forms (but not when they are expressed in other coordinates) which seems somewhat indicative

Just to recap:

$$\text{Jac}\left([\boldsymbol{\theta}_a, \boldsymbol{\theta}_b]^{\nu} e_{\nu c} \right) = 0 \Longrightarrow \omega^{\mu}_{ab} = \partial_{[a} \theta^{\mu}_{b]} \qquad \text{Double Copy} \tag{126}$$

A Baker-Campbell-Hausdorff

A translational gauge transformation on an element $Z := X^a \boldsymbol{P}_a + Y^a \boldsymbol{P}_a$ of the ISO(2,1) algebra will be of the form

$$\begin{aligned} Z \longrightarrow Z' &= e^{-\xi^a \boldsymbol{P}_a}\, \boldsymbol{Z}\, e^{\xi^a \boldsymbol{P}_a} \\ &= \left(X^a + \epsilon^a{}_{bc} Y^b \xi^c \boldsymbol{P}_a \right) + Y^a \boldsymbol{J}_a \end{aligned} \tag{127}$$

B Non-linear realisations of group actions

The approach we take to coupling point particles to the Poincaré gauge field is in line with the work of Balachandran [BBS78; Bal+17] on Lagrangian descriptions of particles charged under groups with non-linear realisations (NLRs) of their group action. The intent is to write down a Lagrangian whose variation yields Wong's equations. One attaches an element of a faithful unitary representation of the group to the worldline of the particle

$$L = \left\langle K, U^{-1} D_\tau U \right\rangle \tag{128}$$

which is parametrised by a set of coordinates on the *group manifold* $U = U(X)$. Variations will be performed implicitly with respect to these coordinates using the following;

$$\begin{aligned} e^{-\kappa^B \boldsymbol{T}_B} \boldsymbol{U}(X) &:= \boldsymbol{U}[f(\kappa, X)] \qquad f^A(0, X) = X^A \\ \frac{d}{d\kappa^A} \left(e^{-\kappa^B \boldsymbol{T}_B} \boldsymbol{U}(X) \right) \Big|_{\kappa^A = 0} &= -\boldsymbol{T}_A \boldsymbol{U}(X) = \frac{\partial \boldsymbol{U}(X)}{\partial X^B} \frac{\partial f^B}{\partial \kappa^A} \end{aligned} \tag{129}$$

Expanding the Lagrangian in terms of the internal coordinates

$$L = \left\langle K, \boldsymbol{U}^{-1} \dot{X}^A \frac{\partial \boldsymbol{U}}{\partial X^A} + \boldsymbol{U}^{-1} \boldsymbol{A}_\tau \boldsymbol{U} \right\rangle \tag{130}$$

Varying with respect to X^A we have

$$\begin{aligned}
0 \stackrel{!}{=} & \frac{d}{d\tau}\left[\frac{\partial L}{\partial \dot{X}^A}\right] - \frac{\partial L}{\partial X^A} \\
= & \frac{d}{d\tau}\left[\left\langle K, \boldsymbol{U}^{-1}\frac{\partial \boldsymbol{U}}{\partial X^A}\right\rangle\right] \\
& - \dot{X}^B\left\langle K, \frac{\partial \boldsymbol{U}^{-1}}{\partial X^A}\frac{\partial \boldsymbol{U}}{\partial X^B} + \boldsymbol{U}^{-1}\frac{\partial^2 \boldsymbol{U}}{\partial X^A X^B}\right\rangle - \left\langle K, \frac{\partial \boldsymbol{U}^{-1}}{\partial X^A}\boldsymbol{A}_\tau \boldsymbol{U} + \boldsymbol{U}^{-1}\boldsymbol{A}_\tau \frac{\partial \boldsymbol{U}}{\partial X^A}\right\rangle \\
= & \left\langle \dot{K}, \boldsymbol{U}^{-1}\frac{\partial \boldsymbol{U}}{\partial X^A}\right\rangle + \dot{X}^B\left\langle K, \frac{\partial \boldsymbol{U}^{-1}}{\partial X^B}\frac{\partial \boldsymbol{U}}{\partial X^A} + \boldsymbol{U}^{-1}\frac{\partial^2 \boldsymbol{U}}{\partial X^A \partial X^B}\right\rangle \\
& - \dot{X}^B\left\langle K, \frac{\partial \boldsymbol{U}^{-1}}{\partial X^A}\frac{\partial \boldsymbol{U}}{\partial X^B} + \boldsymbol{U}^{-1}\frac{\partial^2 \boldsymbol{U}}{\partial X^A X^B}\right\rangle - \left\langle K, \frac{\partial \boldsymbol{U}^{-1}}{\partial X^A}\boldsymbol{A}_\tau \boldsymbol{U} + \boldsymbol{U}^{-1}\boldsymbol{A}_\tau \frac{\partial \boldsymbol{U}}{\partial X^A}\right\rangle \\
= & \left\langle \dot{K}, \boldsymbol{U}^{-1}\frac{\partial \boldsymbol{U}}{\partial X^A}\right\rangle + \dot{X}^B\left\langle K, \frac{\partial \boldsymbol{U}^{-1}}{\partial X^B}\frac{\partial \boldsymbol{U}}{\partial X^A} - \frac{\partial \boldsymbol{U}^{-1}}{\partial X^A}\frac{\partial \boldsymbol{U}}{\partial X^B}\right\rangle - \left\langle K, \frac{\partial \boldsymbol{U}^{-1}}{\partial X^A}\boldsymbol{A}_\tau \boldsymbol{U} + \boldsymbol{U}^{-1}\boldsymbol{A}_\tau \frac{\partial \boldsymbol{U}}{\partial X^A}\right\rangle \\
= & \langle\rangle
\end{aligned} \tag{131}$$

Multiplying through by $\frac{\partial f^A}{\partial \kappa^{A'}}$ we have

$$\left\langle \dot{K}, \boldsymbol{U}^{-1}\boldsymbol{T}_A\boldsymbol{U}\right\rangle + \left\langle \boldsymbol{U}K\boldsymbol{U}^{-1}, \left[\boldsymbol{T}_A, \dot{X}^B\frac{\partial \boldsymbol{U}}{\partial X^B}\boldsymbol{U}^{-1} + \boldsymbol{A}_\tau\right]\right\rangle = 0 \tag{132}$$

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