When is a double copy not a double copy?

Nathan Moynihan[1], Michael Reichenberg Ashby[2] and Chris D. White[3]

Centre for Theoretical Physics, School of Physical and Chemical Sciences,
Queen Mary University of London, 327 Mile End Road, London E1 4NS, UK

Abstract

The double copy relates scattering amplitudes and classical solutions in non-abelian gauge theories and gravity. As such, it is usually expressed in the conventional second-order formalisms in both theories, corresponding to the Einstein-Hilbert action in General Relativity. In this paper, we instead consider alternative formulations of gravity, which are known to terminate at finite order in the coupling at Lagrangian level. We argue that for certain formulations, gravity becomes a doppelgänger of gauge theory, allowing straightforward replacement of generators and structure constants in both theories. Translation to the more standard formulations then replaces the doppelgänger relationship with a double copy, such that the former formulation can be seen as a possible justification for the latter. We present results in both 3 and d spacetime dimensions, and relate our conclusions to previous studies in the literature.

1 Introduction

The double copy is a by now well-established relationship between quantities in gauge, gravity and related theories. Inspired by previous work in string theory [1], its original incarnation applied to scattering amplitudes in perturbation theory [2–4], before it was extended to classical solutions [5–55]. Potential non-perturbative aspects have been addressed in ref. [48, 56–69], and recent pedagogical reviews may be found in ref. [70–75]. Despite the above progress, a full understanding of the scope and remit of the double copy remains lacking, due mainly to a failure to understand its operation at the level of Lagrangians or, equivalently, equations of motion. This is chiefly due to the fact that a certain relationship between the colour and kinematic degrees of freedom [2] – known as BCJ duality – must be made manifest in order to double-copy non-linear amplitudes or solutions. This can typically be achieved only order-by-order in perturbation theory, resulting in a gauge theory Lagrangian that involve increasingly complicated higher-order interactions (see e.g. ref. [4]). Upon performing the double copy, the structure of such interactions is such as to reproduce the all-order form of the conventional Einstein-Hilbert gravitational action. Although the latter looks simple in that it involves a single power of the Ricci scalar, one finds an infinite

[1]n.moynihan@qmul.ac.uk
[2]m.l.reichenbergashby@qmul.ac.uk
[3]christopher.white@qmul.ac.uk

tower of higher-order graviton interactions upon expanding the action (together with the covariant spacetime volume measure) in terms of the graviton field.

As is well-known, the Einstein-Hilbert formulation is not the only way to write down General Relativity and related theories. Many alternative approaches exist, in which the role of the graviton is replaced by dependence on other quantities of geometric interest, such as the vierbein and / or spin connection[4]. The action then terminates at finite order in the gravitational coupling constant, and the question then naturally arises of what the double copy looks like in such formulations. Is it still a "double copy" in the usual sense, or are other replacement rules or structures at play? Answering such questions may prove fruitful in elucidating the origins and scope of the double copy, as well as the mysterious *kinematic algebras* that are known to be a consequence of BCJ duality. Both the double copy and BCJ duality strongly suggest that our traditional language for describing field theories is hiding a common underlying structure. The use of alternative formulations for describing gravity could help uncover this structure, and in any case having multiple languages to express the same physical idea – where the "right" language depends upon the question being asked – is often a fruitful way to proceed in physics.

The main theme of our paper will be to adopt formulations of gravity in which the action and / or equations of motion look identical in form to those of (non-abelian) gauge theory. In such formulations, the double copy ceases to look like a "doubling", and we instead (following ref. [46]) refer to the gravity theory as the doppelgänger of the gauge theory. To go from one theory to the other, one simply reinterprets a colour index in terms of an internal index in gravity theory, which is typically associated with a generator of the Lorentz or Poincaré algebra. This offers potential geometric insights into the double copy, which we comment on in what follows. Furthermore, upon translating to the usual Einstein-Hilbert formulation of gravity, the conventional double copy emerges. For the cases we study, we may therefore regard the doppelgänger relationship as an underlying explanation for the double copy.

We note that our work is in a similar spirit to – and indeed inspired by – the so-called *covariant colour-kinematics duality* of ref. [46]. This also considered manifest doppelgänger relationships at the level of the equations of motion, and such that types of index were reinterpreted in order to proceed from one theory to another. Our results are distinct from those in ref. [46], however, in that we consider different formalisms of gravity, and also show how explicit solutions can be mapped between theories, as well as considering equations of motion. Furthermore, we will furnish our relationships with a geometric interpretation, where this is possible.

We will begin in the following section by reviewing the various formulations of gravity that we will use throughout the paper, in particular the well-known Einstein-Cartan formalism for General Relativity. Unlike the conventional Einstein-Hilbert description of gravity in terms of the metric tensor, the Einstein-Cartan approach instead introduces two dynamical quantities known as the *vielbein* (in d dimensions) and the *spin connection*, from which the conventional metric and curvature tensors can be obtained. Advantages of this formalism include the fact that the Lagrangian is manifestly polynomial, and also that one may furnish the theory with a suggestive geometric interpretation that mimics the structure of (non-abelian) gauge theories in terms of principal fibre bundles. In 2+1 dimensions, this idea can be taken further, such that gravity can be entirely written as a gauge theory of the Poincaré group. This is the so-called Chern-Simons-Witten (CSW) formulation of three-dimensional gravity [77], and we also discuss its coupling to

[4]A detailed and highly pedagogical review of alternative formulations of General Relativity may be found in ref. [76].

matter particles. We will argue in detail that in this language, the double copy between gauge theory and gravity can be reinterpreted as a simple replacement of colour indices with gravitational internal indices (corresponding to the gauged Poincaré symmetry). Furthermore, coupling to matter introduces degrees of freedom living on the worldline in both the gauge and gravity cases. These degrees of freedom can themselves be put in a similar form in both the gauge and gravity theory, again such that internal indices get reinterpreted in going from one theory to the other.

Our analysis could be taken as an underlying explanation for the double copy in 2+1 dimensions, with the existence of a gauge-theoretic formulation of gravity naturally implying a double-copy structure when translated into the conventional Einstein-Hilbert description. Indeed, we will show how the conventional double copy of Wilson lines [62] emerges from the Chern-Simons-Witten theory. More importantly, though, our ideas can be used to resolve a key open question regarding the scope of the double copy itself: given an exact solution in a gauge theory, is it always possible to convert this into an exact solution in gravity? We will show precisely how this mapping works for pointlike sources in CSW theory, including how indices in the CSW gauge field get reinterpreted in terms of colour degrees of freedom. Crucial to this analysis is that one can identify a subset of generators of the Poincaré group in gravity that get mapped to a given subalgebra of the gauge group in the single copy theory. It is unlikely such a mapping exists for arbitrary solutions, thereby diminishing previous hopes that one might be able to double copy arbitrary non-abelian solutions.

Following our detailed analysis of 2+1 dimensions, we examine how our analysis can be generalised to four and higher spacetime dimensions. We will in particular make use of higher-dimensional formulations of gravity as a Poincaré gauge theory (see e.g. ref. [78] for a review). We will focus in particular on the theory of ref. [79], which is naturally related to the 2+1 gravity case described above. However, whilst the actions and matter couplings on both sides of the double copy correspondence can be put in a doppelgänger-like form, the nature of the field constraints are different. Nevertheless, we can still consider how to map solutions between theories, by first casting them into a common gauge theory language. The latter also allows for novel geometric interpretations of the double copy, that are difficult to arrive at directly in the conventional formulation. We hope that our study inspires further work on using alternative formulations of gravity or gauge theory in order to explore the double copy and related ideas, as this is a relatively unexplored area. Our present analysis is certainly not the last word on this matter.

The structure of our paper is as follows. In section 2, we review relevant formulations of gravity needed for the rest of the paper. In section 3, we study the double copy in 2+1 dimensions in detail, showing how this can be reinterpreted as a set of straightforward replacements between doppelgänger theories. We also show how classical solutions can be explicitly mapped, and how this can fail. In section 4, we see how to generalise our remakrs to higher spacetime dimensions. Finally, we discuss our results and conclude in section 5.

2 Alternative formulations of General Relativity

Einstein's formulation of General Relativity in terms of a metric tensor is not the only one. Various formulations of the theory exist by now, which use a variety of mathematical languages. Different ways of thinking about gravity can offer calculational advantages, but also conceptual advantages, particularly in relating gravitational physics to well-studied geometric structures such as fibre bundles. Here, we review salient details of formulations needed for for the rest of the paper, where our presentation follows that of ref. [76] in places.

2.1 Einstein-Cartan Theory

Perhaps the most well-known alternative formulation of gravity is the *Einstein-Cartan formalism.* As stressed in ref. [76], this can itself be introduced in different ways, and we here adopt the geometric point of view adopted in that reference. Given a (spacetime) manifold M, we may construct a vector bundle $E \to M$, namely a space E which is locally isomorphic to $M \times V$ for some vector space V. For Einstein-Cartan theory, one requires the vector space V to be equipped with an inner product, and for the bundle E to be associated to the principal frame bundle such that it is isomorphic to the tangent bundle TM. The latter is achieved by introducing d linearly independent one-forms

$$e^a = e^a{}_\mu dx^\mu, \tag{1}$$

known as the *vielbein* in d spacetime dimensions. Here the vector on the left-hand side lives in the internal space V, and the coefficients on the right-hand side can be used to convert vectors $v = v^\mu \partial_\mu$ from the tangent space of M to the internal space via

$$e^a(v) = e^a{}_\mu v^\mu. \tag{2}$$

Let $\langle \cdot, \cdot \rangle$ denote the inner product between two vectors in the internal space, which we may write in components as

$$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = \eta_{ab} v_1^a v_2^b. \tag{3}$$

One may then pullback this metric to obtain the corresponding metric on the spacetime manifold M. That is, if u and v denote vectors in the tangent space of M, then

$$g(u, v) = \langle e(u), e(v) \rangle, \quad g_{\mu\nu} = e^a{}_\mu e^b{}_\nu \eta_{ab}. \tag{4}$$

Note that the choice of vielbein is not unique, but may be subjected to local transformations that preserve the internal space metric. Choosing this metric to be the Minkowski metric, these transformations will be Lorentz:

$$e^a = \Lambda^a{}_b e^b, \quad \eta_{ab} = \Lambda^c{}_a \Lambda^d{}_b \eta_{cd}. \tag{5}$$

Furthermore, in order that the spacetime metric be non-degenerate, it is required that the vielbein have a suitable inverse $e^\mu{}_a$ such that

$$e^a{}_\mu e^\mu{}_b = \delta^a_b, \; e^\mu{}_a e^a{}_\nu = \delta^\mu_\nu \tag{6}$$

We may then define a connection on the bundle E, which in turn is used to form covariant derivatives of vector fields:

$$DV^a = dV^a + \omega^a{}_b \wedge V^b. \tag{7}$$

The one-form $\omega^a{}_b$ is known as the *spin connection*, and is required to be *metric-compatible* meaning vanishing covariant derivative of the internal space metric:

$$D\eta^{ab} = \omega^a{}_c \eta^{cb} + \omega^b{}_c \eta^{ac} = 0. \tag{8}$$

The conventional covariant derivative ∇ that acts on TM can be obtained by the pullback relation

$$e^a_\mu \nabla u^\mu = Du^a, \tag{9}$$

from which one may obtain the following relation between the spin connection and Levi-Civita connection $\Gamma^{\nu}_{\mu\lambda}$:

$$\Gamma^{\nu}_{\mu\lambda} = e^{\nu}{}_{a}\partial_{\mu}e_{\lambda}{}^{a} + e^{\nu}{}_{a}e_{\lambda}{}^{b}\omega_{\mu}{}^{a}{}_{b}. \tag{10}$$

This is already to sufficient to tell us that, in contrast the usual Einstein equations that depend only upon the spacetime metric $g_{\mu\nu}$, the Einstein-Cartan formalism is instead phrased in terms of two dynamical quantities, namely the vielbein and spin connection. In order to state the equations of motion, one may introduce the *torsion*, or covariant derivative of the vielbein:

$$T^{a} = De^{a} = de^{a} + \omega^{a}{}_{b} \wedge e^{b} \equiv T^{a}{}_{\mu\nu}dx^{\mu} \wedge dx^{\nu}. \tag{11}$$

Converting the internal index to a spacetime one and using eq. (10) yields

$$T^{\lambda}{}_{\mu\nu} = e^{\lambda}{}_{a}T^{a}{}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu}, \tag{12}$$

such that eq. (11) is indeed related to the usual torsion in terms of the Christoffel symbol. Vanishing of the torsion is then expressed by the equation

$$de^{a} + \omega^{a}{}_{b} \wedge e^{b} = 0. \tag{13}$$

We may also introduce an analogue of the Riemann curvature, by considering the commutator of two covariant derivatives on a vector in the internal space. In components:

$$[D_{\mu}, D_{\nu}]u^{a} \equiv R^{a}{}_{b\mu\nu}u^{b}, \tag{14}$$

which defines a two-form

$$R^{a}{}_{b} = d\omega^{a}{}_{b} + \omega^{a}{}_{c} \wedge \omega^{c}{}_{b}. \tag{15}$$

We will refer to this simply as the curvature (of the spin connection), and its components can be related to the conventional Riemann curvature components via

$$R^{\alpha}{}_{\rho\mu\nu} = R^{a}{}_{b\mu\nu}e^{\alpha}_{a}e^{b}_{\rho}. \tag{16}$$

Armed with these ingredients, we may write the action for Einstein-Cartan gravity in d spacetime dimensions, which is given by

$$S_{\rm EC}[e,\omega] = \frac{1}{32\pi G_{N}}\int_{M} \epsilon_{a_1 a_2 a_3 \ldots a_{d-2} bc}\, e^{a_1} \wedge e^{a_2} \wedge \ldots \wedge e^{a_{d-2}} \wedge \left(R^{bc}(\omega) - \lambda e^{c} \wedge e^{d}\right), \tag{17}$$

where G_N is Newton's constant, and λ is related to the cosmological constant. We obtain two field equations from this action, given that we may vary with respect to the vielbein or spin connection. Restricting to four spacetime dimensions, for example, varying with respect to $\omega^{a}{}_{b}$ yields the zero torsion condition of eq. (13). Instead varying with respect to e^{a} yields

$$\epsilon_{abcd}e^{b} \wedge R^{cd} = 2\lambda\epsilon_{abcd}e^{b} \wedge e^{c} \wedge e^{d}, \tag{18}$$

which is equivalent to the vacuum Einstein equation.

2.2 Gravity in 2+1 dimensions

In 2+1 dimensions, the Einstein-Cartan action of eq. (17) implies the equation

$$R^{ab} = 3\lambda e^a \wedge e^b. \tag{19}$$

Thus, the curvature of an Einstein metric is constant, such that there are no propagating degrees of freedom in three-dimensional gravity[5]. Nevertheless, solutions can be globally non-trivial (see e.g. ref. [80] for a useful compendium), and three-dimensional massless or massive gravity has previously proven useful as a testing ground for ideas relating to the double copy. [10,12,13,68,81–86]. Previous studies have focused on the Einstein-Hilbert formulation, and we here wish to provide new insights based on alternative formulations, in which gravity looks much more similar to a suitable non-abelian gauge theory. Indeed, there is a well-known formulation of gravity in 2+1 dimensions as a Chern-Simons theory, where the gauge symmetry corresponds to the Poincaré group ISO(2,1). This is known as *Chern-Simons-Witten (CSW) theory*, and the relevant gauge field is then valued in the Lie algebra of the Poincaré group [77,87]. The latter has conventional momentum and Lorentz generators $\{\mathbf{P}_a\}$ and $\{\mathbf{J}_{ab}\}$ respectively, satisfying the commutation relations

$$\begin{aligned}
[\boldsymbol{P}_a, \boldsymbol{P}_b] &= 0, \\
[\boldsymbol{J}_{ab}, \boldsymbol{P}_c] &= \eta_{ac}\boldsymbol{P}_b - \eta_{bc}\boldsymbol{P}_a \\
[\boldsymbol{J}_{ab}, \boldsymbol{J}_{cd}] &= \eta_{ac}\boldsymbol{J}_{bd} + \eta_{bd}\boldsymbol{J}_{ac} - \eta_{ad}\boldsymbol{J}_{bc} - \eta_{bc}\boldsymbol{J}_{ad},
\end{aligned} \tag{20}$$

and where $\mathbf{J}_{ab} = -\mathbf{J}_{ba}$. As shown in ref. [77], it is possible to equip this Lie algebra with an invariant bilinear form $\langle \cdot, \cdot \rangle$, whose action on the generators is

$$\langle \boldsymbol{P}_a, \boldsymbol{P}_b \rangle = 0, \quad \langle \boldsymbol{J}_{ab}, \boldsymbol{P}_c \rangle = \epsilon_{abc}, \quad \langle \boldsymbol{J}_{ab}, \boldsymbol{J}_{cd} \rangle = 0. \tag{21}$$

Denoting an abstract generator $\mathbf{P}_a$ or $\mathbf{J}_{ab}$ by $\mathbf{T}_I$, the gauge field then has generic form

$$\mathbf{A} = A^I \mathbf{T}_I = e^a \mathbf{P}_a + \frac{1}{2}\omega^{ab}\mathbf{J}_{ab}, \tag{22}$$

where we have included a conventional factor of 1/2 in the second term on the right-hand side. The action for the theory can then be written as

$$S_{\text{CSW}} = \frac{1}{2}\int_M \left\langle \boldsymbol{A} \wedge d\boldsymbol{A} + \frac{2}{3}\boldsymbol{A} \wedge \boldsymbol{A} \wedge \boldsymbol{A} \right\rangle. \tag{23}$$

This may be shown to be equivalent to the three-dimensional case of the Einstein-Cartan action of eq. (17), upon substituting eq. (22) and using eqs. (20, 21). We see that the CSW gauge field unifies the two fields (vielbein and spin connection) of the Einstein-Cartan theory into a single object. Furthermore, the cubic term in the action will give rise to the quadratic term on the right-hand side of eq. (19), which is in turn associated with the cosmological constant. Thus, solutions with no cosmological constant arise as solutions of an *abelian* Chern-Simons theory, a fact that will prove useful later on.

[5] An extreme case of eq. (19) is that of vanishing cosmological constant, for which one finds that spacetime must be locally flat.

Equation (23) may be compared with the usual action for non-abelian Chern-Simons theory, with a compact gauge group G:

$$S_{\rm CS} = \int_M {\rm Tr}\left(\mathbf{A}\wedge d\mathbf{A} + \frac{2}{3}\mathbf{A}\wedge\mathbf{A}\wedge\mathbf{A}\right). \tag{24}$$

This is identical in form (up to irrelevant normalisation factors) with eq. (23), where the relationship between the theories amounts to reinterpreting adjoint indices in the gravity theory in terms of colour indices, and replacing the invariant bilinear form of ISO(2,1) with the Killing form of the non-abelian gauge group. Thus, in the CSW formulation of gravity, there can be no "double copy" relationship with the corresponding gauge theory. Rather, the gravity theory becomes a precise doppelgänger of the gauge theory, allowing us to simply reinterpret indices in proceeding from one theory to another. We can then ask how the conventional double copy emerges out of this correspondence, and we explore this in more detail in the following section.

Before doing so, let us note that the vacuum field equations of Chern-Simons theory demand that the field strength

$$\mathbf{F} = d\mathbf{A} + \mathbf{A}\wedge\mathbf{A} \tag{25}$$

associated with the gauge field $\mathbf{A}$ vanishes. In the gravity case, an explicit calculation yields

$$\begin{aligned}\mathbf{F} &= \left(de^a + \omega^a{}_b\wedge e^b\right)\mathbf{P}_a + \frac{1}{2}\left(d\omega^{ab} + \omega^a{}_c\wedge\omega^{cb}\right)\mathbf{J}_{ab}\\ &= T^a[e,\omega]\mathbf{P}_a + \frac{1}{2}R^{ab}[\omega]\mathbf{J}_{ab}.\end{aligned} \tag{26}$$

Here we have recognised the torsion and curvature of eqs. (11, 15), and the vanishing of the field strength then implies the zero torsion and curvature conditions found before.

3 Gravity as a doppelgänger in 2+1 dimensions

In the previous section, we have reviewed the CSW formulation of gravity, in which the gravitational action can itself be cast in the form of a gauge theory. In this section, we wish to interpret how this gives rise to the conventional three-dimensional double copy between gauge theory and gravity. For this to work, however, we must consider the coupling of gravity to matter. We note that it is not known in general how to couple arbitrary matter sources to gauge theory and gravity and preserve the double copy (see e.g. refs. [7, 9, 24, 88] for important works). Likewise, equivalence between different formulations of gravity can itself be disrupted when trying to include matter. Here, we will restrict ourselves to sources which are highly localised, such that we may consider gravity coupled to a single worldline. Our aim is to cast gauge theory and gravity couplings in a form that looks almost identical, and to see how this gives rise to known double copy properties.

3.1 Wilson lines and loops

Given a gauge field $\mathbf{A}_\mu$, there is a natural way to couple it to worldlines, namely through the *Wilson line operator*

$$\Phi_{\mathcal{C}}[\mathbf{A}] = \mathcal{P}\exp\left[g\int_{\mathcal{C}} dx^\mu \mathbf{A}_\mu\right] = \mathcal{P}\exp\left[g\mathbf{T}^a\int_{\mathcal{C}} d\tau \dot{x}^\mu A^a_\mu\right], \tag{27}$$

where g is the coupling constant, τ the proper time along a given contour $\mathcal{C}$, and the dot denotes differentiation with respect to τ. The exponent is matrix-valued in the vector space corresponding to a particular representation of the gauge group, which is fixed in eq. (27) by the representation of the generators $\{\mathbf{T}^a\}$. The exponential itself is defined by its Taylor expansion, and the $\mathcal{P}$ symbol in eq. (27) denotes that the generators should then be *path-ordered* i.e. ordered according to their parameter distance along $\mathcal{C}$. In gauge theories, Wilson lines are associated with the transportation of gauge information between different points in the underlying manifold, and are thus a crucial ingredient for constructing gauge-covariant or gauge-invariant observables. They also arise as dressing factors for scattering amplitudes in certain kinematic limits, such as when fast-moving particles emit low energy ("soft") radiation. In gravity, the relevant operator describing soft graviton emission is given by [89, 90]

$$\Phi_{\mathcal{C}}^{\text{grav.}} = \exp\left[\frac{\kappa}{2} m \int_{\mathcal{C}} dx^\mu \dot{x}^\nu h_{\mu\nu},\right], \tag{28}$$

where $h_{\mu\nu}$ is the graviton field defined by

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}, \tag{29}$$

$\kappa = \sqrt{32\pi G_N}$ the gravitational coupling in terms of Newton's constant G_N, and m the mass of the particle. For a straight-line contour such that

$$x^\mu = \beta^\mu \tau, \tag{30}$$

where β^μ is the 4-velocity, one finds that the operator of eq. (28) is obtained from eq. (27) by: (i) replacing the field A_μ^a with $h_{\mu\nu}$ (itself involving replacing a colour index with a spacetime one); (ii) replacing coupling constants according to $g \to \kappa/2$; (iii) replacing the colour generator $\mathbf{T}^a$ with the momentum $p^\mu = m\beta^\mu$. As argued in refs. [62, 91, 92], these replacements – which also generalise to the case of massless particles – correspond to the known double copy for scattering amplitudes (see also ref. [93] for a non-Wilson line treatment of soft radiation and the double copy). In these previous studies, eq. (27) has only been considered in the context of non-abelian gauge theory with a compact gauge group. Here, however, we have reviewed that gravity can itself be written as a gauge theory in 2+1 dimensions. Thus, we must be able to take $\mathbf{A}_\mu$ in eq. (27) as the Chern-Simons-Witten gravity field. In that language, the form of the coupling to a worldline looks identical to that in non-abelian gauge theory, such that there is a manifest doppelgänger relationship between the two theories. However, one must then show that the CSW coupling reproduces the known Wilson line operator double copy implicit in eq. (28).

In fact, there are numerous ways to show that the CSW Wilson line is consistent with eq. (28). For concreteness, we will focus on the explicit case of a gauge-invariant *Wilson loop*, in which the contour $\mathcal{C}$ is taken to be a closed curve. Given that both non-abelian Chern-Simons theory and three-dimensional gravity are topological field theories, Wilson loops and their expectation values have been widely studied due to their ability to encode topological invariants (see e.g. ref. [94] for a comprehensive review). Physically, Wilson loops describe the phase experienced by a particle traversing a closed curve, and to be scalar-valued and gauge-invariant must then include a trace in the given representation $\mathcal{R}$ of the gauge group:

$$W_{\mathcal{C}}[\mathbf{A}] = \text{Tr}_{\mathcal{R}} \mathcal{P} \exp\left[i \oint_{\mathcal{C}} dx^\mu \mathbf{A}_\mu\right]. \tag{31}$$

In order to make contact with the gravitational Wilson loop, the relevant representation of the Poincaré group will be that of a massive spinless particle. This will necessarily be infinite dimensional which complicates the direct implementation of the trace, so we follow an alternative approach suggested in [87] and employed in [95, 96] within the context of AdS CSW gravity. The basic idea is to consider the test particle on the wordline as a quantum mechanical system whose Hilbert space provides a representation space for the gauge group. That is, we may introduce coordinates $\{q^a\}$ and their conjugate momenta $\{p^a\}$ associated with an internal space at each point on the worldline, and such that a local Poincaré transformation at each point corresponds to

$$q^a \to \Lambda^a{}_b q^b + \rho^a, \quad p^a \to \Lambda^a{}_b q^b, \tag{32}$$

where $\Lambda^a{}_b$ is a Lorentz transformation matrix. We can couple a background CSW gauge field to the internal Poincaré degrees of freedom by introducing the worldline covariant derivative

$$Dq^a = dq^a + \omega^a{}_b q^b + e^a, \tag{33}$$

where $\omega^a{}_b$ and e^a are the background spin connection and vielbein. In coordinate notation this reads

$$D_\mu q^a = \partial_\mu q^a + (\omega_\mu)^a{}_b q^b + e^a_\mu, \tag{34}$$

and the locally gauge invariant action coupling the test particle to the CSW field is given by

$$\begin{aligned} S[q^a, p_a; \mathbf{A}] &= \int_{\mathcal{C}} d\tau \left[p_a D_\tau q^a - \xi(p_a p^a - m^2) \right] \\ &= \int_{\mathcal{C}} d\tau \left[p_a \dot{x}^\mu (\partial_\mu q^a + (\omega_\mu)^a{}_b q^b + e^a_\mu) - \xi(p_a p^a - m^2) \right], \end{aligned} \tag{35}$$

where the second term, involving a Lagrange multiplier ξ, implements the mass-shell constraint. If we have an open Wilson line with contour $\mathcal{C}$, there will be an initial state of the $|i\rangle$ of the system $\{q^a, p_a, \xi\}$ at one end of the contour, and a final state $|f\rangle$ at the other. The expectation value of the Wilson line sandwiched between these states can then be interpreted as a transition amplitude from $|i\rangle$ to $|f\rangle$ in the presence of the background CSW gauge field, which has the path-integral representation

$$\left\langle f \left| \mathcal{P} \exp\left(i \int_{\mathcal{C}} dx^\mu \mathbf{A}_\mu \right) \right| i \right\rangle = \int \mathcal{D}q^a \mathcal{D}p_a \mathcal{D}\xi \, e^{iS[q^a, p_a; \mathbf{A}]}. \tag{36}$$

On the right-hand side, one integrates over all possible values of the dynamical degrees of freedom at all points on the worldline, subject to the appropriate boundary conditions at the end-points corresponding to $|i\rangle$ and $|f\rangle$. For a closed loop $\mathcal{C}$, one may integrate over all values of the field at *all* points in the loop. If one fixes a point to be the common start and end of the loop, this amounts to summing over all states at that point, and thus to carrying out the trace in eq. (31). That is, one has [87, 95, 96]

$$\mathrm{Tr}_{\mathcal{R}} \mathcal{P} \exp\left[i \oint_{\mathcal{C}} dx^\mu \mathbf{A}_\mu \right] = \int \mathcal{D}q^a \mathcal{D}p_a \mathcal{D}\xi \, \exp\left[i \oint_{\mathcal{C}} d\tau p_a \dot{x}^\mu (\partial_\mu q^a + (\omega_\mu)^a{}_b q^b + e^a_\mu) - \xi(p_a p^a - m^2) \right], \tag{37}$$

and the usefulness of this expression is that one may explicitly carry out the path integrals appearing on the right-hand side. Given that the Wilson line corresponds to the phase experienced by a particle following a classical trajectory, we may carry out the integrals using the saddle point approximation, commencing with the integral over q^a. Varying the action with respect to the latter gives the the equation of motion

$$\dot{x}^\mu \left[\partial_\mu p^a + (\omega_\mu)^a{}_b p^b\right] = 0, \tag{38}$$

which when substituted back into the path integral yields

$$W[C] = \int [\mathcal{D}p\mathcal{D}\xi] \exp\left[i \oint_C d\tau \Big(\dot{x}^\mu p_a e^a_\mu - \xi \left(p_a p^a + m^2\right)\Big)\right]. \tag{39}$$

Next, the equation of motion for p^a is found to be

$$p^a = \frac{1}{2\xi} e^a_\mu \dot{x}^\mu, \tag{40}$$

such that we are left with

$$W[C] = \int [\mathcal{D}\xi] \exp\left[i \oint_C d\tau \Big(\frac{1}{4\xi}\dot{x}^\mu \dot{x}^\nu e^a_\mu e^b_\nu \eta_{ab} - \xi m^2\Big)\right]. \tag{41}$$

The field e^a_μ on the worldline is the pullback of the vielbein field associated with the background bulk CSW gauge field. From eq. (4), we may thus recognise the conventional metric tensor, such that eq. (41) becomes

$$W[C] = \int [\mathcal{D}\xi] \exp\left[i \oint_C d\tau \left(\frac{1}{4\xi} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu - \xi m^2\right)\right]. \tag{42}$$

At this stage, one may gauge-fix ξ and substitute the graviton definition of eq. (29) to yield the Wilson line operator of eq. (28), where the advantage of this approach is that the operator works manifestly for either massless or massive test particles. Alternatively, for non-zero mass one may eliminate ξ using its equation of motion to get

$$W[C] = \exp\left[-im \oint_C d\tau \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}\right]. \tag{43}$$

This makes clear that the Wilson line operator is associated with the path-length of the worldline. Again, substituting eq. (29) and expanding in κ yields the known Wilson line operator of eq. (28), as noted previously in e.g. ref. [62].

Above, we have argued that the known double copy of Wilson line operators for soft gluon / graviton radiation emerges from casting both theories in the same language – that of a non-abelian Chern-Simons gauge theory. It is possible, however, to make the common structure of the coupling of gauge fields to matter much more explicit in the two theories, as we now see.

3.2 Point particle couplings in gauge theory and gravity

As reviewed in e.g. ref. [41], the current for a pointlike colour charge in a non-abelian gauge theory can be written (in D spacetime dimensions) as

$$j^{\mu\, a}(y^\mu) = \int d\tau c^a(\tau) \dot{x}^\mu \delta^D(y^\mu - x^\mu(\tau)), \tag{44}$$

where the delta function localises onto a particular worldline $(x^\mu(\tau))$, and $c^a(\tau)$ is a colour vector associated with each point along the worldline. Covariant current conservation $D_\mu j^{\mu\, a}$ then implies

$$\frac{dc^a}{d\tau} = g f^{abc} \dot{x}^\mu(\tau) A^b_\mu(\tau) c^c(\tau), \tag{45}$$

where $\{f^{abc}\}$ are the structure constants of the gauge group. The particle trajectory itself obeys the non-abelian generalisation of the Lorentz force law:

$$m\frac{d^2 x^\mu}{d\tau^2} = g c^a F^{a\,\mu}{}_\nu \dot{x}^\nu. \tag{46}$$

Collectively, eqs. (45, 46) are known as the *Wong equations*, having first been derived in ref. [97]. The question then arises of how to derive these equations from a suitable point particle action, and various prescriptions have appeared in the literature [98, 99].

The approach of ref. [99] uses as its dynamical variable a faithful representation of the gauge group attached to the worldline, such that a group element associated with proper time τ can be written as $\boldsymbol{g}(\tau)$. For a given gauge group G, the quantity

$$\boldsymbol{g}^{-1}\boldsymbol{D}_\mu \boldsymbol{g} = \boldsymbol{g}^{-1}\Big(\partial_\mu + A^b_\mu \boldsymbol{T}_b\Big)\boldsymbol{g} \tag{47}$$

is known as the *gauged Maurer-Cartan form*, and is a one-form valued in the Lie algebra of G. An action for Yang-Mills theory coupled to the worldline can then be obtained by taking the inner product of this form – defined at all points on the worldline – with a Lie algebra element

$$\boldsymbol{K} = K^a \boldsymbol{T}_a, \tag{48}$$

where the form of the action itself is

$$S = \int ds + \int d\tau \dot{x}^\mu \left\langle K^a \boldsymbol{T}_a, \boldsymbol{g}^{-1}(\tau)\Big(\partial_\mu + A^b_\mu[x(\tau)]\boldsymbol{T}_b\Big)\boldsymbol{g}(\tau)\right\rangle. \tag{49}$$

One may think of the role of $\boldsymbol{K}$ as fixing the particular representation of G that acts on the worldline (in other words, the "type" of colour charge that is being coupled to Yang-Mills theory). To make this precise, one may identify

$$c^a \boldsymbol{T}_a = \boldsymbol{g}\boldsymbol{K}\boldsymbol{g}^{-1}, \tag{50}$$

and then obtain equations of motion by varying the action of eq. (49) with respect to both $\boldsymbol{g}$ and the worldline coordinate $x^\mu(\tau)$. As shown in ref. [99], this yields the Wong equatons of eqs. (45, 46). Thus, eq. (50) tells us that the Lie algebra element $\boldsymbol{K}$ in eq. (48) determines the colour vector carried by Wong's moving charged particle.

Given that the above arguments are independent of the gauge group, they may also be applied directly to CSW gravity. In that case, one inserts the Poincaré valued gauge field into the coupling action, such that $\boldsymbol{g} \equiv (\Lambda, q)$ is an element of the Poincaré group (with Lorentz and translation parameters Λ and q respectively), and $\boldsymbol{K}$ a Poincaré algebra element:

$$\boldsymbol{K} = p^a \boldsymbol{J}_a + s^a \boldsymbol{P}_a. \tag{51}$$

In order to concretely compute the action, we use a convenient 4×4 matrix representation of the group elements. Following e.g. ref. [100], we may choose the 4×4 matrix representation

$$\boldsymbol{g} = \begin{pmatrix} \Lambda^i{}_j & q^i \\ 0 & 1 \end{pmatrix} \qquad \boldsymbol{K} = \begin{pmatrix} p^a(\boldsymbol{J}_a)^i{}_j & s^a(\boldsymbol{P}_a)^i \\ 0 & 0 \end{pmatrix}, \tag{52}$$

where the generators are in turn given by

$$\boldsymbol{J}_a = \begin{pmatrix} -\epsilon_a{}^i{}_j & 0 \\ 0 & 0 \end{pmatrix} \qquad \boldsymbol{P}_a = \begin{pmatrix} 0 & \delta^i_a \\ 0 & 0 \end{pmatrix}. \tag{53}$$

The second term in eq. (49) represents the coupling of the gauge field to the worldline, and in the gravity case this now reads

$$\begin{aligned} S_{\text{coupling}} &= \int_\gamma \left\langle \begin{pmatrix} p^a \boldsymbol{J}_a & s^a \boldsymbol{P}_a \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \Lambda^{-1} & -\Lambda^{-1} q \\ 0 & 1 \end{pmatrix} \cdot \left[\begin{pmatrix} d\Lambda & dq \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \omega^a \boldsymbol{J}_a & e^a \boldsymbol{P}_a \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \Lambda & q \\ 0 & 1 \end{pmatrix} \right] \right\rangle \\ &= \int_\gamma \left\langle p^a \boldsymbol{J}_a + s^a \boldsymbol{P}_a, \frac{1}{2} (\Lambda^{-1})^i{}_k \left[d\Lambda^k{}_j + \omega^k{}_l \Lambda^l{}_j \right] (-\epsilon^a)^j{}_i \, \boldsymbol{J}_a + (\Lambda^{-1})^a{}_b D q^b \boldsymbol{P}_a \right\rangle \end{aligned} \tag{54}$$

Applying the invariant bilinear form of (21), the action becomes

$$S_{\text{coupling}} = \int_\gamma p_a (\Lambda^{-1})^a{}_b \left(dq^b + \omega^b{}_c q^c + e^c \right) + \frac{1}{2} s^a \mathrm{Tr} \left[\Lambda^{-1} \left(d\Lambda + \omega\Lambda \right) \boldsymbol{J}_a \right] \tag{55}$$

In a spinless massive representation, the second term of this action associated with s^a vanishes and Λ becomes non-dynamical, allowing us to absorb it into p_a and giving the following action

$$S^{\text{massive, spinless}}_{\text{coupling}} = \int \mathrm{d}\tau \, p_a \left(\partial_\mu q^a + (\omega_\mu)^a{}_b q^b + e^a_\mu \right) \dot{x}^\mu \tag{56}$$

which, when supplied with a Hamiltonian constraint on the m^2 Casimir can be seen to reproduce (35). Combined with the results of the previous section, the second term that appears in eq. (49) thus indeed reproduces the known double copy of Wilson lines that couple gravitons or gluons to worldlines. Here, though, we have cast this relationship in a form that looks the same in both theories: the coupling term in eq. (49) is independent of the gauge group, suggesting that one may simply reinterpret generators and colour indices in proceeding from one theory to another. This is thus an explicit manifestation of the BCJ (or colour-kinematics) duality property, which posits that one must replace a colour algebra by a kinematic algebra upon performing the double copy. Here, this acts as a literal replacement, and this may be further furnished by a geometric interpretation that we will explore later on.

Careful scrutiny of the above analysis shows that it is almost – but not quite – true that the CSW formulation places the gauge and gravity theories on a precisely equal footing. Although the second term in eq. (49) matches up between the two types of theory, the first term on the right-hand side does not. In gauge theory, this term must be inserted by hand in order to generate the dynamics of the free particle. In gravity, however, the total length of the worldline of the particle emerges from the coupling term, given that eq. (56) ends up generating the Wilson line operator of eq. (43). This can be viewed as a counterpart of the fact that in the conventional Einstein-Hilbert formulation of gravity, the double copy applies to the graviton field $h_{\mu\nu}$ rather than the full metric $g_{\mu\nu}$. Furthermore, this observation does not invalidate the fact that the CSW formulation provides a concrete setting in which interaction terms in gravity are a manifest doppelgänger of their analogues in gauge theory, with a strict replacement of colour generators by kinematic ones.

Given that the action of eq. (49) reproduces the Wong equations in gauge theory, it is natural to ask what their counterpart represents in gravity. This is the subject of the following section.

3.3 Doppelgänger dynamics: Wong versus Mathisson–Papapetrou–Dixon

In the previous section, we have seen that at least in in 2+1 dimensions, interaction terms in gauge theory and gravity can be written such that there is a manifest doppelgänger relationship between the two theories. For colour degrees of freedom, this leads to the Wong equations (45, 46), and it is now pertinent to ask what these indices represent when interpreted gravitationally. The Wong equations describe the motion of a coloured particle in a non-Abelian background and, as we will now show, the exact same formalism describes a *spinning* particle moving on a curved background, governed by the Mathisson–Papapetrou–Dixon (MPD) equations. To see this explicitly, it is convenient to rewrite the colour charge and field strength in the adjoint (matrix) representation,

$$Q^{ab} := f^{abc}c^{c}, \qquad F^{ab}_{\mu\nu} := f^{abc}F^{c}_{\mu\nu}, \tag{57}$$

so that the commutator structure of the non–abelian field is explicit. With these definitions eqs. (45, 46) become

$$\frac{dQ^{ab}}{d\tau} = \dot{x}^{\mu}\big[A_{\mu}, Q\big]^{ab} = \dot{x}^{\mu}A^{ac}_{\mu}Q^{cb} - \dot{x}^{\mu}A^{bc}_{\mu}Q^{ca} \tag{58}$$

and

$$m\frac{d^2x^{\mu}}{d\tau^2} = gQ^{ab}F^{ab\mu}{}_{\nu}\dot{x}^{\nu} \tag{59}$$

Notice that the right-hand side of the first equation involves a single contraction of two adjoint indices, leaving a free pair (ab) that mirrors the antisymmetric index pair $(\rho\sigma)$ of a local Lorentz bivector. We are now free to interpret the indices (ab) as belonging to either $SU(N)$ or $SO(D-1,1)$, describing a gauge or gravitational theory, respectively. However, it is notationally convenient to make the explicit identifications

$$Q^{ab} \longrightarrow S^{ab} + J^{ab}, \qquad gF^{ab}_{\mu\nu} \longrightarrow \frac{1}{2}R_{ab\mu\nu}, \qquad A^{ab}_{\mu} \longrightarrow \frac{1}{2}\omega^{ab}_{\mu}, \tag{60}$$

such that we land on the conventional form of the MPD equations

$$\frac{Dp^{\mu}}{d\tau} = -\frac{1}{2}R^{\mu}{}_{\nu ab}S^{ab}\dot{x}^{\nu}, \qquad \frac{dS^{ab}}{d\tau} = \dot{x}^{\mu}(\omega^{ac}_{\mu}S_{c}{}^{b} - \omega^{bc}_{\mu}S_{c}{}^{a}) + p^{a}u^{b} - p^{b}u^{a} \tag{61}$$

where S^{ab} is the intrinsic spin-tensor, J^{ab} the orbital tensor and ω^{ab}_{μ} the spin-connection. This essentially amounts to simply replacing the covariant derivative, but in its slightly unusual adjoint form, i.e.

$$D_{\mu} = \partial_{\mu} + gA^{ab}_{\mu}T^{ab}, \qquad [D_{\mu}, D_{\nu}] = gF^{ab}_{\mu\nu}T^{ab} \tag{62}$$

which, under the replacements above, together with $T^{ab} \to M^{ab}$, simply becomes the gravitational covariant derivative, now identifying the colour indices with tangent bundle indices, i.e.

$$D_{\mu} = \partial_{\mu} + \frac{1}{2}\omega^{ab}_{\mu}M^{ab}, \qquad [D_{\mu}, D_{\nu}] = \frac{1}{2}R^{ab}_{\mu\nu}M^{ab}. \tag{63}$$

Note that the elements T^{ab} and M^{ab} both obey the same algebra

$$[T^{ab}, T^{cd}] = 2(\eta^{a[c}T^{d]b} - \eta^{b[c}T^{d]a}) \tag{64}$$

Thus the colour precession of a gauge charge and the spin precession of a gravitational probe are different faces of the same worldline dynamics. Equation (60) is the dynamical analogue of the colour-to-kinematics replacement rules that produced the operator-level double copy in section 3.1.

Both the Wong and MPD equations can be derived directly from the CSW coupling by extending the action of eq. (35) to include $Q^{\bar{a}\bar{b}}$, which we interpret as the *angular-momentum* bivector, and we use barred indices to make the distinction clear: if we unbar $\bar{a}$, it becomes a local Lorentz (tangent bundle) index, otherwise it should be thought of as a gauge-group colour index. The gauged Maurer–Cartan action of eq. (35) then becomes

$$S\big[q,p,\Sigma;\mathbf{A}\big] := \int_{\mathcal{C}} d\tau \Big[p_a D_\tau q^a + \frac{1}{2} Q^{\bar{a}\bar{b}} \Omega_{\bar{a}\bar{b}} - \xi(p^2+m^2)\Big], \tag{65}$$

where $\Omega_{\bar{a}\bar{b}} = \big(\Lambda^{-1} D_\tau \Lambda\big)_{\bar{a}\bar{b}}$.

The resulting equations of motion are

$$D_\tau Q^{\bar{a}\bar{b}} = 0, \qquad D_\tau p^a = 0, \qquad D_\tau q^a = 2\xi p^a. \tag{66}$$

If we unbar the indices here, and identify Q^{ab} as the *total* spin tensor, then these are the MPD equations. If not, then they are the Wong equations.

From this perspective the same action yields both the Wong and MPD systems, making (100) the parent "master" action whose equations of motion admit two doppelgänger incarnations.

3.4 Point source solutions in 2+1 dimensional gravity

The solutions of the Einstein equations sourced by point masses and spins in 2+1 dimensions have been studied extensively in both metric and first-order formalisms (see e.g. [**?**]). It is well understood that in the absence of a cosmological constant, metric solutions are automatically Riemann flat and do not contain propagating degrees of freedom. Consequently, point masses no longer source Schwarzschild-like potentials and instead introduce topological defects into the spacetime manifold in the form of conical singularities. The metric solution for a single point particle of mass M can be written in polar coordinates as

$$\mathrm{d}s^2 = -\left(\mathrm{d}t + \frac{\kappa S}{2\pi}\mathrm{d}\phi\right)^2 + \frac{\mathrm{d}r^2}{\left(1-\frac{\kappa M}{2\pi}\right)^2} + r^2 \mathrm{d}\phi^2 \qquad t\in[-\infty,\infty], r\in[0,\infty], \phi\in[0,2\pi) \tag{67}$$

Clearly this metric can be brought to a form which is explicitly Minkowski via a coordinate transformation,

$$t \to \tilde{t} = t + \frac{\kappa S}{2\pi}\phi, \qquad r \to \tilde{r} = \frac{r}{1-\frac{\kappa M}{2\pi}}, \qquad \phi \to \tilde{\phi} = \left(1-\frac{\kappa M}{2\pi}\right)\phi, \tag{68}$$

but this transformation has a non-trivial topological effect, altering the boundary conditions on the azimuthal coordinate $\tilde{\phi} \in [0, 2\pi - \kappa M)$. This shows the presence of a conical singularity in the spacetime which has no effect on the curvature but does induce non-trivial phase factors for semi-classical test particles looping around the source. Curiously, the form of the deficit angle implies a maximum mass content of the spacetime of $2\pi/\kappa$, above which the metric is no longer non-degenerate.

We can translate this spacetime geometry into the CSW language by identifying an appropriate dreibein and spin connection which correspond to the metric. There is a Lorentz freedom in making

Figure 1: The conical geometry of a spatial slice Σ induced by a point mass at the origin. The deficit angle $\Delta\phi$ is proportional to the mass. Identified edges are indicated by dashed blue lines.

this identification, and we choose the following form in order to eliminate $\mathcal{O}(1)$ terms in the spin connection.

$$\begin{aligned} e^0 &= \mathrm{d}t + \frac{\kappa}{2\pi} S \mathrm{d}\phi & \omega^1{}_2 &= \frac{\kappa M}{2\pi}\mathrm{d}\phi \\ e^1 &= \frac{\cos\phi\,\mathrm{d}r}{1-\frac{\kappa M}{\pi}} - r\sin\phi\,\mathrm{d}\phi & \omega^2{}_0 &= 0 \\ e^2 &= \frac{\sin\phi\,\mathrm{d}r}{1-\frac{\kappa M}{\pi}} + r\cos\phi\,\mathrm{d}\phi & \omega^0{}_1 &= 0 \end{aligned} \tag{69}$$

We can now make use of the gauged Poincaré symmetry to perform a transformation of the form $\exp \xi^a \boldsymbol{P}_a$, under which $e^a \to e^a - \mathrm{d}\xi^a - \omega^a{}_b \xi^b$. Choosing the transformation parameters which are globally well defined

$$\xi^a = \begin{pmatrix} t & \frac{r\cos\phi}{1-\frac{\kappa M}{2\pi}} & \frac{r\sin\phi}{1-\frac{\kappa M}{2\pi}} \end{pmatrix}^a \tag{70}$$

the full CSW gauge field now has the form

$$\boldsymbol{A} = \frac{\kappa}{2\pi}\left(M\boldsymbol{J}_{12} + S\boldsymbol{P}_0\right)\mathrm{d}\phi = \boldsymbol{U}^{-1}\mathrm{d}\boldsymbol{U} \qquad \boldsymbol{U} = \exp\left[\frac{\kappa}{2\pi}\left(M\boldsymbol{J}_{12} + S\boldsymbol{P}_0\right)\phi\right] \tag{71}$$

This gauge field is pure gauge, but possesses a non-trivial monodromy about the origin. The gauge choice has also decoupled the colour and kinematic parts of the gauge field, yielding a globally constant colour factor $M\boldsymbol{J}_{12} + S\boldsymbol{P}_0$ multiplying a solution to the abelian Chern-Simons equations of motion.

This approach can be extended to a static multipole solution describing an arbitrary number of point masses and spins. We take the solution in homogeneous coordinates as given for example in [80]

$$\mathrm{d}s^2 = -\left(\mathrm{d}t + \frac{\kappa}{2\pi}\sum_n s_n \frac{\epsilon_{ij}\left(r^i - r^i_n\right)\mathrm{d}x^j}{|\vec{r}-\vec{r}_n|^2}\right)^2 + \prod_n |\vec{r}-\vec{r}_n|^{-\frac{\kappa m_n}{\pi}}\left(\mathrm{d}x^2 + \mathrm{d}y^2\right) \tag{72}$$

and re-express it using complex coordinates on the $x-y$ plane

$$\mathrm{d}s^2 = -\left(\mathrm{d}t - g(z)\mathrm{d}z + \bar{g}(\bar{z})\right)^2 + f(z)\bar{f}(\bar{z})\mathrm{d}z\mathrm{d}\bar{z} \tag{73}$$

where

$$f(z) = C_0 \prod_n (z-z_n)^{-\frac{\kappa m_n}{2\pi}} \qquad g(z) = \frac{i\kappa}{4\pi}\sum_n s_n \frac{\mathrm{d}z}{z-z_n} \tag{74}$$

Now we can define a vielbein corresponding to this metric

$$e^a = \begin{pmatrix} \mathrm{d}t - g(z)\mathrm{d}z + \bar{g}(\bar{z})\mathrm{d}\bar{z} \\ f(z)\mathrm{d}z \\ f(\bar{z})\mathrm{d}\bar{z} \end{pmatrix}^a \tag{75}$$

For $\kappa m_n/2\pi \notin \mathbb{Z}$ the functions $f(z)$ and $\bar{f}(\bar{z})$ are multivalued, and so we restrict them with appropriate branch cuts γ_n. The vielbein can then be expressed in an explicitly Minkowski form by defining new coordinates

$$T(t, z, \bar{z}) = t - \int^z g(\zeta)d\zeta + \int^{\bar{z}} \bar{g}(\bar{\zeta})d\bar{\zeta} \qquad Z(z) = \int^z f(\zeta)d\zeta \qquad \bar{Z}(\bar{z}) = \int^{\bar{z}} \bar{f}(\bar{\zeta})d\bar{\zeta} \tag{76}$$

and identifying the edges on either side of the branch cut. In these new coordinates, the spin connection is everywhere zero, and the Poincaré gauged translation which eliminates the vielbein is $\rho^a = (T, Z, \bar{Z})^a$. As in the gase of the monopole, we see that going to explicitly Minkowski coordinates requires the introduction of non-trivial boundary conditions. We now wish to identify a gauge in which the solution is an abelianised gauge field solution on a trivial chart.

MRA: It's completely up to you whether it's worth including some kind of diagram like this, it just gives a nice illustration of what the flattening is doing and how the deficit angles emerge in the multipole case

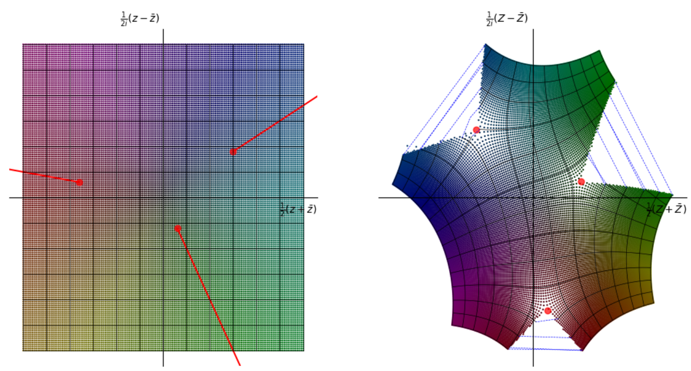

Figure 2: The change of spatial coordinates from left to right makes the metric explicitly Minkowski, but is no longer defined over the entire complex plane. Branch cuts introduce deficit angles around point sources. Identifying edges on either side of the deficit allows the surface to be wrapped into a "multicone".

To do this, we Lorentz transform the vielbein to make its spatial components real and single valued. Separating the magnitude and argument of the function $f(z) := R(z, \bar{z})e^{i\Xi(z,\bar{z})}$, we define the argument function as

$$\Xi(z, \bar{z}) = -\frac{\kappa}{2\pi}\sum_n m_n \arg(z - z_n) \tag{77}$$

Constructing the Lorentz matrix as

$$\begin{aligned} \Lambda &= \exp -i\boldsymbol{J}_{12}\Xi(z, \bar{z}) \\ &= \mathrm{diag}\left[1, e^{-i\Xi(z,\bar{z})}, e^{i\Xi(z,\bar{z})}\right] \end{aligned} \tag{78}$$

The transformed vielbein is

$$\tilde{e}^a = \begin{pmatrix} \mathrm{d}t - g(z)\mathrm{d}z + \bar{g}(\bar{z})\mathrm{d}\bar{z} \\ R(z,\bar{z})\mathrm{d}z \\ R(z,\bar{z})\mathrm{d}z \end{pmatrix}^a \tag{79}$$

and the spin connection is given by

$$\frac{1}{2}\omega^{ab}\boldsymbol{J}_{ab} = \Lambda \mathrm{d}\Lambda^{-1} = -i\mathrm{d}\Xi(z,\bar{z})\boldsymbol{J}_{12} \tag{80}$$

Now under a gauged translation parameterised by

$$\tilde{\rho}^a = \Lambda^a{}_b \rho^b = \begin{pmatrix} t \\ \exp[-i\Xi]\int^z f(\zeta)\mathrm{d}\zeta \\ \exp[+i\Xi]\int^{\bar{z}} \bar{f}(\bar{\zeta})\mathrm{d}\bar{\zeta} \end{pmatrix}^a \tag{81}$$

the full CSW connection will take the form

$$\boldsymbol{A} = \frac{\kappa}{2\pi}\mathrm{d}\left(\sum_n (m_n \boldsymbol{P}_0 + s_n \boldsymbol{J}_{12}) \arg(z - z_n)\right) \tag{82}$$

which is once again an

3.5 Doppelgängers and the conventional double copy

The point-source spacetimes of §3.4 are flat everywhere except at the particle locations, where the geometry carries distributional curvature and non-trivial holonomy. We now recast this same structure in double-copy language.

Consider Einstein's equations with point masses,

$$G_{\mu\nu} = 8\pi G\, T_{\mu\nu} = 8\pi G \sum_n M_n u_{n\mu} u_{n\nu}\, \delta^2(\mathbf{r} - \mathbf{r}_n). \tag{83}$$

The corresponding line element is [101, 102]

$$ds^2 = dt^2 - \prod_n |\mathbf{r} - \mathbf{r}_n|^{-8GM_n} \left(dr^2 + r^2 d\theta^2\right), \tag{84}$$

with scalar curvature

$$R = 16\pi G \sum_n M_n\, \delta^2(\mathbf{r} - \mathbf{r}_n), \tag{85}$$

so the curvature vanishes away from $r = r_n$. In $2+1$ dimensions, the full Riemann tensor is fixed by the Einstein tensor via

$$R_{\mu\nu\rho\sigma} = \varepsilon_{\mu\nu}{}^{\alpha}\varepsilon_{\rho\sigma}{}^{\beta} G_{\alpha\beta} = 8\pi G\, \varepsilon_{\mu\nu}{}^{\alpha}\varepsilon_{\rho\sigma}{}^{\beta} \sum_n M_n u_{n\alpha} u_{n\beta}\, \delta^2(\mathbf{r} - \mathbf{r}_n), \tag{86}$$

and we see that there really is no curvature for $r \neq r_n$. This "flat-off-source" profile suggests that any gauge-theory avatar should also be flat away from sources and topological in character, matching the holonomy picture above. Chern–Simons theory with point charges provides exactly this. Take

$$\mathcal{L}_{\rm CS} = \frac{k}{4\pi}\varepsilon^{\mu\nu\rho}\left(A^a_\mu\partial_\nu A^a_\rho + \frac{1}{3}f^{abc}A^a_\mu A^b_\nu A^c_\rho\right) + A^a_\mu J^{\mu a}, \tag{87}$$

with equations of motion

$$\frac{k}{2\pi}\varepsilon^{\mu\nu\rho}F^a_{\nu\rho} = J^{\mu a}, \qquad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc}A^b_\mu A^c_\nu, \tag{88}$$

and $D_\mu J^{\mu a} = 0$. For static Wong charges with charge g_n, three-vector u^μ_n and colour c^a_n, we have

$$J^{\mu a} = \sum_n g_n u^\mu_n c^a_n, \delta^{(2)}(\vec{r} - \vec{r}_n), \tag{89}$$

eq. (88) gives

$$F^a_{\mu\nu} = \frac{2\pi}{k}\varepsilon_{\mu\nu\rho}\sum_n g_n u^\rho_n c^a_n, \delta^{(2)}(\vec{r} - \vec{r}_n) = \sum_n F^a_{n\,\mu\nu}, \tag{90}$$

which is distributional and pure gauge for $r \neq r_n$, mirroring the gravitational case.

In $2+1$ dimensions the Weyl tensor vanishes identically, so the Weyl double copy doesn't make sense here, however we can identify a double copy at the level of the Riemann tensor (or, relatedly, the Einstein tensor)

$$R_{\mu\nu\rho\sigma} = \frac{k^2}{4\pi^2}\sum_i \frac{F^a_{i\,\mu\nu}F^a_{i\,\rho\sigma}}{\phi_i}\Bigg|_{g_i\to 8\pi G M_i}. \tag{91}$$

Here ϕ is the trace of the zeroth copy, defined by the non-propagating bi-adjoint scalar

$$\mathcal{L} = \phi^{a\bar{a}}\phi^{a\bar{a}} - \phi^{a\bar{a}}J^{a\bar{a}}, \qquad \phi^{a\bar{a}} = J^{a\bar{a}}, \tag{92}$$

so for point sources at r_n,

$$\phi^{a\bar{a}} = \sum_n g_n c^a_n c^{\bar{a}}_n\,\delta^{(2)}(\vec{r} - \vec{r}_n) = \sum_n \phi_n, \qquad \phi = \phi^{a\bar{a}}c^a c^{\bar{a}}. \tag{93}$$

As reviewed in §2, gravity in $2+1$ dimensions admits a Chern–Simons formulation. Writing

$$S_{CSW} = \int d^3x\ \eta_{ab}\left(\frac{k}{4\pi}\varepsilon^{\mu\nu\rho}e^a_\mu R^b_{\nu\rho}(\omega) + e^a_\mu T^{\mu b}\right),$$

with $R^a_{\mu\nu} = \partial_\mu\omega^a_\nu - \partial_\nu\omega^a_\mu + \varepsilon^a{}_{bc}\omega^b_\mu\omega^c_\nu$ and

$$T^{\mu a} = \sum_n M_n u^\mu_n u^a_n\,\delta^{(2)}(\vec{r} - \vec{r}_n),$$

the equations of motion (after contracting with $\varepsilon_{\mu\nu\alpha}$) read

$$R^a_{\nu\rho} = 8\pi G\,\varepsilon_{\mu\nu\alpha}T^{\alpha a} = 8\pi G\sum_n M_n\varepsilon_{\mu\nu\alpha}u^\alpha_n u^a_n\,\delta^{(2)}(\vec{r} - \vec{r}_n).$$

In this guise, performing the same double copy as above amounts to a doppelgänger identification: instead of 'squaring' the field strengths, one chooses the gauge algebra to be either $\mathfrak{g}$ (for the Yang–Mills Chern–Simons system) or $\mathfrak{iso}(2,1)$ (for the gravitational Chern–Simons system). The two descriptions are flat away from sources and encode identical holonomy data around each defect, matching the construction in §3.4.

NM: We could also point out that Einsteins equation also double copies directly

$$G_{\mu\nu} = 8\pi G \frac{F^a_\mu F^a_\nu}{\phi},$$

where $F^a_\mu = \frac{1}{2}\varepsilon_{\mu\nu\rho}F^{\nu\rho a}$.

4 Extensions to higher dimensions

5 Conclusion

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A Hidden Momentum and Spin

In §3.3, we identified Q^{ab} with the *total* (local) spin tensor, incorporating both intrinsic spin and spin-orbit contributions. We could have instead chosen to identify Q^{ab} only with the *intrinsic* spin, leading to a form of the MPD equations that are missing the term $p^a u^b - p^b u^a$. In effect, this sets $p^a \propto u^a$ and effectively ensures that any "hidden" momentum vanishes. This hidden momentum is defined as [103, 104]

$$p^a = m u^a + p^a_{\text{hidden}}, \tag{94}$$

and is closely related to the spin-supplementary condition (SSC) used. For the covariant (Tulczyjew-Dixon) SSC, given by $p_a S^{ab} = 0$, the hidden momentum only enters at order $S^{ab}S_{ab}$. This means that, making the identification of Q^{ab} with the the intrinsic spin only gives the MPD equations valid to linear in spin, but not beyond. However, it should be stressed that to properly consider spins beyond leading order, one typically includes many higher-dimensional operators involving contractions of the Riemann and spin tensors, which we will not consider here.

To see this more clearly, we can derive the hidden momentum directly from the covariant SSC by taking the derivative along the worldline

$$\begin{aligned}
\frac{D}{d\tau}(p_\mu S^{\mu\nu}) &= \frac{Dp_\mu}{d\tau}S^{\mu\nu} + p_\mu \frac{DS^{\mu\nu}}{d\tau} \\
&= -\frac{1}{2}R_{\mu\alpha ab}u^\alpha S^{ab}S^{\mu\nu} + p_\mu(p^\mu u^\nu - u^\mu p^\nu) \\
&= -\frac{1}{2}R_{\mu\alpha ab}u^\alpha S^{ab}S^{\mu\nu} + M^2 u^\nu - m p^\nu \\
&= 0
\end{aligned} \tag{95}$$

where we have used the MPD equations. Rearranging for p^μ, we discover that the hidden momentum is given by

$$p^\mu_{\text{hidden}} = \frac{1}{2m}P^\mu{}_\nu S^{\alpha\nu}R_{\alpha\beta ab}u^\beta S^{ab}, \tag{96}$$

where

$$P^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} - u^{\mu} u_{\nu}. \tag{97}$$

This confirms that p^{μ}_{hidden} is of order S^2.

To see this from the perspective of the Maurer–Cartan action, it is useful to express (p, q) in a particular body-fixed frame, where the covariant derivative can be written in terms of Ω_{ab}. Defining

$$\hat{q}^a = \Lambda^a{}_b q^b, \qquad \hat{p}_a = p_b (\Lambda^{-1})^b{}_a, \tag{98}$$

we can write

$$\mathcal{D}_\tau \hat{q}^a = D_\tau \hat{q}^a + \Omega^a{}_b \hat{q}^b \tag{99}$$

such that

$$S\big[q, p, \Sigma; \mathbf{A}\big] := \int_{\mathcal{C}} d\tau \Big[\hat{p}_a \mathcal{D}_\tau \hat{q}^a + \frac{1}{2} Q^{ab} \Omega_{ab} - \xi(p^2 + m^2)\Big] = \int_{\mathcal{C}} d\tau \Big[\hat{p}_a D_\tau \hat{q}^a + \frac{1}{2}(Q^{ab} + J^{ab}) \Omega_{ab} - \xi(p^2 + m^2)\Big], \tag{100}$$

where $J^{ab} = \hat{p}^a \hat{q}^b - \hat{p}^b \hat{q}^a$.

From this action, we get the EOM

$$D_\tau (Q^{ab} + J^{ab}) = 0, \qquad D_\tau \hat{p}^a = 0, \qquad D_\tau \hat{q}^a = 2\xi \hat{p}^q, \tag{101}$$

which is again the MPD equations for a, b as local Lorentz indices, or the Wong equations for a, b as $SU(N)$ indices. We note that we are free to choose $q^a = f(\tau_0) p^a$ at some instant τ_0, which renders $J^{ab}(\tau_0) = 0$. The difference between the spacetime and $SU(N)$ cases is that in the case of $SU(N)$ this colinearity can be chosen for all time τ, whereas in the spacetime case it is tied to the SSC, and can be chosen to vanish only at linear order, as described above.

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