Introduction to Googology

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2010 *Mathematics Subject Classification.* Primary
Key words and phrases. Googology, Mathematics

Abstract.

Contents

CHAPTER 1

The Beginning of Googology

1.1. The Simplest Unary Operation

1.1.1. The Positive Integers. The set of positive integers, $\mathbb{N}^*$, is defined as satisfying the following conditions:

(1) For any arbitrary positive integer a, its successor, a^+ is a positive integer as well.
(2) There exists a positive integer such that no positive integer's successor is that integer. Notate this as 1.
(3) Except for 1, every positive integer is the successor of a positive integer.
(4) If $1 \in S$, and whenever $n \in S$, $n^+ \in S$ as well; then $S = \mathbb{N}^*$.

But problems arise. Until now, we might have wondered how to express things like "1's successor," "1's successor's successor," and so on. Could they be expressed as 1^+, 1^{++}, etc?

For a long time, people defined [TODO] as follows: $1^+ = 2, 2^+ = 3, 3^+ = 4, 4^+ = 5, 5^+ = 6, 6^+ = 7, 7^+ = 8, 8^+ = 9, 9^+ = 10$. We use two digits in "10" to refer to the successor of the number 9.

Then, $10^+ = 11, 11^+ = 12, \ldots, 19^+ = 20, \ldots, 99^+ = 100, \ldots$ Thus, we now have a way to theoretically express any positive integer.

However, this simple operation by itself is practically useless. The really useful stuff is the following...

1.1.2. Addition. Addition is defined as follows, expressed using the symbol "+".

$$a + 1 = a^+$$

$$a + b^+ = (a + b)^+$$

very simple indeed! But this definition is too abstract, in fact we can define it like this:

$$a + b = a^{++++\ldots+}$$

(with b "+"es). This new definition is clearer.

However, sometimes addition isn't enough. Then we need...

1.1.3. Multiplication. Multiplication is defined as follows, expressed using the symbol "×".

$$a \times 1 = a$$

$$a \times b^+ = a \times b + a$$

This is also a very simple definition. A bit more accessible definition:

$$a \times b = a + a + a + \ldots + a$$

(with "b" many "a"s.) From this, one can find that $10 \times 10 = 100, 100 \times 10 = 1000, 100\ldots0$ (with n zeroes) $\times 10 = 100\ldots0$ (with $n+1$ zeroes.) This leads us into another operation:

1.1.4. Exponentiation. Exponentiation is usually denoted by the symbol "^", and defined as:

$$a\hat{\ }1 = a$$
$$a\hat{\ }b^{+} = a\hat{\ }b \times a$$

Exponentiation is neither commutative nor associative. Usually, it is right-associative, so $a\hat{\ }b\hat{\ }c$ gets interpreted as $a\hat{\ }(b\hat{\ }c)$.

Sometimes, we don't use any symbol to express exponentiation, instead choosing to write a^b where the b being on top signifies exponentiation. Then, $a^b = a\hat{\ }b$.

A more intuitive definition is: $a^b = a \times a \times \ldots \times a$ (with b "a"s). However, higher level operaations were not developed for a long time, until recently. People used to "create" many numbers, name them, and use them to express any positive integer.