Project Euler - Problem 103

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June 5, 2024

1 important

Theorem 1.1. *there are no non-trivial solutions to the equation $S * C = \mathbf{0}$, where*

$$C = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

and $*$ denotes the convolution operator with wrapping around the edges.

Proof. To start, we note 2 properties of $*$:

$$A * C + B * C = (A + B) * C,$$

$$\begin{bmatrix} n & \cdots & n \\ \vdots & \ddots & \vdots \\ n & \cdots & n \end{bmatrix} * C = \mathbf{0}$$

these combined, imply that we can "reduce" a move such that there exists at least one 0 entry, and all entries are nonegative. now, let's say that there exists a non-trivial (not every entry being equal) solution. Then we can reduce it. now we get a move where:

1. all of its entries are nonnegative and there exist at least one 0 entry. (since we reduced it)
2. at least one of its entry is positive. (otherwise it's trivial)

these 2 imply there exists a 0 entry such that at least one of its non-diagonal neighbours is positive. now, we can see that to be a solution of the equation, it must be that the convolution of the 0 entry along with its non-diagonal neighbours must be 0, which basically means

$$a + b + c + d = 0,$$

where a, b, c, and d are the neighbouring entries of the 0 entry but this is impossible, as at least one of them is positive, and the rest is non-negative, so the sum is always positive. $\square$

Proof. Let there be a toppling sequence $p_1, \cdots, p_n$ of cells and a specific cell p_i. Let's say it has k of its 4 orthogonal neigbours already toppled, giving it an advantage of k, making the minimum required number of stones in it before the cycle be $4 - k$. If we move it to the front of the sequence so that it's the first cell to be toppled, its minimum requirement now becomes 4, increasing the total requirement by k.
However, each of its k neighbours now get an additional advantage of 1 from another one of their neighbours (itself, more specifically p_i) getting toppled, hence decreasing the total minimum requirement by $1 \cdot k = k$, thus making the net change from the ordering change 0 $\square$