

# Understanding Orbital Resonance and Planetary Rings by Python Simulations

Po-Han Chen, Shi-Shen Fang

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## Abstract

This assay goes through some papers and try to understand orbital resonance and Saturn's ring by Python simulation. We checked the properties of large/small eccentricity orbital resonance. We tried to build a program of Saturn's ring system. We consider one of Saturn's moon, Mimas. Also self-gravity was taken into consideration. We found that high-dense region would form, propagate outward and disappear. We hoped that in the future we could add collision into our program.

## 1 Introduction

Gravity is an astonishing mechanism of the universe. It has short-term effect such as free falling on Earth, or long-term phenomenon like orbital resonance. It describes the simplest integer ratio(commensurability) of the orbit periods of objects. In a forced oscillation model, when the period of the force matches the natural period, the amplitude of the oscillation will increase. Orbital resonance shares the same idea about this. For example, three of Jupiter's moons, Io, Europa, Ganymede, hold a 1:2:4 orbital resonance, which means the orbital period is 1:2:4. Moreover, the effect contributes to the Neptune-Pluto system and Jupiter-asteroids belt.

## 2 Orbit-orbit resonance of satellites

The main cause of orbital resonance is one object periodically increase or decrease the angular momentum of the other object[Pea76].

### 2.1 Large eccentricity stabilization process

This process is about finite objects stabilize their orbits. It demands a non-zero eccentricity orbit. And the eccentricity of the orbit, where the conjunction happens can lead to different results.

#### 2.1.1 Stable equilibrium at apocenter

We can consider two satellites orbit counter-clock-wisely around a planet. The inner satellites, primary, has a circular orbit, with its mass  $m$ . The outer one, secondary, has a ecliptic orbit, and its mass  $m' \ll m$ . Let  $\omega$  be the initial phase of pericenter (perihelion). When the two satellites meets a bit before secondary's apocenter, at point A, we can note that the torque of dragging force is bigger than that of pulling force. Hence  $\Delta L < 0$ , and lead to increase in secondary's orbital angular speed. That means the next conjunction will be closer to apocenter. The same mechanism happens to conjunction after apocenter. In that case,  $\Delta L > 0$ , thus decrease in secondary's orbital angular speed. Also leads to a closer conjunction to apocenter. Hence it's a stable equilibrium.

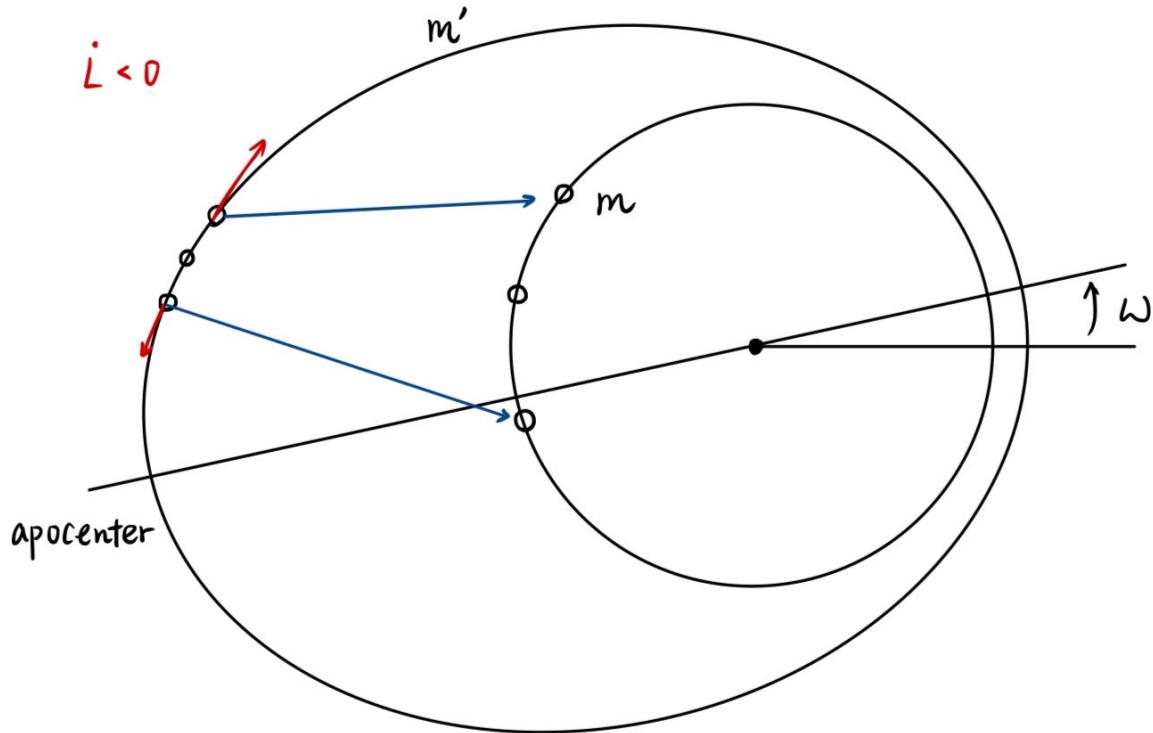


Figure 1: The torque assert on the outer planet by inner planet gradually make the conjunction to apocenter. That's because the orbits are diverging.

### 2.1.2 Unstable equilibrium at pericenter

But when the conjunction is near pericenter, the result is totally different. It's a unstable equilibrium. In Fig.2, we can notice that if the two satellites meets a bit after pericenter, the torque of pulling force in bigger than that of dragging force. Hence  $\Delta L > 0$ , the orbital angular speed of secondary increases, making next conjunction later, further to pericenter. On the contrary, the conjunction before pericenter makes next conjunction earlier.

### 2.1.3 Retrograding of line of apsides

If conjunctions keep occurring at apocenter, the radial force from m pull m' in. The orbit of m' is slightly inner than it used to be. This means that m' will reach its pericenter faster than usual. It comes out that the pericenter is receding, thus retrograding of apsidal line.

## 2.2 Small eccentricity stabilization process

When it comes to small eccentricity orbital, the mechanisms are different. Because of the orbit is almost circular, the location of conjunctions depends on the change of phase angle  $\omega$  and may not fix at the same position due to procession or recession. And the radial force dominates the change of  $\omega$ . But the small eccentricity situation is way less intuitive, hence we are not going to discuss its details here. Overall, it's about eccentricity change leading to the rate change of retrograde or prograde of apsidal line. It makes both apocenter and pericenter vibrates about the conjunction point and both of them are stable equilibrium[BS33].

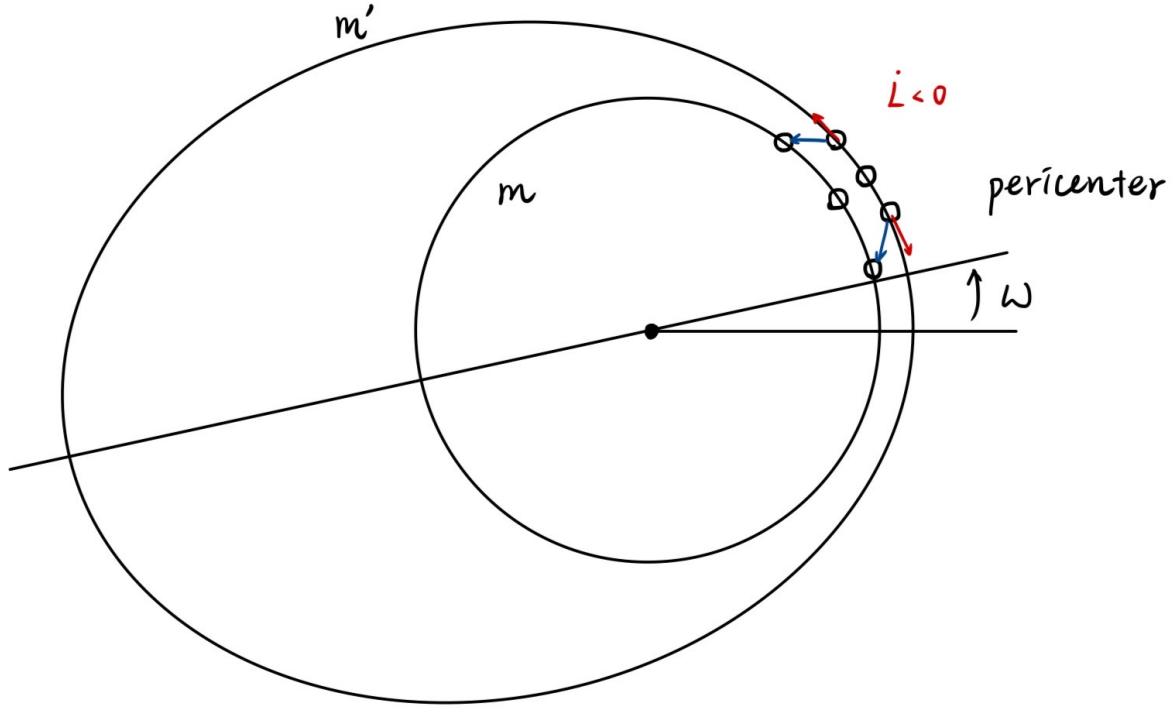


Figure 2: The torque assert on the outer planet by inner planet make the conjunction unstable. That's because the orbits are diverging, too.

## 2.3 Simulations

### 2.3.1 Setting up

We take Jupiter's two satellites' as our model, which are Io(Primary) and Europa(Secondary). We consider Io's orbital a perfect circle with orbital period 42.5 hours. And we use Kepler's third law to determine the value of Europa's semi-major axes.  $a_{Europa} = 2^{2/3}r_{Io}$ . Next by the equation of elliptical orbit:

$$r = \frac{a(1-e^2)}{1+e\cos\theta},$$

where  $a$ ,  $e$ ,  $\theta$  are length of semi-major axes, eccentricity, phase of the orbit(if  $\theta = 0$  its at its pericenter). And let  $\omega$  be the initial angle of pericenter.

### 2.3.2 Conjunction at pericenter or apocenter

At eccentricity=0.1, according to Fig.3, when the conjunction happens near apocenter, the secondary's orbital period is well fixed at about 85 hours. We think that's a significant resonance system. But when conjunction is near pericenter, secondary' orbital period changes a lot, the resonance seems not to exist.

At eccentricity=0.05, according to Fig.4, when the conjunction happens near both apocenter and pericenter, the secondary's orbital periods are roughly fixed at about 85 hours, with  $|P_i - \mu| < 0.25$ . But interestingly, the conjunction at apocenter holds a resonance at 85.25 hours. We still need to dig further into the orbit pattern to see what happened. Among above, we think that 0.1 is suitable for large eccentricity model, 0.05 for small eccentricity model.

### 2.3.3 Different eccentricity

This simulation shows the conjunction at apocenter and  $\omega = 0.1rad$  and at different eccentricity(Fig.5). For the eccentricity of 0.1, 0.05, 0.009 ,the period is all about 85 hours. And the amplitudes of fluctuation are 0.1, 0.2, 0.05 hour. Hence we thought that all the three conditions are in orbit-orbit resonance. But we can't explain why when eccentricity is 0.05 the period fluctuate the most.

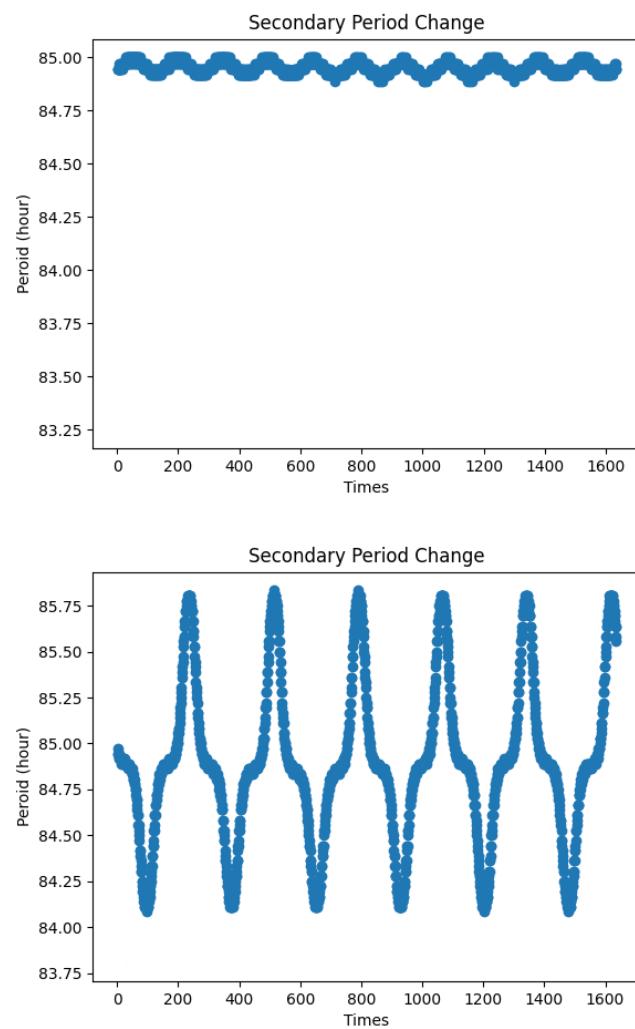


Figure 3: The  $n$ -th times orbital period(hr) of eccentricity=0.1, conjunctions near apocenter(up) and pericenter(down)

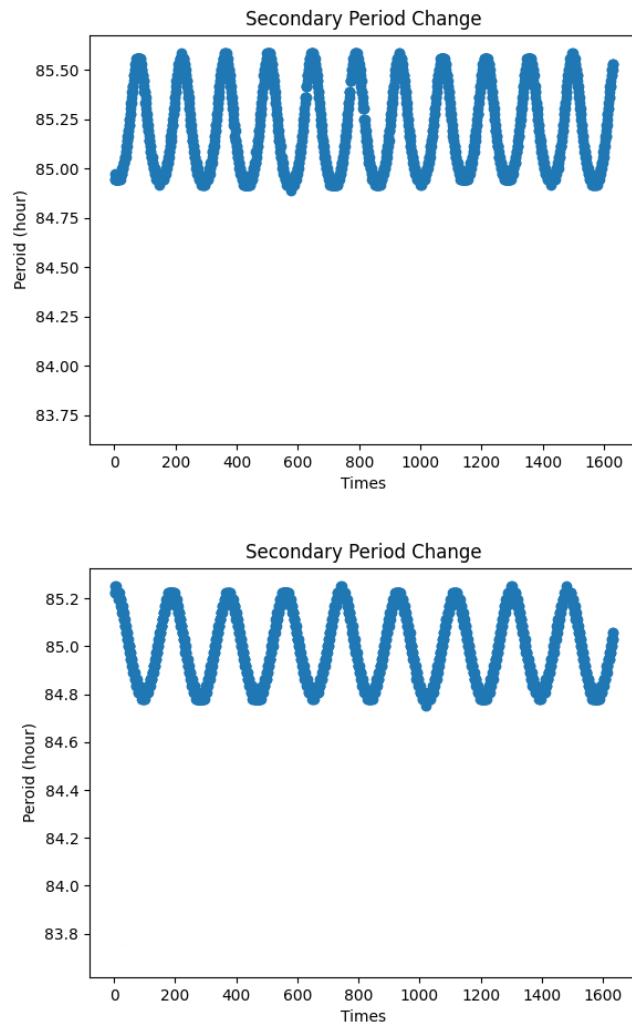


Figure 4: The  $n$ -th times orbital period(hr) of eccentricity=0.05, conjunctions near apocenter(up) and pericenter(down)

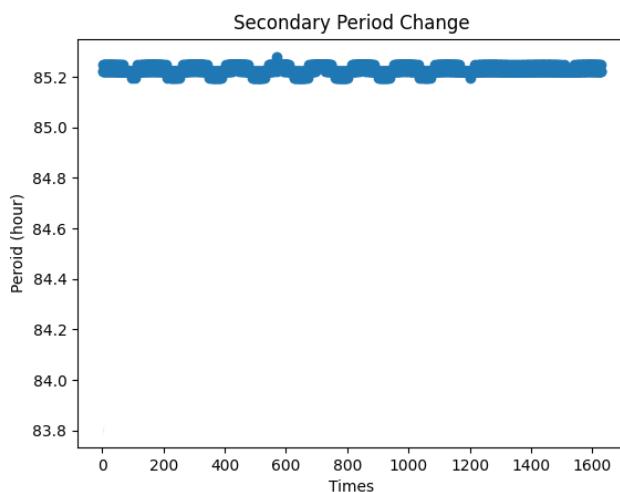
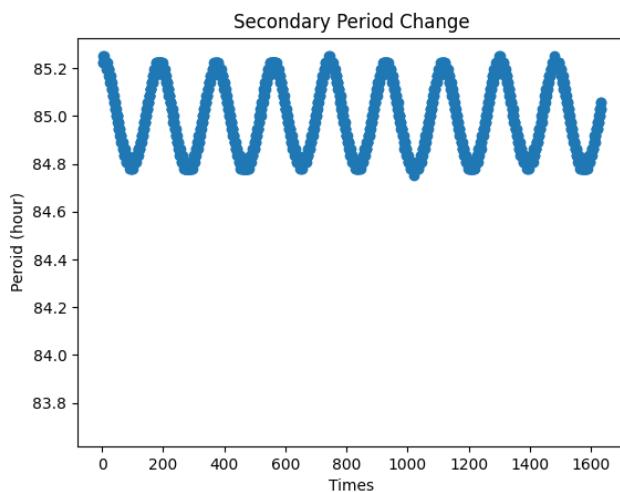
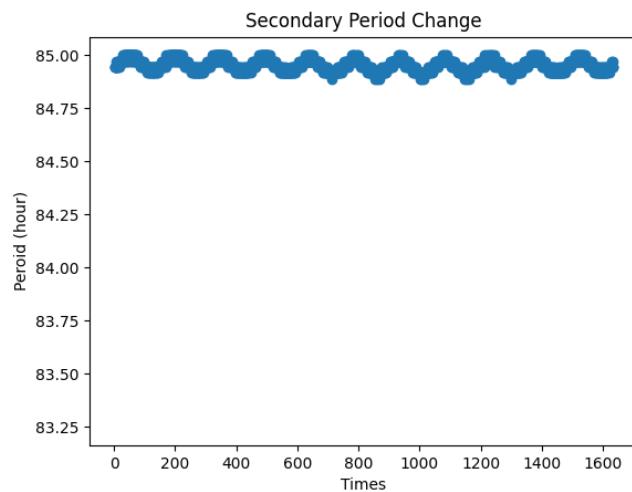


Figure 5: The above one is  $e=0.1$ , middle is  $e=0.05$ , below is  $e=0.09$

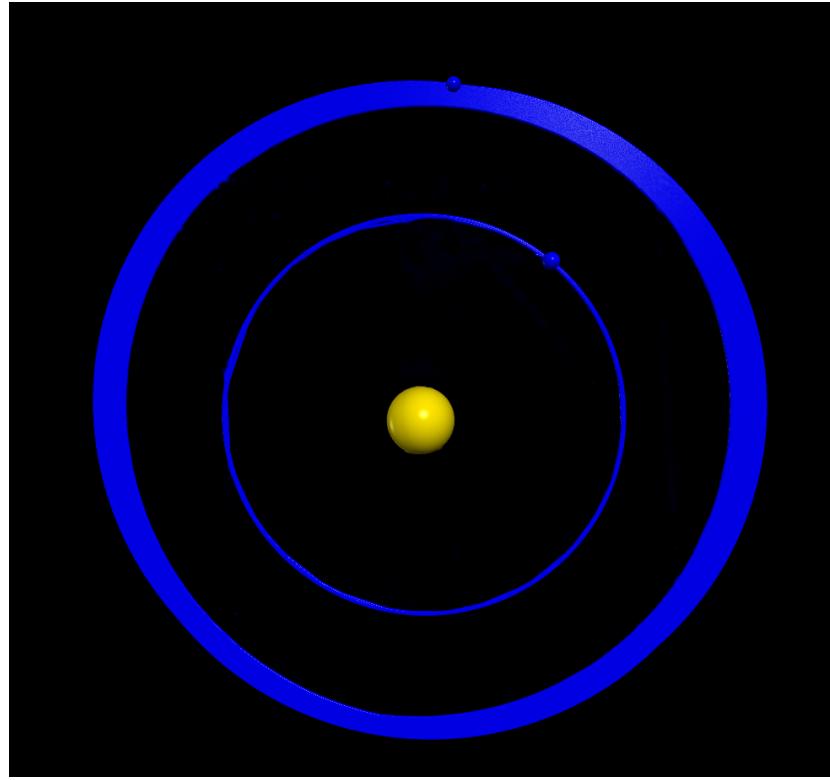


Figure 6: Secondary's retrograde motion of apsidal line is obvious in this figure

#### 2.3.4 Retrograde apsidal line

During the programming, we found that most of the tests feature retrograding apsidal line. Fig.6 is the test of eccentricity = 0.05 and conjunction near apocenter.

### 3 Saturn's rings

#### 3.1 Composition and structure

Saturn's rings consist of many little rocks and water ice. The structure of it is very complex. There are gaps, waves and web-like patters. For the biggest structure, Cassini Division is at the location of 1/2 period of Mimas(a Saturn's moon). The formation of this division is due to orbital resonance of Mimas. When the ring particles go in to a higher eccentric orbit by resonance. The particle density surges. More collisions happens between particles at Cassini Division. And collision takes away energy hence the particles turn out leaving the orbit. The huge gap thus formed.

#### 3.2 Collision

When a region is dense enough, the collision becomes significant. Former research found that when the relative speed rate is low, the collision is almost elastic and when the relative speed goes up the coefficient of restitution drops[BGT84].

#### 3.3 Resonances

Like forced oscillation, when the small particle's natural frequency of vertical, horizontal or matches the frequency that the moons pass by. It will suffer resonance. The horizontal resonance is also known as Lindblad resonance. The strongest inner horizontal resonances correspond to period ratio  $n : n - 1$ . The strongest inner vertical resonances correspond to period ratio  $n + 1 : n - 1$ [Shu84][Esp14].

### 3.4 Simulations without self-gravity

Each simulation is at  $3 \times 10^6 s$  scale, and each costs about half day to run.

#### 3.4.1 Setting up

We choose Saturn's B to F ring, from  $9.2 \times 10^7$  to  $1.4 \times 10^8 m$ , including the location of Cassini Division. We randomly put 5000 particles in this area, and use Mimas to affect ring structure. Self-gravity and collisions are invalid in this code. We record every ring particle's radius distribution every 1000 seconds.

#### 3.4.2 High-dense area propagate outward

From the start(t=0)(Fig.7), the particles are randomly distributed. But at around t=46,000 particles start to gather around at the inner location, and start to propagate outward(t=70,000). It repeats again and again. No specific a location with resonance is shown. The difference of number density start to be significant at around t=200,000. But since t=900,000, the high-dense regions have been eaten little by little and finally disappear. And we don't know what mechanism is behind this.

### 3.5 Simulations with self-gravity

Each simulation is at  $5 \times 10^5 s$  scale, and each costs about 1 day to run.

#### 3.5.1 Setting up

A same model was adapted. But self-gravity of particles is validated, collisions are still invalid in this code.

#### 3.5.2 Self-gravity seems not putting effects on forming dense region

Because simulation with self-gravity is way slower, we compare from t=0 to t=300,000. All the shape of radius distribution(Fig.8) is about the same as no self-gravity. The only subtle difference is that when t=200,000, the one with self-gravity gathers the particles better, with more particles at same distance. We suppose that's because self-gravity amplifies the effect of gathering.

## 4 Conclusion

- At large eccentricity, conjunction near pericenter is not stable. But conjunction near apocenter is stable.
- At small eccentricity, conjunctions near both pericenter and apocenter are stable.
- For conjunctions at different eccentricity,  $e=0.05$  has biggest fluctuation.
- When collision is not valid it does not seem to have stably fixed high-dense region.
- Self-gravity does not contribute that much.

## 5 Future prospects

Because underestimating the time of programming/running and analysis, we still had some simulations want to make. As for the first part of this assay, two satellites' orbit-orbit resonance, we wanted to build a simulation of outer planet affecting inner planet and to see at different orbit eccentricity the behavior is the same or not. As for the second part of this assay, we wanted to modify the code, adding 2 dimension collision process. But the running time of code would multiply 2 times than only self-gravity validated. Hence we didn't have time to do such try. Last but not least, we both agreed that we still need to learn more. Because if we want to do the analysis of resonance system, the ability of some advanced mechanics is demanded. We hoped after some times, we could learn more and move on!

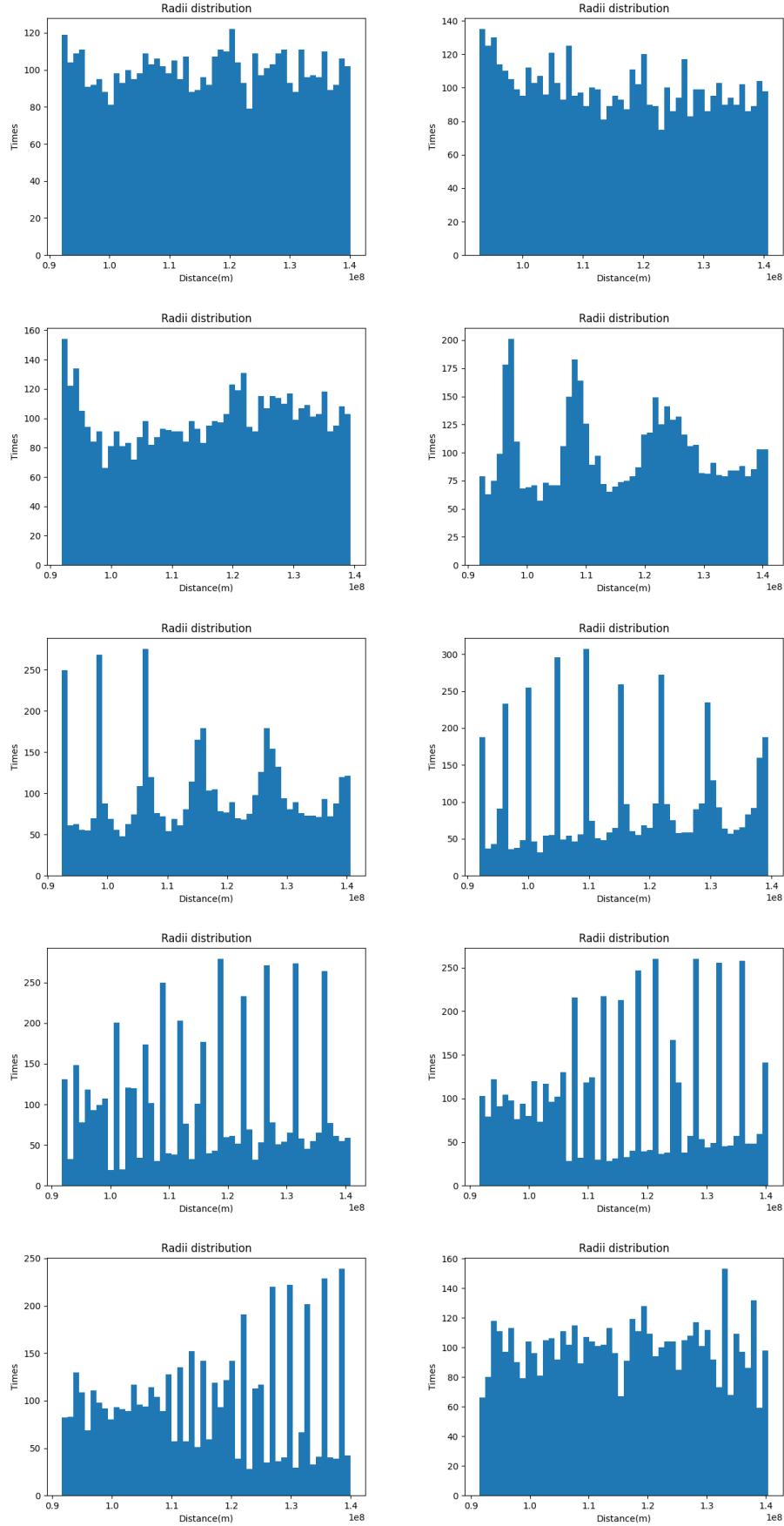


Figure 7: The high-dense area gradually formed and disappeared. The y-axis should be "density". up-to-right, up-to-down:  $t=0$ ,  $t=46000$ ,  $t=70000$ ,  $t=200000$ ,  $t=300000$ ,  $t=500000$ ,  $t=900000$ ,  $t=1100000$ ,  $t=1500000$ ,  $t=2700000$ ,

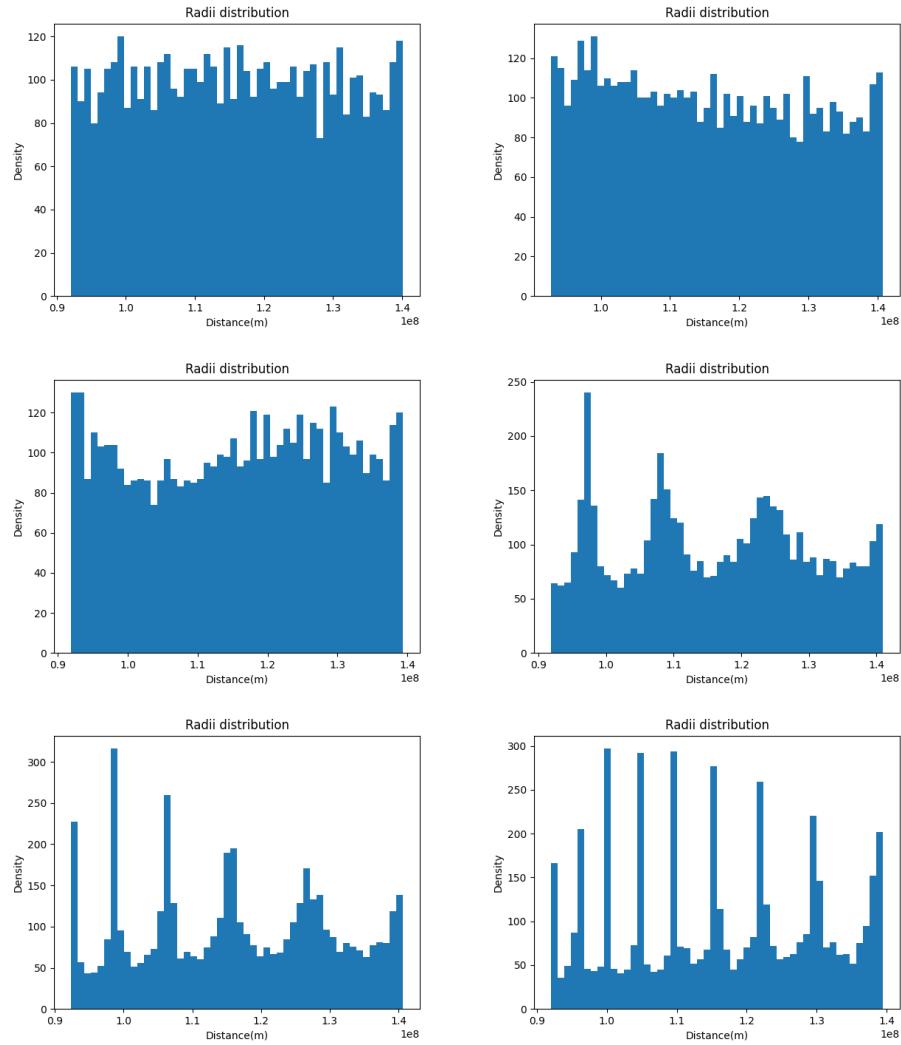


Figure 8: Validate self-gravity. up-left  $t=0$ , up-right  $t=46000$ , mid-left  $t=70000$ , mid-right  $t=200000$ , down-left  $t=300000$ , down-right  $t=500000$

## 6 Acknowledgements

Sincerely thanks to my friend and schoolmate in high school Fang, Shi-Shen. He taught me how to build a proper program and always is there when I am stuck. The origin program of planet rings was built by Fang. He indeed helped a lot. And that's why I put him in the co-author. Besides, this is our first English assay. Thanks to our courage and perseverance. Although this assay isn't good enough, We are satisfied with the results!

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