10th
Bangladesh Mathematical Olympiad: Selected Problems and Solutions

Editor: *Masum Billal*

Special Thanks:
Nur Muhammad Shafiullah
Md Sanzeed Anwar
Asif-E-Elahi
Nayeemul Islam Swad

Prologue

This booklet contains selected problems used in the training and selection process of the IMO team that participated in IMO 2015. Many of the problems are taken from IMO Shortlisted problems. And many other are taken from other olympiads. We are grateful to the problem setters of those problems. They helped a lot in our training process. We hope it wouldn't raise any legal issue related to copyrights for using those problems since they were by no means used for any commercial gain. So we apologize in advance if it's any inconvenience for anyone. Also, thanks to anyone who contributed to the training process in any way, including our **MOVER**s(Math Olympiad Volunteers), who took care of the participants in the camp.

I am very grateful to Mahi, Sanzeed, Asif and Swad for their time and contribution. At first, I wanted to create this document all by myself. But later I realized I don't have enough time for that. So, I invited Mahi. Later on I had to invite others too because we both got busy. Whereas it should have been published in 2015, I couldn't do it until now. Therefore, it goes without saying that they had a lot to do with it.

Another point I should mention is that, all problems may not have solution right now or some might contain typos. Probably in a later version, we will update it. If there are any typos or errors in solutions or any suggestions, feel free to email me: billalmasum93@gmail.com

Masum Billal

You can use this document in any form as long as you don't benefit commercially. Moreover, one of my primary motivations to create this document was to encourage other countries to publish their booklets as well. Because many countries tend to keep their training problems and materials secret. Therefore, you can share it as much as you want, and also enable others to share their booklet too.

Bangladesh Mathematical Olympiad

In Bangladesh, students face at least twelve stages of primary, secondary and higher secondary education. Excluding pre-school studies, one has to study in classes $1-12$. Grades 1 to 5 are considered primary, $6-10$ secondary and $11-12$ is higher secondary. Mathematical competitions in Bangladesh are divided into four categories:

1. **Primary** Students of class $1-5$.
2. **Junior** Students of class $6-8$.
3. **Secondary** Students of class $9-10$.
4. **Higher Secondary** Students of class $11-12$.

It is to be noted that, we treat the participants of secondary and higher secondary category almost equally. Therefore, most problems posed for these two categories are about same.

Two contests are held: one on a regional level and the other on a national level. At first, regional contests are held in different districts, 21 this year. In a district, a school provides the venue of the regional olympiad. Participants who are awarded gets to participate in the national olympiad. The olympiads take place in a festive manner and the national level olympiad is known as **BdMO**(Bangladesh Mathematical Olympiad). Around 40 participants are chosen as campers of the *national math camp*, where some exams are held in order to determine the team for the IMO. Sometimes, there is an extension camp, where around 20 campers are called for in order to take part in mock exams of **Team Selection Tests**. Finally a pool of at most six students is selected to represent Bangladesh at the International Mathematical Olympiad.

IMO Contestants of 2015

From right to left in figure (1), the members are:

- Asif E Elahi
  2015 Bronze, 2014 HM
- Nayeemul Islam Swad
  2015 HM
- Adib Hasan
  2015, 14, 13 Bronze, 2012 HM
- Sazid Akhter Turzo
  2015 Bronze, 2014 HM
- Sanzeed Anwar
  2015 Silver, 2014 HM
- Sabbir Rahman Abir
  2015 Bronze

Figure 1: Bangladesh IMO Team 2015, at the IMO Camp

Trainer Panel of 2015

This year the following trainers contributed in the math camps by taking classes and setting problemsets.

1. Dr. Mahbub Majumdar (coach of BdMO and leader of our IMO team)
2. Masum Billal
3. Nur Muhammad Shafiullah

Special thanks to Muhammad Milon(A BIG THANK YOU to him. He cheered up and entertained everyone throughout his classes when all the campers were in the ICU called national math camp) and Zadid Hasan.

Contents

Notations

- a divides b is denoted by $a|b$
- $(a, b) = \gcd(a, b)$ is the greatest common divisor of a and b.
- $[a, b] = \operatorname{lcm}(a, b)$ is the least common multiple of a and b.
- $\tau(a)$ is the number of divisors of a.
- $\sigma(n)$ is the sum of divisors of n.
- $\varphi(n)$ is the number of positive integers less than or equal to n which are co-prime to n.
- $\pi(n)$ is the number of primes less than or equal to n.
- $\nu_p(n) = \alpha$ is the largest positive integer so that $p^\alpha|n$ but $p^\alpha \not| n$.
- $\Lambda(n)$ is the *Van Mangoldt Function*.

Chapter 1

National Olympiad Problems

1.1. Primary Category

Problem 1.1.1. Write down all the prime numbers in the range of 1 to 50.

Solution. $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47$

Problem 1.1.2. Four people A, B, C and D have an average monthly income of 10000 taka. First three of them have an average monthly income of 12000 taka. Average income of first two of them is 15000 taka. Find the monthly income of B, C and D if A has a monthly income of 20000 taka.

Solution. Let a, b, c, d denote their respective incomes. Then the given conditions are:

$$\frac{a+b+c+d}{4} = 10000 \Rightarrow a+b+c+d = 40000 \tag{1.1.1}$$

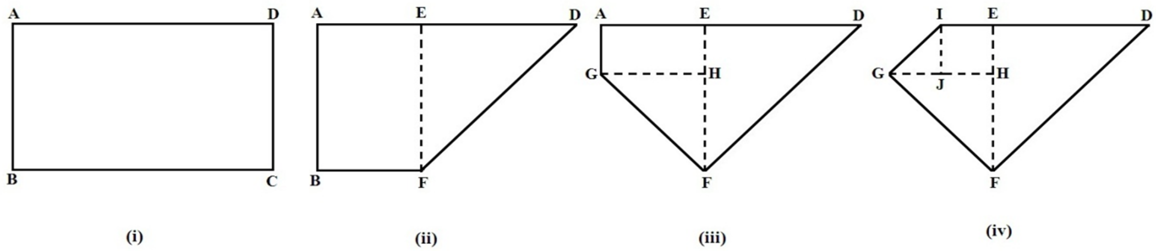
$$\frac{a+b+c}{3} = 12000 \Rightarrow a+b+c = 36000 \tag{1.1.2}$$

$$\frac{a+b}{2} = 15000 \Rightarrow a+b = 30000 \tag{1.1.3}$$

$$a = 20000 \tag{1.1.4}$$

(3) and (4) $\Rightarrow b = 10000$
(2) and (3) $\Rightarrow c = 6000$
(1) and (2) $\Rightarrow d = 4000$

Problem 1.1.3. In the following figures a rectangular piece of paper $ABCD$ has been folded several times. First, the side CD was made to fall on the line AD. Point E in figure (ii) represents the point C after folding. In the next figure the portion BF was made to fall on EF. Lastly, the side AG was made to fall on GH. Find the lengths of GJ, IJ, IE, ED, EH and HF. It is given that $AB = 8$ and $BC = 15$.



Solution.

Solution $ED = EF = AB = 8$

$HF = BF = AE = AD - ED = BC - ED = 15 - 8 = 7$

$EH = EF - HF = AB - HF = 8 - 7 = 1$

$GJ = GA = EH = 1$

$IJ = EH = 1$

$IE = AE - AI = BF - AI = HF - AI = HF - GJ = 7 - 1 = 6$

Problem 1.1.4. A circus party has the same number of lions as tigers. You asked to the owner of the circus the number of lions and tigers. He gave you the following information:

i. An elephant is enough to feed all the tigers and lions in the circus.

ii. Eighteen deers produce the same amount of meat as an elephant does.

iii. A lion eats twice as much as a tiger.

iv. One buffalo is enough to feed a lion and a tiger.

v. A tiger will eat exactly the same amount of meat a deer has.

Find the number of tigers and lions in that circus party.

Solution. Let the number of tigers(and lions)be x.

1. All of $2x$ animals eat in total $3x$(a single tiger's food).

2. $3x$(a single tiger's food) = an elephant.

3. $3x$(a single tiger's food) = 18 deer.

4. $3x$(a single tiger's food) $= 18$(a single tiger's food)

So, $3x = 18 \Rightarrow x = 6$.

Problem 1.1.5. Surjo is four years old and he is learning to write numbers. His math notebook looks like a square grid with 20 rows and 20 columns. He usually writes the numbers from top to bottom and when one column is finished he starts writing along the next column. One day he starts writing the numbers from left to right (along the rows). How many of the numbers will be placed in exactly the same place where it would have appeared if he had written along the columns?

Solution. Let n be such a number which remained in the position in both of the writing methods.
Let x and y be the row and column number of n, respectively, $1 \leq x, y \leq 20$.
Then following the order of the numbers in the vertical writing method,

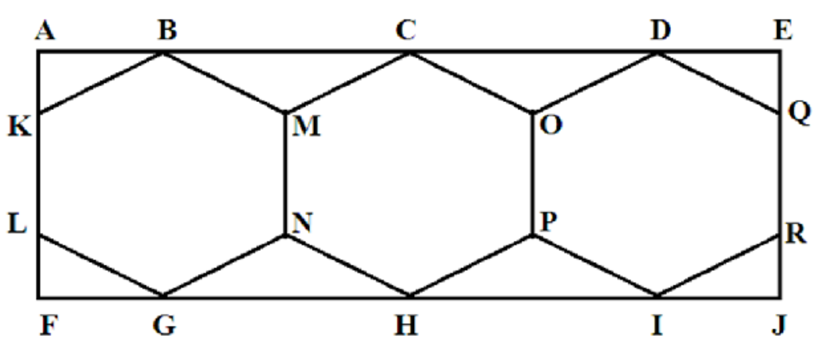
$$n = 20(y-1) + x$$

Again by the horizontal writing method,

$$\begin{aligned} & n = 20(x-1) + y \\ \therefore\ & 20(y-1) + x = n = 20(x-1) + y \\ \Rightarrow\ & x = y \end{aligned}$$

So, x must be equal to y and there are 20 such pairs. So they correspond to 20 possible values for n.

Problem 1.1.6. In the following figure $BKLGNM, CMNHPO$ and $DOPIRQ$ are regular hexagons (all six sides of each hexagon are equal and so are the angles). $BKLGNM$ has an area of 24 square units. What is the area of the rectangle $AFJE$?



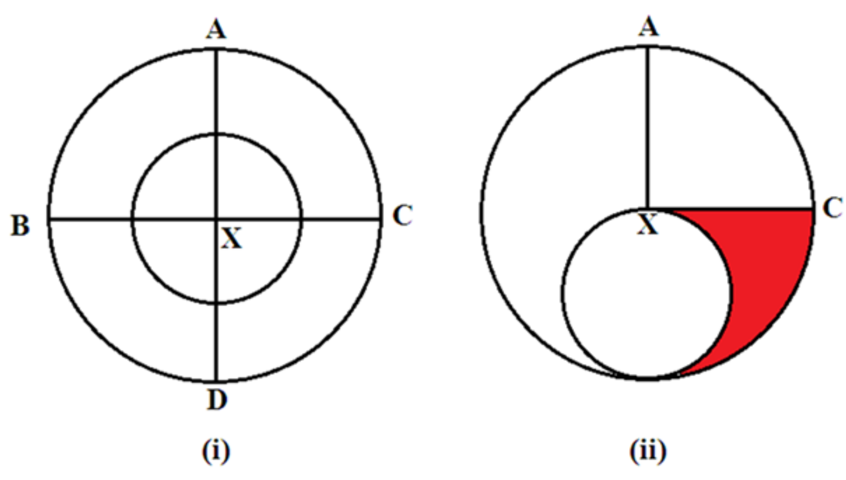
Solution. Let the center of the hexagon $BKLGNM$ be O and
$OB = OG = \frac{AF}{2} = a$. Then
$\text{area}[BKLGNM] = 6\times \text{ area}[OBK]$
$\Rightarrow \text{area}[OBK] = \frac{24}{6} = 4$
$\Rightarrow \frac{\sqrt{3}a^2}{4} = 4$
$\Rightarrow a = \frac{4}{\sqrt[4]{3}}$
$\Rightarrow AF = \frac{8}{\sqrt[4]{3}}$
Again, $\triangle OKL$ equilateral and with side-length a, so, altitude $= \frac{\sqrt{3}a}{2} = 2\sqrt[4]{3}$
So, $FJ = 6 \times$ altitude of $\triangle OKL = 12\sqrt[4]{3}$
$\therefore \text{area}[AFJE] = AF \times FJ = 96$

1.2. Junior Category

Problem 1.2.1. A small country has a very simple language. People there have only two letters and all their words have exactly seven letters. Calculate the maximum number of words people can use in that country.

Solution. There are two possibilities for each letter. So 2^7 possibilities for the 7 letters. So they can use at most 2^7 words.

Problem 1.2.2. In the following figures, the larger circles are identical and so are the smaller ones. In (i) the circles have a common center and the lines AD and BC divide both the circles in four equal halves. The larger circle has an area of 100 square meters. Find the area of the shaded region in figure(ii).



Solution. area[circle$ABCD$] $= 100 \Rightarrow$ area$[XDC] = 25$

radius of ABCD $= \sqrt{\dfrac{\text{area}[ABCD]}{\pi}} = \sqrt{100/\pi} = 2\times$radius of small circle

So, area[small circle] $= \pi \left(\dfrac{5}{\pi}\right)^2 = \frac{25}{\pi}$

$\therefore$ area of the shaded region $=$ area$[XDC] - \dfrac{\text{area[small circle]}}{2}$

$$= 25 - \frac{\frac{25}{\pi}}{2}$$

Problem 1.2.3. A circus party has the same number of lions as tigers. You asked to the owner of the circus the number of lions and tigers. He gave you the following information:

i. An elephant is enough to feed all the tigers and lions in the circus.

ii. Eighteen deers produce the same amount of meat as an elephant does.

iii. A lion eats twice as much as a tiger.

iv. One buffalo is enough to feed a lion and a tiger.

v. A tiger will eat exactly the same amount of meat a deer has.

Find the number of tigers and lions in that circus party.

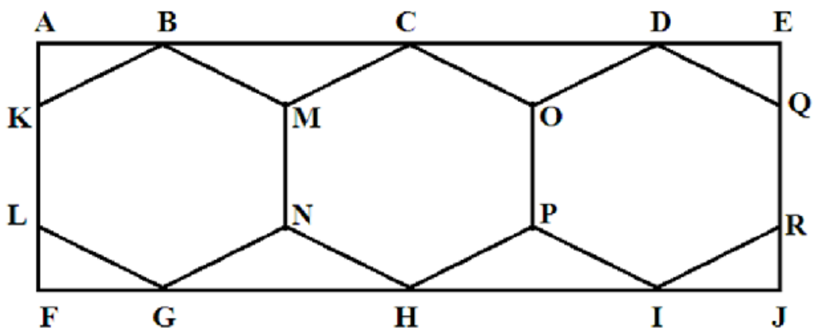
Solution. Let the number of tigers(and lions)be x.

1. All of $2x$ animals eat in total $3x$(a single tiger's food).
2. $3x$(a single tiger's food) = an elephant.
3. $3x$(a single tiger's food) = 18 deer.
4. $3x$(a single tiger's food) = 18(a single tiger's food)

So, $3x = 18 \Rightarrow x = 6$.

Problem 1.2.4. In the following figure $BKLGNM, CMNHPO$ and $DOPIRQ$ are regular hexagons (all six sides of each hexagon are equal and so are the angles). $BKLGNM$ has an area of 24 square units. What is the area of the rectangle $AFJE$?

Solution. Let the center of the hexagon $BKLGNM$ be O and
$OB = OG = \dfrac{AF}{2} = a$. Then
area$[BKLGNM] = 6\times$ area$[OBK]$



$\Rightarrow$ area$[OBK] = \dfrac{24}{6} = 4$
$\Rightarrow \dfrac{\sqrt{3}a^2}{4} = 4$
$\Rightarrow a = \dfrac{4}{\sqrt[4]{3}}$
$\Rightarrow AF = \dfrac{8}{\sqrt[4]{3}}$
Again, $\triangle OKL$ equilateral and with side-length a, so, altitude $= \dfrac{\sqrt{3}a}{2} = 2\sqrt[4]{3}$
So, $FJ = 6 \times$ altitude of $\triangle OKL = 12\sqrt[4]{3}$
$\therefore$ area$[AFJE] = AF \times FJ = 96$

Problem 1.2.5. In a party, boys shake hands with girls only but each girl shakes hands with everyone else. If there are total 40 handshakes, find the number (more than one) of boys and girls in the party.

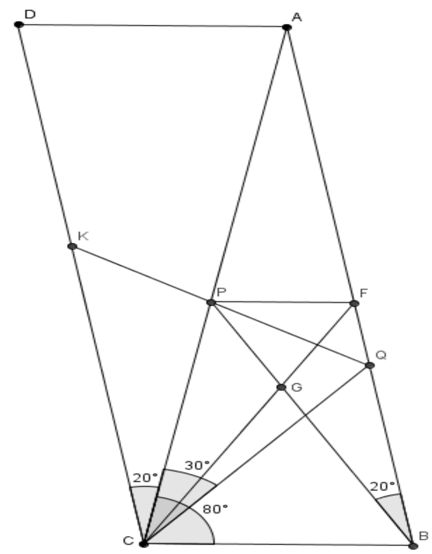
Solution. Let the number of boys in the party be x and the number of girls be y. Then each boy shakes hands exactly y times and each girl shakes hands $y + (x - 1)$ times. So the total number of handshakes will be $xy + y(y + x - 1) = y(2x + y - 1)$ $\therefore y(2x + y - 1) = 40$ Now a little checking for y over the factors of 40 shows us that only for $y = 5 (y > 1)$ we get a positive integral value for $x (= 8)$.

Problem 1.2.6. $ABCD$ is a parallelogram, where $\angle ACB = 80°$, $\angle ACD = 20°$. P is a point on AC such that, $\angle ABP = 20°$ and Q is a point on AB such that $\angle ACQ = 30°$. Find the magnitude of the angle determined by the lines CD and PQ.

Solution. Let PQ meet CD at K and the parallel from P to BC meet AB at F. Let CF meet BP at G. Since $\triangle BCG$ is equilateral, $BG = BC$. Since $\triangle CBQ$ is isosceles $BQ = BC$. Hence $\triangle BGQ$ is isosceles,

$$\angle BGQ = 80°, \angle FGQ = 40°$$

Since $\angle QFG = 40°$, $\triangle FQG$ is isosceles and $FQ = QG$. Also $PF = PG$. Hence $\triangle GPQ \cong \triangle FPQ, PQ$ bisects $\angle FPG$, and $\angle QPB = 30°$. Now $\angle CKQ = \angle CPQ - \angle KCP = (\angle CPB + \angle BPQ) - \angle KCP = (40° + 30°) - 20° = 50°$.



1.3. Secondary Category

Problem 1.3.1. A crime is committed during the hortal. There are four witnesses. The witnesses are logicians and make the following statements.

- Witness one says exactly one of the witnesses are liar
- Witness one says exactly two of the witnesses are liar
- Witness one says exactly three of the witnesses are liar
- Witness one says exactly four of the witnesses are liar

Assume that each of the statements are true or false. Find the number of liar witnesses.

Solution. All the 4 witnesses provided 4 different kind of informations and any two of them cannot be true at the same time. So there can be at most 1 truthful. Again all 4 of them cannot be liar otherwise the 4th person will be truthful. So there are exactly 3 liars.

Problem 1.3.2. There were 36 participants in a BdMO event. Some of the participants shook hand with each other. No two of them shook hands with each more than once. It was found that no two participants with the same number of handshakes made, had shaken hands each other. Find the maximum number of handshakes at the party.

Solution. Suppose that the number of participants who shook hands with exactly i other participants is $f(i)$. Then, due to the given condition, $f(i) \leq 36 - i$. Now, the total number of handshakes is $\frac{1}{2}\sum_{i=0}^{35} if(i)$. Thus,

$$\frac{1}{2}\sum_{i=0}^{35} i \cdot f(i) \leq \frac{1}{2}\sum_{i=0}^{35} i(36-i) = 3885$$

Thus the maximum number of handshakes at the party is 3885. It's left to the reader to find an appropriate construction with 3885 handshakes.

Problem 1.3.3. A tetrahedron is a polyhedron composed of 4 triangular faces. Faces ABC and BCD of tetrahedron $ABCD$ meet at and angle of $\frac{\pi}{6}$. The area of $\triangle ABC$ and $\triangle BCD$ are 120 and 80 resp. where $BC = 10$. What is the volume of the tetrahedron? (The volume of a tetrahedron is one third the area of it's base times its height)

Solution. Let P and Q be the projection of A on the plane BCD and line BC resp.

Then
$$(ABC) = \frac{1}{2} \times BC \times AQ \Longrightarrow AQ = \frac{2 \times 120}{10} = 24$$

Again $\angle AQP = 30°$ and $\angle APQ = 90°$. So $AP = AQ \times \sin 30° = \frac{24}{2} = 12$

$\therefore$ volume of tetrahedron $ABCD = \frac{1}{3} \times (BCD) \times AP = \frac{1}{3} \times 80 \times 12 = 320.$

Problem 1.3.4. Trapezoid $ABCD$ has sides $AB = 92, BC = 50, CD = 19, AD = 70$.The side AB is parallel to CD. A circle with center P on AB is drawn tangent to BC and AD. Given that $AP = \frac{m}{n}$ where m and n are coprime positive integers. Find $m+n$?

Solution. Let the circle touches AC and BD at Q and R resp and $AD \cap BC = S$.

Then $PQ \perp BC$ and $PR \perp AD$. So

$$\begin{aligned}
PQ = PR &\implies PB.\sin\angle PBQ = PA.\sin\angle PAR \\
&\implies \frac{PB}{PA} = \frac{\sin\angle BAS}{\sin\angle ABS} \\
&\implies \frac{AB - PA}{PA} = \frac{BS}{AS} \\
&\implies \frac{92}{PA} - 1 = \frac{BC}{AD} \\
&\implies \frac{92}{PA} = 1 + \frac{50}{70} = \frac{12}{7} \\
&\implies PA = \frac{92 \times 7}{12} = \frac{161}{3}
\end{aligned}$$

$\therefore m + n = 164.$

Problem 1.3.5. In $\triangle ABC$, A', B', C' are on sides BC, CA, AB resp. Also AA', BB', CC" are concurrent at O. Also, $\frac{AO}{OA'} + \frac{BO}{OB'} + \frac{CO}{OC'} = 92$. Find $\frac{AO}{OA'}\frac{BO}{OB'}\frac{CO}{OC'}$.

Solution. Let $(BOC) = p, (COA) = q$ and $(AOC) = r$.

$$\frac{AO}{OA'} = \frac{(ABO)}{(OBA')} = \frac{(ACO)}{(OCA')} = \frac{(ABO) + (ACO)}{(OBA') + (OCA')} = \frac{q + r}{p}$$

Similarly $\frac{BO}{OB'} = \frac{r+p}{q}$ and $\frac{CO}{OC'} = \frac{p+q}{r}$.

Therefore $\frac{AO}{OA'} + \frac{BO}{OB'} + \frac{CO}{OC'} = 92$ implies

$$\frac{q+r}{p} + \frac{r+p}{q} + \frac{p+q}{r} = \frac{\sum_{cyc} q^2r + qr^2}{pqr} = 92$$

So

$$\begin{aligned}
\frac{AO}{OA'}\frac{BO}{OB'}\frac{CO}{OC'} &= \frac{q+r}{p} \times \frac{r+p}{q} \times \frac{p+q}{r} \\
&= \frac{(\sum_{cyc} q^2r + qr^2) + 2pqr}{pqr} \\
&= 92 + 2 \\
&= 94
\end{aligned}$$

1.4. Higher Secondary Category

Problem 1.4.1. A crime is committed during the hortal. There are four witnesses. The witnesses are logicians and make the following statements.

- Witness one says exactly one of the witnesses are liar
- Witness one says exactly two of the witnesses are liar
- Witness one says exactly three of the witnesses are liar
- Witness one says exactly four of the witnesses are liar

Assume that each of the statements are true or false. Find the number of liar witnesses.

Solution. All the 4 witnesses provided 4 different kind of informations and any two of them cannot be true at the same time. So there can be at most 1 truthful. Again all 4 of them cannot be liar otherwise the 4th person will be truthful. So there are exactly 3 liars.

Problem 1.4.2. Let N be the number of pairs (m, n) of integers that satisfy the equation $m^2 + n^2 = m^3$. Is N finite or infinite. If N is finite, find the cardinality of N.

Solution. $m^2 + n^2 = m^3 \implies n^2 = m^2(m-1)$. Now if we take $m = k^2 + 1$ where $k \in \mathbb{N}$, then $(m, n) = (k^2 + 1, k(k^2 + 1))$ is a valid solution. As there are infinite choices of k, it has infinite solutions. Hence N is infinite.

Problem 1.4.3. Let n be a positive integer. Consider the polynomial $p(x) = x^2 + x + 1$. What is the remainder of x^3 when divided by x^3. For what $n \in \mathbb{N}$ is $x^{2n} + x^n + 1$ divisible by $p(x)$?

Solution.

$$\begin{aligned} x^3 - 1 &= (x-1)(x^2+x+1) \\ &\equiv 0 \pmod{p(x)} \\ x^3 &\equiv 1 \pmod{p(x)} \end{aligned}$$

Notice that,

$$x^{2n} + x^n + 1 \equiv \begin{cases} (x^3)^{\frac{2n}{3}} + (x^3)^{\frac{n}{3}} + 1 \equiv 1 + 1 + 1 \equiv 3 \pmod{p(x)} \text{ if } 3|n \\ x^2 + x + 1 \equiv 0 \pmod{p(x)} \text{ if } 3 \nmid n \end{cases}$$

The second case is true because $\{2n, n\} \equiv \{1, 2\} \pmod 3$.

Problem 1.4.4. There were 36 participants in a BdMO event. Some of the participants shook hand with each other. No two of them shook hands with each more than once. It was found that no two participants with the same number of handshakes made, had shaken hands each other. Find the maximum number of handshakes at the party.

Solution. Same as (1.3.2).

Problem 1.4.5. A tetrahedron is a polyhedron composed of 4 triangular faces. Faces ABC and BCD of tetrahedron $ABCD$ meet at and angle of $\frac{\pi}{6}$. The area of $\triangle ABC$ and $\triangle BCD$ are 120 and 80 resp. where $BC = 10$. What is the volume of the tetrahedron? (The volume of a tetrahedron is one third the area of it's base times its height)

Solution. Let P and Q be the projection of A on the plane BCD and line BC respectively. Then

$$(ABC) = \frac{1}{2} \times BC \times AQ \implies AQ = \frac{2 \times 120}{10} = 24$$

Again $\angle AQP = 30°$ and $\angle APQ = 90°$. So $AP = AQ \times \sin 30° = \dfrac{24}{2} = 12$

$\therefore$ volume of tetrahedron $ABCD = \dfrac{1}{3} \times (BCD) \times AP = \dfrac{1}{3} \times 80 \times 12 = 320.$

Problem 1.4.6. Trapezoid $ABCD$ has sides $AB = 92, BC = 50, CD = 19, AD = 70$.The side AB is parallel to CD. A circle with center P on AB is drawn tangent to BC and AD. Given that $AP = \frac{m}{n}$ where m and n are coprime positive integers. Find $m + n$?

Solution. Let the circle touches AC and BD at Q and R resp and $AD \cap BC = S$.
Then $PQ \perp BC$ and $PR \perp AD$. So

$$\begin{aligned}
PQ = PR \quad &\implies \quad PB. \sin \angle PBQ = PA. \sin \angle PAR \\
&\implies \quad \frac{PB}{PA} = \frac{\sin \angle BAS}{\sin \angle ABS} \\
&\implies \quad \frac{AB - PA}{PA} = \frac{BS}{AS} \\
&\implies \quad \frac{92}{PA} - 1 = \frac{BC}{AD} \\
&\implies \quad \frac{92}{PA} = 1 + \frac{50}{70} = \frac{12}{7} \\
&\implies \quad PA = \frac{92 \times 7}{12} = \frac{161}{3}
\end{aligned}$$

$\therefore m + n = 164.$

Problem 1.4.7. In $\triangle ABC$, A', B', C' are on sides BC, CA, AB resp. Also AA', BB', CC" are concurrent at O. Also, $\dfrac{AO}{OA'} + \dfrac{BO}{OB'} + \dfrac{CO}{OC'} = 92$. Find $\dfrac{AO}{OA'}\dfrac{BO}{OB'}\dfrac{CO}{OC'}$.

Solution. Let $(BOC) = p, (COA) = q$ and $(AOC) = r$.

$$\frac{AO}{OA'} = \frac{(ABO)}{(OBA')} = \frac{(ACO)}{(OCA')} = \frac{(ABO) + (ACO)}{(OBA') + (OCA')} = \frac{q + r}{p}$$

Similarly $\dfrac{BO}{OB'} = \frac{r+p}{q}$ and $\dfrac{CO}{OC'} = \dfrac{p + q}{r}$.

Therefore $\dfrac{AO}{OA'} + \dfrac{BO}{OB'} + \dfrac{CO}{OC'} = 92$ implies

$$\frac{q+r}{p} + \frac{r+p}{q} + \frac{p+q}{r} = \frac{\sum_{cyc} q^2r + qr^2}{pqr} = 92$$

So

$$\begin{aligned}
\frac{AO}{OA'}\frac{BO}{OB'}\frac{CO}{OC'} &= \frac{q+r}{p} \times \frac{r+p}{q} \times \frac{p+q}{r} \\
&= \frac{(\sum_{cyc} q^2r + qr^2) + 2pqr}{pqr} \\
&= 92 + 2 \\
&= 94
\end{aligned}$$

Chapter 2

National Math Camp

2.1. Geometry

Problem 2.1.1. A point P is chosen in the interior of $\triangle ABC$ so that when lines are drawn through P parallel to the sides of $\triangle ABC$, the resulting smaller triangles t_1, t_2, t_3 in $\triangle ABC$ have areas $4, 9$ and 49 respectively. Find the area of $\triangle ABC$.

Solution. Let the line through P parallel to BC intersect AB, AC at D, E respectively. Again, let the line through P parallel to CA intersect BC, AB at F, G respectively. Finally, let the line through P parallel to AB intersect BC, CA at K, L respectively. Assume that $\triangle PKF = t_1, \triangle PEL = t_2, \triangle PDG = t_3$.

Now, $\triangle PKF \sim \triangle LPE \sim \triangle GDP \sim \triangle ABC$, and $AGPL, BDPK, CEPF$ are all parallelograms.

Next, $\dfrac{EC}{LE} = \dfrac{PF}{LE} = \sqrt{\dfrac{(KPF)}{(PLE)}} = \sqrt{\dfrac{4}{9}} = \dfrac{2}{3}$. Similarly $\dfrac{AL}{LE} = \dfrac{7}{3}$.

So,

$$\begin{aligned}\frac{AC}{LE} &= \frac{AL + LE + EC}{LE}\\ &= \frac{AL}{LE} + \frac{LE}{LE} + \frac{EC}{LE}\\ &= 4\end{aligned}$$

So $\dfrac{(ABC)}{(LPE)} = \left(\dfrac{AC}{LE}\right)^2 = 16$ which implies $(ABC) = 144$.

Problem 2.1.2. A convex hexagon $ABCDEF$ is inscribed in a circle such that $AB = CD = EF$ and diagonals AD, BE and CF are concurrent. Let P be the intersection of AD and

CE. prove that,

$$\frac{CP}{PE} = \left(\frac{AC}{CE}\right)^2$$

Solution. Let Q be the concurrency point of the diagonals Ad, BE, CF.

Lemma 2.1.1. *In ΔABC, if P is on BC then*

$$\frac{BP}{PC} = \frac{AB\angle BAP}{AC\angle PAC}$$

We can prove it using sine law on triangles ΔABP and ΔACP. Now, note that according to lemma (2.1.1)

$$\frac{CP}{PE} = \frac{CA \cdot \sin\angle CAD}{BF \cdot \sin\angle DAE}$$

Next, since $AB = EF$, $ABEF$ must be an isosceles trapezoid, which means $AE = BF$.Similarly, $DF = CE$. Now,

$$\begin{aligned}
\frac{CE}{BF} &= \frac{CQ}{DQ} \\
&= \frac{CQ}{DQ} \cdot \frac{DQ}{BQ} \\
&= \frac{CQ}{DQ} \cdot \frac{DE}{AB} \\
&= \frac{CQ}{DQ} \cdot \frac{DE}{CD} \\
&= \frac{CA}{DF} \cdot \frac{\sin\angle CAD}{\sin\angle DAE} \\
&= \frac{CA}{CE} \cdot \frac{\sin\angle CAD}{\sin\angle DAE}
\end{aligned}$$

From the previous relations we have

$$\begin{aligned}
\frac{CP}{PE} &= \frac{CA \cdot \sin\angle CAD}{BF \cdot \sin\angle DAE} \\
&= \frac{CA}{CE} \cdot \frac{CE}{BF} \cdot \frac{\sin\angle CAD}{\sin\angle DAE} \\
&= \left(\frac{CA}{CE}\right)^2
\end{aligned}$$

Problem 2.1.3. Let $ABCD$ be a convex quadrilateral such that diagonals AC and BD intersect at right angles, and let E be their intersection. Prove that the reflections of E across AB, BC, CD, DA are concyclic.

Solution. Let the reflections of E across AB, BC, CD, DE be P, Q, R, S respectively. Now, $AP = AE = AS$, i.e., A is the circumcenter of $\triangle PSE$. So, $\angle SPE = \frac{1}{2}\angle SAE = \angle DAE$. Similarly, $\angle EPQ = \angle EBC, \angle ERQ = \angle ECB, \angle ERS = \angle EDA$. So

$$\begin{aligned}\angle SPQ + \angle SRQ &= \angle SPE + \angle EPQ + \angle ERQ + \angle ERS \\ &= \angle DAE + \angle EBC + \angle ECB + \angle EDA \\ &= 180^\circ - \angle AED + 180^\circ - \angle BEC \\ &= 180^\circ\end{aligned}$$

since $\angle AED = \angle BEC = 90^\circ$. So $PQRS$ is cyclic.

Problem 2.1.4. Let O be the circumcenter of a triangle $\triangle ABC$ and let ℓ be the line going through the midpoint of the side BC and which is perpendicular to the bisector of $\angle BAC$. Find the value of $\angle BAC$ if the line ℓ goes through the midpoint of the line segment AO.

Solution. There are two parts in this solution, actually. The first part is to prove that $\angle BAC$ is obtuse. The second part is using this information to get the correct figure and evaluate the desired angle.

For the first part, note that unless $\angle BAC$ is obtuse, the line ℓ can't intersect the segment AO.

For the second part, let M, L be the midpoints of BC, AO respectively. Then ML is the line ℓ. Again, let A' be the midpoint of arc BC that does not contain A. Then AA' is the bisector of $\angle BAC$. Let N be the mispoint of AA'. And let ML intersect AA' at K. So, $MK \perp AA', ON \perp AA'$.

Now, clearly L is the center of $\odot AON$. So, $LA = LN$. But $LK \perp AN$. So $AK = KN$. This means $KL \parallel ON \Rightarrow LM \parallel NO$. Again, $\angle LNA = \angle LAN = \angle OAA' = \angle OA'A \Rightarrow LN \parallel MO$. So $LMON$ is a parallelllogram. Now, $OM = NL = LA = \frac{1}{2}OA = \frac{1}{2}OC$.

Now, in $\triangle OCM$, $\angle OMC = 90^\circ$ and $OM = \frac{1}{2}OC$. From these, it is an easy drill to prove that $\angle OCM = 30^\circ$. A little angle chase from there yields $\angle BAC = 120^\circ$.

Problem 2.1.5. An old IMO problem: A triangle $\triangle A_1A_2A_3$ and a point P_0 are given in the plane. We define

$$A_s = A_{s-3} \forall s \geq 4$$

We construct a sequence of points $P_1, P_2, ...$ such that P_{k+1} is the image of P_k under rotation with center A_{k+1} through an angle 120 degree clockwise (for $k = 0, 1, 2, ...$).

Prove that if $P_{1986} = P_0$, then the triangle $\triangle A_1A_2A_3$ is equilateral.

Solution. A composition of three 120 rotations is a rotation of $120+120+120 = 360$, i.e. a translation. Thus, $P_0\vec{P}_3 = P_3\vec{P}_6 = \cdots = P_{1983}\vec{P}_{1986}$. But $P_0 = P_{1986}$, so the vector is null and $P_0 = P_3 = \cdots = P_{1986}$. Since P_0 had no restrictions, we can say that any point in the plane gets mapped to itself after the three rotations. In particular, let's examine the behavior of

A_0. After the first rotation, A_0 remains A_0. After the second, it gets mapped to some point B. Finally, by our previous result, the third rotation takes B to A_0 again. Now noting that $\angle A_0A_2B = \angle BA_1A_0 = 120$, and that $BA_2 = A_2A_0$ and $BA_1 = A_1A_0$, it is easy to deduce that $A_0A_1A_2$ is equilateral.

2.2. Number Theory

Problem 2.2.1 (Masum Billal)**.** An integer is called square-free if it doesn't have any divisor that is a perfect square greater than 1. Prove that $a^{a-1} - 1$ is never square-free for $a > 2$.

Solution (**First**)**.** *Lifting the Exponent Lemma* totally kills this problem.

Lemma 2.2.1 (Lifting the Exponent Lemma(LTE))**.** *If p is an odd prime divisor of $x - y$ where* $\gcd(x, y) = 1$*, then*

$$\nu_p(x^n - y^n) = \nu_p(x - y) + \nu_p(n)$$

See [2] for details on this topic. Assume p is a prime divisor of $a - 1$. Then, by the lemma,

$$\begin{aligned} \nu_p(a^{a-1} - 1) &= \nu_p(a - 1) + \nu_p(a - 1) \\ &= 2\nu_p(a - 1) \\ &\geq 2 \end{aligned}$$

Therefore, $p^2 | a^{a-1} - 1$ and it's not square-free. We are left with the case $p = 2$. It is easy so we will leave it to the readers.

Solution (**Second**)**.** This is a better solution that uses nothing.

$$a^{a-1} - 1 = (a - 1)(a^{a-2} + \ldots + a + 1)$$

Let $m = a - 1$. Then, $a \equiv 1 \pmod{m}$ and

$$\begin{aligned} a^{a-2} + \ldots + a + 1 &\equiv 1^{a-2} + \ldots + 1 + 1 \pmod{m} \\ &\equiv m \equiv 0 \pmod{m} \end{aligned}$$

Therefore, $a^{a-1} - 1$ is divisible by m^2.

Note. The second solution also provides a stronger claim.

Problem 2.2.2. Determine if $2^{2015} + 3^{2015} + 4^{2015} + 5^{2015}$ is a prime.

Solution. Well, this was a problem so everyone solves at least two(paired with problem (2.2.4)). No solution provided for this one.

Problem 2.2.3. For a prime $p > 3$, prove that $\binom{2p-1}{p-1} - 1$ is divisible by p^3.

Solution.

Theorem 2.2.1 (Wolstenholme's Theorem)**.** *For a prime $p > 3$,*

$$\binom{ap}{bp} \equiv \binom{a}{b} \pmod{p^3}$$

Set $a = 2, b = 1$. We have,

$$\binom{2p}{2} \equiv \binom{2}{1} \equiv 2 \pmod{p^3}$$

Remember that, $\binom{n}{k} = \dfrac{n}{k}\binom{n-1}{k-1}$, so

$$\binom{2p}{p} = 2\binom{2p-1}{p-1}$$

Therefore, p^3 divides $2\binom{2p-1}{p-1} - 2 = 2\left(\binom{2p-1}{p-1} - 1\right)$. Since $(p^3, 2) = 1$, we can say p^3 divides $\binom{2p-1}{p-1} - 1$.

Problem 2.2.4. For integers a, b, prove that $a^p b - ab^p$ is divisible by p.

Solution.

Theorem 2.2.2 (Fermat's Little Theorem)**.** *For any prime p and an integer a, p divides $a^p - a$. Particularly, if p doesn't divide a i.e. $(a, p) = 1$,*

$$a^{p-1} \equiv 1 \pmod{p}$$

Write $a^p b - ab^p = ab(a^{p-1} - b^{p-1})$. If one of a or b is divisible by p, we are done. If neither of them is divisible by p,

$$a^{p-1} \equiv 1 \equiv b^{p-1} \pmod{p}$$

Thus, p divides $a^{p-1} - b^{p-1}$.

Problem 2.2.5 (Masum Billal)**.** Find the number of positive integers d so that d divides $a^n - a$ for all integer a where n is a fixed natural number.

Solution. Let's assume $n > 1$.

Lemma 2.2.2. *d is square-free.*

Proof. Let p be a prime so that p^2 divides d. Then setting $a = p$, we get $p^2 | p^n - p$ or $p^2 | p$, which is a contradiction. Thus, no square of a prime divides d i.e. d is square-free. $\square$

Lemma 2.2.3. *If n has k distinct prime factors, it has at least 2^k divisors.*

Proof. Let $n = \prod_{i=1}^{k} p_i^{e_i}$. Then since $e_i \geq 1$,

$$\begin{aligned} \tau(n) &= \prod_{i=1}^{k}(e_i+1) \\ &\geq \prod_{i=1}^{k} 2 \\ &= 2^k \end{aligned}$$

□

Theorem 2.2.3. *For a prime p, there are $\varphi(p)$ primitive roots. In particular, a prime p has a primitive root.*

Theorem 2.2.4. *If $h = ord_n(a)$ and n divides $a^k - 1$, then h divides k.*

Lemma 2.2.4. *$p-1$ divides $n-1$.*

Proof. Without loss of generality, p must divide $a^{n-1}-1$ for integer $(a,p)=1$. Since we are free to choose a, we choose a primitive root g of p. Then p divides $g^{n-1}-1$ and p divides $g^{p-1}-1$. Because $\text{ord}_p(g) = p-1$, we have by theorem (2.2.4) that $p-1$ divides $n-1$. □

Finally, notice that, we only need to find the largest d such that which satisfies this property since other such integers would be divisors of the max d. From the lemma above, such d is square-free and has prime factors p for which $p-1$ divides $n-1$. Therefore, if

$$l = \sum_{p-1|n-1} 1$$

and $p_1, \cdots, p_l$ are the primes such that $p_i - 1 | n-1$ then $d = p_1 \cdots p_l$. By the first lemma, d has 2^l divisors.

Note. The function $C(n) = \sum_{p-1|n-1} 1$ is very interesting. You can study on it if you are intrigued.

Problem 2.2.6 (Masum Billal)**.** For a positive real number $c > 0$, call a positive integer n, $c - good$ if for all positive integer $m < n$, $\frac{m}{n}$ can be written as

$$\frac{m}{n} = \frac{a_0}{b_0} + \ldots + \frac{a_k}{b_k}$$

for some non-negative integers $k, a_0, ..., a_k, b_0, ..., b_k$ with $k < \frac{n}{c}, 2b_k < n$ and $0 \leq a_i < \min(b_j), 0 \leq j < k$. Show that, for any positive real c there are infinite $c - good$ numbers.

Solution. Consider a prime $p \geq 3$. Then any number can be written in p-base as

$$m = a_k p^k + \ldots + a_1 p + a_0$$

where $0 \leq a_i \leq p-1$Therefore, if $n = p^r$ with $r > k$,

$$\frac{m}{n} = \frac{a_k}{p^{r-k}} + \ldots + \frac{a_1}{p^{r-1}} + \frac{a_0}{p^r}$$

$a_i < p \leq \min(b_j) = p^{r-k}$, $2p^{r-k} < p^r$ and $k \leq \log_p m < \log_p n < \dfrac{n}{c}$ since n can be arbitrary large but c is fixed. Fixing c, since we can choose any odd prime, we have infinite such c-good number.

Note. There was one more problem. But I decided to omit it since it was more like an analytic number theory problem than an elementary one.

2.3. Combinatorics

Problem 2.3.1. In a picnic, let there be 1^2 student from Class One, 2^2 students from Class Two, 3^2 students from Class Three, 4^2 students from Class Four and 5^2 students from Class Five. A teacher is picking students for a game at random. How many students must he pick to make sure that there are at least 10 students from the same class?

Solution. Each of class one, two and three contains less than 10 students and 14 students in total. Now if 19 students are taken from class three and four, then by pigeonhole principle one of the chosen classes will contain at least 10 students. So taking $14 + 19 = 33$ ensures at least 10 students in some class.

Again if we choose 1 student from class one, 4 student from class two, 9 student from each of class three, four and five, then there will be 32 students in total with less than 10 students from each class. So taking 32 students is not enough. So the answer is 33.

Problem 2.3.2. In a party, there are n people and their shoes are in n lockers. After the party, electricity went out and everyone forgot the number of locker his/her shoe was in. So they take the shoes randomly. What's the probability that all of them got their own shoes?

Solution. This is a straightforward derangement problem. Derangement of $S = \{1, 2, \cdots, n\}$ is the number of permutations of S such that no element of S appears in its original position. Let the ith person has taken the $\sigma(i)$th left shoe and $\pi(i)$th right shoe where σ and π are two permutations of $\{1, 2, ...n\}$. Now for a fixed permutation σ, we can choose π in $n!$ ways and exactly D_n of them don't have any common point with σ where D_n denotes the derangement number of n objects. So the probability that σ and π don't have any common point is $\dfrac{D_n}{n!}$. The probability remains the same for every choice of σ. So the probability is

$$\frac{D_n}{n!} = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}$$

Problem 2.3.3. You have n jewels, but exactly one of them is a fake. You know that the fake jewel is lighter. With a scale balance, how many measurements are sufficient to find the fake jewel?

Solution. We prove that for every n, if $3^k \geq n > 3^{k-1}$ then we need to do at least k measurements.
We use strong induction. The base case $n = 2$ is trivial. Let it is true for every natural number less than n. Let $n = 3^{k-1} + r$ where $0 < r \leq 2.3^{k-1}$. If we put different number of jewels in the sides of the balance and the balance shows that the side with more jewels is heavier, nothing can be deduced from the result. So suppose in the first measurement, we have put a jewels in the left pan, a jewels in the right pan and b jewels are left aside.

Now $2a + b = n = 3^{k-1} + r$. So at least one of a and b is grater or equal ot $\lceil \frac{n}{3} \rceil = \lceil \frac{3^{k-1}+r}{3} \rceil = 3^{k-2} + q$ where $0 < q \leq 2.3^{k-2}$. If the two sides don't have equal weight, the light one contains the fake jewel. Otherwise the rest b jewels contain the fake one. So if we consider the worst case, we may get at least $3^{k-2} + q$ jewels containing the fake one and by our induction hypothesis, it will take at least $k - 1$ measurements to find the fake jewel. So in total $(k - 1) + 1 = k$ measurements.

Now we prove that k measurements are sufficient. We again apply induction.

- If $n = 3s + 1$, then we take s jewels in both side.
- If $n = 3s + 2$ or $3s + 3$, then we take $s + 1$ jewels in both side.

In all 3 cases, we can reduce the number of jewels containing the fake one to $s + 1$. As $3^k \geq n > 3^{k-1}$, we have $3^{k-1} \geq s+1 > 3^{k-2}$. Now we can do the rest by $k-1$ measurements. Therefore, we can do it using $k = \lceil \log_3 n \rceil$ measurements.

Problem 2.3.4. There are n ants on a p meter rope, on which each walks on a vm/s speed. It is known that

- when two ants collide on the rope, they turn around and continue to move the way they came from at the same speed
- when an ant reaches the end of a rope they fall off from it

Find the greatest amount of time after which every single ant must fall off the rope, and find the arrangement for which that is possible.

Solution. The key observation is that the problem doesn't change if we alter it as: when two ants moving in opposite directions meet, they simply pass through each other and continue moving at the same speed. Thus instead of rebounding, if the ants pass through each other, the only difference from the original problem is that the identities of the ants get exchanged, which is inconsequential. Now the statement is obvious each ant is unaffected by the others, and so each ant will fall of the stick of length one unit in at most p/v second.

Problem 2.3.5. We have 2015 points in the plane such that any three are not collinear. Prove that there is a circle which contains 1007 points in its interior and another 1007 points in its exterior.

Solution. Let's say we have already found the circle and it has center O and radius R. 1007 points are strictly outside and 1007 are inside, this means the other point must be on the boundary. This is quite useful, which tells us to consider the distances of the points from the center. Call the points $P_1, ...P_n$ where $n = 2015$. Without loss of generality, we can assume that $P_1$008 lies on the boundary and the points $OP_1, ..., OP_{1007}$ are inside the circle of radius OP_{1008}. Then $OP_{1009}, ..., OP_{2015}$ are outside the circle. If OP_i is inside the circle then we must have $OP_i < OP_{1008}$, otherwise $OP_i > OP_{1008}$. This should tell you to sort the distances somehow. In other words, we need a construction for the center O so that the distances of P_i are sorted. We are done if we can find O so that all the distances are distinct. In order to find such a construction, we can think the opposite. When will two distances be equal? $OP_i = OP_j$ is possible only if O lies on the perpendicular bisector of P_iP_j. Since we want all the distances distinct, we need to take O so that it doesn't lie on any perpendicular bisector of P_iP_j for all i, j. And obviously there are infinite such points. Now, we can sort the points according to distances i.e. $OP_1 < OP_2 < ... < OP_{2015}$. Therefore, we make O center and draw a circle with radius OP_{1008} and we are done.

Problem 2.3.6. Can you choose 1983 pairwise distinct integers each less than 100000 such that no three are in an arithmetic progression?

Solution. We consider the set S so that for $x \in S$, we have $x \leq 100000$ and the base-3 representation of x consists of only 0 and 1. We prove that S doesn't contain 3 numbers in arithmetical progression.

We assume the contrary. So there exists $a, b, c \in S$ so that $a + b = 2c$ and a, b, c are pairwise different.

Let $a = \overline{a_k a_{k-1} ....a_1}_{(3)}$,$b = \overline{b_k b_{k-1} ....b_1}_{(3)}$ and $c = \overline{c_k c_{k-1} ....c_1}_{(3)}$
Then $a + b$ has $a_i + b_i$ as their ith digit because $a_i + b_i \leq 2$ for all i. As $a \neq b$, there exists some j for which $a_j \neq b_j$. Hence $a_j + b_j = 1$. But all of the digits of $2c$ are either 0 or 2, so it's ith digit cannot be 1. So S doesn't contain 3 numbers in arithmetical progression.

Now for every n, there are exactly 2^n numbers which are less or equal to 3^n and have digits only 0 and 1.

As $3^{12} \leq 100000$, S contains more than 2^{12} digits. As $2^{12} > 1007$,we are done.

Problem 2.3.7. Show that for $n > 2$, there is a set of 2^{n-1} points in the plane, no three collinear such that no 2n form a convex $2n$-gon.

Solution. Let S_2 be $\{(0,0),(1,1)\}$. Given Sn, take $T_n = \{(x+2^{n-1}, y+M_n) : (x,y) \in S_n\}$, where M_n is chosen sufficiently large that the gradient of any segment joining a point of S_n to a point of T_n is greater than that of any segment joining two points of S_n. Then put $S_{n+1} = S_n \cup T_n$.

Clearly S_n has $2n-1$ points. The next step is to show that no three are collinear. Suppose not. Then take k to be the smallest n such that S_k has 3 collinear points. They cannot all be in S_{k-1}. Nor can they all be in T_{k-1}, because then the corresponding points in S_{k-1} would also be collinear. So we may assume that P is in S_{k-1} and Q in T_{k-1}. But now if R is in S_{k-1}, then the gradient of PQ exceeds that of PR. Contradiction. Similarly, if R is in T_{k-1}, then the gradient of QR equals that of the two corresponding points in S_{k-1} and is therefore less than that of PQ. Contradiction.

Finally, we have to show that S_n does not contain a convex $2n$-gon. Suppose it does. Let k be the smallest n such that S_k contains a convex $2k$-gon. Let P be the vertex of the $2k$-gon with the smallest x-coordinate and Q be the vertex with the largest. We must have $P \in S_{k-1}, Q \in T_{k-1}$, otherwise all vertices would be in S_{k-1} or all vertices would be in T_{k-1}, contradicting the minimality of k. Now there must be at least $(k-1)$ other vertices below the line PQ, or at least $(k-1)$ above it. Suppose there are at least $(k-1)$ below it. Take them to be $P = P_0, P_1, ..., P_k = Q$, in order of increasing x-coordinate. These points must form a convex polygon, so gradiant of $P_{i-1}P_i <$ gradiant of P_iP_{i+1}. But the greatest gradient must occur as we move from S_{k-1} to T_{k-1}, so all but Q must belong to S_{k-1}. Thus we have k vertices in S_{k-1} with increasing x-coordinate and all lying below the line joining the first and the last. We can now repeat the argument. Eventually, we get 3 vertices in S_2. Contradiction.

The case were we have k-1 vertices above the line PQ is similar. By convexity, all but P must lie in T_{k-1}. We now take their translates in S_{k-1} and repeat the argument, getting the same contradiction as before.

2.4. Mock Exam 1

Problem 2.4.1. Let x, y be integers and p be a prime for which

$$x^2 - 3xy + p^2y^2 = 12p$$

Find all triples (x, y, p).

Solution. The equation can be rewritten as $x(x-3y) = p(12 - py^2)$. If $p = xd$ then $d(pd - 3y) = 12 - py^2 \implies p(d^2 + y^2) = 3(4 + yd)$. If $p = 3$ then $d^2 - yd + y^2 - 4 = 0$ so we get that $16 - 3y^2$ is a perfect square so $y = 2$ or $y = -2$ then $d \in 0, 2$ so $(x, y, p) \in \{(0, 2, 3), (6, 2, 3), (0, -2, 3), (6, -2, 3)\}$. If p isn't 3 then $d, y \equiv 0(\mod 3)$. then $4 \equiv 0(mod\ 3)$,contradiction. We approach similarly when $x - 3y = pd$.

Problem 2.4.2. In a convex quadrilateral $ABCD$, the diagonals are perpendicular to each other and they intersect at E. Let P be a point on the side AD which is different from A such that $PE = EC$. The circumcircle of triangle BCD intersects the side AD at Q where Q is also different from A. The circle, passing through A and tangent to line EP at P, intersects the line segment AC at R. If the points B, R, Q are concurrent then show that $\angle BCD = 90°$.

Solution. Let $\odot ARD$ meet BD at F. The power of E with respect to $(ARFD)$ is $ER \cdot AE = EF \cdot ED$. The power of E with respect to (ARP) is $ER \cdot AE = EP^2 = EC^2$. So $EF \cdot ED = EC^2$ yields that $\odot FCD$ is tangent to CE or in other words $\angle ECF = \angle EDC$. Also we have $\angle ADE = \angle ERF$. Since $\angle QDB + \angle BDC = \angle FRC + \angle RCF$, we have $\angle RBC = \angle RFC$. This yields $BCFR$ is deltoid. (If you cannot see this easily, take reflection of B with respect to RC. Call it B' . Since $\angle RBC = \angle RB'C = \angle RFC$, B' is on BD, $F = B'$.) So $\angle BCR = \angle RCF = \angle BDC$. Since $\angle BEC = 90°$, $\angle BCD = 90°$.

Problem 2.4.3. We want to place 2012 pockets, including variously colored balls, into k boxes such that

i) For any box, all pockets in this box must include a ball with the same color or ii) For any box, all pockets in this box must include a ball having a color which is not included in any other pocket in this box

Find the smallest value of k for which we can always do this placement whatever the number of balls in the pockets and whatever the colors of balls.

Solution. The answer is 62. We can assume no pocket has two same color ball. It does not change the problem at all. We will use induction, assume the answer is k for $\frac{k(k+1)}{2} \leq n < \frac{(k+1)(k+2)}{2}$. Let $1, 2, , s$ be different colors. Let $a_1, a_2, ..., a_s$ be number of balls of different colors. Assume $a_1 \geq a_2 \geq ... \geq a_s$. If a pocket has color-$p$ ball, we will say this pocket is type-p.(A type-p pocket can also type-q.) If $a_1 \geq k+1$, we will put type-1 pockets into same box. Now we have $\dfrac{(k+1)(k+2)}{2}a_1 \leq \dfrac{k(k+1)}{2}$ and by induction we can put the other pockets into $(k-1)$ boxes. So assume $a_1 < k+1$. Put all type-1 pockets in different boxes. Now start to put remaining type-2 pockets with (ii) statement. If we cannot put all type-2 pockets, this means $a_2 \geq a_1$. Because if we cant add type-2 to a box, it means existing type-1 is also type-2, it means every box has color-2 ball. So we conclude we can place all type-2 pockets. Same strategy for type-3,...,type-m and we are done. The example for 62: $a_1 = 63, a_2 = 62, ..., a_{59} = 5, a_{60} = 3, a_{61} = 2, a_{62} = 1$. (All pockets contain only one ball) Proof: Assume we can place pockets into 61 boxes. We have 63 type-1 pocket by pigeonhole principle we have 2 type-1 pocket in the same box. This box cannot contain another type pocket. After that we have 60 boxes and 62 type-2 pockets. Similarly we can find another box which only has type-2. Then we need at least 62 boxes, contradiction.

2.5. Mock Exam 2

Problem 2.5.1. Determine all triples of positive integers (k, m, n) so that $2^k + 3^m + 1 = 6^n$.

Solution. It is easy to observe that $a \geq 3, c \geq 3 \implies 8|2^a - 6^c \implies 8|3^b + 1$ which is impossible since all possible residues of 3^b modulo 8 are $1, 3$.

For $a \geq 3, c = 2$ we have $2^a + 3^b = 35 \implies (a, b) = (3, 3), (5, 1)$

For $a \geq 3, c = 1$ there's no solution.

For $a = 2$ there's no solution and for $a = 1$ the only is $(1, 1, 1)$

Thus $(a, b, c) = (1, 1, 1), (3, 3, 2), (5, 1, 2)$.

Problem 2.5.2. Let Γ be the circumcircle of a triangle ΔABC. Let ℓ be a line tangent to Γ at point A.Let D, E be interior points of the sides AB, AC respectively, which satisfy the condition $\dfrac{BD}{DA} = \dfrac{AE}{EC}$. Let F, G be the two points of intersection of line DE and circle Γ. Let H be the point of intersection of the line ℓ and the line parallel to AC and going through point D. Let I be the point of intersection of the line ℓ and the line parallel to AB and going through E. Prove that the four points F, G, H, I lie on the circumference of a circle which is tangent to line BC.

Solution. Let $HD \cap BC = P$. So, $\dfrac{BP}{PC} = \dfrac{BD}{DA} = \dfrac{AE}{EC} \implies EP \parallel AB \implies P \in EI$. Now, $\measuredangle CPI = \measuredangle CBA = \measuredangle IAC \implies EI.EP = EA.EC = EF.EG \implies I \in \odot PFG$. Similarly, $H \in \odot PFG$. $\therefore FGHIP$ cyclic. Now, $\measuredangle CPI = \measuredangle PBA = \measuredangle PHI \implies CP, i.e., BC$ touches $\odot FGHI$ at P.

Problem 2.5.3. Let n be a positive integer. For every pair of students enrolled in a certain school having n students, either the pair are mutual friends or not mutual friends. Let N be the smallest possible sum, a + b, of positive integers a and b satisfying the following two conditions concerning students in this school.

1. It is possible to divide students into a teams in such a way that any pair of students belonging to the same team are mutual friends
2. It is possible to divide students into b teams in such a a way that any pair of students belonging to the same team are not mutual friends.

Assume that every student will belong to one and only one team when the students are divided into teams that satisfy the conditions above. A team may consist of only one student, in which case this team is assumed to satisfy both of the conditions: that any pair of students in this team are mutual friends; are not mutual friends. Determine in terms of n the maximum possible value that N can take.

Solution. We can prove by induction on n that $N \leq n + 1$. This is trivial for $n = 1$. Consider a graph G with $|V(G)| = n + 1$, and a vertex $v \in V(G)$ with $\deg_G v = d$. Also consider the graph $G' = G - v$, with $|V(G')| = n$.

Say $a(G') > n - d$; then there cannot exist a vertex v_i in each of the $a(G')$ cliques so that vv_i is not an edge in G, since $\deg_{\overline{G}} v = n - d$. We can then add v to one of these cliques, so $a(G) = a(G')$. Since we may take $\{v\}$ as an independent set, we have $b(G) \leq b(G') + 1$, and so $a(G) + b(G) \leq a(G') + b(G') + 1 \leq (n + 1) + 1$ (by the induction step).

Say $b(G') > d$; then there cannot exist a vertex v_i in each of the $b(G')$ independent sets so that vv_i is an edge in G, since $\deg_G v = d$. We can then add v to one of these independent sets, so $b(G) = b(G')$. Since we may take $\{v\}$ as a clique, we have $a(G) \leq a(G') + 1$, and so $a(G) + b(G) \leq a(G') + b(G') + 1 \leq (n + 1) + 1$ (by the induction step).

We are left with $a(G') \leq n - d$ and $b(G') \leq d$, but then we may take $\{v\}$ as both a clique and an independent set, so we have $a(G) \leq a(G') + 1$ and $b(G) \leq b(G') + 1$, and so $a(G) + b(G) \leq a(G') + b(G') + 2 \leq n + 2 = (n + 1) + 1$.

Since easily it can be seen that for $G = K_n$ we have $a(G) = 1$ and $b(G) = n$, therefore $N = n + 1$, it follows this is the best bound, i.e. $\max N = n + 1$.

Chapter 3

Extension Camp

3.1. Exam One

Problem 3.1.1. Find the number of k tuples $(a_1, ..., a_k)$ with $1 \leq a_i \leq n$ so that their greatest common divisor with n is 1 i.e. $(a_1, ..., a_k, n) = 1$.

Solution. We consider the case when $n = p^m$ for some prime p and natural number m.
We call a $k-tuple$ n good if it satisfies the given condition. Then if $\{a_1, a_2...a_k\}$ is not a good $k-tuple$, all of $a_1, a_2...a_k$ must be divisible by p. So there are $\frac{p^m}{p} = p^{m-1}$ choices for every a_i. So there are $p^{m-1}.p^{m-1}.........p^{m-1} = p^{k(m-1)}$ not good $k-tuples$. So the number of good $k-tuples$ is

$$(p^m)^k - p^{k(m-1)} = p^{mk}(1 - \frac{1}{p^k})$$

Now we solve it for any general n. Let the answer is $f(n)$. Let d be any divisor of$n$. If $gcd(a_1, a_2...a_k, n) = d$,

$$gcd\left(\frac{a_1}{d}, ...\frac{a_k}{d}, \frac{n}{d}\right) = 1$$

So there are exactly $f(\frac{n}{d})$ $k-tuples$ with $gcd(a_1, a_2...a_k, n) = d$. On the other hand, the number of $k-tuples$ is n^k in total. Therefore,

$$\sum_{d|n} \left(\frac{n}{d}\right) = n^k$$

Let $\mathcal{F}$ be the summation function of f. We have $\mathcal{F}(n) = n^k$ which is a multiplicative function. We use the following theorem.

Theorem 3.1.1 (Reverse Multiplicativity Theorem)**.** *If* $F(n) = \sum_{d|n} f(n)$ *is the summation function of* f*, then* f *is multiplicative if* F *is multiplicative*

Here $\mathcal{F}$ is the summation function of f. So f must be a multiplicative function. Let $n = \prod_{i=1}^{r} p_i^{e_i}$. Then

$$\begin{aligned}
f(n) &= f\left(\prod_{i=1}^{r} p_i^{e_i}\right) \\
&= \prod_{i=1}^{r} f(p_i^{e_i}) \\
&= \prod_{i=1}^{r} p^{e_i k}\left(1 - \frac{1}{p_i^k}\right) \\
&= n^k \prod_{i=1}^{r}\left(1 - \frac{1}{p_i^k}\right)
\end{aligned}$$

Problem 3.1.2. Let $1 \leq k \leq n$. Consider all sequences of positive integers with sum n. If the term k appears $\mathcal{F}(n,k)$ times, find $\mathcal{F}(n,k)$ in terms of n and k.

Solution. Let X_n be the set of sequences with sum n. For a set A of sequences, let $f(A)$ denote the total number of appearances of k's in the elements of A. We have $\mathcal{F}(n,k) = f(X_n)$.
Now we show that $X_n = 2^{n-1}$. To prove this we consider n points in a row.There are $n-1$ free spaces among them. So we can partition the n points in 2^{n-1} ways and there is a bijection between the set of sequences with sum n and the set of partitions of n points. So we have $X_n = 2^{n-1}$.
We partition X_n into n disjoint subsets $Y_{1,n}, Y_{2,n}.....Y_{n,n}$ where every sequence in $Y_{i,n}$ has it's first element i. Let $(i, a_2, ...a_m) \in Y_{i,n}$ for some $1 \leq i \leq n$. Then $(a_2, a_3........a_m) \in X_{n-i}$. Now $f(Y_{i,n}) = f(X_{n-i})$ if $i \neq k$ and $f(Y_{i,n}) = f(X_{n-i}) + 2^{n-i}$ if $i = k$. So

$$\begin{aligned}
f(X_n) &= f(Y_{1,n}) + ...f(Y_{n,n}) \\
&= f(X_{n-1}) + ...f(X_{n-k}) + 2^{n-k-1} + f(X_{n-k-1}) + ... + f(X_{k,k}) \\
&= f(X_{n-1}) + ... + f(X_{k,k}) + 2^{n-k-1}
\end{aligned}$$

Similarly $f(X_{n-1}) = f(X_{n-2}) + ... + f(X_k) + 2^{n-k-2}$
Combining these two equations we get

$$\begin{aligned}
f(X_n) &= 2f(X_{n-1}) + 2^{n-k-1} + 2^{n-k-2} \\
\frac{f(X_n)}{2^n} &= \frac{f(X_{n-1})}{2^{n-1}} + \frac{3}{2^{k+2}}
\end{aligned}$$

Therefore by induction

$$\frac{f(X_n)}{2^n} = \frac{f(X_k)}{2^k} + \frac{3(n-k)}{2^{k+2}}$$

$\therefore f(X_n) = 2^{n-k-2}(3n - 3k + 4)$

Problem 3.1.3. A *lattice point* is a point with integer coordinates. There is a block in every lattice point. Decide if there are 100 lattice points $P_1, ..., P_{100}$ so that

- P_i is visible to P_{i+1} for $1 \leq i < 99$.
- P_1 is visible to P_{100}.
- P_i is not visible to P_j is $|j - i| > 1$.

Hint. The following theorem is necessary, and it is a very useful one.

Theorem 3.1.2. *The segment with endpoints $P(x, y)$ and $Q(a, b)$ has $(|x - a|, |y - b|) + 1$ lattice points on it including P and Q.*

To prove it, we need the following facts.

Theorem 3.1.3. *A point $P(x, y)$ is visible from origin if and only if $(x, y) = 1$.*

Proof. The if part is easy. If P is visible then we must have $(x, y) = 1$. If not, assume that $g = (x, y)$ and $g > 1$. Consider the segment joining origin and P. Since P is visible from O, there is no other lattice point between O and P by definition. But note that $(\frac{x}{g}, \frac{y}{g})$ is a lattice point since g divides both x and y. Moreover, this point lies on OP, between O and P, a contradiction.

Let's prove the only if part now. Assume that $(x, y) = 1$. We need to show there is no other lattice point on OP. For the sake of contradiction, assume that $Q(a, b)$ lies between O and P. Then, the slope of O and Q is $\dfrac{b}{a}$. Again, the slope between O and P is $\dfrac{y}{x}$. According to theorem (**??**), we have $\dfrac{b}{a} = \dfrac{y}{x}$. We have, $ay = bx$ and $0 < a < x, 0 < b < y$. The equation also says that $x|ya$. Since $(x, y) = 1$, $x|a$, which gives us $a \geq x$, contradiction. So, there is no other lattice point on this segment.

$\square$

Theorem 3.1.4. *Two points $P(x, y)$ and $Q(a, b)$ are visible from one another if and only if $(x - a, y - b) = 1$.*

Proof. It actually follows from the theorem above. Just notice that, if we translate a segment to an integer distance, the number of lattice points and all properties of that line is preserved, except that it will be below or above the previous line since it has been translated. See (3.1) for better understanding. So we can translate the point $Q(a, b)$ to $(0, 0)$ without loss of generality. Then the translated new P (which is now A)has coordinates $(x-a, y-b)$.[1] After the translation, note that, P is visible to Q if and only if A is visible to origin. Then using the previous theorem, we get that A is visible from origin if and only if $(x-a, y-b) = 1$. $\square$

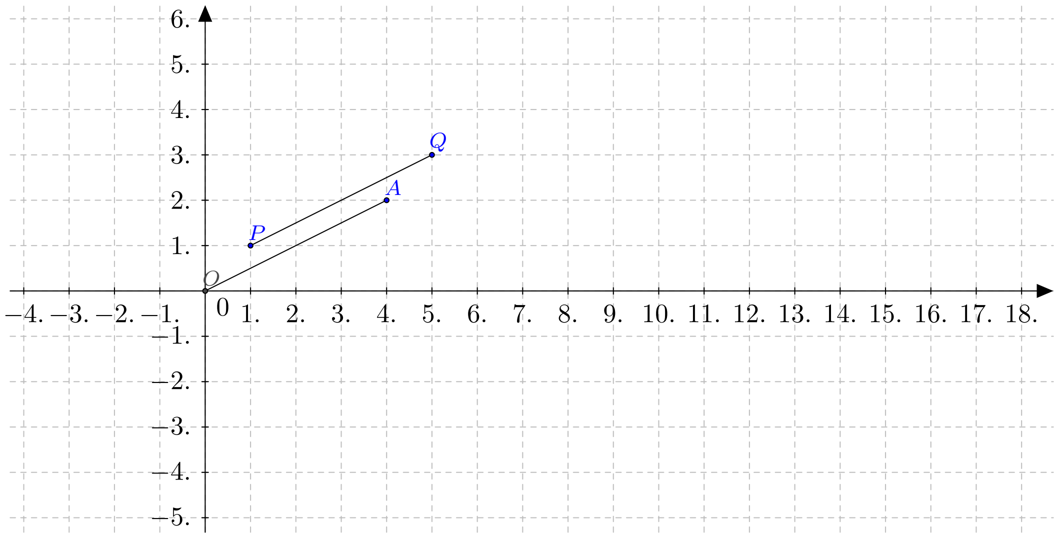


Figure 3.1: Translation preserves the number of lattice points on a segment, and the slope

Let's try to find the number of lattice points on a lattice segment. **Problem** Find the number of lattice points the segment PQ contains.

Proof of the main theorem. First we will modify the figure as we need, kinda like the previous one. Let's translate (x, y) to $(0, 0)$, so (a, b) is translated to $(x-a, y-b) = (m, n)$. Now, reflect this line with respect to Y axis and then translate by $(m, 0)$. The endpoints are $(0, n)$ and $(m, 0)$ now but the number of lattice points is same. If $m = 0$, the result is trivial since the only lattice points are $(0, 0), \cdots, (0, n)$. Similarly, if $n = 0$, the points are $(0, 0), \cdots, (m, 0)$. Both of them support our claim.

Without loss of generality, we can assume $m, n > 0$. Now, the number of lattice points on the segment is actually the number of nonnegative integer solutions that satisfies the equation of this segment:

$$\begin{aligned} \frac{x}{a} + \frac{y}{b} &= 1 \\ \iff bx + ay &= ab \end{aligned}$$

[1] we should use absolute value here, but the result is same

Let $g = (a, b)$ and $a = gu, b = gv$ with $(u, v) = 1$. Then

$$\begin{aligned} vx + uy &= guv \\ v(gu - x) &= uy \\ u(gv - y) &= vx \end{aligned}$$

From these equations, we get v divides uy. But $(u, v) = 1$ so v divides y. Similarly, u divides x. Assume that $y = vk$ and $x = ul$, we have $k + l = g$. The number of nonnegative integer solutions to this equation is $g + 1$. So, our claim is proved.

□

Now, try to use *Chinese Remainder Theorem.*

Note. We can find n such points explicitly as well. Coordinates of such points may have coordinates involving factorials.

Problem 3.1.4. Two students A and B are playing the following game: Each of them writes down on a sheet of paper a positive integer and gives the sheet to the referee. The referee writes down on a blackboard two integers, one of which is the sum of the integers written by the players. After that, the referee asks student A," Can you tell the integer written by the other student?". If A answers "the referee puts the same question to student B. If B answers "no," the referee puts the question back to A, and so on. Assume that both students are intelligent and truthful. Prove that after a finite number of questions, one of the students will answer "yes."

Solution. Let the two numbers on blackboard be $X < Y$. Also use A and B to represent the number from students A and B, respectively.

i. Suppose no "yes" in round 1. A knows $B < X$, otherwise B would have said "yes" and solve $A = Y - B$. Similarly B knows $A < X$.

ii. Suppose no "yes" in round 2. If B saw $Y - B >= X$, he would have known A could not be $Y - B$ since he knew $A < X$ and then he would have said "yes' by solving $A = X - B$. Hence, A knows $Y - B < X$, or $B > Y - X$. Similarly B knows $A > Y - X$.

iii. Suppose no "yes" in round 3. A knows $B < 2X - Y$. B knows $A < 2X - Y$.

Each round without "yes" will tighten A's knowledge on B, also B's knowledge on A. Here knowledge means both upper bound and lower bound. Let us call the series of upper bounds x_n and lower bounds y_n. We see that $x_{n+1} = X - y_n$ and $y_{n+1} = Y - x_n$. Obviously x_n are strictly decreasing and y_n are strictly increasing. So in a finite number of rounds, A or B have to answer yes. The stopping rule is one of the following four:

i. $X - A < x_n \leq Y - A$, A say yes and solve $B = X - A$.

ii. $X - A \leq y_n < Y - A$, A say yes and solve $B = Y - A$.

iii. $X - B < x_n \leq Y - B$, B say yes and solve $A = X - B$.

iv. $X - B \leq y_n < Y - B$, B say yes and solve $A = Y - B$.

Problem 3.1.5 (Masum Billal)**.** Define two sequences $F_0 = 0, F_1 = 1, G_0 = u, G_1 = v$ and

$$\begin{aligned} F_n &= aF_{n-1} + bF_{n-2} \\ G_n &= aG_{n-1} + bG_{n-2} \end{aligned}$$

where a, b, u, v are integers. Prove that,

$$S_{m,n} = \frac{G_{m+n+1} - G_{m+1}F_{n+1}}{G_m F_n}$$

is an integer independent of m or n for natural m, n.

Hint.

$$\begin{aligned} G_{m+n+1} &= G_{m+1}F_{n+1} + bG_m F_n \\ G_{m+n} &= G_{m+1}F_n + bG_m F_{n-1} \end{aligned}$$

You can use induction or prove it combinatorially. The official solution was the combinatorial proof and that's what I had in mind when I posed this in the camp after some examples of **Counting In Two Ways**. But some campers used induction and it was quite easy with that approach. But if anyone is still interested in the combinatorial proof, they can consult with [1]. Be aware that there maybe typos or errors in the paper, but the result should be correct.

3.2. Geometry

Problem 3.2.1. (a) Let ABC be an acute triangle with altitude AD from A to BC. Let P be a point on AD. Line PB meets AC at E and PC meets AB at F. Suppose that $AEDF$ is the inscribed quadrilateral. Prove that $PA/PD = (tanB + tanC)cot(A/2)$.
(b) Let ABC be an acute triangle with orthocenter H and P be a point moving on line AH. The line perpendicular to AC at C cuts BP at M and the line perpendicular to AB at B cuts CP at N. Let K be the projection of A on line MN. Prove that the value of $\angle BKC + \angle MAN$ does not depend on the point P.

Solution. (a)Let $EF \cap BC = K, AD \cap EF = M$ and $\odot AFDE \cap BC = L$.
Now using *Ceva* and *Menelau's* theorem in $\triangle ABC$,we can derive that $\frac{BK}{CK} = \frac{BD}{CD}$. So B, C, D, K are in harmonic order. Then AB, AC, AD, AK is a harmonic pencil and EF imtersects these 4 lines at F, E, M, K resp. Which implies F, E, M, K are in harmonic order.Again $\angle KDM = 90°$. So $\angle FDM = \angle EDM$. AS $AFDLE$ is cyclic and $\angle ADL = 90°$ we have $\angle AFL = \angle AEL = 90°$ and $\angle FDA = \angle EDA \Rightarrow \angle ALF = \angle AEL$. So $\angle FAL = \angle EAL$ and $\angle LAC = \frac{\angle A}{2}$.
Now,

$$\frac{AE}{CE} = \frac{LE.cot\frac{A}{2}}{LE.cotC} = \frac{cot\frac{A}{2}}{cotC}$$

$$\begin{aligned}\frac{AP}{DP} &= \frac{AB.sin\angle ABP}{DB.sin\angle DBP} \\ &= \frac{sin\angle ABE}{sin\angle CBE} \cdot \frac{AB}{BD} \\ &= \frac{\frac{AE}{CE}}{\frac{AB}{CB}}.secB \\ &= \frac{AE}{CE} \cdot \frac{CB}{AB} \cdot secB \\ &= \frac{cot\frac{A}{2}.sinA}{cotC.sinC}.secB\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{AP}{DP} &= \frac{A}{2} \cdot \frac{sin(B+C)}{cosB.cosC} \\ &= \frac{A}{2} \cdot \frac{sinB.cosC + sinC.cosB}{cosB.cosC} \\ &= \frac{A}{2} \cdot \frac{sinB.cosC + sinC.cosB}{cosB.cosC} \\ &= \frac{A}{2} \cdot (tanB + tanC)\end{aligned}$$

(b) Easy to see that $ABKN$ and $ACKM$ are cyclic. So

$$\begin{aligned}\angle BKC &= \angle BKA + \angle CKA \\ &= \angle BNA + \angle CNA \\ &= 90^\circ - \angle A + 90^\circ - \angle A \\ &= 180^\circ - 2\angle A\end{aligned}$$

So $\angle BKC + \angle MAN = 180^\circ - 2\angle A + \angle A = 180^\circ - \angle A$

Problem 3.2.2. Let $\triangle ABC$ be an acute triangle inscribed in circle O. Two points P, Q lie on segments AB, AC and do not coincide with the vertices of $\triangle ABC$. The circumcircle of $\triangle APQ$ intersects O at M at a point different from A. The point N is the point symmetric to M about the line PQ. Prove that
(a) $(AQP) + (BPN) + (CNQ) < (ABC)$ where (X) is the area of triangle X.

(b) If the point N lies on BC, then MN passes through a certain fixed point.

Solution. (a) If N lies inside $\triangle ABC$,then the result is obvious. So we assume N is outside $\triangle ABC$.
Let $PQ \cap BC = T, \angle MTQ = \angle NTQ = x, \angle BTQ = y$ and U, V be the feet of perpendicular from N to BC and PQ resp. Easy to see that M, Q, N are collinear Now $\angle MBP = \angle MBA = \angle MCA = \angle MCQ$ and $\angle MPB = 180^\circ - \angle MPA = 180^\circ - \angle MQA = \angle MQC$. So $\triangle MPB \sim \triangle MQC$ which implies $\triangle MPQ \sim \triangle MBC$. Again M is the *Miquel point* of $BPQC$. So $TCQM$ are cyclic with $\angle MTQ = x$ and $\angle CTQ = y$.

$$\therefore \frac{BC}{PQ} = \frac{MC}{MQ} = \frac{sin(x+y)}{sinx}$$

And

$$\frac{NU}{NV} = \frac{sin(x-y)}{sinx}$$

Now

$$\begin{aligned}
\frac{(NBC)}{(NPQ)} &= \frac{\frac{1}{2}NU.BC}{\frac{1}{2}.NV.PQ} \\
&= \frac{NU}{NV}\frac{BC}{PQ} \\
&= \frac{sin(x-y)}{sinx}\frac{sin(x+y)}{sinx} \\
&= \frac{cos2y - cos2x}{sin^2x} \\
&= \frac{cos2y - 1 + 2sin^2x}{2sin^2x} \\
&= \frac{cos2y - 1}{2sin^2x} + 1 \\
&\leq 1
\end{aligned}$$

So

$$\begin{aligned}
(NBC) \leq (NPQ) &\Rightarrow (ABNC) - (NBP) - (NQC) - (APQ) \geq (NBC) \\
&\Rightarrow (NBP) + (NOC) + (APQ) \leq (ABNC) - (NBC) = (ABC)
\end{aligned}$$

(b) Let $MN \cap \bigodot ABC = D$. As N lies on BC, $\angle BTP = \angle MTP$. So $PB = PM = PN$. Let S be the projection of P on BC. Now $\angle BPS = \frac{\angle BPN}{2} = \angle BMN = \angle BMD = \angle BAD$. So $AD \parallel PS$ which implies $AD \perp BC$. So D is a fixed point and MN passes through it.

Problem 3.2.3. For a sequence $x_1, x_2, ..., x_n$ of real numbers. We define the price as $max_{1in}|x_1 + x_2 + \ + x_i|$. Given n real numbers, Dada and Gadha want to arrange them into a sequence with a low price. Diligent Dada checks all possible ways and finds the minimum possible price D. Greedy Gadha, on the other hand, chooses x_1 such that $|x_1|$ is as small as possible; among the remaining numbers, he chooses x_2 such that $|x_1 + x_2|$ is as small as possible and so on. Thus in the ith step, he chooses x_i among the remaining numbers so as to minimize the value of $|x_1 + x_2 + \ + x_i|$. In each step, if several numbers provide the same value, Gadha chooses one at random. Finally, he gets a sequence with price G. Find the least possible constant c such that for every positive integer n, for every collection of n real numbers, and for every possible sequence that Gadha might obtain, the resulting values satisfy $G \leq cD$.

Solution. We claim that $c = 2$. As mentioned above, us $1, -1, 2, -2$ as a construction. Now we will prove that $G \leq 2D$. Suppose George's sequence goes like $x_1, x_2, ..., x_n$. Now, since by definition, Dave's price is the minimum possible price, then $G \leq 2D$ iff $G \leq 2 \cdot \text{price for any permutation}$. And since $G \geq |x_1 + x_2 + \cdots + x_i|$ for any $1 \leq i \leq n$, we have that if for every i,

$$|x_1 + x_2 + \cdots + x_i| \leq 2 \cdot \text{price for any permutation}$$

then we're good to go.

Lemma 1: If $|a| > 2|b|$ then $|a+b| > |b|$. Proof: From triangle inequality $|a+b|+|-b| \geq |a|$ so $|a+b| \geq |a| - |b| > |b|$. $\square$ Lemma 2: If $ab < 0$ then $|a+b| \leq \max\{|a|, |b|\}$ Proof: WLOG $|a| < |b|$. So we have to show that $|a+b| \leq |b|$. Squaring both sides yields $a^2 + 2ab + b^2 \leq b^2$ iff $a^2 + 2ab \leq 0$.

Let our arbitrary permutation be $y_1, y_2, ..., y_n$ and let the price be $P = |y_1 + y_2 + \cdots + y_p|$ for some $1 \leq p \leq n$. Let $S_0 = 0$ and $S_i = y_1 + y_2 + \cdots + y_i$. First of all, we can prove that $|x_j| \leq 2P$. Assume that $|x_j| > 2P$, and we have $y_i = x_j$ for some i. Then

$$2P \geq |S_i| + |S_{i-1}| \geq |S_i - S_{i-1}| = |x_j|$$

contradiction. Then we can use induction.

Base case: We prove that $|x_1| \leq 2P$. Already done. Inductive step: Assume that $|x_1 + \cdots + x_k| \leq 2P$. We want to prove that $|x_1 + \cdots + x_{k+1}| \leq 2P$. Now let's not forget the definition of G. We certainly know that $|x_1 + x_2 + \cdots + x_{k+1}| \leq |x_1 + x_2 + \cdots + x_k + x_j|$ for some $j > k$. Let's select an x_j such that $x_j(x_1 + x_2 + \cdots + x_k) < 0$. Then we're done by lemma 2. If we cannot find an x_j like that, that means $x_1 + x_2 + \cdots + x_k, x_{k+1}, x_{k+2}, ..., x_n$ all have the same sign. But that means

$$|x_1 + \cdots + x_{k+1}| \leq |x_1 + \cdots + x_n| \leq P \leq 2P$$

So by induction we are done.

3.3. Number Theory

Problem 3.3.1. Let $n \geq 2$ be an integer, and let A_n be the set

$$A_n = \{2^n - 2^k \mid k \in \mathbb{Z},\, 0 \leq k < n\}.$$

Determine the largest positive integer that cannot be written as the sum of one or more (not necessarily distinct) elements of A_n .

Solution. Note that some odd a can be written as the sum of some elements of A_n iff so can be $a - 2^n + 1$ because $2^n - 1$ is the only odd number in the set. Let T_n be the answer for n. It follows that T_n must be odd. Also, if a can be written as the sum of some elements of A_n, $2a$ can be written as the sum of some elements of A_{n+1}. It follows that all numbers $> 2T_n + 2^{n+1} - 1$ can be written as the sum of some elements of A_{n+1}. I claim that $2T_n + 2^{n+1} - 1$ cannot be written as the sum of some elements of A_{n+1}. Suppose $2T_n + 2^{n+1} - 1 = t(2^{n+1} - 1) + q$, where the representation of q doesn't contain $2^{n+1} - 1$. Note that q must be even, and thus, t odd. This implies $T_n = \dfrac{t+1}{2}(2(2^n - 2^{n-1}) + 2^n) + \dfrac{q}{2}$. Note that $\dfrac{q}{2}$ can be written as the sum of some elements of A_n (just divide its representation in A_{n+1} by 2), so T_n can be written as the sum of some elements of A_n. Contradiction.

Thus, we get that $T_{n+1} = 2T_n + 2^{n+1} - 1$. From here, we easily get that $T_n = (n-1)2^n + 1$.

Problem 3.3.2. Determine all pairs (x, y) of positive integers such that

$$\sqrt[3]{7x^2 - 13xy + 7y^2} = |x - y| + 1.$$

Solution. let $x \geq y$ than we have

$7x^2 - 13xy + 7y^2 = (x - y + 1)^3$

now let $x - y = a$ and hence we get

$7a^2 + x(x - a) = (a + 1)^3 \Longrightarrow x^2 - ax - a^3 + 4a^2 - 3a - 1 = 0$

now as x, y are positive int. so discriminant of above quadratic in x must be perfect square.

hence $D = 4a^3 - 15a^2 + 12a + 4 = (4a + 1)(a - 2)^2 = m^2$ so $4a + 1 = k^2$. and thus

$x = \frac{k^2-1 \pm k(k^2-9)}{8}$ and $y = x - \frac{k^2-1}{4} = \frac{k^2-1 \pm k(k^2-9)}{8} - \frac{k^2-1}{4}$

so we get family of solution for different values of k.

Problem 3.3.3. Let $n > 1$ be a given integer. Prove that infinitely many terms of the sequence $(a_k)_{k\geq 1}$, defined by

$$a_k = \left\lfloor \frac{n^k}{k} \right\rfloor,$$

are odd. (For a real number x, $\lfloor x \rfloor$ denotes the largest integer not exceeding x.)

Solution. If n is odd just choose n^u for $u > 1$. It is easy to see that this produces odd integers.

If $n-1$ is odd and $n-1 \neq 1$, consider a prime factor p of $n-1$. Now consider p^l, where $l > 1$,

$$\lfloor \frac{n^{p^l}}{p^l} \rfloor = \frac{n^{p^l} - 1}{p^l}$$

This is an integer because $v_p(n^{p^l} - 1) = v_p(p^l) + v_p(n-1) \geq l$ by LTE, and it is obviously odd.

Now consider $n = 2$. In this case, I claim $k = 3 \cdot 2^{2j}$, for arbitrary $j \neq 0$ works. Indeed

$$\lfloor \frac{2^{3(2^{2j})}}{3(2^{2j})} \rfloor = \lfloor \frac{2^{3(2^{2j})-2j}}{3} \rfloor$$

.

Observe that $3(2^{2j}) - 2j$ is always even, so then this quotient becomes

$$\frac{2^{3(2^{2j})-2j} - 1}{3}$$

, which is clearly odd, so we are done.

3.4. Combinatorics

Problem 3.4.1. There are n cars, numbered from 1 to n and a row with n parking spots, numbered from 1 to n. Each car i has its favorite parking spot a_i. When it is its time to park, it goes to its favorite parking spot. If it is free, it parks and if it is taken, it advances until the next free parking spot and parks there. If it cannot find a parking spot this way, it leaves and never comes back. First car 1 tries to park, then car number 2 tries to park and so on until car number n. Find the number of lists of favorite spots $a_1, ..., a_n$ such that all the cars park. Note, different cars may have the same favorite spot.

Solution. We call an $n - tuple$ $(a_1, a_2 \ldots a_n)$ good if all of the cars can park according to their choices where $1 \leq a_i \leq n+1$ for all i.

We consider $n+1$ parking spots around a circle and number them from 1 to $n+1$ in counterclockeise direction. Suppose every car has a parking choice and if the parking spot is occupied by some other car when it's his time to park, he moves counterclockwisely and parks in the next free spot. As the parking spots are situated sround a circle, all of the cars wil be able to park and there will be exacty one empty parking spot.

Now we call an $n - tuple$ k empty, if after parking, the kth spot is left empty where all the elements of the $n - tuple$ are integers between 1 and $n+1$. Let $f(k)$ be the number of k empty tuples. By symmetry,$f(k) = f(n+1)$ for all k and $\sum_{i=1}^{n+1} f(i) = (n+1)^n$ which implies $f(n+1) = (n+1)^{n-1}$.

Again if $(x_1, x_2 \ldots x_n)$ is an $n+1$ empty tuple, obviously none of the x_i's is equal to $n+1$. It's easy to see that $(x_1, x_2 \ldots x_n)$ is an $n - tuple$ as the $(n+1)$th spot remains empty and none of the cars has to cross the $(n+1)$th spot to find their parking spot. Again all of the good $n - tuples$ are $(n+1)$ empty.

So number of good $n - tuples = f(n+1) = (n+1)^{n-1}$.

Problem 3.4.2. Given a 2007-gon, find the smallest integer k such that among any k vertices of the polygon there are 4 vertices with the property that the convex quadrilateral they form share 3 sides with the polygon.

Solution. Note that,among any k vertices,there exist a convex quadrilateral sharing 3 sides with polygon if and only if it contains 3 consecutive vertices of the polygon. Let $A_1A_2 \ldots\ldots A_{2007}$ be the polygon. If we take the vertices A_i where $i \equiv 1, 2, 3 (mod 4)$ and $1 \leq 2006$,then there are 1505 points in total with no 4 consecutive points. So we must have $k \geq 1506$. We prove that $k = 1506$.
Suppose we can choose a set X of 1506 points in such a way that there are no4 consecutive points. WLOG X contains the point A_1.

Let $B_i = \{A_{4(i-1)+1}, A_{4(i-1)+2}, A_{4(i-1)+3}, A_{4(i-1)+4}\}$ where $i = 1, 2, ...501$.Then X can contain at most 1503 points from $A_1, A_2..........A_{2004}$ and all of $A_{2005}, A_{2006}, A_{2007}$ can't be in X as X contains 1.So $|X| \leq 1505$,a contradiction.

So the minimum value of k is 1506.

Problem 3.4.3. The entries of a $2 \times n$ matrix are positive real numbers. The sum of the numbers in each of the n columns sum to 1. Show that we can select one number in each column such that the sum of the selected numbers in each row is at most $\dfrac{n+1}{4}$.

Solution. We denote the numbers from the first row by a_1, a_2, ..., a_n in increasing order: $a_1 \leq a_2 \leq ... \leq a_n$. Then, the corresponding numbers from the second row are obviously $1 - a_1$, $1 - a_2$, ..., $1 - a_n$.

Now, let k be the largest index satisfying $a_1 + a_2 + ... + a_k \leq \dfrac{n+1}{4}$. Then, of course, $a_1 + a_2 + ... + a_{k+1} > \dfrac{n+1}{4}$ (else, k wouldn't be the largest index). Now, we are going to prove that $(1 - a_{k+1}) + (1 - a_{k+2}) + ... + (1 - a_n) \leq \dfrac{n+1}{4}$.

In fact, the arithmetic mean of the numbers a_{k+1}, a_{k+2}, ..., a_n is surely greater or equal than the number a_{k+1} (the smallest of the numbers a_{k+1}, a_{k+2}, ..., a_n). In other words,
$\dfrac{a_{k+1} + a_{k+2} + ... + a_n}{n-k} \geq a_{k+1}$.

On the other hand, the arithmetic mean of the numbers a_1, a_2, ..., a_{k+1} is surely smaller or equal than the number a_{k+1} (the greatest of the numbers a_1, a_2, ..., a_{k+1}). In other words,
$\dfrac{a_1 + a_2 + ... + a_{k+1}}{k+1} \leq a_{k+1}$.

Thus,
$\dfrac{a_{k+1} + a_{k+2} + ... + a_n}{n-k} \geq a_{k+1} \geq \dfrac{a_1 + a_2 + ... + a_{k+1}}{k+1}$,
and thus

$$a_{k+1} + a_{k+2} + ... + a_n \geq (n-k) \cdot \frac{a_1 + a_2 + ... + a_{k+1}}{k+1} \geq (n-k) \cdot \frac{\left(\dfrac{n+1}{4}\right)}{k+1}$$

(since $n - k \geq 0$ and $a_1 + a_2 + ... + a_{k+1} > \dfrac{n+1}{4}$). In other words,

$$a_{k+1} + a_{k+2} + ... + a_n \geq (n-k) \cdot \frac{\left(\dfrac{n+1}{4}\right)}{k+1} = \frac{(n+1)(n-k)}{4(k+1)}.$$

Hence,

$(1 - a_{k+1}) + (1 - a_{k+2}) + ... + (1 - a_n) = (n-k) - (a_{k+1} + a_{k+2} + ... + a_n) \leq (n-k) - \dfrac{(n+1)(n-k)}{4(k+1)}$.

Thus, in order to show that $(1 - a_{k+1}) + (1 - a_{k+2}) + ... + (1 - a_n) \leq \dfrac{n+1}{4}$, it will be

enough to prove that $(n-k)-\frac{(n+1)(n-k)}{4(k+1)} \leq \frac{n+1}{4}$.
This, however, is straightforward

$$(n-k)-\frac{(n+1)(n-k)}{4(k+1)} \leq \frac{n+1}{4}$$

Therefore,

$$\begin{aligned} n-k &\leq \frac{n+1}{4}+\frac{(n+1)(n-k)}{4(k+1)} \\ &\leq \frac{n+1}{4}\left(1+\frac{n-k}{k+1}\right) \\ &\leq \frac{n+1}{4}\cdot\frac{n+1}{k+1} \\ &\leq \left(\frac{n+1}{2}\right)^2\cdot\frac{1}{k+1} \\ (n-k)(k+1) &\leq \left(\frac{n+1}{2}\right)^2 \end{aligned}$$

But this is clear from AM-GM: $(n-k)(k+1) \leq \left(\frac{(n-k)+(k+1)}{2}\right)^2 = \left(\frac{n+1}{2}\right)^2$.

So we have proved the inequality $(1-a_{k+1})+(1-a_{k+2})+...+(1-a_n) \leq \frac{n+1}{4}$. Together with $a_1+a_2+...+a_k \leq \frac{n+1}{4}$, this shows that if we choose the numbers a_1, a_2, ..., a_k from the first row and the numbers $1-a_{k+1}$, $1-a_{k+2}$, ..., $1-a_n$ from the second row, then the sum of the chosen numbers in each row is $\leq \frac{n+1}{4}$. And the problem is solved.

3.5. Mock Exam 1

Problem 3.5.1. Let ABC be a triangle. The points K, L and M lie on the segments BC, CA and AB respectively such that the lines AK, BL and CM intersect in a common point. Prove that it is possible to choose two of the triangles ALM, BMK and CKL whose inradius sum up to at least the inradius of the triangle ABC.

Solution. Denote $a = \dfrac{BK}{CK}, b = \dfrac{CL}{AL}, c = \dfrac{CM}{AM}$ By Ceva's theorem, $abc = 1$, so we may, without loss of generality, assume that $a \geq 1$. Then at least one of the numbers b or c is not greater than 1. Therefore at least one of the pairs $(ab), (b, c)$ has its first component not less than 1 and the second one not greater than 1. Without loss of generality, assume that $1 \leq a$ and $b \leq 1$. Therefore, we obtain $bc \leq 1$ and $1 \leq ca$, or equivalently

$$\frac{AM}{MB} \leq \frac{LA}{CL} \text{ and } \frac{MB}{AM} \leq BKKC\ .$$

The first inequality implies that the line passing through M and parallel to BC intersects the segment AL at a point X (see Figure 1). Therefore the inradius of the triangle ALM is not less than the inradius r_1 of triangle AMX. Similarly, the line passing through M and parallel to AC intersects the segment BK at a point Y , so the inradius of the triangle BMK is not less than the inradius r_2 of the triangle BMY . Thus, to complete our solution, it is enough to show that $r_1 + r_2 \geq r$, where r is the inradius of the triangle ABC. We prove that in fact $r_1 + r_2 = r$.

Since $MX \parallel BC$, the dilation with centre A that takes M to B takes the incircle of the triangle AMX to the incircle of the triangle ABC. Therefore

$$\frac{r_1}{r} = \frac{AM}{AB} \text{ and similarly } \frac{r_2}{r} = \frac{BM}{AB}$$

Adding these equalities gives $r_1 + r_2 = r$, as required.

Problem 3.5.2. We have 2^m sheets of paper with the number 1 written on each of them. We perform the following operation. In every step, we choose two distinct sheets. If the two numbers on the two sheets are a and b, then we erase the numbers and write the number $a + b$ on both sheets. Prove that after $m2^{m-1}$ steps that the sum of the numbers on all of the sheets is at least 4^m.

Solution. consider an operation that we erase a, b and write $a+b$ instead of them. let S be the sum of other sheets(other than a, b) then the sum of all the sheets is $2a+2b+S$. without loss of generality we can erase a, b and replace them by $2a, 2b$; the sum of the sheets after this operation is also $2a+2b+S$ so we can do this operation instead of the original operation (because only the sum of the sheets is important for us). thus after $m2^{m-1}$ operations the numbers $2^{k_1}, 2^{k_2}, \cdots, 2^{k_{2^m}}$ are written on the sheets where $\sum_{i=1}^{2^m} k_i = m2^m$ so using AM-GM inequality we get $2^{k_1} + 2^{k_2} + \cdots + 2^{k_{2^m}} \geq \sqrt[2^m]{2^{\sum_{i=1}^{2^m} k_i}} = 4^m$

Problem 3.5.3. Find all triples (p, x, y) consisting of a prime number p and two positive integers x and y such that $x^{p-1} + y$ and $x + y^{p-1}$ are both powers of p.

Solution. Set $x^{p-1} + y = p^a$, $x + y^{p-1} = p^b$. If $p = 2$, then $x + y = 2^a = 2^b$, so $x + y$ is any power of 2. Now assume $p > 2$. Notice that both $a, b \geq 1$ since x, y are positive integers. Now by Fermat, the second number is either congruent to x or $x+1$ modulo p, depending on if $p|y$. If $p|y$, we get that $p|x$, and if p doesn't divide y, then $x \equiv -1 \pmod p$ which implies that $y \equiv -1 \pmod p$ too. So we have two cases.

Case 1: $x \equiv y \equiv 0 \pmod p$. Set $v_p(x) = m$ and $v_p(y) = n$. Since $m, n \geq 1$ and $p > 2$, we can't have both $m(p-1) = n$ and $n(p-1) = m$. WLOG suppose $m(p-1) \neq n$. Then we have $v_p(x^{p-1} + y) = \min(m(p-1), n)$, so $x^{p-1} + y = p^{\min(m(p-1),n)}$. But we have

$$\min(x^{p-1}, y) \geq \min(p^{m(p-1)}, p^n) = p^{\min(m(p-1),n)}$$

which is a contradiction. So there are no solutions in this case. Case 2: $x \equiv y \equiv -1 \pmod p$. Set $k = \min(a, b)$. It is easy to see that $x \neq y$, since if $x = y$ then x would divide a power of p, which we ruled out. Claim: $x + 1$ and $y + 1$ are multiples of p^{k-1}. Proof: We prove this for $x + 1$; the proof for $y + 1$ is similar. We have $y = p^a - x^{p-1}$, so

$$x + (p^a - x^{p-1})^{p-1} = p^b$$

Taken modulo p^k, the equation above becomes $p^k | x + (-x^{p-1})^{p-1} = x + x^{(p-1)^2}$ (since p is odd). Since p doesn't divide x, this reduces to $p^k | 1 + x^{p(p-2)}$. This is $v_p(x^{p(p-2)} + 1) \geq k$; LTE reduces this to $v_p(x + 1) \geq k - 1$, which is what we wanted to show so we have proved the claim.

Note that $x^{p-1} + y, x + y^{p-1} > 1^{p-1} + p - 1 = p$, so we get that $a, b \geq 2$ and thus $k \geq 2$. Now, since $p^{k-1}|x + 1$ and $p^{k-1}|y + 1$, we get $x, y \geq p^{k-1} - 1$, and thus

$$(p^{k-1} - 1)^{p-1} + p^{k-1} - 1 \leq p^k$$

Claim: The above inequality must be false if $p \geq 5$. Proof: This is quite boring and simple. We have $(p^{k-1} - 1)^{p-1} + p^{k-1} - 1 = (p^{k-1} - 1)[(p^{k-1} - 1)^{p-2} + 1] > (p^{k-1} - 1)[(p^{k-1} - 1)^2 + 1] \geq (p^{k-1} - 1)[p^{k-1} + 2]$ which is equal to $p^{2k-2} + p^{k-1} - 2$. Now $p^{2k-2} \geq p^k$ and $p^{k-1} > 2$, so $p^{2k-2} + p^{k-1} - 2 > p^k$ as desired.

Thus $p = 3$, so $(3^{k-1} - 1)^2 + 3^{k-1} - 1 \leq 3^k$. Claim: The above inequality must be false if $k \geq 3$. Proof: This is also pretty simple. The LHS of the above is $3^{2k-2} - 3^{k-1} = 3^{k-1}(3^{k-1} - 1) \geq 3^{k-1}(8) > 3^k$.

Thus $p = 3$ and $k = 2$, so we have one of $x^2 + y$ and $x + y^2$ equal to 9. The only solution of $x^2 + y = 9$ with $3|x + 1$ and $3|y + 1$ is $(x, y) = (2, 5)$; in this case we do have $2 + 5^2 = 27 = 3^3$. If $x + y^2 = 9$, we get $(x, y) = (5, 2)$.

So the only solutions are $(2, x, y)$, $(3, 2, 5)$ and $(3, 5, 2)$, where $x + y$ is any power of 2.

3.6. Mock Exam 2

Problem 3.6.1. Let Ω and O be the circumcircle and the circumcentre of an acute-angled triangle ABC with $AB > BC$. The angle bisector of $\angle ABC$ intersects Ω at $M \neq B$. Let Γ be the circle with diameter BM. The angle bisectors of $\angle AOB$ and $\angle BOC$ intersect Γ at points P and Q, respectively. The point R is chosen on the line PQ so that $BR = MR$. Prove that $BR \parallel AC$

Solution. Let $X = \Gamma \cup MO$, and let D, E be the midpoints of BC and AB respectively. Let T be the midpoint of BM. Since BM is a diameter of $\Gamma \implies MOX \perp BX \implies BX \parallel AC$. Observe that E, T, D are the midpoints of chords of Ω with center O . $\implies OE \perp BE$, $OT \perp BT$ and $BD \perp OD$. Therefore, E, O, T, D and B are cyclic

From the above result $\angle EOR{=}\angle EBT{=}\angle TBC{=}\angle TOD \implies T$ lies on the external angle bisector of POQ. On the other hand, $T \epsilon$ perpendicular bisector of PQ. Hence P, O, T, Q are cyclic. Hence R is the radical center of $\odot(POTQ)$, $\odot(BXEOTD)$ and Γ. $\implies B, X, R$ are collinear. So, $BR \parallel AC$ and we are done

Problem 3.6.2. Define the function $f : (0, 1) \to (0, 1)$ by

$$f(x) = \begin{cases} x + \frac{1}{2} & \text{if } \ x < \frac{1}{2} \\ x^2 & \text{if } \ x \geq \frac{1}{2} \end{cases}$$

Let a and b be two real numbers such that $0 < a < b < 1$. We define the sequences a_n and b_n by $a_0 = a, b_0 = b$, and $a_n = f(a_{n-1})$, $b_n = f(b_{n-1})$ for $n > 0$. Show that there exists a positive integer n such that

$$(a_n - a_{n-1})(b_n - b_{n-1}) < 0.$$

Solution. Suppose that the conclusion is false, and let $g(n) = b_n - a_n$. If $a_i, b_i < \dfrac{1}{2}$, we have

$$g(i + 1) = b_{i+1} - a_{i+1} = \left(b_i + \frac{1}{2}\right) - \left(a_i + \frac{1}{2}\right) = g(i)$$

If $a_i, b_i \geq \dfrac{1}{2}$, we have

$$g(i+1) = b_i^2 - a_i^2 = (b_i - a_i)(b_i + a_i) = g(i)(b_i - a_i + 2a_i) \geq g(i)(g(i) + 1) \geq g(i)(g(0) + 1)$$

Because $a_i, b_i \geq \dfrac{1}{2}$ for infinitely many i, we have that for any $n \in \mathbb{N}$, we find k such that $g(k) \geq g(0)(g(0) + 1)^n$. As $g(0)(g(0) + 1)^n$ doesn't have any upperbound, we have reached a contradiction.

Problem 3.6.3. Let n points be given inside a rectangle R such that no two of them lie on a line parallel to one of the sides of R. The rectangle R is to be dissected into smaller rectangles with sides parallel to the sides of R in such a way that none of these rectangles contains any of the given points in its interior. Prove that we have to dissect R into at least $n + 1$ smaller rectangles.

Solution. Notice that there must be at least n line segments inside the big rectangle.

Lemma 3.6.1. *Each vertical/horizontal line segment inside the big rectangle must "stop" at 2 horizontal/vertical segments.*

Proof. The only way a line segment does not "stop" at two other horizontal segments is if two perpendicular segments "stop" when they meet. However, this is not possible as there is no way to "rectangulate" the region if this happens, so the lemma is true.

$\square$

Thus, for each vertical/horizontal segment, there are 2 corners. The total number of corners is then at least $4n$ plus the four corners on the big rectangle for a total of $4n + 4$. However, each rectangle has 4 corners for a total of at least $n + 1$ rectangles, as desired.

Bibliography

[1] Masum Billal, *Remarks On General Fibonacci Numbers*, https://arxiv.org/pdf/1502.06869v1.pdf

[2] Amir Hossein Parvardi, *Lifting The Exponent, Version* 6, http://s3.amazonaws.com/aops-cdn.artofproblemsolving.com/resources/articles/lifting-the-exponent.pdf