INTRODUCTION TO ELEMENTARY ANALYTIC NUMBER THEORY

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Contents

Notations

$\gcd(a, b)$	Greatest common divisor of a and b
$\operatorname{lcm}(a, b)$	Least common multiple of a and b
$\varphi(n)$	Euler's totient function of n
$\tau(n)$	Number of divisors of n
$\sigma(n)$	Sum of divisors of n
$\omega(n)$	Number of distinct prime divisors of n
$\Omega(n)$	Number of total prime divisors of n
$\lambda(n)$	Liouville function of n
$\mu(n)$	Möbius function of n
$\vartheta(x)$	Tchebycheff function of the first kind
$\psi(x)$	Tchebycheff function of the second kind
$\zeta(s)$	Zeta function of the complex number s

Chapter 1

Arithmetic Functions

In this chapter, we will discuss some generalized arithmetic functions and their asymptotic behavior. We will skip discussing the basic definitions since they are common in most introductory number theory texts. $f : \mathbb{N} \to \mathbb{N}$.

Summatory function. For an arithmetic function f, the *summatory function* of f is defined as

$$F(n) = \sum_{d|n} f(d)$$

Note that the number of divisor function $\tau(n)$ is the summatory function of the unit function $u(n) = 1$. The sum of divisor function $\sigma(n)$ is the summatory function of the invariant function $f(n) = n$. An interesting property that we will repeatedly use is that

$$\begin{aligned} \sum_{i=1}^{n} F(i) &= \sum_{i=1}^{n} \sum_{d|i} f(d) \\ &= \sum_{i=1}^{n} \left\lfloor \frac{n}{i} \right\rfloor f(i) \end{aligned}$$

Here, the last equation is true because there are $\lfloor n/i \rfloor$ multiples of i not exceeding n. We can

Generalized sum of divisor. The *generalized sum of divisor* function can be defined as

$$\sigma_k(n) = \sum_{d|n} d^k$$

We can write $\tau(n) = \sum_{ab=n} 1$.

Generalized number of divisor. The *generalized number of divisor* function is defined as

$$\tau_k(n) =$$

We will focus on the following function for now.

$$\sum_{n \leq x} \tau_k(n)$$

THEOREM 1. *A test theorem*

This is a theorem[1]. Now I am citing Apostol[2]

[1]This is a theorem

[2]Tom M. Apostol: *Introduction to analytic number theory*. In: *Undergraduate Texts in Mathematics* (1976). **DOI**: `10.1007/978-1-4757-5579-4`.

Chapter 2

Dirichlet Convolution and Generalization

In this chapter, we will discuss Dirichlet convolution and its generalization, use Dirichlet derivative to prove the Selberg identity, establish some results using generalized convolution and finally, prove the fundamental identity of Selberg.

Dirichlet product. For two arithmetic functions f and g, the *Dirichlet product* or *Dirichlet convolution* of f and g is defined as

$$f * g = \sum_{d|n} f(d)g\left(\frac{n}{d}\right)$$

It is not so easy to see how the idea of Dirichlet convolution originates even though it is highly used in number theory. We can connect its origin with the zeta function.

$$\begin{aligned}\zeta(s) &= \frac{1}{1^s} + \frac{1}{2^s} + \dots \\ &= \sum_{i\geq 1} \frac{1}{i^s}\end{aligned}$$

Chapter 3

A Modest Introduction to Sieve Theory

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