INTRODUCTION TO ELEMENTARY ANALYTIC NUMBER THEORY

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Notations

$\gcd(a, b)$	Greatest common divisor of a and b
$\operatorname{lcm}(a, b)$	Least common multiple of a and b
$\phi(n)$	Euler's totient function of n
$\tau(n)$	Number of divisors of n
$\sigma(n)$	Sum of divisors of n
$\omega(n)$	Number of distinct prime divisors of n
$\Omega(n)$	Number of total prime divisors of n
$\lambda(n)$	Liouville function of n
$\mu(n)$	Möbius function of n
$\vartheta(x)$	Tchebycheff function of the first kind
$\psi(x)$	Tchebycheff function of the second kind
$\zeta(s)$	Zeta function of the complex number s

Chapter 1

Arithmetic Functions

In this chapter, we will discuss some generalized arithmetic functions and their asymptotic behavior. We will skip discussing the basic definitions since they are common in most introductory number theory texts. But for the sake of completeness, check Möbius function,Erdős in section 4.1

Summatory function. For an arithmetic function f, the *summatory function* of f is defined as

$$F(n) = \sum_{d|n} f(d)$$

Note that the number of divisor function $\tau(n)$ is the summatory function of the unit function $u(n) = 1$. The sum of divisor function $\sigma(n)$ is the summatory function of the invariant function $f(n) = n$. An interesting property that we will repeatedly use is that

$$\begin{aligned} \sum_{i=1}^{n} F(i) &= \sum_{i=1}^{n} \sum_{d|i} f(d) \\ &= \sum_{i=1}^{n} \left\lfloor \frac{n}{i} \right\rfloor f(i) \end{aligned}$$

Here, the last equation is true because there are $\lfloor n/i \rfloor$ multiples of i not exceeding n. We can

Generalized sum of divisor. The *generalized sum of divisor* function can be defined as

$$\sigma_k(n) = \sum_{d|n} d^k$$

Recall that the number of divisor function $\tau(n) = \sum_{ab=n} 1$. We can generalize this as follows.

Generalized number of divisor. The *generalized number of divisor* function is defined as

$$\tau_k(n) = \sum_{d_1 \cdots d_k = n} 1$$

So $\tau_k(n)$ is the number of ways to write n as a product of k positive integers. At this point, we will define some asymptotic notions.

Big O. Let f and g be two real or complex valued functions. We say that

$$f(x) = O(g(x))$$

if there is a constant C such that

$$|f(x)| \leq Cg(x)$$

for all sufficiently large x. It is also written as $f(x) \ll g(x)$ or $g(x) \gg f(x)$. Some trivial examples are $x^2 = O(x^3)$, $x + 1 = O(x)$ and $x^2 + x = O(x^2)$. We usually want $g(x)$ to be as small as possible to avoid triviality.

Small O. Let f and g be two real or complex valued functions. Then the following two statements are equivalent

$$\begin{aligned} & f(x) = o(g(x)) \\ \iff & \lim_{x\to\infty} \frac{f(x)}{g(x)} = 0 \end{aligned}$$

Some trivial examples are $1/x = o(1)$, $x = o(x^2)$ and $2x^2 \neq o(x^2)$.

Equivalence. Let f and g be two real or complex valued functions. We say that they are **asymptotically equivalent** if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$$

and we denote it by $f \sim g$. An example is $x^2 \sim x^2 + x$. Note the following.

$$\begin{aligned} & f \sim g \\ \iff & |f(x) - g(x)| = o(g(x)) \end{aligned}$$

We will use these symbols extensively throughout the book. It is advised that the reader properly familiarize themselves with these notations. These symbols make our lives a lot easier very often. See Erdös

Chapter 2

Dirichlet Convolution and Generalization

In this chapter, we will discuss Dirichlet convolution and its generalization, use Dirichlet derivative to prove the Selberg identity, establish some results using generalized convolution and finally, prove the fundamental identity of Selberg.

Dirichlet product. For two arithmetic functions f and g, the *Dirichlet product* or *Dirichlet convolution* of f and g is defined as

$$f * g = \sum_{d|n} f(d) g\left(\frac{n}{d}\right)$$

It is not so easy to see how the idea of Dirichlet convolution originates even though it is highly used in number theory. We can connect its origin with the zeta function.

$$\begin{aligned} \zeta(s) &= \frac{1}{1^s} + \frac{1}{2^s} + \dots \\ &= \sum_{i \geq 1} \frac{1}{i^s} \end{aligned}$$

Chapter 3

Bertrand to Tchebycheff

Definition. Tchebycheff function of the first kind or *Tchebycheff's theta function* is defined as

$$\vartheta(x) = \sum_{p \le x} \log p$$

Definition. Tchebycheff function of the second kind or *Tchebycheff's psi function* is defined as

$$\begin{aligned}\psi(x) &= \sum_{n \le x} \Lambda(n) \\ &= \sum_{p^e \le x} \log p \\ &= \sum_{p \le x} \left\lfloor \frac{\log x}{\log p} \right\rfloor \log p\end{aligned}$$

Chapter 4

A Modest Introduction to Sieve Theory

4.1 Glossary

Erdős	He is Erdős
Möbius function	$\mu(n)$ is a very important function in number theory.

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