INTRODUCTION TO ELEMENTARY ANALYTIC NUMBER THEORY

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Notations

$\gcd(a,b)$ Greatest common divisor of a and b

$\operatorname{lcm}(a,b)$ Least common multiple of a and b

$n^a \parallel k$ $n^a \mid k, n^{a+1} \nmid k$

$\varphi(n)$ Euler's totient function of n

$\tau(n)$ Number of divisors of n

$\sigma(n)$ Sum of divisors of n

$\omega(n)$ Number of distinct prime divisors of n

$\Omega(n)$ Number of total prime divisors of n

$\lambda(n)$ Liouville function of n

$\mu(n)$ Möbius function of n

$\vartheta(x)$ Tchebycheff function of the first kind

$\psi(x)$ Tchebycheff function of the second kind

$\zeta(s)$ Zeta function of the complex number s

Chapter 1

Arithmetic Functions

In this chapter, we will discuss some generalized arithmetic functions and their asymptotic behavior. We will skip discussing the basic definitions since they are common in most introductory number theory texts.

Summatory function. For an arithmetic function f, the *summatory function* of f is defined as

$$F(n) = \sum_{d\in\mathbb{D}} f(d)$$

where $\mathbb{D}$ is some set possibly dependent on n. When $\mathbb{D}$ is the set of divisors of n,

the number of divisor function $\tau(n)$ is the summatory function of the unit function $u(n) = 1$ and the sum of divisor function $\sigma(n)$ is the summatory function of the invariant function $f(n) = n$. Another interesting summatory function we will see are functions of the form

$$\sum_{n\leq x} f(n)$$

for a real number x. Associated with this is the *average order* of an arithmetic function.

$$\lim_{x\to\infty} \frac{\sum_{n\leq x} f(x)}{x}$$

is the average order of the arithmetic function f.

An interesting property that we will repeatedly use is that

$$\begin{aligned}\sum_{i=1}^{n} F(i) &= \sum_{i=1}^{n}\sum_{d|i} f(d)\\ &= \sum_{i=1}^{n}\left\lfloor\frac{n}{i}\right\rfloor f(i)\end{aligned}$$

Here, the last equation is true because there are $\lfloor n/i \rfloor$ multiples of i not exceeding n. Recall that the number of divisor function $\tau(n) = \sum_{ab=n} 1$. We can generalize this as follows.

Generalized number of divisor. The *generalized number of divisor* function is defined as

$$\tau_k(n) = \sum_{d_1\cdots d_k=n} 1$$

So $\tau_k(n)$ is the number of ways to write n as a product of k positive integers. Similarly, we can take the sum of divisor function and generalize it.

Generalized sum of divisor. The *generalized sum of divisor* function can be defined as

$$\sigma_k(n) = \sum_{d|n} d^k$$

At this point, we should discuss some asymptotic notions.

Big O. Let f and g be two real or complex valued functions. We say that

$$f(x) = O(g(x))$$

if there is a positive real constant C such that

$$|f(x)| \leq Cg(x)$$

for all sufficiently large x. It is also written as $f(x) \ll g(x)$ or $g(x) \gg f(x)$. When we say g is an asymptotic estimate of f, we mean that

$$f(x) = g(x) + O(h(x))$$

for two functions g and h. Here, h is the *error term* which obviously should be of lower magnitude than g. In particular, $f(x) = O(1)$ means that f is bounded above by some positive constant. Some trivial examples are $x^2 = O(x^3)$, $x+1 = O(x)$ and $x^2 + 2x = O(x^2)$. We usually want $g(x)$ to be as small as possible to avoid triviality. A useful example is

$$\lfloor x \rfloor = x + O(1)$$

since $x = \lfloor x \rfloor + \{x\}$ and $0 \leq \{x\} < 1$.

Small O. Let f and g be two real or complex valued functions. Then the following two statements are equivalent

$$f(x) = o(g(x)) \qquad (\ddagger\ 1.1)$$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0 \qquad (\ddagger\ 1.2)$$

Some trivial examples are $1/x = o(1)$, $x = o(x^2)$ and $2x^2 \neq o(x^2)$. It should be evident that having an estimate with respect to O asymptotic formulas is more desirable than o formulas. By nature, O formulas give us a better understanding and a specific estimate whereas o does not always say as much. Moreover, working with O is a lot easier than working with o. For example,

$$\sum O(f(x)) = O\left(\sum f(x)\right)$$
$$\int O(f(x))dx = O\left(\int f(x)dx\right)$$

Or consider the possibility that we can very easily deal with constants that would otherwise pop up here and there unnecessarily. With the help of O,

$$O(1) + c = O(1)$$
$$O(cf(x)) = O(f(x))$$

and so on.

Equivalence. Let f and g be two real or complex valued functions. We say that they are *asymptotically equivalent* if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$$

and we denote it by $f \sim g$. So, we can say that g is an asymptotic formula for f. An example is $x^2 \sim x^2 + x$. Note the following.

$$f \sim g \iff |f(x) - g(x)| = o(g(x))$$

We will use these symbols extensively throughout the book. It is of utmost importance that the reader gets well familiarized with these notions since they will be crucial in understanding much of this book. The primary motivation behind these asymptotic notions is to get an as precise as possible idea about the *order of magnitude* of a certain function. This is why we will be leaning more towards $x^2 + 2x = O(x^2)$ than $x^2 + 2x = O(x^3)$ even though both are mathematically correct. The reason is, even though $x^2 + 2x = O(x^3)$ is true, it is taking away a great portion of the accuracy to which we suppose $x^2 + 2x$ should be measured with. On the other hand, we easily see that we cannot have $x^2 + 2x = O(x^\epsilon)$ for $\epsilon < 2$. Under the same philosophy, we define the order of magnitude equivalence.

Definition. If f and g are functions such that both $f(x) \ll g(x)$ and $g(x) \ll f(x)$ hold, then we write $f \asymp g$ and say that f and g have the *same order of magnitude*.

Now, we are interested in the general number of divisors and general sum of divisors. Let us define the cumulative sum of these functions.

$$\begin{aligned} S_k(x) &= \sum_{n \le x} \sigma_k(n) \\ T_k(x) &= \sum_{n \le x} \tau_k(n) \end{aligned}$$

Notice the following.

$$\begin{aligned} S_k(x) &= \sum_{n \le x} \sum_{d \mid n} d^k \\ &= \sum_{n \le x} \left\lfloor \frac{x}{n} \right\rfloor n^k \\ &= \sum_{n \le x} \left(\frac{x}{n} + O(1) \right) n^k \\ &= x \sum_{n \le x} n^{k-1} + O\left(\sum_{n \le x} n^k \right) \end{aligned}$$

We can use this to establish an asymptotic for $T_k(x)$ if we can establish the asymptotic of $A_2(x)$. We will get to that in a moment. First, let us take care of the summation within the big O bracket. We have the trivial inequality that

$$\begin{aligned}\sum_{n\le x} n^k &\le \sum_{n\le x} x^k \\ &= x^k \sum_{n\le x} 1 \\ &= \lfloor x\rfloor x^k \\ &= (x+O(1))x^k \\ &= x^{k+1} + O(x^k)\end{aligned}$$

We have that $S_k(x) = x\left(x^k + O(x^{k-1})\right) + O(x^{k+1}) = O(x^{k+1})$. Although weak, we get an estimate this way. On this note, an interested reader can try and prove that

$$(n+1)^{k+1} - 1 = \sum_{i=0}^{k} \binom{k+1}{i} \mathfrak{S}(n,k)$$

where $\mathfrak{S}(x,k) = \sum_{n\le x} n^k$. This is known as the *Pascal identity* (see Pascal,* for an English translation, see Knoebel et al.†). Lehmer‡ proves that

$$\mathfrak{S}(x,k) = \frac{x^{k+1}}{k+1} + \Delta \tag{‡ 1.3}$$

where $|\Delta| \le x^k$. The reader may also be interested in MacMillan and Sondow.§

*Blaise Pascal. "Sommation des puissances numériques". In: *Oeuvres complètes, Jean Mesnard, ed., Desclée-Brouwer, Paris* 3 (1964), pp. 341–367.

†Arthur Knoebel et al. "Sums of numerical powers". In: *Mathematical Masterpieces: Further chronicles by the explorers*. Springer-Verlag, 2007, pp. 32–37.

‡Derrick Norman Lehmer. "Asymptotic evaluation of certain Totient Sums". In: *American Journal of Mathematics* 22.4 (1900), pp. 293–335. DOI: 10.2307/2369728, Chapter II, Theorem 1.

§Kieren MacMillan and Jonathan Sondow. "Proofs of power sum and binomial coefficient congruences via Pascal's identity". In: *The American Mathematical Monthly* 118.6 (2011), pp. 549–551. DOI: 10.4169/amer.math.monthly.118.06.549.

We shall try to estimate T in a similar fashion. First, see that

$$\begin{aligned}\tau_k(n) &= \sum_{d_1 \cdots d_k = n} 1 \\ &= \sum_{d_k \mid n} \sum_{d_1 \cdots d_{k-1} = n/d_k} 1 \\ &= \sum_{d \mid n} \tau_{k-1}\left(\frac{n}{d}\right)\end{aligned}$$

Note that the two sets $\{d : d \mid n\}$ and $\{n/d : d \mid n\}$ are actually the same. So, we get

$$\tau_k(n) = \sum_{d \mid n} \tau_{k-1}(d)$$

Using this for T,

$$\begin{aligned}T_k(x) &= \sum_{n \le x} \tau_k(n) \\ &= \sum_{n \le x} \sum_{d \mid n} \tau_{k-1}(d) \\ &= \sum_{n \le x} \left\lfloor \frac{x}{n} \right\rfloor \tau_{k-1}(n) \\ &= \sum_{n \le x} \left(\frac{x}{n} + O(1)\right) \tau_{k-1}(n) \\ &= x \sum_{n \le x} \frac{\tau_{k-1}(n)}{n} + O\left(\sum_{n \le x} \tau_{k-1}(n)\right)\end{aligned}$$

Thus, we have the recursive result

$$T_k(x) = x \sum_{n \le x} \frac{\tau_{k-1}(n)}{n} + O(T_{k-1}(x))$$

It gets nontrivial how to proceed from here. Consider the *harmonic sum*

$$H(x) = \sum_{n \le x} \frac{1}{n}$$

It does not seem easy to calculate H accurately, however, we can make a decent attempt to estimate H. The tool that is best suited for carrying out such an estimation is the *Abel partial summation formula*, which today is a cornerstone of analytic number theory.

THEOREM 1.1 (Abel partial summation formula). *Let $\{a_n\}$ be a sequence of real numbers and f be a continuous differentiable function in $[y, x]$. If the partial sums of $\{a_n\}$ is*

$$A(x) = \sum_{n \le x} a_n$$

are known, then

$$\sum_{y<n\le x} a_n f(n) = A(x)f(x) - A(y)f(y) - \int_y^x A(t)f'(t)dt$$

In particular, if f is an arithmetic function,

$$\sum_{n\le x} a_n f(n) = A(x)f(x) - \int_1^x A(t)f'(t)dt$$

Proof. □

It is not straightforward to realize how such a formula can be as influential as we are describing it to be. Notice that, the formula essentially converts a discreet sum into an integral, which occasionally may be calculable. If the integral is not calculable, we may be able to estimate its value sometimes. We will demonstrate these ideas next. It is worth mentioning that Ramanujan also used such a technique. For example, in Aiyangar et al.,[¶] we can definitely see what can only be described as the formula itself. It is unclear whether Ramanujan simply knew about this. Considering he shows the calculation instead of just mentioning the formula, it is certainly possible he came up with the idea on his own, possibly before he had been working on that particular paper. He essentially derives the partial summation formula while trying to express a sum of the form $\sum_{p\le x} \phi(p)$ with respect to $\pi(x)$, $\phi(x)$ and an integral where $\pi(x)$ is the number of primes not exceeding x. A consequence of Abel partial summation formula is the celebrated *Euler's summation formula*.

[¶]Ramanujan Srinivasa Aiyangar et al. "Highly Composite Numbers". In: *Collected papers of Srinivasa Ramanujan*. Cambridge University Press, 1927, pp. 78–128, Page 83, §4.

THEOREM 1.2 (Euler's summation formula). *Let f be a continuous differentiable function in $[y, x]$. Then*

$$\sum_{y<n\leq x} f(n) = \int_y^x f(t)dt + \int_y^x \{t\}f(t)dt + \{y\}f(y) - \{x\}f(x)$$

where $t = t - \lfloor t \rfloor$ is the fractional part of t.

Proof. □

As an application of Euler's summation formula, we can derive a result similar to ‡ 1.3 taking $f(n) = n^k$ for $k \geq 0$.

$$\begin{aligned}
\mathfrak{S}_k(x) &= \sum_{n\leq x} n^k \\
&= \int_1^x t^k dt + k\int_1^x t^{k-1}(t - \lfloor t \rfloor)dt + 1 - (x - \lfloor x \rfloor)x^k \\
&= \frac{x^{k+1}}{k+1} - \frac{1}{k+1} + O\left(k\int_1^x t^{k-1}dt\right) + O(x^k) \\
&= \frac{x^{k+1}}{k+1} + O(x^k)
\end{aligned}$$

Setting $a_n = \tau_{k-1}(n)$ and $f(n) = 1/n$ in Abel partial summation formula, we get

$$\sum_{n\leq x} \frac{\tau_{k-1}(n)}{n} = \frac{T_{k-1}(x)}{x} - \int_1^x -\frac{T_{k-1}(t)}{t^2}dt$$

Thus, we have a result where we can inductively get to the final expression. First, let us see the case $k = 2$.

$$\sum_{n\leq x} \tau(n) = \sum_{n\leq x} \left\lfloor \frac{x}{n} \right\rfloor$$

Clearly, this is just the number of pairs (a, b) such that $ab \leq x$. We can divide the pairs in two classes. In the first class, $1 \leq a \leq \sqrt{x}$ and in the second one, $a > \sqrt{x}$. In the first case, for a fixed a, there are $\lfloor x/a \rfloor$ possible choices for a valid value of b. So, the number of pairs in the first case is

$$\sum_{a\leq\sqrt{x}} \left\lfloor \frac{x}{a} \right\rfloor$$

In the second case, since $a > \sqrt{x}$ and $b \leq x/a$, we must have $b \leq \sqrt{x}$. For a fixed b, there are $\lfloor x/b \rfloor - \sqrt{x}$ choices for a valid value of a, the choices namely are

$$\lfloor x \rfloor + 1, \dots, \left\lfloor \frac{x}{b} \right\rfloor$$

Then the number of pairs in this case is

$$\sum_{b \leq \sqrt{x}} \left\lfloor \frac{x}{b} \right\rfloor - \sqrt{x}$$

Thus, the total number of such pairs is

$$\sum_{a \leq \sqrt{x}} \left\lfloor \frac{x}{a} \right\rfloor + \sum_{b \leq \sqrt{x}} \left(\left\lfloor \frac{x}{b} \right\rfloor - \sqrt{x} \right) = 2 \sum_{n \leq \sqrt{x}} \left\lfloor \frac{x}{n} \right\rfloor - \lfloor x \rfloor^2 \qquad (\ddagger\ 1.4)$$

For getting past this sum, we have to deal with the sum

$$\begin{aligned} \sum_{n \leq \sqrt{x}} \lfloor x/n \rfloor &= \sum_{n \leq \sqrt{x}} \left(\frac{x}{n} + O(1) \right) \\ &= x \sum_{n \leq \sqrt{x}} \frac{1}{n} + O(\sqrt{x}) \\ &= xH(\sqrt{x}) + O(\sqrt{x}) \end{aligned}$$

Setting $a_n = 1$ and $f(n) = 1/n$ in Abel partial summation formula, we get

$$H(x) = \frac{A(x)}{x} - \int_1^x -\frac{A(t)}{t^2} dt$$

Here, $A(x) = \lfloor x \rfloor = x + O(1)$. Using this,

$$\begin{aligned} H(x) &= 1 + O\left(\frac{1}{x}\right) + \int_1^x \left(\frac{1}{t} + \frac{O(1)}{t^2} \right) dt \\ &= 1 + O\left(\frac{1}{x}\right) + \int_1^x \frac{1}{t} dt + O\left(\int_1^x \frac{1}{t^2} dt \right) \\ &= 1 + O\left(\frac{1}{x}\right) + \log x + O\left(1 - \frac{1}{x}\right) \end{aligned}$$

Thus, we have the following result.

THEOREM 1.3.

$$H(x) = \log x + C + O\left(\frac{1}{x}\right)$$

where C is a constant.

We get a more precise formulation of $H(x)$ by considering the limit $x \to \infty$ which removes $O(1/x)$ from the expression since this limit would be 0.

THEOREM 1.4. *There is a constant γ such that*

$$\gamma = \lim_{x\to\infty} (H(x) - \log x)$$

This constant γ is now known as *Euler's constant* or *Euler-Mascheroni's constant*, although, neither Euler nor Mascheroni used the notation γ for this constant. Euler (republished in Euler[**]) used C and O in his original paper. Mascheroni[††] used A and a. Today it is not known whether γ is even irrational. For now, we will not require the use of γ, so we will use Theorem 1.3. Applying this, we have

$$\begin{aligned}
\sum_{n\le\sqrt{x}} \left\lfloor \frac{x}{n} \right\rfloor &= xH(\sqrt{x}) + O(\sqrt{x}) \\
&= x\left(C + \log\sqrt{x} + O\left(\frac{1}{\sqrt{x}}\right)\right) + O(\sqrt{x}) \\
&= \frac{1}{2}x\log x + Cx + O\left(\frac{x}{\sqrt{x}}\right) + O(\sqrt{x}) \\
&= \frac{1}{2}x\log x + O(x)
\end{aligned}$$

We can now use this to get

$$\begin{aligned}
\sum_{n\le x} \tau(n) &= 2\sum_{n\le\sqrt{x}} \lfloor x/n \rfloor - \lfloor\sqrt{x}\rfloor^2 \\
&= x\log x + O(x)
\end{aligned}$$

Leonhard Euler. "De Progressionibus Harmonicis Observationes". In: *Commentarii academiae scientiarum Petropolitanae* 7 (1740), pp. 150–161.

[**]Leonhard Euler. "E-43: De Progressionibus Harmonicis Observationes". In: *Spectrum* (2020), pp. 133–141. DOI: 10.1090/spec/098/23.

[††]Lorenzo Mascheroni. *Adnotationes ad calculum Integralem Euleri*. Galeatii, 1790.

Thus, we get the following result.

$$\frac{\sum_{n\leq x}\tau(n)}{x} = \log x + O(1)$$

Dirichlet[‡‡] actually proves the more precise result given below.

Theorem 1.5 (Dirichlet's average order of τ theorem).

$$\frac{\sum_{n\leq x}\tau(n)}{x} = \log x + 2\gamma - 1 + O\left(\frac{1}{\sqrt{x}}\right)$$

where γ is the Euler-Mascheroni constant.

Then Dirichlet's theorem on τ can be restated as *the average order of τ is $O(\log x)$*. Aiyangar et al.[§§] points out in his paper that the error term $O(1/\sqrt{x})$ in Dirichlet's theorem can be improved to $O\left(x^{-\frac{2}{3}+\epsilon}\right)$ or $O\left(x^{-2/3}\log x\right)$ as Landau[¶¶] shows.

We can now get back to estimating T. Using Abel partial summation formula, we were able to deduce

$$T_k(x) = O\left(T_{k-1}(x)\right) + x\int_1^x \frac{T_{k-1}(t)}{t^2}dt$$

Using Dirichlet's average order of τ theorem, $T(x) = x\log x + O(x)$, so

$$\begin{aligned}
T_3(x) &= O(T(x)) + x\int_1^x \frac{T(t)}{t^2}dt\\
&= O(x\log x) + x\int_1^x \frac{\log t + O(1)}{t}dt\\
&= O(x\log x) + x\int_1^x \frac{\log t}{t}dt + xO\left(\int_1^x \frac{1}{t}dt\right)\\
&= O(x\log x) + x\int_1^x \frac{\log t}{t}dt
\end{aligned}$$

‡‡G. Lejeune Dirichlet. "Über Die Bestimmung Der Mittleren Werthe". In: *G. Lejeune Dirichlet's Werke*. Ed. by L. Kronecker and L. Fuchs. Vol. 2. Druck Und Verlag Von Georg Reimer., 1897, pp. 49–66.

§§Aiyangar et al., "Highly Composite Numbers".

¶¶Edmund Landau. "Über die Anzahl der Gitterpunkte in geweissen Bereichen". In: *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse* 19 (1912), pp. 687–772, Page 689.

Using integration by parts,

$$\begin{aligned}\int \frac{\log t}{t}dt &= \log t \int \frac{1}{t} - \int \left(\frac{1}{t}\int \frac{1}{t}dt\right)dt \\ &= \log^2 t - \int \frac{\log t}{t}dt\end{aligned}$$

Thus, we get

$$\int_1^x \frac{\log t}{t}dt = \frac{1}{2}\log^2 x$$

which in turn gives

$$T_3(x) = \frac{1}{2}x\log^2 x + O(x\log x)$$

We leave it as an exercise for the reader to prove the following (from what we have already developed, induction is one way to go about it).

Theorem 1.6. *Let k be a positive integer. Then*

$$T_k(x) = \frac{1}{(k-1)!}x\log^{k-1} x + O\left(x\log^{k-2} x\right)$$

The reason we do not write $T_k(x)$ as $O\left(x\log^{k-1} x\right)$ directly is because in this case, we already know what the constant multiplier of $x\log^{k-1} x$ is. Usually, we write $O(f(x))$ when we do not know what the constant multiplier of $f(x)$ is. Landau*** states a sharper result.

$$T_k(x) = x\left(\sum_{m=0}^{k-1} b_m \log^m x\right) + O\left(x^{1-\frac{1}{k}}\right) + O\left(x^{1-\frac{1}{k}}\log^{k-2} x\right)$$

***Edmund Landau. "Über eine idealtheoretische funktion". In: *Transactions of the American Mathematical Society* 13.1 (1912), pp. 1–21. DOI: 10.1090/s0002-9947-1912-1500901-6, Page 2.

Let us now turn our attention to improving the asymptotic of $S_k(x)$.

$$\begin{aligned}
S_k(x) &= \sum_{n\le x}\sum_{d|n} d^k \\
&= \sum_{n\le x}\sum_{m\le x/n} m^k \\
&= \sum_{n\le x} \mathfrak{S}_k\left(\frac{x}{n}\right) \\
&= \sum_{n\le x} \frac{x^{k+1}}{(k+1)n^{k+1}} + O\left(\frac{x^k}{n^k}\right) \\
&= \frac{x^{k+1}}{k+1}\sum_{n\le x}\frac{1}{n^{k+1}} + O\left(x^k\sum_{n\le x}\frac{1}{n^k}\right)
\end{aligned}$$

Here, we can see that the function

$$\sum_{n\le x}\frac{1}{n^k}$$

occurs repeatedly. It is in fact, the partial sum of the famous Euler's *zeta function*.

Zeta function. For a complex number s, the zeta function $\zeta(s)$ is defined as

$$\zeta(s) = \sum_{n\ge 1}\frac{1}{n^s}$$

The function ζ has quite a rich history. Today ζ is mostly called Riemann's zeta

function, however, Euler is the first one to investigate this function. Euler started working on ζ around 1730. During that period, the value of $\zeta(2)$ was unknown and of high interest among prominent mathematicians.Ayoub[†††] is a very good read on this subject. Euler's first contribution in this matter is Euler[‡‡‡] where he proves that $\zeta(2) \approx 1.644934$. The paper was first presented to the St. Petersburg Academy on

[†††]Raymond Ayoub. "Euler and the zeta function". In: *The American Mathematical Monthly* 81.10 (1974), pp. 1067–1086. DOI: 10.2307/2319041.

[‡‡‡]Leonhard Euler. "De summatione innumerabilium progressionum". In: *Commentarii academiae scientiarum Petropolitanae* 5 (1738), pp. 91–105.

March 5, 1731 and republished in Euler.[§§§] Euler[¶¶¶] (republished in Euler[17]) proves the following fundamental result which essentially gives a new proof of infinitude of primes.

Theorem 1.7 (Euler's identity). *Let s be a positive integer. Then*

$$\zeta(s) = \prod_p \frac{p^s}{p^s - 1}$$

where p extends over all primes.

One of the results in Euler[18] is the following which we shall prove later.

$$\sum_{n \le x} \frac{1}{p} \sim \log \sum_{n \le x} \frac{1}{x}$$

Here, $\sim$ is the asymptotic equivalence we have already defined. Even though Euler is the main architect behind the development of ζ, there are compelling reasons why it is called Riemann's zeta function. Riemann[19] (which is the only paper on number theory by Riemann) first considered ζ for complex s instead of real s only. This also gave new insight of primes and Riemann conjectured that the non-trivial zeros of ζ lie in the *critical line*

$$\{s \in \mathbb{C} : Re(s) = 1/2\}$$

which is now known as the *Riemann hypothesis*. Riemann hypothesis is considered to be one of the greatest unsolved mysteries in mathematics. This topic is right now out of the scope of this book, so let us back to establishing a result similar to Dirichlet's average order of τ theorem for partial sums of ζ. Setting $f(n) := n^{-s}$ and $a_n = 1$ in

§§§ Leonhard Euler. "E-20: De summatione innumerabilium Progressionum". In: *Spectrum* (2020), pp. 52–64. DOI: 10.1090/spec/098/10.

¶¶¶ Leonhard Euler. "Variae Observationes circa series infinitas". In: *Commentarii academiae scientiarum Petropolitanae* 9 (1744), pp. 160–188.

17 Leonhard Euler. "E-72: Variae Observationes circa series Infinitas". In: *Spectrum* (2020), pp. 249–260. DOI: 10.1090/spec/098/41.

18 Euler, "Variae Observationes circa series infinitas".

19 Bernhard Riemann. "Ueber die anzahl der primzahlen unter einer gegebenen grösse". In: *Monatsberichte der Berliner Akademie* (Nov. 1859), pp. 136–144. DOI: 10.1017/cbo9781139568050.008.

Abel partial summation formula, $A(x) = \lfloor x \rfloor = x + O(1)$ and

$$\begin{aligned}
\sum_{n \le x} \frac{1}{n^s} &= \lfloor x \rfloor x^{-s} - \int_1^x (t + O(1)) f'(t) dt \\
&= x^{1-s} + O\left(x^{-s}\right) + s \int_1^x t^{-s} dt + O\left(s \int_1^x t^{-s-1} dt\right) \\
&= x^{1-s} + \frac{s}{1-s}\left(x^{1-s} - 1\right) + O\left(\int_1^x t^{-s-1}\right) \\
&= \frac{x^{1-s}}{1-s} + C + O(x^{-s})
\end{aligned}$$

Similar to γ, we can take $x \to \infty$ and get the following result.

THEOREM 1.8. *Let s be a positive real number other than 1. Then*

$$\sum_{n \le x} \frac{1}{n^s} = \frac{x^{1-s}}{1-s} + C + O(x^{-s})$$

where C is a constant similar to Euler-Mascheroni constant dependent on s and

$$C = \lim_{x \to \infty} \left(\sum_{n \le x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s} \right)$$

Furthermore, if $0 < s < 1$, then $C = \zeta(s)$ since $x^{1-s} \to 0$.

We can now get back to estimating $S_k(x)$.

$$\begin{aligned}
S_k(x) &= \frac{x^{k+1}}{k+1} \sum_{n \le x} \frac{1}{n^{k+1}} + O\left(x^k \sum_{n \le x} \frac{1}{n^k}\right) \\
&= \frac{x^{k+1}}{k+1}\left(\frac{x^{-k}}{-k} + \zeta(k+1) + O(x^{-k-1})\right) + O\left(x^k\left(\frac{x^{1-k}}{1-k} + \zeta(k) + O(x^{-k})\right)\right) \\
&= \frac{x}{-k(k+1)} + \frac{x^{k+1}}{k+1}\zeta(k+1) + O(x^{k+1-k-1}) + O\left(\frac{x}{1-k} + x^k \zeta(k) + O(1)\right) \\
&= \frac{x^{k+1}}{k+1}\zeta(k+1) + O(x) + O(1) + O(x + x^k)
\end{aligned}$$

From this, we finally get the following.

THEOREM 1.9. *Let k be a positive integer. Then*

$$S_k(x) = \frac{x^{k+1}}{k+1}\zeta(k+1) + O(x^{\max(1,k)})$$

We leave the case when k is a negative integer as an exercise. For our next function, let us consider a generalization of the Euler's totient function $\varphi(n)$.

$$\varphi(x,a) = \sum_{\substack{n\le x\\ \gcd(n,a)=1}} 1$$

For a positive integer n, $\varphi(n) = \varphi(n,n)$ and *Jordan function* is a generalization of φ.

Jordan function. Let n and k be positive integers. Then the Jordan function $J_k(n)$ is the number of k tuples of positive integers not exceeding n that are relatively prime to n.

$$J_k(n) = \sum_{\substack{1\le a_1,\ldots,a_k\le n\\ \gcd(a_1,\ldots,a_k,n)=1}} 1$$

Lehmer[20] used the notation $\varphi_k(n)$ but today $J_k(n)$ is used more often. Jordan[21] first discussed this function and Lehmer[22] developed some asymptotic results. Jordan totient function is interesting not only because it is a generalization of Euler's totient function but also because it has many interesting properties. For example, similar to φ, we can show that

$$J_k(n) = \prod_{p^e\| n} p^{k(e-1)}(p-1)$$
$$J_k(n^m) = n^{k(m-1)}J_k(n)$$

Lehmer[23] proves the following which he calls the *fundamental theorem.*

$$J_k(mn) = J_k(n)\prod_{p^e\| m}\left(p^{ke} - p^{k(e-1)}\lambda(n,p)\right) \tag{‡ 1.5}$$

[20] Lehmer, "Asymptotic evaluation of certain Totient Sums".

[21] Camille Jordan. *Traiteé des substitutions et des équations algébriques*. Gauthier-Villars, Paris, 1870, Page $95-97$.

[22] Lehmer, "Asymptotic evaluation of certain Totient Sums".

[23] Lehmer, "Asymptotic evaluation of certain Totient Sums", Theorem VI.

where $\lambda(n,p)=0$ if $p \mid n$ otherwise $\lambda(n,p)=1$. We leave the proof of this result and the following to the reader.

$$\sum_{d|n} J_k(d) = n^k$$

Like $\sigma_k(n)$, $J_k(n)$ is also related to the sum $\mathfrak{S}(x,k)$. But before we further look into $J_k(n)$, we will have to discuss Möbius inversion as well as generalized inversion which we shall do in the next chapter.

Works Cited In Arithmetic Functions

Aiyangar, Ramanujan Srinivasa et al. "Highly Composite Numbers". In: *Collected papers of Srinivasa Ramanujan*. Cambridge University Press, 1927, pp. 78–128.

Ayoub, Raymond. "Euler and the zeta function". In: *The American Mathematical Monthly* 81.10 (1974), pp. 1067–1086. DOI: 10.2307/2319041.

Dirichlet, G. Lejeune. "Über Die Bestimmung Der Mittleren Werthe". In: *G. Lejeune Dirichlet's Werke*. Ed. by L. Kronecker and L. Fuchs. Vol. 2. Druck Und Verlag Von Georg Reimer., 1897, pp. 49–66.

Euler, Leonhard. "De Progressionibus Harmonicis Observationes". In: *Commentarii academiae scientiarum Petropolitanae* 7 (1740), pp. 150–161.

— "De summatione innumerabilium progressionum". In: *Commentarii academiae scientiarum Petropolitanae* 5 (1738), pp. 91–105.

— "E-20: De summatione innumerabilium Progressionum". In: *Spectrum* (2020), pp. 52–64. DOI: 10.1090/spec/098/10.

— "E-43: De Progressionibus Harmonicis Observationes". In: *Spectrum* (2020), pp. 133–141. DOI: 10.1090/spec/098/23.

— "E-72: Variae Observationes circa series Infinitas". In: *Spectrum* (2020), pp. 249–260. DOI: 10.1090/spec/098/41.

— "Variae Observationes circa series infinitas". In: *Commentarii academiae scientiarum Petropolitanae* 9 (1744), pp. 160–188.

Jordan, Camille. *Traiteé des substitutions et des équations algébriques*. Gauthier-Villars, Paris, 1870.

Knoebel, Arthur et al. "Sums of numerical powers". In: *Mathematical Masterpieces: Further chronicles by the explorers*. Springer-Verlag, 2007, pp. 32–37.

Landau, Edmund. "Über die Anzahl der Gitterpunkte in geweissen Bereichen". In: *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse* 19 (1912), pp. 687–772.

— "Über eine idealtheoretische funktion". In: *Transactions of the American Mathematical Society* 13.1 (1912), pp. 1–21. DOI: 10.1090/s0002-9947-1912-1500901-6.

Lehmer, Derrick Norman. "Asymptotic evaluation of certain Totient Sums". In: *American Journal of Mathematics* 22.4 (1900), pp. 293–335. DOI: 10.2307/2369728.

MacMillan, Kieren and Jonathan Sondow. "Proofs of power sum and binomial coefficient congruences via Pascal's identity". In: *The American Mathematical Monthly* 118.6 (2011), pp. 549–551. DOI: 10.4169/amer.math.monthly.118.06.549.

Mascheroni, Lorenzo. *Adnotationes ad calculum Integralem Euleri*. Galeatii, 1790.

Pascal, Blaise. "Sommation des puissances numériques". In: *Oeuvres complètes, Jean Mesnard, ed., Desclée-Brouwer, Paris* 3 (1964), pp. 341–367.

Riemann, Bernhard. "Ueber die anzahl der primzahlen unter einer gegebenen grösse". In: *Monatsberichte der Berliner Akademie* (Nov. 1859), pp. 136–144. DOI: 10.1017/cbo9781139568050.008.

Chapter 2

Dirichlet Convolution and Generalization

In this chapter, we will discuss Dirichlet convolution and its generalization, use Dirichlet derivative to prove the Selberg identity, establish some results using generalized convolution and finally, prove the fundamental identity of Selberg.

Dirichlet product. For two arithmetic functions f and g, the *Dirichlet product* or *Dirichlet convolution* of f and g is defined as

$$f * g = \sum_{d|n} f(d) g\left(\frac{n}{d}\right)$$

It is not so easy to see how the idea of Dirichlet convolution originates even though it is highly used in number theory. We can connect its origin with the zeta function.

$$\zeta(s) = \sum_{i \geq 1} \frac{1}{i^s}$$

Chapter 3

Bertrand to Tchebycheff

Definition. Tchebycheff function of the first kind or *Tchebycheff's theta function* is defined as

$$\vartheta(x) = \sum_{p \leq x} \log p$$

Definition. Tchebycheff function of the second kind or *Tchebycheff's psi function* is defined as

$$\begin{aligned}\psi(x) &= \sum_{n \leq x} \Lambda(n) \\ &= \sum_{p^e \leq x} \log p \\ &= \sum_{p \leq x} \left\lfloor \frac{\log x}{\log p} \right\rfloor \log p\end{aligned}$$

Chapter 4

A Modest Introduction to Sieve Theory

Chapter 5

Prime Number Theorems

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