Average Order. For an arithmetic function f,

$$\lim_{x\to\infty} \frac{\sum_{n\leq x} f(x)}{x}$$

is the *average order*. In this context, a very interesting way of analyzing growth is the *normal order* of f. The concept of normal numbers arises from Hardy and Aiyangar.[1]

Normal Order. Let f and F be arithmetic functions such that

$$(1-\epsilon)F(n) < f(n) < (1+\epsilon)F(n) \tag{‡ 1.1}$$

holds for *almost* all $n \leq x$ as $x \to \infty$. Then we say that F is the *normal order* of f. A trivial(?) example of normal order is that almost all positive integers not exceeding x are composite if x is sufficiently large. We should probably elaborate on what we mean by *almost* here. One interpretation is that the number of positive integers not exceeding x which are prime is very small compared to x. Similarly, f is of order F means that the number of positive integers n not exceeding x which do not satisfy ‡ 1.1 is very small compared to x.

An interesting property in summatory functions is that

$$\begin{aligned}\sum_{i=1}^{n} F(i) &= \sum_{i=1}^{n}\sum_{d\mid i} f(d)\\ &= \sum_{i=1}^{n}\left\lfloor\frac{n}{i}\right\rfloor f(i)\end{aligned}$$

Here, the last equation is true because there are $\lfloor n/i \rfloor$ multiples of i not exceeding n.

§§1.1 Order of Some Arithmetic Functions

Recall that the number of divisor function

$$\tau(n) = \sum_{ab=n} 1$$

We can generalize this as follows.

Generalized Number of Divisors. The *generalized number of divisor* function is defined as

$$\tau_k(n) = \sum_{d_1\cdots d_k = n} 1$$

So $\tau_k(n)$ is the number of ways to write n as a product of k positive integers. Similarly, we can take

[1] Godfrey Harold Hardy and Srinivasa Ramanujan Aiyangar, "The normal number of prime factors of a number n", *Quarterly Journal of Mathematics*, vol. xlviii (1917), pp. 76-92.