Notice the following.

$$\begin{aligned}
S_k(x) &= \sum_{n\le x}\sum_{d|n} d^k\\
&= \sum_{n\le x}\left\lfloor \frac{x}{n}\right\rfloor n^k\\
&= \sum_{n\le x}\left(\frac{x}{n}+O(1)\right)n^k\\
&= x\sum_{n\le x} n^{k-1} + O\left(\sum_{n\le x} n^k\right)
\end{aligned}$$

We can use this to establish an asymptotic for $T_k(x)$ if we can establish the asymptotic of $A_2(x)$. We will get to that in a moment. First, let us take care of the summation within the big O bracket. We have the trivial inequality that

$$\begin{aligned}
\sum_{n\le x} n^k &\le \sum_{n\le x} x^k\\
&= x^k\sum_{n\le x} 1\\
&= \lfloor x\rfloor x^k\\
&= (x+O(1))x^k\\
&= x^{k+1}+O(x^k)
\end{aligned}$$

We have that $S_k(x) = x\left(x^k + O(x^{k-1})\right) + O(x^{k+1}) = O(x^{k+1})$. Although weak, we get an estimate this way. On this note, an interested reader can try and prove that

$$(n+1)^{k+1}-1 = \sum_{i=0}^{k}\binom{k+1}{i}\mathfrak{S}(n,i)$$

where $\mathfrak{S}(x,k) = \sum_{n<x} n^k$. This is known as the *Pascal identity* (see Pascal,[6] for an English translation, see Knoebel et al.[7]). Lehmer[8] proves that

$$\mathfrak{S}(x,k) = \frac{x^{k+1}}{k+1} + \Delta \tag{‡ 1.4}$$

where $|\Delta| \le x^k$. The reader may also be interested in MacMillan and Sondow.[9]

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[6] Blaise Pascal, "Sommation des puissances numériques", *Oeuvres complètes, Jean Mesnard, ed., Desclée-Brouwer, Paris*, vol. III (1964), pp. 341-367.

[7] Arthur Knoebel et al., "Sums of numerical powers", in, *Mathematical Masterpieces: Further chronicles by the explorers*, Springer-Verlag, 2007, pp. 32-37.

[8] Derrick Norman Lehmer, "Asymptotic evaluation of certain Totient Sums", *American Journal of Mathematics*, vol. XXII, no. 4 (1900), pp. 293-335, DOI: 10.2307/2369728, Chapter II, Theorem 1.

[9] Kieren MacMillan and Jonathan Sondow, "Proofs of power sum and binomial coefficient congruences via Pascal's identity", *The American Mathematical Monthly*, vol. CXVIII, no. 6 (2011), pp. 549-551, DOI: 10.4169/amer.math.monthly.118.06.549.