INTRODUCTION TO ELEMENTARY ANALYTIC NUMBER THEORY

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Notations

$\gcd(a, b)$ Greatest common divisor of a and b

$\text{lcm}(a, b)$ Least common multiple of a and b

$n^a \parallel k$ $n^a \mid k, n^{a+1} \nmid k$

$\varphi(n)$ Euler's totient function of n

$\tau(n)$ Number of divisors of n

$\sigma(n)$ Sum of divisors of n

$\omega(n)$ Number of distinct prime divisors of n

$\Omega(n)$ Number of total prime divisors of n

$\lambda(n)$ Liouville function of n

$\mu(n)$ Möbius function of n

$\vartheta(x)$ Tchebycheff function of the first kind

$\psi(x)$ Tchebycheff function of the second kind

$\zeta(s)$ Zeta function of the complex number s

Chapter 1

Arithmetic Functions

In this chapter, we will discuss some generalized arithmetic functions and their asymptotic behavior. The primary reason to analyze asymptotic behavior is to see how a function grows. A very common way to understand the growth of a function f is to find out how the sum

$$\sum_{n\leq x} f(n)$$

behaves as $x \to \infty$. This may seem like a random thing to do at first. But as we will see, analyzing growth of arithmetic functions give us insight to many questions we have. A prominent example of one such question is, how often do the primes occur? We will attempt to answer this question in Chapter 4.

1.1 Some Summatory functions

We will skip discussing the basic definitions since they are common in most introductory number theory texts.

Summatory function. For an arithmetic function f, the *summatory function* of f is defined as

$$F(n) = \sum_{d\in\mathbb{D}} f(d)$$

where $\mathbb{D}$ is some set possibly dependent on n. When $\mathbb{D}$ is the set of divisors of n, the number of divisor function $\tau(n)$ is the summatory function of the unit function $u(n) = 1$ and the sum of divisor function $\sigma(n)$ is the summatory function of the invariant function $f(n) = n$. Another interesting summatory function we will see are functions of the form

$$\sum_{n\leq x} f(n)$$

for a real number x. Associated with this is the *average order* of an arithmetic function.

$$\lim_{x\to\infty} \frac{\sum_{n\le x} f(x)}{x}$$

is the average order of the arithmetic function f.

An interesting property that we will repeatedly use is that

$$\begin{aligned}\sum_{i=1}^{n} F(i) &= \sum_{i=1}^{n} \sum_{d|i} f(d)\\ &= \sum_{i=1}^{n} \left\lfloor \frac{n}{i} \right\rfloor f(i)\end{aligned}$$

Here, the last equation is true because there are $\lfloor n/i \rfloor$ multiples of i not exceeding n. Recall that the number of divisor function $\tau(n) = \sum_{ab=n} 1$. We can generalize this as follows.

Generalized number of divisor. The *generalized number of divisor* function is defined as

$$\tau_k(n) = \sum_{d_1\cdots d_k=n} 1$$

So $\tau_k(n)$ is the number of ways to write n as a product of k positive integers. Similarly, we can take the sum of divisor function and generalize it.

Generalized sum of divisor. The *generalized sum of divisor* function can be defined as

$$\sigma_k(n) = \sum_{d|n} d^k$$

At this point, we should discuss some asymptotic notions.

Big O. Let f and g be two real or complex valued functions. We say that

$$f(x) = O(g(x))$$

if there is a positive real constant C such that

$$|f(x)| \le Cg(x)$$

for all sufficiently large x. It is also written as $f(x) \ll g(x)$ or $g(x) \gg f(x)$. When we say g is an asymptotic estimate of f, we mean that

$$f(x) = g(x) + O(h(x))$$

for two functions g and h. Here, h is the *error term* which obviously should be of lower magnitude than g. In particular, $f(x) = O(1)$ means that f is bounded above by some positive constant. Some trivial examples are $x^2 = O(x^3)$, $x + 1 = O(x)$ and $x^2 + 2x = O(x^2)$. We usually want $g(x)$ to be as small as possible to avoid triviality. A useful example is

$$\lfloor x \rfloor = x + O(1)$$

since $x = \lfloor x \rfloor + \{x\}$ and $0 \leq \{x\} < 1$.

Small O. Let f and g be two real or complex valued functions. Then the following two statements are equivalent

$$f(x) = o(g(x)) \qquad (\ddagger\ 1.1)$$

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0 \qquad (\ddagger\ 1.2)$$

Some trivial examples are $1/x = o(1)$, $x = o(x^2)$ and $2x^2 \neq o(x^2)$. It should be evident that having an estimate with respect to O asymptotic formulas is more desirable than o formulas. By nature, O formulas give us a better understanding and a specific estimate whereas o does not always say as much. Moreover, working with O is a lot easier than working with o. For example,

$$\sum O(f(x)) = O\left(\sum f(x)\right)$$

$$\int O(f(x))dx = O\left(\int f(x)dx\right)$$

Or consider the possibility that we can very easily deal with constants that would otherwise pop up here and there unnecessarily. With the help of O,

$$O(1) + c = O(1)$$

$$O(cf(x)) = O(f(x))$$

and so on. Landau* states that the symbol O had been first used by Bachmann.† Hardy and Riesz‡ adopted the notations small o and big O.

*Edmund Landau. *Handbuch der Lehre von der Verteilung der Primzahlen*. Vol. 2. 1909, Page 883.

†Paul Gustav Heinrich Bachmann. *Analytische zahlentheorie*. Vol. 2. 1894, Page 401.

‡G. H. Hardy and Marcel Riesz. *The general theory of Dirichlet's series*. Cambridge University Press, 1915.

Equivalence. Let f and g be two real or complex valued functions. We say that they are *asymptotically equivalent* if

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=1$$

and we denote it by $f\sim g$. So, we can say that g is an asymptotic formula for f. An example is $x^2\sim x^2+x$. Note the following.

$$f\sim g \iff |f(x)-g(x)|=o(g(x))$$

We will use these symbols extensively throughout the book. It is of utmost importance that the reader gets well familiarized with these notions since they will be crucial in understanding much of this book. The primary motivation behind these asymptotic notions is to get an as precise as possible idea about the *order of magnitude* of a certain function. This is why we will be leaning more towards $x^2+2x=O(x^2)$ than $x^2+2x=O(x^3)$ even though both are mathematically correct. The reason is, even though $x^2+2x=O(x^3)$ is true, it is taking away a great portion of the accuracy to which we suppose x^2+2x should be measured with. On the other hand, we easily see that we cannot have $x^2+2x=O(x^\epsilon)$ for $\epsilon<2$. Under the same philosophy, we define the order of magnitude equivalence.

Definition. If f and g are functions such that both $f(x)\ll g(x)$ and $g(x)\ll f(x)$ hold, then we write $f\asymp g$ and say that f and g have the *same order of magnitude*.

Now, we are interested in the general number of divisors and general sum of divisors. Let us define the cumulative sum of these functions.

$$S_k(x)=\sum_{n\leq x}\sigma_k(n)$$
$$T_k(x)=\sum_{n\leq x}\tau_k(n)$$

Notice the following.

$$\begin{aligned}
S_k(x)&=\sum_{n\leq x}\sum_{d|n}d^k\\
&=\sum_{n\leq x}\left\lfloor\frac{x}{n}\right\rfloor n^k\\
&=\sum_{n\leq x}\left(\frac{x}{n}+O(1)\right)n^k\\
&=x\sum_{n\leq x}n^{k-1}+O\left(\sum_{n\leq x}n^k\right)
\end{aligned}$$

We can use this to establish an asymptotic for $T_k(x)$ if we can establish the asymptotic of $A_2(x)$. We will get to that in a moment. First, let us take care of the summation within the big O bracket. We have the trivial inequality that

$$\begin{aligned}\sum_{n\le x} n^k &\le \sum_{n\le x} x^k\\ &= x^k\sum_{n\le x}1\\ &= \lfloor x\rfloor x^k\\ &= (x+O(1))x^k\\ &= x^{k+1}+O(x^k)\end{aligned}$$

We have that $S_k(x) = x\left(x^k + O(x^{k-1})\right) + O(x^{k+1}) = O(x^{k+1})$. Although weak, we get an estimate this way. On this note, an interested reader can try and prove that

$$(n+1)^{k+1}-1 = \sum_{i=0}^{k}\binom{k+1}{i}\mathfrak{S}(n,k)$$

where $\mathfrak{S}(x,k) = \sum_{n\le x} n^k$. This is known as the *Pascal identity* (see Pascal,[§] for an English translation, see Knoebel et al.[¶]). Lehmer proves that

$$\mathfrak{S}(x,k) = \frac{x^{k+1}}{k+1} + \Delta \tag{‡ 1.3}$$

where $|\Delta| \le x^k$. The reader may also be interested in MacMillan and Sondow.[**]

We shall try to estimate T in a similar fashion. First, see that

$$\begin{aligned}\tau_k(n) &= \sum_{d_1\cdots d_k=n} 1\\ &= \sum_{d_k\mid n}\ \sum_{d_1\cdots d_{k-1}=n/d_k} 1\\ &= \sum_{d\mid n}\tau_{k-1}\left(\frac{n}{d}\right)\end{aligned}$$

[§] Blaise Pascal. "Sommation des puissances numériques". In: *Oeuvres complètes, Jean Mesnard, ed., Desclée-Brouwer, Paris* 3 (1964), pp. 341–367.

[¶] Arthur Knoebel et al. "Sums of numerical powers". In: *Mathematical Masterpieces: Further chronicles by the explorers*. Springer-Verlag, 2007, pp. 32–37.

Derrick Norman Lehmer. "Asymptotic evaluation of certain Totient Sums". In: *American Journal of Mathematics* 22.4 (1900), pp. 293–335. doi: `10.2307/2369728`, Chapter II, Theorem 1.

[**] Kieren MacMillan and Jonathan Sondow. "Proofs of power sum and binomial coefficient congruences via Pascal's identity". In: *The American Mathematical Monthly* 118.6 (2011), pp. 549–551. doi: `10.4169/amer.math.monthly.118.06.549`.

Note that the two sets $\{d : d \mid n\}$ and $\{n/d : d \mid n\}$ are actually the same. So, we get

$$\tau_k(n) = \sum_{d|n} \tau_{k-1}(d)$$

Using this for T,

$$\begin{aligned} T_k(x) &= \sum_{n\leq x} \tau_k(n) \\ &= \sum_{n\leq x} \sum_{d|n} \tau_{k-1}(d) \\ &= \sum_{n\leq x} \left\lfloor \frac{x}{n} \right\rfloor \tau_{k-1}(n) \\ &= \sum_{n\leq x} \left(\frac{x}{n} + O(1)\right) \tau_{k-1}(n) \\ &= x \sum_{n\leq x} \frac{\tau_{k-1}(n)}{n} + O\left(\sum_{n\leq x} \tau_{k-1}(n)\right) \end{aligned}$$

Thus, we have the recursive result

$$T_k(x) = x \sum_{n\leq x} \frac{\tau_{k-1}(n)}{n} + O(T_{k-1}(x))$$

It gets nontrivial how to proceed from here. Consider the *harmonic sum*

$$H(x) = \sum_{n\leq x} \frac{1}{n}$$

It does not seem easy to calculate H accurately, however, we can make a decent attempt to estimate H. The tool that is best suited for carrying out such an estimation is the *Abel partial summation formula*. Abel[††] states this formula which today is a cornerstone of analytic number theory.

THEOREM 1.1 (Abel partial summation formula). *Let $\{a_n\}$ be a sequence of real numbers and f be a continuous differentiable function in $[y, x]$. If the partial sums of $\{a_n\}$ is*

$$A(x) = \sum_{n\leq x} a_n$$

[††]Niels Henrik Abel. "Untersuchungen über die Reihe: $1+(m/1)x+m\cdot(m-1)/(1\cdot2)\cdots x^2+m\cdot(m-1)\cdot(m-2)/(1\cdot2\cdot3)\cdots x^3+\ldots$". In: *Journal für Math.* 1 (1826), pp. 311–339. DOI: 10.1515/9783112347386-030.

are known, then

$$\sum_{y<n\leq x} a_n f(n) = A(x)f(x) - A(y)f(y) - \int_y^x A(t)f'(t)dt$$

In particular, if f is an arithmetic function,

$$\sum_{n\leq x} a_n f(n) = A(x)f(x) - \int_1^x A(t)f'(t)dt$$

Proof. □

It is not straightforward to realize how such a formula can be as influential as we are describing it to be. Notice that, the formula essentially converts a discreet sum into an integral, which occasionally may be calculable. If the integral is not calculable, we may be able to estimate its value sometimes. We will demonstrate these ideas next. It is worth mentioning that Ramanujan also used such a technique. For example, in Aiyangar,[‡‡] we can definitely see what can only be described as the formula itself. It is unclear if Ramanujan simply knew about this. Considering he shows the calculation instead of just mentioning the formula, it is certainly possible he came up with the idea on his own, possibly before he had been working on that particular paper. He essentially derives the partial summation formula while trying to express a sum of the form $\sum_{p\leq x}\phi(p)$ with respect to $\pi(x), \phi(x)$ and an integral where $\pi(x)$ is the number of primes not exceeding x. A consequence of Abel partial summation formula is the celebrated *Euler's summation formula*.

Theorem 1.2 (Euler's summation formula). *Let f be a continuous differentiable function in $[y, x]$. Then*

$$\sum_{y<n\leq x} f(n) = \int_y^x f(t)dt + \int_y^x \{t\}f(t)dt + \{y\}f(y) - \{x\}f(x)$$

where $t = t - \lfloor t \rfloor$ is the fractional part of t.

Proof. □

[‡‡]Ramanujan Srinivasa Aiyangar. “Highly Composite Numbers”. In: *Collected papers of Srinivasa Ramanujan*. Ed. by Godfrey Harold Hardy, Ve katesvara Seshu Aiyar P, and Bertram Martin Wilson. Cambridge University Press, 1927, pp. 78–128, Page 83, §4.

As an application of Euler's summation formula, we can derive a result similar to ‡ 1.3 taking $f(n) = n^k$ for $k \geq 0$.

$$\begin{aligned}
\mathfrak{S}_k(x) &= \sum_{n \leq x} n^k \\
&= \int_1^x t^k dt + k \int_1^x t^{k-1}(t - \lfloor t \rfloor) dt + 1 - (x - \lfloor x \rfloor) x^k \\
&= \frac{x^{k+1}}{k+1} - \frac{1}{k+1} + O\left(k \int_1^x t^{k-1} dt\right) + O(x^k) \\
&= \frac{x^{k+1}}{k+1} + O(x^k)
\end{aligned}$$

Setting $a_n = \tau_{k-1}(n)$ and $f(n) = 1/n$ in Abel partial summation formula, we get

$$\sum_{n \leq x} \frac{\tau_{k-1}(n)}{n} = \frac{T_{k-1}(x)}{x} - \int_1^x -\frac{T_{k-1}(t)}{t^2} dt$$

Thus, we have a result where we can inductively get to the final expression. First, let us see the case $k = 2$.

$$\sum_{n \leq x} \tau(n) = \sum_{n \leq x} \left\lfloor \frac{x}{n} \right\rfloor$$

Clearly, this is just the number of pairs (a, b) such that $ab \leq x$. We can divide the pairs in two classes. In the first class, $1 \leq a \leq \sqrt{x}$ and in the second one, $a > \sqrt{x}$. In the first case, for a fixed a, there are $\lfloor x/a \rfloor$ possible choices for a valid value of b. So, the number of pairs in the first case is

$$\sum_{a \leq \sqrt{x}} \left\lfloor \frac{x}{a} \right\rfloor$$

In the second case, since $a > \sqrt{x}$ and $b \leq x/a$, we must have $b \leq \sqrt{x}$. For a fixed b, there are $\lfloor x/b \rfloor - \sqrt{x}$ choices for a valid value of a, the choices namely are

$$\lfloor x \rfloor + 1, \ldots, \left\lfloor \frac{x}{b} \right\rfloor$$

Then the number of pairs in this case is

$$\sum_{b \leq \sqrt{x}} \left\lfloor \frac{x}{b} \right\rfloor - \sqrt{x}$$

Thus, the total number of such pairs is

$$\sum_{a\leq\sqrt{x}}\left\lfloor\frac{x}{a}\right\rfloor+\sum_{b\leq\sqrt{x}}\left(\left\lfloor\frac{x}{b}\right\rfloor-\sqrt{x}\right)=2\sum_{n\leq\sqrt{x}}\left\lfloor\frac{x}{n}\right\rfloor-\lfloor x\rfloor^2 \tag{‡ 1.4}$$

For getting past this sum, we have to deal with the sum

$$\begin{aligned}\sum_{n\leq\sqrt{x}}\lfloor x/n\rfloor &= \sum_{n\leq\sqrt{x}}\left(\frac{x}{n}+O(1)\right)\\ &= x\sum_{n\leq\sqrt{x}}\frac{1}{n}+O(\sqrt{x})\\ &= xH(\sqrt{x})+O(\sqrt{x})\end{aligned}$$

Setting $a_n = 1$ and $f(n) = 1/n$ in Abel partial summation formula, we get

$$H(x)=\frac{A(x)}{x}-\int_1^x-\frac{A(t)}{t^2}dt$$

Here, $A(x) = \lfloor x\rfloor = x + O(1)$. Using this,

$$\begin{aligned}H(x) &= 1+O\left(\frac{1}{x}\right)+\int_1^x\left(\frac{1}{t}+\frac{O(1)}{t^2}\right)dt\\ &= 1+O\left(\frac{1}{x}\right)+\int_1^x\frac{1}{t}dt+O\left(\int_1^x\frac{1}{t^2}dt\right)\\ &= 1+O\left(\frac{1}{x}\right)+\log x+O\left(1-\frac{1}{x}\right)\end{aligned}$$

Thus, we have the following result.

Theorem 1.3.

$$H(x)=\log x+C+O\left(\frac{1}{x}\right)$$

where C is a constant.

We get a more precise formulation of $H(x)$ by considering the limit $x\to\infty$ which removes $O(1/x)$ from the expression since this limit would be 0.

THEOREM 1.4. *There is a constant γ such that*

$$\gamma = \lim_{x\to\infty}(H(x) - \log x)$$

This constant γ is now known as *Euler's constant* or *Euler-Mascheroni's constant*, although, neither Euler nor Mascheroni used the notation γ for this constant. Euler[§§] (republished in Euler[¶¶]) used C and O in his original paper. Mascheroni[***] used A and a. Today it is not known whether γ is even irrational. For now, we will not require the use of γ, so we will use Theorem 1.3. Applying this, we have

$$\begin{aligned}\sum_{n\le\sqrt{x}}\left\lfloor\frac{x}{n}\right\rfloor &= xH(\sqrt{x}) + O(\sqrt{x})\\ &= x\left(C + \log\sqrt{x} + O\left(\frac{1}{\sqrt{x}}\right)\right) + O(\sqrt{x})\\ &= \frac{1}{2}x\log x + Cx + O\left(\frac{x}{\sqrt{x}}\right) + O(\sqrt{x})\\ &= \frac{1}{2}x\log x + O(x)\end{aligned}$$

We can now use this to get

$$\begin{aligned}\sum_{n\le x}\tau(n) &= 2\sum_{n\le\sqrt{x}}\lfloor x/n\rfloor - \lfloor\sqrt{x}\rfloor^2\\ &= x\log x + O(x)\end{aligned}$$

Thus, we get the following result.

$$\frac{\sum_{n\le x}\tau(n)}{x} = \log x + O(1)$$

Lejeune[†††] actually proves the more precise result given below.

THEOREM 1.5 (Dirichlet's average order of τ theorem)**.**

$$\frac{\sum_{n\le x}\tau(n)}{x} = \log x + 2\gamma - 1 + O\left(\frac{1}{\sqrt{x}}\right)$$

where γ is the Euler-Mascheroni constant.

[§§]Leonhard Euler. "De Progressionibus Harmonicis Observationes". In: *Commentarii academiae scientiarum Petropolitanae* 7 (1740), pp. 150–161.

[¶¶]Leonhard Euler. "E-43: De Progressionibus Harmonicis Observationes". In: *Spectrum* (2020), pp. 133–141. DOI: 10.1090/spec/098/23.

[***]Lorenzo Mascheroni. *Adnotationes ad calculum Integralem Euleri*. Galeatii, 1790.

[†††]Dirichlet Peter Gustav Lejeune. "Über Die Bestimmung Der Mittleren Werthe". In: *G. Lejeune Dirichlet's Werke*. Ed. by L. Kronecker and L. Fuchs. Vol. 2. Druck Und Verlag Von Georg Reimer., 1897, pp. 49–66.

Then Dirichlet's theorem on τ can be restated as *the average order of τ is $O(\log x)$*. Aiyangar[‡‡‡] points out in his paper that the error term $O(1/\sqrt{x})$ in Dirichlet's theorem can be improved to $O\left(x^{-\frac{2}{3}+\epsilon}\right)$ or $O\left(x^{-2/3}\log x\right)$ as Landau[§§§] shows.

We can now get back to estimating T. Using Abel partial summation formula, we were able to deduce

$$T_k(x) = O\left(T_{k-1}(x)\right) + x\int_1^x \frac{T_{k-1}(t)}{t^2}dt$$

Using Dirichlet's average order of τ theorem, $T(x) = x\log x + O(x)$, so

$$\begin{aligned} T_3(x) &= O(T(x)) + x\int_1^x \frac{T(t)}{t^2}dt \\ &= O(x\log x) + x\int_1^x \frac{\log t + O(1)}{t}dt \\ &= O(x\log x) + x\int_1^x \frac{\log t}{t}dt + xO\left(\int_1^x \frac{1}{t}dt\right) \\ &= O(x\log x) + x\int_1^x \frac{\log t}{t}dt \end{aligned}$$

Using integration by parts,

$$\begin{aligned} \int \frac{\log t}{t}dt &= \log t\int \frac{1}{t} - \int\left(\frac{1}{t}\int\frac{1}{t}dt\right)dt \\ &= \log^2 t - \int \frac{\log t}{t}dt \end{aligned}$$

Thus, we get

$$\int_1^x \frac{\log t}{t}dt = \frac{1}{2}\log^2 x$$

[‡‡‡] Aiyangar, "Highly Composite Numbers".

[§§§] Edmund Landau. "Über die Anzahl der Gitterpunkte in geweissen Bereichen". In: *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse* 19 (1912), pp. 687–772, Page 689.

which in turn gives

$$T_3(x) = \frac{1}{2}x\log^2 x + O(x\log x)$$

We leave it as an exercise for the reader to prove the following (from what we have already developed, induction is one way to go about it).

THEOREM 1.6. *Let k be a positive integer. Then*

$$T_k(x) = \frac{1}{(k-1)!}x\log^{k-1} x + O\left(x\log^{k-2} x\right)$$

The reason we do not write $T_k(x)$ as $O\left(x\log^{k-1} x\right)$ directly is because in this case, we already know what the constant multiplier of $x\log^{k-1} x$ is. Usually, we write $O(f(x))$ when we do not know what the constant multiplier of $f(x)$ is. Landau[¶¶¶] states a sharper result.

$$T_k(x) = x\left(\sum_{m=0}^{k-1} b_m \log^m x\right) + O\left(x^{1-\frac{1}{k}}\right) + O\left(x^{1-\frac{1}{k}}\log^{k-2} x\right)$$

Let us now turn our attention to improving the asymptotic of $S_k(x)$.

$$\begin{aligned}
S_k(x) &= \sum_{n\le x}\sum_{d|n} d^k\\
&= \sum_{n\le x}\sum_{m\le x/n} m^k\\
&= \sum_{n\le x}\mathfrak{S}_k\left(\frac{x}{n}\right)\\
&= \sum_{n\le x}\frac{x^{k+1}}{(k+1)n^{k+1}} + O\left(\frac{x^k}{n^k}\right)\\
&= \frac{x^{k+1}}{k+1}\sum_{n\le x}\frac{1}{n^{k+1}} + O\left(x^k\sum_{n\le x}\frac{1}{n^k}\right)
\end{aligned}$$

Here, we can see that the function

$$\sum_{n\le x}\frac{1}{n^k}$$

[¶¶¶] Edmund Landau. "Über eine idealtheoretische funktion". In: *Transactions of the American Mathematical Society* 13.1 (1912), pp. 1–21. DOI: 10.1090/s0002-9947-1912-1500901-6, Page 2.

occurs repeatedly. It is in fact, the partial sum of the famous Euler's *zeta function*.

Zeta function. For a complex number s, the zeta function $\zeta(s)$ is defined as

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}$$

We will discuss zeta function in details in Section 1.2. For now, let us establish a result similar to Dirichlet's average order of τ theorem for partial sums of ζ. Setting $f(n) := n^{-s}$ and $a_n = 1$ in Abel partial summation formula, $A(x) = \lfloor x \rfloor = x + O(1)$ and

$$\begin{aligned}
\sum_{n\leq x} \frac{1}{n^s} &= \lfloor x \rfloor x^{-s} - \int_1^x (t + O(1)) f'(t)dt \\
&= x^{1-s} + O\left(x^{-s}\right) + s\int_1^x t^{-s}dt + O\left(s\int_1^x t^{-s-1}dt\right) \\
&= x^{1-s} + \frac{s}{1-s}\left(x^{1-s} - 1\right) + O\left(\int_1^x t^{-s-1}\right) \\
&= \frac{x^{1-s}}{1-s} + C + O(x^{-s})
\end{aligned}$$

Similar to γ, we can take $x \to \infty$ and get the following result.

THEOREM 1.7. *Let s be a positive real number other than 1. Then*

$$\sum_{n\leq x} \frac{1}{n^s} = \frac{x^{1-s}}{1-s} + C + O(x^{-s})$$

where C is a constant similar to Euler-Mascheroni constant dependent on s and

$$C = \lim_{x\to\infty} \left(\sum_{n\leq x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s}\right)$$

Furthermore, if $0 < s < 1$, then $C = \zeta(s)$ since $x^{1-s} \to 0$.

We can now get back to estimating $S_k(x)$.

$$\begin{aligned}
S_k(x) &= \frac{x^{k+1}}{k+1}\sum_{n\leq x}\frac{1}{n^{k+1}} + O\left(x^k\sum_{n\leq x}\frac{1}{n^k}\right)\\
&= \frac{x^{k+1}}{k+1}\left(\frac{x^{-k}}{-k}+\zeta(k+1)+O(x^{-k-1})\right)+O\left(x^k\left(\frac{x^{1-k}}{1-k}+\zeta(k)+O(x^{-k})\right)\right)\\
&= \frac{x}{-k(k+1)}+\frac{x^{k+1}}{k+1}\zeta(k+1)+O(x^{k+1-k-1})+O\left(\frac{x}{1-k}+x^k\zeta(k)+O(1)\right)\\
&= \frac{x^{k+1}}{k+1}\zeta(k+1)+O(x)+O(1)+O(x+x^k)
\end{aligned}$$

From this, we finally get the following.

THEOREM 1.8. *Let k be a positive integer. Then*

$$S_k(x) = \frac{x^{k+1}}{k+1}\zeta(k+1)+O(x^{\max(1,k)})$$

We leave the case when k is a negative integer as an exercise. For our next function, let us consider a generalization of the Euler's totient function $\varphi(n)$.

$$\varphi(x,a) = \sum_{\substack{n\leq x\\ \gcd(n,a)=1}} 1$$

For a positive integer n, $\varphi(n)=\varphi(n,n)$ and *Jordan function* is a generalization of φ.

Jordan function. Let n and k be positive integers. Then the Jordan function $J_k(n)$ is the number of k tuples of positive integers not exceeding n that are relatively prime to n.

$$J_k(n) = \sum_{\substack{1\leq a_1,\ldots,a_k\leq n\\ \gcd(a_1,\ldots,a_k,n)=1}} 1$$

Lehmer[17] used the notation $\varphi_k(n)$ but today $J_k(n)$ is used more often. Jordan[18] first discussed this function and Lehmer[19] developed some asymptotic results. Jordan totient function is interesting not only because it is a generalization of Euler's totient

[17] Lehmer, "Asymptotic evaluation of certain Totient Sums".

[18] Camille Jordan. *Traiteé des substitutions et des équations algébriques*. Gauthier-Villars, Paris, 1870, Page 95 – 97.

[19] Lehmer, "Asymptotic evaluation of certain Totient Sums".

function but also because it has many interesting properties. For example, similar to φ, we can show that

$$J_k(n) = \prod_{p^e \| n} p^{k(e-1)}(p-1)$$
$$J_k(n^m) = n^{k(m-1)} J_k(n)$$

Lehmer[20] proves the following which he calls the *fundamental theorem*.

$$J_k(mn) = J_k(n) \prod_{p^e \| m} \left(p^{ke} - p^{k(e-1)}\lambda(n,p)\right) \tag{‡ 1.5}$$

where $\lambda(n,p) = 0$ if $p \mid n$ otherwise $\lambda(n,p) = 1$. We leave the proof of this result and the following to the reader.

$$\sum_{d|n} J_k(d) = n^k$$

Like $\sigma_k(n)$, $J_k(n)$ is also related to the sum $\mathfrak{S}(x,k)$. But before we further look into $J_k(n)$, we will have to discuss Möbius inversion as well as generalized inversion.

1.2 Zeta Function, Dirichlet Series and Dirichlet Product

We encountered ζ when we tried to develop an asymptotic for $S_k(x)$. The function ζ has quite a rich history. Today ζ is mostly called Riemann's zeta function, however, Euler is the first one to investigate this function. Euler started working on ζ around 1730. During that period, the value of $\zeta(2)$ was unknown and of high interest among prominent mathematicians. Ayoub[21] is a very good read on this subject. Euler's first contribution in this matter is Euler[22] where he proves that $\zeta(2) \approx 1.644934$. The paper was first presented to the St. Petersburg Academy on March 5, 1731 and republished in Euler.[23] Euler[24] (republished in Euler[25]) proves the following fundamental result which essentially gives a new proof of infinitude of primes.

[20]Lehmer, "Asymptotic evaluation of certain Totient Sums", Theorem VI.

[21]Raymond Ayoub. "Euler and the zeta function". In: *The American Mathematical Monthly* 81.10 (1974), pp. 1067–1086. DOI: 10.2307/2319041.

[22]Leonhard Euler. "De summatione innumerabilium progressionum". In: *Commentarii academiae scientiarum Petropolitanae* 5 (1738), pp. 91–105.

[23]Leonhard Euler. "E-20: De summatione innumerabilium Progressionum". In: *Spectrum* (2020), pp. 52–64. DOI: 10.1090/spec/098/10.

[24]Leonhard Euler. "Variae Observationes circa series infinitas". In: *Commentarii academiae scientiarum Petropolitanae* 9 (1744), pp. 160–188.

[25]Leonhard Euler. "E-72: Variae Observationes circa series Infinitas". In: *Spectrum* (2020), pp. 249–260. DOI: 10.1090/spec/098/41.

THEOREM 1.9 (Euler's identity). *Let s be a positive integer. Then*

$$\zeta(s) = \prod_{p} \frac{p^s}{p^s - 1}$$

where p extends over all primes.

One of the results in Euler[26] is the following which we shall prove later.

$$\sum_{n \leq x} \frac{1}{p} \sim \log \sum_{n \leq x} \frac{1}{x}$$

Here, $\sim$ is the asymptotic equivalence we have already defined. Even though Euler is the main architect behind the development of ζ, there are compelling reasons why it is called Riemann's zeta function. Riemann[27] first considered ζ for complex s instead of real s only. For a complex s, we usually write $s = \omega + it$ where $\omega = \Re(s)$ is the real part of s and $t = \Im(s)$ is the imaginary part of s.

Dirichlet series. For a complex number s, a *Dirichlet series* is a series of the form

$$\mathfrak{D}_a(s) = \sum_{i \geq 1} \frac{a(n)}{n^s}$$

So, ζ is a special case of $\mathfrak{D}$ when $a(n) = 1$ for all n. Hardy and Riesz[28] considers the following as *general Dirichlet series*

$$\sum_{i \geq 1} a_n e^{-\lambda_n s} \qquad (\ddagger\ 1.6)$$

where (λ_n) is an increasing sequence of real numbers. Following this, Hardy and Riesz[29] calls $\mathfrak{D}$ the *ordinary Dirichlet series* when $\lambda_n = \log n$. Lejeune[30] considers real values of s and proves a number of important theorems. Jensen[31] discuss the first theorems where s is complex involving the nature of convergence of ‡ 1.6. Cahen[32] makes the *first attempt to construct a systematic theory of the function* $\mathfrak{D}_f(s)$

[26] Euler, "Variae Observationes circa series infinitas".

[27] Bernhard Riemann. "Ueber die anzahl der primzahlen unter einer gegebenen grösse". In: *Monatsberichte der Berliner Akademie* (Nov. 1859), pp. 136–144. DOI: 10.1017/cbo9781139568050.008.

[28] Hardy and Riesz, *The general theory of Dirichlet's series*, §1, Page 1.

[29] Hardy and Riesz, *The general theory of Dirichlet's series*.

[30] Dirichlet Peter Gustav Lejeune. *Vorlesungen Über Zahlentheorie*. Ed. by R. Dedekind. Cambridge University Press, 1879.

[31] J. L. W. V. Jensen. "OM RÆKKERS KONVERGENS". in: *Tidsskrift for mathematik*. 5th ser. 2 (1884), pp. 63–72. ISSN: 09092528, 24460737. URL: http://www.jstor.org/stable/24540057; J. L. W. V. Jensen. "Sur une généralisation d'un théorème de Cauchy". In: *Gomptes Rendus* (Mar. 1888).

[32] E. Cahen. "Sur la fonction $\zeta(s)$ de Riemann et sur des fonctions analogues". fr. In: *Annales scientifiques de l'École Normale Supérieure* 3e série, 11 (1894), pp. 75–164. DOI: 10.24033/asens.401. URL: http://www.numdam.org/articles/10.24033/asens.401/.

which, *although much of the analysis which it contains is open to serious criticism, has served—and possibly just for that reason—as the starting point of most of the later researches in the subject.* Jensen[33] proves the following theorem of fundamental importance.

Theorem 1.10. *If $\mathfrak{D}_f(s)$ is convergent for the complex number $s = \omega + it$, then it is convergent for any value of s for which $\Re(s) > \omega$.*

Dirichlet series is the origin of *Dirichlet product*. Consider the Dirichlet series for two arithmetic functions f and g.

$$\mathfrak{D}_f(s) = \sum_{i\geq 1}\frac{f(n)}{n^s}$$
$$\mathfrak{D}_g(s) = \sum_{i\geq 1}\frac{g(n)}{n^s}$$

Then we have

$$\mathfrak{D}_f(s)\mathfrak{D}_g(s) = \sum_{i\geq 1}\frac{f(n)}{n^s}\sum_{i\geq 1}\frac{g(n)}{n^s}$$

Now, imagine we want to write this product as another Dirichlet series. Then it would be of the form

$$\mathfrak{D}_h(s) = \sum_{i\geq 1}\frac{h(n)}{n^s}$$

The coefficients $h(n)$ of $\mathfrak{D}_h(s)$ is determined as follows.

$$h(n) = \sum_{de=n} f(d)g(e)$$

After a little observation, it seems quite obvious that this is indeed correct. In fact, this is what is known as the Dirichlet product.

Dirichlet product. For two arithmetic functions f and g, the *Dirichlet product* or *Dirichlet convolution* of f and g is defined as

$$f * g = \sum_{d|n} f(d)g\left(\frac{n}{d}\right)$$

[33] Jensen, "OM RÆKKERS KONVERGENS".

1.3 General Convolution and Inversion

In this chapter, we will discuss Dirichlet convolution and its generalization, use Dirichlet derivative to prove the Selberg identity, establish some results using generalized convolution and finally, prove the fundamental identity of Selberg.

1.4 Dirichlet Hyperbola Method

Chapter 2

Bertrand to Tchebycheff

Definition. Tchebycheff function of the first kind or *Tchebycheff's theta function* is defined as

$$\vartheta(x) = \sum_{p \leq x} \log p$$

Definition. Tchebycheff function of the second kind or *Tchebycheff's psi function* is defined as

$$\begin{aligned}\psi(x) &= \sum_{n \leq x} \Lambda(n) \\ &= \sum_{p^e \leq x} \log p \\ &= \sum_{p \leq x} \left\lfloor \frac{\log x}{\log p} \right\rfloor \log p\end{aligned}$$

Chapter 3

A Modest Introduction to Sieve Theory

Chapter 4

Prime Number Theorems

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