Some Random Things

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Problem 1. *Let X be a metric space and A, B be two separated subsets. For $a \in A, b \in B$, define*

$$f(t) = ta + (1-t)b$$

for a real number t. Prove that there is a $t \in (0,1)$ such that $f(t) \notin A \cup B$.

Solution. Consider all pairs (a, b) such that $a \in A$ and $b \in B$. By definition, $A \cap \bar{B} = \varnothing$ and $\bar{A} \cap B = \varnothing$ where $\bar{X}$ is the closure of X. Take $(a, b) = (p, q)$ such that $d(p, q)$ is minimum. Let N_p and N_q be the neighborhoods of p and q respectively and r_p and r_q be the radius of N_p and N_q respectively. Due to separation of N_p and N_q, $d(p, q) > r_p + r_q$. Note that $f(t)$ represents the line joining p and q as t varies. Since $r_p + r_q < d(p, q)$, there exists positive real number t such that

$$\frac{r_p}{d(p,q)} < t < 1 - \frac{r_q}{d(p,q)}$$

For such t, $f(t) \notin A$ and $f(t) \notin B$. Thus, $f(t) \notin A \cup B$.

1.1 Prime Number Theorem

Let f any arithmetic function. Consider it's partial sum

$$F(x) = \sum_{n \le x} f(n)$$

We want to estimate this sum by reducing x to a lower number. A good way to do that would be to consider multiplicative f so that for $p \nmid m$, $f(p^e m) =$

$f(p^e)f(m)$. Then

$$\begin{aligned}
F(x) &= \sum_{\substack{n\le x\\ p\nmid x}} f(n) + \sum_{\substack{n\le x\\ p\mid x}} f(n)\\
&= \sum_{\substack{n\le x\\ p^i\|n}} f(n)\\
&= \sum_{\substack{p^i\le x\\ p^i\|n}} f(p^i)f(n/p^i)\\
&= \sum_{p^i\le x} f(p^i) \sum_{\substack{n\le x/p^i\\ p\nmid n}} f(n)\\
&= \sum_{p^i\le x} f(p^i) \left(F(x/p^i) - \sum_{\substack{n\le x/p^i\\ p\mid n}} f(n) \right)
\end{aligned}$$

Letting

$$\begin{aligned}
S_p^f(x) &= \sum_{\substack{n\le x\\ p\nmid n}} f(n)\\
T_p^f(x) &= \sum_{\substack{n\le x\\ p\mid n}} f(n)\\
&= \sum_{p^i\le x} f(p^i) S_p^f(x/p^i)
\end{aligned}$$

we have $F(x) = S_p^f(x) + T_p^f(x)$ for any prime $p \le x$. So, F can be recursively calculated using

$$F(x) = \sum_{p^i\le x} f(p^i) S_p^f(x/p^i)$$

Notice that the choice of p is free. So letting p run for all $p \le x$,

$$\pi(x)F(x) = \sum_{p\le x} \sum_{i=0}^{\lfloor \log_p x\rfloor} f(p^i) S_p^f(x/p^i)$$

Choosing $f(n) = \tau(n)$, the number of divisors of n, we know that

$$\sum_{n\le x} \tau(n) = x\log x + (2\gamma - 1)x + O(\sqrt{x})$$

So, $F(x) \sim x \log x$. Since, $\tau(p^i) = i + 1$, if we can show somehow that

$$\sum_{p \leq x} \sum_{i=0}^{\lfloor \log_p x \rfloor} (i+1) S_p^{\tau}(x/p^i) \sim x^2$$

then we would have

$$\pi(x) F(x) \sim x^2$$

Since $\pi(x) F(x) \sim \pi(x) x \log x$, this would give us

$$\pi(x) \sim \frac{x}{\log x}$$