On the r-th Divisors of a Number.

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Introduction. Associated with every positive integer n is its set of distinct divisors (including 1 and n). Let us designate this set by δ_1. We have also a set of divisors associated with each number in the set δ_1. In this way the set δ_1 gives rise to a new set δ_2 of numbers (not all distinct) which are in fact the divisors of the divisors of n itself. If we take the divisors of the numbers in the set δ_2 we obtain a new set δ_3 and so on. The elements of the set δ_r we call the rth divisors of n. If $n > 1$ it is clear that the number of elements in the set δ_r increases with r. The order in which the elements in each set are arranged is of no consequence.

With the number n are also associated its proper divisors, those divisors of n which are less than n. This set of numbers we call d_1. Each number (> 1) in this set has one or more proper divisors. The set of all proper divisors of the numbers in the set d_1 we call d_2. The proper divisors of the numbers in the set d_2 form a set d_3 and so on. The numbers in the set d_r we call the rth proper divisors of n. The number of rth proper divisors of n ultimately becomes zero although it may increase for the first few values of r. It is convenient to think of sets δ_0 and d_0 each consisting of a single element, namely n itself.

The following example illustrates the nature of these sets of numbers in the case $n = 24$, and will be of use later.

$\delta_1 = 1, 2, 3, 4, 6, 8, 12, 24.$

$\delta_2 = 1 \quad 1, 2 \quad 1, 3 \quad 1, 2, 4 \quad 1, 2, 3, 6 \quad 1, 2, 4, 8 \quad 1, 2, 3, 4, 6, 12$
$\quad 1, 2, 3, 4, 6, 8, 12, 24.$

$\delta_3 = 1 \quad 1 \quad 1, 2 \quad 1 \quad 1, 3 \quad 1 \quad 1, 2 \quad 1, 2, 4 \quad 1 \quad 1, 2 \quad 1, 3 \quad 1, 2, 3, 6$
$\quad 1 \quad 1, 2 \quad 1, 2, 4 \quad 1, 2, 4, 8 \quad 1 \quad 1, 2 \quad 1, 3 \quad 1, 2, 4 \quad 1, 2, 3, 6$
$\quad 1, 2, 3, 4, 6, 12 \quad 1 \quad 1, 2 \quad 1, 3 \quad 1, 2, 4 \quad 1, 2, 3, 6 \quad 1, 2, 4, 8$
$\quad 1, 2, 3, 4, 6, 12 \quad 1, 2, 3, 4, 6, 8, 12, 24.$

. . . .

$d_1 = 1, 2, 3, 4, 6, 8, 12.$

$d_2 = 1 \quad 1 \quad 1, 2 \quad 1, 2, 3 \quad 1, 2, 4 \quad 1, 2, 3, 4, 6.$

$d_3 = 1 \quad 1 \quad 1 \quad 1 \quad 1, 2 \quad 1 \quad 1 \quad 1, 2 \quad 1, 2, 3.$

$d_4 = 1 \quad 1 \quad 1 \quad 1$

For $r \geqq 5$ the sets d_r are empty.

We first study the connection between the same arbitrary function of δ_r and of d_r which leads to certain inversion formulas. These we apply to the determination of the product and the sum of the kth powers of the rth divisors of n. Solutions of certain problems in the theory of probability can be obtained as a by-product of this investigation. If r is negative the definition of the rth divisors of n is meaningless. Nevertheless if we put negative values of r in the expressions for the above sums we obtain a sequence of functions different in nature but essentially belonging to the same class. Among these new functions we find Euler's totient function $\phi(n)$ and Merten's inversion function $\mu(n)$. In fact the whole investigation can be considered as a simultaneous extension of these two classical functions which thus appear to be closely connected.

Inversion of Numerical Integrals. The following investigation is simplified by the introduction of the concept of numerical integration. If two numerical functions $F(n)$ *and* $f(n)$ *are connected by the relation*

$$F(n) = \Sigma f(\delta_1) \tag{1}$$

where δ_1 ranges over the divisors of n, then $F(n)$ is said to be the numerical integral of $f(n)$; conversely $f(n)$ is the numerical derivative of $F(n)$. This concept is due to Bougaief * and has led to remarkable relations existing between the standard discontinuous functions of the theory of numbers. There are various inversion formulas by means of which we can express $f(n)$ in (1) in terms of $F(n)$. The most compact expression is perhaps

$$f(n) = \Sigma \mu(n/\delta_1) F(\delta_1) \tag{2}$$

which is due to Laguerre.† The function $\mu(n)$ is Merten's inversion function which is zero if n contains a square factor and $(-1)^t$ if n is a product of t distinct primes and is unity if $n = 1$. Another form of (2) is due to Dedekind: ‡

$$f(n) = F(n) - \Sigma F(n/p_1) + \Sigma F(n/p_1p_2) - \Sigma F(n/p_1p_2p_3) + \cdots$$

where $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}$. We assume throughout the article that n is expressed in this form. We also make use of the quantity

$$\omega = \alpha_1 + \alpha_2 + \cdots + \alpha_t,$$

which is called the order of n.

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* *Matematicheskii Sbornik*, Vol. 5 (1870-72), pp. 1-63; *Bulletin des Sciences Mathématiques et Astronomiques*, Vol. 10, pp. 13-32.

† *Bulletin de la Société Mathématique de France*, Vol. 1 (1872-3), pp. 77-81.

‡ *Journal für Mathematik*, Vol. 54 (1857), p. 21.

Introducing our sets d_r we can give another expression for $f(n)$ as follows:

$$(4) \quad f(n) = F(n) - \Sigma F(d_1) + \Sigma F(d_2) - \Sigma F(d_3) + \cdots (-1)^\omega \Sigma F(d\omega).$$

To prove this formula we take for values of n the numbers in the set d_1 and n itself and sum. The result is

$$\begin{aligned}\Sigma f(d_1) + f(n) = \Sigma F(d_1) - \Sigma F(d_2) + \Sigma F(d_3) - \Sigma F(d_4) + \cdots \\ + F(n) - \Sigma F(d_1) + \Sigma F(d_2) - \Sigma F(d_3) + \Sigma F(d_4) - \cdots .\end{aligned}$$

Hence $\Sigma f(\delta_1) = F(n)$.

Formula (4) gives us new expressions for certain important functions. For example, if $\phi(n)$ is Euler's totient function we can write, in view of Gauss' theorem $n = \Sigma\phi(\delta_1)$, that

$$\phi(n) = n - \Sigma(d_1) + \Sigma(d_2) - \Sigma(d_3) + \cdots .$$

THEOREM 1. *Let $F(n)$ be an arbitrary function of n then for every integer r we have*

$$(5) \quad \begin{aligned}(-1)^r \Sigma F(d_r) = F(n) - \tbinom{r}{1}\Sigma F(\delta_1) \\ + \tbinom{r}{2}\Sigma F(\delta_2) - \cdots (-1)^r \Sigma F(\delta_r).\end{aligned}$$

In fact $\Sigma F(d_1) = \Sigma F(\delta_1) - F(n)$.

To obtain $\Sigma F(d_2)$ we sum over the proper divisors of δ_1 and n, namely over the set $\delta_2 - \delta_1$ and the set $\delta_1 - n$. That is,

$$\Sigma F(d_2) = \Sigma F(\delta_2) - 2\Sigma F(\delta_1) + F(n).$$

Similarly

$$\Sigma F(d_3) = \Sigma F(\delta_3) - 3\Sigma F(\delta_2) + 3\Sigma F(\delta_1) - F(n).$$

From the relation

$$(6) \quad \binom{m-1}{l-1} + \binom{m-1}{l} = \binom{m}{l}$$

we have by a simple induction

$$\Sigma F(d_r) = \Sigma F(\delta_r) - \tbinom{r}{1}\Sigma F(\delta_{r-1}) + \tbinom{r}{2}\Sigma F(\delta_{r-2}) - \cdots + (-1)^r F(n),$$

which is the theorem. In the same way it can be shown that

$$(7) \quad \Sigma F(\delta_r) = \Sigma F(d_r) + \tbinom{r}{1}\Sigma F(d_{r-1}) + \tbinom{r}{2}\Sigma F(d_{r-2}) + \cdots + F(n).$$

If we sum equations of the type (5) for $r = 0, 1, 2, \cdots, \omega$ we obtain in view of (4)

$$(8) \quad \begin{aligned}f(n) = \tbinom{\omega+1}{1}\ F(n) - \tbinom{\omega+1}{2}\ \Sigma F(\delta_1) \\ + \tbinom{\omega+1}{3}\Sigma(\delta_2) - \cdots + (-1)^\omega \Sigma F(\delta\omega),\end{aligned}$$

which gives us a new inversion formula. As a matter of fact, if the index r in (5) exceeds ω, then $\Sigma F(d_r) = 0$. So that in (8) we may take in place of ω any larger number and the result will be the same. As an example of (8) we may put $f(n) = n$ and get another expression for Euler's totient function in the form

$$(9) \qquad \phi(n) = \tbinom{r}{1} n - \tbinom{r}{2}\Sigma\delta_1 + \tbinom{r}{3}\Sigma\delta_2 - \cdots + (-1)^{r-1}\Sigma\delta_{r-1},$$

where $r \geqq \omega + 1$.

For instance, if $n = 6$ we have

$\delta_1 = 1, 2, 3, 6.$
$\delta_2 = 1, \quad 1, 2, \quad 1, 3, \quad 1, 2, 3, 6.$
$\delta_3 = 1, \quad 1, \quad 1, 2, \quad 1, \quad 1, 3, \quad 1, \quad 1, 2, \quad 1, 3, \quad 1, 2, 3, 6.$
$\delta_4 = 1, \quad 1, \quad 1, \quad 1, 2, \quad 1, \quad 1, \quad 1, 3, \quad 1, \quad 1, \quad 1, 2, \quad 1, \quad 1, 3, \quad 1, \quad 1, 2, \quad 1, 3, \quad 1, 2, 3, 6.$
$\Sigma\delta_1 = 12, \quad \Sigma\delta_2 = 20, \quad \Sigma\delta_3 = 30, \quad \Sigma\delta_4 = 42.$

Applying (9) we have for $r = 3, 4, 5$,

$$\phi(6) = 3.6 - 3.12 + 20 = 2,$$
$$\phi(6) = 4.6 - 6.12 + 4.20 - 30 = 2,$$
$$\phi(6) = 5.6 - 10.12 + 10.20 - 5.30 + 42 = 2.$$

If F and f are related not as in (1) but as follows

$$(1') \qquad F(n) = \Pi f(\delta_1),$$

or what is the same

$$\log F(n) = \Sigma \log f(\delta_1),$$

the above inversion formulas (4) and (8) go over into

$$(4') \qquad f(n) = \frac{F(n) \cdot \Pi F(d_2) \cdot \Pi F(d_4) \cdots}{\Pi F(d_1) \cdot \Pi F(d_3) \cdots}$$

$$(8') \qquad f(n) = \frac{[F(n)]^{\binom{\omega+1}{1}} \cdot [\Pi F(\delta_2)]^{\binom{\omega+1}{3}} \cdots}{[\Pi F(\delta_1)]^{\binom{\omega+1}{2}} \cdot [\Pi F(\delta_3)]^{\binom{\omega+1}{4}} \cdots}$$

Successive Numerical Integrals. Let us consider a sequence of functions

$$(10) \qquad \cdots f_{-3}, \quad f_{-2}, \quad f_{-1}, \quad f_0, \quad f_1, \quad f_2, \quad f_3, \cdots$$

each of which is the numerical integral of its predecessor. That is

$$(11) \qquad f_r(n) = \Sigma f_{r-1}(\delta_1).$$

If we define one of the functions, say $f_0(n)$, for all values of n, then every function $f_r(n)$ is completely determined.

These functions f_r have certain properties in common. For instance

$$f_r(1) = f_0(1) \qquad \text{for every } r. \tag{12}$$

The following multiplicative property is important for our purposes:

$$f(m) \cdot f(n) = f(mn), \tag{13}$$

where m and n are relatively prime integers.

THEOREM 2. *If any function f_r of the set* (10) *possesses the multiplicative property* (13), *so does every function of the set.*

The proof is by induction. We shall first show that f_{r+1} has the property. From (11) we have

$$f_{r+1}(n) \cdot f_{r+1}(m) = \sum_{\delta_1, \delta_1'} f_r(\delta_1) \cdot f_r(\delta_1'), \tag{14}$$

where δ_1' ranges over the divisors of m. Since m and n are prime to each other and since (13) holds for f_r, the right side of (14) becomes $\Sigma f_r(\delta_1\delta_1')$. As δ_1 and δ_1' run over their respective values their product ranges over the divisors of mn without repetition. Hence

$$f_{r+1}(n) \cdot f_{r+1}(m) = f_{r+1}(mn).$$

To prove that (13) holds for f_{r-1} we make use of (2) and write

$$\begin{aligned} f_{r-1}(n) \cdot f_{r-1}(m) &= \sum_{\delta_1, \delta_1'} \mu(n/\delta_1) f_r(\delta_1) \cdot \mu(n/\delta_1') f_r(\delta_1') \\ &= \Sigma\, \mu(mn/\delta_1\delta_1') f_r(\delta_1\delta_1'), \end{aligned}$$

since $\mu(n)$ has the multiplicative property.* Thus

$$f_{r-1}(n) \cdot f_{r-1}(m) = f_{r-1}(mn).$$

Hence the theorem is proved.

The expression (11) can be generalised at once to give the relation between f_r and f_{r+s}.

$$\begin{aligned} f_{r+1}(n) &= \Sigma f_r(\delta_1), \\ f_{r+2}(n) &= \Sigma f_{r+1}(\delta_1) = \Sigma f_r(\delta_2), \\ &\cdots\cdots\cdots\cdots \\ f_{r+s}(n) &= \Sigma f_r(\delta_s) = \Sigma f_s(\delta_r). \end{aligned} \tag{15}$$

Comparing (15) and (7) we have

$$f_{r+s}(n) = f_r(n) + \binom{s}{1} \Sigma f_r(d_1) + \binom{s}{2} \Sigma f_r(d_2) + \cdots + \Sigma f_r(d_s). \tag{16}$$

This expression for the integral of order s of f_r is to be preferred to (15) for computational purposes. After the finite number of sums

* This simple fact follows at once from the definition of $\mu(n)$.

$$\Sigma f_r(d_i) \qquad (i = 0, 1, 2, 3, \cdots, \omega)$$

have been calculated, the integral of any order may be obtained as the linear combination of these with the proper binomial coefficients.

Example. Let $f_r(n) = P_0(n)$, the number of primes less than or equal to n. The problem of getting an explicit formula even for the first integral is a very difficult one. Nevertheless we can easily calculate the numerical value of the integral of any order for a given n. For instance for $n = 24$ referring to the sets d_r already given in the introduction, we have the following values for the sums:

$$P_0(n) = 9, \quad \Sigma P_0(d_1) = 17, \quad \Sigma P_0(d_2) = 15, \quad \Sigma P_0(d_3) = 5, \quad \Sigma P_0(d_4) = 0.$$

Hence

$$P_s(24) = 9 + 17\binom{s}{1} + 15\binom{s}{2} + 5\binom{s}{3}.$$

We can express $f_r(n)$ in (15) in terms of $f_{r+s}(n)$ by making successive applications of (2) with the result

$$(17) \qquad f_r(n) = \Sigma\, \mu(n/\delta_s) f_{r+s}(\delta_s),$$

which gives us a way of calculating isolated derivatives of a given function f_{r+s}. Successive applications of the formula (4) give an expression similar to (16), namely

$$(18) \quad f_r(n) = f_{r+s}(n) - \binom{s}{1} \Sigma f_{r+s}(d_1) + \binom{s+1}{2} \Sigma f_{r+s}(d_2) - \cdots + (-1)^{\omega}\binom{\omega+s-1}{\omega} \Sigma f_{r+s}(d_\omega).$$

Putting $n = 24$ and $s = 5$ we calculate $P_5(\delta_1)$ from (16) with the result

$$P_5(24) = 294, \quad P_5(12) = 85, \quad P_5(8) = 29, \quad P_5(6) = 18,$$
$$P_5(4) = 7, \quad P_5(3) = 2, \quad P_5(2) = 1.$$

Using these values in (18) we have

$$P_0(24) = 294 - \binom{5}{1} \cdot 145 + \binom{6}{2} \cdot 40 - \binom{7}{3} \cdot 5 = 9,$$

which is, in fact, the number of primes less than 24.

In view of relation (15) we can transform (5) to read

$$(19) \quad (-1)^s \Sigma f_r(d_s) = f_r(n) - \binom{s}{1} f_{r+1}(n) + \binom{s}{2} f_{r+2}(n) - \cdots (-1)^s f_{r+s}(n) = \sum_{i=0}^{s} (-1)^i \binom{s}{i} f_{r+i}(n).$$

This last sum is zero for all values of $s > \omega$ regardless of what sequence of functions f_r we select.

In the same way the inversion formula (8) may be written

$$f_r(n) = \binom{h}{1} f_{r+1}(n) + \binom{h}{2} f_{r+2}(n) + \binom{h}{3} f_{r+3}(n) - \cdots + (-1)^{h-1} f_{r+h}(n), \tag{20}$$

where $h > \omega$.

The Function $\sigma_{k,r}(n)$. We shall consider as a special case of (10) the sequence $\sigma_{k,r}(n)$ determined by putting $f_0(n) = n^k$. From (15) it follows that

$$\sigma_{k,r}(n) = \Sigma\, \sigma_{k,0}(\delta_r) = \Sigma\, (\delta_r)^k,$$

which shows that $\sigma_{k,r}(n)$ is the sum of the k-th powers of the r-th divisors of n.

THEOREM 3. *If p is a prime, then*

$$\sigma_{k,r}(p^\alpha) = p^{k\alpha} + \binom{r}{1} p^{k(\alpha-1)} + \binom{r+1}{2} p^{k(\alpha-2)} + \cdots = \sum_{j=0}^{\alpha} \binom{j+r-1}{j} p^{k(\alpha-j)}.$$

The theorem is evidently true for $r = 0$. The proof of the theorem consists in showing that if it is true for the index r, it is also true for both the indices $r - 1$ and $r + 1$, the first part of the proof being required for the discussion of negative indices. First we may write

$$\sigma_{k,r}(p^\alpha) = \sigma_{k,r-1}(p^\alpha) + \sum_{j=0}^{\alpha-1} \sigma_{k,r-1}(p^j), \tag{21}$$
$$\sigma_{k,r-1}(p^\alpha) = \sigma_{k,r}(p^\alpha) - \sigma_{k,r}(p^{\alpha-1}).$$

Supposing the theorem to be true for the index r we have

$$\sigma_{k,r-1}(p^\alpha) = p^{k\alpha} + \binom{r}{1} p^{k(\alpha-1)} + \binom{r+1}{2} p^{k(\alpha-2)} + \binom{r+2}{3} p^{k(\alpha-3)} + \cdots - p^{k(\alpha-1)} - \binom{r}{1} p^{k(\alpha-2)} - \binom{r+1}{2} p^{k(\alpha-3)} - \cdots.$$

In view of (6)

$$\sigma_{k,r-1}(p^\alpha) = p^{k\alpha} + \binom{r-1}{1} p^{k(\alpha-1)} + \binom{r}{2} p^{k(\alpha-2)} + \binom{r+1}{3} p^{k(\alpha-3)} + \cdots.$$

Hence the theorem holds true for the index $r - 1$.

To prove that the theorem is true for $r + 1$ we use an induction on the exponent α. Writing (21) in the form

$$\sigma_{k,r+1}(p^\alpha) = \sigma_{k,r}(p^\alpha) + \sigma_{k,r+1}(p^{\alpha-1}), \tag{22}$$

and putting $\alpha = 1$ we have

$$\sigma_{k,r+1}(p) = \sigma_{k,r}(p) + \sigma_{k,r+1}(1).$$

Using property (12) and the hypothesis of the induction we have

$$\sigma_{k,r+1}(p) = p^k + \binom{r}{1} + 1 = p^k + \binom{r+1}{1}.$$

Hence the theorem is true for $\sigma_{k,r+1}(p)$. To complete the proof we must show that if the theorem holds for $\sigma_{k,r+1}(p^{a-1})$ and $\sigma_{k,r}(p^a)$ it is true for $\sigma_{k,r+1}(p^a)$. With these hypotheses equation (22) becomes

$$\begin{aligned}\sigma_{k,r+1}(p^a) = p^{ka} + \tbinom{r}{1}p^{k(a-1)} + \tbinom{r+1}{2}p^{k(a-2)} + \cdots \\ + p^{k(a-1)} + \tbinom{r+1}{1}p^{k(a-2)} + \cdots \\ = p^{ka} + \tbinom{r+1}{1}p^{k(a-1)} + \tbinom{r+2}{2}p^{k(a-2)} + \cdots.\end{aligned}$$

This establishes the theorem for the index $r+1$ and hence for all indices.

It is obvious that $\sigma_{k,0}(n)$ has the multiplicative property (13). By Theorem 2 all σ's have this property. This leads us at once to the general expression for $\sigma_{k,r}(n)$ namely

$$(23) \qquad \sigma_{k,r}(n) = \prod_{i=1}^{t} \sum_{j=0}^{a_i} \tbinom{j+r-1}{j} p_i^{k(a_i-j)}.$$

The function $\mu_r(n)$. We call $\mu_r(n)$ the special case of the function $\sigma_{k,r}(n)$ for which $k=0$. From the last equation,

$$(24) \qquad \mu_r(n) = \prod_{i=1}^{t} \sum_{j=0}^{a_i} \tbinom{j+r-1}{j} = \prod_{i=1}^{t} \tbinom{r+a_i}{r}.$$

The function $\mu_r(n)$ gives the number of r-th divisors of n, that is the number of elements in the set δ_r. For $r=1$, $\mu_1(n) = \prod_{i=1}^{t}(\alpha_i+1)$, which the well known formula for the number of first divisors of n. The function $\mu_r(n)$ does not depend on the actual prime factors of n but upon their number and multiplicity. In case n is a product of t distinct primes $\mu_r(n) = (r+1)^t$.

Example. Let $n = p_1^3 p_2^2 p_3$, then $\mu_r(n) = (r+1)^3(r+2)^2(r+3)/12$, which gives for the first five values of r:

$$\begin{array}{lll}\mu_0(n) = 1, & \mu_1(n) = 24, & \mu_2(n) = 180, \\ \mu_3(n) = 800, & \mu_4(n) = 2625, & \mu_5(n) = 7056.\end{array}$$

THEOREM 4. *The number of r-th proper divisors of n is given by*

$$(25) \qquad M_r(n) = \mu_r(n) - \tbinom{r}{1}\mu_{r-1}(n) + \tbinom{r}{2}\mu_{r-2}(n) - \cdots + (-1)^r\mu_0\cdot(n).$$

This follows at once from equation (19). If n is a product of t distinct primes, (25) becomes

$$M_r(n) = (r+1)^t - \tbinom{r}{1}r^t + \tbinom{r}{2}(r-1)^t - \tbinom{r}{3}(r-2)^t + \cdots + (-1)^r.$$

If $r > t$ this expression is zero since the sets d_r are empty. The vanishing of this expression for $r > t$ has been noted in connection with binomial series.*

If we introduce negative values of the index, the function $\mu_r(n)$ becomes †

$$(26) \quad \mu_{-r}(n) = \prod_{i=1}^{t} \binom{\alpha_i - r}{\alpha_i} = \prod_{i=1}^{t} (-1)^{\alpha_i} \binom{r-1}{\alpha_i} = (-1)^{\omega} \prod_{i=1}^{t} \binom{r-1}{\alpha_i}.$$

The function $\mu_{-1}(n)$ is zero if $n > 1$ and unity if $n = 1$. The function $\mu_{-2}(n)$ is zero if n contains a square factor, is $(-1)^t$ if n is a product of t distinct primes and is unity if $n = 1$. In short $\mu_{-2}(n)$ is Merten's inversion function. This proves that if a function $f(n)$ is such that $\Sigma f(\delta_1)$ is zero for $n > 1$ and is unity for $n = 1$, then $f = \mu$.‡ The function $\mu_{-3}(n)$ vanishes if n is divisible by a cube. Otherwise its value is $(-2)^{\lambda}$ where λ is the number of prime factors of n which appear to the first power only. In general $\mu_{-r}(n)$ is zero if n is divisible by a perfect r-th power.

Example. If we take again $n = p_1^3 p_2^2 p_3$, we have
$\mu_{-1} = \mu_{-2} = \mu_{-3} = 0, \quad \mu_{-4} = 9, \quad \mu_{-5} = 96, \quad \mu_{-6} = 500, \quad \mu_{-7} = 1800.$

The function $\phi_r(n)$. We define the function

$$(27) \qquad \phi_r(n) \equiv \sigma_{1,r}(n) = \prod_{i=1}^{t} \sum_{j=0}^{\alpha_i} \binom{j+r-1}{r-1} p_i^{\alpha_i - j}$$

which gives the sum of the r-th divisors of n. If $r = 1$ we have

$$\phi_1(n) = \prod_{i=1}^{t} (p_i^{\alpha_i} + p_i^{\alpha_i - 1} + \cdots + 1) = \prod_{i=1}^{t} (p_i^{\alpha_i + 1} - 1)/(p_i - 1),$$

which is well known. For $r = 2$ we have

$$\phi_2(n) = \prod_{i=1}^{t} [p_i^{\alpha_i + 2} - p_i(\alpha_i + 2) + \alpha_i + 1]/(p_i - 1)^2,$$

which gives the sum of the numbers in the set δ_2. The function

$$\phi_3(n) = \prod_{i=1}^{t} [p_i^{\alpha_i + 3} - \binom{\alpha_i + 3}{2} p_i^2 + (\alpha_i + 1)(\alpha_i + 3) p_i - \binom{\alpha_i + 2}{2}]/(p_i - 1)^3,$$

gives the sum of the third divisors of n, and so on.

For negative indices, (27) becomes §

$$\phi_{-r} = \prod_{i=1}^{t} \sum_{j=0}^{\alpha_i} (-1)^j p^{\alpha_i - j} \binom{r}{j}.$$

* See *Chrystal's Algebra*, Vol. 2, p. 183 (3).

† Adopting the usual definition $\binom{-m}{l} = (-1)^l \binom{m+l-1}{l}$ so that (6) holds.

‡ This is the converse of Merten's theorem. *Journal für Mathematik*, Vol. 78 (1874), p. 53.

§ N. V. Bervi, ***Matematicheskii Sbornik***, Vol. 18, pp. 525-8.

If $r = 1$ we have

$$\phi_{-1} = \prod_{i=1}^{t} (p^{a_i} - p^{a_i - 1}),$$

which is Euler's totient function.* This converse of Gauss' theorem has been given by Lucas.† The functions ϕ_{-r} for $r > 1$ have certain similarities to Euler's $\phi(n)$. For instance

$$(28) \qquad \begin{aligned} \phi_{-r}(p^a) &= p^{a-r}(p-1)^r \qquad \alpha \geqq r \\ \phi_{-r}(p) &= p - r. \end{aligned}$$

THEOREM 5. *If n contains the prime factor p to the first but no higher power, then $\phi_{-p}(n) = 0$.*

This follows at once from (28).

The following is a list of values of $\phi_r(n)$ for $n = 42$ and $n = 56$

$r =$	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3
$\phi_r(42) =$	-168	-84	-30	0	12	12	6	0	0	12	42	96	180	300
$\phi_r(56) =$	186	80	24	0	-6	-4	0	4	10	24	56	120	234	420

The function $\pi_r(n)$. In conclusion we shall develop a formula for the product $\pi_r(n)$ of the r-th divisors of n. We shall use the following lemma

LEMMA. The number of times that a certain divisor Δ of n appears in the set δ_r is $\mu_{r-1}(n/\Delta)$.

Evidently the lemma is true for $r = 1$ since δ_1 consists of distinct elements and $\mu_0 = 1$. Let us assume that the lemma is true for the set δ_{r-1}. If we form the set δ_r from the divisors of the set δ_{r-1} the element Δ can arise only from numbers of the form $\Delta\delta'$ where δ' are the divisors of (n/Δ) and from each of these it can arise but once. By the hypothesis of our induction the number of times each divisor of the form $\Delta\delta'$ appears in δ_{r-1} is $\mu_{r-2}(n/\Delta\delta')$. The total number of Δ's appearing in δ_r is therefore

$$\Sigma\mu_{r-2}(n/\Delta\delta') = \Sigma\mu_{r-2}(\delta') = \mu_{r-1}(n/\Delta)$$

since δ' ranges over the divisors of (n/Δ). Thus the lemma is true for all positive values of r.

Let $f_s(n)$ be any function of an arbitrary sequence. Then from the lemma it follows that

$$(29) \qquad f_{r+s}(n) = \Sigma f_s(\delta_r) = \Sigma\mu_{r-1}(n/\delta_1) f_s(\delta_1).$$

* In general $\sigma_{k,-1}(n) = n^k \prod_{i=1}^{t} (1 - P_i^{-k})$. This function has been the subject of a number of investigations. See D. N. Lehmer, *American Journal of Mathematics*, Vol. 22 (1900), pp. 293-335.

† *Théorie des Nombres*, p. 401.

This gives another short method of calculating the actual values of isolated integrals and derivatives. In this method it is unnecessary to determine the numbers in the sets δ_k for $k > 1$. Thus to find $P_5(n)$ in Example 1 we put $f_s(n) = P_0(n)$ and $r = 5$. The results are as follows

$$\begin{array}{rcccccccc}
\delta_1 = & 1, & 2, & 3, & 4, & 6, & 8, & 12, & 24, \\
P_0(\delta_1) = & 0, & 1, & 2, & 2, & 3, & 4, & 5, & 9, \\
\mu_4(n/\delta_1) = & 175, & 75, & 35, & 25, & 15, & 5, & 5, & 1, \\
P_5(n) = & 0 & +\,75 & +\,70 & +\,50 & +\,45 & +\,20 & +\,25 & +\,9 = 294,
\end{array}$$

which agrees with our previous result.

In (29) r and s are interchangable and since we chose s arbitrarily r is similarly at our disposal that is (29) will hold for $r \leq 0$, although the expression $\Sigma f_s(d_r)$ fails to have any meaning in this case. If we put $r = -1$ we have

$$f_{s-1}(n) = \Sigma \mu_{-2}(n/\delta_1) f_s(\delta_1).$$

Since μ_{-2} is Merten's function this expression is simply Leguerre's inversion formula (2). From this standpoint μ_{-r} may be thought of as an inversion function of order $r-1$. Putting $f_s(n) = n$ in (29) we have the following compact expression for $\phi_r(n)$

$$\phi_r(n) = \Sigma \delta_1 \mu_{r-1}(n/\delta_1).$$

If we wish to find the number of times a divisor Δ appears in the set d_r we use equation (5) with $F(n) = 0$ for n different from Δ and $F(\Delta) = 1$. This gives the result

$$\mu_{r-1}(n/\Delta) - \binom{r}{1}\mu_{r-2}(n/\Delta) + \binom{r}{2}\mu_{r-3}(n/\Delta) - \cdots + (-1)^r \mu_{-1}(n/\Delta).$$

THEOREM 6. *The product $\pi_r(n)$ of the r-th divisors of n is $n^{\mu_r(n)/(r+1)}$.* Let us select a certain prime factor p of n and write $n = p^a m$. We shall determine the power to which p appears in the desired product. Let us first consider the numbers of the set δ_r which contain p to the first but not to any higher power. Such numbers are of the form pm' where m' are the divisors of m. By our lemma the number of such elements is

$$\Sigma \mu_{r-1}(p^{a-1}m/m') = \mu_{r-1}(p^{a-1}) \Sigma \mu_{r-1}(m') = \mu_{r-1}(p^{a-1}) \mu_r(m).$$

This is the contribution to the power of p from the set of elements pm'. The corresponding contributions from the numbers of the form pm' are

$$j\mu_r(m)\mu_{r-1}(p^{a-j}) \qquad (j = 1, 2, 3 \cdots \alpha).$$

Summing these contributions we have

$$\mu_r(m)\sum_{j=1}^{a} j\mu_{r-1}(p^{a-j})=\mu_r(m)[\mu_r(p^{a-1})+\mu_r(p^{a-2})+\cdots+1]$$
$$=\mu_r(m)\,\mu_{r+1}(p^{a-1}).$$

Since
$$\mu_{r+1}(p^{a-1})=\mu_r(p^{a})\cdot\alpha/(r+1)$$

the power to which p appears in the final product is

$$[\alpha/(r+1)]\mu_r(m)\,\mu_r(p^{a})=[\alpha/(r+1)]\mu_r(n).$$

This result being true for all primes dividing n we have

$$\pi_r(n)=\prod_{i=1}^{t} p_i^{a_i\mu_r(n)/(r+1)}=n^{\mu_r(n)/(r+1)},$$

which is the theorem.

For $r=1$ we have $\pi_1(n)=n^{\mu_1(n)/2}$. This well known result follows at once by associating with every divisor $\delta_1\leqq n^{1/2}$ its co-divisor. This method fails however for $r>1$.

The product of the numbers in the set d_r may be readily obtained from inversion formula (5) putting $\Sigma F(\delta_k)=[\mu_k(n)/(k+1)]\log n$.

Corollary. The geometric mean of the numbers in the set δ_r is $n^{1/(r+1)}$. This follows immediately from the theorem since $\mu_r(n)$ gives the number of elements in the set δ_r. It is interesting to notice that the mean can be calculated without reference to the prime factors of n and two numbers such as 8191 and 8192 have practically the same mean although their compositions are widely different.

Although the foregoing discussion was carried out in number-theoretic terminology it is of much wider application. We might consider a collection of ω things of which α_1 are of one sort, α_2 of a second sort and so on. It is well known that the number of selections that can be made from such a collection is the same as the number of first divisors of $n=p_1^{a_1}p_2^{a_2}\cdots p_t^{a_t}$. The function $\mu_r(n)$ has immediate application to a similar problem in which r successive selections are made. If we count as inadmissible the selection in which all things are rejected, the r-th proper divisors of n must be considered. If we assign different weights to the different sorts of things the function $\phi_r(n)$ becomes essential to the discussion. In a future paper we hope to apply the theory of r-th divisors to certain problems in combinatory analysis.