PERIODIC RECURRING SERIES

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1. Theorem. Let x be restricted to rational integral values, and let $a_1, \ldots, a_r$ be constant rational numbers not all zero. Let $f(x)$ be a solution of the linear difference equation

$$f(x + r) + a_1 f(x + r - 1) + \ldots + a_r f(x) = 0 \qquad (1.1)$$

of order r, whose characteristic equation

$$y^r + a_1 y^{r-1} + \ldots + a_r = 0 \qquad (1.2)$$

is irreducible (in the rational domain). *Then $f(x)$ has the proper period $m, > 0$, which is unique,*

$$f(x + m) = f(x),\ f(x + n) \neq f(x)\ (n \neq hm,\ h \text{ integer}) \qquad (1.3)$$

when and only when $r = \phi\ (m)$ (the number of integers $\leq m$ and prime to m) *and the roots of the equation* (1.2) *are the primitive mth roots of unity.*

As the proof is short it is given in §3. For the moment we note the following.

Corollary. *When $f(x)$ has the proper period m, the coefficients $a_1, \ldots, a_r$ are integers.*

Corollary. *When $f(x)$ has the proper period $2m$, $f(x + m) = -f(x)$.*

Corollary. *If $f(x)$ has the proper period $m > 2$, the order r is even, (since $\phi\ (m)$ is even when $m > 2$).* The only periodic recurring series of odd order are defined by

$$f(x + 1) - f(x) = 0, f(x + 1) + f(x) = 0,$$

with the respective periods 1,2.

Corollary. *If $f(x)$ has the proper period m, the characteristic equation* (1.2) *is*

$$\Pi\ (y^a - 1)^{\mu(b)} = 0, \qquad (1.4)$$

where μ is Möbius' (or Mertens') function, and the product refers to all pairs (a,b) of positive divisors of m such that $m = ab$.

Let $f_0(x), \ldots, f_{r-1}(x)$ be the linearly independent solutions of (1.1) determined by the initial conditions $f_i(j) = \delta_{ij}$ $(i,j = 0, \ldots, r - 1)$, where $\delta_{ii} = 1$, $\delta_{ij} = 0$, $i \neq j$, and let $f(0), \ldots, f(r - 1)$ be arbitrary constants. Then the general solution of (1.1) is

$$f(x) = f(0)\, f_0(x) + \ldots + f(r - 1) f_{r-1}\,(x). \qquad (1.5)$$

COROLLARY. *By* (1.5) *there are* $\infty^{\phi(m)}$ *recurring series having the proper period* m. *The terms of such a series are integers when and only when* $f(j)$ ($j = 0, \ldots, \phi(m) - 1$) *are integers.*

2. *Remarks on Distribution.*—The values of the possible orders r and periods m raise some interesting questions in distribution. It has been noted that there are precisely two equations (1.1) whose solutions are periodic recurring series of odd order ($r = 1$). All cases are included in the following.

COROLLARY. *If* $\phi^{-1}(r)$ *denotes the (necessarily finite) number of integers whose* ϕ*-function has the constant value* r, *there exist precisely* $\phi^{-1}(r)$ *equations* (1.1) *whose solutions are periodic recurring series of order* r; *the periods of these series are the solutions* m *of* $\phi(m) = r$.

From the table (p. 395) in Lucas' *Théorie des Nombres*, we see therefore that the only $2r \leq 100$ for which no such equation of order $2r$ exists are the 13 numbers

$$14, 26, 34, 38, 50, 62, 68, 74, 76, 86, 90, 94, 98.$$

In the same range the maximum number of such equations of order $2r$ occurs for $r = 36, 48$, and is 17; the minimum number 2 occurs for even orders precisely 12 times, the largest order being 78; the longest period is 420, for the order 96. The distribution of the values of $\phi^{-1}(r)$ is extremely irregular. It is possible, however, to express $\phi^{-1}(r)$ in terms of known arithmetical functions; this will be done in another note. The total number of equations (1.1) of orders ≤ 100 having periodic solutions is 198.

Reuschle's *Tafeln complexer Prinzahlen welche aus Wurzeln der Einheit gebildet sind* (Berlin, 1875), contains incidentally (pp. 193–440, 467–671) all the equations (1.4) worked out fully for odd composite $m \leq 105$ and for $m = 4n$, n composite, $3 \leq n \leq 30$. It is interesting to notice that the only one of these equations (in the ranges indicated) having a coefficient other than 0,1 or -1 occurs when $m = 105$, when the coefficient -2 appears once.

3. *Proof.*—Let $y_1, \ldots, y_r$ be the roots of the equation (1.2). Then any solution of (1.1) is of the form

$$f(x) = c_1 y_1^x + \ldots + c_r y_r^x,$$

where the c's are constants. If this $f(x)$ has the period m, $f(x + m) = f(x)$, and

$$f(x) = c_1 y_1^{x+m} + \ldots + c_r y_r^{x+m}.$$

Hence

$$c_1(y_1^m - 1)y_1^x + \ldots + c_r(y_r^m - 1)y_r^x = 0.$$

In this take $x = 0, \ldots, r - 1$. The Vandermonde determinant $D(y_1,$

$\ldots, y_r) \neq 0$, since $y_1, \ldots, y_r$ are distinct by the irreducibility of (1.2). Hence the set of r homogeneous equations just constructed has the unique set of solutions

$$c_1(y_1^m - 1) = 0, \ldots, c_r(y_r^m - 1) = 0. \tag{3.1}$$

If therefore none of $c_1, \ldots, c_r$ vanish, $y_1, \ldots, y_r$ are mth roots of unity. We shall show that the hypothesis that precisely h of $c_1, \ldots, c_r$ vanish, where $0 < h \leq r$, leads to a contradiction. For, if precisely h vanish, $f(x)$ is a solution of an equation of type (1.1) of order $r - h$. By hypothesis the characteristic equation of this equation is irreducible and has rational coefficients. But it has $r - h$ roots in common with (1.2), which is irreducible. Since $r - h < r$, we have a contradiction. Hence $y_1, \ldots, y_r$ are mth roots of unity.

It remains to be proved that $y_1, \ldots, y_r$ are primitive. Let $y^m - 1 = P_0(y) \ldots P_s(y)$ be the resolution of $y^m - 1$ into factors irreducible in the rational domain, and let the roots of $P_0(y) = 0$ be the $\phi\ (m)$ primitive mth roots of unity. Then the roots of $P_1(y) = 0, \ldots, P_s(y) = 0$ are imprimitive mth roots of unity. But since m is by hypothesis a proper period, m is the least positive integer for which the equations (3.1) hold. Hence $y_1, \ldots, y_r$ are not nth roots of unity, $n < m$, and therefore they are the roots of $P_0(y) = 0$.

NOTE ON SINGULARITIES OF POWER SERIES

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THEOREM: If

(1) the power series $\sum_{n=0}^{\infty} a_n z^n$, $\sum_{n=0}^{\infty} p_n z^n$, and $\sum_{n=0}^{\infty} a_n p_n z^n$ have radii of convergence equal to one; and

(2) $\sum_{n=0}^{\infty} a_n z^n$ has on its closed circle of convergence a single singularity at the point $z = 1$; and

(3) the coefficients p_n are non-negative, $p_n \geqq 0$;

then the power series $\sum_{n=0}^{\infty} a_n p_n z^n$ is necessarily singular at the point $z = 1$.

This theorem is on one hand a generalization of the theorem of Pringsheim on singularities of power series with non-negative coefficients. (Pringsheim's theorem is obtained by putting all $a_n = 1$.) On the other