On a fundamental inequality in number theory

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1. Introduction

Many number-theoretical problems can be reduced to the study of an inequality of the form

$$0 < |\beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n| < e^{-\delta H}, \tag{1}$$

where $\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n$ denote non-zero algebraic numbers, $\delta > 0$, and H is some number at least as large as the heights of the β's, that is the maximum of the absolute values of the coefficients in their minimal defining polynomials. It was shown some four years ago (see [1]) that one can give an explicit upper bound for H in terms of A, d, n, and δ, where the heights of the α's are supposed not to exceed A and the degrees of the α's and β's are supposed not to exceed d; and the success of the resolution of the problems in question depends on the precise form of this bound as a function of the parameters. It was proved in [1, IV] that, in the case when $\beta_1, \cdots, \beta_n$ are rational integers, we have $H < C(\log A)^{(2n+1)^2}$, where C was given explicitly in terms of n, d, and δ; the result was derived principally for computational purposes and it would seem to be the most useful general assertion of this character established to date. Frequently, however, it is known that one of the α's, say α_n, has relatively large height and interest attaches mainly to the magnitude of H with respect to the latter. In this connection it was shown in [2] that if the heights of $\alpha_1, \cdots, \alpha_{n-1}$ do not exceed A' and if $\kappa > n + 1$ then $H < C(\log A)^\kappa$, where C is effectively computable in terms of A', n, d, δ, and κ. The condition $\kappa > n + 1$ could readily be relaxed to $\kappa > n$, and a further relaxation to $\kappa > n - 1$ was obtained by Feldman [4]. In the present note we show that the condition can in fact be relaxed to $\kappa > 1$. We prove:

THEOREM. *Let $\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n$ be non-zero algebraic numbers with degrees at most d, let $\alpha_1, \cdots, \alpha_{n-1}$ have heights at most A' and let α_n and $\beta_1, \cdots, \beta_n$ have heights at most A and B respectively. If $\varepsilon > 0$, $\delta > 0$ and (1) holds for some $H > e^{\sqrt{\log B}}$, then $H < C(\log A)^{1+\varepsilon}$, where $C = C(n, d, \varepsilon, \delta, A')$ is effectively computable.*

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It has been assumed here that the logarithms have their principal values but the conclusion would remain valid for any choice of logarithms if one allowed C to depend on their determinations. The proof involves further developments of the techniques described in [1] and [2]. Some special cases of the theorem were recently derived in connection with certain class-number problems (see [3,5]), and the work here rests on a combination of the methods employed therein. A particular novel feature in the argument is the utilization of the algebraic lemmas given in § 2; their object is to show that the conditions on the auxiliary function derived from the extrapolation procedure imply that a certain multiplicative combination of the α's is a high power of an element of the field which they and the β's generate over the rationals. An induction argument shows that this is in fact untenable and the contradiction establishes the theorem. It will be clear from our exposition that the condition $H > e^{\sqrt{\log B}}$ can be relaxed to $H > (\log B)^{cn^2/\varepsilon}$ for a sufficiently large absolute constant c, provided that $\varepsilon < 1$.

It was remarked above that special cases of the theorem have led to the resolution of certain class-number problems; more precisely, they have yielded an effective determination of all the imaginary quadratic fields with class-numbers 1 and 2. The theorem also admits an application concerning rational approximations to algebraic numbers. It was proved in [2] that if α is any algebraic number with degree $n \geqq 3$ and if $\kappa > n + 1$ then one can specify explicitly a number $c = c(\alpha, \kappa) > 0$ such that

$$|\alpha - p/q| > cq^{-n}e^{(\log q)^{1/\kappa}} \tag{2}$$

for all rationals p/q, $(q > 0)$; this provided the first effective improvement on Liouville's inequality. Again the condition $\kappa > n + 1$ could easily be relaxed to $\kappa > n$ (cf. the footnote on p. 174 of [2]), and Feldman [4] obtained a further improvement to $\kappa > n - 1$. It will be apparent from the arguments of [2] that, in view of our theorem, the condition can now be relaxed to $\kappa > 1$. It would, of course, be of much interest to obtain a result of this nature in which the number on the right of (2) is replaced by $cq^{-\kappa}$ for some $\kappa < n$; and this would correspond to a strengthening of our theorem to the extent of the elimination of ε. But such a result does not seem to be readily derivable with our present techniques. Our method would, however, allow one to replace $(\log A)^{\varepsilon}$ by some power of $\log \log A$.

2. Algebraic lemmas

In this section we give four lemmas of an algebraic nature preliminary to the proof of the theorem. K will signify an arbitrary algebraic number field.

LEMMA 1. *If K contains the p^{th} roots of unity, where p is a prime, and if $\alpha \neq 0$ is in K but $\alpha^{1/p}$ is not in K, then $K(\alpha^{1/p})$ is an extension of K of degree p.*

Proof. If $K(\alpha^{1/p})$ were not an extension of K of degree p then the equation $x^p - \alpha = 0$ would be reducible over K and so a product of $r < p$ of the numbers $\alpha^{1/p}e^{2\pi ij/p}$ $(1 \leqq j \leqq p)$ would be in K. This would imply that $\alpha^{r/p}$ is in K. But $(r, p) = 1$ since $r < p$ and thus there exist integers s, t such that $rs + pt = 1$. It follows that $\alpha = \{(\alpha^{r/p})^s\alpha^t\}^p$ is a p^{th} power in K, contrary to hypothesis.

LEMMA 2. *If α, β are non-zero elements of K and if $\alpha^{1/p}, \beta^{1/p}$ are any fixed p^{th} roots for some prime p, then either $K(\alpha^{1/p}, \beta^{1/p})$ is an extension of $K(\alpha^{1/p})$ of degree p or $\beta = \alpha^j\gamma^p$ for some γ in K and some integer j with $0 \leqq j < p$.*

Proof. We suppose throughout that $K(\alpha^{1/p}, \beta^{1/p})$ is not an extension of degree p and we assume first that K contains $\zeta = e^{2\pi i/p}$. Then, by Lemma 1, $\beta^{1/p}$ is in $K(\alpha^{1/p})$ and so can be expressed in the form

$$\sum_{l=j}^{p-1} a_l\alpha^{l/p}$$

with a_l in K and $a_j \neq 0$; this expression is unique if we suppose, as we may, that $\alpha^{1/p}$ is not in K; otherwise $K(\alpha^{1/p}) = K$ and the lemma holds with $j = 0$. If now $\gamma = \beta^{1/p}\alpha^{-j/p}$ were not in K then, by Lemma 1 again, γ would be of degree p over K and the roots of $x^p - \gamma = 0$ would be given by the expressions

$$\sum_{l=j}^{p-1} a_l(\alpha\zeta^r)^{(l-j)/p} \qquad (0 \leqq r < p)\ .$$

But the sum of these is plainly $pa_j \neq 0$ contrary to the equation Trace $\gamma = 0$.

Now assume that K does not contain ζ and write $K' = K(\zeta)$. Since the irreducibility of $x^p - \beta$ over $K'(\alpha^{1/p})$ implies the irreducibility over $K(\alpha^{1/p})$ we see that $K'(\alpha^{1/p}, \beta^{1/p})$ is not an extension of $K'(\alpha^{1/p})$ of degree p. Thus from the first part we have $\beta = \alpha^j\gamma^p$ for some j with $0 \leqq j < p$ and some γ in K'. But if d is the degree of K' over K then $d < p$ and so $(d, p) = 1$; on taking the relative norm from K' to K we get $(\beta\alpha^{-j})^d = (N\gamma)^p$ and, on noting that $N\gamma$ is an element of K, it follows as at the end of the proof of Lemma 1 that $\beta\alpha^{-j}$ is itself a p^{th} power of an element of K. This proves the lemma.

LEMMA 3. *Let $\alpha_1, \cdots, \alpha_n$ be non-zero elements of K and let $\alpha_1^{1/p}, \cdots, \alpha_n^{1/p}$ denote fixed p^{th} roots for some prime p. Further let $K' = K(\alpha_1^{1/p}, \cdots, \alpha_{n-1}^{1/p})$. Then either $K'(\alpha_n^{1/p})$ is an extension of K' of degree p or we have*

$$\alpha_n = \alpha_1^{j_1} \cdots \alpha_{n-1}^{j_{n-1}}\gamma^p$$

for some γ in K and some integers $j_1, \cdots, j_{n-1}$ with $0 \leqq j_l < p$.

Proof. By the argument used at the end of the proof of Lemma 2, with α_n in place of β and $\alpha_1^{j_1} \cdots \alpha_{n-1}^{j_{n-1}}$ in place of α^j, we see that there is no loss of generality in assuming that K contains the p^{th} roots of unity.* Further we can assume that $\alpha_l^{1/p}$ $(l = 1, 2, \cdots, n-1)$ generates an extension of $K(\alpha_1^{1/p}, \cdots, \alpha_{l-1}^{1/p})$ of degree p; for if this did not hold for some l then, by Lemma 1, $\alpha_l^{1/p}$ would be an element of the latter field and so K' would be generated over K by the elements $\alpha_m^{1/p}$ with $m \neq l$; since the lemma is true for $n = 2$ by Lemma 2 (and hence also for $n = 1$), we can now argue inductively and appeal to the required conclusion with n replaced by $n - 1$.

Now suppose that $K'(\alpha_n^{1/p})$ is not an extension of K' of degree p. Then, by Lemma 1, $\alpha_n^{1/p}$ is in K' and so, by Lemma 2, we obtain

$$\alpha_n^{1/p} = \alpha_l^{j_l/p}\gamma_l \qquad (1 \leqq l < n)$$

for some integers j_l with $0 \leqq j_l < p$ and some γ_l in the field K_l generated over K by the elements $\alpha_m^{1/p}$ $(1 \leqq m < n,\ m \neq l)$. But clearly

$$\gamma = \alpha_n^{1/p}\alpha_1^{-j_1/p} \cdots \alpha_{n-1}^{-j_{n-1}/p}$$

lies in each K_l and, in view of the second assumption made above, we see that the intersection of all the K_l is just K. This completes the proof of the lemma.

LEMMA 4. *Suppose that α, β are elements of an algebraic number field of degree D and that for some positive integer p we have $\alpha = \beta^p$. If $a\alpha$ is an algebraic integer for some rational integer $a > 0$ and if b is the leading coefficient in the minimal defining polynomial of β then $b \leqq a^{D/p}$.*

Proof. Let $\alpha^{(1)}, \cdots, \alpha^{(D)}$ and $\beta^{(1)}, \cdots, \beta^{(D)}$ be the field conjugates of α, β respectively. Further let B be the minimal positive integer such that

$$f(x) = B(x - \beta^{(1)}) \cdots (x - \beta^{(D)})$$

has rational integer coefficients. Since (by Gauss's Lemma) $f(x)$, the field polynomial of β, is some power of the minimal polynomial, it clearly suffices to prove that $B^p \leqq a^D$.

We write

$$g(x) = a^D(x^p - \alpha^{(1)}) \cdots (x^p - \alpha^{(D)}), \quad h(x) = \prod_{j=1}^{p} f(xe^{2\pi ij/p}) \,.$$

Since, by hypothesis, $\alpha = \beta^p$ we have

$$B^p g(x) = (-1)^{D(p+1)} a^D h(x) \,.$$

* Under this assumption, the lemma can be derived easily from the results on pp. 42-3 of H. Hasse, Bericht über neuere Untersuchungen und Probleme aus der Theorie der algebraischen Zahlkörper. Teil II: Reziprozitätsgesetz (Würzburg, 1965). However, the latter work involves an appeal to Galois theory which we prefer to avoid.

The coefficients of $h(x)$ are plainly algebraic integers and in fact rational integers since the coefficients of $g(x)$ are rational. Further, the coefficients of $f(x)$ are relatively prime rational integers, and so the coefficients of $f(xe^{2\pi ij/p})$ are also relatively prime when considered as ideals in the cyclotomic field. It follows from the algebraic generalization of Gauss's Lemma that the coefficients of $h(x)$ are again relatively prime. But $g(x)$ also has rational integer coefficients and so B^p divides a^D; hence $B^p \leqq a^D$ as required.

3. On the logarithms of algebraic numbers

Our object here is to develop further the work of [3, § 2] so as to obtain a precise analogue (Lemma 9 below) of the main result established there (Lemma 4) in which reciprocals of integers occur in the exponents in place of the integers themselves (see also Lemma 12 of [5]).

We adopt the notation of [3, § 2] without change. Thus $\alpha_1, \cdots, \alpha_n$ will denote $n \geqq 2$ non-zero algebraic numbers with degrees at most d. The heights of $\alpha_1, \cdots, \alpha_{n-1}$ and α_n will be supposed not to exceed A' and $A \geqq 2$ respectively. By $\log \alpha_1, \cdots, \log \alpha_n$ will be meant the principal values of the logarithms. We shall denote by $\beta_1, \cdots, \beta_{n-1}$ algebraic numbers with degrees at most d and heights at most $H^{(\log H)^3}$. We shall suppose that $0 < \varepsilon < 1$, $\delta > 0$, and we shall assume that

$$|\beta_1 \log \alpha_1 + \cdots + \beta_{n-1} \log \alpha_{n-1} - \log \alpha_n| < e^{-\delta H} \tag{3}$$

and that

$$H > C(\log A)^{1+\varepsilon}$$

for a sufficiently large number $C = C(n, d, \varepsilon, \delta, A')$. By $c_1, c_2, \cdots$ we signify numbers greater than 1 which can be specified in terms of $n, d, \varepsilon, \delta, A'$ only. We put $\eta = \varepsilon/(4n)^2$ and we write

$$k = [H^{1/(1+\frac{1}{2}\varepsilon)}],\ h = [k^{\frac{1}{4}\eta}]\ ,$$

$$L = L_1 = \cdots = L_{n-1} = [k^{1-\eta}],\ L_n = [k^{n\eta}]\ ,$$

where, as usual, $[x]$ denotes the integral part of x. Also we put

$$f_{m_1,\cdots,m_{n-1}}(z_1, \cdots, z_{n-1}) = \frac{\partial^{m_1+\cdots+m_{n-1}}}{\partial z_1^{m_1} \cdots \partial z_{n-1}^{m_{n-1}}} f(z_1, \cdots, z_{n-1})\ .$$

LEMMA 5. *There are integers $p(\lambda_1, \cdots, \lambda_n)$, not all 0, with absolute values at most e^{hk}, such that the function*

$$\Phi(z_1, \cdots, z_{n-1}) = \sum_{\lambda_1=0}^{L_1} \cdots \sum_{\lambda_n=0}^{L_n} p(\lambda_1, \cdots, \lambda_n)\alpha_1^{\gamma_1 z_1} \cdots \alpha_{n-1}^{\gamma_{n-1} z_{n-1}}\ ,$$

where $\gamma_r = \lambda_r + \lambda_n\beta_r$, $(1 \leqq r < n)$, satisfies

$$(4) \qquad |\Phi_{m_1,\cdots,m_{n-1}}(l, \cdots, l)| < e^{-\frac{1}{2}\delta H}$$

for all integers l with $1 \leqq l \leqq h$ and all non-negative integers $m_1, \cdots, m_{n-1}$ with $m_1 + \cdots + m_{n-1} \leqq k$.

Proof. This is Lemma 1 of [3, § 2].

LEMMA 6. *For any non-negative integers $m_1, \cdots, m_{n-1}$ with $m_1 + \cdots + m_{n-1} \leqq k$ and any complex number z we have*

$$|\Phi_{m_1,\cdots,m_{n-1}}(z, \cdots, z)| < e^{2hk}c_1^{L|z|} .$$

Proof. This is the first part of Lemma 2 of [3].

LEMMA 7. *Let J be any integer satisfying $0 \leqq J \leqq 8n^2$. Then* (4) *holds for all integers l with $1 \leqq l \leqq h^{J+1}$ and each set of non-negative integers $m_1, \cdots, m_{n-1}$ with $m_1 + \cdots + m_{n-1} \leqq k/2^J$.*

Proof. This is Lemma 3 of [3].

LEMMA 8. *We have*

$$|\phi(1/q)| < \exp(-kh^{8n^2})$$

for any integer $q > 2$, where

$$\phi(z) = \Phi_{m_1,\cdots,m_{n-1}}(z, \cdots, z)$$

and $m_1, \cdots, m_{n-1}$ are any non-negative integers between 0 and L inclusive.

Proof. As in the proof of Lemma 4 of [3] we see that, by Lemma 7,

$$(5) \qquad |\phi_m(r)| < n^k e^{-(1/2)\delta H}$$

for all integers r, m with $1 \leqq r \leqq X$ and $0 \leqq m \leqq Y$, where

$$X = h^{8n^2+1}, \quad Y = [k/2^{8n^2+1}] .$$

On writing, for brevity,

$$E(z) = \{(z-1)\cdots(z-X)\}^{Y+1}$$

and denoting by Γ the circle in the complex plane, described in the positive sense, with centre at the origin and radius Xh, we obtain from Cauchy's Residue Theorem

$$\frac{1}{2\pi i}\int_\Gamma \frac{\phi(z)}{(z-1/q)E(z)}dz = \frac{\phi(1/q)}{E(1/q)} + \frac{1}{2\pi i}\sum_{r=1}^{X}\sum_{m=0}^{Y}\frac{\phi_m(r)}{m!}\int_{\Gamma_r}\frac{(z-r)^m dz}{(z-1/q)E(z)} ,$$

where Γ_r denotes the circle in the complex plane described in the positive sense with centre r and radius 1/2. Since, for z on Γ_r,

$$|(z-r)^m/E(z)| < 8^{Y+1} ,$$

it follows from (5) that the absolute value of the double sum on the right of the above equation is at most

$$X(Y+1)8^{Y+2}n^k e^{-(1/2)\delta H} < k^{2n^2\eta+2}(8n)^k e^{-(1/2)\delta H} < e^{-(1/4)\delta H} .$$

Further, by virtue of Lemma 6, we have, for any z on Γ,

$$|\phi(z)| < e^{2hk} c_1^{LXh} ,$$

and, furthermore, it is clear that

$$|E(z)| > \big((1/2)Xh\big)^{X(Y+1)} \quad \text{and} \quad |E(1/q)| < (2X)^{X(Y+1)} .$$

Thus we obtain

$$|\phi(1/q)| < 2e^{2hk}c_1^{LXh}\big((1/4)h\big)^{-X(Y+1)} + e^{-(1/4)\delta H}|E(1/q)| .$$

The lemma now follows at once on noting that $Lh < k$ and that $X(Y+1)$ exceeds kh^{8n^2} but is less than

$$h^{8n^2+1}k < k^{(2n^2+1)\eta+1} < H .$$

LEMMA 9. *For all integers q, $m_1, \cdots, m_{n-1}$ with*

$$2 < q \leqq 2L_n, \quad 0 \leqq m_r \leqq L \qquad (1 \leqq r < n)$$

we have

$$\sum_{\lambda_1=0}^{L_1} \cdots \sum_{\lambda_n=0}^{L_n} p(\lambda_1, \cdots, \lambda_n)\alpha_1^{\lambda_1/q} \cdots \alpha_n^{\lambda_n/q}\gamma_1^{m_1} \cdots \gamma_{n-1}^{m_{n-1}} = 0 . \tag{6}$$

Proof. Let Q' denote the expression on the left of (6) and let $Q = PQ'$ where

$$P = a_1^{L_1} \cdots a_n^{L_n} b_1^{m_1} \cdots b_{n-1}^{m_{n-1}}$$

as in [3]. Clearly Q is an algebraic integer with degree at most $D = d^{2n-1}q^n$, and each conjugate has absolute value at most $c_2^L e^{2hk}$. Thus if $Q \neq 0$ we have

$$|Q| \geqq (e^{2hk}c_2^L)^{-D} . \tag{7}$$

But from (3) we have*

$$|\log(\alpha_1^{\beta_1} \cdots \alpha_{n-1}^{\beta_{n-1}}) - \log \alpha_n| < e^{-\delta H}$$

and thus (cf. [3, proof of Lemma 1])

$$|\alpha_1^{\lambda_1/q} \cdots \alpha_n^{\lambda_n/q} - \alpha_1^{\gamma_1/q} \cdots \alpha_{n-1}^{\gamma_{n-1}/q}| < c_3^L e^{-(3/4)\delta H} .$$

Since also

$$|\gamma_1^{m_1} \cdots \gamma_{n-1}^{m_{n-1}}| < e^{hk}$$

it follows that

$$|P'^{-1}\phi(1/q) - Q'| < c_4^L e^{hk-(3/4)\delta H} < e^{-(5/8)\delta H} ,$$

where

* Note that the first logarithm is not necessarily principal-valued.

$$P' = (\log \alpha_1)^{m_1} \cdots (\log \alpha_{n-1})^{m_{n-1}} .$$

Further we have $|P| < c_5^L e^{hk}$, $|P'| < c_6^k$, and hence Lemma 8 gives

$$|Q| < \exp(-(1/2)kh^{8n^2}) .$$

This together with (7) implies that

$$(2hk + c_7 L)d^{2n-1}q^n > (1/2)kh^{8n^2} ,$$

and since $L < k$, $h > (1/2)k^{(1/4)\eta}$ and, by hypothesis, $q < 2k^{n\eta}$, we plainly have a contradiction if k is sufficiently large. The contradiction shows that $Q' = 0$, as required.

4. Proof of the Theorem

We shall assume that all the hypotheses recorded at the beginning of § 3 are satisfied and indeed we shall commence with the more stringent condition that the heights of $\beta_1, \cdots, \beta_{n-1}$ are at most $H^{(\log H)^2}$. We shall further suppose that the linear form on the left of (3) does not vanish. We shall proceed to establish a contradiction and this will suffice to prove the Theorem. For the hypotheses of the Theorem plainly imply that

$$0 < |\beta_1' \log \alpha_1 + \cdots + \beta_{n-1}' \log \alpha_{n-1} - \log \alpha_n| < |\beta_n|^{-1} e^{-\delta H} ,$$

where $\beta_j' = -\beta_j/\beta_n$, $(1 \leqq j < n)$; and, since $B \leqq H^{\log H}$, the observations recorded in [2, I, § 6] show that $\beta_1', \cdots, \beta_{n-1}'$ have heights at most $H^{(\log H)^2}$ and that $|\beta_n|^{-1} \leqq e^{(1/2)\delta H}$, provided that H exceeds a sufficiently large number $C = C(d, \delta)$. We shall assume also, as we may without loss of generality, that $\alpha_1 = -1$.

It is well known that there exists at least one prime p between L_n and $2L_n$ exclusive, and we take $q = p$ in Lemma 9. On writing (6) in the form

$$\sum_{\lambda_n=0}^{L_n} \left(\sum_{\lambda_1=0}^{L_1} \cdots \sum_{\lambda_{n-1}=0}^{L_{n-1}} p(\lambda_1, \cdots, \lambda_n)\alpha_1^{\lambda_1/p} \cdots \alpha_{n-1}^{\lambda_{n-1}/p}\gamma_1^{m_1} \cdots \gamma_{n-1}^{m_{n-1}}\right)\alpha_n^{\lambda_n/p} = 0 ,$$

we see that Lemma 4 implies that either each of the expressions in parentheses is 0 for all $m_1, \cdots, m_{n-1}$ with $0 \leqq m_r \leqq L$ or

$$\alpha_n = \alpha_1^{j_1} \cdots \alpha_{n-1}^{j_{n-1}}\alpha_n^{(1)p} \tag{8}$$

for some integers $j_1, \cdots, j_{n-1}$ with $0 \leqq j_l < p$ and some element $\alpha_n^{(1)}$ in the field K generated by the α's and β's over the rationals. In fact we cannot have all the above expressions 0, for this would imply that

$$\sum_{\lambda_{n-1}=0}^{L_{n-1}} \left(\sum_{\lambda_1=0}^{L_1} \cdots \sum_{\lambda_{n-2}=0}^{L_{n-2}} p(\lambda_1, \cdots, \lambda_n)\alpha_1^{\lambda_1/p} \cdots \alpha_{n-1}^{\lambda_{n-1}/p}\gamma_1^{m_1} \cdots \gamma_{n-2}^{m_{n-2}}\right)\gamma_{n-1}^{m_{n-1}} = 0$$
$$(0 \leqq m_{n-1} \leqq L_{n-1})$$

and the coefficients of the sums in parenthesis form a Vandermonde determinant; hence these sums would all vanish and repeated application of the argument shows that $p(\lambda_1, \cdots, \lambda_n)$ would be 0 for all $\lambda_1, \cdots, \lambda_n$, contrary to specification. Recalling that, by assumption, $\alpha_1 = -1$, we obtain from (8)

$$\log \alpha_n = (j_1 + j) \log \alpha_1 + \cdots + j_{n-1} \log \alpha_{n-1} + p \log \alpha_n^{(1)} ,$$

where all the logarithms have their principal values and j is a rational integer with absolute value at most np. It follows, on substituting for $\log \alpha_n$ in (3), that

$$0 < |\beta_1^{(1)} \log \alpha_1 + \cdots + \beta_{n-1}^{(1)} \log \alpha_{n-1} - \log \alpha_n^{(1)}| < e^{-\delta H} ,$$

where

$$\beta_1^{(1)} = (\beta_1 - j_1 - j)/p, \beta_r^{(1)} = (\beta_r - j_r)/p \qquad (2 \leqq r < n) .$$

Now $\beta_1^{(1)}, \cdots, \beta_{n-1}^{(1)}$ lie in K, which clearly has degree $D \leqq d^{2n-1}$. Further, the heights of $p\beta_1^{(1)}, \cdots, p\beta_{n-1}^{(1)}$ are plainly at most

$$(D + 1)!\, (2np)^D H^{(\log H)^2}$$

and, since $p \leqq H^{n\eta}$, it follows that the heights of $\beta_1^{(1)}, \cdots, \beta_{n-1}^{(1)}$ are at most

$$(4nD)^{2D} p^{2D} H^{(\log H)^2} \leqq H^{(\log H)^2 + D} .$$

Furthermore, each conjugate of

$$\alpha_n^{(1)p} = \alpha_n \alpha_1^{-j_1} \cdots \alpha_{n-1}^{-j_{n-1}}$$

has absolute value at most $(DA')^{n\eta}DA$, and the same estimate holds for some integer a such that $a\alpha_n^{(1)p}$ is an algebraic integer. Thus, on appealing to Lemma 4, we deduce that the height of $\alpha_n^{(1)}$ is at most $(2DA')^{4nD}A^{2D/p}$. Since now $p > k^{n\eta}$, it is clear that all the hypotheses recorded at the beginning of § 3 are satisfied with $\alpha_n, \beta_1, \cdots, \beta_{n-1}$, and d replaced by $\alpha_n^{(1)}, \beta_1^{(1)}, \cdots, \beta_{n-1}^{(1)}$, and D respectively. We can therefore conclude, as before, that

$$\alpha_n^{(1)} = \alpha_1^{j_1'} \cdots \alpha_{n-1}^{j_{n-1}'} \alpha_n^{(2)p}$$

for some integers $j_1', \cdots, j_{n-1}'$ with $0 \leqq j_r' < p$ and some element $\alpha_n^{(2)}$ in K; and this gives an inequality

$$0 < |\beta_1^{(2)} \log \alpha_1 + \cdots + \beta_{n-1}^{(2)} \log \alpha_{n-1} - \log \alpha_n^{(2)}| < e^{-\delta H} ,$$

where $\beta_1^{(2)}, \cdots, \beta_{n-1}^{(2)}$ and $\alpha_n^{(2)}$ are elements of K with heights at most $H^{(\log H)^2 + 2D}$ and $(2DA')^{4nD(1+2D/p)}A^{(2D/p)^2}$ respectively.

On repeating this argument $s = [2/(n\eta)]$ times we obtain

$$0 < |\beta_1^{(s)} \log \alpha_1 + \cdots + \beta_{n-1}^{(s)} \log \alpha_{n-1} - \log \alpha_n^{(s)}| < e^{-\delta H} ,$$

where $\beta_1^{(s)}, \cdots, \beta_{n-1}^{(s)}$ and $\alpha_n^{(s)}$ are elements of K with heights at most $H^{(\log H)^2 + sD}$ and $(2DA')^{4nD\{1+(2D/p)+\cdots+(2D/p)^{s-1}\}}A^{(2D/p)^s}$ respectively. But clearly

$$p \geqq (1/2)H^{n\eta/(1+(1/2)\epsilon)} > 2D(\log A)^{n\eta}$$

and so $(p/2D)^s > \log A$. Hence the height of $\alpha_n^{(s)}$ is bounded above by $(4DA')^{8nD}$, independent of A. The impossibility of the inequality for H sufficiently large is now immediately apparent from the work of [1, III].

Alternatively we can argue that if we continue the above algorithm then, since there are only finitely many algebraic numbers with bounded degree and height, there exists a number c independent of A such that $\alpha_n^{(j)} = \alpha_n^{(j')}$ for distinct $j, j' \leqq c$. This gives

$$\alpha_n^{(j)q} = \alpha_1^{p_1} \cdots \alpha_{n-1}^{p_{n-1}} ,$$

where $p_1, \cdots, p_{n-1}, q$ denote rational integers with absolute values at most p^c; on substituting for $\log \alpha_n^{(j)}$ in the j^{th} linear form we obtain an inequality of the type (1) but with $\beta_n = 0$ and with the heights of $\beta_1, \cdots, \beta_{n-1}$ at most $H^{2(\log H)^2}$; after repeating the argument n times we derive an inequality involving only one logarithm and, by virtue of the hypothesis that the original linear form does not vanish, this is plainly untenable. The contradiction proves the Theorem.

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