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TRANSCENDENTAL & ALGEBRAIC NUMBERS

BY A. O. GELFOND

Translated from the First Russian Edition

LEO F. BORON

(continued on inside back cover)

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FOREWORD

IT WAS NOT until the twentieth century that the theory of transcendental numbers was formulated as a theory having its own special methods and a sufficient supply of solved problems. Isolated formulations of the problems of this theory existed long ago and the first of them, as far as we know, is due to Euler. The problem of approximating algebraic numbers by rational fractions or, more generally, by algebraic numbers may also be included in the theory of transcendental numbers, regardless of the fact that the study of approximations to algebraic numbers by rational fractions was stimulated by problems in the theory of Diophantine equations. The object of the present monograph is not only to point out the content of the modern theory of transcendental numbers and to discuss the fundamental methods of this theory, but also to give an idea of the historical course of development of its methods and of those connections which exist between this theory and other problems in number theory.

Since the proofs of the fundamental theorems in the theory of transcendental numbers are rather cumbersome and depend on a large number of auxiliary propositions, each such proof is prefaced by a brief discussion of its scheme, which, in our opinion, should facilitate the understanding of the basic ideas of the corresponding method.

The author's articles *Approximation of algebraic irrationalities and their logarithms* [11], *On the algebraic independence of transcendental numbers of certain classes* [15] are included in this monograph in their entirety, and use was made of the author's article *The approximation of algebraic numbers by algebraic numbers and the theory of transcendental numbers* [17].

Siegel's method is discussed in this monograph in the form given by Siegel in his book *Transcendental Numbers*, Princeton, 1949 [5].

Moscow A. O. GELFOND

TRANSCENDENTAL AND ALGEBRAIC NUMBERS

CHAPTER I

The Approximation of Algebraic Irrationalities

§1. Introduction

An *algebraic number* is a root of an algebraic equation with rational integral coefficients; in other words, it is any root of an equation of the form

$$a_0x^n + a_1x^{n-1} + \cdots + a_n = 0, \tag{1}$$

where all the numbers $a_0, a_1, \ldots, a_n$ are rational integers and $a_0 \neq 0$. A number which is not algebraic is said to be *transcendental.*

If equation (1) is irreducible, i.e. its left member is not the product of two polynomials with rational integral coefficients, then its degree will be the *degree of the algebraic number* α which satisfies it. A root of equation (1) in the case $a_0 = 1$ is called an *integral algebraic number* or an *algebraic integer.*

The reader can find the elementary arithmetic properties of algebraic numbers which are required for understanding the following material in any book on algebraic numbers, for example the books *Vorlesungen über die Theorie der algebraischen Zahlen* by Hecke [1] and *The Theory of Algebraic Numbers* by Pollard [1]. Here we shall be occupied only with the problem of the approximation of algebraic irrationalities and various applications of this theory.

All methods of proof of the transcendence of a number in either the explicit or implicit form depend on the fact that algebraic numbers cannot be very well approximated by rational fractions or, more generally, by algebraic numbers. Therefore, the approximation of algebraic numbers by algebraic numbers will be considered in this chapter. This problem, as will be shown, is closely related to the problem of solving algebraic and transcendental equations in integers, and to other problems in number theory. Analytic methods in transcendental number theory may be utilized, in turn, in integral solutions of equations, and in the sequel certainly in the solution of problems dealing with the approximation of algebraic irrationalities.

We note first of all that the existence of transcendental numbers may also be proved without knowledge of the nature of the approximation of algebraic numbers by algebraic numbers. In fact, since the coefficients of equation (1) can be rational integers only, there can be only a countable number of equations of type (1) with prescribed degree n. From this it follows that there exists only a countable set of algebraic numbers of degree n inasmuch as every equation of degree n has only n roots. Therefore, the set of all algebraic numbers is countable. But the set of all complex numbers (or real numbers) is not countable, from which it follows that the transcendental numbers form the major part of all complex and real numbers. Despite this fact, the proof of the transcendence of any concrete, prescribed numbers, for example π or $2^{\sqrt[3]{2}}$, is rather difficult.

The question of the arithmetic nature of an extensive class of numerical expressions was first formulated by Euler. In his book *Introductio in analysin infinitorum* [1], 1748, he makes the assertion that for rational base a the logarithm of any rational number b which is not a rational power of a cannot be an irrational number (in modern terminology, algebraic) and must be counted among the transcendentals. Besides this assertion, which was proved only recently, he also formulated other problems dealing directly with transcendental number theory. Almost a century after Euler, Liouville [2] was the first, in 1844, to give a necessary condition that a number be algebraic and, by the same token, a sufficient condition that the number be transcendental. He showed that if α is a real root of an irreducible equation of degree $\nu \geqslant 2$, and p, q are any rational integers, then the inequality

$$\left|\alpha-\frac{p}{q}\right| > \frac{c}{q^{\nu}}, \quad c > 0,$$

is satisfied, where the constant C does not depend on p and q.

The proof of this inequality is quite straightforward. Suppose α is a real root of an irreducible equation

$$f(x) = a_0x^{\nu} + \cdots + a_{\nu} = 0,$$

where all the a_i $(i = 0, 1, \ldots, \nu)$ are rational integers. Then, using the mean value theorem, we get

$$\left|f\left(\frac{p}{q}\right)\right| = \left|\alpha - \frac{p}{q}\right| |f'(\xi)| \geqslant \frac{1}{q^\nu}; \quad \xi = \alpha + \tau\left(\frac{p}{q} - \alpha\right),$$
$$|\tau| \leqslant 1,$$

from which the Liouville theorem follows directly. This criterion for the transcendence of a number permitted the first construction of examples of transcendental numbers. In fact, it follows from the Liouville transcendence criterion, for example, that the number

$$\eta = \sum_{n=1}^{\infty} \frac{1}{2^{n!}}$$

is transcendental.

Thus, Liouville established that algebraic numbers cannot be very well approximated by rational fractions. In connection with this fact, the problem arose of determining a constant $\vartheta = \vartheta(\nu)$ such that for an arbitrary algebraic number α of degree ν the inequality

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{\vartheta+\varepsilon}}, \tag{2}$$

where p, q are integers, will have only a finite number of solutions when $\varepsilon > 0$ and an infinite number of solutions when $\varepsilon < 0$. We remark that the numbers α for which inequality (2) has an infinite number of solutions for arbitrary ϑ are called *Liouville numbers.* Thue [1] was the first, at the beginning of the present century, to be able to decrease the magnitude of this constant. He showed that $\vartheta \leqslant \nu/2 + 1$. To prove this proposition, Thue constructed a polynomial in two variables x and y with rational integral coefficients having the form

$$f(x, y) = (y-\alpha)f_1(x, y) + (x-\alpha)^m f_2(x, \alpha), \tag{3}$$

where $f_1(x, y)$ and $f_2(x, \alpha)$ are polynomials.

Assuming that inequality (2) has two solutions p_1/q_1 and p_2/q_2, with sufficiently large denominators q_1 and q_2, then setting $m \approx \dfrac{\ln q_2}{\ln q_1}$ in relation (3) and proving that the left member of (3) does not vanish for a suitable choice of $f(x, y)$ when $x = p_1/q_1$ and $y = p_2/q_2$, he obtained his assertion in a manner analogous to the way

Liouville's theorem was proved. This method, which enabled one to essentially decrease the Liouville constant, is inseparably related to the assumption that there exist two sufficiently large solutions of inequality (2). Therefore, this method enables one to establish only a bound for the number of solutions of inequality (2) and not for the magnitude of their denominators.

In fact, it follows from Thue's line of reasoning that if inequality (2) has a sufficiently large number of solutions for $\vartheta = \nu/2+1$ and $\varepsilon > 0$ with denominators $q_1 > q_1'(\alpha, \varepsilon)$, then there are no solutions with denominators $q_2 \geqslant q_2'(\alpha, \varepsilon, q_1)$. This at once enables one to establish, in particular, the finiteness of the number of solutions of the equation

$$(4) \qquad y^n f\left(\frac{x}{y}\right) = c_0 y^n + c_1 y^{n-1} x + \cdots + c_n x^n = c, \quad n \geqslant 3$$

in integers x and y if the coefficients $c, c_0, c_1, \ldots, c_n$ are rational integrals.

In fact, equation (4) implies the relations

$$(5) \qquad f\left(\frac{x}{y}\right) = \left(\alpha - \frac{x}{y}\right) f'(\xi) = \frac{c}{y^n},$$

$$\xi = \alpha + \tau\left(\frac{x}{y} - \alpha\right), \quad 0 \leqslant \tau \leqslant 1,$$

from which, under the condition that the polynomial $f(t)$ is irreducible in the rational field, it follows immediately that we have a contradiction with inequality (2) when $\vartheta + \varepsilon < n$, provided only that we assume the existence of an infinite number of solutions of equation (4).

This method was generalized and made precise by Siegel [1] who showed, using, as did Thue, the existence of two sufficiently large solutions, that the inequality

$$(6) \qquad \vartheta \leqslant \min_{1 \leqslant s \leqslant \nu - 1} \left[\frac{\nu}{s+1} + s\right] < 2\sqrt{\nu}$$

holds. Not only did Siegel make Thue's method more precise; he generalized it to the case of the approximation of an algebraic number α by another algebraic number ζ of height H and degree n. The height of an algebraic number ζ is the maximum of the absolute values of the coefficients of that equation, irreducible in the rational field, which is satisfied by ζ where all the coefficients of this equation are integers and their greatest common divisor equals 1. He showed that the inequality

$$(7) \qquad |\alpha-\zeta| < H^{-n(\vartheta+\varepsilon)}, \quad \vartheta = \min_{1\leqslant s\leqslant \nu-1} \left[\frac{\nu}{s+1}+s\right], \quad \varepsilon > 0$$

has only a finite number of solutions in algebraic numbers ζ if α is an algebraic number of degree ν.

Furthermore, he also gave other variants of inequality (7). Further attempts by Siegel [2] and his students to decrease the magnitude of the constant ϑ in inequalities (2) and (7), assuming the existence not only of two, but of an arbitrary number of sufficiently large solutions of inequalities (2) and (7), led Siegel to a theorem which was sharpened by his student Schneider [1] and which in the sharpened form reads as follows: If $q_1, q_2, \ldots, q_n, \ldots$ are the denominators of all sequences of solutions of inequality (2) for $\vartheta=2$ and $\varepsilon>0$, then either $\overline{\lim\limits_{n\to\infty}} \dfrac{\ln q_{n+1}}{\ln q_n} = \infty$ or $n<n_0$. This so-called Siegel-Schneider theorem, as we see, not only does not make it possible to establish a bound for the magnitudes of the denominators of the solutions of inequality (2) for $2<\vartheta<\vartheta_0$, $\vartheta_0 = \min\limits_{1\leqslant s\leqslant \nu} \left[\dfrac{\nu}{s+1}+s\right]$, but it also does not even assert their finiteness.

The last theorem given above generalizes naturally to the case of inequality (7). From the first generalization of the Thue theorem, based on the consideration of two sufficiently large solutions, it follows, in particular, that the equation

$$(8) \qquad c_0y^n + c_1y^{n-1}x + \cdots + c_nx^n = P_m(x, y), \quad n \geqslant 3,$$

for rational integers $c_0, c_1, \ldots, c_n$ and $P_m(x, y)$ a polynomial with rational integral coefficients of degree m, has only a finite number of solutions in rational integers x and y when

$$n-m > \min_{1\leqslant s\leqslant n-1} \left[\frac{n}{s+1}+s\right]$$

and the left member of the equation is irreducible. From the Siegel-Schneider theorem it only follows that for $n>m+2$ the integral solutions of equation (8) are very rare. We note that the question whether the number of solutions of equation (8) with $n\geqslant m+1$ is finite or infinite is answered completely by another means.

Further generalizations of the Siegel-Schneider theorem and its applications can be found in the works of Mahler [2–5]. One ought also to note that some results in the area of approximations of algebraic irrationalities were obtained by Morduhai-Boltovskoi [1, 4–6], Kuzmin [2], Gelfond [10, 11], and other authors.

Results, analogous to the Thue-Siegel theorem, dealing with the problem of the simultaneous approximation of several algebraic numbers by rational fractions with the same denominators were obtained by Hasse [1].

The most interesting direct application of theorems of Siegel-Schneider type in the theory of transcendental numbers is the following. Suppose $p(x)$ is an integral polynomial which is positive for $x \geqslant 1$. We write down its values for $x=1, 2, 3, \ldots$ in the number system with radix q. We write the infinite q-nary fraction as

$$\eta = 0.q_1 q_2 \ldots q_{\nu_1} \ldots q_{\nu_2} \ldots,$$

where $q_1, q_2, \ldots, q_{\nu_1}$ are the "digits" in the q-nary expansion of $p(1)$, $q_{\nu_1+1}, \ldots, q_{\nu_2}$ are the "digits" in the q-nary expansion of $p(2)$, and so on. Then the number η will be transcendental but it is not a Liouville number. In particular, for $p(x)=x$ and $q=10$, it will be the transcendental number

$$\eta = 0.123456789101112\ldots.$$

This theorem was proved by Mahler [5] with the aid of the theorem on the approximation of algebraic irrationalities by rational fractions, which was a sharpening of Schneider's theorem for the case when the numerators and denominators of the approximating fractions are of a special sort. It also follows from this theorem that the numbers

$$\eta = \sum_0^\infty \frac{a_k}{a^{\lambda_k}}, \quad \lambda_{k+1} > (1+\varepsilon)\lambda_k + \frac{\ln a_{k+1}}{\ln a}, \quad \varepsilon > 0,$$

where $a > 1$, $a_0, a_1, \ldots, \lambda_1, \lambda_2, \ldots$ are positive rational integers, are transcendental. In particular, this assertion holds for the number

$$\eta = \sum_0^\infty \frac{1}{a^{[2^{\alpha n}]}}, \quad \alpha > 0.$$

In connection with the status of the problem of the approximation of algebraic irrationalities which was briefly discussed above, the first question that naturally arises is whether it is possible to decrease the magnitude of the constant ϑ in comparison with the quantity obtained by Siegel using only two solutions of inequality (2). Further, taking into consideration the noneffectiveness of the results, obtained by Thue's method, noneffectiveness in the sense that it is impossible to establish by this method the bounds of the magnitudes of the denominators of the solutions of inequality (2) for $\vartheta < \nu$, the problem how the theorem on the approximation of algebraic numbers which would be a limiting case in the sense of effectiveness, using two solutions of inequality (2), should be worded, also arises naturally. In this formulation of the problem, one must speak of only two solutions inasmuch as by using a larger number of solutions one encounters difficulties which have not been eliminated up to the present time and which are related to the general theory of elimination.

We shall now formulate the theorem, which will give the answer to the above question, by introducing, in anticipation, the concept of *measure of an algebraic number*. Suppose ζ is a number in an algebraic field K of degree σ, and let the numbers $\omega_1, \omega_2, \ldots, \omega_\sigma$ be a basis for the ring of integers in this field. The number ζ we have taken can be represented in an infinite number of ways in the form

$$(9) \quad \zeta = \frac{p_1\omega_1 + \cdots + p_\sigma\omega_\sigma}{q_1\omega_1 + \cdots + q_\sigma\omega_\sigma}, \quad q[p_1, \ldots, q_\sigma] = \max\,[|p_1|, \ldots, |q_\sigma|],$$

where $p_1, p_2, \ldots, p_\sigma, q_1, \ldots, q_\sigma$ are rational integers with greatest common divisor 1. We shall call the number q the measure of the number ζ if it is defined by the relation

$$(10) \qquad q = \min q[p_1, \ldots, p_\sigma, q_1, \ldots, q_\sigma]$$

where the minimum in the right member is taken over all possible representations of the number ζ. It is not difficult to note that when $\zeta = p/q$ is a rational number then its measure equals $\max\,[|p|, |q|]$, i.e. to within a nonessential constant factor, it coincides with its denominator q if ζ is an element of the sequence of fractions which converges to the number $\alpha \neq 0$, 1 as q increases. We can now formulate our general theorem, which we shall call Theorem I in the sequel. Suppose α and β are two arbitrary numbers in the algebraic field K_0 of degree ν (where the case $\alpha=\beta$ is not excluded). Suppose, further, that ζ and ζ_1 are numbers in an algebraic field K, whose measures with respect to a fixed integral basis $\omega_1, \omega_2, \ldots, \omega_\sigma$ of this field are q and q_1, respectively, and that ϑ and ϑ_1 are two real numbers subjected to the conditions $\vartheta \leqslant \vartheta_1 \leqslant \nu$, $\vartheta\vartheta_1 = 2\nu(1+\varepsilon)$, where $\varepsilon>0$ is an arbitrarily small, fixed number. Then, if the inequality

$$(11) \qquad |\alpha-\zeta| < q^{-\sigma\vartheta}$$

has the solution ζ with measure $q>q'[K_0, K, \alpha, \beta, \varepsilon, \delta]$, the inequality

$$(12) \qquad |\beta-\zeta_1| < q_1^{-\sigma\vartheta_1}$$

cannot have solutions with measure q_1 under the condition that

$$(13) \qquad \ln q_1 \geqslant \left[\frac{\vartheta-1}{2(\sqrt{1+\varepsilon}-1)}+\delta\right] \ln q,$$

where δ is any arbitrarily small positive constant. [The special case of this theorem, when $\alpha=\beta$, ζ a rational fraction, $\vartheta=\vartheta_1$ and without inequality (13), was proved independently by Dyson.]

The $\wp$-adic analogue of the above theorem is formulated in a similar manner. It also follows from the above theorem, setting $\vartheta=\vartheta_1$ in it, that inequality (2) has only a finite number of solutions for $\varepsilon>0$, when $\vartheta=\sqrt{2\nu}$. That our general theorem is the best possible from the point of view of effectiveness can be directly established in the case when ζ and ζ_1 are rational fractions and $\alpha=\beta$. In fact, if one could replace $\varepsilon>0$ by $-\varepsilon<0$ in the condition $\vartheta\vartheta_1=2\nu(1+\varepsilon)$ of the theorem, then it would have the form $\vartheta\vartheta_1=2\nu(1-\varepsilon)$ and we could have set $\vartheta=2\sqrt{1-\varepsilon}<2$ and $\vartheta_1=\nu\sqrt{1-\varepsilon}<\nu$. But inequality (11) would indeed have an infinite number of solutions for ζ rational, $\sigma=1$ and $\vartheta<2$, which means that for solutions of inequality (12) with rational denominators, we should find an effective bound in the form of a function of K_0, α, ε. It would already follow directly from this that there exists an effective bound for the magnitudes of the solutions of equation (4). Finally, one can say that our general theorem retains its validity if the measures of the numbers ζ are replaced by the heights of the numbers ζ.

The proof of this theorem is based on a somewhat stronger form of Thue's theorem. Using our general Theorem I, with the aid of some additional considerations, one can prove Theorem II: Suppose $\alpha, \zeta_1, \zeta_2, \ldots, \zeta_s$ are algebraic numbers in the field K.

Suppose also that the product of any integral powers of the numbers $\zeta_1, \zeta_2, \ldots, \zeta_s$ cannot be equal to 1. Then the inequality

$$(14) \qquad |\alpha - \zeta_1^{x_1}\zeta_2^{x_2} \ldots \zeta_s^{x_s}| < e^{-\varepsilon x}, \quad x = \max_{1 \leqslant i \leqslant s} |x_i|$$

and the congruence

$$(15) \qquad \alpha \equiv \zeta_1^{y_1}\zeta_2^{y_2} \ldots \zeta_s^{y_s} \bmod \wp^m, \quad m = [\delta y], y = \max_{1 \leqslant i \leqslant s} |y_i|$$

can have only a finite number of solutions in rational integers $x_1, x_2, \ldots, x_s$ and $y_1, y_2, \ldots, y_s$ provided the numbers $\varepsilon > 0$ and $\delta > 0$ are small; $\wp$ is a prime ideal in the field K. We now state two corollaries to Theorems I and II. We first of all introduce an application of Theorem I to the theory of algebraic equations. Suppose the system of homogeneous forms $P_1(x, y), P_2(x, y), \ldots, P_n(x, y)$ possess the following properties: their degrees are greater than one, all the coefficients of the polynomials $P_1(x, y), \ldots, P_n(x, y)$ are rational integers, these polynomials do not have linear divisors in the rational field, every real zero of the polynomial $t^{-m_k}P_k(t, tx) = R_k(x)$ belongs to the algebraic field K of degree not greater than ν and all such zeros are distinct. We shall also say that the degree of the polynomial $P(x_1, y_1, \ldots, x_n, y_n)$ in $2n$ variables, having rational integral coefficients, is the set of numbers $(s_1, s_2, \ldots, s_n)$, where s_i is the degree of the polynomial P in the variables x_i, y_i. Then the following theorem holds: The equation

$$(16) \qquad P_1(x_1, y_1) \ldots P_n(x_n, y_n) = P(x_1, y_1, x_2, y_2, \ldots, x_n, y_n)$$

has only a finite number of solutions in rational integers $x_1, y_1, x_2, y_2, \ldots, x_n, y_n$, provided the inequalities $m_k - s_k > \sqrt{2\nu}$, $k = 1, 2, \ldots, n$; $\nu \geqslant 3$, are satisfied simultaneously.

One may also obtain a number of corollaries to Theorem II, but these are already for exponential functions. Suppose, for example, that the numbers $\zeta_1, \ldots, \zeta_n, \psi_1, \ldots, \psi_m, \eta_1, \ldots, \eta_p$ are integers in the field K; none is an algebraic unit; A, B, C, $ABC \neq 0$, are numbers in the same field K; and the numbers

$$\zeta = \zeta_1 \ldots \zeta_n, \quad \psi = \psi_1 \ldots \psi_m, \quad \eta = \eta_1 \ldots \eta_p$$

are relatively prime. Then the equation

$$(17) \qquad A\zeta_1^{x_1} \ldots \zeta_n^{x_n} + B\psi_1^{y_1} \ldots \psi_m^{y_m} + C\eta_1^{z_1} \ldots \eta_p^{z_p} = 0$$

can have only a finite number of nonnegative solutions in rational integers $x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_n$. Further, almost directly from Theorem II, one can obtain a theorem on the behavior of linear forms with rational integral coefficients in the logarithms of algebraic numbers.

If the algebraic numbers $\alpha_1, \alpha_2, \ldots, \alpha_s$ have the property that the product of any integral powers of these numbers is not equal to unity, then the inequality

$$(18) \qquad |x_1 \ln \alpha_1 + \cdots + x_s \ln \alpha_s| < e^{-\varepsilon x}, \quad \varepsilon > 0, \quad x = \max_{1 \leqslant i \leqslant s} |x_i|,$$

where the logarithms have arbitrary fixed values, has only a finite number of solutions in rational integers $x_1, x_2, \ldots, x_s$.

As was shown by Linnik [1], the latter theorem, for $s=3$, enables one to carry out a new proof of the finiteness of the number of quadratic fields of class number one which proof does not depend on the analytic theory of $L(s, \chi)$ series. If it is possible to establish the least upper bound of the magnitudes of the solutions of inequality (1) for arbitrary fixed $\varepsilon > 0$, then the solution of the classic problem on effectivization in quadratic fields would follow directly from this new proof, in other words, the possibility of expressing the bounds of the discriminants of fields of class number one in explicit form. This establishes a profound connection between the noneffectiveness of number theory problems which appear different at first glance. The finiteness of the number of solutions of inequality (18) also enables one to draw conclusions concerning the behavior of the solutions of algebraic equations of higher degree in many variables. In fact, for example, the equation in units of the field K under the condition $x_k = 0$ follows from the equation $N(x_1\omega_1 + \cdots + x_s\omega_s) = 1$, where the numbers $\omega_1, \omega_2, \ldots, \omega_s$ form a basis for the ring of integers. This equation will have the form (17) with a large number of terms. All theorems introduced in this section are of a noneffective nature; in other words, because two solutions of the corresponding approximating inequalities have been used, they are fundamentally not in a position to give the magnitudes of the bounds of the solutions of inequalities or equalities, and they yield only bounds for the number of solutions.

In the following sections of this chapter we shall discuss the proofs of the theorems formulated above. It appears at the present time it is only with the aid of analytic methods of transcendental number theory that one can expect the effective solution of the approximating inequalities for algebraic numbers, and by the same token the effective solution of the problems of the theory of Diophantine equations and fields.

§2. Auxiliary lemmas

In this section we shall prove several lemmas which will be needed in the sequel.

LEMMA I. *Suppose* $L_1, L_2, \ldots, L_m$ *are linear forms with real coefficients* $a_{i,k}$; $i=1, 2, \ldots, m$; $k=1, \ldots, n (n>m)$ *in the* n *variables* $x_1, x_2, \ldots, x_n$; *in other words,*

$$L_i = a_{i,1}x_1 + \cdots + a_{i,n}x_n, \quad \max_{i,k} |a_{i,k}| \leqslant a,$$

where a *is an integer.*

Then rational integers $x_1, x_2, \ldots, x_n$, $|x_k| \leqslant x$, *all different from zero, can be found such that*

$$(19) \qquad |L_i| \leqslant \frac{1}{N}, \quad i = 1, 2, \ldots, m; \quad x < 2(naN)^{\frac{m}{n-m}}$$

for N *integral and* $N>1$.

Proof. We note that $|L_i| \leqslant nax$, $i=1, 2, \ldots, m$. In m-dimensional space with coordinates $L_1, L_2, \ldots, L_m$ we shall have $(2x+1)^n$ points, if all the $x_1, x_2, \ldots, x_n$ are given integral values, independently of one another, and which are in absolute value no greater than x. All these points will lie in a cube with side $2nax$. We subdivide each edge of this cube into $2Nanx$ equal parts. The length of each part will be N^{-1}. The whole cube will then decompose into $[2anxN]^m$ subcubes. Setting $x=[(naN)^{\frac{m}{n-m}}]$, we can verify directly that the number of points $L_1, \ldots, L_m$, which we have constructed, is greater than the number of subcubes with side $1/N$. This means that at least one of the subcubes contains two of the points, $(L_1', L_2', \ldots, L_m')$ and $(L_1'', L_2'', \ldots, L_m'')$, the difference between whose coordinates is not greater than N^{-1}. But then

$$L_i' - L_i'' = \sum_1^n a_{i,k}(x_k' - x_k''), \quad i = 1, 2, \ldots, m,$$

$$|x_k' - x_k''| \leqslant 2(anN)^{\frac{m}{n-m}},$$

which completes the proof of our lemma.

As before, we shall call the *height of the algebraic number* ζ the maximum of the absolute values of the coefficients of that irreducible equation with rational integral coefficients having greatest common divisor equal to 1, which is satisfied by ζ; in other words, if ζ is a root of the irreducible equation

$$H_0+H_1\zeta+\cdots+H_h\zeta^h = 0, \quad (H_0, H_1, \ldots, H_h) = 1;$$
$$H = \max_{0\leqslant i\leqslant h} |H_i|,$$

where all the $H_0, H_1, \ldots, H_h$ are rational integers, then H is the height of ζ.

LEMMA II. *If $\alpha_1, \ldots, \alpha_s$ are algebraic numbers in the field K of degree ν, whose measures with respect to a fixed basis for the ring of integers in the field K are $q_1, q_2, \ldots, q_s$, and $P(x_1, x_2, \ldots, x_s)$ is a polynomial of degree n_i in x_i, $1\leqslant i\leqslant s$, $n=\sum_1^s n_i$, then either*

$$P(\alpha_1, \ldots, \alpha_s) = 0$$

or

$$|P(\alpha_1, \ldots, \alpha_s)| > H^{-\nu+1}\prod_{i=1}^{s} q_i^{-\nu n_i}e^{-\gamma n}, \tag{20}$$

where H is the height of $P(x_1, x_2, \ldots, x_s)$ and γ depends only on the field K and a fixed basis for the ring of integers in K. (The *height of a polynomial* is the maximum absolute value of its coefficients.)

Proof. Since, by the definition of the measure of an algebraic number, the number

$$\alpha\sum_{n=1}^{\nu} q_{n,i}\omega_n = \sum_{n=1}^{\nu} p_{n,i}\omega_i, \quad |p_{n,i}| \leqslant q_i, \quad |q_{n,i}| \leqslant q_i,$$

where the $p_{n,i}$ and $q_{n,i}$ are integers, and $\omega_1, \ldots, \omega_\nu$ is the chosen basis for the ring of integers in K, will be an algebraic integer, the product

$$C_1 = P(\alpha_1, \ldots, \alpha_n)\prod_{i=1}^{s}\left(\sum_{n=1}^{\nu} q_{n,i}\omega_n\right)^{n_i}$$

will be a nonzero algebraic integer in the field K having norm greater than or equal to unity. Therefore, we obtain the inequality

$$\left|C_1\prod_2^{\nu} C_k\right| \geqslant 1, \quad |C_k| < H\prod_{i=1}^{s} q_i^{n_i}e^{\gamma_0 n}, \quad 2\leqslant k\leqslant \nu,$$

where C_k, $k=2, 3, \ldots, \nu$, are the numbers conjugate to C_1 and γ_0 depends only on K. Now inequality (20) follows directly from these inequalities.

LEMMA III. *Suppose H and q are the height and measure respectively of the algebraic number ζ in the field K in which $\omega_1, \omega_2, \ldots, \omega_s$ is a basis for the ring of integers. The measure of an algebraic number is defined by relation* (10), §1. *Then the relation*

$$H > cq^{1/2} \tag{21}$$

holds between the height and measure, where c does not depend on q and H, but depends only on $\omega_1, \omega_2, \ldots, \omega_s$.

Proof. In fact, we shall have, first of all, the following inequality for the number $\zeta=\zeta_o$ and all its conjugates ζ_k, $k=1, 2, \ldots, s-1$, independently of the degree of the irreducible equation satisfied by ζ,

$$|\zeta_k|^s \leqslant H(|\zeta_k|^{s-1}+ \cdots +1), \quad k = 0, 1, \ldots, s-1,$$

from which it follows directly that

$$|\zeta_k| \leqslant H+1. \tag{22}$$

On the other hand, according to the definition of the measure q of the number

$$\zeta_k = \frac{p_1\omega_1{}^{(k)}+ \cdots +p_s\omega_s{}^{(k)}}{q_0}, \quad k = 0, 1, \ldots, s-1, \tag{23}$$

$$\max\,[|p_1|, \ldots, |p_s|]=p, \; p+q_0>q, \quad (p_1, p_2, \ldots, p_s, q_0)=1,$$

where $\omega_1{}^{(k)}, \ldots, \omega_s{}^{(k)}$ is a basis of the field K_k, conjugate to K. Eliminating the ratios p_i/q_0 from relations (23) and using inequality (22), we see that

$$H > c_0\frac{p}{q_0}, \tag{24}$$

where c_0 depends only on the bases of the fields $K, K_1, \ldots, K_{s-1}$. By the property of a basis of a field K and in virtue of the absence of a common divisor for the integers $p_1, p_2, \ldots, p_s$ and q_0 we may assert that the coefficient of the highest degree term in the irreducible equation which ζ satisfies must be divisible by q_0. It follows that $H \geqslant q_0$. Comparing the inequalities

$$p+q_0 \geqslant q, \quad H \geqslant q_0 > 0, \quad H > c_0\frac{p}{q_0},$$

we immediately obtain the fundamental inequality of our lemma.

Suppose $f(x)$ is any polynomial of degree n. We define its height $H(f)$ by the relation

$$f(x) = \sum_{k=0}^{n} A_kx^k, \quad H(f) = \max_{0\leqslant k\leqslant n} |A_k|. \tag{25}$$

LEMMA IV. *Suppose $f_1(x)$ and $f_2(x)$ are polynomials of degree n_1 and n_2 and heights H_1 respectively H_2. Then*

$$H(f_1{}^m) \geqslant \frac{1}{2mn} H_1{}^m \tag{26}$$

and

$$H(f_1 f_2) \geqslant 2^{-3n} H_1 H_2, \quad n = n_1 + n_2. \tag{27}$$

Proof. We shall first prove inequality (26). The inequalities

$$\begin{cases} \max\limits_{0 \leqslant \varphi \leqslant 1} |f(e^{2\pi i \varphi})|^2 \geqslant \displaystyle\int_0^1 |f(e^{2\pi i \varphi})|^2 \, d\varphi, \\ (n+1) H^2(f) \geqslant \displaystyle\int_0^1 |f(e^{2\pi i \varphi})|^2 \, d\varphi \geqslant H^2(f) \end{cases} \tag{28}$$

are true for every polynomial of degree n and height H. Besides these obvious inequalities, we have

$$\begin{aligned} |f(e^{2\pi i \varphi})|^2 &= \left| \sum_{k=-n}^{n} \int_0^1 |f(e^{2\pi i \alpha})|^2 e^{2\pi i (\varphi - \alpha) k} \, d\alpha \right| \\ &\leqslant (2n+1) \int_0^1 |f(e^{2\pi i \alpha})|^2 \, d\alpha \end{aligned} \tag{29}$$

which follows directly from the expansion of f in a Fourier series.

From inequalities (28) and (29) we successively obtain

$$\begin{aligned} H^2(f_1{}^m) &\geqslant \frac{1}{mn_1 + 1} \int_0^1 |f^m(e^{2\pi i \varphi})|^2 \, d\varphi \geqslant \frac{\max |f^m(e^{2\pi i \varphi})|^2}{(mn_1+1)(2mn_1+1)} \\ &\geqslant \frac{1}{4n_1{}^2 m^2} \left[\int_0^1 |f(e^{2\pi i \varphi})|^2 \, d\varphi \right]^m \geqslant \frac{1}{4n_1{}^2 m^2} H_1{}^{2m}, \end{aligned}$$

from which inequality (26) of our lemma follows. In order to prove inequality (27), we shall first prove the inequality

$$M = \max_{|z|=1} |f_1(z) f_2(z)| > 2^{-2n} \max_{|z|=1} |f_1(z)| \max_{|z|=1} |f_2(z)|, \tag{30}$$

where $n = n_1 + n_2$ and n_1, n_2 are the degrees of the polynomials $f_1(z), f_2(z)$. We may assume, without loss of generality, that

$$\max_{|z|=1} |f_1(z)| = \max_{|z|=1} |f_2(z)| = 1. \tag{31}$$

Assume the inequality $|f_1(z) f_2(z)| \leqslant 2^{-2n}$ holds. Then at least one of the two inequalities

$$|f_1(e^{2\pi i \frac{k}{n+1}})| \leqslant 2^{-n}, \quad |f_2(e^{2\pi i \frac{k}{n+1}})| < 2^{-n} \tag{32}$$

must hold for every $k=0, 1, 2, \ldots, n$. Since k assumes n_1+n_2+1 values, it is obvious that either the first inequality is true for n_1+1 values of k or the second for n_2+1 values of k. Suppose then, for example, that the first inequality is true for n_1+1 values of k. We set

$$\alpha_\nu = e^{2\pi i \frac{k_\nu}{n+1}}, \ \nu = 0, 1, \ldots, n_1; \quad \alpha_\mu' = e^{2\pi i \frac{k_\mu'}{n+1}}, \ \mu = 1, 2, \ldots, n_2,$$

where k_ν are those values of k for which our inequality is satisfied, and the k_μ' are all the remaining values of k. Then, by the Lagrange interpolation formula we have

$$(33) \quad f_1(z) = \sum_{\nu=0}^{m} f(\alpha_\nu) \frac{(z-\alpha_0)(z-\alpha_1) \ldots (z-\alpha_{n_1})}{(\alpha_\nu-\alpha_0)(\alpha_\nu-\alpha_1) \ldots (\alpha_\nu-\alpha_{n_1})(z-\alpha_\nu)}.$$

Estimating directly the right member of (33), multiplying the numerator and denominator of the ν-th term of the right member by $(\alpha_\nu-\alpha_1') \ldots (\alpha_\nu-\alpha_{n_2}')$, we obtain that

$$|f_1(z)| \leqslant (n_1+1)2^{-n}\cdot 2^{n_1+n_2}(n+1)^{-1}$$

for $|z|=1$, which contradicts condition (31). We have thus proved inequality (30).

Further, from inequalities (28), (29), and (30) we successively obtain

$$\begin{aligned} H^2(f_1,f_2) &\geqslant \frac{1}{n+1}\int_0^1 |f_1(e^{2\pi i\varphi})|^2 |f_2(e^{2\pi i\varphi})|^2 \, d\varphi \\ &\geqslant \frac{1}{(n+1)(2n+1)} \max_{|z|=1} |f_1(z)f_2(z)|^2 \\ &> \frac{2^{-4n}}{(n+1)(2n+1)} \max_{|z|=1} |f_1(z)|^2 \max_{|z|=1} |f_2(z)|^2 \\ &\geqslant 2^{-6n}\int_0^1 |f_1(e^{2\pi i\varphi})|^2 \, d\varphi \int_0^1 |f_2(e^{2\pi i\varphi})|^2 \, d\varphi \geqslant 2^{-6n}H_1^2H_2^2, \end{aligned}$$

from which the assertion of our theorem follows.

Lemma V. *Suppose α and β are two arbitrary numbers in an algebraic field of degree ν. Suppose also that n and $\mu \geqslant n$ are arbitrary positive integers and that η, $0<\eta\leqslant\nu/2$ is an arbitrary real number. Then we can* *construct a polynomial in x and y with nonzero rational integral coefficients $C_{k_1,k}$ not all equal to zero,*

$$(34)\qquad P(x, y) = \sum_{k=0}^{\tau} \sum_{k_1=0}^{\tau_1} C_{k_1,k} x^{k_1} y^k, \quad \max |C_{k_1,k}| < e^{n^{3/4}\mu},$$

where we have set

$$(35)\qquad \tau = \left[(n+\sqrt{n})\sqrt{\frac{\eta\nu}{2}}\right] - 1, \quad \tau_1 = \left[\frac{\nu(n+\sqrt{n})}{2(\tau+1)}\mu\right],$$

which is subject to the conditions

$$(36)\qquad \frac{\partial^{k_1+k}}{\partial x^{k_1}\partial y^k} P(\alpha, \beta) = 0, \quad \frac{k_1}{\mu} + \frac{k}{n} < 1$$

for $n \geqslant n_0(\alpha, \beta)$.

Proof. Suppose $P(x, y)$ is a polynomial with integral coefficients of degree τ and τ_1 in x respectively y. The number of its coefficients is then equal to $(\tau+1)(\tau_1+1)$ and

$$(37)\qquad (\tau_1+1)(\tau+1) \geqslant \frac{\nu}{2}(n+\sqrt{n})\mu.$$

A simple computation shows that the number of solutions of the inequality in condition (36) is not greater than the quantity

$$(\mu+1)\frac{n+1}{2}.$$

If d is a rational integer such that $d\alpha$ and $d\beta$ are algebraic integers, then

$$\frac{d^{\tau_1+\tau}}{k_1!k!} \frac{\partial^{k_1+k}}{\partial x^{k_1}\partial y^k} P(\alpha, \beta) = \sum_{s=1}^{\nu} B_{k_1,k,s}\omega_s,$$

where $\omega_1, \omega_2, \ldots, \omega_r$ is a basis for the ring of integers of the field to which α and β belong, and the $B_{k_1,k,s}$ are linear forms, in the coefficients $C_{k_1,k}$, with rational integral coefficients. The coefficients of these linear forms $B_{k_1,k,s}$, as can easily be seen, taking the quantities τ_1 and τ into consideration, are not greater than the quantity

$$e^{\gamma\mu}, \quad \gamma = \gamma(\alpha, \beta) > 0.$$

It is always possible, according to Lemma I, to find rational integral values not all equal to zero for the $C_{k_1,k}$, for which the inequalities

$$|B_{k_1,k,s}| \leqslant \tfrac{1}{2}; \quad \frac{k_1}{\mu} + \frac{k}{n} < 1; \quad 1 \leqslant s \leqslant \nu,$$

$$\max |C_{k_1,k}| < 2(2re^{\gamma\mu})^{\frac{m}{r-m}},$$

are satisfied, where $r=(\tau+1)(\tau_1+1)$ is the number of variables and m is the number of inequalities.

It is not difficult to see that

$$m \leqslant \nu(\mu+1)\,\frac{n+1}{2}$$

and

$$\frac{\nu}{2}\,(n+\sqrt{n})\mu \;<\; r \;<\; \nu n\mu.$$

It follows that

$$\frac{m}{r-m} < \frac{2\nu n\mu}{\nu(\sqrt{n}-1)\mu} \leqslant 4\sqrt{n}$$

with $n \geqslant 2$, and hence we obtain inequality (34) of our lemma.

We have thus completed the proof of the theorem.

LEMMA VI. *Suppose $P(x, y)$ is a polynomial constructed under the conditions of Lemma V, where we have set $\mu = \left[\eta n\,\frac{\ln q_1}{\ln q}\right]$ and ζ_1 and ζ are two numbers in the algebraic field K with integral basis $\omega_1, \omega_2, \ldots, \omega_s$, whose measures with respect to this basis are q_1 respectively q.*

We shall also assume that the inequalities

$$\ln q \geqslant n^3, \quad \ln q_1 \geqslant \ln q, \quad n \geqslant n_1 \geqslant n_0, \tag{38}$$

are satisfied, where n_1 depends only on n_0 and on the basis of the field K. Then there exists an integer λ, $0 \leqslant \lambda \leqslant \left[\frac{\ln q_1}{\ln q}\right]+\tau+1$, such that

$$\frac{\partial^\lambda}{\partial x^\lambda}\,P(\zeta, \zeta_1) \neq 0. \tag{39}$$

Proof. We shall agree that in the proof of this lemma $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ will denote quantities which do not depend on n, q_1, q. The polynomial $P(x, y)$ can be represented, first of all, in the form

$$P(x, y) = \sum_{k=0}^{\tau} \bar{P}_k(x) y^k, \quad \bar{P}_k(x) = \sum_{k_1=0}^{\tau_1} C_{k_1,k} x^{k_1}. \tag{40}$$

We choose r linearly independent polynomials $\bar{P}_k(x)$ in terms of which the remaining ones can be expressed linearly. Then we shall have the representation

$$P(x, y) = \sum_{k=1}^{r} P_k(x) Q_k(y), \quad 1 \leqslant r \leqslant \tau+1. \tag{41}$$

All the $Q_k(y)$ are also linearly independent in this representation because that power of y necessarily occurs in $Q_k(y)$ which had $\bar{P}_k(x)$ as its coefficient in representation (40) and this power of y does not appear in any other polynomial $Q_i(y)$ except $Q_k(y)$. The coefficients of the polynomials $P_k(x)$ and $Q_k(y)$ are rational numbers.

We now consider the following two polynomials $\Delta(x)$ and $D(y)$,

$$(42)\quad \begin{cases} \Delta(x) = \dfrac{1}{1!2!\dots(r-1)!}\begin{vmatrix} P_1(x) & & P_r(x) \\ P_1'(x) & \dots & P_r'(x) \\ \dots & \dots & \dots \\ P_1^{(r-1)}(x) & \dots & P_r^{(r-1)}(x) \end{vmatrix}, \\ \\ D(y) = \dfrac{1}{1!2!\dots(r-1)!}\begin{vmatrix} Q_1(y) & & Q_r(y) \\ Q_1'(y) & \dots & Q_r'(y) \\ \dots & \dots & \dots \\ Q_1^{(r-1)}(y) & \dots & Q_r^{(r-1)}(y) \end{vmatrix}. \end{cases}$$

The coefficients of these polynomials are rational numbers and therefore we can set

$$(43)\quad \begin{cases} \Delta(x) = \dfrac{a_1}{b_1} R(x), \quad (a_1, b_1) = 1; \\ \\ D(y) = \dfrac{a_2}{b_2} T(y), \quad (a_2, b_2) = 1 \end{cases}$$

where a_1, b_1 and a_2, b_2 are rational integers, all the coefficients of the polynomials $R(x)$ and $T(y)$ are also rational integers, and the greatest common divisor of the coefficients of $R(x)$ and respectively $T(y)$ is unity.

Each of these polynomials is not identically equal to zero because if one of them is equal to zero then we should have that the polynomials $P_k(x)$ or $Q_k(y)$ are linearly dependent.

We shall use the following obvious identity

$$(44)\quad \Delta(x)D(y) = \begin{vmatrix} \dfrac{P(x,y)}{0!0!} & \dfrac{P_y'(x,y)}{0!1!} & & \dfrac{P^{(r-1)}_{y^{r-1}}(x,y)}{0!(r-1)!} \\ \dfrac{P_x'(x,y)}{1!0!} & \dfrac{P_{xy}''(x,y)}{1!1!} & & \dfrac{P^{(r)}_{xy^{r-1}}(x,y)}{1!(r-1)!} \\ \dots & \dots & \dots & \dots \\ \dfrac{P^{(r-1)}_{x^{r-1}}(x,y)}{(r-1)!0!} & \dfrac{P^{(r)}_{x^{r-1}y}(x,y)}{(r-1)!1!} & & \dfrac{P^{(2r-2)}_{x^{r-1}y^{r-1}}(x,y)}{(r-1)!(r-1)!} \end{vmatrix}.$$

The right member of this identity is a polynomial in x and y with integral coefficients. We shall estimate the height, in other words, the maximum absolute value H of its coefficients.

We must first of all estimate the height $H_{k_1,k}$ of every element of this determinant. Since $|C_{k_1,k}| < \exp[n^{3/4}\mu]$, it is directly obvious that

$$(45) \quad H_{k_1,k} = H\left(\frac{\partial^{k_1+k}P(x,y)}{\partial x^{k_1}\partial y^k k_1!k!}\right)$$

$$\leqslant e^{n^{3/4}\mu}\left|\frac{\frac{d^{k_1}}{dx^{k_1}}(1+x)^{\tau_1}}{k_1!}\,\frac{\frac{d^k}{dy^k}(1+y)^{\tau}}{k!}\right|_{x=y=1},$$

from which it follows further that

$$(46) \qquad H_{k_1,k} < e^{n^{3/4}\mu}\cdot 2^{2(\tau_1+\tau)} < e^{\sigma_0 n^{3/4}\mu}$$

because $\tau_1+\tau<2\nu\mu$. But then

$$(47) \qquad H(\Delta(x)\cdot D(y)) < (\tau+1)^{\tau+1}e^{\sigma_0 n^{7/4}\mu} < e^{\sigma_1 n^{7/4}\mu}.$$

Furthermore, we have

$$(48) \qquad R(x)\cdot T(y) = \frac{b_1}{a_1}\frac{b_2}{a_2}\Delta(x)D(y) = \frac{b}{a}\Delta(x)D(y),$$

$$(b, a) = 1,$$

but the coefficients of $\Delta(x)D(y)$ are rational integers and the coefficients of $R(x)$ and $T(y)$ are not only rational integers, but they also are without common divisor. Therefore $b=1$. This means the inequalities

$$(49) \qquad H_1 = H(R(x)) < e^{\sigma_1 n^{7/4}\mu}; \quad H_2 = H(T(y)) < e^{\sigma_1 n^{7/4}\mu}$$

hold. We shall now show that $T(\zeta_1)\neq 0$. In fact, if $T(\zeta_1)=0$, then

$$T(y) = A(y)\cdot B(y),$$

where $A(y)$ is an irreducible polynomial of height h and degree $s_0\leqslant s$, which is satisfied by ζ, and $B(y)$ is a polynomial with integral coefficients of degree not greater than τ^2-s_0, whose height is not less than unity. By Lemma III, $h>cq_1{}^{1/2}$, and by Lemma IV and condition (38),

$$(50) \qquad H(T(y)) > 2^{-3\tau^2}h > e^{-3\nu^2h^2}e^{\frac{1}{2\nu}n^2\mu} > e^{\sigma_3 n^2\mu}, \quad \sigma_3 > 0.$$

Comparing inequalities (49) and (50), we see that for $n > n_0'(c, \nu, n_0)$ these inequalities are contradictory and this means that $T(\zeta_1) \neq 0$ when $n > n_0'$.

It follows directly from the fact that $T(\zeta_1) \neq 0$ that the numbers $Q_1(\zeta_1), Q_2(\zeta_1), \ldots, Q_r(\zeta_1)$ are all different from zero.

We shall now assume that $P_x^{(\lambda)}(\zeta, \zeta_1) = 0;\ 0 \leqslant \lambda \leqslant \left[\dfrac{\ln q_1}{\ln q}\right] + \tau + 1$.

Then we have the system of equations

$$P_1^{(\lambda)}(\zeta)Q_1(\zeta_1) + \cdots + P_r^{(\lambda)}(\zeta)Q_r(\zeta_1) = 0;$$

$$0 \leqslant \lambda \leqslant \left[\frac{\ln q_1}{\ln q}\right] + \tau + 1.$$

Since all the $Q_i(\zeta_1)$ cannot be equal to zero it follows from this system that

$$(51) \qquad \Delta^{(\lambda)}(\zeta) = R^{(\lambda)}(\zeta) = 0, \quad 0 \leqslant \lambda \leqslant \left[\frac{\ln q_1}{\ln q}\right] = m.$$

As a consequence of this we have the representation

$$R(x) = A_1^{m+1}(x) \cdot B(x),$$

where $A_1(x)$ is an irreducible polynomial of degree $s_1 \leqslant s$ and height h, which is satisfied by ζ, and $B(x)$ is a polynomial with integral coefficients, whose degree is not greater than $\tau\tau_1 - s_1(m+1)$.

By Lemma III, $h > cq^{1/2}$, and by Lemma IV and condition (38),

$$(52) \qquad H(R(x)) > \frac{2^{-3\tau\tau_1}}{2\tau_1\tau} c^{m+1} q^{\frac{m+1}{2}} > e^{\sigma_4 n^2 \mu}, \quad \sigma_4 > 0.$$

Again comparing inequalities (49) and (52) we immediately see that they are contradictory when $n > n_0''(n_0', c, \nu)$.

This completes the proof of our lemma.

The following lemma, Lemma VI′, which is a generalization of Lemmas V and VI, is proved in exactly the same way without changing the course of the arguments and with the same estimates.

LEMMA VI′. *Suppose K is an algebraic field of degree s, that the numbers α and β belong to the field K_0, which is an extension of the field K obtained by adjoining to K an algebraic number of relative degree ν, and that the degree of the field K_0 is νs. The numbers n, $q > e^{n^3}$, $q_1 \geqslant q$ are rational* *integers, η is a real number, $0<\eta\leqslant\nu/2$. Then a polynomial $P(x, y)$ with integral coefficients in the field K,*

$$P(x, y) = \sum_{k_1=0}^{\tau_1} \sum_{k=0}^{\tau} C_{k_1,k}x^{k_1}y^k$$

can be constructed, satisfying the following conditions:

$$a) \quad \tau = \left[\sqrt{\frac{\nu\eta}{2}}\,(n+\sqrt{n})\right]-1; \quad \tau_1 = \left[\frac{\nu}{2}\,\frac{n+\sqrt{n}}{\tau+1}\,\mu\right];$$

$$b) \quad C_{k_1,k} = \sum_{1}^{s} C_{k_1,k,m}\omega_m; \quad |C_{k_1,k,m}| < e^{n^{3/4}\mu},$$

$$\mu = \left[\eta n\,\frac{\ln q_1}{\ln q}\right],$$

where $\omega_1, \ldots, \omega_s$ is a basis for the ring of integers in the field K;

$$c) \quad \frac{\partial^{k_1+k}}{\partial x^{k_1}\partial y^k}\,P(\alpha, \beta) = 0, \quad \frac{k_1}{\mu}+\frac{k}{n} < 1;$$

$$d) \quad \text{there exists a } \lambda, \quad 0 \leqslant \lambda \leqslant \frac{\ln q_1}{\ln q}+\tau+1, \text{ such that } \frac{\partial^\lambda P(\zeta, \zeta_1)}{\partial x^\lambda} \neq 0$$

where ζ and ζ_1 are two numbers in the field with measures q and q_1 respectively.

Finally, we note that if α and β belong to the field obtained by adjoining to the Gauss field a root of an arbitrary polynomial of degree ν which is irreducible in this field, then all the preceding lemmas remain valid, where the coefficients of the polynomial $P(x, y)$ in Lemmas V and VI are to be chosen in the form of integral complex numbers in the Gauss field.

An analogous theorem will hold in the more general case when the Gauss field is replaced by *any* imaginary quadratic field.

§3. Fundamental theorems

Now the lemmas proved above enable us to prove some general theorems concerning the approximation of algebraic numbers. In these theorems, α and β can be arbitrary, in particular rational, but the field K_0 to which they belong is assumed to have degree no less than 3.

THEOREM 1. *Suppose α and β are two arbitrary numbers in an algebraic field K_0 of degree ν (they may coincide). Suppose further that ζ and ζ_1 are numbers in an algebraic field K, whose measures with respect to a fixed basis $\omega_1, \omega_2, \ldots, \omega_s$ of the integers in this field are respectively q and q_1, and that ϑ and ϑ_1 are two real constants, $2 \leqslant \vartheta \leqslant \vartheta_1 \leqslant \nu$, $\vartheta\vartheta_1 = 2\nu(1+\varepsilon)$, $\varepsilon > 0$ where ε is an arbitrarily small fixed number. Then, if the inequality*

$$(53) \qquad |\alpha - \zeta| < q^{-s\vartheta}$$

has a solution ζ with measure $q > q'(K_0, K, \alpha, \beta, \varepsilon, \delta)$, the inequality

$$(54) \qquad |\beta - \zeta_1| < q_1^{-s\vartheta_1}$$

cannot have solutions with measure q_1 under the condition that

$$(55) \qquad \ln q_1 \geqslant \left[\frac{\vartheta - 1}{2(\sqrt{1+\varepsilon} - 1)} + \delta\right] \ln q,$$

where δ is any arbitrarily small fixed positive number.

There is a $\wp$-adic analogue to this theorem.

Suppose A is an integral ideal in some field of algebraic numbers and that α is a number in this field. In the sequel, we shall agree to write $\alpha \equiv 0 \pmod{A}$ if α can be represented in the form $\alpha = B/C$, where B and C are integral ideals, where the ideal B is divisible by the ideal A, and the ideal C is relatively prime to the ideal A.

Suppose α and β belong to the field K_0 of degree ν; ζ and ζ_1 belong to a field K of degree s, and K_1 is a field of degree $g \leqslant \nu s$, obtained from K by adjoining the numbers α and β. Suppose $\wp_1, \wp_2, \ldots, \wp_\sigma$ are ideals in the field K_1, all of whose divisors belong respectively to distinct prime rational numbers $p_1, p_2, \ldots, p_\sigma$, and the numbers $f_1, f_2, \ldots, f_\sigma$ are such that the p_i are not divisible by a power of $\wp_i$ greater than f_i. Suppose also, as before, that the measures of the numbers ζ and ζ_1 with respect to the field K are q and q_1. We now set

$$m_i = \left[\lambda_i s\vartheta f_i \frac{\ln q}{\ln p_i}\right], \quad m_i' = \left[\lambda_i s\vartheta_1 f_i \frac{\ln q_1}{\ln p_i}\right], \quad i = 1, \ldots, \sigma,$$

with all the λ_i not less than zero and $\sum_{i=1}^{\sigma} \lambda_i = 1$, where the λ_i do not depend on q, q_1, n;

$$A_q = \prod_1^{\sigma} \wp^{m_i}, \quad A_{q_1} = \prod_1^{\sigma} \wp^{m_i'}.$$

THEOREM II. *Retaining the sense of the notation in Theorem I, one can assert that if the congruence*

$$\alpha-\zeta \equiv 0 \pmod{A_q} \tag{56}$$

has a solution for $q \geqslant q'(K_0, K, \alpha, \beta, \varepsilon, \delta)$, *then the congruence*

$$\beta-\zeta_1 \equiv 0 \pmod{A_{q_1}} \tag{57}$$

cannot have a solution under the condition that

$$\ln q_1 \geqslant \left[\frac{\vartheta-1}{2(\sqrt{1+\varepsilon}-1)}+\delta\right] \ln q, \tag{58}$$

where $\delta>0$ *is fixed, but arbitrarily small.*

Naturally, congruences (56) and (57) are considered as congruences in the field K_1. It is also true that none of the divisors of the ideals $\wp_1, \ldots, \wp_\sigma$ occurs in α either to a positive or to a negative power.

In all proofs in the sequel, the numbers $\gamma_0, \gamma_1, \ldots, \gamma_r$ depend only on K, K_0, α, β, ε, δ and do not depend on n, q, q_1.

Proof of Theorem I. We take the numbers α, β, ζ, ζ_1, ϑ, ϑ_1, δ, ε and assume they satisfy all the conditions of our theorem, except one, namely, we assume that $\ln q_1 > \left[\frac{\vartheta-1}{2(\sqrt{1+\varepsilon}-1)}+\delta\right] \ln q$. By Lemmas V and VI, assuming also that $\ln q > n^3$, $n > n_1(K_0, K, \alpha, \beta)$, $\eta = \vartheta_1/\vartheta$, we construct a polynomial $P(x, y)$ and consider its λ-th order derivative, which is different from zero for $x=\zeta$, $y=\zeta_1$. Then

$$J = \frac{1}{\lambda!}\frac{\partial^\lambda}{\partial x^\lambda} P(\zeta, \zeta_1) = \sum_{k_1=\lambda}^{\tau_1} \sum_{k=0}^{\tau} \frac{k_1!}{\lambda!(k_1-\lambda)!} C_{k_1,k} \zeta^{k_1-\lambda} \zeta_1{}^k. \tag{59}$$

On the other hand, by Lemma V,

$$J = \sum \frac{k_1!}{\lambda!(k_1-\lambda)!} \frac{\partial^{k_1+k} P(\alpha, \beta)}{\partial x^{k_1} \partial y^k} \frac{(\zeta-\alpha)^{k_1-\lambda}}{k_1!} \frac{(\zeta_1-\beta)^k}{k!} \tag{60}$$

$$\frac{k_1}{\mu}+\frac{k}{n} \geqslant 1, \quad k_1 \geqslant \lambda.$$

From this last equality, using estimate (46), we immediately obtain

$$(61)\qquad |J| < e^{\gamma_0 n^{3/4}\mu} \max_{\frac{k_1}{\mu}+\frac{k}{n}\geqslant 1} (|\alpha-\zeta|^{k_1-\lambda}, |\beta-\zeta_1|^{k}),$$

and with the aid of inequalities (53) and (54) that

$$(62)\qquad |J| < e^{\gamma_0 n^{3/4}\mu} \max_{\frac{k_1}{\mu}+\frac{k}{n}\geqslant 1} (q^{-s\vartheta(k_1-\lambda)}, q_1^{-s\vartheta_1 k}).$$

Transforming the right member of this inequality, using the fact that

$$\lambda \leqslant \left[\frac{\ln q_1}{\ln q}\right]+\tau+1$$

and that for $r=\dfrac{\ln q_1}{\ln q}$

$$(63)\qquad \frac{1}{r}s\vartheta k_1+s\vartheta_1 k \geqslant s\vartheta_1 n\left(\frac{k_1}{\mu+1}+\frac{k}{n}\right) > s\vartheta_1 n\left(1-\frac{2}{n}\right),$$

$$s\vartheta\lambda \ln q < ns\left(\frac{1}{r}\sqrt{\frac{\nu\vartheta\vartheta_1}{2}}+\gamma_1 n^{-1/2}\right)\ln q_1; \quad e^{\gamma_0 n^{3/4}\mu} < q_1^{\gamma_2 sn^{-1}},$$

we shall finally have that

$$(64)\qquad |J| < q_1^{-ns\left[\vartheta_1-\frac{1}{r}\sqrt{\frac{\nu\vartheta\vartheta_1}{2}}-\gamma_3 n^{-1/2}\right]}.$$

From relation (59), by Lemma II, we shall have, on the other hand, that

$$(65)\qquad |J| > q_1^{-s\tau}q^{-s(\tau_1-\lambda)}e^{-\gamma_5 n^{3/4}\mu}.$$

It follows, using inequalities analogous to inequalities (63), that

$$(66)\qquad |J| > q_1^{-sn\left[2\sqrt{\frac{\nu\vartheta_1}{2\vartheta}}-\frac{1}{r}\sqrt{\frac{\nu\vartheta_1}{2\vartheta}}+\gamma_4 n^{-1/2}\right]}.$$

Comparing inequalities (66) and (64) and taking into consideration that

$$(67)\qquad 2\sqrt{\frac{\nu\vartheta_1}{2\vartheta}}-\frac{1}{r}\sqrt{\frac{\nu\vartheta_1}{2\vartheta}} < \vartheta_1-\frac{1}{r}\sqrt{\frac{\nu\vartheta\vartheta_1}{2}}$$

since

$$(68)\qquad \vartheta\vartheta_1 = 2\nu(1+\varepsilon), \quad r = \frac{\ln q_1}{\ln q} \geqslant \frac{\vartheta-1}{2(\sqrt{1+\varepsilon}-1)}+\delta,$$

we see that these inequalities are contradictory when $n > n'(K, K_0, \alpha, \beta, \varepsilon, \delta)$. This completes the proof of the theorem.

The proof of Theorem II is carried out in a manner very similar to that of Theorem I.

We construct a polynomial $p(x, y)$, according to Lemmas V and VI, under the same assumptions as in the proof of Theorem I. In the sequel, $N(J)$ will denote the norm of the number J in the field K.

From representation (60), analogous to inequality (61), we obtain the congruence

$$N(J) \equiv 0 \; (\mathrm{mod}\, p_1{}^{\eta_1} p_2{}^{\eta_2} \cdots p_\sigma{}^{\eta_\sigma}), \tag{69}$$

where we have set

$$\eta_i = \min_{\frac{k_1}{\mu}+\frac{k}{n}\geqslant 1} \left(\left[\lambda_i \vartheta_s \frac{\ln q}{\ln p_i}\right](k_1 - \lambda) + \left[\lambda_i \vartheta_1 s \frac{\ln q_1}{\ln p_i}\right] k \right),$$

$$i = 1, 2, \ldots, \sigma.$$

The number λ is the number defined by Lemma VI, and $\lambda_1, \lambda_2, \ldots, \lambda_\sigma$, as was already determined, will be positive numbers whose sum equals unity.

From congruence (69), we obtain directly, after carrying out transformations analogous to transformations (62) and (63), that

$$N(J_1) \geqslant \prod_{i=1}^{\sigma} p_i{}^{\eta_i} > q_1{}^{ns\left[\vartheta_1 - \frac{1}{r}\sqrt{\frac{\nu\vartheta\vartheta_1}{2}} + \gamma_s n^{-1/2}\right]}, \tag{70}$$

$$J_1 = \zeta''^{(\tau_1 - \lambda)} \zeta_1''^{\tau} J,$$

where we have set

$$\zeta = \frac{\zeta'}{\zeta''}, \quad \zeta_1 = \frac{\zeta_1'}{\zeta_1''};$$

all the quantities ζ', ζ'', ζ_1', ζ_1'' are integers and the measures of the corresponding representations of ζ and ζ_1 are q and q_1. First of all, from representation (59) we shall have that

$$N(\zeta''^{(\tau_1-\lambda)}\zeta_1''^{\tau} J) = N(J_1) \tag{71}$$

$$= N\left(\sum_{k_1=\lambda}^{\tau_1} \sum_{k=0}^{\tau} \frac{k_1!}{\lambda!\,(k-\lambda)!} C_{k_1,k} \zeta^{k_1-\lambda} \zeta''^{\tau_1-\lambda} \zeta_1^{k} \zeta_1''^{\tau} \right).$$

Using Lemma II and the estimates for the quantities τ, τ_1, λ, q_1, q and $C_{k_1,k}$, we obtain from representation (71) the inequality

$$N(J_1) < e^{\gamma_6 n^{3/4}\mu} q_1^{s\tau} q^{s(\tau_1-\lambda)} \tag{72}$$

or, analogous to inequality (66):

$$N(J_1) < q_1^{sn\left[2\sqrt{\frac{\nu\vartheta_1}{2\vartheta}}-\frac{1}{r}\sqrt{\frac{\nu\vartheta_1}{2\vartheta}}+\gamma_7 n^{-1/2}\right]}. \tag{73}$$

But since ζ'' and ζ_1'' are integers. inequality (70) yields

$$N(J_1) > q_1^{ns\left[\vartheta_1-\frac{1}{r}\sqrt{\frac{\nu\vartheta\vartheta_1}{2}}+\gamma_5 n^{-1/2}\right]}. \tag{74}$$

Comparing inequalities (73) and (74) and using inequality (67) and relations (68), we again obtain that for $n > n''(K, K_0, \alpha, \beta, \varepsilon, \delta)$ our inequalities are contradictory, which completes the proof of Theorem II. Setting $\alpha=\beta$, $\vartheta_1=\vartheta$ in our theorems, we obtain Theorem I′, below, in this particular case.

THEOREM I′. *The inequality*

$$|\alpha-\zeta| < q^{-s(\sqrt{2\nu}+\varepsilon)}, \quad \varepsilon > 0 \tag{75}$$

and the congruence

$$\alpha-\zeta \equiv 0 \pmod{A_q}, \quad A_q = \prod \wp_i^{m_i}, \tag{76}$$

where

$$m_i = \left[\lambda_i s f_i(\sqrt{2\nu}+\varepsilon)\,\frac{\ln q}{\ln p_i}\right]; \tag{77}$$

$$i = 1, \ldots, \sigma; \quad \sum \lambda_i = 1; \quad \lambda_i \geqslant 0,$$

and $\wp_i$ and f_i have their former meanings, i.e. those of Theorem II, have only a finite number of solutions in numbers ζ with measure q in the field K.

In exactly the same way, using Lemma VI′, one can obtain generalizations of Theorems I and II and, in particular, prove the next theorem.

THEOREM I″. *The inequality*

$$|\alpha-\zeta| < q^{-s(\sqrt{2\nu}+\varepsilon)}, \quad \varepsilon > 0$$

and the congruence

$$\alpha-\zeta \equiv 0 \pmod{A_q}, \quad A_q = \prod_1^{\sigma} \wp_i^{m_i}$$

have only a finite number of solutions if ζ is a number in the field K of degree s, having measure q, α is a root of an equation of degree ν irreducible in the *field K with coefficients in the same field, $\wp_1, \ldots, \wp_\sigma$ are ideals in the field K_0 which is the extension of the field K obtained by adjoining the number α to it, where every ideal belongs to one prime rational number, f_i is the maximal power of $\wp_i$ which divides the corresponding prime p_i, $\lambda_1, \lambda_2, \ldots, \lambda_\sigma$ are any fixed nonnegative numbers, $\sum_1^\sigma \lambda_i = 1$ and*

$$m_i = \left[\lambda_i f_i(\sqrt{2\nu}+\varepsilon)\frac{\ln q}{\ln p_i}\right].$$

The number α is fixed and the number ζ is variable.

The proof of this theorem is a repetition of the proof of Theorems I and II for the particular case $\vartheta=\vartheta_1$ and $\alpha=\beta$.

Basing arguments on the fact that it is possible to consider the roots of equations which are irreducible in the Gauss field without changing the formulation of these lemmas, one can also prove the next theorem.

THEOREM I‴. *The formulation of the theorem remains unchanged if α is a number in a field of degree ν with respect to the Gauss field, and ζ is a number of degree s with respect to the same Gauss field, the prime ideals $\wp_i$ are defined as before, the number f_i is the highest power of the ideal $\wp_i$ which divides a prime number in the Gauss field, the measure q of the number ζ is taken with respect to the basis $\omega_1, \ldots, \omega_s$ in the Gauss field and, finally,*

$$m_i = \left[\lambda_i f_i(\sqrt{2\nu}+\varepsilon)\frac{\ln q}{\ln |p_i|}\right], \quad i = 1, 2, \ldots, \sigma.$$

The proof of this theorem is merely a repetition of the proofs of Theorems I and II, again for the particular case $\alpha=\beta$ and $\vartheta=\vartheta_1$.

Two theorems on the approximation of algebraic irrationalities by numbers consisting of powers of the same given algebraic numbers are a consequence of the theorems already proved.

We shall say that the algebraic numbers $\zeta_1, \zeta_2, \ldots, \zeta_n$ are *multiplicatively independent* if the relation

$$\zeta_1^{x_1} \ldots \zeta_n^{x_n} = 1$$

is possible for rational integral $x_1, x_2, \ldots, x_n$ if and only if $x_1=x_2=\cdots=x_n=0$.

THEOREM III. *Suppose* $\alpha, \zeta_1, \ldots, \zeta_n$ *are algebraic numbers in the field* K_0 *and that* $\zeta_1, \ldots, \zeta_s$ *are multiplicatively independent. Then the inequality*

$$|\alpha - \zeta_1^{x_1} \ldots \zeta_s^{x_s}| < e^{-\varepsilon x}, \quad x = \max_{1 \leqslant i \leqslant s} |x_i| \tag{78}$$

and the congruence

$$\alpha \equiv \zeta_1^{y_1} \ldots \zeta_s^{y_s} (\mathrm{mod}\ \wp^m), \quad m = [\delta y], \quad y = \max_{1 \leqslant i \leqslant s} |y_i| \tag{79}$$

have only a finite number of solutions in rational integers $x_1, \ldots, x_s$ *and* $y_1, \ldots, y_s$, $\wp$ *is a prime ideal in the field* K_0, *no matter how small the prescribed numbers* $\varepsilon > 0$ *and* $\delta > 0$.

Proof. For the sake of simplification, we can assume without loss of generality in the proof, that the prime ideal $\wp$ does not occur in α to either a positive or a negative power.

We shall also assume that $\wp$ is a divisor of the prime p.

In order to prove the first part of the theorem, we shall assume that inequality (78) has an infinite set of solutions

$$\psi_k = \zeta_1^{x_1^{(k)}} \ldots \zeta_s^{x_s^{(k)}}, \quad x^{(k)} = \max_{1 \leqslant i \leqslant s} |x_i^{(k)}| \tag{80}$$

which we may assume to be arranged in order of increasing $x^{(k)}$.

The measures of the numbers ψ_k increase indefinitely (we fix any integral basis in the field K) as the $x^{(k)}$ increase, because in the contrary case we should have identical terms in the sequence of numbers ψ_k which would lead to the multiplicative dependence of the numbers $\zeta_1, \ldots, \zeta_s$.

We now take any arbitrarily large integer N and consider the congruences

$$\left\{ \begin{array}{l} x_1^{(k)} \equiv a_1^{(k)} \ (\mathrm{mod}\ N), \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ x_s^{(k)} \equiv a_s^{(k)} \ (\mathrm{mod}\ N), \end{array} \right. \quad 0 \leqslant a_i < N. \tag{81}$$

There are at most N^s distinct sets of numbers $a_1^{(k)}, \ldots, a_s^{(k)}$, and the number k runs through the set of natural numbers. Therefore, at least one of the sets of residues must repeat an infinite number of times.

We now take only those solutions of inequality (78) in which $x_1, \ldots, x_s$ have the same remainders upon division by N. There is an infinite number of such solutions and they will have the form

$$[\zeta_1^{z_1^{(k)}} \cdots \zeta_s^{z_s^{(k)}}]^N \zeta_1^{a_1} \cdots \zeta_s^{a_s} = \sigma \eta_k^N \tag{82}$$

$$\max_{1 \leqslant i \leqslant s} |z_i^{(k)}| = z^{(k)} \leqslant \frac{x^{(k)}}{N} - \vartheta_k, \quad 0 \leqslant \vartheta_k \leqslant 1.$$

We note that the measure of the number η

(83) $$\eta = \zeta_1^{z_1}\zeta_2^{z_2}\cdots\zeta_s^{z_s}, \quad z = \max|z_i|,$$

is not greater than the number $e^{\gamma_0 z}$, where γ_0 depends only on the field K, the integral basis chosen in K, and on the numbers $\zeta_1, \zeta_2, \ldots, \zeta_s$.

For the sequence of solutions (82), inequality (78) assumes the form

(84) $$|\alpha'-\eta^N| < \frac{1}{|\sigma|}e^{-\varepsilon Nz}, \quad \eta = \zeta_1^{z_1}\cdots\zeta_s^{z_s}, \quad z = \max_{1\leqslant i\leqslant s}|z_i|,$$

where σ is fixed, and $\alpha'=\alpha/\sigma$. This inequality must have, as we have already established, an infinite number of solutions in rational integers $z_1, \ldots, z_s$. Inequality (84) can be rewritten in the form

(85) $$\prod_1^N \left|\eta-\omega_k\alpha'^{\frac{1}{N}}\right| < \frac{1}{|\sigma|}e^{-\varepsilon Nz},$$

where $\alpha'^{\frac{1}{N}}$ is any N-th root of the number α', and $\omega_1, \ldots, \omega_N$ are all the N-th roots of unity.

For each of the solutions of this last inequality, at least one of the factors will be minimal in absolute value. For this factor, since it is the least, we have

(86) $$|\alpha'^{\frac{1}{N}}\omega_n-\eta| < |\sigma|^{-\frac{1}{N}}e^{-\varepsilon z}.$$

Because of this, for every $k\neq n$,

(87) $$\left|\alpha'^{\frac{1}{N}}\omega_k-\eta\right| \geqslant \left|\alpha'^{\frac{1}{N}}\right||\omega_n-\omega_k|-\left|\alpha'^{\frac{1}{N}}\omega_n-\eta\right|$$
$$> 2\sin\frac{\pi}{N}\left|\alpha'^{\frac{1}{N}}\right|-\sigma^{-\frac{1}{N}}e^{-\varepsilon z},$$

from which we finally obtain that

(88) $$\left|\alpha'^{\frac{1}{N}}\omega_k-\eta\right| > \sin\frac{\pi}{N}\left|\alpha'^{\frac{1}{N}}\right|, \quad k\neq n,$$

with $z\geqslant z_0(N)$.

Comparing inequalities (85) and (88), we obtain that for every solution η of inequality (85), for sufficiently large z, there will exist a corresponding n for which the inequality

(89) $$\left|\alpha'^{\frac{1}{N}}\omega_n-\eta\right| < \frac{e^{-\varepsilon Nz}}{|\sigma||\alpha'|^{1-\frac{1}{N}}\sin^{N-1}\frac{\pi}{N}} < Ae^{-\varepsilon Nz}$$

is satisfied. Since inequality (85) has an infinite number of solutions and n can assume only N values, n is the same for an infinite number of solutions. Setting

$$\alpha'^{\frac{1}{N}}\omega_n = \beta \tag{90}$$

for this n, we arrive at the fact that the inequality

$$|\beta-\eta| < Ae^{-\varepsilon Nz} < e^{-\frac{\varepsilon}{2}Nz}, \quad z > z_1(N, A), \tag{91}$$

has an infinite number of solutions.

We shall assume, for definiteness, that the degree of the field K, to which the numbers $\alpha, \zeta_1, \ldots, \zeta_s$ belong, is μ. Then, obviously, the degree ν of the field K_1, to which β must belong, is not greater than μN. Since the measure of η is not greater than $e^{\gamma_0 z}$, inequality (91) can finally be rewritten as

$$|\beta-\eta| < q^{-\frac{\varepsilon}{2\gamma_0}N}, \tag{92}$$

where q is the measure of η in the field K. But by Theorem I′, the inequality

$$|\beta-\eta| < q^{-\mu(\sqrt{2\nu}+1)} \tag{93}$$

can have only a finite number of solutions for arbitrary numbers η in the field K with measure q. Since N is an arbitrary integer, then, taking N so that it satisfies the inequality

$$N > \frac{32\gamma_0{}^2\mu^3}{\varepsilon^2}$$

we obtain that the following inequality

$$\frac{\varepsilon}{2\gamma_0} N > 2\mu\sqrt{2\mu N} > \mu(\sqrt{2\mu N}+1) \geqslant \mu(\sqrt{2\nu}+1) \tag{94}$$

holds, from which it follows that inequality (92) has only a finite number of solutions. Since we have arrived at a contradiction, the first assertion of Theorem III has been proved.

In order to prove the second assertion of the theorem, we again take any arbitrarily large $N > p$ prime, and we chose from the sequence of solutions of congruence (79), which we assume to be infinite, an infinite subsequence of solutions which are such that the corresponding $x_1, \ldots, x_s$ yield the same remainders $a_1, \ldots, a_s$ upon division by N. We then obtain from congruence (79) that the congruence

$$\alpha' - \eta^N \equiv 0 \pmod{\wp^m}, \quad m = [\delta Nz], \tag{95}$$

where we have set

(96) $$\eta = \zeta_1^{z_1} \cdots \zeta_s^{z_s}, \quad \alpha' = \alpha\zeta_1^{-a_1} \cdots \zeta_s^{-a_s}, \quad z = \max_{1 \leqslant i \leqslant s} |z_i|,$$

has an infinite number of solutions. We note that α' is relatively prime to the prime ideal $\wp$, because this ideal does not occur, as can be assumed without loss of generality, in any of the ζ.

We shall assume the opposite, namely that the prime ideal $\wp$ appears in the system $\zeta_1, \ldots, \zeta_k$; in other words, that

(97) $$\zeta_1 = \lambda_1 \wp^{\tau_1}, \ldots, \zeta_k = \lambda_k \wp^{\tau_k},$$

and that it does not appear in any other ζ.

From congruence (79) and the fact that $\wp$ does not appear in α, it follows that $\wp$ does not appear in the number

(98) $$\zeta_1^{x_1} \cdots \zeta_k^{x_k} = \lambda_1^{x_1} \cdots \lambda_k^{x_k} \wp^{x_1\tau_1 + \cdots + x_k\tau_k}$$

for any solution of this congruence, where $\lambda_1, \ldots, \zeta_k$ are certain ideals in which $\wp$ does not occur. It follows from equality (98) that for every solution of congruence (79) the relation

(99) $$x_1\tau_1 + \cdots + x_k\tau_k = 0$$

is satisfied. One cannot always find an integer n such that the ideals $\lambda_1^n, \ldots, \lambda_k^n$ are principal. Choosing from the solution of the congruence an infinite subsequence for which $x_1, \ldots, x_k$ will yield the same remainders $a_1, \ldots, a_k$ upon division by n, we shall have for these solutions that

(100) $$\zeta_1^{x_1} \cdots \zeta_k^{x_k} = \lambda_1^{nz_1} \cdots \lambda_k^{nz_k} \zeta_0, \quad \zeta_0 = \lambda_1^{a_1} \cdots \lambda_k^{a_k}.$$

Setting $\lambda_1^n = \zeta_1', \ldots, \lambda_k^n = \zeta_k'$ and noting that ζ_0 is an algebraic number, we again arrive at a congruence of the type (79) with $\zeta_1, \zeta_2, \ldots, \zeta_s$ in which $\wp$ does not appear, and $\alpha = \alpha'\zeta_0^{-1}$.

We now return to congruence (95). The numbers $\alpha, \zeta_1, \ldots, \zeta_s$ and the prime ideal $\wp$ belong to the field K_0 of degree μ_0. Suppose p is a rational integer which is divisible by $\wp$, and that f_0 is the highest power of $\wp$ which divides p. We denote the N-th roots of unity by $\omega_1 = 1, \omega_2, \ldots, \omega_N$, and suppose $K_1, \ldots, K_N$ are algebraic fields, formed respectively by adjoining the numbers

$$\alpha'^{\frac{1}{N}}\omega_1 \ldots, \alpha'^{\frac{1}{N}}\omega_N$$

to the field K_0. All the fields $K_1, \ldots, K_N$ are relatively conjugate fields. The number N, which was assumed to be arbitrarily large, prime and larger than p, naturally, can be chosen in an infinite number of ways in such a way that the degree of the field K_i, $i=1, 2, \ldots, N$ equals $\mu_0 N$, where μ_0 is the degree of K_0. In the field K_i the prime ideal $\wp$ has the representation

$$\wp = \wp_{1.i}^{f_1} \cdots \wp_{\sigma.i}^{f_\sigma}, \quad i = 1, \ldots, N,$$

where $\wp_{1,i}, \ldots, \wp_{\sigma,i}$ are prime ideals in the field K_i. In the field K_1 congruence (95) assumes the form

$$(101) \quad (\alpha'^{\frac{1}{N}} - \eta)(\alpha'^{\frac{N-1}{N}} + \eta\alpha'^{\frac{N-2}{N}} + \cdots + \eta^{N-1}) \equiv 0 \bmod \wp_{1.0.}^{f_{1m}}, \ldots, \wp_{\sigma.0}^{f_{\sigma m}}.$$

If $\wp_0$ is one of the prime ideals in the modulus of the congruence, only one of the factors of the left member of this congruence is divisible by it. In fact, assuming the contrary, we would by the same token assume that the two congruences

$$(102) \quad \begin{cases} \alpha'^{\frac{1}{N}} \equiv \eta \bmod \wp_0, \\ \alpha'^{\frac{N-1}{N}} + \eta\alpha'^{\frac{N-2}{N}} + \cdots + \eta^{N-1} \equiv 0 \bmod \wp_0 \end{cases}$$

hold simultaneously. Eliminating η from both congruences, we obtain the congruence

$$(103) \quad N\alpha'^{\frac{N-1}{N}} \equiv 0 \bmod \wp_0.$$

This last congruence is impossible because $\wp_0$ is a divisor of $\wp$, $\wp$ divides p, and the numbers N and norm of α' are not divisible by p by assumption.

If we assume that

$$(104) \quad \alpha'^{\frac{1}{N}} - \eta \not\equiv 0 \bmod \wp_{i,1}, \quad i = 1, 2, 3, \ldots, \sigma,$$

then it will follow that

$$(105) \quad \alpha'^{\frac{1}{N}}\omega_k - \eta \not\equiv \bmod \wp_{i,k}; \ 1 \leqslant i \leqslant \sigma, \ 1 < k \leqslant N$$

in every relatively conjugate field. But from the fact that the congruence is impossible it must follow that also

$$(106) \quad N_{K_0}(\alpha'^{\frac{1}{N}} - \eta) = \alpha' - \eta^N \not\equiv 0 \bmod \wp,$$

where N_{K_0} is the relative norm of the number $\alpha'^{\frac{1}{N}} - \eta$ in the field K_0. This means that the congruence

$$(107) \quad \alpha'^{\frac{1}{N}} - \eta \equiv 0 \bmod \wp_{i.0}^{f_{im}}$$

holds for at least one value of i. Finally, setting $\wp_{i,0}=\wp_0, f_i=f_1$, we shall have the congruence

$$\alpha'^{\frac{1}{N}}-\eta \equiv 0 \bmod \wp_0{}^{f_1 m}, \quad m = \left[\frac{\varepsilon N}{\gamma_0}\ln q\right], \tag{108}$$

where $\wp_0$ appears in $\wp$ to the power f_1, q is the measure of the number η, $\varepsilon > 0$ is arbitrarily small. Obviously, this prime ideal $\wp_0$ can vary depending on η. But since by assumption congruence (98) has an infinite set of solutions, it is possible to choose an infinite subsequence of these solutions for which $\wp_0$ is the same because $\wp_0$ can assume no more than σ values. In the sequel, we shall be dealing with only these solutions. We thus obtain, from the assumption that congruence (95) has an infinite number of solutions, the existence of an infinite number of solutions of (108) for the same fixed $\wp_0$.

Since the prime ideal $\wp$ in the field K_0 appears in the prime p to the power f_0 and the prime ideal $\wp_0$ appears in the ideal $\wp$ in the field K_1 to the power f_1, the prime ideal $\wp_0$ in the field K_1 appears in the prime number p to the power $f=f_0 f_1$.

Inasmuch as the degree of the field K_0 is μ_0, the degree ν of the field K_1 is not greater than $N\mu_0$, $\nu \leqslant N\mu_0$, by Theorem I′, setting $\sigma=\lambda_1=\varepsilon=1$ in it, we obtain that the congruence

$$\alpha'^{\frac{1}{N}}\omega_1-\eta \equiv 0 \bmod \wp_0^{\left[\vartheta\mu_0 f\frac{\ln q}{\ln p}\right]}, \tag{109}$$

where $\vartheta=\sqrt{2\nu}+1 \leqslant \sqrt{2\mu_0 N}+1$, can have only a finite number of solutions in numbers η in the field K, if the measure of η with respect to the chosen integral basis in the field K_0 is q. But taking N sufficiently large,

$$N > \frac{\gamma_0{}^2(2\ln p+3\mu_0{}^{3/2}f_0)^2}{\varepsilon^2\ln^2 p}, \; N > p, \quad N \not\equiv 0 \bmod p,$$

we easily obtain that

$$f_1\left[\frac{\varepsilon N}{\gamma_0}\ln q\right] > \left[\vartheta\mu_0 f_0 f_1\frac{\ln q}{\ln p}\right],$$

from which it follows that congruence (108) has only a finite number of solutions.

Since we have arrived at a contradiction, we have by the same token completed the proof of the second assertion in Theorem III.

The fundamental idea in the proof of the first part of Theorem III is due to Mahler, who proved a particular case of the first part of Theorem III, namely, when α is algebraic, and all the $\zeta_1, \ldots, \zeta_s$ are rational numbers.

One can easily obtain two theorems on the order of approximation to zero of linear forms in logarithms of algebraic numbers with rational integral coefficients from Theorem III.

THEOREM IV. *If the algebraic numbers $\alpha_1, \alpha_2, \ldots, \alpha_s$ are multiplicatively independent, the inequality*

$$(110) \quad |x_1 \ln \alpha_1 + \cdots + x_s \ln \alpha_s| < e^{-\varepsilon x}, \quad \varepsilon > 0, \quad x = \max_{1 \leqslant i \leqslant s} |x_i|,$$

where all the $x_1, \ldots, x_s$ are rational integers and the number $\ln \alpha$ may be an arbitrary, but fixed, value of the logarithm, has only a finite number of solutions in rational integers $x_1, x_2, \ldots, x_s$.

Proof. In fact, if inequality (110) had an infinite number of solutions, then as is easily seen, the inequality

$$(111) \qquad |1 - \alpha_1^{x_1} \cdots \alpha_s^{x_s}| = |1 - e^{\sum_1^s x_k \ln \alpha_k}| < 2e^{-\varepsilon x}$$

would also have an infinite number of solutions which is impossible by Theorem III.

THEOREM V. *If $\alpha_1, \ldots, \alpha_s$ are algebraic multiplicatively independent numbers and their logarithms are defined in a $\wp$-adic extension of the field K to which they all belong, then the inequality*

$$(112) \quad |x_1 \ln \alpha_1 + \cdots + x_s \ln \alpha_s|_\wp < \wp^m, \quad m = [\varepsilon x], \quad x = \max_{1 \leqslant i \leqslant s} |x_i|,$$

can have only a finite number of solutions in rational integers $x_1, x_2, \ldots, x_s$.

Proof. In fact, if this inequality had an infinite number of solutions, then congruence (79) of Theorem III, as is easily seen, would also have an infinite number of solutions, which is impossible.

The author of this monograph proved a sharper theorem than Theorem IV for the particular case $s=2$: the inequality

$$(113) \qquad |x_1 \ln \alpha_1 + x_2 \ln \alpha_2| < e^{-\ln^{2+\varepsilon} x}$$

has only a finite number of solutions in rational integers x_1, x_2.

An analogous theorem for $s=2$ was also proved for the case of Theorem V.

The proof of these theorems depends not on the method of Thue but on another, effective method, which enables one, for example, to establish the least upper bound, in the form of a function of α_1, α_2 and ε, of the magnitudes of the possible solutions of inequality (113). This method will be discussed in Chapter III.

The following theorem also follows directly from Theorem III.

THEOREM VI. *If* $\alpha, \eta_1, \ldots, \eta_\nu$ *are fixed numbers in the algebraic field* K_1; $\eta_1, \ldots, \eta_\nu$ *are multiplicatively independent; and* $\zeta_1, \ldots, \zeta_s$ *are arbitrary and integral ideals in this field, none of which is a unit ideal, then the congruences*

$$(114)\quad \alpha-\eta_1^{x_1}\cdots\eta_\nu^{x_\nu}\equiv 0 \bmod \zeta_1^{m_1}\cdots\zeta_s^{m_s},\quad m_i=[\lambda_i\varepsilon x],\quad \varepsilon>0,$$

where $x=\max_{1\leqslant i\leqslant \nu}|x_i|$, *the numbers* λ_i *are not negative and* $\sum_{i=1}^{s}\lambda_i=1$, *where the* λ_i *are arbitrary and are such that any number of them may be equal to zero, have in all only a finite number of solutions in integers* $x_1, \ldots, x_\nu$.

In the contrary case, in the presence of an infinite number of solutions of the set of congruences (114), obviously, one can find an infinite subsequence of these solutions for which any one definite λ_n will always be greater than or equal to $1/s$ and we thus arrive at Theorem III.

§4. Applications of the fundamental theorems

We shall agree to say that a system of homogeneous forms $P_1(x,y), P_2(x,y), \ldots, P_n(x,y)$ having degrees $m_1, \ldots, m_n$, respectively, satisfy conditions (A_ν), if the following conditions are satisfied: All the forms are polynomials, without linear divisors in the field of rationals, of degree higher than the first with rational integral coefficients, and every real root of the equation $t^{-m_k}P_k(t, tx)=0$ belongs to an algebraic field K of degree not higher than ν, and these roots are distinct for each $P(x,y)$. We shall also say that the degree of an arbitrary polynomial in $2n$ variables $P(x_1, y_1, \ldots, x_n, y_n)$ is the set of numbers $(s_1, s_2, \ldots, s_n)$, where s_i is the degree of the polynomial P in x_i and y_i together, in other words, it is the maximum k_1+k_2 for all terms $x_i^{k_1}y_i^{k_2}$ which appear in P.

THEOREM VII. *The equation*

$$(115)\qquad P_1(x_1,y_1)\ldots P_n(x_n,y_n)=P(x_1,y_1,\ldots,x_n,y_n),$$

where the system of polynomials $P_1, \ldots, P_n$ *have degrees* $p_1, \ldots, p_n$, *respectively, satisfying conditions* (A_ν), *the degree of the polynomial* P *is* $(s_1, \ldots, s_n)$ *and the inequalities* $p_k-s_k\geqslant\vartheta$, $k=1, 2, \ldots, n$ *are satisfied, cannot have an infinite number of solutions in integers* $x_1, y_1, \ldots, x_n, y_n$, *if* $\vartheta=\sqrt{2\nu}+\varepsilon$, *with* $\nu\geqslant 3$ *and* $\varepsilon>0$.

Proof. We shall assume that our equation has an infinite number of solutions. Naturally, we are speaking here about solutions which are such that

$$(x_1^2+y_1^2)\dots(x_n^2+y_n^2) \neq 0. \tag{116}$$

We now consider equation (115). A definite system of inequalities

$$|x_1| \lesseqqgtr |y_1|, \dots, |x_n| \gtreqqless |y_n| \tag{117}$$

will hold for each of its solutions.

Since there are only 2^n such systems of inequalities, and the number of solutions is infinite, there exists an infinite number of solutions for which any one of the systems is true. Interchanging the x and y in a suitable way, we obtain an infinite system of solutions such that $|y_1| \leqslant |x_1|, \dots, |y_n| \leqslant |x_n|$. Therefore, we can assume from the very beginning the existence of an infinite set of such solutions of equation (115) and in the sequel we shall consider only these solutions. If $P(x, y)$ is a homogeneous polynomial with distinct zeros $\alpha_1, \alpha_2, \dots, \alpha_n$; $|x| \geqslant |y|$, then, as can easily be seen,

$$|P(x, y)| > c\left|\alpha-\frac{y}{x}\right| |x|^n, \tag{118}$$

where n is the degree of the polynomial, c does not depend on x and y, and α is that zero of the polynomial for which the quantity $|x\alpha-y|$ is the least. In virtue of this discussion, for every solution of equation (115) the inequality

$$\prod_{k=1}^{n}\left|\alpha_k-\frac{y_k}{x_k}\right| < C\left|\frac{P(x_1, y_1, \dots, x_n, y_n)}{x_1^{p_1}x_2^{p_2}\dots x_n^{p_n}}\right| < \frac{c}{|x_1x_2\dots x_n|^{\vartheta}} \tag{119}$$

holds due to the conditions of the theorem, where c does not depend on $x_1, \dots, x_n$ and α_k is one of the zeros of the polynomial $P_k(x, y)$.

Since the number of solutions of inequality (119) is infinitely large, the set $|x_1|, \dots, |x_n|$ must increase indefinitely. Therefore, we can choose an infinite subsequence of solutions of (119) in which some one of the $|x|$, say $|x_k|$ increases monotonically. From this subsequence, one can in turn choose an infinite subsequence in which either all the remaining $|x_i|$ have the same value a or still another $|x_{k_1}|$ increases monotonically. Continuing this process, which must terminate after at most n steps, we arrive at a subsequence, which is infinite, of solutions of (119), in which $|x_k|, |x_{k_1}|, \dots, |x_{k_{s-1}}|$ increase monotonically and the remaining $x_q=a$.

By varying the indices of α, x, y, we thus obtain, from inequality (119), the inequality

$$(120) \qquad \prod_{k=1}^{s} \left| \alpha_k - \frac{y_k}{x_k} \right| \prod_{k=s+1}^{n} \left| \alpha_k - \frac{y_k}{x_k} \right| < \frac{c}{a^{(n-s)\vartheta} |x_1 \ldots x_s|^{\vartheta}}$$

having an infinite number of solutions. But all the α are irrational, and each y, $|y| < |a|$ can assume at most $2|a|+1$ values. Therefore

$$(121) \qquad \prod_{s+1}^{n} \left| \alpha_k - \frac{y_k}{a} \right| \geqslant d > 0$$

for an arbitrary system $y_{s+1}, \ldots, y_n$, where d is a constant.

As a consequence we have the inequality

$$(122) \qquad \prod_{k=1}^{s} \left| \alpha_k - \frac{y_k}{x_k} \right| < \frac{c_1}{|x_1 \ldots x_s|^{\vartheta}}$$

which has an infinite number of solutions and in which all $|x_1|, \ldots, |x_s|$ increase monotonically. From this it follows that for every system of solutions at least one of the inequalities

$$(123) \qquad \left| \alpha_k - \frac{y_k}{x_k} \right| \leqslant \frac{c^{1/s}}{|x_k|^{\vartheta}}$$

is true. But the set of numbers α_k cannot have more than $p_1 \ldots p_n$ values, and the number of solutions of inequality (122) is infinite. This means that for at least one of the zeros of our polynomials the inequality (ν is the degree of α)

$$(124) \qquad \left| \alpha_k - \frac{y_k}{x_k} \right| < \frac{c_0}{|x_k|^{\vartheta}}, \quad \vartheta = \sqrt{2\nu} + \varepsilon,$$

has an infinite number of solutions. But it follows from Theorem I′ that if we set $s=1$ in it, this inequality has only a finite number of solutions and our theorem is proved for $\nu \geqslant 3$. When $\nu = 2$, it is clear that it is impossible to have an infinite number of solutions of (124).

THEOREM VIII. *Suppose the numbers* $\zeta_1, \ldots, \zeta_n, \psi_1, \ldots, \psi_m, \eta_1, \ldots, \eta_p$ *are integers in the field K none of which is an algebraic unit, and suppose A, B, C,* $ABC \neq 0$ *are numbers in the same field K, and the numbers*

$$(125) \qquad \zeta = \zeta_1 \cdots \zeta_n, \quad \psi = \psi_1 \cdots \psi_m, \quad \eta = \eta_1 \cdots \eta_p$$

are mutually relatively prime. Then the equation

$$(126) \qquad A\zeta_1^{x_1} \cdots \zeta_n^{x_n} + B\psi_1^{y_1} \cdots \psi_m^{y_m} + C\eta_1^{z_1} \cdots \eta_p^{z_p} = 0$$

can have only a finite number of solutions in nonnegative integers $x_1, \ldots, x_n, y_1, \ldots, y_m, z_1, \ldots, z_p$.

Proof. In fact, let us assume that this equation has an infinite number of solutions. We may then choose an infinite subsequence from this set, in which the same variable, for example z_1, will appear in the given solution no less frequently than in all the remaining. From equation (126) we shall then have the congruence

$$\frac{A}{B} \equiv \frac{\psi_1^{y_1} \cdots \psi_m^{y_m}}{\zeta_1^{x_1} \cdots \zeta_n^{x_n}} \left(\operatorname{mod} \frac{C}{B} \eta_1^{z_1}\right). \tag{127}$$

But this congruence can have only a finite number of solutions by Theorem III. This completes the proof of our theorem.

Inequality (110) enables us to give a new proof of the fact that the number of fields with class number one is finite.

THEOREM IX. *The least upper bound of the absolute values of the fundamental discriminants* $-D<0$, *for which the field* $K(\sqrt{-D})$ *has class number one, is finite.*

We shall now present a short proof of this theorem. Suppose $D>163$ is a fundamental class one discriminant. It is well known that D is a prime. We shall consider the primitive real character of the absolute value of D, and we shall denote it by $\chi(n)$. Then $\chi(-1)=-1$. Suppose $D_1<D^{1/3}$ is any prime and that $\chi_1(n)$ is the real character of the absolute value of D_1. Then the relation

$$\begin{aligned} L(s, \chi_1)L(s, \chi_1\chi) &= \zeta(2s)\left(1-\frac{1}{D_1^{2s}}\right) \\ &\quad + \frac{\sqrt{\pi}}{D_1} \frac{D_1^2 - D_1^{2s}}{D_1^{2s}} \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \left(\frac{D}{4}\right)^{1/2-s} \zeta(2s-1) + R(s) \end{aligned} \tag{128}$$

holds, where $|R(s)|<e^{-c_1 D^{1/2}/D_1}$ with $|s|<3$ (in the sequel $c_1, c_2, \ldots, k_1, k_2, \ldots$ are positive constants which can be effectively computed). Relation (128) is a direct generalization of the Deuring formula with $\zeta(s)L(s, \chi)=\frac{1}{2}\zeta_Q(s)$ replaced by $L(s, \chi_1)L(s, \chi_1\chi)$.

Setting $s=1$ in (128) and making use of the well-known relations

$$\lim_{s \to 1} (D_1^2 - D_1^{2s})\zeta(2s-1) = -D_1^2 \ln D_1, \quad \Gamma(\tfrac{1}{2}) = \sqrt{\pi},$$

we obtain that

$$L(1, \chi_1)L(1, \chi_1\chi) = \zeta(2)\left(1-\frac{1}{D_1^2}\right) - \frac{2\pi \ln D_1}{D_1\sqrt{D}} + R_1, \tag{129}$$

where $|R_1|<e^{-c_1 D^{1/2}/D_1}$.

It follows that, dividing by $L(1, \chi_1) = c_2$, we find the estimate

$$(130)\qquad |L(1, \chi_1\chi)| < K_1$$

which will be needed in the sequel. In order to eliminate the factor $\zeta(2)$ we take $D_2 < D^{1/3}$, $D_2 \neq D_1$, and we write for this number D_2 an equality, analogous to (129), we multiply it and equality (129) by $(1 - 1/D_1^2)$ respectively $(1 - 1/D_2^2)$ and we then subtract the relations thus obtained one from the other. We shall then have that

$$(131)\qquad \left(1 - \frac{1}{D_2^2}\right)L(1, \chi_1)L(1, \chi_1\chi) - \left(1 - \frac{1}{D_1^2}\right)L(1, \chi_2)L(1, \chi_2\chi)$$
$$= \frac{2\pi}{D_2}\left(1 - \frac{1}{D_1^2}\right)\frac{\ln D_2}{\sqrt{D}} - \frac{2\pi}{D_1}\left(1 - \frac{1}{D_2^2}\right)\frac{\ln D_1}{\sqrt{D}} + R_3,$$
$$|R_3| \leqslant |R_2| + |R_1|.$$

We shall now assume that D_1 and D_2 are fixed, $D_1 < k_2$, $D_2 < k_2$ in such a way that $\chi_1(-1) = \chi_2(-1) = 1$ (for example, $D_1 = 5$, $D_2 = 13$). We now use the well-known relations

$$L(1, \chi_1\chi) = \frac{\pi}{2\sqrt{DD_1}}H_1, \quad L(1, \chi_2\chi) = \frac{\pi}{2\sqrt{DD_2}}H_2,$$

from number theory, where H_1 and H_2 are rational integers which satisfy, as a consequence of (130), the inequalities

$$(132)\qquad |H_i| < k_3\sqrt{D}, \quad i = 1, 2.$$

Further, if $\alpha_i = t_i + u_i\sqrt{D}$ are fundamental Pell units in the field $K(\sqrt{D_i})$, $i = 1, 2$, then, as is known,

$$L(1, \chi_i) = \frac{h_i \ln \alpha_i}{2\sqrt{D_i}}; \qquad 0 < h_i < k_4, \quad i = 1, 2,$$

where the h_i are integers.

Substituting the expressions found for the values of the function L into relation (131), cancelling out $\pi/\sqrt{D}$ in the resultant equation, and multiplying it by the common rational integral denominator, we arrive at the inequality

$$(133)\qquad |x_1 \ln \alpha_1 + x_2 \ln \alpha_2 + x_3 \ln \alpha_3| < e^{-c_3\sqrt{D}},$$

where α_1 and α_2 are fixed algebraic numbers, α_3 is a rational number with numerator and denominator bounded in absolute value, and x_1, x_2, x_3 are rational integers,

$$(134) \qquad |x_i| < k_5\sqrt{D}, \quad i = 1, 2, \quad |x_3| < k_5,$$

where c_3 and k_5 do not depend on D.

Assuming that D can take on arbitrarily large values, we arrive at a contradiction with Theorem IV, which completes the proof of Theorem IX.

In concluding this chapter, one may note that the criteria for transcendence of numbers which depend on the character of approximating numbers by rational fractions or algebraic numbers can be used directly only for the proof of the transcendence of numbers which can be represented by rapidly convergent expansions, whose terms possess sufficiently good arithmetic properties. In order to prove the transcendence of numbers defined as the values of analytic functions with algebraic values of the argument, rather complicated analytic methods are already necessary, which we shall discuss in the following chapters.

In view of the constructive complications of these methods, we shall precede their precise discussion in each case with a brief scheme of the structure of the corresponding discussions, which, in our opinion, ought to facilitate the understanding of the basic ideas of the method.

CHAPTER II

Transcendence of Values of Analytic Functions whose Taylor Series have Algebraic Coefficients

§1. Introduction. The Hermite and Lindemann theorems

As was already mentioned above (§1, Chapter I), Euler was the first to formulate a number of problems on the transcendence, or in other words the nonalgebraic nature, of some general classes of numbers. The first examples of transcendental numbers were constructed using the Liouville inequality

$$\left|\alpha - \frac{p}{q}\right| > \frac{c}{q^{\nu}}, \quad (p, q) = 1,$$

where α is an algebraic number of degree ν and c is independent of q.

The questions concerning the arithmetic nature of the classical constants e and π were present long ago, long before the work of Liouville. The question of the arithmetic nature of the number π is so much the more interesting because of the fact that the affirmative or negative solution of the problem of the quadrature of the circle, with which even ancient Greek mathematics occupied itself, depended on the arithmetic nature of the number π. The problem of the quadrature of the circle, i.e. the problem of constructing with straight edge and compasses a square whose area equals that of a prescribed circle, would be solved in the negative in case the number π were transcendental, since, as was shown in the last century, only the roots of certain classes of algebraic equations, whose coefficients are integers, can be constructed with the use of a straight edge and compasses. The numbers π and e are interrelated by the remarkable Euler formula. It was only in the second half of the last century, in 1873, that Hermite [1] related the arithmetic nature of the values of a function at an algebraic point with its analytic behavior and the arithmetic nature of its coefficients.

Hermite's relationship is given in a very particular form and was not recognized by him in all its generality. Hermite proved the transcendence of the base of natural logarithms, that is of the number e.

Hermite's proof is based on a certain identity for the function e^x. Suppose $f(x)$ is any polynomial in x; we set

$$F(x) = \sum_{k=0}^{\infty} f^{(k)}(x).$$

It is clear that $F(x)$ is also a polynomial of the same degree as $f(x)$. Then, integrating by parts, we immediately arrive at the identity

$$e^x F(0) - F(x) = e^x \int_0^x e^{-t} f(t)\,dt. \tag{1}$$

In fact, integrating by parts, we get

$$e^x \int_0^x e^{-t} f(t)\,dt = -e^{-t} f(t)\Big]_0^x e^x + e^x \int_0^x e^{-t} f'(t)\,dt$$
$$= f(0)e^x - f(x) + e^x \int_0^x e^{-t} f'(t)\,dt.$$

Integrating by parts further, we obviously arrive at relation (1) after a finite number of steps. We shall now assume that the number e satisfies an algebraic equation with rational integral coefficients

$$a_0 + a_1 e + \cdots + a_n e^n = 0, \quad a_0 \neq 0.$$

Setting $x = k$ in identity (1), multiplying the equation thus obtained by a_k and then adding these equations, we arrive at the relation

$$F(0) \sum_{k=0}^{n} a_k e^k - \sum_{k=0}^{n} a_k F(k) = \sum_{k=0}^{n} a_k e^k \int_0^k e^{-t} f(t)\,dt,$$

or, as a consequence of the fact that e satisfies our equation, at the equation

$$a_0 F(0) + \sum_{k=1}^{n} a_k F(k) = -\sum_{k=0}^{n} a_k e^k \int_0^k e^{-t} f(t)\,dt. \tag{2}$$

We now choose $f(t)$ by setting

$$f(t) = \frac{1}{(p-1)!}\, t^{p-1} \prod_{k=1}^{n} (k-t)^p, \quad p > n + |a_0|,$$

where p is a prime. We shall show first of all that the left member of equation (2) is integral and nonzero. Since $f(t)$ has a zero of multiplicity $p-1$ at $t=0$, making use of the formula for the derivative of a product, we shall have that

$$(3) \quad f^{(p-1)}(0) = (n!)^p; \qquad f^{(k)}(0) = 0, \quad k < p-1;$$

$$f^{(k)}(0) = \frac{k!}{(p-1)!(k-p+1)!} \frac{d^{k-p+1}}{dt^{k-p+1}} \prod_{k=1}^{n} (k-t)^p \Big]_{t=0}, \quad k \geqslant p.$$

Hence, we see that $f^{(k)}(0)$ is integral for arbitrary k, and that $f^{(p-1)}(0)$ is not divisible by p since p is a prime, $p>n$, but $f^{(k)}(0)$ is divisible by p for $k \geqslant p$ since p necessarily appears as a factor of the integer $f^{(k)}(0)$ upon differentiation of the product $\prod_{k=1}^{n} (k-t)^p$. Therefore, the number $F(0)$ will be integral and since all the terms in the sum which represents $F(0)$, except $f^{(k)}(0)$, are divisible by p, it will be nonzero.

We shall prove that the second term in the left member of equality (2) is integral also and that it is divisible by p. In fact, since $f(t)$ has a zero of multiplicity p at $t=m$, $1 \leqslant m \leqslant n$, m integral, we have that $f^{(k)}(m)=0$ for $0 \leqslant k \leqslant p-1$, and that

$$(4) \qquad f^{(k)}(m) = -p \frac{k!}{p!(k-p)!} \frac{d^{k-p}}{dt^{k-p}} \left[\frac{t^{p-1} \prod_{s=1}^{n} (s-t)^p}{(m-t)^p} \right] \Bigg]_{t=m}$$

for $k \geqslant p$. Our assertion follows directly from the last formula since all the terms of $F(m)$ are integral multiples of p. But since $|a_0| < p$, $a_0 F(0)$ is an integer which is not divisible by the prime p, and the sum

$$\sum_{1}^{n} a_k F(k)$$

is an integer divisible by p, the left member of equality (2) is also integral and is not divisible by p, in other words, it is a nonzero number. Since the left member of (2) is integral and nonzero, it must be in absolute value greater than or equal to unity for arbitrary sufficiently large p, $p > n + |a_0|$.

But the right member of (2), as can easily be shown, tends to zero as the prime p increases. In fact,

$$(5) \qquad \left|\sum_0^n a_k e^k \int_0^k e^{-t} f(t)\,dt\right| < e^n\left(\sum_{k=1}^n |a_k|\right)\int_0^n |f(t)|\,dt$$

$$< \frac{ne^n \sum_{k=1}^n |a_k|}{(p-1)!}(2n)^{(n+1)p} = \frac{Ca^p}{(p-1)!},$$

where C and a are independent of p. But it is already clear from this inequality that the right member of equality (2) tends to zero as p increases, since $(p-1)!$ increases more rapidly than any constant to the power p. Choosing p so large that the right member of (2) is in absolute value less than $\frac{1}{2}$, we arrive at a contradiction because the left member of this equality must not be less than 1 in absolute value. Hence, the contradiction at which we arrived by assuming that e is an algebraic number proves it is transcendental.

Some time passed after the appearance of Hermite's work before Lindemann [1], [2], in 1882, used Hermite's identity to prove a general theorem on the nature of the values of the function e^x for algebraic values of its argument. This is the so-called Lindemann theorem. Suppose $\alpha_1, \alpha_2, \ldots, \alpha_s$ are arbitrary distinct algebraic numbers and that $A_1, A_2, \ldots, A_s$ are arbitrary nonzero algebraic numbers. Then the relation

$$(6) \qquad A_1 e^{\alpha_1} + A_2 e^{\alpha_2} + \cdots + A_s e^{\alpha_2} = 0$$

cannot hold. The transcendence of the number π follows at once from this theorem and at the same time also the negative solution of the problem of the quadrature of the circle because the assumption that π is algebraic leads, in virtue of Euler's identity, $e^{2\pi i} = 1$, to a relation of the type (6).

The course of the proof of this general theorem is the same as that of the proof of Hermite's theorem and differs from it only in its technical complications.

We shall now prove Lindemann's theorem. We note first of all that if all the numbers A_k, $1 \leqslant k \leqslant n$, and B_s, $1 \leqslant s \leqslant m$, are different from zero, and the numbers α_k, $1 \leqslant k \leqslant n$ and β_s, $1 \leqslant s \leqslant m$ are distinct, then the equality

$$\left(\sum_{k=1}^{n} A_k e^{\alpha_k x}\right)\left(\sum_{s=1}^{m} B_s e^{\beta_s x}\right) = \sum_{r=1}^{p} C_r e^{\gamma_r x}, \tag{7}$$

$$\gamma_r = \alpha_k + \beta_s,$$

where similar terms have been combined and all γ_r are distinct, has at least one nonzero C_r. In fact, we may assume, without loss of generality, that the numbers α_k, as well as the β_s, are arranged in order of increasing magnitudes of their real parts, and in case their real parts are equal, in order of increasing magnitudes of their coefficients of i. We may also assume that the numbers γ_r are arranged in the same order. Then $\gamma_p = \alpha_n + \beta_m$ and this is the only term in the right member of (7) having exponent γ_p inasmuch as any sum of the form $\alpha_k + \beta_s$, where either k or s is respectively less than n or m, has either a real part which is less than the real part of $\alpha_n + \beta_m$, or, with identical real parts, its coefficient of i is less than that of $\alpha_n + \beta_m$. Therefore, $C_p = A_n B_m \neq 0$.

We shall now assume that Lindemann's theorem is not true, in other words that relation (6) holds for nonzero algebraic A_k and distinct algebraic α_k. The numbers A_k may always be assumed to be algebraic integers because in the contrary case we could multiply the left member of (6) by a rational integer A so that the numbers AA_k, $k = 1, 2, \ldots, s$ are algebraic integers. The algebraic integers A_k, $k = 1, \ldots, s$ will be elements in the ring of integers of some algebraic field K_0 of degree ν. Denoting by $A_k^{(q)}$ an element of the field K_q, conjugate to K_0, which is conjugate to $A_k = A_k^{(0)}$, we shall have that the coefficients $B_{k_1, \ldots, k_s}$ in the product

$$\prod_{n=0}^{\nu-1} \sum_{k=1}^{s} A_k^{(n)} x_k = \sum_{k_1 + \cdots + k_s = \nu} \cdots \sum B_{k_1, \ldots, k_s} x_1^{k_1} \ldots x_s^{k_s} \tag{8}$$

will be rational integers since all the $A_k^{(n)}$ are algebraic integers and all the $B_{k_1, \ldots, k_s}$ are symmetric functions of the roots of an equation which is satisfied by an element which generates the algebraic field K_0. Setting $x_k = e^{\alpha_k}$, $k = 1, 2, \ldots, s$, into identity (8), we obtain, in virtue of relation (6) and the fact that $C_p \neq 0$ in equality (7), that

$$\prod_{n=0}^{\nu-1} \sum_{k=1}^{s} A_k^{(n)} e^{\alpha_k} = \sum_{k=1}^{m} B_k e^{\beta_k} = 0, \tag{9}$$

where all the β_k are algebraic and distinct, at least one of the B_k is nonzero, and all the B_k are rational integers.

Thus, assuming that equality (6) holds for algebraic A_k, we have arrived at the same equality (9) for integral B_k. Therefore we now make the hypothesis, assuming that the Lindemann theorem is not true, that equality (6) holds for nonzero rational integers A_k and for distinct algebraic α_k, $k=1, 2, \ldots, s$. Moreover, if not all the α_k in equality (6) are algebraic integers, then one can find a positive integer q such that the numbers $\beta_k = q\alpha_k$, $1 \leqslant k \leqslant s$, will be algebraic integers. Therefore it will be sufficient to prove that the equality

$$\sum_{1}^{s} A_k e^{\beta_k/q} = 0 \tag{10}$$

is impossible with rational integral nonzero A_k and distinct integral algebraic β_k. We shall again assume that equality (10) holds. All the distinct numbers β_k, $1 \leqslant k \leqslant s$, can be considered as the roots of the same algebraic equation

$$\sum_{k=0}^{m} a_k z^k = 0, \quad a_m = 1, \tag{11}$$

with rational integral coefficients a_k. We can also assume that this equation does not have equal roots since a sufficient condition that all the distinct algebraic integers β_k satisfy such an equation is that its left member, being represented in the form of a product of irreducible polynomials, contain only the first powers of distinct irreducible polynomials having the same roots, which, as is well known, cannot hold. We shall form all possible permutations of m roots of this equation taking s elements at a time. The number of such permutations will be $m(m-1)\ldots(m-s+1)$.

Denoting the roots of equation (11) by $z_1, \ldots, z_m$, we shall consider the product of $m(m-1)\ldots(m-s+1) = \mu$ factors

$$\prod \sum_{k=1}^{s} A_k e^{\frac{1}{q} z_{n_k}} = \sum_{\sum_{k=1}^{m} h_k = \mu} B_{h_1, \ldots, h_m} e^{\frac{h_1 z_1 + \cdots + h_m z_m}{q}} = 0, \tag{12}$$

taken over all possible permutations of the roots $z_{n_1}, \ldots, z_{n_s}$ of equation (11). This product is indeed equal to zero because the permutation $\beta_1, \ldots, \beta_s$ is also contained among the permutations of the roots, and this means that one of the factors of the product coincides with the left member of (11).

We now note that the coefficient B of the term e to the power $\dfrac{h_1z_1+\cdots+h_mz_m}{q}$ coincides with the coefficient of e to the power $(h_{n_1}z_1+\cdots+h_{n_m}z_m)\dfrac{1}{q}$, where $n_1, n_2, \ldots, n_m$ is any permutation of the numbers $1, 2, \ldots, m$. This is a simple consequence of the fact that the left member of (12) remains invariant upon replacing z_k by z_i and z_i by z_k. Therefore the left member of equality (12) can be written in the form

$$\sum_{k=1}^{r} C_k \sum e^{\frac{1}{q}(h_1z_{n_1}+\cdots+h_mz_{n_m})} = 0, \tag{13}$$

where the C_k are rational integers and r is the number of systems $h_1, \ldots, h_m$

$$\sum_{i=1}^{m} h_i = \mu = m(m-1)\ldots(m-s+1),$$

which cannot be gotten one from another by a simple permutation of the numbers $h_1, h_2, \ldots, h_m$, the outside sum is taken over all systems $h_1, \ldots, h_m$, distinct in the above sense, enumerated in any order, and the interior sum is taken over all the permutations of the m roots of equation (11). The left member of equation (13) can, obviously, also be written in the form

$$\sum_{k=1}^{N} D_k e^{\lambda_k/q} = 0, \tag{14}$$

where all the D_k are rational integers which are different from zero and the λ_k are distinct algebraic integers. The fact that the left member of equation (13) will, after a transformation, contain at least one member $D_k e^{\lambda_k/q}$ with $D_k \neq 0$, is a direct consequence of the remark made in connection with equation (7).

Equation (13) shows that if an algebraic integral power λ_k, satisfying an irreducible equation $\varphi(x)=0$ of degree ν, appears in equality (14), then all the remaining roots of this equation also appear as powers in the left member of (14) with the same $D_k \neq 0$. In fact, since $\lambda_k = h_1z_1+\cdots+h_mz_m$ and the product

$$\prod [z-(h_1z_1+\cdots+h_mz_m)],$$

taken over all permutations of the roots of equation (11), is a polynomial with rational integral coefficients, all the conjugates of λ_k will be roots of this equation and, by the same token, they will appear as powers in the interior sum of equality (13) with the same frequency as the λ_k. It already follows from this that

$$f(z) = q_N \prod_{k=1}^{N} \left(z - \frac{\lambda_k}{q}\right)$$

will be a polynomial with rational integral coefficients and that it will not have equal zeros because all the λ_k are distinct and its right member is a symmetric function of all the roots of a certain number of irreducible polynomials. We shall now set

$$(15) \qquad f_i(z) = \frac{1}{(p-1)!} \frac{[f(z)]^p}{z - \dfrac{\lambda_i}{q}} = \frac{1}{(p-1)!} \sum_{r,s} b_{r,s} \lambda_i^r z^s,$$

where $p > q$ is any prime. Obviously the coefficients $b_{r,s}$ in equality (15) will be rational integers. One can verify this by simple division. Replacing $f(t)$ in the Hermite identity by $f_i(t)$, we obtain the identity

$$(16) \qquad \begin{cases} e^x F_i(0) - F_i(x) = e^x \displaystyle\int_0^x e^{-t} f(t)\,dt; \\ F_i(x) = \displaystyle\sum_0^{\infty} f_i^{(k)}(x). \end{cases}$$

Setting $x = \lambda_k/q$ in this identity, multiplying both members by D_k and then summing with respect to k, we obtain, in virtue of (14), the identity

$$(17) \qquad \sum_{k=1}^{N} D_k F_i(\lambda_k/q) = -\sum_{1}^{N} e^{\lambda_k/q} \int_0^{\lambda_k/q} e^{-t} f_i(t)\,dt.$$

But, in a manner analogous to equalities (3), we have

$$(18) \qquad \begin{cases} f_i^{(p-1)}(\lambda_i/q) = q^p \displaystyle\prod_{\substack{k=1 \\ k \neq i}}^{N} (\lambda_i - \lambda_k)^p; \quad f_i^{(k)}(\lambda_i/q) = 0; \; k < p-1; \\ f_i^{(k)}(\lambda_i/q) = \dfrac{q^{Np} k!}{(p-1)!(k-p+1)!} \dfrac{d^{k-p+1}}{dt^{k-p+1}} \displaystyle\prod_{\substack{k=1 \\ k \neq i}}^{N} \left(t - \frac{\lambda_k}{q}\right)^p \Bigg|_{t = \frac{\lambda_i}{q}}, \\ \hfill k \geqslant p, \end{cases}$$

from which it immediately follows that $f_i^{(p-1)}(\lambda_i/q)$ and $(1/p) f_i^{(k)}(\lambda_i/q)$ will be polynomials in λ_i with rational integral coefficients for $k \neq p-1$. This means that

$$(19) \qquad F_i(\lambda_i/q) = \sum_{k=0}^{\infty} f_i^{(k)}(\lambda_i/q) = f_i^{(p-1)}(\lambda_i/q) + p\gamma(\lambda_i),$$

where $\gamma(x)$ is a polynomial in x with rational integral coefficients.

Further, analogous to relations (4), we shall have $f_i^{(k)}(\lambda_s/q)=0$; $0 \leqslant k \leqslant p-1$; $s \neq i$, and

$$(20) \quad \frac{1}{p} f_i^{(k)}(\lambda_s/q) = \frac{k!\,(p-1)!}{p!\,(k-p)!} \frac{d^{k-p}}{dt^{k-p}} \frac{f_i(t)}{\left(t-\dfrac{\lambda_s}{q}\right)^p} \Bigg|_{t=\frac{\lambda_s}{q}} = P_k(\lambda_s, \lambda_i)$$

for $k \geqslant p$, where $P_k(x, y)$ is a polynomial with rational integral coefficients. Therefore

$$(21) \qquad F_i(\lambda_s/q) = \sum_{k=0}^{\infty} f_i^{(k)}(\lambda_s/q) = pQ(\lambda_s, \lambda_i),$$

where $Q(x, y)$ is a polynomial with rational integral coefficients. But if $F_i(\lambda_k/q)$ appears in the left member of (17) with a coefficient $D_k \neq 0$, then, as was established above, $F_i(\bar{\lambda}_k/q)$ will appear in the left member of (17) with the same coefficient D_k, with $\bar{\lambda}_k$ any other root of an irreducible equation which is satisfied by the algebraic integer λ_k. But $F_i(x)$ is a polynomial in x and λ_i with rational coefficients. Therefore the equality

$$(22) \qquad \sum_{k=1}^{N} D_k F_i(\lambda_k/q) = \sum D_k \sum F_i(\lambda_k/q) = R(\lambda_i),$$

where the interior sum is taken over all roots of the same irreducible equation, shows that $R(\lambda_i)$ is a polynomial in λ_i with rational coefficients. On the other hand, as we have already established,

$$(23) \qquad R(\lambda_i) = f_i^{(p-1)}(\lambda_i/q) + p\omega_i,$$

where ω_i is an algebraic integer, because $(1/p)f_i^{(k)}(\lambda_s/q)$, $(k-p+1)^2+(s-i)^2 \neq 0$ is algebraic integral, $f_i^{(p-1)}(\lambda_i/q)$ is an algebraic integer, and all the D_k are rational integrals.

It follows immediately from identity (7) that

$$|R(\lambda_i)| < \frac{Ca^p}{(p-1)!}, \quad 1 \leqslant i \leqslant N,$$

where C and a are positive constants which are independent of p, i. It follows from this that

$$(24) \qquad |L| = \left| \prod_{i=1}^{N} R(\lambda_i) \right| < \frac{C^N a^{Np}}{[(p-1)!]^N}.$$

But the product $\prod_1^N R(\lambda_i)$, being the product of polynomials with rational coefficients in all the roots of a certain number of irreducible equations, must be a rational integer. On the other hand, the number L, being the product of algebraic integers, must be an algebraic integer. This means that L, being simultaneously rational and algebraic integral, must be a rational integer.

Finally, if we set

$$L_0 = \prod_1^N f_i^{(p-1)}(\lambda_i/q)$$

and note that L_0 as the product of polynomials with rational integral coefficients, taken over all roots of irreducible equations, must be an integer, we can easily verify that the integer $L-L_0$,

$$L-L_0 = \prod_1^N [f_i^{(p-1)}(\lambda_i/q)+p\omega_i] - \prod_1^N f_i^{(p-1)}(\lambda_i/q) \tag{25}$$

must be divisible by p. In fact, it is immediately clear from (25) that $(1/p)[L-L_0]$ will be an algebraic integer and, as the ratio of integers, it is simply a rational integer. But, since $\lambda_i \neq \lambda_k$, it is clear from (18) that the integer L_0 is not zero. Therefore, we can assert that if the prime $p > N+|L_0{}^{1/p}|$, then $L_0{}^{1/p}$ cannot be divisible by p. Since $L-L_0$ is divisible by p, it follows from this that the integer L cannot be zero and this means that $|L| \geqslant 1$. Inequality (24) now goes over into the inequality

$$1 < \frac{C^N a^{Np}}{[(p-1)!]^N}$$

which is true for $p > N+|L_0{}^{1/p}|$ and which no longer holds for sufficiently large prime p because C and a do not depend on p. Hence, assuming that the equality (6) holds, we have arrived at a contradiction and have, by the same token, proved the Lindemann theorem.

§2. Further development of the ideas of Hermite and Lindemann

After the works of Hermite and Lindemann, there appeared a number of papers, by very prominent mathematicians, which gave different new proofs of the Hermite and Lindemann theorems, but which did not essentially alter the basic ideas of the method of proof. Of these works, one can single out the paper by Markov [1] because the proof of the Lindemann theorem in this paper is given very precisely and the basic idea is stated very comprehensively. The above proof is a revision of Markov's proof. The Hermite identity, which lies at the base of the general Lindemann theorem, is specific for the function e^z; it is impossible, apparently, to construct an analogous identity for other functions of this type, for example for the Bessel function. A more general method which enables one to investigate the arithmetic nature of the values of a rather large class of entire functions having algebraic coefficients in their Taylor series about 0 and satisfying algebraic differential equations with polynomial coefficients, was published by Siegel [3] in 1929–1930. This method is a natural extension of the works of Hermite and Lindemann. The general theorem on algebraic independence proved by Siegel is a direct generalization of the Lindemann theorem. In this method, an essential role is played by one general idea concerning the determination of the greatest lower bounds of linear forms with integral or algebraic coefficients in powers of one or several numbers, which is a development of the idea of Thue in the theory of approximation of algebraic numbers by rational fractions.

In order to explain the basic ideas of this general method, we shall present the proof of the Lindemann theorem by this method in very general outline.

We shall first set down two lemmas which we shall need in our further discussion.

Suppose the number a belongs to some algebraic field K of degree ν and that the numbers $a_1, a_2, \ldots, a_{\nu-1}$ are the conjugates of a. We shall agree to denote in the sequel the maximum of the numbers $|a|, |a_1|, \ldots, |a_{\nu-1}|$ by the symbol $\overline{|a|}$.

LEMMA I. *If $\alpha_1, \alpha_2, \ldots, \alpha_s$ are given numbers in an algebraic field K of degree ν, and $P(x_1, x_2, \ldots, x_s)$ is a polynomial of degree n in all the variables, with integral coefficients in the field K, then either $P(\alpha_1, \ldots, \alpha_s) = 0$, or*

$$(26) \qquad |P(\alpha_1, \alpha_2, \ldots, \alpha_s)| > H^{-\nu+1}e^{-\gamma n}, \quad \gamma > 0,$$

where $H \geqslant \overline{|A_{k_1,\ldots,k_s}|}$ and $A_{k_1,\ldots,k_s}$ are the coefficients of $P(x_1, \ldots, x_s)$ and the constant γ is independent of H and n.

This lemma is a trivial consequence of the fact that the norm of an algebraic integer is not less than 1 and is a simple generalization of Lemma II, §2, Chapter I.

LEMMA II. *Suppose the coefficients $a_{k,s}$ of the linear forms*

$$L_k = a_{k,1}x_1 + \cdots + a_{k,q}x_q, \quad 1 \leqslant k \leqslant p$$

are integers in an algebraic field K of degree ν and that $\overline{|a_{k,s}|} < A$. Then there exists a solution of the system of equations $L_k = 0$, $1 \leqslant k \leqslant p$, in nonzero integers of the field K, with

$$\overline{|x_k|} < C(CqA)^{p/(q-p)}, \tag{27}$$

where C is a positive constant which does not depend on A, p and q.

Proof. Suppose $\omega_1{}^{(0)}, \omega_2{}^{(0)}, \ldots, \omega_\nu{}^{(0)}$ is any basis for the ring of integers of the field K and that $\omega_1{}^{(m)}, \omega_2{}^{(m)}, \ldots, \omega_\nu{}^{(m)}$ is a basis for the ring of integers of the field K_m, conjugate to the field K, $m = 1, 2, \ldots, \nu - 1$. Setting

$$\left\{ \begin{aligned} a_{k,s}^{(m)} &= \sum_{r=1}^{\nu} a_{k,s,r}\omega_r{}^{(m)}, \quad a_{k,s}^{(0)} = a_{k,s}; \\ x_s{}^{(m)} &= \sum_{r=1}^{\nu} x_{s,r}\omega_r{}^{(m)}, \quad x_s{}^{(0)} = x_s; \end{aligned} \right. \tag{28}$$

$$1 \leqslant k \leqslant p, \quad 1 \leqslant s \leqslant q, \quad 0 \leqslant m \leqslant \nu - 1,$$

where $a_{k,s}^{(m)}$ and $x_s{}^{(m)}$ are numbers conjugate to $a_{k,s}$ and x_s, respectively, and $a_{k,s,r}$ are given, and the $x_{s,r}$ are arbitrary rational integers, we shall have

$$\left| \sum_{r=1}^{\nu} a_{k,s,r}\omega_r{}^{(m)} \right| \leqslant A, \qquad 0 \leqslant m \leqslant \nu - 1,$$

as a consequence of the conditions of the lemma. Since the determinant $|\omega_r{}^{(m)}|$ of the system is different from zero, it follows immediately from this that

$$|a_{k,s,r}| < C_0 A \tag{29}$$

for arbitrary k, s and r, where C_0 depends only on K.

Setting the expressions for $a_{k,s}$ and x_s from (28) into the forms L_k, we shall have that

$$L_k = \sum_{r=1}^{\nu} \left[\sum_{s=1}^{q} b_{k,s,r}x_{s,r} \right] \omega_r, \quad 1 \leqslant k \leqslant p, \tag{30}$$

where $b_{k,s,r}$ are rational integers satisfying, in virtue of (29), the inequality

$$|b_{k,s,r}| < C_1 A \tag{31}$$

for arbitrary k, s and r, where C_1 depends only on K. Representation (30) shows that a necessary and sufficient condition that the conditions $L_k=0$, $1 \leqslant k \leqslant p$ be satisfied for rational integral $x_{s,r}$ is that the equations

$$L_{k,r} = \sum_{s=1}^{q} b_{k,s,r} x_{s,r} = 0; \qquad 1 \leqslant k \leqslant p, \quad 1 \leqslant r \leqslant \nu, \tag{32}$$

be satisfied. The number of equations in this system is νp, and the number of variables is $\nu q > \nu p$. Setting $N=2$, $n=\nu q$, $m=\nu p$ in Lemma I, §2, Chapter I, we can assert that there exists a system of integers $x_{s,r}$, $1 \leqslant s \leqslant q$, $1 \leqslant r \leqslant \nu$, all different from zero, satisfying the inequalities

$$|x_{s,r}| < 2[C_1 \nu q A]^{p/(q-p)} \tag{33}$$

and which also satisfy the system of inequalities

$$|L_{k,r}| \leqslant \tfrac{1}{2}, \ 1 \leqslant k \leqslant p, \ 1 \leqslant r \leqslant \nu. \tag{34}$$

But since $b_{k,s,r}$ and $x_{s,r}$ are integers, it follows immediately from inequalities (34) that $L_{k,r}=0$, $1 \leqslant k \leqslant p$, $1 \leqslant r \leqslant \nu$, for these values of $x_{s,r}$. Finally, inequality (27) of our theorem with constant C independent of A, p and q, follows directly from inequalities (33) and relation (28).

Suppose the numbers $\omega_1, \omega_2, \ldots, \omega_\nu$ form a basis for the ring of integers in the algebraic field K. We shall assume that the relation

$$T_1 = \sum_{k_1=0}^{\sigma} \cdots \sum_{k_\nu=0}^{\sigma} A_{k_1,\ldots,k_\nu} e^{\frac{k_1\omega_1+\cdots+k_\nu\omega_\nu}{d}} = 0, \tag{35}$$

$$|A_{k_1,\ldots,k_\nu}| < A,$$

holds, where d, $A_{k_1,\ldots,k_\nu}$ are nonzero rational integers. If we prove that such a relation is impossible, the Lindemann theorem will follow, as we saw above, with the aid of very simple purely algebraic arguments.

We shall subdivide the course of the proof of the impossibility of relation (35) into a sequence of five steps.

Step One. We enumerate the $q=(p+1)^\nu$ numbers

$$k_1\omega_1 + \cdots + k_\nu\omega_\nu, \quad 0 \leqslant k_i \leqslant p, \quad 1 \leqslant i \leqslant \nu,$$

where p is any sufficiently large number, in any order and we write them in the form of a sequence $d\lambda_1, d\lambda_2, \ldots, d\lambda_q$, with the first $\tau=(\sigma+1)^\nu$ of them, say, coinciding with the set of numbers $k_1\omega_1+\cdots+k_\nu\omega_\nu$; $0\leqslant k_i\leqslant\sigma$; $1\leqslant i\leqslant\nu$. Then relation (35) can be rewritten in the form

$$(36)\qquad T_1=\sum_1^q A_{k,1}e^{\lambda_k},\quad A_{k,1}=0,\quad k>\tau,\quad |A_{k,1}|\leqslant A.$$

It is easily verified that if we multiply equality (35) by

$$e^{\frac{1}{d}(k_1\omega_1+\cdots+k_\nu\omega_\nu)},\quad 0\leqslant k_i\leqslant p-\sigma,\quad 1\leqslant i\leqslant\nu,$$

we shall obtain a new relation of the type (36),

$$(37)\quad T_s=\sum_1^q A_{k,s}e^{\lambda_k}=0,\quad |A_{k,s}|\leqslant A,\quad 1\leqslant s\leqslant(p+1-\sigma)^\nu$$

where, obviously, the linear forms T_s in the quantities e^{λ_k} will be linearly independent. The number of linear forms is $(p+1-\sigma)^\nu$. Hence, assuming that relation (35) is true, we have obtained many linear forms in the quantities e^{λ_k} which are linearly independent.

Step Two. Suppose $N>e^{q^2\ln^2 q}$ is a sufficiently large integer. We shall consider the function

$$(38)\qquad f(z)=\sum_1^q P_m(z)e^{\lambda_m z},\quad P_m(z)=N!\sum_{k=0}^N C_{k,m}\frac{z^k}{k!}.$$

An almost direct consequence of Lemma II is that there exist nonzero algebraic integers in the field K, $C_{k,m}$, $\overline{|C_{k,m}|}<e^{\gamma_0 Nq^2}$, where γ_0 is independent of both N and q, such that the expansion of the function $f(z)$ in a Maclaurin series will begin with z to a power not less than $N(q-1)+2q$. Considering the function $f(z)$ chosen in this way, we easily obtain that

$$(39)\qquad f(z)=z^{N(q-1)+2q}\varphi(z),\quad |\varphi(z)|<e^{-\gamma_1 N(q-2)\ln N}$$

with $|z|\leqslant 4$, where γ_1 is again independent of N and q.

Step Three. Differentiating the function $f(z)$, we obtain a system of linear forms in $e^{\lambda_k z}$, $k=1,2,\ldots,q$,

$$(40)\qquad \begin{cases} L_s(z)=f^{(s)}(z)=\sum\limits_1^q U_{m,s}(z)e^{\lambda_m z},\\[2ex] U_{m,s}(z)=d^{-s}\sum\limits_{k=0}^N C_{m,k,s}z^k\end{cases}$$

where all the numbers $C_{m,k,s}$ are algebraic integers with absolute value $|\overline{C_{m,k,s}}|$ less than or equal to N^{3N} for $s \leqslant N$ and $N > N_0$. Since the function $f(z)$ has a zero of order at least $N(q-1)+2q$ at the origin, then, as can easily be verified by successive differentiation and factoring out the exponential functions $e^{\lambda_k z}$ none of the polynomials $P_m(z)$ in the representation (38) of the function $f(z)$ can vanish identically. It follows from this and from the fact that the function, consisting of the sum of derivatives of polynomials in the various $e^{\lambda_m z}$, can never vanish identically, we may assert that the linear forms $L_0(z), \ldots, L_{q-1}(z)$ are linearly independent; in other words, we may assert that the determinant

$$(41)\quad \Delta(z) = \begin{vmatrix} U_{1,0}(z) & \cdots & U_{q,0}(z) \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ U_{1,q-1}(z) & & U_{q,q-1}(z) \end{vmatrix}$$

$$= \begin{vmatrix} f(z)U_{2,0}(z) & \cdots & U_{q,0}(z) \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ f^{(q-1)}(z)U_{2,q-1}(z) & \cdots & U_{q,q-1}(z) \end{vmatrix} = z^{N(q-1)+q+1}\psi(z)$$

does not vanish identically. Since the degree of this determinant with respect to z is less than or equal to Nq, then, having a zero of order less than or equal to $N(q-1)+q+1$ at the origin, it cannot have a zero of order higher than $N-q-1$ for any nonzero value of z.

Step Four. If we choose arbitrarily q of the forms with indexes $s_1, s_2, \ldots, s_q$ from the $N+1$ linear forms $L_0(z), \ldots, L_N(z)$, and consider the determinants of such systems $\Delta(z, s_1, \ldots, s_q)$, we may assert that at least one of these determinants will be different from zero for $z=1$. For in the contrary case, $\Delta(z)$ would have a zero of order greater than or equal to $N-q$ at the point $z=1$, which is impossible. Taking into consideration the estimate for the coefficients of the forms $L_s(z)$ with $s \leqslant N$ introduced above, i.e. the estimate (39), and changing the enumeration of our forms, we may now assert that there exists a system of q linear forms in the q quantities e^{λ_k}, $k=1, 2, \ldots, q$, whose determinant is not equal to zero and such that

$$L_i = \sum_{s=1}^{q} C_{i,s}e^{\lambda_s}, \quad |\overline{C_{i,s}}|d^N < N^{4N}, \quad |d^N L_i| < e^{-\gamma_2 N(q-3)\ln N},$$

where all the $C_{i,s}d^N$ are algebraic integers and the $\gamma_2 > 0$ are independent of q and N. Recalling that we have $r = (p+1-\sigma)^\nu$ linearly independent forms $T_1, \ldots, T_r$ in the same quantities e^{λ_k} at our disposal, we may obviously choose $q-r$ of the q forms L, which together with the r forms T form a system of linearly independent forms. Inasmuch as it is possible to change the enumeration of the forms L, we may assume that the forms $L_1, L_2, \ldots, L_{q-r}$, $T_1, \ldots, T_r$ are linearly independent. The determinant D of this system of forms is different from zero and will have the form

$$D = \begin{vmatrix} B_{1,1} & B_{1,2} & \ldots & B_{1,q} \\ \ldots & \ldots & \ldots & \ldots \\ B_{q-r,1} & B_{q-r,2} & & B_{q-r,q} \\ A_{1,1} & A_{2,1} & & A_{q,1} \\ \ldots & \ldots & \ldots & \ldots \\ A_{1,r} & A_{2,r} & & A_{q,r} \end{vmatrix}, \quad B_{i,s} = C_{n_i,s}.$$

Step Five. This nonzero determinant D can be estimated in absolute value from below by Lemma I and the estimates for the quantities A and $|\overline{C}|$, and it can be estimated from above since we know the estimates for the absolute values of the forms L. These estimates give us the inequalities

$$(42) \quad A^{-\nu r}e^{-4\nu(q-r)N\ln N-\gamma_3 Nq} < |D| < [e^{-\gamma_2 N(q-3)\ln N} + |T_1|]e^{\gamma_4(q-r)N\ln N}A^{\nu r},$$

where γ_3 and γ_4 depend only on K. Since $q-r = (p+1)^\nu - (p+1-\sigma)^\nu < \nu\sigma q^{\frac{\nu-1}{\nu}}$, one can choose p so large that the inequality $(\gamma_4 + 5\nu)(q-r) < \gamma_2(q-3)$ holds. Choosing p in this manner and setting $T_1 = 0$ (this is our basic assumption), for sufficiently large N, which can be increased indefinitely, we arrive at a contradiction, inasmuch as then the right member of inequality (42) becomes less than the left. This completes the proof of the Lindemann theorem.

Inequality (42) enables us not only to prove this latter theorem but also to establish the greatest lower bound of the form (35) in terms of its height A and its degree σ. We shall return later to the problem of finding such greatest lower bounds.

Considering the various steps into which we subdivided the proof of the Lindemann theorem, we may make the following assertion. The first step consists in this that from one algebraic relation with integral coefficients for the powers $e^{\omega_1}, e^{\omega_2}, \ldots, e^{\omega_\nu}$ we obtain many relations also with integral coefficients. We obtain here, in this way, not less than $q-\sigma\nu q^{\frac{\nu-1}{\nu}}$ linearly independent forms in the q numbers e^{λ_k}, whose height is fixed. The second step consists in the construction of a function formed from functions which generate our numbers e^{λ_k} for $z=1$, which function, having relatively small algebraic coefficients in the field K, small in the algebraic sense, since its Taylor expansion begins with a high power of z, and so is in absolute value at a finite distance from the origin. This function serves as the basis for the construction of a system of linearly independent functional forms. The third step consists in the construction of a system of functional linear forms where the fact that the function so constructed is small algebraically serves as the basis for the presence of all powers $e^{\lambda_k z}$ in these forms and by the same token for the linear independence of q consecutive functions. The determinant of this system cannot be divisible by $z-1$ to a power greater than $N-q-1$ for the same reason. The most essential point in this connection is the fact that in the differentiation of these forms we again obtain forms in the previous functions with algebraic coefficients. This is a consequence of the fact that our fundamental functions satisfy a system of algebraic equations with polynomial coefficients, whose coefficients in turn are algebraic numbers. When we are dealing with functions which do not possess this last property, this method is essentially not applicable, and the problem of investigating the arithmetic nature of their values becomes immeasurably more difficult. Step four consists in the choice of q numerical, linearly independent forms from the N functional forms obtained in the preceding step by putting $z=1$. Combining these forms with those we had previously, we again obtain q numerical, linearly independent forms of which there are no less than $q-\sigma\nu q^{\frac{\nu-1}{\nu}}$, i.e. our fundamental set, which are equal to zero in absolute value, and have fixed coefficients. The fifth, and last, step consists in the estimation of the determinant of this combined system of forms from above as well as from below, which leads to a contradiction for sufficiently large values of the parameters q and N. This method, whose limits of applicability are already sufficiently clear, enabled Siegel to prove a number of transcendence theorems. We shall formulate several of them.

Suppose $\zeta, \alpha_1, \ldots, \alpha_n$ are algebraic numbers where, say, ζ is nonzero, and the numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ are distinct. Suppose also that $P_1(x, y), \ldots, P_n(x, y)$ are polynomials with algebraic coefficients not all identically zero. Then

$$P_1[J_0(\zeta), J_0'(\zeta)]e^{\alpha_1} + \cdots + P_n[J_0(\zeta), \quad J_0'(\zeta)]e^{\alpha_n} \neq 0,$$

where $J_0(x)$ is the well-known Bessel function. In the case where the left member consists of one polynomial $P(x, y)$, ζ is an algebraic number of degree m, and the height and degree of the polynomial $P(x, y)$ are H and p, respectively, then the inequality

$$|P[J_0(\zeta), \quad J_0'(\zeta)]| > CH^{-123p^2m^3} \tag{43}$$

holds, where the coefficients of $P(x, y)$ are rational integers. The class of functions to which this method is applicable may, in particular, be described in the following way: The function $y = \sum_{n=0}^{\infty} \frac{a_n}{b_n} \frac{z^n}{n!}$ belongs to this class if 1) all the numbers a_n are integers in the same field K, where the absolute values of the numbers a_n as well as of their conjugates increase more slowly than any arbitrarily small power of $n!$; 2) all the denominators b_n are rational integers where the least common multiple of the first n denominators increases more slowly than any arbitrarily small power of $n!$; and 3) the function $y = y(z)$ must satisfy a linear differential equation with polynomial coefficients, which in their turn have algebraic numbers as coefficients. If such a function y satisfies a linear differential equation of degree s then every step in the proof of the Lindemann theorem is applicable to it, and, in general, the functions $y^{k_1}[y']^{k_2} \cdots [y^{(s-1)}]^{k_s}$ play the role of the function $e^{\lambda_k z}$.

Purely qualitative facts on the transcendence of a number α or the algebraic independence of a system of numbers $\alpha_1, \alpha_2, \ldots, \alpha_s$ over the field of rational numbers, take on a quantitative character if we introduce the so-called *measure of transcendence of the number* α or, more generally, the measure of mutual transcendence of a system of numbers $\alpha_1, \alpha_2, \ldots, \alpha_s$. The measure of mutual transcendence of the system of numbers $\alpha_1, \alpha_2, \ldots, \alpha_s$ is the function $\Phi(H, n_1, \ldots, n_s; \alpha_1, \ldots, \alpha_s)$ where

$$\Phi(H, n_1, \ldots, n_s; \ \alpha_1, \ldots, \alpha_s) = \min | P(\alpha_1, \ldots, \omega_s)|, \tag{44}$$

with $P(x_1, \ldots, x_s)$ a polynomial with rational integral coefficients, whose height is less than or equal to H, its degree in $x_1, x_2, \ldots, x_s$ is less than or equal to $n_1, n_2, \ldots, n_s$, respectively, and the minimum in the right member is taken over all polynomials satisfying these conditions. It is sometimes convenient to consider the degree of the polynomial $P(x_1, \ldots, x_s)$ with respect to all the variables $x_1, \ldots, x_s$. Denoting this degree by n, we shall then write our transcendence measure in the form $\Phi(H, n; \alpha_1, \ldots, \alpha_s)$. The fundamental inequalities, which the measure of transcendence will always satisfy will be the following:

$$\Phi(H, n_1, \ldots, n_s; \alpha_1, \ldots, \alpha_s) < e^{\lambda \sum_1^s n_i} H^{-\tau(n_1+1) \ldots (n_s+1)+1} \tag{45}$$

and, analogously

$$\Phi(H, n; \alpha_1, \ldots, \alpha_s) < e^{\lambda n} H^{-\tau \frac{(n+s)!}{n!s!}+1}, \tag{46}$$

where $\lambda > 0$ does not depend on the height H and the degrees $n_1, \ldots, n_s$ or n, and τ equals 1 if all the numbers $\alpha_1, \alpha_2, \ldots, \alpha_s$ are real and τ equals $\frac{1}{2}$ if at least one of the α_i is complex. These inequalities are proved directly with the aid of Lemma I (the Dirichlet principle), §2, Chapter I.

Various classifications of transcendental numbers are based on the behavior of the transcendence measure, as for example, the Morduhai-Boltovskoi or Mahler classifications. We shall not stop to consider these classifications further in view of the insufficient effectiveness or the absence of real criteria that a number belong to one class or another. Here we shall only state the definition of a Liouville number with the aid of the concept of transcendence measure. The number α is called a *Liouville number* if it is irrational and

$$\lim_{H \to \infty} \frac{\ln \Phi(H, 1; \alpha)}{\ln H} = -\infty. \tag{47}$$

A great number of mathematicians undertook the investigation of the behavior of the measure of transcendence, in particular, Morduhai-Boltovskoi [1, 2], who was the first to give an estimate of the measure of transcendence $\Phi(H, n; e^{\omega_1}, \ldots, e^{\omega_\nu})$ and $\Phi(H, n; \ln \alpha)$, Siegel, Mahler, Gelfond, Feldman, and many others. Here we shall introduce only some of the more precise of the known results concerning the greatest lower bound of the measure of transcendence. Mahler [2] proved that if $\alpha_1, \alpha_2, \ldots, \alpha_s$ are algebraic numbers which are linearly independent over the field of rationals, then the inequality

$$\Phi(H, n_1, \ldots, n_s; e^{\alpha_1}, \ldots, e^{\alpha_s}) > H^{-\lambda n_1 \cdots n_s}, \quad H > H_0 \tag{48}$$

holds, where λ is independent of $H, n_1, \ldots, n_s$. Mahler also gave the more precise inequalities

(49) $\Phi(H, n;\ e) > H^{-n-\frac{C_0 n^2 \ln(n+1)}{\ln\ln H}};\quad \Phi(H, n; \ln\alpha) > H^{-C^n},$

where C_0 is an absolute constant, α is an algebraic number, and $C>1$ is an arbitrary number. The latter inequalities were essentially obtained for fixed n and increasing H. The method of obtaining all these inequalities for this measure is the quantitative formulation of the ideas of Hermite-Siegel. The corresponding estimate of the measure for the values of cylindrical functions at algebraic points was introduced above (inequality (43)). An inequality for the measure of transcendence of the number π which is significantly more precise than inequality (49) when n increases with H, was obtained relatively recently by Feldman. His method of obtaining the inequality is essentially different from the methods discussed in the present section, and we shall discuss this method further in the following chapter. Feldman [1] established the inequalities

(50) $\Phi(H, n;\ \pi) > e^{-\gamma_0 n N},\ N = \max\,[\ln H \ln\ln H, n \ln^2 \mathrm{n}]$

and

(51) $\Phi(H, n; \ln\alpha) > e^{-\gamma n^2(1+\ln n)N},\ N = \max\,[\ln H \ln\ln H, \mathrm{n} \ln^2 n]$

for algebraic $\alpha \neq 0, 1$, where $\gamma_0 > 0$ is an absolute constant, and γ depends only on $\ln\alpha$. For $n \gg \ln\ln H$, inequalities (50) and (51) are essentially more precise than all the previously existing inequalities and come significantly closer to the least upper bound for the transcendence measure given by inequalities (45) and (46). The essential advantage of inequalities (50) and (51) over those previously available is the fact that they hold for arbitrary n and H.

In the following sections we shall give a complete discussion of the method developed by Siegel which is applicable to the proof of the transcendence and the algebraic independence of the values of an E-function, the definition of which will be given below, for algebraic values of the argument.

§3. Auxiliary propositions and definitions

An entire analytic function $f(z)$,

$$f(z) = \sum_0^\infty C_n \frac{z^n}{n!}$$

is called an E-function if the following conditions are satisfied:

1) *all the coefficients C_n of the function $f(z)$ belong to a finite algebraic field K;*

2) $\overline{|C_n|} = O(n^{n\varepsilon})$ *for all $\varepsilon > 0$;*

3) *if $q_n > 0$ is the smallest rational integer for which the numbers $C_k q_n$, $k = 0, 1, \ldots, n$, are algebraic integers, then $q_n = O(n^{n\varepsilon})$ for arbitrary $\varepsilon > 0$.*

Obviously, the function $e^{\omega z}$ will be an E-function when ω is algebraic. The function

$$f(z) = \sum_{n=k}^{\infty} \frac{a_n}{([n/k]!)^k} z^n$$

will also be an E-function if a_n is rational integral and $a_n = O(n^{n\varepsilon})$ for arbitrary $\varepsilon > 0$.

An arbitrary polynomial with algebraic coefficients will obviously be an E-function because all the conditions for an E-function are satisfied trivially in this case. If in the sequel we shall consider a finite set of E-functions, we shall assume in advance that their coefficients belong to the same finite algebraic field K, in which the coefficient fields of all these functions are embedded.

If $f_k(z) = \sum_{s=0}^{\infty} a_{k,s} \frac{z^s}{s!}$, $1 \leqslant k \leqslant m$, is an E-function and $q_{k,n}$ are integers such that $q_{k,n} a_{k,s}$ are algebraic integers for $s \leqslant n$, then clearly we shall have $q_n = O(n^{n\varepsilon})$ and that $q_n a_{k,s}$ are algebraic integers for $1 \leqslant k \leqslant m$, $s \leqslant n$ if we set $q_n = q_{1,n} \cdots q_{m,n}$. In other words, if we are dealing with a finite number of E-functions, then the sequence of numbers q_n, which are such that if the coefficients $a_{k,s}$, $s \leqslant n$ are multiplied by them, become algebraic integers, can be chosen as the standard sequence, which does not depend on k.

We may also assert that the sum and product of a finite number of E-functions is also an E-function.

It suffices to prove this assertion only for the sum and product of two E-functions. In fact, if $f_1(z)$ and $f_2(z)$,

$$f_1(z) = \sum_{0}^{\infty} \frac{a_n}{n!} z^n, \quad f_2(z) = \sum_{0}^{\infty} \frac{b_n}{n!} z^n,$$

are E-functions, then there exist two sequences of rational integers q_n and p_n such that the numbers $q_n a_k$, $0 \leqslant k \leqslant n$, and $p_n b_k$, $0 \leqslant k \leqslant n$, will be algebraic integers. But then the numbers $r_n = p_n q_n = O(n^{\varepsilon n})$ will be such that the numbers $r_n(a_k + b_k)$, $0 \leqslant k \leqslant n$, and the numbers $r_n a_s b_k$, $0 \leqslant s$, $k \leqslant n$, turn out to be algebraic integers.

Therefore the functions

$$F_1(z) = \sum_0^\infty \frac{A_n}{n!} z^n = f_1(z) + f_2(z), \quad A_n = a_n + b_n,$$

$$F_2(z) = \sum_0^\infty \frac{B_n}{n!} z^n = f_1(z) f_2(z), \quad B_n = \sum_{k=0}^n \frac{n!}{k!(n-k)!} a_k b_{n-k}$$

will be E-functions since

$$\overline{|A_n|} \leqslant \overline{|a_n|} + \overline{|b_n|} = O(n^{\varepsilon n}),$$

$$|B_n| \leqslant \sum_0^n \frac{n!}{k!(n-k)!} \overline{|a_k|}\,\overline{|b_{n-k}|} = 2^n O(n^{2\varepsilon}) = O(n^{\delta n})$$

for arbitrary $\delta > 0$.

Finally, one can assert that an arbitrary derivative of an E-function is an E-function. To prove this assertion, it suffices to show that the first derivative of an E-function is again an E-function. In fact, the coefficients a_n of the function $f'(z)$, where

$f(z) = \sum_0^\infty \frac{c_n}{n!} z^n$, with $f(z)$ an E-function, will have the form

$a_n = c_{n+1}$, from which it follows directly that they satisfy all the conditions for an E-function. Summarizing the above discussion, we may now assert that if

$$F(z) = \sum_{k=1}^n P_k(z) f_k(z),$$

where the $P_k(z)$ are polynomials with algebraic coefficients and the $f_k(z)$ are E-functions, then $F^{(s)}(z)$ will also be an E-function.

In the sequel we shall consider and study the arithmetic properties of only those sets of E-functions $f_1(z), \ldots, f_m(z)$ which satisfy a system of first order differential equations

$$f_s'(z) = \sum_{k=1}^m Q_{k,s}(z) f_k(z), \quad 1 \leqslant s \leqslant m, \tag{52}$$

where $Q_{k,s}(z)$, $1 \leqslant k \leqslant m$, $1 \leqslant s \leqslant m$, are rational functions with algebraic coefficients in their numerators and denominators.

As we have already seen in a particular example, when we considered the course of the proof of the Lindemann theorem by Siegel's method, a very essential role in the investigation of the arithmetic nature of an E-function is played by linear forms of E-functions with polynomial coefficients, whose Maclaurin series expansion begins with a very high power of z. The possibility of constructing such linear forms, which we shall call approximating forms, is given in the general case by Lemma III.

LEMMA III. *Suppose $f_1(z), f_n(z), \ldots, f_m(z)$ are E-functions with coefficients in a finite algebraic field K and n is an arbitrary large integer. Then there exist m polynomials $P_1(z), \ldots, P_m(z)$, whose degrees are less than or equal to $2n-1$, such that the following conditions are satisfied:*

1) *the coefficients of the polynomials $P_1(z), \ldots, P_m(z)$ are nonzero integers in a field K, where these coefficients $A_{i,k}$, $P_k(z) = \sum_{i=0}^{2n-1} A_{i,k} z^i$, satisfy the conditions*

$$\overline{|A_{i,k}|} = O[n^{(2+\varepsilon)n}], \quad \varepsilon > 0, \tag{53}$$

for arbitrary $\varepsilon > 0$ uniformly in i and k, $k \leqslant m$, $i \leqslant 2n-1$;

2) *the coefficients a_ν of the linear form*

$$R(z) = \sum P_k(z) f_k(z) = \sum_{\nu=0}^{\infty} a_\nu \frac{z^\nu}{\nu!} \tag{54}$$

vanish for $\nu \leqslant 2mn-n-1$; in other words,

$$a_\nu = 0, \quad 0 \leqslant \nu \leqslant 2mn-n-1 \tag{55}$$

and

$$|a_\nu| = \nu^{\varepsilon\nu} O(n^{2n}), \quad \varepsilon > 0,\ n > n_0(\varepsilon) \tag{56}$$

uniformly in ν, for $\nu \geqslant 2mn-n$, where ε is arbitrarily small.

Proof. Suppose

$$\begin{gathered} f_k(z) = \sum_{\nu=0}^{\infty} C_{k,\nu} \frac{z^\nu}{\nu!}, \\ P_k(z) = (2n-1)! \sum_{\nu=0}^{2n-1} g_{k,\nu} \frac{z^\nu}{\nu!},\ k = 1, 2, \ldots, m, \end{gathered} \tag{57}$$

where the $g_{k,\nu}$ are certain integers in the field K. Introducing the notation

$$(58)\qquad \begin{cases} P_k(z)f_k(z) = (2n-1)!\sum\limits_{\nu=0}^{\infty} d_{k,\nu}\dfrac{z^\nu}{\nu!}; \\ d_{k,\nu} = \sum\limits_{p=0}^{\nu} \dfrac{\nu!}{p!(\nu-p)!} g_{k,p}C_{k,\nu-p}, \end{cases}$$

where $g_{k,p}=0$ for $p>2n-1$, we obtain

$$a_\nu = (2n-1)!\sum_{s=1}^{m} d_{s,\nu}.$$

Since all the functions $f_k(z)$ belong to the class of E-functions, there exists a sequence of positive rational integers q_n which are such that the products $q_nC_{k,\nu}$, $\nu\leqslant n$, are algebraic integers. It follows from the definition of an E-function that

$$(59)\qquad \overline{|C_{k,\nu}|} = O(\nu^{\varepsilon\nu}),\quad q_n = O(n^{\varepsilon n}).$$

In order that the conditions (55) of our lemma be satisfied, it is necessary that the system

$$(60)\qquad \frac{q_\nu a_\nu}{(2n-1)!} = \sum_{k=0}^{m}\sum_{s=0}^{\nu}\frac{\nu!}{s!(\nu-s)!}q_\nu C_{k,\nu-s}g_{k,s} = 0,$$
$$0 \leqslant \nu \leqslant 2mn-n-1$$

of equations, linear in $g_{k,\nu}$, be satisfied. In virtue of Lemma II, §2, this chapter, this system of equations with $2mn$ unknowns $g_{k,s}$ can be solved in algebraic integers $g_{k,s}$.

According to this lemma, there always exists a set of algebraic integers $g_{k,\nu}$, $1<k\leqslant m$, $0\leqslant\nu\leqslant 2n-1$, which are such that conditions (60) are satisfied and

$$(61)\qquad \overline{|g_{k,\nu}|} < C[CmnA]^{\frac{p}{q-p}} = O(n^{2\varepsilon mn}) = O(n^{\delta n}),$$

for arbitrary $\delta>0$, because in our case $p=2mn-n$, $q=2mn$, $\dfrac{p}{q-p}=2m-1$ and $A=O(n^{\delta n})$ as a consequence of the inequalities

$$q_\nu\frac{\nu!}{s!(\nu-s)!}\overline{|C_{k,\nu-s}|} < O(\nu^{\varepsilon\nu})2^\nu O(\nu^{\varepsilon\nu}) = O(\nu^{3\varepsilon\nu})$$
$$\leqslant O[(2mn)^{2\varepsilon mn}] = O(n^{\delta n}),\ \nu < 2mn,$$

for arbitrary $\delta>0$ and $n>n_0$. Since the numbers $\frac{(2n-1)!}{s!}g_{k,s}$ will be the coefficients of $P_k(z)$, and $(2n-1)!=O(n^{2n})$, condition (53) of our lemma follows from inequality (61). These same inequalities and the estimate $\overline{|C_{k,\nu}|}=O(\nu^{\varepsilon\nu})$ also prove condition (56).

Suppose the m E-functions $f_1(z), \ldots, f_m(z)$ are a solution of the system of differential equations

$$f_k'(z) = \sum_{s=1}^{n} Q_{k,s}(z)f_s(z), \quad 1 \leqslant k \leqslant m, \tag{62}$$

where the $Q_{k,s}(z)$ are rational functions of z with integral coefficients in an algebraic field K of degree ν. We shall also assume that the coefficients of our E-functions are numbers in K. Suppose also that $T(z)$ is a polynomial of least degree with algebraic integral coefficients in the field K which is such that all the products $T(z)Q_{k,s}(z)$ are polynomials with algebraic integral coefficients. We shall retain these notations and conditions in this and the next section of the present chapter.

We set

$$\left\{\begin{aligned} R(z) &= R_1(z) = \sum_{k=1}^{m} P_k(z)f_k(z), \\ R_s(z) &= \sum_{k=1}^{m} P_{s,k}(z)f_k(z), \end{aligned}\right. \tag{63}$$

where the $P_k(z)=P_{1,k}(z)$ are prescribed polynomials with coefficients in the field K and

$$\begin{aligned} R_s(z) &= T(z)R'_{s-1}(z) \\ &= T(z)\left[\sum_{k=1}^{m} P'_{s-1,k}(z)f_k(z) + \sum_{k=1}^{m} P_{s-1,k}(z)f_k'(z)\right] \\ &= \sum_{k=1}^{m}\left[T(z)P'_{s-1,k}(z) + \sum_{q=1}^{m} P_{s-1,q}(z)T(z)Q_{q,k}(z)\right]f_k(z) \end{aligned} \tag{64}$$

for $s>1$, from which it follows directly that the $P_{s,k}(z)$ are polynomials with integral coefficients in K, since $f_1(z), \ldots, f_m(z)$ is a solution of system (62). Considering the functions $R_1(z)$, $R_2(z), \ldots, R_m(z)$ as linear forms in the functions $f_1(z), \ldots, f_m(z)$, it will be very important in the sequel for us to know under what conditions the determinant $\Delta(z)$ of this system

$$\Delta(z) = \begin{vmatrix} P_{1,1}(z) & P_{1,2}(z) & \ldots & P_{1,m}(z) \\ \ldots & \ldots & \ldots & \ldots \\ P_{m,1}(z) & P_{m,2}(z) & & P_{m,m}(z) \end{vmatrix} \tag{65}$$

will not vanish identically, assuming in this connection that not all the polynomials $P_1(z), \ldots, P_m(z)$ are identically zero. To answer this question, we shall consider the coefficient matrix of the system (62), $Q = \|Q_{k,s}(z)\|$. After a suitable interchange of rows and columns, which we shall assume to have been done, this matrix Q may appear split into blocks along the principal diagonal, where all the elements of Q outside the blocks are zero. Each of the blocks will be a matrix W_t of order m_t, $\sum_{t=1}^{r} m_t = m$, where r is the number of independent block matrices. Assuming that r is the largest possible, we immediately see that with such a largest r, the decomposition of the matrix Q into blocks will be unique to within a rearrangement of the blocks along the diagonal. Of course, in this connection we do not exclude the case $r = 1$; in other words, Q itself is, as we shall say, a primitive matrix. Hence, we shall assume that Q is decomposed into r primitive matrices of orders $m_1, m_2, \ldots, m_r$, where r is the largest possible. Corresponding to this decomposition of Q, our linear system of the first order (62) decomposes into r independent subsystems

$$U'_{k,t}(z) = \sum_{s=1}^{m_t} Q_{k,s,t}(z) U_{s,t}(z), \quad 1 \leqslant k \leqslant m_t, \tag{66}$$

where t assumes the values $1, 2, \ldots, r$, $\sum_{t=1}^{r} m_t = m$, and the functions $Q_{k,s,t}(z)$ are just the $Q_{k,s}(z)$ enumerated differently. Taking into consideration the fundamental system of solutions of the linear system (66), whose matrix we shall denote by $\bar{W}_t$,

$$\bar{W}_t = \begin{vmatrix} U_{1,1,t}(z) & & U_{m_t,1,t}(z) \\ \ldots & \ldots & \ldots \\ U_{1,m_t,t}(z) & \ldots & U_{m_t,m_t,t}(z) \end{vmatrix},$$

we can, on the basis of known theorems, assert, concerning the general solution of system (66), that

$$f_{k,t}(z) = \sum_{s=1}^{m_t} C_{s,t} U_{k,s,t}(z), \quad 1 \leqslant k \leqslant m_t, \quad 1 \leqslant t \leqslant r, \tag{67}$$

where the $f_{k,t}(z)$, $1 \leqslant k \leqslant m_t$, $1 \leqslant t \leqslant r$, are our E-functions enumerated in another way and the $C_{s,t}$ are certain constants,

We shall say that the matrices $W_1, W_2, \ldots, W_r$ are linearly independent over the field of rational functions, or simply, for the sake of brevity, that they are independent, if the relation

$$\sum_{t=1}^{r} \sum_{s=1}^{m_t} \sum_{k=1}^{m_t} p_{k,t}(z) C_{s,t} U_{k,s,t}(z) = 0, \tag{68}$$

where the $p_{k,t}(z)$ are arbitrary polynomials and the $C_{s,t}$ are arbitrary constants, is possible if and only if all the products $C_{s,t} p_{k,t}(z)$ vanish identically. Obviously, this definition of the independence of the matrices W_k does not depend on the choice of the fundamental system of solutions of equations (66). It follows directly from this definition and relation (67) that the function

$$R(z) = \sum_{k=1}^{m} P_k(z) f_k(z),$$

where the $P_k(z)$ are polynomials, all different from zero, and $f_1(z), f_2(z), \ldots, f_m(z)$ is any solution of the system (62) which does not vanish identically. Moreover, the independence of the primitive matrices of system (62) makes it possible to give a very simple formulation of the condition that the determinant (65) not vanish identically.

LEMMA IV. *If all the primitive matrices of the system* (66) *are independent and at least one of the polynomials* $P_{k,t}(z)$ *is not identically equal to zero for every* $t = 1, 2, \ldots, r$, *then the determinant* $\Delta(z) = |P_{s,k}(z)|$, $s, k = 1, 2, \ldots, m$, *of the system*

$$R_s(z) = \sum_{k=1}^{m} P_{s,k}(z) f_k(z), \quad 1 \leqslant s \leqslant m; \tag{69}$$

$$R_1(z) = R(z) = \sum_{k=1}^{m} P_k(z) f_k(z) = \sum_{t=1}^{r} \sum_{k=1}^{m_t} P_{k,t}(z) f_{k,t}(z),$$

$$R_{s+1}(z) = T(z) R_s'(z),$$

where the $P_{k,t}(z)$ *are the polynomials* $P_k(z) = P_{1,k}(z)$ *enumerated differently, corresponding to the decomposition of the matrix* Q *of the system* (62) *into primitive matrices, and the* $\{f_{k,t}(z)\}$ *are a solution of system* (62) *which is not identically equal to zero.*

Proof. Suppose $f_1(z), \ldots, f_m(z)$ is any solution of system (62). We shall assume that $\Delta(z) \equiv 0$. Then, as is known from linear algebra, polynomials $q_1(z), \ldots, q_\nu(z)$, $\nu \leqslant m$, can be found such that the relation

$$\sum_{s=1}^{\nu} q_s(z) R_s(z) = 0, \quad q_\nu(z) \not\equiv 0$$

is satisfied identically. From this it follows, in virtue of relations (69), that

$$(70) \qquad \begin{aligned} B_1(z) R^{(\nu-1)}(z) + \cdots + B_\nu(z) R(z) &\equiv 0, \\ B_1(z) &= [T(z)]^{\nu-1} q_\nu(z), \end{aligned}$$

where the $B_1(z), \ldots, B_\nu(z)$ are polynomials, $B_1(z) \not\equiv 0$.

These coefficients $B_k(z)$, obviously, do not depend on the arbitrary solution $f_1(z), \ldots, f_m(z)$ we have chosen and depend only on the coefficients of the system (62) and the polynomials $P_k(z)$, $k=1, 2, \ldots, m$. Since by hypothesis system (62) decomposes into r independent systems with matrices $W_1, \ldots, W_r$, our arbitrary solution $f_1(z), f_2(z), \ldots, f_m(z)$ of system (62) turns out to be the set of r arbitrary solutions $U_{s,t}(z)$, $1 \leqslant s \leqslant m_t$ of the independent systems (66), $1 \leqslant t \leqslant r$. Upon differentiation, the functions $U_{s,t}(z)$ go over into linear combinations of the functions $U_{s,t}(z)$ for the same t.

Therefore, since

$$R(z) = \sum_{k=1}^{m} P_k(z) f_k(z) = \sum_{t=1}^{r} \sum_{k=1}^{m_t} P_{k,t}(z) U_{k,t}(z),$$

we obtain, upon setting $U_{k,s}(z) \equiv 0$, $s \neq t$, $s=1, 2, \ldots, r$; $k=1, 2, \ldots, m_t$, that the functions

$$R_t(z) = \sum_{k=1}^{m_t} P_{k,t}(z) U_{k,t}(z), \quad 1 \leqslant t \leqslant r,$$

must satisfy equation (70) if the $U_{k,t}(z)$, $1 \leqslant k \leqslant m_t$, are any solution of system (66). But, replacing $U_{k,t}(z)$, $1 \leqslant k \leqslant m_t$ by each of the solutions $U_{k,s,t}(z)$, $1 \leqslant k \leqslant m_t$ from the fundamental system of solutions of (66), we conclude from this that the functions

$$R_{s,t}(z) = \sum_{k=1}^{m_t} P_{k,t}(z) U_{k,s,t}(z), \quad 1 \leqslant s \leqslant m_t, \quad 1 \leqslant t \leqslant r,$$

must satisfy equation (70). Since we have $m = \sum_{t=1}^{r} m_t$ solutions $R_{s,t}(z)$ of the linear equation (70), and that the degree of this equation is $\nu-1<m$, these solutions must be linearly dependent in the usual sense; in other words, a relation

$$\sum A_{s,t}R_{s,t}(z) = \sum_{k,s,t} A_{s,t}P_{k,t}(z)U_{k,s,t}(z) \equiv 0 \tag{71}$$

must hold for nonzero constants $A_{s,t}$, $1\leqslant s\leqslant m_t$, $1\leqslant t\leqslant r$. In this sum, at least one of the products $A_{s,t}P_{k,t}(z)\not\equiv 0$ because if $A_{q,t}\neq 0$ we can find a k by the condition of the lemma, such that $P_{k,t}(z)\not\equiv 0$. But then relation (71) contradicts the independence condition on the primitive matrices of system (62) and this means that $\Delta(z)\not\equiv 0$.

We now state the definition of a normal system of E-functions.

A system of m E-functions $f_1(z), f_1(z), \ldots, f_m(z)$ is said to be *normal* if:

1) *none of the functions* $f_i(z)$ *vanishes identically*;

2) *this system of functions is a solution of the linear system of differential equations* (62) *under the conditions that all the functions* $Q_{k,s}(z)$ *of this system of equations are rational functions in* z *with numerical coefficients belonging to a finite algebraic field, and that all the primitive matrices of this system are independent.*

Suppose the system of E-functions $f_1(z), \ldots, f_m(z)$ is normal. We construct, with the aid of Lemma III, an approximating form

$$R(z) = \sum_{k=1}^{m} P_k(z)f_k(z), \tag{72}$$

where the $P_k(z)$ are polynomials, all different from zero, whose degrees are less than or equal to $2n-1$, and $R(z)$ has a zero at $z=0$ of order greater than or equal to $2nm-n$.

If the matrix of the system of linear equations (62) decomposes into primitive matrices $W_1, \ldots, W_r$ of ranks $m_1, \ldots, m_r$, then equation (72) can be written in the form

$$R(z) = \sum_{t=1}^{r}\sum_{k=1}^{m_t} P_{k,t}(z)f_{k,t}(z), \tag{72'}$$

where the set of all $f_{k,t}(z)$, $1\leqslant k\leqslant m_t$ forms a solution of the system with primitive matrix W_t, $t=1, 2, \ldots, r$, and the set of polynomials $P_{k,t}(z)$ coincides with the set of polynomials $P_k(z)$.

We shall now show that if $R(z)$ is an approximating form, in other words if it satisfies the conditions stated above, then for at least one k, $1\leqslant k\leqslant m_t$, $P_{k,t}(z)\not\equiv 0$, for any t, $1\leqslant t\leqslant r$, provided $n\geqslant p+q\,\dfrac{m(m-1)}{2}$.

In fact, let us assume, which can be done without loss of generality, that for every t, $1 \leqslant t \leqslant \mu$, $\mu < m$, $P_{k,t}(z) \not\equiv 0$ for at least one k, and that for $t > \mu$ all the $P_{k,t}(z) \equiv 0$, $1 \leqslant k \leqslant m_t$. Then the functions $f_{k,t}(z)$, $1 \leqslant t \leqslant \mu$, $1 \leqslant k \leqslant m_t$, form a normal system of E-functions and

$$R(z) = \sum_{t=1}^{\mu} \sum_{k=1}^{m_t} P_{k,t}(z) f_{k,t}(z).$$

If we set $m_0 = \sum_{1}^{\mu} m_t$, $m_0 < m$, then by Lemma IV, the determinant $\Delta_0(z)$ of the system of m_0 linear forms $R_1(z), \ldots, R_{m_0}(z)$; $R_{k+1}(z) = T(z) R_k'(z)$; $R_1(z) = R(z)$, will not vanish identically. We denote by q the largest of all the degrees of the m^2+1 polynomials $T(z)$ and $T(z) Q_{k,s}(z)$; $1 \leqslant k \leqslant m$; $1 \leqslant s \leqslant m$, where the $Q_{k,s}(z)$ are the coefficients in system (62). Then, if

$$R_k(z) = \sum_{s=1}^{m} P_{k,s}(z) f_s(z),$$

the highest possible degree of $P_{k,s}(z)$, $1 \leqslant s \leqslant m$, is less than or equal to the quantity $2n-1+(k-1)q$. It follows from this that the degree of the determinant $\Delta_0(z)$, which is a polynomial in z, will be less than or equal to the quantity

$$(2n-1)m_0 + q\,\frac{m_0(m_0-1)}{2}. \tag{73}$$

On the other hand, since $R(z)$ has a zero of order not less than $2mn-n$ at the origin, then, as a consequence of the relation $R_{k+1}(z) = T(z) R_k'(z)$, $R_k(z)$ must have a zero at the origin of order not less than $2m-n-k+1$. Multiplying the first column of $\Delta_0(z)$ by $f_{1,1}(z)$ and combining it with the remaining, multiplied by the $f_{k,s}(z)$ respectively, we obtain that all the elements of the first column of $f_{1,1}(z)\Delta_0(z)$ are divisible by z to the power $2mn-n-m_0+1$. If we now denote by p the largest order of the zero which the functions of our system $f_1(z), \ldots, f_m(z)$ can have at the origin, we see that $\Delta_0(z)$ must be divisible throughout by z^k, $k = 2mn-n-m-p+1$. Therefore, as a consequence of the fact that $\Delta_0(z) \not\equiv 0$, the inequality

$$(2n-1)m_0 + q\,\frac{m_0(m_0-1)}{2} \geqslant 2mn-n-m_0-p+1,$$

or the inequality

$$(74) \qquad p+q\frac{m_0(m_0-1)}{2} \geqslant [2(m-m_0)-1]n+1$$

holds. If we now assume that

$$(75) \qquad n \geqslant p+q\frac{m(m-1)}{2},$$

which is possible since n can be taken arbitrarily large independently of m, we see that if $m_0<m$ then inequality (74) is not true. This means that if inequality (75) is satisfied, then relation (72′) has at least one $P_{k,t}(z)\not\equiv 0$, $1\leqslant k\leqslant m_t$, for every t, $1\leqslant t\leqslant r$. From this it already follows directly that if the system of E-functions $f_1(z),\ldots,f_m(z)$ is normal, $R(z)$ is an approximating form satisfying conditions (55) of Lemma III, and the degrees of the polynomials in it are less than or equal to $2n-1$, then if the inequalities (75) are satisfied, the conditions of Lemma IV will be satisfied and this means that the determinant of the system of linear forms in $f_1(z),\ldots,f_m(z)$

$$R_k(z) = T(z)R'_{k-1}(z),\ R_1(z) = R(z),\ k = 1, 2, \ldots, m,$$

will not vanish identically. In other words, if $R(z)$ is an approximating form and inequality (75) is satisfied, then $\Delta(z)\not\equiv 0$. The latter fact enables one to go over from linear functional approximating forms for the functions $f_1(z),\ldots,f_m(z)$ to numerical linear approximating forms in $f_1(\alpha),\ldots,f_m(\alpha)$, where $\alpha\neq 0$ is a prescribed number.

LEMMA V. *Suppose $f_1(z),\ldots,f_m(z)$ form a normal system of E-functions and that they satisfy system* (62). *Suppose further that $T(z)$ is the least common multiple of the denominators of the $Q_{k,s}(z)$ in system* (62), *q is the least upper bound of the degrees of $T(z)$ and the $T(z)Q_{k,s}(z)$, $1\leqslant k\leqslant m$, $1\leqslant s\leqslant m$, and that p is the highest order of the zero that the functions $f_1(z), f_2(z),\ldots,f_m(z)$ have at $z=0$. Finally, assume that*

$$R(z) = \sum_{1}^{m} P_k(z)f_k(z),$$

where the polynomials $P_k(z)$ having degree less than or equal to $2n-1$ are chosen in such a way that the conditions of Lemma III are satisfied and that

$$R_s(z) = \sum_{k=1}^{m} P_{s,k}(z)f_k(z),$$

$$R_{s+1}(z) = T(z)R_s'(z),\ R_1(z) = R(z),\ P_{1,k}(z) = P_k(z).$$

Then, if α is not zero and does not coincide with any zero of $T(z)$, and inequality (75) *is satisfied, then the rank of the matrix*

$$\left\|\begin{matrix} P_{1,1}(\alpha) & P_{1,2}(\alpha) & \dots & P_{1,m}(\alpha) \\ \dots & \dots & \dots & \dots \\ P_{m+t,1}(\alpha) & P_{m+t,2}(\alpha) & & P_{m+t,m}(\alpha) \end{matrix}\right\|, \tag{76}$$

where $t=n+p+q\dfrac{m(m-1)}{2}-1$, is m.

Proof. As we have already seen above, the determinant $\Delta(z)$ of the system of forms $R_1(z), \dots, R_m(z)$ does not vanish identically, provided the conditions of our Lemma are satisfied. The degree of the polynomial $\Delta(z)$, as we have already explained, is less than or equal to $(2n-1)m+q\dfrac{m(m-1)}{2}$ for $m_0=m$ by formula (73). On the other hand, by the conditions of the choice of the $P_k(z)$, $k=1, 2, \dots, m$ as we have already seen, $\Delta(z)$ must have a zero of order greater than or equal to $2mn-n-m-p+1$ for $z=0$. Therefore,

$$\Delta(z) = z^{2mn-n-m-p+1}\Delta_0(z),$$

where $\Delta_0(z)$ is a polynomial of degree not greater than t,

$$\begin{aligned} t &= (2n-1)m+q\frac{m(m-1)}{2}-2mn+n+m+p-1 \\ &= n+p+q\frac{m(m-1)}{2}-1. \end{aligned}$$

Therefore $\Delta(z)$ cannot be a zero of order higher than t at $z=\alpha\neq 0$. In other words, $\Delta^{(\mu)}(\alpha)\neq 0$ for some μ, $0\leqslant\mu\leqslant t$.

If we introduce the notation

$$\Delta(z;\, n_1, \dots, n_m) = \begin{vmatrix} P_{n_1,1}(z) & P_{n_1,2}(z) & \dots & P_{n_1,m}(z) \\ \dots & \dots & \dots & \dots \\ P_{n_2,1}(z) & P_{n_2,2}(z) & & P_{n_2,m}(z) \\ \dots & \dots & \dots & \dots \\ P_{n_m,1}(z) & P_{n_m,2}(z) & \dots & P_{n_m,m}(z) \end{vmatrix},$$

where $1\leqslant n_1<n_2<\dots<n_m$, and use formula (64), we immediately obtain the relation

$$\begin{aligned} &T(z)\Delta'(z;\, n_1, \dots, n_m) \\ &\quad= \sum_{k=1}^{m}\Delta(z;\, n_1, \dots, n_{k-1},\, n_k+1,\, n_{k+1}, \dots, n_m) \\ &\qquad - T(z)\sum_{k=1}^{m}\sum_{q=1}^{m}P_{n_k,q}(z)\sum_{r=1}^{m}Q_{q,r}(z)\Delta_{k,r}(z;\, n_1, \dots, n_m), \end{aligned}$$

where $\Delta_{k,r}(z;\ n_1, \ldots, n_m)$ is the algebraic complement of the element $P_{n_k,r}(z)$ in $\Delta(z;\ n_1, n_2, \ldots, n_m)$. Interchanging the order of summation in the last sum, and summing first with respect to k, and discarding determinants having equal columns, we finally obtain the relation

$$\begin{aligned}(77)\quad \Delta'(z;\ n_1, \ldots, n_m) &\\ = \frac{1}{T(z)} &\sum_{k=1}^{m} \Delta(z;\ n_1, \ldots, n_{k-1},\ n_k+1,\ n_{k+1}, \ldots, n_m)\\ &-\Delta(z;\ n_1, \ldots, n_m) \sum_{r=1}^{m} Q_{r,r}(z).\end{aligned}$$

If we assume that the rank of the matrix (76) is less than m, then $\Delta(\alpha;\ n_1, \ldots, n_m)=0$ for $n_m \leqslant m+t$. It then follows from relation (77), as a consequence of $T(\alpha)\neq 0$, that $\Delta'(\alpha;\ n_1, \ldots, n_m)=0$ for $n_m \leqslant m+t-1$. Differentiating relation (77), we successively arrive from the relations obtained in this process at the fact that $\Delta^{(\mu)}(\alpha;\ n_1, \ldots, n_m)=0$ for $n_m \leqslant m+t-\mu$. But

$$\Delta^{(\mu)}(\alpha) = \Delta^{(\mu)}(\alpha;\ 1, 2, \ldots, m) \neq 0$$

for some μ, $\mu \leqslant t$.

This means that for some $\mu \leqslant t$ the inequality $m > m+t-\mu$ must be satisfied, which is impossible in virtue of the inequality $\mu \leqslant t$. This completes the proof of our lemma.

We shall now agree to assume that in all lemmas in the sequel and in the main theorem of this chapter all the definitions and properties of the various quantities appearing in the formulation of Lemma V are retained.

We shall prove the estimates of the quantities $|R_k(\alpha)|$ and $\overline{|P_{k,s}(\alpha)|}$ which will be necessary in the sequel, assuming that the number $\alpha \neq 0$ and that all the coefficients of the functions $f_1(z), \ldots, f_m(z)$, $T(z)$, $Q_{k,s}(z)$, $1 \leqslant k \leqslant m$, $1 \leqslant s \leqslant m$, and of the polynomials $P_{1,s}(z)=P_s(z)$, $1 \leqslant s \leqslant m$, in virtue of their choice according to Lemma III, belong to some algebraic field K of degree ν.

LEMMA VI. *Suppose α belongs to the field K and that $k \leqslant m+t$. Then*

$$(78)\quad \begin{aligned} R_k(\alpha) &= O[n^{-(2m-5-\varepsilon)n}],\\ \overline{|P_{k,s}(\alpha)|} &= O[n^{(3+\varepsilon)n}], \quad 1 \leqslant s \leqslant m,\end{aligned}$$

for arbitrary $\varepsilon > 0$.

Proof. If the inequalities $|a_k| \leqslant b_k$, $b_k \geqslant 0$, $k=0, 1, 2, \ldots$, are satisfied simultaneously, we shall agree to write $f_1(z) \ll f_2(z)$ provided

$$f_1(z) = \sum_{k=0}^{\infty} a_k z^k \quad \text{and} \quad f_2(z) = \sum_{k=0}^{\infty} b_k z^k,$$

and if

$$f(z) = \sum_{0}^{\infty} a_k z^k,$$

then

$$\bar{f}(z) = \sum_{k=0}^{\infty} |a_k| z^k.$$

First of all, since $R(z)$ is chosen according to Lemma III, as a consequence of conditions (54), (55) and (56) of this lemma, we have

$$R(z) \ll \bar{R}(z) = \sum_{0}^{\infty} |a_k| \frac{z^k}{k!} = \sum_{\nu=(2m-1)n}^{\infty} \nu^{\varepsilon\nu} O(n^{2n}) \frac{z^\nu}{\nu!} \tag{79}$$

for arbitrary $\varepsilon > 0$ and $n > n_0(\varepsilon)$.

Further,

$$T(z) \ll C(1+z)^q; \quad T(z) Q_{k,s}(z) \ll C(1+z)^q,$$

where C and q are constants which do not depend on n.

It follows that

$$T'(z) \ll Cq(1+z)^q$$

and that, as a consequence of the relations

$$\begin{aligned} R_{k+1}(z) &= T(z) R_k'(z), \\ \bar{R}_2(z) &\ll C(1+z)^q \bar{R}'(z); \\ \bar{R}_3(z) &\ll C^2(1+z)^{2q}[q\bar{R}'(z) + \bar{R}''(z)] \\ &= C^2(1+z)^{2q} \frac{d}{dz}\left(q + \frac{d}{dz}\right)\bar{R}(z), \end{aligned}$$

we have, by induction

$$\bar{R}_{k+1}(z) \ll C^k(1+z)^{kq} \prod_{s=0}^{k-1} \left(sq + \frac{d}{dz}\right)\bar{R}(z). \tag{80}$$

In exactly the same way, as a consequence of the relation

$$P_{k+1,s}(z) = T(z) P'_{k,s}(z) + \sum_{r=1}^{m} P_{k,r}(z) T(z) Q_{r,s}(z),$$

by induction one can prove the inequality

$$(81)\quad P_{k+1,s}(z) \ll C^k(1+z)^{kq+2n-1}\prod_{r=0}^{k-1}(rq+m+2n-1)O[n^{(2+\varepsilon)n}],$$

since the coefficients of the $P_{k,s}(z)$, $s=1, \ldots, m$, satisfy the conditions of Lemma III. We note that in virtue of the conditions of this lemma, our last inequality remains valid if the coefficients of all the polynomials $P_{k,s}(z)$, $T(z)$ and $T(z)Q_{k,s}(z)$ from the field K are all replaced by their conjugates from the field K', a conjugate field of K. Since, for $k \leqslant m+t=n+n_0$, $n_0=O(1)$,

$$\begin{aligned}\prod_{s=0}^{k-1}\left(sq+\frac{d}{dz}\right)\bar{R}(z) &\ll (kq)^k\left(1+\frac{d}{dz}\right)^k\bar{R}(z)\\ &\ll (kq)O^k(n^{2n})\left(1+\frac{d}{dz}\right)^k\sum_{s=(2m-1)n}^{\infty} s^{\varepsilon s}\frac{z^s}{s!}\\ &\ll (2kq)^kO(n^{2n})\sum_{s=(2m-2)n-n_0}^{\infty}(s+n+n_0)^{\varepsilon(s+n+n_0)}\frac{z^s}{s!}\\ &\ll O[n^{(3+\varepsilon)n}]\sum_{s=(2m-2)n-n_0}^{\infty}\frac{z^s}{s^{(1-3\varepsilon)s}},\end{aligned}$$

for arbitrary ε, $\frac{1}{4}>\varepsilon>0$ and $n>n_1(\varepsilon)$, then setting $|\alpha|=\varrho$, we finally obtain from inequality (80) that

$$\begin{aligned}|R_{k+1}(\alpha)| &< C^k(1+\varrho)^{kq}O[n^{(3+\varepsilon)n}]\sum_{\nu=(2m-2)n-n_0}^{\infty}\frac{\varrho^\nu}{\nu^{(1-3\varepsilon)\nu}}\\ &< O[n^{(3+2\varepsilon)n}]n^{-(1-4\varepsilon)(2m-2)n} = O[n^{-(2m-5-\varepsilon')n}]\end{aligned}$$

for arbitrary $\varepsilon'>8m\varepsilon$ and $n>n_2(\varepsilon')$, since ε is arbitrarily small.

Further, from inequality (81), setting $z=\alpha'$ in it, where α' is a conjugate of α in the field K', a conjugate of K, and replacing the coefficients of $P_s(z)$, $T(z)$ and $Q_{k,s}(z)$ by their conjugates from K', we shall have for all $k \leqslant m+t=n+n_0$, $n_0=O(1)$, that

$$\overline{|P_{k+1,s}(\alpha)|} < n^{\varepsilon n}n^{(1+\varepsilon)n}O[n^{(2+\varepsilon)n}] < O[n^{(3+\delta)n}]$$

for arbitrary $\delta>0$ and $n>n_3(\delta)$, since ε can be taken arbitrarily small. This completes the proof of our lemma.

We shall now introduce the definition of the rank of a set of, in general complex, numbers $\omega_1, \ldots, \omega_m$ with respect to (or over) a finite algebraic field K. We shall say that the set of m numbers $\omega_1, \ldots, \omega_m$ has rank r over the field K if among these m numbers there are r and only r numbers which are linearly independent over the field K. In other words, if the system $\omega_1, \ldots, \omega_m$ has rank r over the field K, then its elements satisfy $m-r$, and only $m-r$, linearly independent relations

$$\sum_{s=1}^{m} C_{k,s}\omega_s = 0, \quad k = 1, 2, \ldots, m-r,$$

where all the $C_{k,s}$ are elements in the field K.

LEMMA VII. *Assume that α and all the coefficients of the m E-functions $f_1(z), \ldots, f_m(z)$ belong to a finite algebraic field K of degree ν and the system of these E-functions is normal. Then, if $\alpha T(\alpha) \neq 0$, the rank of the numerical system $f_1(\alpha), \ldots, f_m(\alpha)$ with respect to K is not less than $m/2\nu$.*

Proof. Since, by hypothesis, α is not a pole of any of the functions $Q_{k,s}(z)$, α is a regular point of the system (62) and at least one of the numbers $f_1(\alpha), \ldots, f_m(\alpha)$ must be nonzero; in other words, without loss of generality one can assume that $f_1(\alpha) \neq 0$. If the rank of the system $f_1(\alpha), \ldots, f_m(\alpha)$ with respect to the field K is r, then the following relations must hold

$$L_k = \sum_{s=1}^{m} A_{k,s} f_s(\alpha) = 0, \quad 1 \leqslant k \leqslant m-r, \tag{82}$$

$$|A_{k,s}^{(i)}| \leqslant A; \quad 1 \leqslant i \leqslant \nu; \quad 1 \leqslant k \leqslant m-r; \quad 1 \leqslant s \leqslant m,$$

where the L_k are linearly independent forms in the quantities $f_1(\alpha), \ldots, f_m(\alpha)$, the numbers $A_{k,s}$ are integers in the field K, the numbers $A_{k,s}^{(i)}$ are the conjugates of $A_{k,s} = A_{k,s}^{(1)}$, which belong to the field K_i, conjugate to the field K, and A is some constant. By Lemma V, we can construct the linear forms

$$U_k = \sum_{s=1}^{m} P_{k,s}(\alpha) f_s(\alpha), \quad 1 \leqslant k \leqslant m+t, \tag{83}$$

where the $P_{k,s}(z)$ are polynomials of degree not higher, as can easily be calculated, than $2n-1+q(k-1)$, which coefficients are integers in the field K, where the rank of the matrix of system (83) is m. Therefore, the system (83) contains m linearly independent forms

$$U_{k_\sigma} = P_{k_\sigma,s}(\alpha) f_s(\alpha), \quad 1 \leqslant \sigma \leqslant m, \tag{84}$$

where $k_\sigma \leqslant m+t$, $1 \leqslant \sigma \leqslant m$. Since the $m-r$ forms (82) are linearly independent, the m linearly independent forms U_{k_σ} contain r forms which together with all the forms (82) already form a system of m linearly independent forms in the quantities $f_1(\alpha), \ldots, f_m(\alpha)$. Suppose these are the forms $U_{k_1}, \ldots, U_{k_r}$. Setting $N = 2n-1+q(m+t-1) = (q+2)n + O(1)$ and

$$a^N P_{k_\sigma, s}(\alpha) = A_{m-r+\sigma, s}, \quad 1 \leqslant \sigma \leqslant r, \quad 1 \leqslant s \leqslant m,$$

where a is an integer, $a > 0$, such that $a\alpha$ is an algebraic integer, we shall have that the linear forms

$$L_k = \sum_{s=1}^{m} A_{k,s} f_s(\alpha), \quad 1 \leqslant k \leqslant m,$$

will be linearly independent. All the numbers $A_{k,s}$, $m-r+1 \leqslant k \leqslant m$, will now be algebraic integers in the field K, and as a consequence of Lemma VI the inequalities

$$\overline{|A_{k,s}|} < a^N O[n^{(3+\varepsilon)n}] = O[n^{(3+\delta)n}] \tag{85}$$

will hold for arbitrary $\delta > 0$ and $1 \leqslant s \leqslant m$, $m-r+1 \leqslant k \leqslant m$.

Since the forms L_k are linearly independent with respect to $f_1, f_2, \ldots, f_m$, the determinant D of this system

$$D = D_1 = \begin{vmatrix} A_{1,1} & A_{1,2} & \ldots & A_{1,m} \\ A_{2,1} & A_{2,2} & \ldots & A_{2,m} \\ \ldots & \ldots & \ldots & \ldots \\ A_{m,1} & A_{m,2} & \ldots & A_{m,m} \end{vmatrix} \tag{86}$$

is nonzero and will be an algebraic integer. Multiplying the first column by $f_1(\alpha) \neq 0$ and combining it with the remaining columns, multiplied by the corresponding $f_k(\alpha)$, we shall obtain the equality

$$f_1(\alpha) D_1 = \begin{vmatrix} L_1 & A_{1,2} & \ldots & A_{1,m} \\ L_2 & A_{2,2} & \ldots & A_{2,m} \\ \ldots & \ldots & \ldots & \ldots \\ L_m & A_{m,2} & \ldots & A_{m,m} \end{vmatrix} \neq 0.$$

But the minors of the elements of the first column, for $k > m-r$, are, by (85), in absolute value less than or equal to the quantity $m^m A^{m-r} O[n^{(3+\delta)(r-1)n}]$. Therefore, by condition (78) of Lemma VI, the inequality

$$\begin{aligned} |D_1| &< \lambda m^m A^{m-r} O[n^{(3+\delta)(r-1)n}] \sum_{k=m-r+1}^{m} |L_k| \\ &< \lambda m^{m+1} A^{m-r} O[n^{(3+\delta)(r-1)n}] O[n^{-(2m-5-\varepsilon)n}] \\ &< O[n^{-(2m-3r-2-\varepsilon_1)n}] \end{aligned} \tag{87}$$

holds, where $\varepsilon_1 > 0$ is arbitrarily small and $\lambda = 1/|f_1(\alpha)|$. If one replaces all the elements in the determinant D_1 by their conjugates, belonging to the field K_i, conjugate to the field K, then we obtain the determinant D_i, conjugate to D_1,

$$D_i = \begin{vmatrix} A_{1,1}^{(i)} & \dots & A_{1,m}^{(i)} \\ \dots & \dots & \dots \\ A_{m,1}^{(i)} & \dots & A_{m,m}^{(i)} \end{vmatrix}$$

with $D_i \neq 0$ an algebraic integer. Estimating the absolute value of D_i we obtain, in virtue of inequalities (82) and (85), that

$$(88) \qquad |D_i| < m^m A^{m-r} O[n^{(3+\delta)rn}] = O[n^{(3+\delta_1)rn}],$$

where $\delta_1 > 0$ is arbitrarily small and $2 \leqslant i \leqslant \nu$.

But the norm of a nonzero algebraic integer is never less than 1. Therefore, by inequalities (87) and (88) we have

$$1 \leqslant \prod_{i=1}^{\nu} |D_i| < O[n^{(3+\delta_1)(\nu-1)rn}] O[n^{-(2m-3r-2-\varepsilon_1)n}]$$
$$< O[n^{(3\nu r+2-2m+\varepsilon_0)n}],$$

where $\varepsilon_0 > 0$ is arbitrarily small. It already follows from this inequality, which must be true for sufficiently large n, that we have the inequality $3\nu r - 2m + 2 + \varepsilon_0 > 0$, or, since ε_0 can be taken arbitrarily small, the inequality

$$r \geqslant \frac{2m-2}{3\nu}.$$

We have thus proved our lemma for $m \geqslant 4$, since in this case $2m - 2 \geqslant (3/2)m$. If $m = 3$, then $r \geqslant 4/3\nu$, in other words, $r \geqslant 1$, $\nu > 1$ and $r \geqslant 2$, $\nu = 1$. But in the case when $m = 1, 2$; $r \geqslant 1$. These same bounds yield our lemma. This completes the proof of the lemma.

§4. General theorem on the algebraic independence of values of an E-function and consequences of it

It is readily noted that if the m functions $f_1(z), \dots, f_m(z)$ are a solution of the linear system (62) with rational $Q_{k,s}(z)$, whose coefficients are algebraic numbers, then the μ functions $F(z)$, $\mu = (m+N)!/m!\,N!$,

$$F(z) = f_1^{N_1}(z) \dots f_m^{N_m}(z), \sum_{k=1}^{m} N_k \leqslant N,$$

where the $N_1, \ldots, N_m$ are nonnegative integers, also satisfy a system of type (62) for $m=\mu$ and certain rational coefficients $Q_{k,s}(z)$, whose numerical coefficients in turn belong to a finite algebraic field. In fact, differentiating $F(z)$ we shall have the relations

$$F'(z) = \sum_{k=1}^{m} N_k f_1^{N_1}(z) \ldots f_m^{N_m}(z) \frac{f_k'(z)}{f_k(z)},$$

from which our assertion will follow if we replace the $f_k'(z)$ with the aid of (62). Using this fact and the last lemma of the preceding section, we can now prove a general theorem.

FUNDAMENTAL THEOREM. *Assume that the m E-functions $f_1(z), \ldots, f_m(z)$ form a normal system and that $\alpha T(\alpha) \neq 0$, where α is an algebraic number, and $T(z)$ is the least common multiple of the denominators of the rational $Q_{k,s}(z)$ of the system* (62), *whose solution is our E-functions. Then, if for arbitrary N, $\mu=(m+N)!/m!N!$ functions $f_1^{N_1}(z), \ldots, f_m^{N_m}(z)$, $N_i \geqslant 0$, $1 \leqslant i \leqslant m$; $\sum_{i=1}^{m} N_i \leqslant N$, form a normal system of E-functions, then the numbers $f_1(\alpha), f_2(\alpha), \ldots, f_m(\alpha)$ cannot be interrelated by means of any algebraic relationship with algebraic coefficients, all different from zero.*

Proof. Suppose $S[f_1, f_2, \ldots, f_m]$ is a polynomial with respect to $f_1, f_2, \ldots, f_m$ of degree s, in all the variables, with coefficients in the algebraic field K of degree ν, to which all the coefficients of the functions $f_1(z), \ldots, f_m(z)$ also belong. Suppose N is an integer, $N \geqslant s$. We shall consider the $\mu_{N-s}=(m-s+N)!/m!(N-s)!$ polynomials

$$L(z) = f_1^{N_1}(z) \ldots f_m^{N_m}(z) S[f_1(z), \ldots, f_m(z)], \tag{88'}$$

$$\sum_{i=1}^{m} N_i \leqslant N-s,$$

where N_i is an arbitrary nonnegative integer. If we consider these polynomials L as linear forms with respect to the variables $f_1^{N_1} \ldots f_m^{N_m} = t_{N_1,\ldots,N_m}$, then, obviously, these μ_{N-s} linear forms will be linearly independent. Since the μ_N functions $f_1^{N_1}(z) \ldots f_m^{N_m}(z)$, $\sum_{k=1}^{m} N_k \leqslant N$, form a normal system of E-functions and α satisfies the conditions of Lemma VII, the rank r of the system of μ_N numbers $f_1^{N_1}(\alpha) \ldots f_m^{N_m}(\alpha)$ with respect to the field K of degree

ν cannot be less than $\mu_N/2\nu$; in other words, these numbers cannot be interrelated by more than $\mu_N - r \leqslant \left(1-\frac{1}{2\nu}\right)\mu_N$ linearly independent relations with coefficients in the field K. But if the equality

$$S[f_1(\alpha), \ldots, f_m(\alpha)] = 0 \tag{89}$$

holds, then we obtain from (88′) exactly μ_{N-s} linearly independent relations among the μ_N numbers. Therefore, the inequality $\mu_{N-s} \leqslant \left(1-\frac{1}{2\nu}\right)\mu_N$ or

$$\frac{(N+m-s)!}{m!(N-s)!} \leqslant \left(1-\frac{1}{2\nu}\right)\frac{(N+m)!}{m!N!}$$

holds. But since N can be taken arbitrarily large by the condition of the theorem, it follows from this inequality that the inequality

$$\frac{1}{m!}N^m + O(N^{m-1}) \leqslant \left(1-\frac{1}{2\nu}\right)\frac{1}{m!}N^m + O(N^{m-1})$$

holds for arbitrarily large N, which obviously is impossible. This proves that relation (89) is impossible.

The proof of the algebraic independence of the values of an E-function in a finite algebraic field for algebraic values of the argument reduces, in virtue of the fundamental theorem proved above, to the proof of the normality of a finite set of products of these functions. The Lindemann theorem follows directly from this theorem. In fact, suppose the algebraic numbers $\alpha_1, \ldots, \alpha_m$ of a finite field K are linearly independent in the rational field. Then the m functions $e^{\alpha_1 z}, \ldots, e^{\alpha_m z}$ which, obviously, will be E-functions, form a normal system of E-functions, since they are the solution of the decomposed system

$$f_k'(z) = \alpha_k f_k(z), \quad 1 \leqslant k \leqslant m$$

and a linear relation with polynomial coefficients among them is impossible. Moreover, $\mu_N = (N+m)!/m!N!$ functions $e^{\gamma z}$, $\gamma = \sum_{k=1}^{m} N_k\alpha_k$, $\sum_{1}^{m} N_k \leqslant N$ will also be a solution of such a decomposed system and in view of the fact that the γ are distinct, a linear relation with polynomial coefficients is not possible among them. Therefore, the conditions of the fundamental theorem will be satisfied for the functions $e^{\alpha_1 z}, \ldots, e^{\alpha_m z}$ for $\alpha = 1$, which proves the Lindemann theorem.

We shall now show that the transcendence of the values of a cylindrical function follows from this general theorem.

We shall consider the function

$$(90) \qquad K_\lambda(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(\lambda+1)\dots(\lambda+n)} \left(\frac{z}{2}\right)^{2n}$$

with rational $\lambda \neq -m$, $m=1, 2, \dots$. We shall assume also that $\lambda \neq \frac{2m-1}{2}$ where m is an integer, since in the contrary case, as can easily be shown, $K_\lambda(z)$ reduces to a simple linear combination of exponential functions and polynomials.

We shall show that $K_\lambda(z)$ is an E-function. We shall set $\lambda = m/q$, $(m, q)=1$, $q \geqslant 1$, where m and q are integers. Then, if

$$K_\lambda(z) = \sum (-1)^n a_{2n} \frac{z^{2n}}{(2n)!}$$

we have

$$a_{2n} = \frac{2^{-2n} q^n (2n)!}{n!(m+q)\dots(m+nq)}, \quad n = 0, 1, \dots.$$

If the prime p divides q, then $(p, m)=1$, from which it follows that $(m+nq, p)=1$, $n \geqslant 0$. Therefore, if $2^{2n} a_{2n} = A/Q$, where A and Q are rational integers, $(A, Q)=1$ then $(p, Q)=1$. We shall find the highest power of the prime p, $(p, q) \doteq 1$, which can divide Q. Suppose this power is ν_p. As is well known from elementary number theory, the prime p occurs in the ratio $\frac{(2n)!}{n!}$ to the power μ_p

$$\mu_p = \sum_{k=1}^{\left[\frac{\ln 2n}{\ln p}\right]} \left\{ \left[\frac{2n}{p^k}\right] - \left[\frac{n}{p^k}\right] \right\}.$$

We now consider the congruence

$$m + qk \equiv 0 \bmod p.$$

Suppose k_1 is the least nonnegative solution of this congruence. The number k_1 always exists because $(q, p)=1$ and, clearly, $k_1 < p$. Then, as is known, all positive solutions k of this congruence will have the form $k = k_1 + rp$, $r = 0, 1, \dots$. Therefore, the number of terms in the sequence $m+q$, $m+2q, \dots, m+nq$, divisible by p, will be $\sigma_1 + 1$, where σ_1 is determined by the inequality $k_1 + \sigma_1 p \leqslant n$. It follows that $\sigma_1 = \left[\frac{n-k_1}{p}\right]$. Of course, if $k_1 > n$, then $\sigma_1 = -1$.

In an exactly analogous manner, the number of terms in the sequence $m+q, \ldots, m+nq$, which are divisible by p^ν, which we shall denote by $\sigma_\nu+1$, will be $1+\left[\frac{n-k_\nu}{p^\nu}\right]$, $0 \leqslant k_\nu < p$. Therefore, the number p will occur in the product $(m+q)(m+2q)\ldots(m+nq)$ to a power not greater than

$$\sum_{\nu=1}^{t}\left(1+\left[\frac{n-k_\nu}{p^\nu}\right]\right) < t+\sum_{\nu=1}^{\left[\frac{\ln n}{\ln p}\right]}\left[\frac{n}{p^\nu}\right], \quad t=\left[\frac{\ln(m+nq)}{\ln p}\right].$$

It follows from this that the power of p which divides Q is not greater than

$$\left[\frac{\ln(m+nq)}{\ln p}\right]+\sum_{\nu=1}^{\infty}\left\{2\left[\frac{n}{p^\nu}\right]-\left[\frac{2n}{p^\nu}\right]\right\} \leqslant \left[\frac{\ln(m+nq)}{\ln p}\right]$$

because $2\left[\frac{n}{s}\right]-\left[\frac{2n}{s}\right] \leqslant 0$.

It is now clear that if we set

$$\Omega_n = \prod_{p \leqslant m+nq} p^{\left[\frac{\ln(m+nq)}{\ln p}\right]},$$

then the inequality

$$\Omega_n \leqslant \prod_{p \leqslant m+nq}(m+nq) = e^{\pi(m+nq)\ln(m+nq)} < e^{2(m+nq)}$$

will hold as a consequence of the simple Chebyshev estimate $\pi(x)\ln x < 2x$.

Since $2^{2n}\Omega_n a_{2k}$, $k=0, 1, \ldots, n$ are integers and $2^{2n}\Omega_n = O(n^{\varepsilon n})$, $K_\lambda(z)$ as well $K_\lambda'(z)$ will indeed be an E-function.

Differentiating $K_\lambda(z)$, we shall obtain a second order differential equation

$$K_\lambda''(z)-\frac{2\lambda+1}{z}K_\lambda'(z)+K_\lambda(z) = 0 \tag{91}$$

for this function. Setting $K_\lambda(z)=f_1(z)$, $K_\lambda'(z)=f_2(z)$, we obtain the linear system

$$\left\{\begin{aligned} f_1'(z) &= f_2(z), \\ f_2'(z) &= -\frac{2\lambda+1}{z}f_2(z)-f_1(z) \end{aligned}\right. \tag{92}$$

for the E-functions $f_1(z)$ and $f_2(z)$.

In order to prove the normality of the set of E-functions $f_1^{n_1}(z)f_2^{n_2}(z)$, $n_1+n_2\leqslant N$, we shall need auxiliary purely analytic considerations. First of all, we shall have to prove the absence of any interrelationships among the functions

$$y = J_\lambda(x) = \frac{1}{\Gamma(\lambda+1)}\left(\frac{x}{2}\right)^\lambda K_\lambda(x), \quad y' \quad \text{and} \quad x.$$

The absence of such relationships among y', y, x is equivalent, in virtue of the rationality of λ, to the absence of algebraic relationships among x, $K_\lambda(x)$ and $K_\lambda'(x)$, and it will be most convenient to carry out all the arguments with the functions $J_\lambda(x)$. The function $y=J_\lambda(x)$, being a Bessel function, satisfies the equation

$$y''+\frac{1}{x}y'+\left(1-\frac{\lambda^2}{x^2}\right)y = 0, \tag{93}$$

with λ not equal to a negative integer or to half an arbitrary odd integer. We shall assume that y is any fixed nontrivial solution of equation (93), which also satisfies the equation

$$P(x,y,y') = \sum_{k=0}^{m} P_k(x,y,y') = 0, \tag{94}$$

where $P(x,y,y')$ is a polynomial in x,y,y' which does not vanish identically, and the $P_k(x,y,y')$ are polynomials in y and y' which are homogeneous of order k. The polynomial $P(x,y,y')$ can, without loss of generality, be assumed irreducible and we can suppose that $P_m(x,y,y')\not\equiv 0$, $m\geqslant 1$. It is not difficult to note that y' must occur in $P(x,y,y')$. For in the contrary case, equation (94) would show that y is an algebraic function. But then y would be expandable at infinity into a series with decreasing rational powers of x with the same denominator. It is easily seen that any such expansion, set in place of y in (93), cannot make the left member identically equal to zero, because its unique highest term will remain the unique highest term after differentiation in the left member of (93). Therefore y cannot be an algebraic function and y' appears in $P(x,y,y')$. It follows from the irreducibility assumption and the last argument that any polynomial $R(x,y,y')$ which vanishes identically for the solution y of equation (93) which we have chosen must be divisible by $P(x,y,y')$ completely and algebraically, as a polynomial in x,y,y'. In the contrary case, we could eliminate y' from the two equations and we would again arrive at the fact that y is an algebraic function. We shall now differentiate equation (94) with respect to x and consider equation (93). We shall obtain a new equation

$$(95)\quad \frac{\partial P}{\partial x}+\frac{\partial P}{\partial y}y'-\frac{\partial P}{\partial y'}\left[\frac{1}{x}y'+\left(1-\frac{\lambda^2}{x^2}\right)y\right]$$

$$=\sum_{k=0}^{m}\left\{\frac{\partial P_k}{\partial x}+\frac{\partial P_k}{\partial y}y'-\frac{\partial P_k}{\partial y'}\left[\frac{1}{x}y'+\left(1-\frac{\lambda^2}{x^2}\right)y\right]\right\}=0$$

where, obviously, the k-th term of the sum is again a homogeneous polynomial in y' and y with homogeneity of degree k. Since the equation (95) again yields an algebraic connection among y, y', x, its left member must be divisible by $P(x, y, y')$, where, inasmuch as the degrees of both polynomials in y and y' are the same, their behavior must depend only on x and, as is easy to see, it must have the form $a+b/x+c/x^2$, where a, b and c are constants. Moreover, since the dimension of the terms of the sums in (94) and (95) are identical for the same k, for $P_m(x, y, y')\not\equiv 0$ we shall have the following identity in x, y, y':

$$(96)\quad \frac{\partial P_m}{\partial x}+\frac{\partial P_m}{\partial y}y'-\frac{\partial P_m}{\partial y'}\left[\frac{y'}{x}+\left(1-\frac{\lambda^2}{x^2}\right)y\right]=\left(a+\frac{b}{x}+\frac{c}{x^2}\right)P_m.$$

Since (96) is an identity in x, y, y', the left member of (96) equals $dP_m(x)/dx$ for any solution y of equation (93). Therefore

$$\frac{dP_m(x, y, y')}{dx}=\left(a+\frac{b}{x}+\frac{c}{x^2}\right)P_m(x, y, y'),$$

if y is any solution of (93) and after integrating

$$(97)\qquad P_m(x, y, y')=\theta x^b e^{ax-c/x}$$

where θ is a constant, depending only on the chosen solution y. Suppose y_1 and y_2 are two arbitrary linearly independent solutions of equation (93). Setting $y=t_1y_1+t_2y_2$ and substituting into equation (97), we obtain that $\theta=\theta(t_1, t_2)$ must be a homogeneous polynomial of degree m in t_1 and t_2, because the left member of (97) is such a polynomial. But then $\theta(t_1, t_2)=t_2{}^m\theta(t_1/t_2)$ from which it follows that t_1' and t_2' can be chosen, both different from zero and such that $\theta(t_1', t_2')$ vanishes. For the solution $y_0=t_1'y_1+t_2'y_2$ chosen in this manner, we obtain the equation

$$(98)\qquad P_m(x, y_0, y_0')=y_0{}^mR(x, t)\equiv 0,$$

where $t=y_0'/y_0$ and $R(x, t)$ is a polynomial because $P(x, y, y')$ is a polynomial which is homogeneous in y, y'. Since $y_0' \not\equiv 0$, equation (98) shows that t is an algebraic function in x. It is not difficult to show that t satisfies an equation of the first order. In fact, using equation (93), we shall have the equation

$$\frac{dt}{dx} = \frac{y_0''y_0 - y_0'^2}{y_0^2} = -\frac{\frac{1}{x}y_0'y_0 + \left(1-\frac{\lambda^2}{x^2}\right)y_0^2 + y_0'^2}{y_0^2}, \tag{99}$$

or

$$\frac{dt}{dx} + \frac{1}{x}t + t^2 + 1 - \frac{\lambda^2}{x^2} = 0. \tag{100}$$

Since t is an algebraic function of x, t must be expandable into a series with decreasing powers of x

$$t = \sum_{k=0}^{\infty} a_k x^{\alpha_k}, \quad a_0 \neq 0, \quad \alpha_k > \alpha_{k+1}, \tag{101}$$

where the α_k are integral or rational with the same denominators, for large values of x.

Substituting this expansion into (100), we should find that its left member vanishes identically. Since t^2 occurs in (100), we have $\alpha_0 \leqslant 0$, and in view of the presence of 1, $\alpha_0 \geqslant 0$. Therefore, comparing the 0-th powers of x, we obtain that $a_0^2+1=0$, or $a_0 = \pm i$. The next two, according to absolute value, powers of x in the left member of (100) will be a_0x^{-1} and $2a_0a_1x^{\alpha_1}$. Since the remaining powers are smaller, we have $\alpha_1 = -1$ and $a_0(1+2a_1)=0$. It follows that $a_1 = -\frac{1}{2}$. Further, if we suppose that all the α_k, $k=1, \ldots, n$, are negative integers, then the term $2a_0a_{n+1}x^{\alpha_{n+1}}$ which occurs in the left member of (100) from t^2, can be reduced only with powers of x having exponents not greater than α_{n+1} and, consequently, they have the form $-2, \alpha_k - 1, \alpha_k + \alpha_s$; $1 \leqslant k \leqslant n$, $0 \leqslant s \leqslant n$. Since all these powers, by hypothesis, are negative integers, the α_{n+1} are also negative integers. Since $\alpha_0 = 0$, $\alpha_1 = -1$, then, by induction, we can assert that all the α_k, $k \geqslant 1$ are negative integers. From this it follows that

$$t = \sum_{0}^{\infty} a_k x^{-k};$$

in other words, $t=t(x)$ is regular at infinity. Since, as can easily be seen, the only singular points of equation (93) are the points

0 and ∞, $t=y_0'/y_0$ cannot have branch points anywhere except at 0 and ∞. But we proved that $t(x)$ is regular at infinity and this means it also cannot have a branch point at zero. But an algebraic function without branch points must be rational and this means that $t(x)$ is a rational function of x. Therefore, $t(x)$ can be expanded in a Laurent series about $x=0$; in other words,

$$t = \sum_{k=-m}^{\infty} C_k x^k.$$

Setting this expansion into equation (100) and combining like terms, we at once obtain, due to the presence of t^2, that $m \leqslant 1$ and, because of the term $-\lambda x^{-2}$, that $m=1$ and $-C_{-1}+C_{-1}{}^2+C_{-1}-\lambda^2=0$. It follows that $C_{-1}=\pm\lambda$. If we make the substitution

$$y = (x-x_0)^m y_1(x), \quad x_0 \neq 0, \quad y_1(x_0) \neq 0,$$

where $y_1(x)$ is regular at the point x_0, into equation (93), it is easy to see that the solution cannot have a pole at x_0 and the order of the zero of y_0 cannot be greater than unity. Therefore the poles of the rational function $t(x)=y_0'/y_0$ will always be poles of the first order with residue 1. Since we already know the form of $t(x)$ at the origin, we now have the right to write $t(x)$ in the form

$$t(x) = \pm i \pm \frac{\lambda}{x} + \sum_{k=1}^{m} \frac{1}{x-x_k},$$

where the x_k are nonzero and distinct. Using this form of $t(x)$, we can write its expansion at infinity in the form

$$t(x) = \pm i + [\pm\lambda + m]x^{-1} + \sum_{k=2}^{\infty} a_k x^{-k}.$$

Comparing the value a_1 thus obtained with that found earlier, we see that λ must satisfy the condition

$$\lambda = \pm(m-\tfrac{1}{2}) = \pm\frac{2m-1}{2},$$

where m is an integer. But by assumption this is not possible. This means that relation (94) is also impossible.

Using the results just obtained, we can prove the following lemma.

LEMMA VIII. *Suppose y_0 and y_1 are two arbitrary linearly independent solutions of equation* (93) *and that λ does not equal half an odd integer. Then an algebraic relation among x, y_0, y_0', and y_1 does not exist.*

Proof. Let us assume the contrary, i.e. that the relation

$$(102) \qquad P(x, y_0, y_0', y_1) = f(x, y_0, y_0')y_1^n + \cdots = 0$$

holds, where $P(x, y_0, y_0', y_1)$ is a polynomial in x, y_0, y_0', y_1, and $f(x, y_0, y_0')$ is the coefficient of the highest power of y_1 which occurs in $P(x, y_0, y_0', y_1)$. The function $f(x, y_0, y_0')$ is, of course, a polynomial. The number n is $\geqslant 1$, because in the contrary case relation (102) would be an algebraic relation among x, y_0, y_0', the impossibility of which was proved above. We shall assume also that $P(x, y_0, y_0', y_1)$ is an irreducible polynomial. If $P(x, y_0, y_0', y_1)$ were the product of more than one irreducible factor, then, setting any one of them equal to zero, we should obtain a relation of type (102) for x, y_0, y_0', y_1 with irreducible P. Therefore, if $R(x, y_0, y_0', y_1) = 0$ is any other algebraic relation among the x, y_0, y_0', y_1, then the polynomial $R(x, y_0, y_0', y_1)$ must be divisible by $P(x, y_0, y_0', y_1)$. For in the contrary case, we could eliminate y_1 from the two relations and obtain an algebraic relation among the x, y_0, y_0', which is impossible. Differentiating (102) with respect to x, we get

$$(103) \qquad \frac{dP}{dx} = \frac{df}{dx}y_1^n + nfy_1^{n-1}y_1' + \cdots = 0.$$

Since y_0 and y_1 are linearly independent and are the solutions of a linear equation (93), by the Liouville theorem there exists a dependence between them,

$$(103') \qquad y_0y_1' - y_1y_0' = \frac{a}{x}, \quad a \neq 0,$$

from which we have

$$(104) \qquad \frac{dP}{dx} = \left[\frac{df}{dx} + nf\frac{y_0'}{y_0}\right]y_1^n + \cdots = 0,$$

where the left member is again a polynomial of degree n with respect to y_1 of the type $y_0^{-1}P(x, y_0, y_0', y_1)$, because y_0'' can be replaced by $-\frac{1}{x}y_0' - \left(1 - \frac{\lambda^2}{x^2}\right)y_0$ in all terms of the left member. Since the left member of (104), multiplied by y_0, must be divisible throughout by P, we can write the relation

$$(105) \qquad \frac{P'}{P} = \frac{1}{f}\left(f' + nf\frac{y_0'}{y_0}\right) = \frac{f'}{f} + n\frac{y_0'}{y_0},$$

which is an identity in x, y_0, y_0', y_1, where P' and f' are derivatives with respect to x, if y_0, y_1 are solutions of (93). But since this is an identity, it will be true whatever be the linearly independent solutions y_0, y_1 of (93), because then (103) will hold, with the identity being independent of a. Therefore, replacing y_1 in (105) by $y_1+\sigma y_0$, where σ is an arbitrary constant, and integrating, we shall have that

$$P(x, y_0, y_0', y_1+\sigma y_0) = \gamma(\sigma) f(x, y_0, y_0') y_0^n$$

where $\gamma(\sigma)$ is a polynomial in σ of degree n with constant coefficients. If we differentiate this relation with respect to σ and set $\sigma=0$, we obtain that

$$(106) \qquad y_0[nfy_1^{n-1}+\cdots]-C_1fy_0^n = 0,$$

where C_1 is a constant and the omitted terms in the square brackets are of degree less than or equal to $n-2$ in y_1. But, as we already know, the left member of (106) must, as a polynomial in x, y_0, y_0', y_1, be divisible by $P(x, y_0, y_0', y_1)$, which is impossible, because the degree of the left member of (106) is not greater than $n-1$ (the degree of P with respect to y_1 is $n \geqslant 1$). Therefore, all the coefficients of $y_1^{n-1}, y_1^{n-2}, \ldots, y_1$ in the left member must vanish. But since equality $f(x, y_0, y_0')=0$ is impossible by what we proved above, we conclude from (106) that $n=1$. Therefore $P(x, y_0, y_0', y_1)$ must be of the first degree in y_1, from which it follows that

$$y_1 = \frac{U_1(x, y_0, y_0')}{U_2(x, y_0, y_0')}$$

where U_1 and U_2 are polynomials in x, y_0, y_0' which have no common divisors. Setting this representation for y_1 in equation (103), we obtain, in virtue of (93), the relation

$$(107) \quad \frac{y_0}{U_2^2}\left[\left(\frac{\partial U_1}{\partial x}+\frac{\partial U_1}{\partial y_0}y_0'-\frac{\partial U_1}{\partial y_0'}\left(\frac{y_0'}{x}+\left(1-\frac{\lambda^2}{x^2}\right)y_0\right)\right)U_2\right.$$
$$\left.-\left(\frac{\partial U_2}{\partial x}+\frac{\partial U_2}{\partial y_0}y_0'-\frac{\partial U_2}{\partial y_0'}\left(\frac{y_0'}{x}+\left(1-\frac{\lambda^2}{x^2}\right)y_0\right)\right)U_1\right]-\frac{U_1}{U_2}y_0' = \frac{a}{x}.$$

This must be an identity in the variables x, y_0, y_0' because it is impossible to have a nontrivial relation among x, y_0, y_0'. Suppose d_1 is the largest sum of exponents of the terms $y_0^{n_1} y_0'^{n_2}$, $n_1+n_2=d_1$, which actually occur in U_1, and that d_2 is the corresponding number for U_2. We combine all terms of degree d_1 occurring in U_1, and all the terms of degree d_2 occurring in U_2, and we denote their sums by V_1 and V_2. The difference $d=d_1-d_2$ will be called the dimension of the function U_1/U_2. Then, clearly, the dimension of the function $U_1/U_2 - V_1/V_2$ will be less than d. If we likewise set $W = V_1/V_2$, we see that W is a ratio of polynomials, which are homogeneous in y_0 and y_0', where the dimension of W is d. Since

$$y_1' = W' + \frac{d}{dx}\left[\frac{U_1}{U_2} - W\right] \tag{108}$$

in virtue of the homogeneity of (93), after the replacement $y_0'' = -\frac{1}{x}y_0' - \left(1 - \frac{\lambda^2}{x^2}\right)y$, the dimension of W', as the ratio of polynomials homogeneous in y_0, y_0', remains equal to d, and the dimension of the other term in (108) is less than d. Since the right member of identity (107) has zero dimension, the term $y_0W' - y_0'W$ in the left member has dimension $d+1$, and the dimension of the remaining terms is less than $d+1$, we have $d+1 \geqslant 0$. For in the contrary case, (107) could not be an identity. If $d+1 > 0$, the term $y_0W' - y_0'W$ in the left member of (107) cannot be cancelled out with any other term which is the ratio of homogeneous polynomials and having higher dimension. From this it follows that $y_0W' - y_0'W = 0$ or $W = Cy_0$, where C is a constant. Since, in virtue of the impossibility of having algebraic relations among x, y_0, y_0', $W = Cy_0$ is an identity in x, y_0, y_0', in this case the dimension of W is 1. But if $d+1 = 0$, then, by the same arguments,

$$y_0W' - y_0'W = \frac{a}{x}, \quad a \neq 0, \tag{109}$$

where this relation is an identity in the variables x, y_0, y_0', and W is the ratio of polynomials which are homogeneous in y_0, y_0', and the dimension of W is $d = -1$. We note that the first case, when $d = 1$, reduces to the second case by the substitution of $y_1 - Cy_0$ for y_1, which is always possible. Thus, there always exists a function W of dimension $d = -1$ having the properties stated above and which satisfies (109) identically in the variables x, y_0, y_0'. But, as is rather easy to prove, if W satisfies (109), then W is linearly independent with the solution y_0 of (93). Since (109) is an identity in the variables x, y_0, y_0' and W' remains the derivative of W with respect to x when y_0 is replaced by any other solution of (93), then (109) remains valid when y_0 is replaced by any other solution of (93). Replacing y_0 by $t_0z_0 + t_1z_1$, where z_0 and z_1 are arbitrary linearly independent solutions of (93), and t_0, t_1 are constants, we may now assert that

$$W = W(x,\, t_0 z_0 + t_1 z_1,\, t_0 z_0' + t_1 z_1') = T_0 z_0 + T_1 z_1, \tag{110}$$

where T_0 and T_1 are certain constants which depend only on t_0 and t_1, since W is a solution of (93), and z_0 and z_1 are linearly independent. Computing the expressions $Wz_0' - W'z_0$ and $Wz_1' - W'z_1$ by means of replacing W by $T_0 z_0 + T_1 z_1$ in them, and using relation (103), we will have the relations

$$Wz_1' - W'z_1 = \frac{C_0}{x} T_1, \quad Wz_0' - W'z_0 = -\frac{C_0}{x} T_0,$$

where C_0 is a constant which does not depend on t_0 and t_1. These relations show that T_0 and T_1 are homogeneous rational functions in t_0 and t_1 of dimension -1. Therefore, one can find nonzero, finite values t_0 and t_1 such that at least one of the numbers T_1 and T_2 becomes infinite. But then it already follows from (110) and the linear independence of z_0 and z_1 that

$$W_1 = \frac{1}{W(x,\, z_2,\, z_2')} = 0, \quad z_2 = t_0' z_0 + t_1' z_1,$$

where W_1 is a homogeneous rational function in z_2, z_2' of dimension 1. But such an algebraic relation is impossible, as was already proved, and this also proves that relation (102) is impossible.

In order to prove the algebraic independence of the numbers $K_\lambda(\alpha)$ and $K_\lambda'(\alpha)$ over an arbitrary finite algebraic field, with $\alpha \neq 0$ algebraic and $\lambda \neq \dfrac{2n-1}{2}$ rational, where n is an integer, it suffices to prove that the system of E-functions

$$f_1^{k_0}(z) f_2^{k_1}(z), \quad k_0 + k_1 \leqslant N, \quad f_1(z) = K_\lambda(z), \quad f_2(z) = K_\lambda'(z)$$

is normal for arbitrary N. This is a simple corollary to the fundamental theorem inasmuch as $f_1(z)$ and $f_2(z)$ is a solution of system (92).

Setting $F_{k,q}(z) = f_1^k(z) f_2^{q-k}(z)$, taking the derivative of $F_{k,q}(z)$ and using system (92), we obtain the equation

$$F_{k,q}'(z) = kF_{k-1,q}(z) - (q-k)\, \frac{2\lambda+1}{x}\, F_{k,q}(z) - (q-k) F_{k+1,q}(z),$$

$$k = 0, 1, \ldots, q, \tag{111}$$

where we agree that $F_{-1,q} = F_{q+1,q} = 0$. The set of $q+1$ functions $F_{0,q}, \ldots, F_{q,q}$ is a solution of this linear system. The set of such systems for $q = 0, 1, \ldots, N$ has as solution the set of functions $F_{k,q}(z)$, $0 \leqslant k \leqslant q$, $0 \leqslant q \leqslant N$, the number of which is $\mu_N = (N+1)$

$(N+2)/2$. We see that the system of linear differential equations for the μ_N functions reduced to $N+1$ primitive systems (111). If we assume that the matrix of the fundamental system of solutions of system (111) consists of the elements $U_{k,s,q}(z)$, $0 \leqslant k \leqslant q$, $0 \leqslant s \leqslant q$, then the proof of the normality of the system of the μ_N functions $F_{k,q}(z)$ will consist in proving that the relation

$$T = \sum_{q=0}^{N} \sum_{k=0}^{q} \sum_{s=0}^{q} P_{k,q}(x) C_{s,q} U_{k,s,q}(x) \equiv 0, \tag{112}$$

where the $P_{k,q}(x)$ are nonzero polynomials and the $C_{s,q}$ are constants which are also all not zero, cannot hold. We shall assume that relation (112) holds. Since the functions $f_1(z)$ and $f_2(z)$ satisfy system (92), replacing $f_1(z)$ by V and $f_2(z)$ by V', where V is any solution of the equation

$$V'' + \frac{2\lambda+1}{x} V' + V = 0, \tag{113}$$

we see that the functions $F_{k,q} = V^k V'^{q-k}$, $0 \leqslant k \leqslant q$ are again a solution of system (111). Setting $V = t_0 V_0 + t_1 V_1$, where V_0 and V_1 are linearly independent solutions of (113), and t_0, t_1 are constants, we can define $q+1$ functions $U_{k,s,q}(z)$ by the following relations

$$V^k (V')^{q-k} = \sum_{s=0}^{q} t_0{}^s t_1{}^{q-s} U_{k,s,q}(x), \quad 0 \leqslant k \leqslant q, \tag{114}$$

which are identities in t_0, t_1. The fact that the functions $U_{k,s,q}$, $0 \leqslant k \leqslant q$, form a solution of system (111) follows from the possibility of differentiating $V^k(V')^{q-k}$ s times with respect to t_0 and the possibility of then setting $t_0=0$, $t_1=1$. These manipulations with the system of functions $V^k(V')^{q-k}$, $k=0, 1, \ldots, q$, in virtue of the linearity of (111), lead to a system of functions which again is a solution of (111). The linear independence of the $q+1$ solutions $U_{0,s,q}, U_{1,s,q}, \ldots, U_{q,s,q}$ for $s=0, 1, \ldots, q$ follows simply from the fact that any linear relation among the $U_{q,s,q} = \dfrac{q!}{s!(q-s)!} V_0{}^s V_1{}^{q-s}$, $0 \leqslant s \leqslant q$, with constant coefficients, would lead to an algebraic equation in V_0/V_1. But this is impossible in virtue of the linear independence of V_0 and V_1.

We set

$$y_0 = x^\lambda V_0, \quad y_1 = x^\lambda V_1.$$

Then y_0 and y_1 will be two linearly independent solutions of equation (93) and, moreover,

$$(115)\qquad V_i' = x^{-\lambda}\left(y_i' - \frac{\lambda}{x} y_i\right), \quad i = 0, 1; \quad y_0 y_1' - y_1 y_0' = \frac{a}{x},$$

where a is a nonzero constant. After straightforward transformations, we also obtain that

$$t_0 V_0 + t_1 V_1 = x^{-\lambda}(t_0 y_0 + t_1 y_1),$$

$$t_0 V_0' + t_1 V_1' = x^{-\lambda}\left[t_0 y_0' + t_1 y_1' - \frac{\lambda}{x}(t_0 y_0 + t_1 y_1)\right]$$

$$= x^{-\lambda}\left[\left(\frac{y_0'}{y_0} - \frac{\lambda}{x}\right)(t_0 y_0 + t_1 y_1) + \frac{a t_1}{x y_0}\right]$$

in virtue of (115), from which it already follows that

$$(116)\quad \sum_{s=0}^{q} t_0^s t_1^{q-s} U_{k,s,q}(x)$$
$$= x^{-\lambda q}(t_0 y_0 + t_1 y_1)^k \left[\left(\frac{y_0'}{y_0} - \frac{\lambda}{x}\right)(t_0 y_0 + t_1 y_1) + \frac{a t_1}{x y_0}\right]^{q-k}.$$

It follows that $x^{q(p+1)} y_0^{q-k} U_{k,s,q}(x)$ is a polynomial in y_0, y_0', y_1 and $x^{1/r}$, if $\lambda = p/r$. Multiplying the left member of (112) by $y_0^N x^{N(p+1)}$, assuming, naturally, that not all the products $P_{k,q} C_{s,q} = 0$, and replacing the $U_{k,s,q}(x)$ by their values from (116), we obtain a relation among $x^{1/r}, y_0, y_0', y_1$. If in this connection the left member of (112) does not vanish identically in the variables $x^{1/r}$, y_0, y_0', y_1 then, replacing $x^{1/r}$ by the product $\omega_k x^{1/r}$ where ω_k is an r-th root of unity, $\omega_1 = 1$, and multiplying the expressions obtained for $k = 1, 2, \ldots, r$, we arrive at a nontrivial algebraic relation among x, y_0, y_0', y_1, which is impossible by Lemma VIII. Thus, the left member of (112) must vanish identically in the variables x, y_0, y_0', y_1 after replacing the $U_{k,s,q}$ by their values. We shall now assume that all the products $P_{k,q} C_{s,q} = 0$ for k, s; $1 \leqslant k \leqslant q$, $1 \leqslant s \leqslant q$ and $q = n+1, \ldots, N$.

We shall now determine the form of $U_{k,s,n}(x)$ from relations (116). We shall have

$$U_{k,s,n}(x) = \frac{n!}{s!(n-s)!} x^{-\lambda n} \left(\frac{y_0'}{y_0} - \frac{\lambda}{x}\right)^{n-k} y_0^s y_1^{n-s}.$$

Substituting these expressions into the left member of (112) and combining all terms of degree n with respect to all the variables y_0, y_0', y_1, we obtain that their sum must vanish identically, because

(112) is an identity in all the variables which appear in it, and the first degree terms in the variables y_0, y_0', y_1 cannot interfere with terms of other powers. We thus obtain the identity

$$y_0^{-n}\sum_{k=0}^{n}\sum_{s=0}^{n}P_{k,n}(x)C_{s,n}\frac{n!}{s!\,(n-s)!}\left(\frac{y_0'}{y_0}-\frac{\lambda}{x}\right)^{n-k}\left(\frac{y_1}{y_0}\right)^{n-s}\equiv 0$$

in the variables y_0'/y_0 and y_1/y_0. It follows directly from this that

$$P_{k,n}(x)C_{s,n}=0 \quad \text{for} \quad 0\leqslant k\leqslant n, \quad 0\leqslant s\leqslant n.$$

Thus we have proved that relation (112) cannot hold if at least one of the products $C_{s,q}P_{k,q}(x)$ is nonzero.

This proves the normality of the system of E-functions $F_{k,q}(z)$, $0\leqslant k\leqslant q$, $0\leqslant q\leqslant N$, for arbitrary N, from which it follows on the basis of the fundamental theorem of this section that the numbers $K_\lambda(\alpha)$ and $K_\lambda'(\alpha)$ are algebraically independent over an arbitrary finite algebraic field and, in particular, that they are transcendental if $\alpha\neq 0$ is algebraic, and λ is rational and not equal to half an odd integer or to a negative integer. In the same way, one can prove that the $2m$ numbers

$$K_\lambda(\alpha_1),\ K_\lambda'(\alpha_1),\ \ldots,\ K_\lambda(\alpha_m),\ K_\lambda'(\alpha_m)$$

with distinct, nonzero, and algebraic $\alpha_1, \alpha_2, \ldots, \alpha_m$, and λ rational, but not equal to half an odd number, are algebraically independent over an arbitrary finite algebraic field. We note that the proof of the normality of the system of E-functions is very difficult if these functions are the products of powers of some function and its derivatives, where the fundamental function satisfies a linear differential equation with polynomial coefficients of degree higher than two.

§5. Problems concerning transcendence and algebraic independence, over the rational field, of numbers defined by infinite series or which are roots of algebraic or transcendental equations

We shall say that the number η is not algebraically expressible in terms of the numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ if no algebraic relation in the rational field exists among the numbers $\eta, \alpha_1, \alpha_2, \ldots, \alpha_n$. Introducing the concept of algebraic nonexpressibility, Morduhai-Boltovskoi was the first to formulate the problem of criteria enabling one to determine the algebraic nonexpressibility of the number η in terms of the numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ which was in nature the approximation of the number η by rational fractions or, more generally, by algebraic numbers. We shall agree that (in the following discussion this applies to the entire section) $P(x, y_1, \ldots, y_n)$ is always a polynomial with rational integral coefficients, irreducible in the rational field and actually containing x and at least one y. Morduhai-Boltovskoi proved [1, 2] the following two theorems on algebraic nonexpressibility.

If η is a root of the equation

$$P(x, e^{\alpha_1}, \ldots, e^{\alpha_n}) = 0, \tag{117}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are algebraic numbers which are linearly independent over the rational field, or η is a root of the equation

$$P(x, \ln \alpha) = 0, \tag{118}$$

where $\alpha \neq 0, 1$ is algebraic, then there exists an integer $\nu > 0$ such that the inequality

$$\left| \eta - \frac{p}{q} \right| > \frac{1}{q^{\nu}!} \tag{119}$$

is satisfied for $q > q_0$, where p and q are rational integers, and ν does not depend on p and q. Both theorems generalize to the case of approximation of the number η by algebraic numbers.

The fact that inequality (119) is not satisfied is also a condition that the number η not be expressible in terms of the numbers $e^{\alpha_1}, e^{\alpha_2}, \ldots, e^{\alpha_n}$ or the number $\ln \alpha$. Hermite's identity and the mean value theorem for functions lie at the base of the proof of the above theorems.

In these theorems, inequality (119) can be significantly sharpened with the aid of estimates of the measure of transcendence of the corresponding numbers, which estimates were made much later. It follows almost directly from inequalities (48) and (49), §2, that inequality (119) can be replaced by the inequality

$$\left| \eta - \frac{p}{q} \right| > \frac{1}{q^{\nu}}, \quad \nu = \nu(\eta), \tag{120}$$

in the Morduhai-Boltovskoi theorems. It follows from the last inequality that Liouville numbers cannot be expressed algebraically in terms of $e^{\alpha_1}, \ldots, e^{\alpha_n}$ or $\ln \alpha$. An example of such a number is

$$\omega = \sum_{1}^{\infty} 2^{-n!}.$$

The definition of a Liouville number was given in §2 (definition (47)).

Morduhai-Boltovskoi also gave some generalizations of the theorems stated above.

Using various inequalities for the measure of transcendence of various numbers, one may give other criteria that a number not be expressible algebraically.

Suppose η is some real number, p and q, $(p, q)=1$, are rational integers, and $\varphi(t)$ is a positive monotonically increasing function of t. Assume that the condition

$$\lim_{H\to\infty} \frac{\ln \Phi(H, 1, \eta)}{\varphi(H)} = \lim_{q\to\infty} \frac{\min\limits_{q_0<q} \ln \left|\eta - \frac{p}{q_0}\right|}{\varphi(q)} = -\infty \tag{121}$$

is satisfied for the number η. If $\varphi(q) = \ln q[\ln \ln q]^{\lambda}$, $\lambda > 2$, then η cannot be algebraically expressed in terms of α^{β}, and if $\varphi(q) = [\ln q]^{\lambda}$, $\lambda > 2$, then η cannot be expressed in terms of $\dfrac{\ln \alpha}{\ln \beta}$ with α and β algebraic. This fact is a direct consequence of inequalities (116), (117) of Chapter III. If $\varphi(q) = \ln q$, in other words η is a Liouville number, then η cannot be expressed algebraically in terms of $J_0(\alpha)$, $J_0'(\alpha)$, where $J_0(x)$ is a Bessel function, and α is an algebraic number. This is a direct consequence of inequality (43), §2.

Finally, if $\lim \dfrac{\ln \varphi(t)}{\ln t} = \infty$, then, as was proved by Morduhai-Boltovskoi [6], the number η cannot be a root of the equation

$$P(x, a_1^x, \ldots, a_n^x) = 0, \tag{122}$$

where $a_1, a_2, \ldots, a_n$ are algebraic numbers. He proved the existence of an integer $\nu > 0$ such that $\left|\eta - \dfrac{p}{q}\right| > \dfrac{1}{q^{\nu}!}$. This can be obtained with the aid of Lemma I, §2.

In the same paper, Morduhai-Boltovskoi asserts that the theorem also holds when the a_i $(i = 1, 2, \ldots, n)$ are, more generally, the roots of an equation of type (118) with the same $\ln \alpha$. In fact, a calculation shows that it follows from inequality (51), §2, which is true for arbitrary n and H, that the inequality

$$\left|\eta - \frac{p}{q}\right| > \exp\left[-q^{3n} \ln q\right]$$

holds for η, where n is the number of exponential functions in equation (122) and $q > q_0$.

Finally, Morduhai-Boltovskoi introduced the concept and proved the existence of hypertranscendental numbers.

A transcendental value of every function $F(x)$ with algebraic values of the argument, which is the solution of any algebraic differential equation with constant integral coefficients, determined by algebraic initial conditions, will be called a *simply transcendental number*. All other complex numbers will be called *hypertranscendental numbers*. The existence of hypertranscendental numbers is a consequence of the countability of the set of simply transcendental numbers and the noncountability of the set of all real or complex numbers.

We note further that the transcendence of the roots of the equation

$$P(x, e^{\alpha_1 x}, \ldots, e^{\alpha_n x}) = 0,$$

with algebraic and linearly independent $\alpha_1, \alpha_2, \ldots, \alpha_n$, is a consequence of Lindemann's general theorem.

CHAPTER III

Arithmetic Properties of the Set of Values of an Analytic Function whose Argument Assumes Values in an Algebraic Field; Transcendence Problems

§1. Integrity of analytic functions

There is a very essential relationship between the growth of an entire analytic function and the arithmetic nature of its values for an argument which assumes values in a given algebraic field. If we assume in this connection that the values of the function also belong to some definite algebraic field, where all the conjugates of every value do not grow too rapidly in this field, then this at once places a restriction on the growth of the function from below, in other words, it cannot be too small. This situation and its analogues for meromorphic functions can be used with success to solve transcendence problems. The first theorem concerning the relationship between the growth and the arithmetic value of a function was the Pólya theorem [1]. Pólya showed that if an entire analytic function assumes integral values for positive rational integral values of its argument, and its growth is bounded by the inequality

$$|f(z)| < C2^{\alpha|z|}, \quad \alpha < 1,$$

then it must be a polynomial. Of further results in this direction, a number of which are due to the author of this book, we introduce only one general theorem, to which we shall restrict ourselves, inasmuch as we are not concerned directly with these problems in this book.

We shall assume that the values of an entire analytic function for an argument assuming values in some countable set E belong to a finite algebraic field. Then it is even not necessary to assume that the set E itself, for example, consists of all the integers in a finite algebraic field, in order that the growth of an entire function be subject to definite limitations, the nonfulfillment of which would imply that the function belongs to a subclass of some class of entire functions. To this end, it suffices to assume that E possesses some group-lattice properties. Suppose the maximum of the absolute values of an entire function is $M(r)$ and that the set E has infinity as its only limit point. We shall call an entire function $f(z)$ a *normal integral function* if the values of $f(z)$ for $z \in E$ are integers in an algebraic field K of degree ν and if, in this connection, $\alpha \in E$ implies that

$$(1) \qquad \overline{|f(\alpha)|} < C_0[M(|\alpha|)]^{1+\delta}, \quad C_0 = C_0(\delta),$$

for arbitrary $\delta > 0$ and C_0 which does not depend on α. The number ν is called the *degree of integrity of* $f(z)$. We shall also assume that the set E consists of all sums of the form $\alpha_k + \beta_k$ where α_k, $|\alpha_k| \leqslant |\alpha_{k+1}|$, $k = 0, 1, \ldots$, are points in a countable set E_1, and β_k, $|\beta_k| \leqslant |\beta_{k+1}|$, are points in a set E_2; we shall define the functions $N_1(r)$, $N_2(r)$ and $N(r)$ by setting

$$(2) \qquad N_1(r) = \sum_{|\alpha_k| \leqslant r} 1, \quad N_2(r) = \sum_{|\beta_k| \leqslant r} 1,$$
$$N(r) = \min [N_1(r), N_2(r)].$$

We shall call $N(r)$ the *additive density of the set E*. We note that the sets E_1 and E_2 may coincide. Then the following theorem can be proved.

THEOREM I. *If the entire function $f(z)$ with maximum modulus $M(r)$ is normal integral on the set E, $E = E_1 \cup E_2$, with additive density $N(r)$, and its values on E belong to a field K of degree ν, then two numbers θ and λ, for example $\theta > 2 + \sqrt{2}$ and $\lambda < \frac{1}{8\nu} \ln \frac{(\theta-1)^2 - 1}{2\theta - 2}$, where ν is the degree of integrity, can be found such that if the inequality*

$$(3) \qquad \ln M(\theta r) < \lambda N(r)$$

is satisfied then $f(z)$ is a solution of the functional equation

$$(4) \qquad \sum_{k=0}^{m} A_k f(z + \beta_k) = 0, \quad m > 1,$$

where $A_1, A_2, \ldots, A_m$ are algebraic numbers, which are all nonzero, in the field K.

Proof. In the following arguments, the numbers $a_1, a_2, \ldots$ will be constants which are independent of ϱ; we shall assume the real number $\varrho>0$ is sufficiently large and whose magnitude will be determined later.

We consider the auxiliary function $F(z)$ defined by

$$F(z) = \sum_{k=0}^{n} A_k f(z+\beta_k), \quad n = N(\varrho). \tag{5}$$

By Lemma II, §2, Chapter II, the numbers A_k, $0\leqslant k\leqslant n$, can be chosen to be algebraic integers in the field K, all different from zero, in such a way that the $m+1$ equalities

$$F(\alpha_k) = 0, \quad 0 \leqslant k\leqslant m, \quad m = \left[\frac{n-1}{2}\right] \tag{6}$$

and the $n+1$ inequalities

$$\overline{|A_k|} < a_0 n[M(2\varrho)]^{1+\delta}, \quad 0 \leqslant k \leqslant n, \tag{7}$$

are satisfied for arbitrary fixed $\delta>0$ and a_0 depending only on δ. In fact, since $|\alpha_k+\beta_k|\leqslant 2\varrho$ for $k\leqslant n$, $s\leqslant n$ and $\overline{|f(\alpha_k+\beta_s)|}<$ $a_0\ [M(2\varrho)]^{1+\delta}$ on the basis of inequality (1), our assertion follows directly from Lemma II as a consequence of the fact that the $F(\alpha_k)$, $0\leqslant k\leqslant m$, are linear forms with respect to the A_k.

We now choose a number δ, $0<\delta<1$, in condition (1) and an $\varepsilon>0$ satisfying the inequalities

$$0 < \varepsilon < \frac{(\theta-1)^2-1}{\theta}, \quad 2+\delta < \frac{1}{4\nu\lambda}\ln\frac{(\theta-1)^2-1-\varepsilon}{(\theta-1)(2+\varepsilon)}, \tag{8}$$

which is possible inasmuch as θ and λ satisfy the conditions of our theorem.

Since the entire function $F(z)$ has zeros at the points α_k, $0\leqslant k\leqslant m$, using known properties of the Blaschke product, we shall have, by the maximum principle, that for $|z|\leqslant R$, $R=(\theta-1)\varrho$,

$$|F(z)| < a_2 N^2(\varrho)[M(2\varrho)]^{1+\delta} M(R+\varrho)\prod_{k=0}^{m}\frac{R|z-\alpha_k|}{|R^2-z\bar{\alpha}_k|}, \tag{9}$$

from which it follows, by virtue of (3), that

$$\begin{aligned} |F(z)| &< a_2 N^2(\varrho)[M(\theta\varrho)]^{2+\delta}\left[\frac{(\theta-1)(2+\varepsilon)}{(\theta-1)^2-1-\varepsilon}\right]^m \\ &< a_3 \exp\left\{\left[\lambda(2+\delta)-\tfrac{1}{2}\ln\frac{(\theta-1)^2-1-\varepsilon}{(\theta-1)(2+\varepsilon)}\right]N(\varrho)+2\ln N(\varrho)\right\}. \end{aligned} \tag{10}$$

But condition (1) and inequalities (7) with $|\alpha_k| \leqslant \varrho_1$ imply the inequalities

$$(11) \quad \overline{|F(\alpha_k)|} < a_0 N(\varrho)[M(2\varrho)]^{1+\delta} \sum_{s=0}^{n} \overline{|f(\alpha_k+\beta_s)|} < a_4 N^2(\varrho)[M(2\varrho)M(\varrho_1+\varrho)]^{1+\delta},$$

where a_4 does not depend on either ϱ_1 or ϱ, or

$$(12) \quad \overline{|F(\alpha_k)|} < a_4 N^2(\varrho)\{M[(2+\varepsilon)\varrho]\}^{2+2\delta} < a_4 \exp\,[2\lambda(1+\delta)N(\varrho)+2\ln N(\varrho)]$$

for $\varrho_1=(1+\varepsilon)\varrho$. Since all the numbers $F(\alpha_k)$ are integral algebraic by assumption, if one assumes that $F(\alpha_k)\neq 0$ for any k, then we arrive at the inequality

$$(13) \qquad 1 \leqslant |F(\alpha_k)| \prod_{1}^{\nu-1} |F_s(\alpha_k)|,$$

where the product is taken over all the $\nu-1$ numbers which are conjugate with $F(\alpha_k)$. Making use of inequalities (10) and (12), we obtain from this the inequality

$$(14) \quad 1 \leqslant a_5 \exp\left\{\left[2\lambda\nu(2+\delta) - \tfrac{1}{2}\ln\frac{(\theta-1)^2-1-\varepsilon}{(\theta-1)(2+\varepsilon)}\right] N(\varrho)+2\nu\ln N(\varrho)\right\},$$

where $a_5=a_2a_4{}^{\nu-1}$. But in virtue of inequalities (8), $N(\varrho)$ has a negative coefficient. This means that when $\varrho>\varrho_0$ and $|\alpha_k|\leqslant(1+\varepsilon)\varrho$ the right member of inequality (14) becomes less than 1. Therefore, for $F(\alpha_k)=0$, we have $|\alpha_k|\leqslant(1+\varepsilon)\varrho_0$. But the number of α_k, whose absolute values are not greater than ϱ_1, is not less than $N(\varrho_1)=N[(1+\varepsilon)\varrho]$. Making use again of inequality (9) with $m=m_1=N(\varrho_1)$ and setting $\varrho_2=(1+\varepsilon)\varrho_1$, we obtain that for $|z|\leqslant\varrho_2$, $R=(\theta-1)\varrho_1$ the inequality

$$(15) \quad |F(z)| < a_2 N^2(\varrho_1)[M(\theta\varrho_1)]^{2+\delta}\left[\frac{(\theta-1)(2+\varepsilon)}{(\theta-1)^2-1-\varepsilon}\right]^{m_1} < a^3 \exp\left\{\left[\lambda(2+\delta)-\ln\frac{(\theta-1)^2-1-\varepsilon}{(\theta-1)(2+\varepsilon)}\right]N(\varrho_1)+2\ln N(\varrho_1)\right\}$$

holds for $|z|\leqslant\varrho_2$, $R=(\theta-1)\varrho_1$. Further, inequality (12) will obviously hold for $|\alpha_k|\leqslant\varrho_2$, if ϱ is replaced by ϱ_1 in it. It follows, using inequality (13) with the assumption that $F(\alpha_k) \neq 0$ for any k, $|\alpha_k| \leqslant \varrho_2$, we obtain again inequality (14) with ϱ replaced by ϱ_1. But $\varrho_1 > \varrho \geqslant \varrho_0$, from which it again follows that

$$F(\alpha_k) = 0, \quad |\alpha_k| \leqslant \varrho_2.$$

Continuing this process of accumulating zeros of $F(z)$ indefinitely, which is possible, since the constants in our inequalities do not depend on ϱ and ϱ_1, we arrive at the fact that $F(\alpha_k) = 0$ for all $k = 0, 1, \ldots,$ inasmuch as $\varrho_n = (1+\varepsilon)^n \varrho$. But

$$|F(z)| < C(\varrho) M(r+\varrho), \quad |z| = r.$$

Therefore, making use again of the Blaschke products and setting $R = (\theta - 1)r$, where r is arbitrarily large, we obtain

$$\begin{aligned} |F(z)| &< C(\varrho) M(R+\varrho) \prod_{|\alpha_k| \leqslant |r} \frac{R|z-\alpha_k|}{|R^2 - \bar{\alpha}_k z|} \\ &< C(\varrho) M(\theta r) \left[\frac{(\theta-1)r(1+r)}{(\theta-1)^2 r^2 - r}\right]^{N(r)} \end{aligned}$$

for $|z| \leqslant 1$, from which it already follows that

$$\begin{aligned} \ln |F(z)| &< \ln M(\theta r) - \ln\left(\theta - 1 - \frac{\theta}{r}\right) N(r) + C_1(\varrho) \\ &< -\left[\ln(\theta - 1) - \lambda - \frac{\theta}{r}\right] N(r) + C_1(\varrho). \end{aligned}$$

The last inequality, in which r may be taken arbitrarily large, shows that $F(z) \equiv 0$ because $\lambda < \ln(\theta - 1)$ by the conditions of the theorem. This completes the proof of the theorem. We note that the bounds for the magnitudes of the constants θ and λ can be improved.

By Schürer's theorem, if an entire function $f(z)$ of the first order of minimal type is the solution of equation (4), then $f(z)$ must be a polynomial, and if $f(z)$ is a function of the first order and normal type, then $f(z)$ must be the finite sum of products of polynomials by exponential functions. (See, for example, A. Gelfond [16], p. 290, Theorem VIII.)

Consequently, every entire function, which does not grow faster than the exponential function and which satisfies the conditions of our theorem, must belong either to the class of polynomials or to the class of linear forms in exponential functions with polynomial coefficients. Moreover, if we make any arithmetic assumptions concerning the arithmetic nature of the numbers α_k and β_s, then under known hypotheses, $f(z)$ can only be a constant. The theorem thus introduced shows that the assumption concerning the algebraic nature of the values of an analytic function, in particular of an entire function, with argument assuming values belonging to a set possessing the simplest lattice property and of sufficient density, already places a restriction on the function $f(z)$. Connections of this type can be used successfully in the solution of problems related to the arithmetic nature of numbers. Such connections were first used in the realm of problems on transcendence by the author of this monograph.

§2. The Euler-Hilbert problem

The problem of the transcendence or the rationality of the logarithms, with rational base, of rational numbers, stated by Euler in 1748, was formulated by Hilbert in a significantly more general form and introduced by him as number seven of a set of 23 problems, to the solution of which there appeared to be no suitable approach even at the very end of the nineteenth century. Hilbert stated the proposition on the transcendence or rationality of the logarithms with algebraic base of algebraic numbers, which is equivalent to the proposition on the transcendence of numbers of the form α^β, $\alpha \neq 0, 1$, for algebraic α and algebraic and irrational β.

A partial solution of the Euler-Hilbert problem was given in 1929. The author of the present monograph proved that for α algebraic, the number $\alpha^{i\sqrt{p}}$ is always a transcendental number when $\alpha \neq 0, 1$, $p > 0$, where p is a rational integer which is not a perfect square. In order to clarify the method of proof we shall give a brief proof of the transcendence of $(-1)^{-i} = e^{\pi}$. We enumerate all the points in the ring of integers of the Gauss field; in other words, we enumerate the numbers $m+ni$ with n and m rational integral. This enumeration will be carried out in the order of increasing absolute values of the complex integers, and in the case of equal absolute values, in the order of increase of their arguments. Then we can write down the set of complex integers in the form of a sequence $z_0 = 0, z_1, z_2, \ldots$. We expand the entire analytic function $e^{\pi z}$ in a Newton interpolation series with interpolation points $z_0, z_1, \ldots, z_n, \ldots$. Then we shall have, as is well known, the expansion

$$e^{\pi z} = \sum_{n=0}^{\infty} A_n(z-z_0)\ldots(z-z_{n-1}), \tag{16}$$

$$A_n = \frac{1}{2\pi i}\int_C \frac{e^{\pi\xi}d\xi}{(\xi-z_0)\ldots(\xi-z_n)} = \sum_{k=0}^{n} \frac{e^{\pi z_k}}{(z_k-z_0)\ldots(z_k-z_n)}.$$

It can be proved in an extraordinarily easy fashion that in view of the comparatively small growth of our function and the relatively large density of the points of interpolation, the Newton series converges to this function for arbitrary z. We now consider the least common multiple Ω_n of the complex integers

$$t_{n,k} = (z_k-z_0)\ldots(z_k-z_n), \quad 0 \leqslant k \leqslant n.$$

Using very simple arguments from the theory of the distribution of prime numbers of the form $4n+1$ and $4n+3$, we can easily establish that Ω_n satisfies the conditions

$$|\Omega_n| = e^{\frac{1}{2}n \ln n + O(n)}, \quad |C_{n,k}| = \left|\frac{\Omega_n}{t_{n,k}}\right| = e^{O(n)}. \tag{17}$$

In view of the fact that Ω_n is the least common multiple of the numbers $t_{n,k}$, we may assert that all the numbers $C_{n,k}$ are complex integers. Further, since $e^{\pi z_k} = e^{\pi(m_1+m_2 i)} = \pm e^{\pi m_1}$, $0 \leqslant m_1 < 2\sqrt{k}$, we see that $\Omega_n A_n$ will be a polynomial in e^{π} of degree no higher than $2\sqrt{n}$ with complex integral coefficients. Assuming that e^{π} is algebraic and making use of estimate (17) and of Lemma I, §2, Chapter II, we immediately obtain that either $A_n = 0$ or

$$|\Omega_n A_n| > e^{-O(n)}. \tag{18}$$

On the other hand, considering the integral representation (16) and taking the circumference $|\xi| = n$ as the contour C, we immediately obtain the estimate

$$|\Omega_n A_n| < e^{-\frac{1}{2}n \ln n + O(n)}. \tag{19}$$

For sufficiently large n, estimates (18) and (19) become contradictory, from which it follows that $A_n = 0$ for $n > n_0$. But the right member of expansion (16) will then be a polynomial, and the left member is an entire transcendental function $e^{\pi z}$. Thus, it follows from the assumption that e^{π} is algebraic that $e^{\pi z}$ must be a polynomial. It also follows that e^{π} is transcendental.

Kuzmin [1] carried over this method of proving the transcendence of numbers, with minor changes, to the case of real exponents and proved that for algebraic α, $\alpha \neq 0, 1$, the number $\alpha^{\sqrt{p}}$, where

$p > 0$ is rational integral and not equal to the square of an integer, is a transcendental number. In particular, this applies to the number $2^{\sqrt{2}}$.

Siegel [4] used this method to prove the transcendence of at least one period of the elliptic function $\wp(z)$:

$$[\wp'(z)]^2 = 4\wp^3(z) - g_2\wp(z) - g_3, \tag{20}$$

if its invariants g_2 and g_3 are algebraic numbers. For algebraic exponents β, but of power higher than the second, Boehle [1] established that at least one of the numbers $\alpha^{\beta}, \alpha^{\beta^2}, \ldots, \alpha^{\beta^{\nu-1}}$, where $\alpha \neq 0$, 1 is algebraic and is the root of an irreducible equation with integral coefficients of degree ν, will be a transcendental number. The proof of this proposition was carried out by the same method as that used in the case of a quadratic irrationality. The complete solution of the Euler-Hilbert problem was given by the author of this monograph by another method in 1934 [5]. In this new method, the former interpolation idea was augmented with the idea of analytic-arithmetic continuation. In order to clarify the fundamental elements of this method, we shall present the scheme of a proof of the transcendence of the ratio of the logarithms of two algebraic numbers, when this ratio is irrational. We shall give this proof in a sequence of four steps. Suppose α and β are algebraic numbers in a finite field K, $a = \ln \alpha$, $b = \ln \beta$, $\eta = b/a$ and let the logarithms be arbitrary but determined by the logarithms of the numbers α and β. Suppose also that the number $N > 0$ is arbitrarily large. We shall also assume that η is an irrational algebraic number.

Step One. We set $r_1 = \left[\frac{N^2}{\ln N}\right]$, $r_2 = [\ln \ln N]$. Then there will exist numbers $C_{k,l}$ in a finite algebraic field K, all different from zero, $\overline{|C_{k,l}|} < e^{2N^2}$, $N > N_0$, such that the function $f(z)$, $f(z) \not\equiv 0$,

$$f(z) = \sum_{k=0}^{N} \sum_{m=0}^{N} C_{k,m}\alpha^{kz}\beta^{mz} = \sum_{0}^{N} \sum_{0}^{N} C_{k,m}e^{(ak+bm)z}, \tag{21}$$

will have zeros of multiplicity r_1 at the points $0, 1, \ldots, r_2$; in other words,

$$f^{(s)}(t) = a^s \sum_{0}^{N} \sum_{0}^{N} C_{k,m}(k+\eta m)^s \alpha^{kt}\beta^{mt} = 0, \tag{22}$$

$$0 \leqslant t \leqslant r_2, \quad 0 \leqslant s \leqslant r_1.$$

Since α, β and η are algebraic numbers, this assertion can be easily proved with the aid of Lemma II, §2, Chapter II.

Step Two. In view of the fact that $f(z)$ has a large number of zeros and the integral representation

$$(23)\quad f^{(s)}(t) = \frac{s!}{2\pi i}\int_{C_1}\frac{d\xi}{(\xi-t)^{s+1}}\int_{C_2}\left[\frac{\xi(\xi-1)\ldots(\xi-r_2)}{z(z-1)\ldots(z-r_2)}\right]^{r_1+1}\frac{f(z)}{z-\xi}\,dz,$$

where C_1 is the circumference $|\xi|=N^{3/4}$, and C_2 is the circumference $|z|=N$, we easily obtain the estimate

$$(24)\qquad |f^{(s)}(t)| < e^{-\frac{1}{10}N^2 \ln\ln N}, \quad |t| \leqslant [\sqrt{N}] = r_4, \quad s \leqslant r_1$$

for $N>N_1$. Since α, β and η are algebraic, it follows from these estimates, again with the aid of Lemma I, §2, Chapter II, that

$$(25)\qquad f^{(s)}(t) = 0; \qquad 0 \leqslant t \leqslant r_4, \quad 0 \leqslant s \leqslant r_1.$$

Step Three. We again make use of the integral representation (23), in which we may replace r_2 by r_4, because our function now has more zeros and conditions (25) are satisfied. Estimating, with the aid of this new integral representation of $f^{(s)}(t)$, we find that the inequalities

$$(26)\qquad |f^{(s)}(0)| < e^{-\frac{1}{8}N^{5/2}}, \quad 0 \leqslant s \leqslant (N+1)^2$$

are true. It follows from these last inequalities, again with the aid of Lemma I, §2, Chapter II, that

$$(27)\quad a^{-s}f^{(s)}(0) = \sum_0^N \sum_0^N C_{k,m}(k+\eta m)^s = 0, \quad 0 \leqslant s \leqslant (N+1)^2.$$

Step Four. Since there are $(N+1)^2$ numbers $C_{k,m}$ in all, considering the first $(N+1)^2$ of the equalities (27), we obtain a system of $(N+1)^2$ homogeneous equations with $(N+1)^2$ unknowns $C_{k,m}$. The determinant of this system is the Vandermonde determinant. It can be equal to zero if and only if the equality $\eta m_1+k_1=\eta m_2+k_2$ holds at least once. But in general such an equality is impossible, because η is irrational. This means that the determinant of the system is different from zero and that all the $C_{k,m}$ equal zero. We thus arrive at a contradiction with the condition of choice of the $C_{k,m}$ and by the same token the transcendence of the number η is proved.

The first step consists in the construction of a function which is a linear combination of the powers α^z and β^z; this linear combination has a very large number of zeros for integral coefficients which are not too large. This construction is possible only because α, β and η (by assumption) are algebraic. This function cannot be identically equal to zero because η is irrational.

The second step consists in the proof of the existence of a still larger number of zeros of our function, which zeros are distributed in a larger circle than previously. This is possible due to the fact that a function which is not too large in absolute value in some circle and which has a sufficient number of zeros in this circle, must be small in a large circle. It is still impossible, at this step, in general, to assure a sufficient smallness of the $(N+1)^2$ derivatives at the origin, from which conditions (27) would follow.

The third step consists in a new accumulation of the zeros of our function, the number of which will become sufficient for conditions (27) to hold.

The fourth step consists in the use of the fact that if our function has a large number of zeros at the origin, then it is identically equal to zero. This is, however, impossible in virtue of the choice of its coefficients and the irrationality of the ratio of logarithms. This process of accumulating zeros of a function consists in three steps. The number of steps required for the accumulation can be indefinitely increased if one obtains the most precise results possible concerning the lower bounds of the measure of transcendence of a number.

We shall now give a complete proof of the transcendence of the numbers a^b with $a \neq 0$, 1 algebraic and b irrational algebraic.

Theorem II. *The number a^b is irrational transcendental when $a \neq 0$, 1 and b are algebraic.*

Proof. Let us assume the contrary, namely that the number $c = a^b = e^{b \ln a}$, where $\ln a$ is an arbitrary, but fixed, value of the logarithm, is an algebraic number. Suppose the a, b and c are numbers in an algebraic field of degree ν. We set $\eta = \ln a$ and consider the auxiliary function

$$f(z) = \sum_{k=0}^{q} \sum_{s=0}^{q} A_{k,s} e^{\eta(k+bs)z}, \tag{28}$$

where the $A_{k,s}$ are arbitrary integers in the field K, and $q > q_0$ is any integer. Suppose $d > 0$ is rational integral, with da, db and dc algebraic integers, and $q_1 = \left[\frac{q^2}{\ln q}\right]$. By Lemma II, §2, Chapter II, the numbers $A_{k,n}, 0 \leqslant k \leqslant q$, $0 \leqslant n \leqslant q$, can be chosen all different from zero and satisfying two sets of conditions:

$$(29) \qquad f^{(s)}(t) = 0,$$
$$0 \leqslant s \leqslant q_1 - 1, \quad t = 0, 1, \ldots, t_1, \quad t_1 = [\tfrac{1}{2} \ln q],$$

and

$$(30) \qquad \overline{|A_{k,n}|} = e^{O(q^2)}, \quad 0 \leqslant k \leqslant q, \quad 0 \leqslant n \leqslant q.$$

In fact, considering the expressions

$$(31) \qquad \eta^{-s} d^{3q^2} f^{(s)}(t) = \sum_{k=0}^{q} \sum_{r=0}^{q} A_{k,r} a^{kt} c^{rt} (k+br)^s d^{3q^2}$$

for integral t, as linear forms in the $(q+1)^2$ variables $A_{k,r}$, setting $n=(q+1)^2$, $m=q_1(t_1+1)$ in Lemma II, §2, Chapter II, we immediately obtain the proof of the existence of the numbers $A_{k,n}$ satisfying relations (29) and (30), inasmuch as

$$(32) \qquad \overline{|U_{s,t,r,k}|} = \overline{|d^{3q^2}(k+br)^s a^{kt} c^{rt}|} = e^{O(q^2+s \ln q + qt)} = e^{O(q^2)}$$

with $s \leqslant q_1$, $t \leqslant q$ and $U_{s,t,r,k}$ algebraic integers.

Having chosen the numbers $A_{k,n}$ we have by the same token chosen an entire function $f(z)$ which is not identically zero inasmuch as b is irrational, for which conditions (29) and (30) are satisfied. In virtue of condition (29) we shall have the integral representation

$$(33) \quad f^{(s)}(t) = \frac{s!}{(2\pi i)^2} \int_{|z|=R_1} \frac{dz}{(z-t)^{s+1}} \int_{|\xi|=R_0} \prod_{r=0}^{t_1} \left(\frac{z-r}{\xi-r}\right)^{q_1} \frac{f(\xi)}{\xi - z}\, d\xi$$

for $|t| \leqslant [q^{1/2}] = t_2$ where $R_1 = q^{3/4}$, $R_0 = q$. Estimating the absolute value of the integral in the right member and assuming that $s \leqslant q_1$, we obtain the inequality

$$(34) \quad |\eta^{-s} d^{3q^2} f^{(s)}(t)| < |\eta^{-q_1}| d^{3q^2} q_1^{q_1} (R_1 - t_1)^{-q_1} \left[\frac{R_1 + t_1}{q - t_1}\right]^{q_1 t_1} e^{O(q^2)}$$
$$< e^{-\frac{1}{8} q^2 \ln q + O(q^2)}.$$

Further, it follows from relations (30) and (32) that

$$(35) \quad \overline{|\eta^{-s} d^{3q^2} f^{(s)}(t)|} \leqslant \sum_{k=0}^{q} \sum_{n=0}^{q} \overline{|A_{k,n}|}\, \overline{|U_{s,t,n,k}|}$$
$$\leqslant e^{O(q^2+s \ln q + qt)} \leqslant e^{O(q^2)}$$

for $s \leqslant q_1$ and integral t, $0 \leqslant t \leqslant t_2$. Since the numbers $\eta^{-s} d^{3q^2} f^{(s)}(t)$ are integral algebraic for integral t, $0 \leqslant t \leqslant t_2$, $0 \leqslant s \leqslant q_1$, if we assume that at least one of the $f^{(s)}(t) \neq 0$ for $0 \leqslant s \leqslant q_1$, $0 \leqslant t \leqslant t_1$, we arrive, in virtue of the fact that the norm of an algebraic integer must not be less in absolute value than unity, and relations (34) and (35), at the inequality

$$1 < e^{-\frac{1}{8}q^2 \ln q + O(q^2) + \nu O(q^2)} = e^{-\frac{1}{8}q^2 \ln q + O(q^2)}$$

But this inequality cannot hold for $q > q'$. Therefore, the equalities

$$(36) \qquad f^{(s)}(t) = 0; \quad 0 \leqslant s \leqslant q_1; \quad 0 \leqslant t \leqslant t_2 = [\sqrt{q}]$$

hold, where the t are natural numbers.

We note that for real b and $\eta = \ln a$ equalities (36) already prove our theorem. In fact, in this case the numbers $A_{k,s}$ are also real and $f(z)$ cannot have more than $(q+1)^2 - 1$ real zeros, taking into consideration their multiplicity, which fact can be very easily proved by means of straightforward successive differentiation. But then equalities (36) show that $f(z) \equiv 0$ inasmuch as

$$(q_1+1)(t_2+1) > \sqrt{q}\,\frac{q^2}{\ln q} > (q+1)^2$$

with $q > e^6$. But this contradicts the irrationality of b and the nature of the choice of the numbers $A_{k,s}$. In order to obtain a contradiction from the relations (36) in the general case, it suffices, for example, to prove that $f(z)$ must have a zero at the origin of multiplicity greater than $(q+1)^2$. If conditions (36) are satisfied, the function $f(z)$ will have zeros of multiplicity q_1+1 at the points $z = 0, 1, \ldots, t_2$, from which it follows that the integral representation (33) holds if t_1 is replaced by t_2. Making this change in (33), also setting $R_1 = 1$, $R_0 = q$ in it, and estimating the absolute value of the right member of (33), we obtain the inequality

$$(37) \quad |\eta^{-s} d^{3q^2} f^{(s)}(0)| < e^{s \ln s + O(s) + O(q^2)} \left[\frac{\sqrt{q}+1}{q-\sqrt{q}}\right]^{q^{1/2}}$$
$$= e^{2q^2 \ln q + O(q^2) - \frac{1}{2}q^{5/2}} = e^{-\frac{1}{2}q^{5/2} + O(q \ln q)}$$

with $s \leqslant (q+1)^2$.

But from estimates (30) and (32) the inequality

$$(38) \qquad \overline{|\eta^{-s} d^{3q^2} f^{(s)}(0)|} < \sum_{k=0}^{q} \sum_{n=0}^{q} \overline{|A_{k,n}|}\, \overline{|U_{s,o,n,k}|}$$
$$< e^{O(q^2 + s \ln q)} < e^{O(q^2 \ln q)}.$$

again follows, with $s \leqslant (q+1)^2$. But all the numbers $\eta^{-s} d^{3q^2} f^{(s)}(0)$, $s \leqslant 3q^2$, are algebraic integers, from which the inequality

$$1 \leqslant e^{-\frac{1}{2}q^{5/2} + (\nu+1) O(q^2 \ln q)} \leqslant e^{-\frac{1}{2}q^{5/2} + O(q^2 \ln q)}$$

again follows if $f^{(s)}(0) \neq 0$ and $s \leqslant (q+1)^2$. This last inequality will be contradictory if $q \geqslant q'' > q' > 10$ and this means that for $q \geqslant q'', f^{(s)}(0) = 0, s = 0, 1, \ldots, (q+1)^2 - 1$, or

$$\sum_{k=0}^{q} \sum_{n=0}^{q} A_{k,n}(k+bn)^s = 0, \quad 0 \leqslant s \leqslant (q+1)^2 - 1. \tag{39}$$

The system of equalities (39) is a linear homogeneous system of $(q+1)^2$ equations in the $(q+1)^2$ unknowns $A_{k,n}$. The determinant Δ of this system is the Vandermonde determinant and $\Delta \neq 0$ because $k_1 + n_1 b \neq k_2 + n_2 b$ for $(k_2 - k_1)^2 + (n_2 - n_1)^2 \neq 0$ as a consequence of the irrationality of b. This means that the unique solution of system (33) is the trivial solution

$$A_{k,n} = 0, \quad 0 \leqslant k \leqslant q, \quad 0 \leqslant n \leqslant q,$$

which contradicts the choice of the $A_{k,n}$, which were all chosen different from zero. This contradiction proves our theorem.

Schneider [2] independently gave another proof of this theorem, which in form is much closer to the ideas of Siegel, after the publication of the complete proof of the Euler-Hilbert problem by the author of the present monograph; he applied the method discussed above to prove the transcendence of certain constants connected with elliptic functions and abelian integrals. Here, we present a complete proof of one of the most interesting results obtained by Schneider.

We first help the reader recall certain facts from the theory of elliptic functions which will be necessary in the sequel. If ω_1 and ω_2 are two arbitrary numbers, whose ratio ω_1/ω_2 is not a real number, then the Weierstrass elliptic function $\wp(z)$ is defined by the equation

$$\wp(z) = \frac{1}{z^2} + \sum_{m_1, m_2}' \left[\frac{1}{(z - m_1\omega_1 - m_2\omega_2)^2} - \frac{1}{(m_1\omega_1 + m_2\omega_2)^2} \right], \tag{40}$$

where the sum is taken over all positive as well as negative values of m_1 and m_2 under the assumption that $m_1{}^2 + m_2{}^2 \neq 0$. The function $\wp(z)$ is even and doubly periodic, with periods ω_1 and ω_2; in other words,

$$\wp(z + k_1\omega_1 + k_2\omega_2) = \wp(z),$$

where k_1 and k_2 are arbitrary integers. The function $\wp(z)$ satisfies the differential equation

$$[\wp'(z)]^2 = 4[\wp(z)]^3 - g_2\wp(z) - g_3, \tag{41}$$

where g_2 and g_3 are invariants which are connected with the periods by the relations

$$(42)\quad g_2 = 60 \sum' \frac{1}{(m_1\omega_1+m_2\omega_2)^4}, \quad g_3 = 140 \sum' \frac{1}{(m_1\omega_1+m_2\omega_2)^6},$$

with the sums again taken over all integral values of m_1 and m_2, $m_1^2+m_2^2 \neq 0$. The so-called addition theorem is valid for the function $\wp(z)$,

$$(43)\qquad \wp(x+y)+\wp(x)+\wp(y) = \frac{1}{4}\left[\frac{\wp'(x)-\wp'(y)}{\wp(x)-\wp(y)}\right]^2,$$

where x and y are two arbitrary complex numbers. Further, differentiating (41), we obtain the important relation

$$(44)\qquad \wp''(z) = 6[\wp(z)]^2-\tfrac{1}{2}g_2.$$

It follows directly from relations (43) and (44) that $\wp(kz)$ is a rational function of $\wp(z)$ and $\wp'(z)$ for arbitrary rational integers k, where the coefficients of this rational function are polynomials with rational coefficients in g_2 and g_3. It is also known that

$$(45)\qquad \wp'\left(\frac{\omega_1}{2}\right) = \wp'\left(\frac{\omega_2}{2}\right) = \wp'\left(\frac{\omega_1+\omega_2}{2}\right) = 0.$$

The only singularities of $\wp(z)$ in the finite part of the plane are poles of the second order at the points $z=m_1\omega_1+m_2\omega_2$. If the invariants g_2 and g_3 are prescribed and the discriminant $\Delta=g_2^3-27g_3^2\neq 0$, then there exists an elliptic function $\wp(z)$ with these invariants. If we set $\omega_1'=\lambda\omega_1$, $\omega_2'=\lambda\omega_2$ and denote the invariants of the elliptic function $\wp_1(z)$ by g_2' and g_3' where the periods of $\wp_1(z)$ are ω_1' and ω_2 , then the following relations follow immediately from the formulas introduced above:

$$(46)\qquad \wp_1(z) = \lambda^{-2}\wp\left(\frac{z}{\lambda}\right), \quad g_2' = \lambda^{-4}g_2, \quad g_3' = \lambda^{-6}g_3.$$

We define the elliptic function $\zeta(z)$ with the aid of $\wp(z)$,

$$(47)\qquad \zeta(z) = \frac{1}{z}-\int_0^z \left[\wp(z)-\frac{1}{z^2}\right]dz, \quad \zeta'(z) = -\wp(z),$$

where the integral is taken over an arbitrary path which does not pass through a pole of $\wp(z)$. The addition theorem also holds for this function, namely

$$(48)\qquad \zeta(x+y) = \zeta(x)+\zeta(y)+\frac{1}{2}\,\frac{\wp'(x)-\wp'(y)}{\wp(x)-\wp(y)},$$

from which it again follows directly that $\zeta(kz)$ is a rational function with rational coefficients in the quantities $\zeta(z)$, $\wp(z)$, $\wp'(z)$ and g_2, g_3, for arbitrary rational integral k.

Further, the relations

$$\zeta(z+\omega_1) = \zeta(z)+2\eta_1, \quad \zeta(z+\omega_2) = \zeta(z)+2\eta_2, \tag{49}$$

$$\eta_1 = \zeta\left(\frac{\omega_1}{2}\right), \quad \eta_2 = \zeta\left(\frac{\omega_2}{2}\right),$$

hold, where η_1 and η_2 are constants which are connected by the equality

$$\eta_1\omega_2-\eta_2\omega_1 = (-1)^\mu \frac{\pi i}{2} \tag{50}$$

with $\mu=0$ or $\mu=1$ depending on whether $I\dfrac{\omega_2}{\omega_1} > 0$ or $I\dfrac{\omega_2}{\omega_1} < 0$.

The reader can find all these derivations in any book on the theory of elliptic functions, for example in Ahiezer [1] or Tricomi [1].

We now consider the function $f(z)$ defined by

$$f(z) = t_1 z+t_2\zeta(z), \tag{51}$$

where t_1 and t_2 are any fixed constants, which are not equal to zero simultaneously. On the basis of relation (48), the addition theorem

$$f(x+y) = f(x)+f(y)+\frac{t_2}{2}\frac{\wp'(x)-\wp'(y)}{\wp(x)-\wp(y)} \tag{52}$$

will again hold for $f(z)$. It again follows directly from this theorem that for arbitrary rational integral k the function $f(kz)$ is a rational function in $f(z)$, $\wp(z)$, $\wp'(z)$, g_2, g_3 and t_2, with rational coefficients.

We shall prove the absence of an algebraic relation between $f(z)$ and $\wp(z)$. Let us assume the contrary, namely that

$$[f(z)]^n+A_1(z)[f(z)]^{n-1}+\cdots+A_n(z) = 0 \tag{53}$$

holds identically in z, where $A_1(z), \ldots, A_n(z)$ are rational functions in $\wp(z)$ and $\wp'(z)$.

If k_1 and k_2 are rational integers, we obtain the identities

$$f(z+k_1\omega_1+k_2\omega_2) = f(z)+t_1(k_1\omega_1+k_2\omega_2)+2t_2(k_1\eta_1+k_2\eta_2) \tag{54}$$

from relations (49) and (51). But for t_1 and t_2 which are not zero simultaneously and arbitrary ω_1, ω_2, η_1, η_2, one can choose integers k_1, k_2 such that the number $\theta=t_1(k_1\omega_1+k_2\omega_2)+2t_2(k_1\eta_1+k_2\eta_2)$

is not equal to zero. In the contrary case the numbers t_1 and t_2 would be solutions of the linear system of equations

$$\omega_1 t_1 + 2\eta_1 t_2 = 0, \quad \omega_2 t_1 + 2\eta_2 t_2 = 0,$$

whose determinant in virtue of (50) would be different from zero, and t_1, t_2 would vanish. For prescribed t_1 and t_2, choosing numbers k_1 and k_2 so that $\theta \neq 0$, and setting $\lambda = k_1\omega_1 + k_2\omega_2$, we shall have the identity

$$f(z+\lambda t) = f(z) + t\theta$$

for arbitrary rational integral t in virtue of (54).

Replacing z by $z+\lambda t$ in relation (53), we see that $A_i(z)$, $1 \leqslant i \leqslant n$, does not change because $\wp(z)$ and $\wp'(z)$ have periods ω_1 and ω_2. Therefore, we obtain a new relation

$$[f(z)+t\theta]^n + A_1(z)[f(z)+t\theta]^{n-1} + \cdots + A_n(z) = 0.$$

But this last relation is true for arbitrary integer t. Taking the n-th order finite difference with respect to t of its left member, we arrive at the equality $n!\,\theta^n = 0$, which is impossible, since $\theta \neq 0$ by the choice of k_1 and k_2. Thus, relation (53) is impossible.

The facts concerning the functions $\wp(z)$ and $f(z)$ introduced above enable us to prove the following theorem.

THEOREM III. *If t_1 and t_2 are not equal to zero simultaneously and $z_1 \neq k_1\omega_1 + k_2\omega_2$ with k_1 and k_2 rational integers, then at least one of the six numbers g_2, g_3, t_1, t_2, $\wp(z_1)$ and $f(z_1)$ must be transcendental. The numbers ω_1 and ω_2 are the periods of $\wp(z)$, $f(z) = t_1 z + t_2\zeta(z)$.*

Proof. We shall assume the contrary, namely that the seven numbers

$$g_2, \quad g_3, \quad t_1, \quad t_2, \quad \wp(z_1), \quad \wp'(z_1), \quad f(z_1)$$

are algebraic numbers in the field K of degree ν (the fact that $\wp'(z_1)$ is algebraic follows from (41)). We set $q = 8\nu - 3$ and $z_k = s_k z_1$, $k = 1, 2, \ldots, q$, where $s_k > s_{k-1}$, $s_1 = 1$ and s_k are rational integers, chosen so that $z_k s_k \neq k_1\omega_1 + k_2\omega_2$ with rational integers k_1 and k_2. In particular, if $z_1 \neq m_1\omega_1 + m_2\omega_2$ with rational m_1 and m_2, then $s_k = k$.

Then on the basis of the addition theorem, we can assert, for the functions $\wp(z)$ and $f(z)$, that all $3q$ numbers $\wp(s_k z_1)$, $\wp'(s_k z_1)$, $f(s_k z_1)$ are also numbers in the field K.

These $3q$ numbers, and also the numbers t_1, t_2, g_2 and g_3, naturally, are not required to be integers in the field K. Therefore, we choose two nonzero rational integers a and b such that the $3q+4$ numbers

$$\tfrac{1}{2}a^4g_2, \quad a^6g_3, \quad bat_1, \quad ba^{-1}t_2, \quad a^2\wp(z_k), \quad bf(z_k), \quad a^3\wp'(z_k),$$
$$k = 1, 2, \ldots, q,$$

are integers in the field K. It is always possible to choose such numbers a and b. But now, replacing ω_1, ω_2, t_1 and t_2 respectively by ω_1/a, ω_2/a, bat_1, $ba^{-1}t_2$, we obtain that $\wp(z)$, $\wp'(z)$, $f(z)$ and g_2, g_3 go over in virtue of (46) into $a^2\wp(az)$, $a^3\wp'(az)$, $bf(az)$, a^4g_2 and a^6g_3. Therefore, the values of these new functions for $z=z_k/a$ will not only be algebraic in the field K, but they will be algebraic integers. In virtue of relations (41) and (44) for $\wp_1(z)=a^2\wp(az)$ we see also that $\wp_1^{(s)}(z_k/a)$ will be algebraic integers, and by the equality

$$f_1'(z) = t_1' - t_2'\wp_1(z),$$

where

$$f_1(z) = af(az), \quad t_1' = bat_1, \quad t_2' = ba^{-1}t_2,$$

we can assert that $f_1^{(s)}(z_k/a)$ will also be algebraic integers for arbitrary s and $1 \leqslant k \leqslant q$.

This shows that we can assume the $3q+4$ numbers

$$(55) \quad \tfrac{1}{2}g_2, \quad g_3, \quad t_1, \quad t_2; \qquad \wp(z_k), \quad \wp'(z_k), \quad f(z_k); \qquad 1 \leqslant k \leqslant q,$$

to be integers in the field K right from the beginning, because the general proposition concerning the algebraic nature of these numbers reduces to this particular proposition by means of a simple change in the periods and without changing the algebraic nature of these numbers and the number z_1. Therefore, we shall assume that the numbers (55) are algebraic integers.

We shall set

$$(56) \qquad f_{m,n}(z) = [\wp(z)]^m[f(z)]^n, \quad 0 \leqslant m \leqslant N, \quad 0 \leqslant n \leqslant N,$$

where N is an arbitrary integer. Then, as we have already seen, it follows from the assumption that the numbers (55) are algebraic integers that the numbers $\wp^{(s)}(z_k)$ and $f^{(s)}(z_k)$, and this means also the numbers $f_{m,n}^{(s)}(z_k)$, $1 \leqslant k \leqslant q$, for arbitrary s, will be algebraic integers in the field K.

It is clear from the integral representation

$$(57) \qquad f_{m,n}^{(s)}(z_k) = \frac{s!}{2\pi i}\int_{|z-z_k|=\varepsilon} \frac{f_{m,n}(z)}{(z-z_k)^{s+1}}\,dz,$$

where ε is so small that there are no poles of the function $\wp(z)$ in the circles $|z-z_k|\leqslant\varepsilon$, $1\leqslant k\leqslant q$, that we have the estimate

(58) $$|f^{(s)}_{m,n}(z_k)|\leqslant s^s e^{O(N+s)},\quad 1\leqslant k\leqslant q.$$

We shall agree to denote by $\bar{a}$ a number in the field $\bar{K}$, conjugate to the field K, if $\bar{a}$ is conjugate to the number a in the field K. In order to obtain an estimate of the numbers $\overline{f^{(s)}(z_k)}$ we apply the following method. We fix any field $\bar{K}$, conjugate to K. Since

$$\Delta = g_2{}^3-27g_3{}^2 \neq 0,$$

we also have that the number

$$\bar{\Delta} = \bar{g}_2{}^3-27\bar{g}_3{}^2 \neq 0.$$

But we can construct an elliptic function $\bar{\wp}(z)$ with the periods ω_1' and ω_2' for the invariants $\bar{g}_2$ and $\bar{g}_3$, where the periods ω_1' and ω_2' are in some sense conjugate to ω_1 and ω_2.

We now define q numbers z_k', lying in the fundamental period parallelogram of the function $\bar{\wp}(z)$, by means of the relations

(59) $$\bar{\wp}(z_k') = \overline{\wp(z_k)},\quad \bar{\wp}'(z_k') = \overline{\wp'(z_k)},\quad 1\leqslant k\leqslant q.$$

As is known, relations (59) are defined uniquely by the numbers z_k', $1\leqslant k\leqslant q$, if they must lie in the fundamental period parallelogram of the function $\bar{\wp}(z)$. If we also set

$$\bar{f}_k'(z) = \bar{t}_1-\bar{t}_2\bar{\wp}(z)\quad\text{and}\quad \bar{f}_k(z_k') = \overline{f(z_k)},$$

we define $\bar{f}_k(z)$ uniquely for given k. But then $\bar{f}_k{}^{(s)}(z_k') = \overline{f_k{}^{(s)}(z_k)}$, $\bar{\wp}^{(s)}(z_k') = \overline{\wp^{(s)}(z_k)}$ as a consequence of the fact that the relations connecting $\bar{f}_k{}^{(s)}(z)$ and $\bar{\wp}^{(s)}(z)$ with $\bar{f}(z)$, $\bar{\wp}(z)$ and $\bar{\wp}'(z)$, differ from the relations connecting $f^{(s)}(z)$ and $\wp^{(s)}(z)$ with $f(z)$, $\wp(z)$ and $\wp'(z)$, only in that the constants which appear in them are replaced by the numbers conjugate to them in the field $\bar{K}$. But then we also have

$$\bar{f}^{(s)}_{k,m,n}(z_k') = \overline{f^{(s)}_{k,m,n}(z_k)},\quad \bar{f}_{k,m,n}(z) = \bar{\wp}^m(z)\bar{f}_k{}^n(z).$$

Therefore, since the integral representation (57) is true for the numbers $\bar{f}^{(s)}_{k,m,n}(z)$, and consequently estimate (58) also, we also obtain that

(60) $$\overline{|f^{(s)}_{m,n}(z_k)|}\leqslant s^s e^{O(N+s)},\quad 1\leqslant k\leqslant q,\quad m\leqslant N,\quad n\leqslant N.$$

We recall that $\overline{|a|}$ is the maximum absolute value of a and all the $\nu-1$ numbers conjugate to a, and $O(N+s)<C(N+s)$, where C does not depend on k, s, N.

We now consider the function

$$F(z) = \sum_{n_1=0}^{N} \sum_{n_2=0}^{N} A_{n_1,n_2} f_{n_1,n_2}(z), \tag{61}$$

where the $A_{m,n}$, $0 \leqslant m \leqslant N$, $0 \leqslant n \leqslant N$, are integers in the field K, all different from zero, and the $f_{n_1,n_2}(z)$ are defined by relations (56). By Lemma II, §2, Chapter II, we can choose nonzero numbers A_{n_1,n_2} in the field K, such that the relations

$$F^{(s)}(z_k) = 0, \quad 1 \leqslant k \leqslant q, \quad 0 \leqslant s \leqslant \left[\frac{N^2}{2q}\right] - 1 = t-1 \tag{62}$$

are satisfied, and

$$\overline{|A_{n_1,n_2}|} \leqslant t^t e^{O(t)}. \tag{63}$$

In fact, the number of linear equations in the A_{n_1,n_2} is $q[N^2/2q] \leqslant \frac{1}{2}N^2$ and estimates (60) hold. Substituting this data into Lemma II, we obtain conditions (62) and (63).

The function $F(z)$ chosen in this manner is, finally, not identically equal to zero, because the numbers A_{n_1,n_2} are all different from zero and relation (53) is impossible. Therefore, there exists an integer r such that

$$F^{(s)}(z_k) = 0, \quad 1 \leqslant k \leqslant q, \quad 0 \leqslant s \leqslant r-1$$

and

$$\lambda = F^{(r)}(z_{k_0}) \neq 0, \quad 1 \leqslant k_0 \leqslant q.$$

By the choice of $F(z)$, we have $r \leqslant t$. Since

$$\overline{|F^{(r)}(z_{k_0})|} \leqslant \sum_{n_1=0}^{N} \sum_{n_2=0}^{N} \overline{|A_{n_1,n_2}|}\, \overline{|f^{(r)}_{n_1,n_2}(z_{k_0})|},$$

we have, in virtue of (60) and (63), that

$$\overline{|\lambda|} = \overline{|F^{(r)}(z_{k_0})|} \leqslant r^{2r} e^{O(r)} \tag{64}$$

inasmuch as $r \geqslant t > N$ with $N > N_0$. But taking into consideration that the norm of an algebraic integer which is not equal to zero is not less than 1, we obtain directly from (64) the inequality

$$|\lambda| = |F^{(r)}(z_{k_0})| \geqslant e^{-2(\nu-1)r \ln r + O(r)}. \tag{65}$$

We set

$$n = \left[\sqrt{\frac{qr}{N}}\right], \quad n \geqslant \left[\left(\frac{qt}{N}\right)^{1/2}\right] \geqslant \sqrt{\frac{N}{2}} + O(1) \tag{66}$$

since $r \geqslant t$. Suppose also that N is so large that all points z_k,

$1 \leqslant k \leqslant q$, lie in the parallelogram with boundary C and vertices in the points $\eta_{i,j}$; $i=0, 1, j=0, 1$,

$$\eta_{i,j} = [(-1)^i\omega_1+(-1)^j\omega_2]n+\frac{\omega_1+\omega_2}{2}$$

in the z-plane, where ω_1 and ω_2 are the periods of the function $\wp(z)$. The function $F(z)$ has singularities at the points $k_1\omega_1+k_2\omega_2$, where k_1 and k_2 are integral, so that the singular points of $F(z)$ can only be poles of order not greater than $3N$ in virtue of the definition of $F(z)$, since $\wp(z)$ has poles of the second order and $\zeta(z)$ has poles of the first order at these points. Therefore, in virtue of the definition of the number r, we must also have the integral representation

$$(67) \quad \lambda = F^{(r)}(z_{k_0}) = \frac{r!}{(2\pi i)^2}\int_{C_0}\frac{d\xi}{(\xi-z_{k_0})^{r+1}}\times$$

$$\times\int_C \frac{\prod_{k_1=-n}^{n}\prod_{k_2=-n}^{n}(z-k_1\omega_1-k_2\omega_2)^{3N}\prod_{k=1}^{q}(\xi-z_k)^r}{\prod_{k_1=-n}^{n}\prod_{k_2=-n}^{n}(\xi-k_1\omega_1-k_2\omega_2)^{3N}\prod_{k=1}^{q}(z-z_k)^r}\frac{F(z)}{z-\xi}\,dz,$$

where C is the perimeter of the parallelogram defined above and C_0 is the circumference $|z-z_{k_0}|=\varepsilon$, with the same $\varepsilon>0$ as in integral (57). We shall now estimate the absolute value of integral (67). If z is a point on the contour C, then

$$\sum_{k_1=-n}^{n}\sum_{k_2=-n}^{n}\ln\left|\frac{z-k_1\omega_1-k_2\omega_2}{\xi-k_1\omega_1-k_2\omega_2}\right|$$

$$=\sum_{-n}^{n}\sum_{-n}^{n}\ln\left|\frac{\frac{z}{n}-\frac{k_1}{n}\omega_1-\frac{k_2}{n}\omega_2}{\frac{\xi}{n}-\frac{k_1}{n}\omega_1-\frac{k_2}{n}\omega_2}\right|=O(n^2),$$

$$\prod_{k=1}^{q}|z-z_k|^r = e^{qr\ln n+O(r)}, \quad \prod_{k=1}^{q}|\xi-z_k|^r = e^{O(r)}$$

and, finally,

$$|F(z)| < e^{O(N\ln n+N^2)} = e^{O(\sqrt{r}\ln n+r)}$$

in virtue of the periodicity of $\wp(z)$, relation (54) for $f(z)$, and the inequalities $2qr \geqslant 2qt \geqslant N^2$. Therefore, estimating the absolute value of the right member of (67), we obtain the inequality

$$|\lambda| \leqslant r^r e^{O(r) - qr \ln n + O(\sqrt{r} \ln n + n^2 N + r)}. \tag{68}$$

But since $r \geqslant t > \dfrac{N^2}{2q} - 1$, we have $N < \sqrt{2qr} + 2q$ and this means that

$$\ln n > \ln \sqrt{\frac{qr}{N}} - 1 > \ln \sqrt{\frac{qr}{\sqrt{2qr} + 2q}} - 1 > \tfrac{1}{4} \ln r + O(1),$$

$$Nn^2 \leqslant qr,$$

and hence

$$|\lambda| < r^r e^{-\frac{1}{4} qr \ln r + O(r)}.$$

Comparing this inequality with inequality (65), we obtain the inequality

$$e^{-\frac{1}{4} qr \ln r + O(r)} \geqslant e^{-(2\nu - 1) r \ln r + O(r)}.$$

But since $q = 8\nu - 3$, we have $\frac{1}{4} q = 2\nu - \frac{3}{4} > 2\nu - 1$, which causes the last inequality to be contradictory for sufficiently large N, and this means it will be contradictory also for r sufficiently large since $r > \dfrac{N^2}{2q} - 1$. This contradiction proves Theorem III.

One can obtain a number of more concrete corollaries to Theorem III. Let us assume, for example, that the number $\dfrac{k_1\omega_1 + k_2\omega_2}{2} = z_1$ (k_1 and k_2 rational integers, $k_1{}^2 + k_2{}^2 \neq 0$, $(k_1, k_2, 2) = 1$) is algebraic for algebraic invariants g_2 and g_3. We also set $t_1 = 1$, $t_2 = 0$, where t_1 and t_2 appear in $f(z)$. Then in virtue of (45), (41) and the periodicity of $\wp'(z)$ the conditions of Theorem III will be satisfied, and we will obtain a contradiction with the basic assertion of this theorem. Therefore, the number $k_1\omega_1 + k_2\omega_2$ ($k_1{}^2 + k_2{}^2 \neq 0$ and ω_1, ω_2 are the periods of the function $\wp(z)$) must be transcendental for algebraic invariants. It already follows directly from this that the periods ω_1 and ω_2 are transcendental for algebraic invariants. Again setting $t_1 = 1$, $t_2 = 0$ and assuming that z_1 is algebraic, we obtain that $\wp(z_1)$ is transcendental for algebraic g_2 and g_3. Setting $t_1 = 0$, $t_2 = 1$, we obtain that at least one of the two numbers $\wp(z_1)$ and $f(z_1)$ must be transcendental with $z_1 \neq k_1\omega_1 + k_2\omega_2$ arbitrary, and k_1 and k_2 integral. Further, Schneider [3] proved the transcendence of the values of the Euler Beta function $B(p, q)$ for rational, but not integral, p and q. The method of proof of Theorem III introduced above was used also by Ricci [1] in the proof of the transcendence of numbers of the form α^β with α and β belonging to different classes of Liouville type numbers, and by Mahler in the proof of the $\wp$-adic analogue of the theorem on the transcendence of numbers of the form α^β.

§3. Problems concerning measure of transcendence and further development of methods

In this section we shall first of all show how the method discussed in §2 can be used in problems dealing with the measure of transcendence. As an example, we shall go into detail concerning the question of the approximation of the ratio of logarithms by rational fractions in an extended algebraic $\wp$-adic field. We stop to consider this question in particular, because the possibility or the impossibility of an approximation, in this sense, of the ratio of logarithms by means of rational fractions, corresponds to the solvability or the nonsolvability, in an algebraic field, of exponential congruences with a high power of a prime ideal, and by the same token, this example shows how the analytic methods of transcendental numbers can be used in the classical theory of divisibility.

Suppose $\wp$ is a prime ideal in the algebraic field K of degree ν, p is a prime rational integer, $N(\wp)=p^\sigma$, $p\equiv 0 \bmod \wp^\mu$, with μ the largest possible. If the ideal $\wp$ occurs to the power n in the number a of the field K, in other words, if n is the difference of exponential powers of the ideal $\wp$, to which powers the integers a_1 and a_2, $a=a_1/a_2$, can be divided, then the $\wp$-adic norm of the number a is $\wp^{-n}$, which we write in the form

$$|a|_\wp = \wp^{-n}. \tag{69}$$

The integer n, of course, can be positive, negative, or zero. If $n_1>n_2$ and

$$|a_1|_\wp = \wp^{-n_1}, \quad |a_2|_\wp = \wp^{-n_2},$$

then

$$|a_1|_\wp < |a_2|_\wp.$$

Further, if

$$|a_k|_\wp = \wp^{-n_k} \quad \text{and} \quad \lim_{k\to\infty} n_k = \infty,$$

then

(70) $$\lim_{k\to\infty} |a_k|_\wp = 0.$$

The $\wp$-adic norm satisfies the rules

(71) $$|ab|_\wp = |a|_\wp |b|_\wp, \quad |a+b|_\wp \leqslant \max\,[|a|_\wp, |b|_\wp].$$

In the $\wp$-adic metric, by virtue of (70) and (71), the series

(72) $$a = \sum_0^\infty a_k, \quad \lim |a_k|_\wp = 0,$$

where the a_k are elements in the field K, will converge and the elements a obtained in this formal manner, are not, in general, numbers in the field K, and will satisfy all the operational rules for ordinary complex numbers. If we adjoin all the elements (72) to the field K, then we obtain the $\wp$-adic extension of the field K, which is in its turn a field, which we shall denote by $K(\wp)$. The elements of $K(\wp)$ are called $\wp$-adic numbers. Among the various classes of functions of a $\wp$-adic variable in $K(\wp)$, the class of normal functions considered by Mahler and Skolem is of special interest.

We shall say that $f(z)$ is a normal function of the $\wp$-adic variable z if

(73) $$f(z) = \sum_{k=0}^\infty f_k z^k; \qquad |f_k|_\wp \leqslant 1, k = 0, 1, \ldots; \lim_{k\to\infty} |f_k|_\wp = 0.$$

As a consequence of (72), series (73) will converge for every z, $|z|_\wp \leqslant 1$. Series (73) can be integrated and differentiated because of its uniform convergence in the $\wp$-adic sense, in the region $|z|_\wp \leqslant 1$. Moreover, we also have $|f(z)|_\wp \leqslant 1$ in the region $|z|_\wp \leqslant 1$. It is perfectly clear also that the function

$$F(z) = \frac{1}{k!} f^{(k)}(z+z_0)$$

is normal provided $|z_0|_\wp \leqslant 1$. Since

$$n! = A_n p^{\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\cdots}, \quad (A_n, p) = 1, \quad \sum \left[\frac{n}{p^k}\right] \leqslant \left[\frac{n}{p-1}\right],$$

the function

(74) $$e^{p^k z} = \sum_{n=0}^\infty \frac{p^{kn}}{n!} z^n,$$

where $k=1$ if $\wp$ divides an odd prime, and $k=2$ if $\wp$ divides 2, will be normal and it will retain the algebraic properties of exponential functions. In exactly the same way,

$$(75) \qquad \ln(1-pz) = -\sum_{1}^{\infty} \frac{p^k}{k} z^k, \quad e^{\ln(1-pz)} = 1-pz,$$

since $\left|\frac{p^k}{k}\right|_{\wp} < 1$, will be a normal function, and the algebraic properties of the logarithm will be retained. If a is an element in $K(\wp)$ and $|a|_{\wp}=1$, then $a=a_0+a_1$, where a_0 is a number in the field K, $|a_0|_{\wp}=1$ and $|a_1|_{\wp}<\wp^{-3\mu}$. By Euler's theorem on algebraic fields, $a_0{}^{\lambda}\equiv 1 \bmod \wp^{3\mu}$. Since $\lambda=p^{3\sigma\mu}-p^{(3\mu-1)\sigma}$, it follows that

$$a^{\lambda} = (a_0+a_1)^{\lambda} \equiv 1 \bmod \wp^{3\mu}.$$

Therefore we can define the natural logarithm η of the number a^{λ} as

$$(76) \qquad \eta = \ln a^{\lambda} = -\sum_{k=1}^{\infty} \frac{(1-a^{\lambda})^k}{k}, \quad \lambda = p^{3\sigma\mu}-p^{(3\mu-1)\sigma},$$

with $|\eta|_{\wp}\leqslant\wp^{-2\mu}$ and $e^{\eta}=a^{\lambda}$ in virtue of (74).

We now note that if a is an arbitrary number in the field K and $\overline{|a|}\leqslant A$, then

$$(77) \qquad |a|_{\wp} \geqslant \wp^{-m}, \quad m = \mu\left[\nu\frac{\ln A}{\ln p}\right].$$

In fact, if a_i is conjugate to a, $a_1=a$, then

$$\prod_{1}^{\nu} |a_i| = n \leqslant A^{\nu},$$

where n is a rational integer. But therefore n cannot be divisible by p to a power higher than $\left[\frac{\nu \ln A}{\ln p}\right]$ which completes the proof of our assertion. Further, if $x_1, x_2, \ldots, x_{r_2}$, $|x_k|_{\wp}\leqslant 1$, $1\leqslant k\leqslant r_2$, are distinct elements in the field $K(\wp)$ and the normal function $f(z)$ satisfies the conditions

$$f^{(s)}(x_k) = 0; \quad 0\leqslant s\leqslant r_1-1; \quad 0\leqslant k\leqslant r_2,$$

then

$$(78) \qquad f(z) = [(z-x_0)\ldots(z-x_{r_2})]^{r_1}f_1(z)$$

where $f_1(z)$ is again a normal function. In fact, in this case,

$$f(z) = \sum_{n=r_1}^{\infty} \frac{f^{(n)}(x_0)}{n!}(z-x_0)^n$$

$$= (z-x_0)^{r_1} \sum_{n=r_1}^{\infty} \frac{f^{(n)}(x_0)}{n!}(z-x_0)^{n-r_1} = (z-x_0)^{r_1} f_1(z),$$

where $f_1(z)$ is a normal function, since

$$|a_n|_\wp = \left|\frac{f^{(n)}(x_0)}{n!}\right|_\wp = \sum_{m=n}^{\infty} \frac{m!}{n!(m-n)!} f_m x_0^{m-n} \leqslant \max_{m \geqslant n} |f_m|_\wp,$$

from which the normality of $f(z)$ also follows. Since $x_i \neq x_k$, $i \neq k$, then continuing the process of factoring out powers of linear functions, we arrive at assertion (78).

Finally, if the polynomial $P(z)$ of degree $r_1 r_2 - 1$ is defined by the conditions

$$(79) \qquad \begin{gathered} P^{(s)}(x_k) = a_{s,k}, \quad |a_{s,k}|_\wp \leqslant \wp^{-\mu m}, \quad x_k = pk, \\ 0 \leqslant s \leqslant r_1 - 1, \quad 0 \leqslant k \leqslant r_2 - 1, \quad r_2 \geqslant 4, \end{gathered}$$

where the $a_{s,k}$ are elements in $K(\wp)$, then

$$(80) \qquad P(z) = p^{m-3r_1r_2} P_1(z),$$

where the polynomial $P_1(z)$ is a normal function in the sense that its coefficients in a power series expansion have $\wp$-adic norm not greater than unity.

We shall write the interpolation polynomial $P(z)$, defined by conditions (79), in the explicit form. Using Hermite's formula (see Gelfond [16], p. 58; or Walsh [1]), we have

$$(81) \quad P(z) = \sum_{k=0}^{r_2-1} \sum_{n=0}^{r_1-1} \sum_{s=0}^{n} \frac{a_{n,s}}{(r_1-n-1)!(n-s)!} \times$$
$$\times \frac{d^{n-s}}{dz^{n-s}} \left[\frac{(z-x_k)^{r_1}}{Q(z)}\right]\Bigg|_{z=x_k} Q_{s,k}(z),$$

$$Q_{s,k}(z) = \frac{Q(z)}{(z-x_k)^{s+1}}, \quad Q(z) = \prod_{k=0}^{r_2-1} (z-x_k)^{r_1},$$

and to prove our assertion we must determine the highest power of $\wp$ which divides all the coefficients of the normal functions $(z-x_k)^{-s-1}Q(z)$. Differentiating, we obtain

$$(82)\quad A_{n,k,s} = \frac{1}{(r_1-n-1)!(n-s)!}\frac{d^{n-s}}{dz^{n-s}}\left[\frac{(z-x_k)^{r_1}}{Q(z)}\right]\Bigg|_{z=x_k}$$
$$= \pm\frac{1}{(r_1-n-1)!(n-s)!}\times$$
$$\times\sum_{\substack{\Sigma\nu_i=n-s\\ \nu_{k+1}=0}}\frac{(n-s)!}{\nu_1!\ldots\nu_{r_2}!}\frac{p^{-(r_2-1)r_1-n+s}}{[k!(r_2-k-1)!]^{r_1}}\times$$
$$\times\frac{(r_1!)^{r_2-1}}{(r_1-\nu_1)!\ldots(r_1-\nu_{r_2})!}\frac{1}{k^{\nu_1}\ldots(k-r_2+1)^{\nu_{r_2}}}.$$

But the highest power of p which divides an integer less than or equal to r_2 is $p^{\left[\frac{\ln r_2}{\ln p}\right]}$. From this it follows that the highest power of p which can divide $k^{\nu_1}\ldots(k-r_2)^{\nu_{r_2}}$ is not greater than $p^{r_1\left[\frac{\ln r_2}{\ln p}\right]}$ since $\sum\limits_{i=1}^{r_2}\nu_i\leqslant r_1$. Further, the highest power of p which divides $[k!(r_2-1-k)!]^{r_1}$ is not greater than the highest power of p which divides $[r_2!]^{r_1}$, which, as we have already seen above, is not greater than $p^{r_1\left[\frac{r_2}{p-1}\right]}<p^{r_1r_2}$. Therefore

$$|p^{3r_1r_2}A_{n,k,s}|_\wp \leqslant 1;\quad 0\leqslant n\leqslant r_1-1;\quad 0\leqslant s\leqslant r_2-1,$$

since $2r_1r_2+r_1\left[\dfrac{\ln r_2}{\ln p}\right]+r_1<3r_1r_2$ for $r_2\geqslant 4$. Condition (80) follows directly from this. Making use of the remarks made above, we can now prove the theorem on the approximation of the ratio of logarithms of algebraic numbers by means of rational fractions in the $\wp$-adic sense.

THEOREM IV. *Suppose a and b are numbers in the algebraic field K, $a^{n_1}\neq b^{n_2}$, $n_1^2+n_2^2\neq 0$, for no rational integral n_1, n_2 and $\wp$ a prime ideal in the field K with $N(\wp)=p^\sigma$, $p\equiv 0 \bmod \wp^\mu$. Then if $|a|_\wp=1$, $|b|_\wp=1$, the inequality*

$$(83)\qquad \left|\ln a^\lambda-\frac{n_1}{n_2}\ln b^\lambda\right|_\wp < \wp^{-m_0},\quad \lambda = p^{3\sigma\mu}-p^{(3\mu-1)\sigma},$$
$$0<|n_1|+|n_2|\leqslant 2N,\quad m_0=[\ln^7 N],\quad (n_1,n_2)=1,$$

where n_1, n_2 *are integers, cannot have solutions for* $N > N_0$, *where* N_0 *can be effectively computed as a function of* a, b *and* $\wp$.

We note that this theorem can also be proved for $m_0 = [\ln^{2+\varepsilon} N]$, where $\varepsilon > 0$ is arbitrary; this gives rise to only certain technical complications in the proof. We shall give a proof of this theorem in terms of $O(x)$, since this will simplify it, and the computation of N_0 can be made completely in this proof also, but this is not necessary for our further discussion.

Proof. We set $\ln a^\lambda = \eta_1$, $\ln b^\lambda = \eta_2$, where the logarithms are understood in the sense of (75), assuming that $|\eta_1|_\wp \leqslant |\eta_2|_\wp$, which can be done without loss of generality in the proof, and we shall assume that inequality (83) is satisfied for some $N > N_0$, where the value of this N_0 will be determined in the course of the proof that (83) is impossible. Suppose also that d is a rational integer such that da^λ and db^λ are integers in the field K.

We shall consider the function

$$f(z) = \sum_{k_1=0}^{q} \sum_{k_2=0}^{q} A_{k_1,k_2} e^{(\eta_1 k_1 + \eta_2 k_2)z}, \quad q = [\ln^{3/2} N], \tag{84}$$

where the A_{k_1,k_2} are integers in the field K, all different from zero. In virtue of the linear independence of η_1 and η_2 over the rational field, the function $f(z)$ cannot be identically zero and will be a normal function on the basis of the arguments set forth above.

We now consider the auxiliary linear forms in A_{k_1,k_2}

$$U_{s,t} = \sum_{k_1=0}^{q} \sum_{k_2=0}^{q} A_{k_1,k_2} \left(\frac{n_1}{n_2} k_1 + k_2 \right)^s e^{(\eta_1 k_1 + \eta_2 k_2) p t}, \tag{85}$$

where n_1 and n_2 are determined by inequality (83) and t is an integer. We note that $(n_2, p) = 1$ follows from the assumption $|\eta_1|_\wp \leqslant |\eta_2|_\wp$, the condition $(n_1, n_2) = 1$ and inequality (83). Since $|n_i| < 2N$, $i = 1, 2$, and $N < e^{O(q^{2/3})}$, we shall obviously have the inequality

$$\overline{\left| n_2^s d^{2pqt} \left(\frac{n_1}{n_2} k_1 + k_2 \right)^s a^{\lambda k_1 p t} b^{\lambda k_2 p t} \right|} \leqslant e^{O(rq)} \tag{86}$$

with $0 \leqslant s \leqslant [q^{1/3}] r$, $0 \leqslant t \leqslant r$, for any rational integral $r \geqslant 1$, where numbers whose absolute values are estimated by these inequalities will be algebraic integers in the field K. On the basis of these inequalities, we can, setting $r = r_0 = \left[\frac{1}{\sqrt{2}} q^{5/6} \right]$, according to Lemma

II, §2, Chapter II, choose numbers A_{k_1,k_2}, $0 \leqslant k_1 \leqslant q$, $0 \leqslant k_2 \leqslant q$, all different from zero, and integers in the field K in such a way that the conditions

$$(87) \qquad U_{s,t} = 0, \quad 0 \leqslant t \leqslant r_0, \quad 0 \leqslant s \leqslant [q^{1/3}]r_0,$$

and

$$(88) \qquad \overline{|A_{k_1,k_2}|} < e^{O(q^{11/6})}, \quad 0 \leqslant k_1 \leqslant q, \quad 0 \leqslant k_2 \leqslant q$$

are satisfied. Having chosen the A_{k_1,k_2} in this manner, we consider the differences

$$f^{(s)}(pt) - \eta_2{}^s U_{s,t}.$$

On the basis of inequality (83) we immediately obtain the inequalities

$$(89) \quad |f^{(s)}(pt) - \eta_2{}^s U_{s,t}|_\wp$$
$$= \left| \sum_{k_1=0}^{q} \sum_{k_2=0}^{q} \left[(\eta_1 k_1 + \eta_2 k_2)^s - \left(\eta_2 \frac{n_1}{n_2} k_1 + \eta_2 k_2\right)^s \right] [a^{k_1} b^{k_2}]^{\lambda p t} \right|_\wp \leqslant \wp^{-m_0},$$

which are true for arbitrary rational integral $s \geqslant 0$ and t, inasmuch as $(n_2, p) = 1$ and $|a|_\wp = |b|_\wp = 1$.

Now suppose that $r > 1$ is an integer such that

$$(90) \qquad U_{s,t} = 0, \quad 0 \leqslant s \leqslant [q^{1/3}]r, \quad 0 \leqslant t \leqslant r,$$

and $U_{s_0,t_0} \neq 0$ for certain t_0 and s_0, $t_0 \leqslant r+1$, $s_0 \leqslant [q^{1/3}](r+1)$. On the basis of (87), $r_0 \leqslant r$ and $r < (q+1)^2$, since the system of $(q+1)^2$ equations, linear in A_{k_1,k_2},

$$U_{0,t} = \sum_{k_1=0}^{q} \sum_{k_2=0}^{q} A_{k_1,k_2} e^{(\eta_1 k_1 + \eta_2 k_2)pt} = 0, \quad 0 \leqslant t \leqslant (q+1)^2 - 1$$

has a nonzero determinant as a consequence of the linear independence of η_1 and η_2 over the field of rational numbers. Making use of conditions (90) and inequalities (89), we shall have the inequalities

$$(91) \qquad |f^{(s)}(pt)|_\wp \leqslant \wp^{-m_0}, \quad 0 \leqslant s \leqslant [q^{1/3}]r, \quad 0 \leqslant t \leqslant r.$$

We now construct the interpolation polynomial $P(z)$, defined by the conditions

$$(92) \qquad P^{(s)}(pt) = f^{(s)}(pt), \quad 0 \leqslant s \leqslant [q^{1/3}]r, \quad 0 \leqslant t \leqslant r.$$

Using relation (80), we obtain in virtue of inequality (91) the representation

$$(93) \qquad P(z) = p^{\left[\frac{m_0}{\mu}\right]-3[q^{1/3}]r^2}P_1(z),$$

where $P_1(z)$ is a normal function. Since $r<(q+1)^2$ and $m_0=[\ln^7 N]$, we have

$$(94) \quad 3[q^{1/3}]r^2 < 3q^{1/3}(q+1)^4 < 4q^{13/3} < 4\ln^{13/2}N < \left[\frac{m_0}{2\mu}\right]-1$$

with $N\geqslant N_1$, from which it follows that

$$(95) \qquad P(z) = p^{\left[\frac{m_0}{2\mu}\right]+1}P_2(z)$$

for $N\geqslant N_1$, where $P_2(z)$ is again a normal function. If we set

$$f_1(z) = f(z)-P(z),$$

then in virtue of condition (92) and representation (78) we have

$$f(z) = P(z)+f_1(z) =$$

$$(96) \quad = p^{\left[\frac{m_0}{2\mu}\right]+1}P_2(z)+[z(z-p)\ldots(z-rp)]^{[q^{1/3}]r+1}f_2(z),$$

where $f_2(z)$ and $P_2(z)$ are normal functions. But then for arbitrary s and t, $s\leqslant[q^{1/3}](r+1)$, $t\leqslant r+1$, the estimate

$$(97) \quad |f^{(s)}(pt)|_\wp < \max\left[\wp^{-\mu\left[\frac{m_0}{2\mu}\right]-\mu},\ \wp^{-\mu[q^{1/3}]r(r+1)+\mu(s-r)}\right]$$

$$\leqslant \max\left[\wp^{-\mu\left[\frac{m_0}{2\mu}\right]-\mu},\ \wp^{-\mu[q^{1/3}]r^2}\right] \leqslant \wp^{-\mu[q^{1/3}]r^2},\quad N\geqslant N_1,$$

holds.

Making use now of inequalities (89), we obtain from inequalities (97) the inequalities

$$(98) \quad |U_{s,t}n_2{}^s d^{2pqt}|_\wp \leqslant |\eta^{-s}|_\wp \wp^{-\mu[q^{1/3}]r^2} < \wp^{-\mu[q^{1/3}]r^2+m_1},\quad m_1 = O(q^{1/3}r),$$

which in virtue of (94) are valid for arbitrary s and t, $s\leqslant[q^{1/3}](r+1)$, $t\leqslant r+1$. It follows directly from inequalities (86) and (88) that

$$(99) \quad \left|\overline{n^s d^{2pqt}U_{s,t}}\right| \leqslant e^{O(rq+q^{11/6})},\quad s\leqslant[q^{1/3}](r+1),\quad 0\leqslant t\leqslant r+1.$$

If $U_{s,t}\neq 0$ for arbitrary s_0 and t_0, $s_0\leqslant[q^{1/3}](r+1)$, $t_0\leqslant r+1$, then the inequality

$$(100) \qquad |U_{s_0,t_0}|_\wp \geqslant \wp^{-m_2},\quad m_2 = O(rq+q^{11/6}) = O(rq)$$

follows from inequalities (99) and (77), since $r\geqslant r_0 = \left[\frac{1}{\sqrt{2}}q^{5/6}\right]$.

Comparing inequalities (98) and (100), we see that the inequality

$$\mu[q^{1/3}]r_2{}^2-m_1 \leqslant m_2 = 0(rq),$$

must be satisfied for $N \geqslant N_1$ or, what amounts to the same thing, the inequality

$$(101) \qquad [q^{1/3}]r^2 < C_0 r[q^{1/3}]^3$$

is satisfied for $N \geqslant N_1$, where C_0 is a constant which is independent of r and q and which is effectively computable. But it follows from (101) that $r < C_0 q^{2/3}$, and on the other hand, $r > r_0 > \frac{1}{2} q^{5/6}$. But these inequalities are contradictory for $N \geqslant N_0 \geqslant N_1$. This leads us to the conclusion that for $N \geqslant N_0$, $U_{s,t} = 0$ for all s and t, $0 \leqslant s \leqslant [q^{1/3}](r+1)$; $0 \leqslant t \leqslant r+1$. We have thus arrived at a contradiction with the choice of the number r, and this completes the proof of our theorem.

It follows directly from the theorem proved above that if a and b are algebraic numbers in the field K, $\wp$ is a prime ideal in this field, $|a|_\wp = |b|_\wp = 1$ and $a^{x_1} \neq b^{x_2}$ for no rational integers x_1 and x_2, which are both different from zero simultaneously, then the congruence

$$(102) \quad a^{x_1} - b^{x_2} \equiv 0 \bmod \wp^m, \quad |x_1| + |x_2| \leqslant x, \quad m = [\ln^7 x]$$

with $x > x_0$, where x_1, x_2 are rational integers and $x_0 = x_0(a, b, \wp)$ is an effectively computable constant, is impossible. Inequalities for the measure of transcendence in the usual, i.e. not in the $\wp$-adic sense, for numbers of the form a^b or $\ln a/\ln b$ with a and b algebraic, will be proved further on. Here, we shall introduce only one particular case of such inequalities. As will be proved below, the inequality

$$(103) \quad |x_1 \ln a + x_2 \ln b| < e^{-\ln^{2+\varepsilon} x}, \quad x \geqslant |x_1| + |x_2|, \quad \varepsilon > 0,$$

does not have a solution in rational integers x_1, x_2 for $x > x_0$, $x_0 = x_0(a, b)$, where x_0 can be effectively computed. This inequality is a particular case of inequality (110), §3, Chapter I, the essential role of which in the theory of Diophantine equations and algebraic fields was established there. In our particular case, for a binary form the bound of the magnitudes of possible solutions of inequality (103), as was already stated, can be found completely effectively by applying completely effective analytic methods. Therefore one can assume the fundamental problem in the analytic theory of transcendental numbers to be that of strengthening the analytic methods in the theory of transcendental numbers, so that it will be possible to apply them to the investigation of the behavior of linear forms in n logarithms of algebraic numbers. But inequality (103) enables one to approach the problem concerning the bounds of the solutions of certain classes of cubic equations, whose fields have one unit. It also follows from our inequalities that the equation

$$\alpha^x + \beta^y = \gamma^z, \tag{104}$$

with algebraic α, β and γ and the condition that at least one of them is not an algebraic unit and γ is not a power of 2, has only a finite number of solutions in rational integers x, y, z. Here, the bound of the magnitudes of possible solutions can be effectively computed. The finiteness of the number of solutions of a rather extensive class of transcendental equations in two variables follows from these same inequalities (83) and (103). Returning again to the question of the measure of transcendence, we introduce still further results concerning the measure of transcendence of certain classes of numbers which are connected with elliptic functions. The first results in this direction were obtained by Feldman [1] who made use of the fundamental method discussed below.

Suppose the invariants g_2 and g_3 of the elliptic function $\wp(z)$ are algebraic numbers, ω is the period of $\wp(z)$, and $\varepsilon > 0$ is an arbitrarily small real number. Then we shall have the inequality

$$\Phi(H, n; \omega) > \exp\{-n^{4+\varepsilon} \max [\ln H (\ln \ln H)^{4+\varepsilon}, \quad n \ln^{5+\varepsilon} n]\},$$
$$\max [H, n^n] \geqslant C(\varepsilon, g_2, g_3).$$

Suppose the invariants of $\wp(z)$ are algebraic numbers as above, and that α is a root of the equation $\wp(z) = a$, where a is an algebraic number. Suppose also that $0 < \delta < 1$ and $\varepsilon > 0$. Then we shall have the inequality

$$\Phi(H, n; \alpha) > \exp\{-\max [\ln H e^{n(\ln \ln H)^{\frac{1}{2}+\delta}}, e^{n^{2+\delta} + n^{2+\frac{\delta}{2}+3\varepsilon}}]\},$$
$$\max [H, e^{e^{n^{2+\delta}}}] > C(g_2, g_3, \alpha, \delta, \varepsilon).$$

The problem of the algebraic independence over the rational field of numbers of the form α^β and, a fortiori, of the logarithms of algebraic numbers, is immeasurably more difficult than the problem of the algebraic independence of the values of E-functions which was discussed in Chapter II.

The difficulty of these problems is apparently connected with the following circumstances. First of all it is clear that in the differentiation of a function of the type α^z, with α algebraic, the powers of the transcendental number $\ln \alpha$ grow rapidly, which hinders the construction of linear forms in products of powers of these logarithms having exponents which are not too large. Further, it is at once clear that the forms obtained by differentiation do not have standard coefficients which are sufficiently simply related to simple functions. Finally, one can note that the interchange of the operation of taking the derivative in the formation of the required linear forms with the operation of adjunction of units or any algebraic numbers to z gives rise to too large a growth of the coefficients of these forms, which also hinders one in obtaining the required result. Because of all these circumstances, no one succeeded in making any progress in the solution of problems of mutual transcendence which problems are considered here. At the present time, it is possible to point out one new method (see Gelfond [12]) which permits one to solve certain of these problems, as yet still of a particular character, although one can expect this method to be significantly strengthened. We shall point out its fundamental features with the example of the proof of one sufficiently general theorem concerning the dual transcendence or, equivalently, the algebraic independence in the rational field of one class of numbers. We shall restrict ourselves here only to the discussion of the course of the proof, subdividing it into separate steps, and we shall prove the theorem that if a is an algebraic number, $a \neq 0$, 1 and α is a cubic irrationality, which can, without loss of generality, be considered an integer, then there is no algebraic relation in the rational field between the numbers a^α and a^{α^2}. Let us assume that such a relation exists, in other words, that the equation $P(\omega, \omega_1) = 0$ holds, where $\omega = a^\alpha = e^{\alpha \ln a}$, $\omega_1 = a^{\alpha^2} = e^{\alpha^2}$ and the polynomial $P(x, y)$ is irreducible in the rational field and has rational integral coefficients. We shall assume that the transcendence of the number ω has already been proved. In the sequel, the number q will be indefinitely large and the constants λ with arbitrary indexes will not depend on q.

Step One. Rational integers C_{k_0,k_1,k_2,k_3}, all different from zero, and a $\lambda_0 > 0$ can be found such that for the function $f(z)$,

$$f(z) = \sum_{k_1=0}^{q} \sum_{k_2=0}^{q} \sum_{k_3=0}^{q} A_{k_1,k_2,k_3} e^{(k_1+k_2\alpha+k_3\alpha^2)\eta z}, \tag{105}$$

$$\eta = \ln a, \quad A_{k_1,k_2,k_3} = \sum_{k_0=0}^{q'} C_{k_0,k_1,k_2,k_3}\omega^{k_0}, \quad q' = [q^{3/2} \ln^{1/4} q],$$

the conditions

$$|C_{k_0,k_1,k_2,k_3}| < e^{2q^{3/2} \ln^{1/4} q} \tag{106}$$

and

$$(107)\ f^{(s)}(t) = 0;\ t = p_1+p_2\alpha+p_3\alpha^2,\ 0 \leqslant p_i \leqslant q_1 = [q^{1/2}\ln^{1/4} q],$$
$$i = 1, 2, 3;\quad 0 \leqslant s \leqslant s_0 = [\lambda_0 q^{3/2}\ln^{-3/4} q],\quad \lambda_0 > 0,$$

are satisfied. This function can easily be constructed with the aid of Lemma II, §2, Chapter II.

Step Two. It can be proved that the function $f(z)$ has the property that the set of numbers $f^{(s)}(t)$,

$$0 \leqslant s \leqslant s_1 = [4q^{3/2}\ln^{-3/4} q];$$

(108)

$$t = \sum_{k=0}^{2} p_k\alpha^k,\quad 0 \leqslant p_i \leqslant q_1;\qquad i = 1, 2, 3$$

contains at least one member which is different from zero.

In order to prove this proposition, we write the function $f(z)$ in the form

$$f(z) = \sum_{0}^{N-1} B_k e^{\tau_k z},\quad \tau_k = k_1+k_2\alpha+k_3\alpha^2,\quad N = (q+1)^3,$$
$$B_n = A_{k_1,k_2,k_3},\quad \max_{0\leqslant k\leqslant N-1} |B_k| = B_\nu| \neq 0,$$

and we set $Q(x) = \prod_{i=0}^{N-1}(x-\tau_i),\ \sum_{s=0}^{N-1} C_{k,s}x^s = Q(x)(x-\tau_k)^{-1}$. We shall assume that all the numbers $f^{(s)}(t)$ in the bounds indicated by inequalities (108) equal zero. Then we shall have the integral representation

$$-B_\nu = \frac{(4\pi^2)^{-1}}{Q'(\tau_\nu)}\sum_{s=0}^{N-1} s!C_{\nu,s}\int_{\Gamma_0}\int_{\Gamma_1}\prod_{0}^{q_1}\prod_{0}^{q_1}\prod_{0}^{q_1}\left[\frac{x-k_1-k_2\alpha-k_3\alpha^2}{z-k_1-k_2\alpha-k_3\alpha^2}\right]^{s_1}\times$$
$$\times\frac{f(z)dzdx}{x^{s+1}(z-x)},$$

where Γ_0 is the circumference $|x|=1$, and Γ_1 is the circumference $|z|=N/q$. Estimating the absolute value of the right member and taking into consideration the fact that $|k_1+k_2\alpha+k_3\alpha^2|>\lambda_1 k^{-2}$, $|k_i|\leqslant k$, $i=1, 2, 3$ and in virtue of the fact that α is a cubic irrationality, we obtain the inequality

$$1 \leqslant \exp[-q^3(\ln q-\lambda_2\ln\ln q)],$$

from which the validity of our assertion follows for $q>q_0$.

Step Three. Using the representation of the numbers

$$(109)\quad f^{(s)}(t) = \frac{-s!}{(2\pi)^2}\int_{\Gamma}\int_{\Gamma_1}\prod_{0}^{q_1}\prod_{0}^{q_1}\prod_{0}^{q_1}\left[\frac{x-k_1-k_2\alpha-k_3\alpha^2}{z-k_1-k_2\alpha-k_3\alpha^2}\right]^{s_0} \times$$

$$\times \frac{f(z)dzdx}{(x-t)^{s+1}(z-x)},$$

where Γ is the circumference $|x| = \sqrt{q \ln q}$, and Γ_1 is the circumference $|z| = q^2$, which holds in virtue of the conditions which the function $f(z)$ satisfies by construction, from the very beginning, we show that for this function the conditions

$$(110)\qquad |\eta^{-s} f^{(s)}(t)| < e^{-\frac{1}{2}\lambda_0 q^3 \ln q},$$

$0 \leqslant s \leqslant [4q^{3/2} \ln^{-3/4} q]$, $t = k_1 + k_2\alpha + k_3\alpha^2$, $0 \leqslant k_i \leqslant q$, $i = 1, 2, 3$, hold. One can easily obtain this if one estimates the absolute value of the integral in the right member of (109).

Step Four. Combining the result obtained in the second step with inequality (110) and eliminating ω with the aid of $P(\omega, \omega_1) = 0$, we prove that, whatever the integer $q > q_0$, there always exists a polynomial $P_q(x)$ with rational integral coefficients of degree n and height H such that

$$(111)\quad 0 < |P_q(\omega)| < e^{-\frac{1}{4}\lambda_0 q^3 \ln q}, \quad \max [n, \ln H] < \lambda_3 q^{3/2} \ln^{1/4} q,$$

where $\lambda_3 > 0$ and $\omega = a^{\alpha}$.

Step Five. We shall establish, from the existence of a polynomial $P(x)$ satisfying conditions (111) for arbitrary q, that there exists an irreducible polynomial $R(\omega)$ for an infinite set of values of q, with integral coefficients having no common divisor, satisfying the conditions

$$0 \leqslant |R(\omega)| < e^{-\lambda_4 \sigma^2 \sqrt{\ln q}}, \quad \max [n_1, \ln H_1] < \sigma < \lambda_5 q^{3/2} \ln^{1/4} q,$$

where $\lambda_4 > 0$, $\lambda_5 > 0$, and n_1 and H_1 are the degree respectively the height of the polynomial $R(x)$. Selecting the polynomial $P(\omega)$ which is suitable in respect to its magnitude, and forming the resultant of the two polynomials $P(\omega)$ and $R(\omega)$, we arrive at a contradiction in view of the fact that $P(x)$ can be chosen so that it is not divisible by $R(x)$ and the smallness of $|P(\omega)|$ and $|R(\omega)|$.

The first step consists in the construction of a function $f(z)$ having very many zeros of high multiplicity at points of the form $k_1 + k_2\alpha + k_3\alpha^2$. The coefficients of the functions are chosen as polynomials of not too high a degree with integral, and in their turn, not too large coefficients. Such a choice is possible in view of the fact that under the assumption that the numbers ω and ω_1 are algebraically independent, the number of conditions defining the coefficients of the polynomials in ω can be taken significantly smaller than the number of these coefficients.

The second step consists in proving that it is impossible for the function $f(z)$ to have too many zeros. This circumstance is connected with the fact that the function $f(z)$ is a linear combination of exponential functions. In this connection, a very essential fact is that the inequality

$$|k_1+k_2\alpha+k_3\alpha^2| > \lambda k^{-2}, \quad |k_i| \leqslant k, i = 1, 2, 3,$$

holds, which in its turn is valid in view of the fact that α is a cubic irrationality.

The third step consists in the proof of the fact that the values of the function $f(z)$ and of its derivatives of a very high order are sufficiently small.

The fourth step consists in the proof of the existence, for a very dense sequence of integers σ, of polynomials $P(z)$, whose heights and degrees are not greater than the quantities e^{σ} respectively σ, which satisfy the inequalities $-\sigma^2\sqrt{\ln q}>\ln|P(\omega)|$. This is possible in view of the facts which were established in steps two and three.

The fifth step consists in the proof of the fact that there does not exist any transcendental number ω which satisfies the conditions obtained in the preceding step.

The line of argument discussed above leads directly, almost without change, to a more general assertion. Suppose α_1, α_2, α_3 are numbers in an algebraic field of degree ν, the α_i are linearly independent over the rational field, and that $\beta_1, \beta_2, \beta_3$ are linearly independent over the rational field. Then no algebraic relation among the nine numbers $e^{\alpha_k\beta_i}$, $k=1, 2, 3$, $i=1, 2, 3$ exists over the rational field, with the aid of which these numbers can be expressed algebraically in terms of one of them.

In particular, if α is an algebraic irrationality of degree not less than three and $a\neq 0, 1$ is algebraic, then the four numbers a^{α}, a^{α^2}, a^{α^3}, a^{α^4} cannot be simultaneously expressed algebraically over the rational field in terms of one of them. Modifying the method slightly, one can, for example, prove that the numbers e^{e^k}, $e^{e^{2k}}$, $e^{e^{3k}}$ cannot simultaneously be expressed algebraically in terms of e and, in particular, at least one of them must be a transcendental number for arbitrary rational integer $k > 0$. By exactly the same reasoning, we conclude that one of the two numbers e^{ie^i}, $e^{ie^{2i}}$ must be a transcendental number.

We shall give a thorough discussion of this method in the following sections.

§4. Formulation of fundamental theorems and auxiliary propositions

In this section we shall give the formulations of three fundamental theorems and prove the auxiliary propositions which will be necessary in their proof.

For the sake of brevity in our discussion, we shall introduce certain notation and definitions which will retain the same sense in the sequel. The polynomial $Q(x_1, x_2, \ldots, x_s)$ in the variables $x_1, x_2, \ldots, x_s$ will always be a polynomial with rational integral coefficients whose greatest common divisor equals unity, and which is irreducible over the rational field. This polynomial will be assumed to be different from a constant. We shall say that two polynomials Q_1 and Q_2 are distinct if $Q_1 \neq \pm Q_2$, and that the numbers $\alpha_1, \ldots, \alpha_s$ are algebraically independent in the rational field, if the equality $Q(\alpha_1, \alpha_2, \ldots, \alpha_s) = 0$ is impossible for all $Q(x_1, x_2, \ldots, x_s) \not\equiv 0$. Further, we shall say that the numbers $\alpha_1, \alpha_2, \ldots, \alpha_s$ are expressible algebraically in terms of one of them, for example, α_i, if the s equalities

$$Q_k(\alpha_k, \alpha_i) = 0, k = 1, 2, \ldots, s,$$

hold, where all the $Q_k(x, \alpha_i) \not\equiv 0$. Finally, we extend the rational field by adjoining one transcendental number to it, and we extend the field R_0 thus obtained by adjoining to it in its turn a root of an algebraic equation with coefficients in the field R_0. Any such field or, simply, a finite algebraic field will be called a field R_1. Obviously, if we extend any algebraic field by adjoining to it the numbers $\alpha_1, \alpha_2, \ldots, \alpha_s$ then we obtain a field R_1 if and only if all the numbers $\alpha_1, \ldots, \alpha_s$ can be expressed algebraically in terms of one of them.

THEOREM I. *Suppose the numbers η_0, η_1, η_2, as well as the numbers $1, \alpha_1, \alpha_2$, are linearly independent over the rational field, and that the inequality*

(112) $\quad |x_0\eta_0+x_1\eta_1+x_2\eta_2| > e^{-\tau x \ln x}, \quad |x_i| \leqslant x, \quad i = 1, 2, 3$

holds, where $\tau>0$ *is some constant, and* $x_0, x_1, x_2, \sum_1^3 |x_i| >0$ *are rational integers, for* $x>x'$. *Then no extension of the rational field by means of adjunction to it of the* 11 *numbers*

(113) $\quad \alpha_1, \quad \alpha_2, \quad e^{\eta_i\alpha_k}, \quad i = 0, 1, 2; \qquad k = 0, 1, 2, \quad \alpha_0 = 1,$

will yield the field R_1.

Corollaries to Theorem I.

1) When $\eta_0=\ln a$, $\eta_1=\alpha \ln a$, $\eta_2=\alpha^2 \ln a$, $\alpha_1=\alpha$, $\alpha_2=\alpha^2$, where α and $a\neq 0, 1$ are algebraic numbers, with α of degree not less than three, we obtain that the numbers a^α, a^{α^2}, a^{α^3}, a^{α^4} cannot be algebraically expressed in terms of one of them. In particular, when α is a cubic irrationality, a^α and a^{α^2} are algebraically independent over the rational field.

2) When $\eta_0=\ln a$, $\eta_1=e^\nu \ln a$, $\eta_2=e^{2\nu} \ln a$, $\alpha_1=e^\nu$, $\alpha_2=e^{2\nu}$, where $a\neq 0, 1$ is algebraic, and $\nu\neq 0$ is rational, we obtain that the four numbers a^{e^ν}, $a^{e^{2\nu}}$, $a^{e^{3\nu}}$, $a^{e^{4\nu}}$, cannot be algebraically expressed in terms of the number e and, in particular, at least one of these numbers must be transcendental.

3) When $\eta_1=\alpha\eta_0$, $\eta_2=\alpha^2\eta_0$, $\alpha_1=\alpha$, $\alpha_2=\alpha^2$, $\eta_0\neq\alpha^{-k}\ln a$, $k=0$, 1, 2, where $a\neq 0, 1$ and α is algebraic with α a cubic irrationality, we obtain that η_0 cannot be a common root of the two equations

$$Q_1(e^\eta, e^{\alpha\eta}, e^{\alpha^2\eta}) = 0, \quad Q_2 = (e^\eta, e^{\alpha\eta}, e^{\alpha^2\eta}) = 0,$$

where Q_1 and Q_2 are distinct.

THEOREM II. *Suppose* $\eta_0\neq 0$, η_1/η_0 *is irrational, the numbers* 1, α_1, α_2 *are linearly independent over the rational field and that for* $x>x'$ *the inequality*

(114) $$\left| x_0+x_1\frac{\eta_1}{\eta_0} \right| > e^{-\tau x^2 \ln x}, \quad \tau > 0, \quad |x_0|+|x_1| \leqslant x$$

holds, where τ *is a constant, and* x_0 *and* x_1 *are rational integers. Then no extension of the rational field by means of adjoining the* 10 *numbers*

(115) $\quad \eta_0, \eta_1, \alpha_1, \alpha_2, e^{\eta_i\alpha_k}; \quad i = 0, 1; \quad k = 0, 1, 2; \quad \alpha_0 = 1$

to it can be an R_1 *field.*

Corollaries to Theorem II.

1) When $\eta_0=1$, $\eta_1=e^\nu$, $\alpha_1=e^\nu$, $\alpha_2=e^{2\nu}$, where $\nu\neq 0$ is a rational number, we obtain that the three numbers e^{e^ν}, $e^{e^{2\nu}}$, $e^{e^{3\nu}}$, cannot be

algebraically expressed in terms of the number e and at least one of them is transcendental.

2) When $\eta_0 = \ln a$, $\eta_1 = \ln^{\nu+1} a$, $\alpha_1 = \ln^{\nu} a$, $\alpha_2 = \ln^{2\nu} a$, where $a \neq 0, 1$ is an algebraic number, and $\nu \neq 0$ is rational, we obtain that the three numbers $a \ln^{\nu} a$, $a \ln^{2\nu} a$, $a \ln^{3\nu} a$ cannot be algebraically expressed in terms of $\ln a$ and at least one of them is transcendental.

A number of other corollaries can be obtained to the preceding two theorems.

Further, we shall prove one general theorem concerning the measure of transcendence of a^b and $\ln a/\ln b$ when a and b are algebraic.

THEOREM III. *Suppose α, β, b and $a \neq 0, 1$ are algebraic numbers, and that b and $\ln \alpha/\ln \beta$ are irrational. Suppose further that the degree and height of the polynomial $P(x)$, having rational integral coefficients, are $s > 0$ and $H \geqslant 1$, respectively. Then the inequalities*

$$(116) \qquad |P(a^b)| > e^{-\frac{s^3}{1+\ln^3 s}(s+\ln H)\ln^{2+\varepsilon}(s+\ln H)}$$

and

$$(117) \qquad \left| P\left(\frac{\ln \alpha}{\ln \beta}\right) \right| > e^{-s^2(s+\ln H)^{2+\varepsilon}}$$

are valid for arbitrary fixed $\varepsilon > 0$ and $H > H_0$.

The proof of these theorems will be given in the following section. We note that inequality (103) of the preceding section is a particular case of inequality (117) when $s = 1$.

An R_1 field, which we already defined above, will always be generated by the numbers ω and ω_1 where ω_1 is a root of an algebraic equation with coefficients in the field R_0; i.e. in the extension of the rational field obtained by adjoining ω to it. The number ω_1 whose degree is ν in R_0 will be assumed to be an integer, where we say that ω_1 is an integer if it is a root of an equation in R_0, with coefficient of the highest power equal 1 and the remaining coefficients are polynomials with rational integral coefficients in ω. In particular, ω_1 may be an algebraic number. Any polynomial with rational integral coefficients in ω and ω_1, whose height, if its degree with respect to ω_1 is not greater than $\nu - 1$, is the maximum of the absolute values of its coefficients and whose degree in this case is the degree with respect to ω, will be called an integer in R_1.

These notations and conditions will be retained in all our further discussions. In the sequel also, the positive integers N, q and p will be assumed to be arbitrarily large numbers, whose magnitudes are bounded below by a finite number of inequalities, and the numbers γ and λ with arbitrary indexes will not depend on these numbers N, q, p. These conditions and fixed notation are supplementary to those conditions and notation which we introduced above.

We shall now prove a number of lemmas, which are necessary for the proof of the fundamental theorems.

LEMMA I. *Suppose we have a system of m linear equations in n unknowns,* $n > m$,

$$L_i = \sum_{k=1}^{n} a_{i,k}x_k = 0, \quad i = 1, 2, \ldots, m, \quad |a_{i,k}| \leqslant a, \tag{118}$$

where all the $a_{i,k}$, $1 \leqslant i \leqslant m$, $1 \leqslant k \leqslant n$ are rational integers. Then there exists a solution of this system in rational integers $x_1, x_2, \ldots, x_n$, all different from zero, and whose magnitudes are bounded above by the inequalities

$$|x_k| < 2(2na)^{\frac{m}{n-m}}, \quad k = 1, 2, \ldots, n. \tag{119}$$

This lemma is a particular case of Lemma II, §2, Chapter II and is proved with the application of Lemma I, §2, Chapter I.

LEMMA II. *Suppose $P_1(x_1, x_2, \ldots, x_s), \ldots, P_m(x_1, x_2 \ldots, x_s)$ are arbitrary polynomials in s variables with heights $H_1, H_2, \ldots, H_m$. Denoting the height and degrees of the polynomial $P(x_1, x_2, \ldots, x_s) = P_1(x_1, \ldots, x_s) \ldots P_m(x_1, \ldots, x_s)$ by H and $n_1, n_2 \ldots, n_s$ in the variables $x_1, x_2, \ldots, x_s$, respectively, we will have the inequality*

$$H \geqslant e^{-n}H_1H_2 \ldots H_m, \quad n = \sum_{i=1}^{s} n_i. \tag{120}$$

The height of the polynomial will be, as always, the maximum of the absolute values of its coefficients. This lemma is a generalization of Lemma III, §2, Chapter I.

Proof. Suppose M is the maximum of $|P(x_1, x_2, \ldots, x_i)|$ when $|x_i| = 1$, $i = 1, 2, \ldots, s$, in other words,

$$M = \max_{|x_i|=1} |P(x_1, x_2, \ldots, x_s)|. \tag{121}$$

We now set $x_k = e^{2\pi i\varphi_k}$, $0 \leqslant \varphi_k \leqslant 1$. Then obviously we have

$$M^2 \geqslant \int_0^1 \cdots \int_0^1 |P(x_1, \ldots, x_s)|^2 \, d\varphi_1 \, d\varphi_2 \ldots d\varphi_s. \tag{122}$$

Suppose now that $P_1(x), P_2(x), \ldots, P_m(x)$ are polynomials of degree $p_1, p_2, \ldots, p_m$ with maxima $M_1, \ldots, M_m$ for $|x|=1$ in one variable x. We set $P(x)=P_1(x)P_2(x)\ldots P_m(x)$. Then the degree of $P(x)$ will be $p=p_1+p_2+\cdots+p_m$. Certain of these p_i can be zero. Suppose any one of the polynomials $R(x)$, $R(0)\neq 0$, of degree n has a maximum equal to unity when $|x|=1$. Then, for $x=e^{2\pi i\varphi}$ we have

$$(123)\quad |R(x)|^2 = \prod_{k=1}^{n} |a_k+b_kx|^2$$

$$= \prod_{k=1}^{n} (|a_k|+|b_k|)^2 \prod_{k=1}^{n} |1-t_k+t_ke^{-2\pi i\alpha_k}x|^2,$$

$$t_k = \frac{|b_k|}{|a_k|+|b_k|},\quad 2\pi i\alpha_k = i\arg\frac{b_k}{a_k},\quad k = 1, 2, \ldots, n.$$

But since

$$(124)\qquad \prod_{k=1}^{n} (|a_k|+|b_k|)^2 \geqslant |R(x)|^2,\quad |x| = 1,$$

we have

$$(125)\qquad \prod_{k=1}^{n} (|a_k|+|b_k|)^2 \geqslant 1$$

and, consequently,

$$(126)\quad |R(x)|^2 \geqslant \prod_{k=1}^{n} |1-t_k+t_ke^{2\pi i(\varphi-\alpha_k)}|^2$$

$$\geqslant \prod_{k=1}^{n} [(1-2t_k)^2+4t_k(1-t_k)\cos^2\pi(\varphi-\alpha_k)].$$

Every factor in the right member of inequality (126) attains its minimum when $t_k=\frac{1}{2}$, from which the inequality

$$(127)\qquad |R(x)|^2 \geqslant 2^{-2n}\prod_{k=1}^{n} |x+x_k|^2,\quad x_k = e^{2\pi i\alpha_k}$$

follows. If $R(0)=0$, then the number n in inequality (127) is replaced by a number $n'<n$. Applying inequalities (127) and (122) to the polynomials $P_k(x)$, we obtain the inequalities

$$(128)\quad |P_k(x)|^2 \geqslant 2^{-2p_k'}M_k^2\prod_{\nu=1}^{p_k'} |x+x_{k,\nu}|^2$$

$$\geqslant 2^{-2p_k'}\prod_{\nu=1}^{p_k'} |x+x_{k,\nu}|^2 \int_0^1 |P_k(x)|^2\,d\varphi,$$

where $p_k' \leqslant p_k$, $x = e^{2\pi i\varphi}$. Further, it follows from inequalities (128), after term-by-term multiplication, that

$$(129)\quad |P(x)|^2 \geqslant 2^{-2p'} \prod_{k=1}^{p'} |x+x_k|^2 \int_0^1 \cdots \int_0^1 |P_1(x_1)\ldots P_m(x_m)|^2\, d\varphi_1 \ldots d\varphi_m,$$

where again $p' \leqslant p$. Integrating both members of inequality (129) with respect to φ, we obtain that

$$(130)\quad \int_0^1 |P(x)|^2\, d\varphi \geqslant 2^{-2p+1} \int_0^1 \cdots \int_0^1 |P_1(x_1)\ldots P_m(x_m)|^2\, d\varphi_1 \ldots d\varphi_m$$

so that if $p' \geqslant 1$, we have

$$\int_0^1 \prod_{k=1}^{p'} |x+x_k|^2\, d\varphi \geqslant 2,$$

and if $p'=0$, then inequality (130) is obvious. It is convenient to rewrite inequality (130) in the form

$$(130')\quad \int_0^1 |P(x)|^2\, d\varphi \geqslant 2^{-2p+1} \prod_{k=1}^{m} \int_0^1 |P_k(x)|^2\, d\varphi.$$

Suppose that now $P_1(x_1, x_2, \ldots, x_s), \ldots, P_m(x_1, x_2, \ldots, x_s)$ are already polynomials in s variables and that

$$P(x_1, x_2, \ldots, x_s) = P_1(x_1, x_2, \ldots, x_s) \ldots P_m(x_1, x_2, \ldots, x_s)$$

is a polynomial of degree $n_1, n_2, \ldots, n_s$ in the variables $x_1, x_2, \ldots, x_s$, respectively. Setting

$$x_{k,\nu} = e^{2\pi i\varphi_{k,\nu}}, \quad 1 \leqslant k \leqslant s, \quad 1 \leqslant \nu \leqslant m,$$
$$x_k = e^{2\pi i\varphi_k}, \quad 1 \leqslant k \leqslant s,$$

and fixing $x_1, x_2, \ldots, x_{p-1}, x_{p+1,\nu}, x_{p+2,\nu}, \ldots, x_{s,\nu}$; $\nu = 1, 2, \ldots, m$ arbitrarily, we obtain from the inequality (130) the inequality

$$(131)\quad \int_0^1 \prod_{\nu=1}^{m} |P_\nu(x_1, x_2, \ldots, x_{p-1}, x_p, x_{p+1,\nu}, \ldots, x_{s,\nu})|^2\, d\varphi_p$$
$$\geqslant 2^{-2n_p+1} \int_0^1 \cdots \int_0^1 \prod_{\nu=1}^{m} |P_\nu(x_1, x_2, \ldots, x_{p-1}, x_{p,\nu}, x_{p+1,\nu}, \ldots, x_{s,\nu})|^2\, d\varphi_{p,1} \ldots d\varphi_{p,m},$$
$$p = 1, 2, \ldots, s,$$

since the degree of the polynomial in the left member of the inequality with respect to x_p cannot be greater than n_p.

We shall now assume that the inequality

$$(132)\quad \int_0^1 \cdots \int_0^1 |P(x_1, \ldots, x_{q-1}, x_q, \ldots, x_s)|^2\, d\varphi_q \ldots d\varphi_s$$
$$\geqslant 2^{-2\sum_{\nu=q}^{s} n_\nu + s - q + 1} \int_0^1 \cdots \int_0^1 \prod_{\nu=1}^{m} |P_\nu(x_1, \ldots$$
$$\ldots, x_{q-1}, x_{q,\nu}, \ldots, x_{s,\nu})|^2 \prod_{k=q}^{s} \prod_{\nu=1}^{m} d\varphi_{k,\nu}$$

is valid for some q, $1 \leqslant q \leqslant s$, where the integral on the right is $(s-q+1)m$-fold.

Integrating both members of inequality (132) with respect to φ_{q-1} and making use of inequality (131) with $p=q-1$, we immediately obtain the inequality

$$(133)\quad \int_0^1 \cdots \int_0^1 |P(x_1, \ldots, x_{q-1}, x_q, \ldots, x_s)|^2\, d\varphi_{q-1}\, d\varphi_q \ldots d\varphi_s$$
$$\geqslant 2^{-2\sum_{\nu=q-1}^{s} n_\nu + s - q + 2} \int_0^1 \cdots \int_0^1 \prod_{\nu=1}^{m} |P_\nu(x_1, \ldots$$
$$\ldots, x_{q-2}, x_{q-1,\nu}, \ldots, x_{s,\nu})|^2 \prod_{k=q-1}^{s} \prod_{\nu=1}^{m} d\varphi_{k,\nu},$$

where the integral on the right is $(s-q+2)m$-fold.

Thus, if inequality (132) holds for $q>1$, then it is also true for $q-1$. When $q=s$, inequality (132) coincides with inequality (131) with $p=s$. It follows that inequality (132) holds for arbitrary $q \geqslant 1$, and, in particular, when $q=1$. Setting $q=1$ in inequality (132), we immediately obtain the fundamental inequality

$$(134)\quad \int_0^1 \cdots \int_0^1 |P(x_1, x_2, \ldots, x_s)|^2\, d\varphi_1\, d\varphi_2 \ldots d\varphi_s$$
$$\geqslant 2^{-2n+s} \prod_{\nu=1}^{m} \int_0^1 \cdots \int_0^1 |P_\nu(x_1, x_2, \ldots, x_s)|^2\, d\varphi_1\, d\varphi_2 \ldots d\varphi_s,$$

since $n = \sum_{k=1}^{s} n_k$. The last inequality is a generalization of inequality (130′) to the case of s variables.

Further, if H_ν is the height of the polynomial $P_\nu(x_1, x_2, \ldots, x_s)$, then it is obvious that

$$(135)\qquad H_\nu^2 \leqslant \int_0^1 \cdots \int_0^1 |P_\nu(x_1, x_2 \ldots, x_s)|^2 \, d\varphi_1 \ldots d\varphi_s;$$

and if H and $n_1, n_2, \ldots, n_s$ are the height and degrees of the polynomial $P(x_1, x_2, \ldots, x_s)$, then it is also clear that

$$(136)\quad H^2 \geqslant \prod_{k=1}^{s} (1+n_k)^{-1} \int_0^1 \cdots \int_0^1 |P(x_1, x_2, \ldots, x_s)|^2 \, d\varphi_1 \ldots d\varphi_s.$$

Comparing inequalities (134), (135) and (136), we arrive at the inequality

$$(137)\qquad H \geqslant 2^{s/2} \prod_{k=1}^{s} (1+n_k)^{-1/2} 2^{-n} H_1 H_2 \ldots H_m,$$

from which, inasmuch as $e^p > \sqrt{\dfrac{p+1}{2}}\, 2^p$ for $p \geqslant 1$, we finally obtain the inequality of our lemma, namely, that

$$(138)\qquad H > e^{-n} H_1 H_2 \ldots H_m.$$

We shall now prove a lemma which supplements Lemma II.

LEMMA II′. *If $P(x_1, \ldots, x_s)$ is a polynomial of degree $n_1, n_2, \ldots, n_s$ in the variables $x_1, x_2, \ldots, x_s$ of height h, and the polynomial $R(x_1, x_2, \ldots, x_s) = P^m(x_1, x_2, \ldots, x_s)$ has height H, then*

$$(139)\qquad H \geqslant \prod_{\nu=1}^{s} (1+2mn_\nu)^{-1} h^m.$$

Proof. In the proof of this lemma, we shall retain the notation we used in the proof of Lemma II. We expand $|R^2(x_1, x_2, \ldots, x_s)|$ in a Fourier series. We obtain the relation

$$|R(x_1, x_2, \ldots, x_s)|^2 = \sum_{k_1=-p_1}^{p_1} \cdots \sum_{k_s=-p_s}^{p_s} e^{2\pi i \sum_{\nu=1}^{s} k_\nu \varphi_\nu} \int_0^1 \cdots$$
$$\cdots \int_0^1 |R(x_1, \ldots, x_s)|^2 e^{-2\pi i \sum_{\nu=1}^{s} k_\nu \varphi_\nu} \, d\varphi_1 \ldots d\varphi_s,$$
$$p_i = mn_i,\ i = 1, 2, \ldots, s,$$

from which the inequality

$$\int_0^1 \cdots \int_0^1 |R(x_1, \ldots, x_s)|^2 \, d\varphi_1 \ldots d\varphi_s \geqslant \prod_{\nu=1}^{s} (1+2p_\nu)^{-1} M^{2m}$$

follows, where M is defined by equality (121). Making use of inequality (136), we obtain from this the inequality

$$H^2 \geqslant \prod_{\nu=1}^{s} (1+2p_\nu)^{-2} M^{2m},$$

and furthermore, with the aid of inequalities (122) and (135), the inequality

$$H^2 \geqslant \sum_{\nu=1}^{s} (1+2p_\nu)^{-2}\left[\int_0^1 |P(x_1, \ldots, x_s)|^2 \, d\varphi_1 \ldots d\varphi_s\right]^m$$

$$\geqslant \prod_{\nu=1}^{s} (1+2p_\nu)^{-2} h^{2m},$$

which completes the proof of the lemma.

It is a curious fact that inequality (137) is asymptotically attainable. We set $n=sm$, $s=[p^{2/3}\ln^{-\varepsilon} p]$, $m=[p^{1/3}\ln^{\varepsilon} p]$,

$$P(x) = x^n - 1, \quad P_k(x) = \prod_{\nu=0}^{s-1} \left(x - e^{2\pi i \frac{ks+\nu}{n}}\right),$$

$$k = 0, 1, \ldots, m-1.$$

Then $H=1$, $H_k = 2^{s+O(\ln n)}$, and consequently,

$$H = 2^{-n+O(n^{1/3}\ln^{1+\varepsilon} n)} H_1 H_2 \ldots H_m.$$

A particular case of this lemma was proved in Chapter I.

LEMMA III. *Suppose p, q, $p>q^\gamma$, r, r_1 are positive rational integers, $\varepsilon>0$ and γ are fixed, and all the numbers $\alpha_1, \alpha_2, \ldots, \alpha_q$, as well as the numbers $\beta_1, \beta_2, \ldots, \beta_r$ are distinct and arranged in order of increasing absolute values, in other words, $|\alpha_k| \leqslant |\alpha_{k+1}|$ and $|\beta_k| \leqslant |\beta_{k+1}|$. We set $|\alpha_q| = \alpha$, $|\beta_r| = \beta$ and suppose that there exist constants $\gamma_0>0$, $\gamma_1>0$, $\gamma_1+\gamma_0<1$ such that $\alpha<(pq)^{\gamma_1}$, $\beta<(pq)^{\gamma_0}$. We also suppose that there exists a constant γ_2, such that the inequalities*

$$\prod_{\substack{k=1 \\ k\neq i}}^{q} |\alpha_i - \alpha_k| > e^{-\gamma_2 q \ln pq}, \quad |\alpha_i - \alpha_k| > e^{-\gamma_2 q \ln pq}, \tag{140}$$

$$1 \leqslant i \leqslant q, \quad 1 \leqslant k \leqslant q$$

are satisfied. Then, if the function $f(z)$ has the form

$$f(z) = \sum_{k=0}^{p-1} \sum_{s=1}^{q} A_{k,s} z^k e^{\alpha_s z}, \tag{141}$$

where the numbers $A_{k,s}$ are all different from zero, at least one of the numbers

$$(142)\qquad f^{(s)}(\beta_k),\quad 0 \leqslant s \leqslant r_1-1,\quad 1 \leqslant k \leqslant r,$$

$$r_1 r \geqslant [\lambda pq],\quad \lambda = \frac{1+\gamma_1+2\gamma_2+\varepsilon}{1-\gamma_1-\gamma_0},$$

is different from zero for sufficiently large pq.

Proof. We note that without loss of generality in our lemma one may assume $|A_{k,s}| \leqslant 1$, where at least one coefficient of our function equals unity in absolute value. To this end, we can always divide all the coefficients by the coefficient whose absolute value is largest, and which is different from zero by assumption. We shall set, for brevity, $m=pq$, $\varepsilon=3\delta$ and assume that

$$(142')\qquad f^{(s)}(\beta_k) = 0,\quad 0 \leqslant s \leqslant r_1-1,\quad 1 \leqslant k \leqslant r,$$

$$r_1 r \geqslant [\lambda m],\quad \lambda = \frac{1+\gamma_1+2\gamma_2+\varepsilon}{1-\gamma_1-\gamma_0}.$$

Then we shall have the representation

$$(143)\quad f^{(s)}(0) = \frac{s!}{(2\pi i)^2}\int_\Gamma \frac{dz}{z^{s+1}}\int_{\Gamma_1}\left[\frac{(z-\beta_1)\ldots(z-\beta_r)}{(\zeta-\beta_1)\ldots(\zeta-\beta_r)}\right]^{r_1}\frac{f(\zeta)}{\zeta-z}\,d\zeta,$$

where the contour Γ is the circumference $|z|=1$, and the contour Γ_1 is the circumference $|\zeta|=m^{1-\gamma_1}$. Estimating the absolute value of the integral in the right member of this equality when $s\leqslant m$, we immediately obtain the inequality

$$(144)\quad |f^{(s)}(0)| < \exp\{[1+\delta-(1-\gamma_1-\gamma_0)]m\ln m\},\quad s \leqslant m.$$

We now construct a polynomial $P(z)$ of degree $m-1=pq-1$, which satisfies the conditions

$$(145)\quad P^{(k)}(\alpha_s) = 0,\quad (k-\nu)^2+(s-n)^2 \neq 0,\quad P^{(\nu)}(\alpha_n) = 1,$$

$$s = 1, 2, \ldots, q;\qquad k = 0, 1, \ldots, p-1;\qquad \nu \leqslant p-1,\quad n \leqslant q.$$

Such a polynomial of degree $pq-1$ is defined uniquely with the aid of the condition (145) from the equation

$$(146)\quad \int_{|\zeta|=2a}\left[\frac{(z-\alpha_1)\ldots(z-\alpha_q)}{(\zeta-\alpha_1)\ldots(\zeta-\alpha_q)}\right]^p\frac{P(\zeta)}{\zeta-z}\,d\zeta = 0,\quad |z| \leqslant \alpha,$$

when all the α_s are distinct. Solving this equation, we obtain that

$$(147)\quad P_{\nu,n}(z) = \frac{[(z-\alpha_1)\ldots(z-\alpha_q)]^p}{\nu!(p-\nu-1)!}\times$$

$$\times\left|\frac{d^{p-\nu-1}}{d\zeta^{p-\nu-1}}\frac{-(\zeta-\alpha_n)^p}{[(\zeta-\alpha_1)\ldots(\zeta-\alpha_q)]^p(\zeta-z)}\right|_{\zeta=\alpha_n},$$

and further that

$$(148)\quad P_{\nu,n}(z) = \frac{(-1)^{p+\nu}}{\nu!}\left[\frac{(z-\alpha_1)\ldots(z-\alpha_q)}{(\alpha_n-\alpha_1)\ldots(\alpha_n-\alpha_q)}\right]^p \times$$
$$\times \sum_{\substack{\nu_1+\cdots+\nu_q\\=p-\nu-1}} (\alpha_n-z)^{-\nu_n-1}\prod_{\substack{k=1\\k\neq n}} \frac{(p+\nu_k-1)!}{(p-1)!\,\nu_k!}(\alpha_n-\alpha_k)^{-\nu_k}.$$

On the other hand, we have the representation

$$(149)\qquad P_{\nu,n}(z) = \sum_{k=0}^{m-1} C_k z^k, \quad m = pq.$$

The coefficients C_k may be immediately estimated in absolute value, if we take condition (140) and the inequality $|\alpha_i| < m^{\gamma_1}$ into consideration. We then obtain the estimate

$$(150)\qquad |C_k| < \exp\,[(\gamma_1+2\gamma_2+\delta)m\ln m]$$

uniformly in ν and n.

Noting now that

$$(151)\quad \left.\frac{d^k}{dz^k}\, z^r e^{\alpha_s z}\right|_{z=0} = k(k-1)\ldots(k-r+1)\alpha_s^{k-r} = \left.\frac{d^r}{dz^r}\, z^k\right|_{z=\alpha_s},$$

we obtain the relation

$$(152)\qquad P_{\nu,n}^{(r)}(\alpha_s) = \sum_{k=0}^{m-1} C_k \left.\frac{d^k}{dz^k}\, z^r e^{\alpha_s z}\right|_{z=0},$$

from which it follows immediately that

$$(153)\qquad \sum_{k=0}^{m-1} C_k f^{(k)}(0) = \sum_{r=0}^{p-1}\sum_{s=1}^{q} A_{r,s}P_{\nu,n}^{(r)}(\alpha_s) = A_{\nu,n}.$$

Since at least one of the $A_{\nu,n}$ is equal to unity in absolute value, estimating the left member of equality (153) in absolute value, with the aid of inequalities (144) and (150) and taking into consideration that $\lambda(1-\gamma_1-\gamma_0) = 1+\gamma_1+2\gamma_2+\varepsilon$, we obtain the inequality

$$(154)\qquad 1 \leqslant me^{-(\varepsilon-2\delta)m\ln m} = me^{-\delta m\ln m},$$

which also leads us to a contradiction for sufficiently large m.

The condition $p > q^{\gamma}$, appearing in the formulation of the lemma, serves only to simplify the proof and may be replaced by a much weaker condition. The magnitude of the number λ may also be made smaller. Lemma IV, which we shall now prove, serves as a supplement to Lemma III.

LEMMA IV. *Suppose the numbers* $\eta \neq 0$, α, ε, $\frac{1}{2} > \varepsilon > 0$, *are prescribed, and that* α *is an irrational number for which the inequality*

$$|x_0\alpha + x_1| > e^{-\tau x}, \quad x = |x_0|, \tag{155}$$

is satisfied when $x > x'$, *where* x_0 *and* x_1 *are rational integers and* $\tau > 0$ *is a constant. Suppose also that* $N > 0$ *and* q, $N \geqslant q \geqslant \ln^{1+\varepsilon} N$ *are rational integers where* N *is sufficiently large. Then if* $f(z)$ *has the form*

$$f(z) = \sum_{k_1=0}^{q} \sum_{k_2=0}^{N} A_{k_1,k_2} e^{\eta(k_1\alpha+k_2)z}, \quad f(z) \not\equiv 0, \tag{156}$$

$$|A_{k_1,k_2}| < e^{Nq}, \quad 0 \leqslant k_1 \leqslant q, \quad 0 \leqslant k_2 \leqslant N,$$

and the inequalities

$$|f^{(s)}(t)| < e^{-Nq \ln q}, \quad 0 \leqslant s \leqslant r_1, \quad t = 0, 1, \ldots, r \tag{157}$$

where either

$$r = [\sqrt{q \ln N}], r_1 = \left[\frac{4}{\varepsilon}\sqrt{N\frac{q}{\ln N}}\right], \quad \text{for} \quad q \leqslant N,$$

or

$$\sqrt{N} \leqslant r \leqslant N^{1-\varepsilon}, \quad r_1 = \left[\frac{4}{\varepsilon}\frac{N^2}{r}\right] \quad \textit{for} \quad q = N,$$

then

$$|A_{k_1,k_2}| < e^{-\frac{1}{4}Nq \ln q}, \quad 0 \leqslant k_1 \leqslant q, \quad 0 \leqslant k_2 \leqslant N. \tag{158}$$

Proof. Condition (155) of this lemma is trivially satisfied if α is either algebraic or equal to an irrational ratio of logarithms of algebraic numbers where the logarithms have the same base. We set $m = (q+1)(N+1)$, $\beta = |\alpha| + 1$ and write $f(z)$ in the form

$$f(z) = \sum_{1}^{m} A_k e^{\alpha_k z}, \quad \alpha_k = \eta(k_1\alpha + k_2), \quad 0 \leqslant k_1 \leqslant q, \quad 0 \leqslant k_2 \leqslant N.$$

We now estimate the quantity $|f^{(s)}(0)|$ for $s \leqslant m$.

We consider first of all the representation $f(z)$ for $|z| = \frac{1}{2} + r$, where r is defined by the conditions of the lemma,

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \left[\frac{z(z-1)\ldots(z-r)}{\zeta(\zeta-1)\ldots(\zeta-r)}\right]^{r_1+1} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

$$-\frac{1}{2\pi i}\sum_{k=0}^{r_1}\sum_{t=0}^{r} \frac{f^{(k)}(t)}{k!}\int_{\Gamma_t}\left[\frac{z(z-1)\ldots(z-r)}{\zeta(\zeta-1)\ldots(\zeta-r)}\right]^{r_1+1}\frac{(\zeta-t)^k}{\zeta-z}\, d\zeta,$$

where the contour Γ is the circumference $|\zeta| = q$, and the contours Γ_t are the circumferences $|\zeta - t| = \frac{1}{4}$, $t = 0, \ldots, r$.

Estimating directly the absolute value of the right member of this equality and taking into consideration inequalities (156), we obtain the inequality

$$(159) \qquad |f(z)| < e^{-\frac{4}{\varepsilon}m \ln \frac{q}{r} + \gamma_3 m} + e^{-m \ln q + \gamma_4 m}, \quad |z| = \tfrac{1}{2} + r.$$

We now consider the polynomial $P_n(z)$

$$(160) \qquad P_n(z) = \frac{(z-\alpha_1)(z-\alpha_2)\ldots(z-\alpha_m)}{(z-\alpha_n)\prod\limits_{\substack{i=1\\ i\neq n}}^{m}(\alpha_n - \alpha_i)} = \sum_{k=0}^{m-1} C_{k,n} z^k;$$

$$P_n(\alpha_k) = \begin{cases} 1, & k = n \\ 0, & k \neq n. \end{cases}$$

Then, as in Lemma III, we have

$$A_n = \sum_{k=0}^{m-1} C_{k,n} f^{(k)}(0).$$

When $s \leqslant m$, it follows from inequality (159) that

$$(161) \qquad |f^{(s)}(0)| = \frac{s!}{2\pi}\left|\int_{|z|=\frac{1}{2}+r} \frac{f(z)}{z^{s+1}}\,dz\right|$$
$$< e^{s \ln s - s \ln r + \gamma_5 m}\left[e^{-\frac{4}{\varepsilon}m \ln \frac{q}{r}} + e^{-m \ln q}\right].$$

We shall now find the least upper bound of the absolute values of the $C_{k,n}$. First of all, since $s!(N-s)! > 2^{-N}N!$, we shall have the estimate

$$(162) \qquad \sum_{k=p}^{N+p} |a-k| > (a) 2^{-N-1} n!$$

for arbitrary a, where p is any integer and the symbol (a) is defined by the relation

$$(163) \qquad (a) = \min |a-k|, \quad k = 0, \pm 1, \pm 2, \ldots.$$

It follows that we have the inequality

$$\prod_{\substack{k=1\\ k\neq n}}^{m} |\alpha_n - \alpha_k| = |\eta|^{m-1} \prod_{\substack{k_1=0\\ (k_1-p_1)^2+(k_2-p_2)^2\neq 0}}^{q} \prod_{k_2=0}^{N} |p_1\alpha + p_2 - k_1\alpha - k_2|$$
$$> |\eta|^{m-1} \prod_{\substack{k_1=0\\ k_1\neq p_1}}^{q} ([p_1-k_1]\alpha) 2^{-m} N!^{q+1},$$

where $0<|p_1-k_1|\leqslant q$. In virtue of condition (155) we obtain finally that

$$\sum_{\substack{k=1\\k\neq n}}^{m} |\alpha_n-\alpha_k| > e^{m\ln N-\gamma_6 m}.$$

Further, we have immediately that

$$(z-\alpha_1)\ldots(z-\alpha_m) \ll (z+\beta N)^m,$$

from which the estimate

$$|C_{k,n}| < e^{(m-k)\ln N-m\ln N+\gamma_7 m} = e^{-k\ln N+\gamma_7 m}; \tag{164}$$
$$0\leqslant k\leqslant m$$

follows. Combining inequalities (161) and (164) we obtain the estimate

$$|A_n| < me^{m\ln q-m\ln r+\gamma_8 m}\left[e^{-\frac{4}{\varepsilon}m\ln\frac{q}{r}}+e^{-m\ln q}\right]. \tag{165}$$

Since, by the conditions of the lemma,

$$\frac{4}{\varepsilon}\,\frac{\ln q-\ln r}{\ln q}+\frac{\ln r}{\ln q}-1 \geqslant \frac{1}{2};\quad \frac{\ln r}{\ln q}\geqslant\frac{1}{2},$$

it follows from inequality (165) that

$$|A_n| < 2me^{-\frac{1}{2}m\ln q} < e^{-\frac{1}{4}m\ln q},$$

which proves our lemma.

We now note that if inequalities (157) are replaced by the conditions $f^{(s)}(t)=0$, where s and t vary in their former intervals, we may divide $f(z)$ by $\max|A_{k_1,k_2}|$ and then inequalities (158) lead us to a contradiction. From this it follows that at least one $f^{(s)}(t)$, for s and t varying in the intervals indicated in the lemma, will be different from zero.

Lemma V. *Suppose α is a fixed transcendental number, the numbers $\omega_1,\omega_2,\ldots,\omega_\nu$ form a basis for the ring of integers in the field K_ν, $P(x)$ and $R(x)$ are polynomials of degree not greater than p and r respectively with coefficients which are integers in the field K_ν; in other words,*

$$P(x)=\sum_{s=0}^{p}H_s x^s;\quad H_s=\sum_{i=1}^{\nu}H_{s,i}\omega_i;\quad \max|H_{s,i}|\leqslant H,$$

$$R(x)=\sum_{s=0}^{r}h_s x^s;\quad h_s=\sum_{i=1}^{\nu}h_{s,i}\omega_i;\quad \max|h_{s,i}|\leqslant h,$$

where all the $H_{s,i}$ *and* $h_{s,i}$ *are rational integers. Then, if the inequality*

$$(166)\qquad |P(\alpha)|+|R(\alpha)| < (a\beta)^{-\nu(p+r)}h^{-\nu p}H^{-\nu r}(p+r)^{-\nu(p+r)},$$
$$a = \nu \max_{1\leqslant k,s\leqslant \nu} |\omega_k^{(s)}|, \quad \beta = 1+|\alpha|,$$

is satisfied, where $\omega_k^{(1)}=\omega_k$, *and* $\omega_k^{(2)}, \ldots, \omega_k^{(\nu)}$ *are its conjugates, the polynomials* $P(x)$ *and* $R(x)$ *must have a common zero when* $p+r\geqslant 2$ *and if* $R(x)$ *is irreducible in the field* K_ν, *then* $P(x)$ *is divisible by* $R(x)$. *In case* $\nu=1$, *the number* $a=1$.

Proof. Without loss of generality, we may assume that $P(0)\neq 0$, $R(0)\neq 0$, $H_p\neq 0$, $h_r\neq 0$, since the right member of (166) is a decreasing function of p and r.

We shall assume that $R(x)$ and $P(x)$ do not have a common zero. Then D_1, the resultant of the two polynomials $R(x)$ and $P(x)$, will be an algebraic integer in the field K_ν, which is different from zero. This resultant is a determinant of order $p+r$ and has the form

$$(167)\quad D_1 = \begin{vmatrix} H_0 & H_1 & \ldots & H_{p-1} & H_p & 0 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \ldots & H_{p-r} & H_{p-r+1} & H_{p-r+2} & \ldots & H_p \\ h_0 & h_1 & \ldots & h_{p-1} & h_p & h_{p+1} & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \ldots & h_0 & h_1 & h_2 & \ldots & h_r \end{vmatrix}$$

$$= \begin{vmatrix} P(\alpha) & H_1 & \ldots & H_{p-r} & H_p & 0 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha^{r-1}P(\alpha) & 0 & \ldots & H_{p-r} & H_{p-r+1} & H_{p-r+2} & \ldots & H_p \\ R(\alpha) & h_1 & \ldots & h_{p-1} & h_p & h_{p+1} & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha^{p-1}R(\alpha) & 0 & \ldots & h_0 & h_1 & h_2 & \ldots & h_r \end{vmatrix},$$

where $H_k=0$ for $k<0$, $k>p$; $h_k=0$ for $k<0$, $k>r$. Estimating the absolute value of D_1, as well as of the numbers $D_2, D_3, \ldots, D_\nu$, which are conjugate to it, with the aid of the Hadamard inequality, we immediately obtain that

$$(168)\quad 1 \leqslant \prod_{k=1}^{\nu} |D_k| \leqslant [|P(\alpha)|+|R(\alpha)|](a\beta)^{\nu(p+r)}H^{\nu r}h^{\nu p}(p+r)^{\frac{p+r}{2}\nu}$$
$$\leqslant (p+r)^{-\frac{p+r}{2}\nu}$$

in virtue of condition (166). This proves the lemma.

LEMMA VI. *Suppose α is fixed and transcendental, and that $P(x)$ is a polynomial with rational integral coefficients, whose common divisor is unity, of height H_0 and degree n_0. If the inequalities*

(169) $|P(\alpha)| < H^{-\lambda n}, \quad \lambda > 6, \quad \ln H \geqslant n, \quad H_0 \leqslant H, \quad n_0 \leqslant n,$

hold, then there exists a divisor $P_1(x)$ of $P(x)$, which is a power of a polynomial irreducible over the rational field, where the degree n_1, the height H_1 and the value at the point α of this divisor $P_1(x)$ satisfy the conditions

(170) $$|P_1(\alpha)| < H^{-(\lambda-6)n}, \quad H_1 \leqslant He^n, \quad n_1 \leqslant n.$$

Proof. We note that if $P(x) = R_1(x)R_2(x)$, where $R_1(x)$ and $R_2(x)$ are relatively prime polynomials with integral coefficients and $|R_1(\alpha)| \geqslant |R_2(\alpha)|$, then

(171) $$H^{-3n} < |R_1(\alpha)|.$$

In fact, if the height and degree of $R_1(x)$ and $R_2(x)$ are h_1, h_2 and n_1, n_2, respectively, then $n_1+n_2=n$ and in virtue of Lemma II, $h_1h_2 < He^n$. From this it follows that for $H > H'(\alpha)$ we have

(172) $$(1+|\alpha|)^n n^n h_1^{n_2} h_2^{n_1} (1+|\alpha|)^n n^n (h_1h_2)^n < \tfrac{1}{2}H^{3n}.$$

This means that if the inequality (171) does not hold, then the conditions of Lemma V are satisfied for $\nu = 1$, and the polynomials $R_1(x)$ and $R_2(x)$ have a common divisor.

We now represent $P(x)$ in the form of a product of powers of distinct irreducible polynomials with integral coefficients

(173) $P(x) = P_1(x) \ldots P_s(x), \quad |P_1(\alpha)| \leqslant |P_2(\alpha)| \leqslant \ldots \leqslant |P_s(\alpha)|.$

The inequality

(174) $$|P_1(\alpha) \ldots P_k(\alpha)| > |P_{k+1}(\alpha) \ldots P_s(\alpha)|$$

cannot hold in virtue of condition (173) as soon as $k \geqslant s-k$. This means that there exists a $\nu < s/2$ such that

(175) $$\begin{cases} |P_1(\alpha) \ldots P_{\nu-1}(\alpha)| \geqslant |P_\nu(\alpha)P_{\nu+1}(\alpha) \ldots P_s(\alpha)|, \\ |P_1(\alpha) \ldots P_\nu(\alpha)| \leqslant |P_{\nu+1}(\alpha) \ldots P_s(\alpha)|. \end{cases}$$

But then, on the one hand, in virtue of condition (169) and inequality (171) we have

(176) $$|P_\nu(\alpha)P_{\nu+1}(\alpha) \ldots P_s(\alpha)| < H^{-(\lambda-3)n},$$

and on the other hand

(177) $$|P_{\nu+1}(\alpha) \ldots P_s(\alpha)| > H^{-3n}.$$

This implies immediately that the inequality

(178) $$|P_1(\alpha)| \leqslant |P_\nu(\alpha)| < H^{-(\lambda-6)n}$$

holds. But it follows from Lemma II that the height of $P_1(x)$ is not greater than He^n which completes the proof of our theorem.

LEMMA VI′. *Under the conditions of Lemma VI, the polynomial $P(x)$ has an irreducible divisor $Q(x)$ (the common divisor of the integral coefficients of $Q(x)$ is unity), satisfying the conditions*

$$(179) \qquad |Q(\alpha)| < H^{-\frac{\lambda-6}{s}n}, \quad H^{\frac{1}{s}}e^{\frac{2n}{s}} > h_2, \quad \frac{n}{s} \geqslant n_2,$$

where h_2 and n_2 are the height and degree of $Q(x)$ and s is some integer.

This lemma follows directly from inequalities (170) and Lemma II since $P_1(x) = Q^s(x)$, where $Q(x)$ is irreducible.

LEMMA VII. *Suppose $\alpha \neq 0$, $a_0 > 1$, $\sigma(x) > x$ and that $\theta(x) > 0$ are given, where $\sigma(x)$ and $\theta(x)$ are monotonic and increase indefinitely with x when $x > x_0 > 0$, and $a_0\sigma(x) \geqslant \sigma(x+1)$. Then, if for every integral $N > N_0 > 0$ there exists a polynomial $P(x) \not\equiv 0$, with rational integral coefficients, of height H and degree n such that the conditions*

$$(180) \qquad |P(\alpha)| < e^{-\sigma^2(N)\theta(N)}, \quad \max[n, \ln H] \leqslant \tfrac{1}{3}\sigma(N)$$

are satisfied, then α must be an algebraic number.

Proof. Let us assume that α is transcendental. Then $P(\alpha) \neq 0$. It follows from Lemma VI′ and condition (180) that for every integral $q > q_0$ there exists an irreducible polynomial $Q_q(x)$ with rational integral relatively prime coefficients, of height H_q and degree $n_q \neq 0$, satisfying the conditions

$$(181) \qquad 0 < |Q_q(\alpha)| < e^{-\frac{1}{2s}\sigma^2(q)\theta(q)},$$
$$\max[n_q, \ln H_q] \leqslant \frac{1}{s}\sigma(q),$$
$$s \leqslant \sigma(q).$$

We now define the number x_q by the equation

$$(182) \qquad s_q = \max[n_q, \ln H_q] = \sigma(x_q), \quad x_q = \sigma_{-1}(s_q),$$

where $\sigma_{-1}[\sigma(x)] = x$, which one can do in a unique manner in virtue of the monotonic growth of $\sigma(x)$ and of the fact that s_q increases indefinitely together with q as a consequence of inequalities (181).

Inequalities (181) can now be replaced by the inequalities

$$(183) \qquad 0 < |Q_q(\alpha)| < e^{-\frac{1}{2}\sigma^2(x_q)\theta(q)},$$
$$\max[n_q, \ln H_q] = \sigma(x_q), \quad x_q \leqslant q.$$

We define the monotonically increasing function $\psi(x)$ in the following way:

(184) $$\psi(x) = \min\{\sqrt{\sigma(x)},\ \sqrt{\theta[\sigma_{-1}(\sqrt{\sigma(x)})]}\}.$$

Further, if $y=y(x)$ is defined by the equation

(185) $$\sigma(y) = \frac{\sigma(x)}{\psi(x)},$$

then $\sigma(y) \geqslant \sqrt{\sigma(x)}$, $y \geqslant \sigma_{-1}[\sqrt{\sigma(x)}]$ as a consequence of (184), and

(186) $$\varphi(x) = \frac{\theta(y)}{\psi(x)} \geqslant \frac{\theta(\sigma_{-1}[\sigma^{1/2}(x)])}{\theta^{1/2}(\sigma_{-1}[\sigma^{1/2}(x)])} = \theta^{1/2}(\sigma_{-1}[\sigma^{1/2}(x)])$$

tends to infinity as x increases.

We shall now find the integer N from the condition

(187) $$\sigma(N-1) < \frac{\sigma(x_q)}{\sqrt{\psi(x_q)}} \leqslant \sigma(N).$$

Then, since $$\theta(N) > \theta(y_q),\ \sigma_{-1}\left[\frac{\sigma(x_q)}{\psi(x_q)}\right] = y_q,$$

(188) $$\sigma^2(N)\theta(N) > \sigma^2(x_q)\frac{\theta(y_q)}{\psi(x_q)} = \sigma^2(x_q)\varphi(x_q),$$

and

$$\sigma(N) \leqslant a_0\sigma(x_q)[\psi(x_q)]^{-1/2}.$$

For this $N=N(x_q)$, with x_q sufficiently large, by the condition of our lemma, there will exist a polynomial with integral coefficients $P_N(x)\not\equiv 0$, which satisfies the conditions

(189) $$0 < |P(\alpha)| \leqslant e^{-\sigma^2(N)\theta(N)} \leqslant e^{-\sigma^2(x_q)\varphi(x_q)},$$

$$\max[n, \ln H] \leqslant \frac{1}{3}\sigma(N) \leqslant \frac{a_0}{3}\frac{\sigma(x_q)}{\sqrt{\psi(x_q)}},$$

where n and H are the degree respectively the height of $P_N(x)$.

From inequalities (183) and (189) we easily obtain the inequalities

(190) $$(1+|\alpha|)^{n+n_q}(n+n_q)^{n+n_q}H_q^nH^{n_q}$$
$$< \exp[3\sigma^{3/2}(x_q)+a_0\sigma^2(x_q)\psi^{-1/2}(x^q)] < e^{\sigma^2(x_q)},$$

for $x_q > x'$, and

(191) $$0 < |P_N(\alpha)|+|Q_q(\alpha)|$$
$$< 2\exp[-\sigma^2(x_q)\min\{\tfrac{1}{2}\theta(x_q),\ \varphi(x_q)\}] < e^{-\sigma^2(x_q)}$$

for $x_q > x''$, since $\theta(x)$ and $\varphi(x)$ increase indefinitely.

Inequalities (190) and (191) show that the conditions of Lemma V are satisfied with $\nu = 1$, in other words, $P_N(x)$ and $Q_q(x)$ must have a common zero. But $Q_q(x)$ is irreducible and the common divisor of its coefficients is unity. This means that $P(x)$ is divisible by $Q(x)$ and $P_N(x) = Q_q(x)R(x)$, where $R(x)$ is a polynomial with integral coefficients and, by the same token, it has height not less than unity.

By Lemma II, it then follows that we have the inequalities

$$(192) \qquad H_q \leqslant He^n \leqslant e^{a\frac{\sigma(x_q)}{\sqrt{\psi(x_q)}}}, \quad n_q \leqslant n \leqslant a_0 \frac{\sigma(x_q)}{\sqrt{\psi(x_q)}},$$

which will contradict equation (182) which defines x_q, when $x_q \geqslant x'''$. And this completes the proof of our lemma.

The conditions of this lemma can be generalized and sharpened. In particular, the function $\theta(x)$ may be replaced by a sufficiently large constant.

LEMMA VIII. *Suppose $P(x)$ is a polynomial, irreducible over the rational field, with rational integral coefficients of degree n and height H, α is a fixed number, and $\omega_1, \omega_2, \ldots, \omega_\nu$ is any fixed basis for the ring of integers in the algebraic field K_ν. If $P(x)$ is reducible in the field K_ν and the inequality*

$$(193) \qquad |P(\alpha)| < T^{-1}$$

holds, then in the field K_ν, $P(x)$ has an irreducible divisor $T(x)$, whose coefficients are integers in the field K_ν, subject to the conditions

$$(194) \qquad |T(\alpha)| < HT^{-\frac{1}{\nu}}; \quad T(x) = \sum_{k=0}^{r} \sum_{s=1}^{\nu} b_{k,s}\omega_s x^k;$$

$$|b_{k,s}| < \lambda_0 e^{2n} H^\nu, \quad \frac{n}{\nu} \leqslant r \leqslant n;$$

where λ_0 depends only on the field K_ν and $\omega_1, \ldots, \omega_\nu$.

Proof. First of all we note that, as is known, $P(x)$ may be represented in the field K_ν as a product of divisors, irreducible in this field, uniquely, up to within a constant multiplier. Further, the number of these irreducible divisors, if one counts each of them as often as its multiplicity indicates, does not surpass the degree ν of the field. In fact, if

$$P(x) = \sum_{k=0}^{n} A_k x^k = A_n \prod_{k=1}^{\mu} R_k(x) = A_n \prod_{k=1}^{\mu} R_k^{(s)}(x), \quad 1 \leqslant s \leqslant \nu - 1,$$

where the $R_k(x)$ are monic polynomials, irreducible in the field

K_ν, and the $R_k^{(s)}(x)$, $s=1,\ldots,\nu-1$, are polynomials whose coefficients are conjugate to the coefficients of $R_k(x)=R_k^{(0)}(x)$ and belong respectively to the fields $K_\nu^{(s)}$, $s=1,\ldots,\nu-1$, conjugate to the field K_ν, then the relations

$$\prod_{s=0}^{\nu-1} R_k^{(s)}(x) = A_n^{-m_k}P^{m_k}(x), \quad m_k < \nu, \quad 1 \leqslant k \leqslant \mu,$$

hold. It follows that n_k, the degree of $R_k(x)$, has the bounds $\frac{n}{\nu} \leqslant n_k < n$. This proves our assertion that $\mu \leqslant \nu$. We shall now set

$$R_k(x) = x^{n_k} + \sum_{s=0}^{n_k-1} a_{k,s}x^s,$$

where the $a_{k,s}$ are, generally speaking nonintegral, numbers in the field K_ν. Suppose $\wp$ is a prime ideal in the field K_ν, r_k is the highest power of $\wp^{-1}$ which divides the numbers $a_{k,s}$, $s=0, 1, \ldots, n_k-1$, and c is an integer in the field K_ν, which is divisible only by the first powers of $\wp$. Then, it follows from the relations

$$P(x)c^r = A_n \prod_{k=1}^{\mu} \left[c^{r_k}x^{n_k} + \sum_{s=0}^{n_k-1} c^{r_k}a_{k,s}x^s \right] = A_nP_1(x);$$

$$r = \sum_1^{\mu} r_k,$$

where now not all the coefficients of $P_1(x)$ are divisible by $\wp$, inasmuch as at least one of the coefficients $c^{r_k}R_k(x)$ is not divisible by $\wp$, and all the remaining ones can contain $\wp$ only to negative powers, that A_n is divisible by $\wp^r$.

It follows (setting $Q_k^{(s)}(x)=A_nR_k^{(s)}(x)$, $A_n=A$), that, since the preceding discussion is true for arbitrary $R_k^{(s)}(x)$,

$$(195) \quad P(x) = A^{-\mu+1}Q_1^{(s)}(x)Q_2^{(s)}(x)\ldots Q_\mu^{(s)}(x), \quad 0 \leqslant s \leqslant \nu-1,$$

where $Q_k^{(s)}(x)$, $k=1, 2, \ldots, \mu$ are polynomials, irreducible in the field $K_\nu^{(s)}$, having integral coefficients in the field $K_\nu^{(s)}$, and A is a rational integer, $|A| \leqslant H$.

Suppose that among the fields $K_\nu^{(0)}, K_\nu^{(1)}, \ldots, K_\nu^{(\nu-1)}$ there are r_1 real and $2r_2$ complex conjugate, and the units of the fields $K_\nu^{(0)}$, $\xi_{k,0}$, $k=1, 2, \ldots, r$; $r=r_1+r_2-1$, form a fundamental system of units. In exactly the same way, let $\xi_{k,s}$, $k=1, 2, \ldots, r$

be a fundamental system of conjugate units for the field $K_\nu^{(s)}$, $s=1, 2, \ldots, \nu-1$. We set

(196) $\qquad \ln|\xi_{k,s}| = \eta_{k,s}; \quad 0 \leqslant s \leqslant \nu-1; \quad 1 \leqslant k \leqslant r.$

We shall also agree that the fields $K_\nu^{(s)}$, $s=1, 2, \ldots, r_1-1$, are real, and that the fields $K_\nu^{(s)}$, $s=r_1, r_1+1, \ldots, r$, are complex, where $K_\nu^{(s)} \neq K_\nu^{(p)}$, $r_1 \leqslant s$, $p \leqslant r$. Suppose also that the logarithms of the heights of conjugate polynomials

$$Q_k^{(0)}(x) = Q_k(x), \quad Q_k^{(1)}(x), \ldots, Q_k^{(\nu-1)}(x), \quad k = 1, 2, \ldots, \mu$$

are respectively

$$q_{k,i}; \quad i = 0, 1, \ldots, \nu-1; \quad k = 1, 2, \ldots, \mu.$$

Fixing the number $k \geqslant 2$, we now solve the system of equations

$$\sum_{s=1}^{r} x_{k,s}\eta_s = q_k - q_{k,m}; \tag{197}$$

$$m = 1, 2, 3, \ldots, r; \quad q_k = \sum_{s=0}^{\nu-1} \frac{1}{\nu} q_{k,s}.$$

The determinant of this system is different from zero in virtue of the fact that the fields $K_\nu^{(s)}$, $s=1, 2, \ldots, r_1-1$ are real and for $r_1 \leqslant s \leqslant r$ they are pairwise not complex conjugate, and the system of units $\xi_{k,s}$ is fundamental. Therefore, our system (197) has a unique solution. Suppose now that in the following discussion, the systems of numbers $x_{k,s}$; $k=2, 3, \ldots, \mu$; $s=1, 2, \ldots, r$, are systems of solutions of the corresponding systems of equations, and by the same token, they are fixed. Furthermore, we shall agree that $K_\nu^{(r_1+s)} = K_\nu^{(r+s+1)}$, $s=0, 1, \ldots, r_2-2$. It follows that either $K_\nu^{(0)} = K_\nu$ is real and then $K_\nu^{(r)} = \bar{K}_\nu^{(\nu-1)}$ or $K_\nu^{(0)} = \bar{K}_\nu^{(r)}$, and then $K_\nu^{(\nu-1)}$ is real.

As a consequence of the fact that $\xi_{k,s}$, $s=0, 1, \ldots, \nu-1$, are a system of conjugate units, we have $\sum_{s=0}^{\nu-1} \eta_{k,s} = 0$. It follows that for the real field $K_\nu^{(\sigma)}$, which does not occur in the series $K_\nu^{(1)}$, $K_\nu^{(2)}, \ldots, K_\nu^{(r)}$,

$$\sum_{s=1}^{r} x_{k,s}\eta_{s,\sigma} = -\sum_{\substack{p=0 \\ p\neq\sigma}}^{\nu-1} \sum_{s=1}^{r} x_{k,s}\eta_{s,p} \tag{198}$$

$$= -(\nu-1)q_k + \sum_{\substack{p=0 \\ p\neq\sigma}}^{\nu-1} q_{k,p} = q_k - q_{k,\sigma},$$

since if $K_\nu^{(m)}=\bar{K}_\nu^{(p)}$, then $\eta_{s,m}=\eta_{s,p}$, $s=1,\ldots,r$, and $q_{k,m}=q_{k,p}$ in virtue of the fact that $Q_k^{(m)}(x)$ and $Q_k^{(p)}(x)$ are complex conjugate.

The immediately preceding discussion shows that for the chosen $x_{k,1}, x_{k,2}, \ldots, x_{k,r}$ the relations

$$\text{(199)} \qquad \sum_{s=1}^{r} x_{k,s}\eta_{s,m} = q_k - q_{k,m},$$
$$m = 0, 1, \ldots, \nu-1, \quad k = 2, 3, \ldots,$$

always hold.

We now set

$$x_{k,s} = y_{k,s} - \varepsilon_{k,s}, \quad |\varepsilon_{k,s}| < 1,$$

where the $y_{k,s}$ are rational integers, and we introduce the supplementary notation

$$\text{(200)} \left\{ \begin{aligned} & a_k^{(s)} = \prod_{m=1}^{r} \xi_{m,s}^{y_{k,m}}; \; k = 2, 3, \ldots, \mu; \; s = 0, 1, \ldots, \nu-1, \\ & a_1^{(s)} = \prod_{k=2}^{\mu} \frac{1}{a_k^{(s)}}; \quad s = 0, 1, \ldots, \nu-1, \\ & \lambda_{k,m} = \sum_{s=1}^{r} \varepsilon_{k,s}\eta_{s,m}; \quad \lambda = 2\nu \max_{\substack{2\leqslant k\leqslant\mu \\ 0\leqslant m\leqslant \nu-1}} |\lambda_{k,m}|, \\ & 2 \leqslant k \leqslant \mu, \quad 0 \leqslant m \leqslant \nu-1, \\ & T_k^{(s)}(x) = a_k^{(s)} Q_k^{(s)}(x); \quad 1 \leqslant k \leqslant m; \quad 0 \leqslant s \leqslant \nu-1. \end{aligned} \right.$$

Since the $\xi_{m,s}$ are units, the $T_k^{(s)}(x)$ are again polynomials with integral coefficients in the field $K_\nu^{(s)}$. Relations (195) may now be rewritten in the form

$$\text{(201)} \quad A^{\mu-1}P(x) = T_1^{(s)}(x)\,T_2^{(s)}(x)\ldots T_\mu^{(s)}(x); \quad 0 \leqslant s \leqslant \nu-1.$$

Denoting the height of $T_k^{(s)}(x)$ by $h_{k,s}$, we will have, in virtue of relations (199) and (200), that

$$\text{(202)} \quad \ln h_{k,s} = \lambda_{k,s} + q_k; \quad 2 \leqslant k \leqslant \mu; \quad 0 \leqslant s \leqslant \nu-1.$$

Further, since $T_k(x) = \prod_{s=0}^{\nu-1} T_k^{(s)}(x)$ will be a polynomial with rational integral coefficients, the obvious relations

$$\text{(203)} \qquad (1+n_k)^\nu h_{k,0}\ldots h_{k,\nu-1} \geqslant 1, \quad k = 1, 2, \ldots, \mu$$

and the relation (202), imply the inequalities

$$\text{(204)} \quad q_k \geqslant -\ln(1+n_k) - \lambda_k, \quad \lambda_k = \frac{1}{\nu}\sum_{s=0}^{\nu-1} \lambda_{k,s}; \qquad k = 2, \ldots, \mu.$$

In their turn, the inequalities

(205) $\ln h_{k,s} \geqslant -\ln(1+n_k)+\lambda_{k,s}-\lambda_k;$
$$2 \leqslant k \leqslant \mu; \quad 0 \leqslant s \leqslant \nu-1$$

follow from these inequalities and from inequalities (202). Further, with the aid of Lemma II and relations (201) we also obtain the inequalities

$$(206) \qquad n+\mu\ln H \geqslant \sum_{k=1}^{\mu} \ln h_{k,s}; \quad s = 0, 1, \ldots, \nu-1.$$

Inequalities (206) and (205) yield an upper bound for $h_{1,s}$, namely,

$$(207) \quad \ln h_{1,s} \leqslant \mu\ln H+n+\sum_{2}^{\mu}\ln(1+n_k)+(\mu-1)\lambda_k-\sum_{k=2}^{\mu}\lambda_{k,s},$$
$$0 \leqslant s \leqslant \nu-1.$$

Since $T_k(x)$ is a polynomial of degree νn_k, with rational integral coefficients, its height h_k will, according to Lemma II, satisfy the inequality

$$(208) \quad \nu n_k+\ln h_k > \sum_{s=0}^{\nu-1}\ln h_{k,s} = \nu q_k+\nu\lambda_k$$
$$= \nu\ln h_{k,s}+\nu(\lambda_k-\lambda_{k,s}),$$
$$k = 2, 3, \ldots, \mu, \quad 0 \leqslant s \leqslant \nu-1,$$

and as a consequence of the relation

$$(209) \qquad A^{\nu(\mu-1)}P^{\nu}(x) = T_1(x)\ldots T_\mu(x)$$

and Lemma II, it also satisfies the inequalities

$$(210) \qquad n\nu+\nu\mu\ln H > \ln h_k > \nu\ln h_{k,s}-\nu n_k+\nu(\lambda_k-\lambda_{k,s}).$$

This last inequality immediately leads us to the inequality

$$(211) \qquad \ln h_{k,s} < \mu\ln H+2n+\lambda_{k,s}-\lambda_k < \mu\ln H+2n+\lambda,$$
$$2 \leqslant k \leqslant \mu, \quad 0 \leqslant s \leqslant \nu-1.$$

It is not difficult to note that inequality (207) can also be rewritten in the same form inasmuch as $\sum_{2}^{\mu}\ln(1+n_k)<n$. This means that inequalities (211) are also valid for $k=1$. These inequalities also complete the proof of our lemma because if

$$|P(\alpha)| < T^{-1},$$

then at least one of the inequalities

(212) $$|T_k^{(0)}(\alpha)| < HT^{-\frac{1}{\nu}}$$

must be satisfied as a consequence of relation (201).

Further, suppose $b = \sum_{k=1}^{\nu} b_k\omega_k$ is any coefficient of the polynomial $T_k(x)$, for which inequality (212) is satisfied. The algebraic integers $b^{(s)}$,

$$b^{(s)} = \sum_{k=1}^{\nu} b_k\omega_k^{(s)}, \quad s = 1, 2, \ldots, \nu-1,$$

where $\omega_k^{(1)}, \omega_k^{(2)}, \ldots, \omega_k^{(s)}$ are the conjugates of the number $\omega_k = \omega_k^{(0)}$, are the conjugates of the number $b = b^{(0)}$. Then in virtue of inequalities (212) we have the inequalities

$$|b^{(s)}| < \lambda H^{\mu} e^{2n}, \quad s = 0, 1, \ldots, \nu-1.$$

Solving the system of equations

$$\sum_{k=1}^{\nu} \omega_k^{(s)} b_k = b^{(s)}, \quad s = 0, 1, \ldots, \nu-1,$$

with respect to b_k, $k = 1, 2, \ldots, \nu$, which system is solvable inasmuch as its determinant is the discriminant of the field K_ν, we obtain the inequalities

$$|b_k| = \left| \sum_{s=1}^{\nu} C_{k,s} b^{(s)} \right| < \lambda_0 H^{\mu} e^{2n},$$

where all the $C_{k,s}$ depend only on the numbers $\omega_1, \omega_2, \ldots, \omega_\nu$. This means that λ_0 does not depend on n, H and T, which was to be proved.

§5. Proof of the fundamental theorems

Proof of Theorem I. Let us assume that by adjoining to the rational field the 11 numbers

(213) $$\alpha_1, \quad \alpha_2, \quad e^{\eta_i \alpha_k}; \qquad i = 1, 2, 3; \qquad k = 0, 1, 2; \qquad \alpha_0 = 1,$$

where $1, \alpha_1, \alpha_2$, as well as η_1, η_2, η_3 are linearly independent over the rational field, we have obtained the field R_1. This field cannot be a simple algebraic field, because when e^{η_1}, $\eta_1 \neq 0$, is algebraic, then $e^{\eta_1 \alpha_1}$ must be transcendental, as a consequence of the fact that

α_1 is an algebraic irrational number. Then this field R_1 is generated by the numbers ω and ω_1, where the degree of ω_1 is set equal to ν, the properties of which numbers are defined in the beginning of §4 of this chapter. The 11 numbers (213) will coincide, by hypothesis, with the 11 numbers

$$(214) \qquad \frac{S_i}{T_i}, \quad i = 1, 2, \ldots, 11; \qquad T = T_1 T_2 \ldots T_{11},$$

of the field, where all the S_i and T_i are integers in the field R_1 (the definition of an integer and its degree was given above).

Suppose N is an arbitrarily large integer. We set $p = [N^3 \ln^{-3/2} N] + 1 = [(N \ln^{-1/2} N)^3] + 1$ and consider the function

$$(215) \quad f(z) = \sum_{k_0=0}^{p-1} \sum_{k_1=0}^{N} \sum_{k_2=0}^{N} \sum_{k_3=0}^{N} A_{k_0,k_1,k_2,k_3} z^{k_0} e^{(k_1\eta_1+k_2\eta_2+k_3\eta_3)z}$$

$$A_{k_0,k_1,k_2,k_3} = \sum_{k_4=0}^{p_1} C_{k_0,\ldots,k_4} \omega^{k_4}, \quad p_1 = \left[\frac{N^3}{\ln^{1/2} N}\right],$$

where all the C_{k_0,k_1,k_2,k_3,k_4} are rational integers, all different from zero.

Making use of inequalities (112) and (162), we obtain the equality

$$(216) \quad \prod_{-N_1}^{N-N_1} \prod_{-N_2}^{N-N_2} \prod_{-N_3}^{N-N_3} |k_1\eta_1 + k_2\eta_2 + k_3\eta_3|$$

$$> |\eta_1|^{-1} \prod_{\substack{k_2=-N_2 \\ k_2^2+k_3^2\neq 0}}^{N-N_2} \prod_{k_3=-N_3}^{N-N_3} |\eta_1|^N \left(k_2 \frac{\eta_2}{\eta_1} + k_3 \frac{\eta_3}{\eta_1}\right) 2^{-N-1} N!$$

$$> |\eta_1'|^{-(N+1)^3} 2^{-(N+1)^3} (N!)^{(N+1)^2} e^{-\tau(N+1)^3 \ln N} > e^{-\tau N^3 \ln N}$$

for $0 \leqslant N_i \leqslant N$, $N > N'$, $\eta_1' = |\eta_1| + \left|\frac{1}{\eta_1}\right|$. We now apply Lemma III to the function $f(z)$. We note that

$$p = [(N \ln^{-1/2} N)^3] + 1, \quad q = (N+1)^3,$$
$$m = pq = (N+1)^3 [N^3 \ln^{-3/2} N + 1] > N^6 \ln^{-3/2} N,$$
$$\gamma_1 = 2/11, \quad \gamma_2 = \tfrac{1}{5}\tau$$

and set $\gamma_0 = 4/11$, $\lambda = 3 + \tau$. With such a choice of the constants, the conditions of Lemma III are satisfied in virtue of inequalities (112) and (216), as can easily be verified, and we obtain that at least one of the numbers

(217) $f(k_0+k_1\alpha_1+k_2\alpha_2), \quad 0 \leqslant k_i \leqslant \lambda \frac{N^2}{\sqrt{\ln N}}, \quad i = 0, 1, 2$

is not equal to zero. The quantity $f(k_0+k_1\alpha_1+k_2\alpha_2)$ is a polynomial in the 11 quantities (213) and for $k_i \leqslant \lambda N^2 \ln^{-1/2} N$ its degree with respect to each of these numbers does not exceed $(\lambda+1)N^3 \ln^{-1/2} N$. Therefore, the numbers f_{k_0,k_1,k_2}

(218) $$\begin{cases} f_{k_0,k_1,k_2} = T^\mu f(k_0+k_1\alpha_1+k_2\alpha_2), & 0 \leqslant k_i \leqslant \lambda N^2 \ln^{-1/2} N, \\ \mu = [11\lambda N^3 \ln^{-1/2} N], \quad i = 0, 1, 2, & \lambda = 3+\tau \end{cases}$$

will be integers in the field R_1.

Recalling the expression for the numbers A in terms of ω, we obtain the representation

(219) $$f_{k_0,k_1,k_2} = \sum_{q=0}^{\mu_1-1} \sum_{q_1=0}^{\nu-1} \omega^q \omega_1^{q_1} \times$$
$$\times \sum_{n_0=0}^{p-1} \sum_{n_1=0}^{N} \sum_{n_2=0}^{N} \sum_{n_3=0}^{N} \sum_{n_4=0}^{p_1} C_{n_0,n_1,\dots,n_4} B_{n_0,\dots,n_4,k_0,k_1,k_2,q,q_1},$$
$$\mu_1 \leqslant [\lambda_0 N^3 \ln^{-1/2} N], \quad k_i \leqslant \lambda N^2 \ln^{-1/2} N, \quad i = 0, 1\ 2,$$
$$|B_{n_0,\dots,n_4,k_0,k_1,k_2,q,q_1}| < e^{\lambda_1 N^3 \ln^{-1/2} N}, \quad \lambda_0 > 1,$$

where all the numbers B are rational integers. The number of numbers C, which we shall denote by n, satisfies the inequality $n > N^9 \ln^{-2} N$, as can be verified by a simple computation. We set $\lambda_2 = (3\nu\lambda_0)^{-1/3}$. Since a necessary condition that the quantity f_{k_0,k_1,k_2} vanish is that $\nu\mu_1 < \nu\lambda_0\ N^3 \ln^{-1/2} N$ equations be satisfied, then, by the first lemma, the integers C may be chosen all different from zero and such that

(220) $$f_{k_0,k_1,k_2} = 0, \quad 0 \leqslant k_i < [\lambda_2 N^2 \ln^{-1/2} N],$$
$$|C_{n_0,\dots,n_4}| < e^{\lambda_3 N^3 \ln^{-1/2} N}.$$

In fact, the number of equations for $C_{n_0,\dots,n_4}$ will be less than

(221) $$\nu\lambda_0 N^3 \ln^{-1/2} N \cdot \lambda_2^3 N^6 \ln^{-3/2} N < \tfrac{1}{2} N^9 \ln^{-2} N.$$

But then $f(z) = 0$ at the points

(222) $$z = k_0+k_1\alpha_1+k_2\alpha_2, \quad 0 \leqslant k_i \leqslant [\lambda_2 N^2 \ln^{-1/2} N] = q.$$

This situation enables us to make use of the integral representation of $f(z)$,

(223) $$f(z) = \frac{1}{2\pi i} \int_\Gamma \prod_{k_0=0}^{q} \prod_{k_1=0}^{q} \prod_{k_2=0}^{q} \left[\frac{z-k_0-k_1\alpha_1-k_2\alpha_2}{\zeta-k_0-k_1\alpha_1-k_2\alpha_2}\right] \frac{f(\zeta)}{\zeta-z}\, d\zeta,$$

where the contour Γ is the circumference $|\zeta| = 2N^{9/2}$, and $|z| \leqslant N^3$. From this representation, we obtain immediately, by estimating the integral in the right member in absolute value, the inequality

$$|f(z)| < e^{-\lambda^{3/2} N^6 \ln^{-1/2} N}, \quad |z| \leqslant N^3. \tag{224}$$

But since $N^3 > \lambda N^2$ for $N > \lambda$, the inequalities

$$|f_{k_0, k_1, k_2}| < e^{-\frac{1}{2}\lambda_2{}^3 N^6 \ln^{-1/2} N}, \quad 0 \leqslant k_i \leqslant \lambda N^2 \ln^{-1/2} N, \tag{225}$$

are satisfied for sufficiently large N.

We have already proved that among these numbers there is at least one which is different from zero. This number will be a polynomial with rational integral coefficients, $P(\omega, \omega_1)$, with respect to the numbers ω and ω_1, of degree n and height H, where as a consequence of inequalities (225), (219) and (220) the inequalities

$$|P(\omega, \omega_1)| < e^{-\frac{1}{2}\lambda^{3/2} N^6 \ln^{-1/2} N}, \; n + \ln H < \lambda_4 N^3 \ln^{-1/2} N \tag{226}$$

will be valid. Suppose the numbers $\omega_2, \omega_3, \ldots, \omega_\nu$ are conjugate to ω_1 in the field $R(\omega)$, where $R(\omega)$ is the field obtained by adjoining the number ω to the rational field. Then we shall have that

$$P_0(\omega) = \prod_{k=1}^{\nu} P(\omega, \omega_k) \neq 0, \tag{227}$$

and that $P_0(\omega)$ is a polynomial of degree n_0 and height H_0, with rational integral coefficients, for which the inequalities

$$\begin{gathered} |P_0(\omega)| < e^{-\lambda_5 N^6 \ln^{-1/2} N}; \\ \max\,[n_0, \ln H_0] < \lambda_6 N^3 \ln^{-1/2} N, \quad N > N'' \end{gathered} \tag{228}$$

hold. Setting

$$\sigma(N) = 3\lambda_6 N^3 \ln^{-1/2} N, \quad \theta(N) = \frac{\lambda_5}{q\lambda_6{}^2}\sqrt{\ln N} \tag{229}$$

we see that the conditions of Lemma VII are satisfied for the number ω and this means that ω is an algebraic number. We thus arrive at a contradiction, which completes the proof of our theorem.

Inequality (112) of our theorem holds unconditionally in the case when η_2/η_1 and η_3/η_1 are either algebraic or irrational powers of the same logarithm of an algebraic number, or algebraic powers of the number e. In all these cases, as has long been known, the inequality

$$|x_1\eta_1 + x_2\eta_2 + x_3\eta_3| > e^{-\tau \ln x}, \quad |x_i| \leqslant x \tag{230}$$

will hold, where τ is some constant (see, for example, inequalities (48) and (49), Chapter II). This remark proves the validity of Corollaries 1, 2 and 3 to our theorem.

Proof of Theorem II. The proof of this theorem differs little from the proof of Theorem I. Therefore, we shall present it in a somewhat shortened form. The field R_1 will again be generated by the numbers ω and ω_1, where $\omega_2, \omega_3, \ldots, \omega_\nu$ are the conjugates, in the previous sense, of the number ω_1. The ten numbers

$$(231) \quad \eta_0, \quad \eta_1, \quad \alpha_1, \quad \alpha_2, \quad e^{\eta_i \alpha_k}; \quad i = 0, 1, \quad k = 0, 1, 2, \quad \alpha_0 = 1$$

are equal to ratios of integers in the field R_1, which have the form

$$(232) \qquad \frac{S_i}{T_i}; \qquad i = 1, 2, \ldots, 10, \quad T = T_1 T_2 \ldots T_{10}.$$

The number ω is transcendental inasmuch as the numbers η_0 and e^{η_0} cannot be simultaneously algebraic according to the Lindemann theorem. Suppose N is an arbitrarily large integer. We set $p = [N^{5/3} \ln^{-1} N] + 1$ and consider the function $f(z)$,

$$(233) \quad \begin{cases} f(z) = \displaystyle\sum_{k=0}^{p-1} \sum_{k_0=0}^{N} \sum_{k_1=0}^{N} A_{k,k_0,k_1} z^k e^{(k_0\eta_0 + k_1\eta_1)z}, \\ A_{k,k_0,k_1} = \displaystyle\sum_{k_2=0}^{p_1} C_{k,k_0,k_1,k_2} \omega^{k_2}, \quad p_1 = [N^{5/3}], \end{cases}$$

where the C_{k,k_0,k_1,k_2} are rational integers, all different from zero. Making use of inequality (162), we obtain the inequality

$$(234) \quad \prod_{k_0=-N_0}^{N-N_0} \prod_{\substack{k_1=-N_1 \\ k_0^2+k_1^2 \neq 0}}^{N-N_1} |k_0\eta_0 + k_1\eta_1|$$

$$\geqslant |\eta_0|^{N^2+2N} 2^{-(N+1)^2 N!N} \prod_{\substack{k_1=-N_1 \\ k_1 \neq 0}}^{N-N_1} \left(k_1 \frac{\eta_1}{\eta_0}\right)$$

$$\geqslant |\eta_0|^{(N^2+2N)} 2^{-(N+1)^2} N!^N \prod_{\substack{k_1=-N_1 \\ k_1 \neq 0}}^{N-N_1} \min_{k_0} \left| k_0 + k_1 \frac{\eta_1}{\eta_0} \right|,$$

with $0 \leqslant N_i \leqslant N$, where (x) is the distance of x to the nearest integer.

Suppose $\eta_2 = \eta_1/\eta_2$. Then, if

$$(235) \quad |k_0 + k_1\eta_2| < 2^{-N}, \quad |k_2 + k_3\eta_2| < 2^{-N}, \quad |k_1|, \quad |k_3| \leqslant N$$

for integral k_i, it is clear that the linear forms $k_0+k_1\eta_2$ and $k_2+k_3\eta_2$ are proportional. Suppose, further, that

$$\varrho = \min_{|k_1|\leqslant N} |k_0+k_1\eta_2|$$

is realized for $|k_1|=s\leqslant N$. Then, making use of inequality (114), we obtain that

$$(236)\quad \prod_{\substack{k_1=-N_1\\ k_1\neq 0}}^{N-N_1} \min_{k_0} |k_0+k_1\eta_2| > 2^{-N^2}\varrho^{\left[\frac{N}{s}\right]} > 2^{-N^2}e^{-\tau s^2\left[\frac{N}{s}\right]\ln s}$$
$$> e^{-\tau N^2 \ln N}2^{-N^2}.$$

We now finally have that

$$(237)\quad \prod_{k_0=-N_0}^{N-N_0}\ \prod_{k_1=-N_1}^{N-N_1} |k_0\eta_0+k_1\eta_1| > e^{-\tau N^2\ln N} > e^{-\frac{3}{10}\tau q\ln pq},$$

where $p=[N^{5/3}\ln^{-1}N]+1$, $q=(N+1)^2$. We now apply Lemma III to our function. We note that in this case, $m=pq\geqslant N^{11/3}\ln^{-1}N$, $\gamma_1=\frac{3}{10}$, $\gamma=\frac{3}{10}\tau$ and set $\gamma_0=\frac{1}{5}$ and $\lambda=3+2\tau$. With such a choice of constants, in virtue of inequalities (237) and (114), the conditions of Lemma III are satisfied, as is easily verified, and we obtain that at least one of the numbers

$$(238)\quad f^{(s)}(k_0+k_1\alpha_1+k_2\alpha_2),\ 0\leqslant k_i\leqslant \lambda N^{2/3},\ 0\leqslant s\leqslant p,\ i=0,1,2$$

is different from zero. The quantity $f^{(s)}(k_0+k_1\alpha_1+k_2\alpha_2)$ is a polynomial in the 10 quantities (231), where its degree with respect to each of these quantities does not exceed $\lambda N^{5/3}$.

Therefore, the numbers f_{s,k_0,k_1,k_2}

$$(239)\quad f_{s,k_0,k_1,k_2} = T^{\mu}f^{(s)}(k_0+k_1\alpha_1+k_2\alpha_2),\quad \mu=[10\lambda N^{5/3}],$$
$$0\leqslant s\leqslant N^{5/3}\ln^{-1}N,\quad 0\leqslant k_i\leqslant \lambda N^{2/3},\quad i=0,1,2$$

will be integers in the field R_1.

In a manner analogous to the way we obtained (219), we obtain the expressions

$$(240)\quad f_{s,k_0,k_1,k_2}=\sum_{q=0}^{\mu_1-1}\sum_{q_1=0}^{\nu-1}\omega^q\omega_1^{q_1}\times$$
$$\times\sum_{n=0}^{p-1}\sum_{n_0=0}^{N}\sum_{n_1=0}^{N}\sum_{n_2=0}^{p_1} C_{n,n_0,n_1,n_2}B_{s,k_0,k_1,k_2,n_0,\ldots,n_2,q,q_1}$$
$$\mu_1\leqslant\lambda_0N^{5/3},\quad |B_{s,\ldots,q_1}|<e^{\lambda_1N^{5/3}},\quad 0\leqslant s\leqslant N^{5/3}\ln^{-1}N,$$
$$0\leqslant k_1\leqslant\lambda N^{2/3},\quad i=0,1,2$$

where all the numbers B are rational integers. The number of integers C is not less than $N^{16/3}\ln^{-1}N$. We set $\lambda_2=(3\nu\lambda_0)^{-1/3}$. A sufficient condition that f_{s,k_0,k_1,k_2} vanish is that $\nu\mu_1<\lambda_0\nu N^{5/3}$ equations hold. Therefore, by Lemma I, the numbers C can be chosen to be rational integral and such that

$$(241)\quad f_{s,k_0,k_1,k_2}=0,\quad |C_{n,n_0,n_1,n_2}|<e^{\lambda_3 N^{5/3}},$$
$$k_i\leqslant\lambda_2 N^{2/3},\quad i=0,1,2;\qquad 0\leqslant s\leqslant N^{5/3}\ln^{-1}N,$$

inasmuch as in this case the number of variables is not less than $N^{16/3}\ln^{-1}N$, and the number of equations is not greater than $\lambda_2{}^3\lambda_0\nu N^{16/3}\ln^{-1}N<\frac{1}{2}N^{16/3}\ln^{-1}N$. But for the numbers C chosen in this manner, $f^{(s)}(t)=0$, where

$$(242)\quad t=k_0+k_1\alpha_1+k_2\alpha_2,\quad 0\leqslant k_i\leqslant q=[\lambda_2N^{2/3}],$$
$$s\leqslant r=\left[\frac{N^{5/3}}{\ln N}\right]$$

which yields the integral representation

$$(243)\quad f^{(s)}(t)=\frac{s!}{(2\pi i)^2}\int_{\Gamma_0}\frac{dz}{(z-t)^{s+1}}\times$$
$$\times\int_{\Gamma}\prod_{k_0=0}^{q}\prod_{k_1=0}^{q}\prod_{k_2=0}^{q}\left[\frac{z-k_0-k_1\alpha_1-k_2\alpha_2}{\zeta-k_0-k_1\alpha_1-k_2\alpha_2}\right]^{r+1}\frac{f(\zeta)}{\zeta-z}\,d\zeta,$$
$$s\leqslant r,\quad t=k_0+k_1\alpha_1+k_2\alpha_2,\quad 0\leqslant k_i\leqslant\lambda N^{2/3},$$

where the contour Γ_0 is the circumference $|z|=N$, and the contour Γ is the circumference $|\zeta|=N^{7/3}$. The integral representation (243) yields, upon estimating the absolute values of the integrals in the right member, the inequalities

$$(244)\qquad |f^{(s)}(k_0+k_1\alpha_1+k_2\alpha_2)|<e^{-2\lambda_2N^{11/3}\ln^{-1}N},$$
$$s\leqslant r,\quad 0\leqslant k_i\leqslant\lambda N^{2/3},\quad i=0,1,2$$

which, in its turn, leads directly to the inequalities

$$(245)\quad |f_{s,k_0,k_1,k_2}|<e^{-\lambda_2N^{11/3}\ln^{-1}N},\quad s\leqslant[N^{5/3}\ln^{-1}N],$$
$$0\leqslant k_i\leqslant\lambda N^{2/3},$$

where at least one of the numbers f_{s,k_0,k_1,k_2} is different from zero. This nonzero number is a polynomial in ω and ω_1, not identically equal to zero, with rational integral coefficients, height H and degree n with respect to ω; in virtue of inequalities (240), (241) and (245), the inequalities

(246) $\quad |P(\omega, \omega_1)| < e^{-\lambda_2 N^{11/3} \ln^{-1} N}, \quad \max[n, \ln H] < \lambda_4 N^{5/3}$

hold for this polynomial. Setting

$$P_0(\omega) = \prod_{k=1}^{\nu} P(\omega, \omega_k), \quad P_0(x) \not\equiv 0,$$

where $\omega_2, \ldots, \omega_\nu$ are the numbers conjugate to ω_1, we obtain that for the polynomial $P_0(x)$, having rational integral coefficients, height H_0 and degree n_0, the inequalities

(247) $\quad 0 < |P_0(\omega)| < e^{-\frac{1}{2}\lambda_2 N^{11/3} \ln^{-1} N}, \quad \max[n, \ln H] < \lambda_5 N^{5/3}$

are valid.

Thus, for every integral $N > N'$ there exists a polynomial $P_0(x)$, which is not identically equal to zero, with rational integral coefficients, for which inequalities (247) are valid. It follows, by Lemma VII, that ω is algebraic; we have thus arrived at a contradiction, which proves our theorem. To verify the validity of Corollaries 1, 2 to Theorem II, it suffices to recall the remark to Theorem I and inequality (230).

Proof of Theorem III. We shall first prove the inequality

(248) $\quad |P(\alpha^\beta)| > e^{-\frac{s^3}{1+\ln^3 s}(s+\ln H) \ln^{2+\delta}(s+\ln H)}, \quad \delta > 0,$

where $P(x)$ is a polynomial with rational integral coefficients of degree s and height H, $\alpha \neq 0$, 1 algebraic, β an algebraic irrational number, $\delta > 0$ arbitrarily small but fixed, and $s + \ln H > n(\alpha, \beta, \delta)$.

Suppose N is a sufficiently large integer, q an integer, $N \geqslant q \geqslant \ln^{1+\varepsilon} N$, $\varepsilon > 0$ arbitrarily small, but fixed. We set $r = [\sqrt{q \ln N}]$, $\omega = \alpha^\beta$, $\eta = \ln \alpha$, and consider the function

(249) $\quad f_1(z) = \sum_{k_0=0}^{q} \sum_{k_1=0}^{N} A'_{k_0,k_1} e^{\eta(k_0\beta + k)z},$

$$A'_{k_0,k_1} = \sum_{k=0}^{p} C'_{k,k_0,k_1} \omega^k, \quad p = [q\sqrt{q \ln N}] > qr - 1,$$

where the C'_{k,k_0,k_1} are rational integers, all different from zero. In this case, the numbers A'_{k_0,k_1} are also all different from zero since, as is known, ω is transcendental.

Suppose α and β are numbers in an algebraic field R_ν of degree ν, $a_0\alpha$ and $a_0\beta$, where a_0 is a rational integer, are integers in R_ν, and

$\omega_1, \omega_2, \ldots, \omega_\nu$ is a basis for the ring of integers in the field R_ν. Then, with $p_1 = 2p+2$, we have

$$(250)\quad f_{1,s,t} = \eta^{-s} a_0^{3Nr+s} f_1^{(s)}(t)$$
$$= \sum_{k=0}^{p_1} \sum_{k_1=0}^{\nu} \omega^k \omega_{k_1} \sum_{n=0}^{p} \sum_{n_0=0}^{q} \sum_{n_1=0}^{N} C'_{n,n_0,n_1} B_{s,t,n,n_0,n_1,k,k_1},$$

$$|B_{s,\ldots,k_1}| < e^{\lambda_0 N\sqrt{q \ln N}}, \quad s \leqslant r_1 = \left[N\sqrt{\frac{q}{\ln N}}\right],$$
$$0 \leqslant t \leqslant r, \quad \lambda_0 > 1,$$

where all the numbers B are rational integers. The number of numbers C' equals $(p+1)(q+1)(N+1) > q^{5/2} N \ln^{1/2} N$. A necessary condition that the inequalities

$$(251)\quad f_{1,s,t} = 0, \quad 0 \leqslant s \leqslant r_1, \quad 0 \leqslant t \leqslant r_2 = \left[\frac{1}{6\nu}\sqrt{q \ln N}\right]$$

hold is that the numbers C' satisfy $\nu(2p+3)(r_1+1)(r_2+1)$ equations. Since

$$(252)\quad \nu(2p+3)(r_1+1)(r_2+1)$$
$$< \nu(2q^{3/2} \ln^{-1/2} N + 3)(Nq^{1/2} \ln^{-1/2} N + 1)\left(\frac{1}{6\nu}\sqrt{q \ln N} + 1\right)$$

we may, by Lemma I, choose the numbers C' to be rational integral, all different from zero, in such a way that all the equalities (251) are satisfied. By Lemma I, the inequalities

$$(253)\quad |C'_{n,n_0,n_1}| < e^{2\lambda_0 N\sqrt{q \ln N}}, \quad n \leqslant p, \quad n_0 \leqslant q, \quad n_1 \leqslant N$$

will hold for these numbers, as a consequence of inequalities (250).

Having chosen such a system of numbers C'_{n,n_0,n_1}, we now consider the set of numbers A'_{n_0,n_1} which are polynomials in ω with rational integral coefficients with coefficients C'_{n,n_0,n_1} whose degrees do not exceed $q_0\sqrt{q \ln N}$. If all these polynomials have a greatest common divisor $R(\omega)$, then we set $A'_{n_0,n_1} = R(\omega) A_{n_0,n_1}$, where the A_{n_0,n_1} are again polynomials with rational coefficients, whose heights can be estimated by Lemma II, inasmuch as $R(\omega)$ is also a polynomial with integral coefficients, and the polynomials A_{n_0,n_1} do not have a common zero. We thus arrive at the new function $f(z)$ satisfying the conditions:

$$(254)\quad \begin{cases} f(z) = \dfrac{1}{R(\omega)} f_1(z) = \displaystyle\sum_{n_0=0}^{q} \sum_{n_1=0}^{N} A_{n_0,n_1} e^{\eta(n_0\beta+n_1)z}, \\ A_{n_0,n_1} = \displaystyle\sum_{n=0}^{p} C_{n,n_0,n_1}\omega^n, \quad |C_{n,n_0,n_1}| < e^{3\lambda_0 N\sqrt{q \ln N}}, \end{cases}$$

$$p = [q\sqrt{q \ln N}]$$

and

$$(255)\quad f^{(s)}(t) = 0, \quad 0 \leqslant s \leqslant r_1 = \left[N\sqrt{\frac{q}{\ln N}}\right],$$

$$0 \leqslant t \leqslant r_2 = \left[\frac{1}{6\nu}\sqrt{q \ln N}\right].$$

Conditions (255) enable one to make use of the integral representation

$$(256)\quad f^{(s)}(t) = $$

$$= \frac{s!}{(2\pi i)^2} \int_{\Gamma} \frac{dz}{(z-t)^{s+1}} \int_{\Gamma_1} \left[\frac{z(z-1)\ldots(z-r_2)}{\zeta(\zeta-1)\ldots(\zeta-r_2)}\right]^{r_1+1} \frac{f(\zeta)}{\zeta-z}\, d\zeta,$$

where Γ is the circumference $|z| = \left[\frac{8}{\varepsilon}\sqrt{q \ln N}\right]$ and Γ_1 is the circumference $|\zeta| = q$. A sufficient condition that the contour Γ be interior to the contour Γ_1 is that the inequality $\ln^{\varepsilon/2} N > 10/\varepsilon$ be satisfied.

Estimating the absolute values of the right members of inequalities (256) along the corresponding contours Γ and Γ_1 and taking inequalities (254) into consideration, we obtain the inequalities

$$(257)\quad \begin{gathered} |f^{(s)}(t)| < e^{-\frac{1}{18\nu} qN \ln \frac{q}{\ln N}}, \\ 0 \leqslant s \leqslant \left[\frac{4}{\varepsilon} N\sqrt{\frac{q}{\ln N}}\right], \quad 0 \leqslant t \leqslant [\sqrt{q \ln N}]. \end{gathered}$$

In our case, β is algebraic and obviously condition (155) of Lemma IV is satisfied. This lemma can be applied to our function $f(z)$ if conditions (157) are satisfied, inasmuch as inequalities (254) and (257) are equivalent to the remaining conditions of Lemma IV. It follows that either

$$(258)\qquad |A_{n_0,n_1}| < e^{-\frac{1}{4}Nq\ln q},\quad 0 \leqslant n_0 \leqslant q,\quad 0 \leqslant n_1 \leqslant N,$$

or

$$(259)\qquad |f^{(s)}(t)| > e^{-2Nq\ln q}$$

holds for at least one pair s, t

$$(260)\qquad 0 \leqslant s \leqslant \left[\frac{4}{\varepsilon} N\sqrt{\frac{q}{\ln N}}\right],\quad 0 \leqslant t \leqslant [\sqrt{q}\ln N].$$

From relations (250) and (254) we obtain that

$$(261)\qquad \begin{gathered} P_{s,t}(\omega) = \eta^{-s}a_0{}^{3Nr+s}f^{(s)}(t) = \sum_{k_0=0}^{2p+2}\sum_{k_1=1}^{\nu} g_{k_0,k_1}\omega_{k_1}\omega^{k_0},\\ 0 \leqslant s \leqslant \left[\frac{4}{\varepsilon} N\sqrt{\frac{q}{\ln N}}\right],\quad 0 \leqslant t \leqslant [\sqrt{q}\ln N],\quad p = [q\sqrt{q}\ln N], \end{gathered}$$

where the g_{k_0,k_1} are rational integers which satisfy the conditions

$$(262)\qquad |P_{s,t}(\omega)| < e^{-\frac{1}{20\nu}qN\ln\frac{q}{\ln N}},\quad \max|g_{k_0,k_1}| = g < e^{\lambda_1 Nr}.$$

Suppose $\delta, \frac{1}{8} > \delta > 0$, is arbitrarily small, but fixed, and that $Q(x)$ is a polynomial of degree n and height H, with rational integral coefficients and irreducible over the rational field, such that the inequalities

$$(263)\qquad \left\{\begin{aligned} |Q(\omega)| &< e^{-Nq\ln\frac{q}{\ln N}\ln^{3\delta}\ln q},\\ n &< \sqrt{\frac{q}{\ln N}}\ln\frac{q}{\ln N}\ln^{-3\delta}\ln q,\\ \ln H &< \frac{N}{\sqrt{q\ln N}}\ln\frac{q}{\ln N}\ln^{-3\delta}\ln q \end{aligned}\right.$$

are satisfied. Then, by Lemma VIII, there exists a polynomial $T(x)$, irreducible in the field R_ν, such that

$$(264)\qquad \left\{\begin{aligned} |T(\omega)| &< e^{-Nq\ln\frac{q}{\ln N}\ln^{2\delta}\ln q},\\ T(x) &= \sum_{n_0=0}^{n}\sum_{n_1=1}^{\nu} t_{n_0,n_1}\omega_{n_1}x^{n_0},\\ t_0 = \max_{\substack{0\leqslant n_0\leqslant n\\ 1\leqslant n_1\leqslant\nu}} |t_{n_0,n_1}| &< \exp\frac{N}{\sqrt{q\ln N}}\ln\frac{q}{\ln N}\ln^{-2\delta}\ln q \end{aligned}\right.$$

for $N>N'$, where the t_{n_0,n_1} are rational integers, all different from zero. We now take any one of the polynomials $P_{s,t}(x)$. Since

(265) $$(a\beta_0)^{\nu(n+2p+2)}t_0^{2\nu(p+1)}g^{\nu n}(n+2(p+1))^{\nu(n+2p+2)}$$
$$< \exp\left[2\nu q\sqrt{q\ln N}\frac{N}{\sqrt{q\ln N}}\ln\frac{q}{\ln N}\ln^{-2\delta}\ln q\right.$$
$$\left.+\nu\sqrt{\frac{q}{\ln N}}\ln\frac{q}{\ln N}\ln^{-2\delta}\ln q\lambda_1 N\sqrt{q\ln N}+\lambda_2 q\sqrt{q\ln N}\right]$$
$$< \exp\left[Nq\ln\frac{q}{\ln N}\ln^{-\delta}\ln q\right]$$
$$< \exp\left[\frac{1}{30\nu}qN\ln\frac{q}{\ln N}\right],$$

where a and β_0 are defined in Lemma V, then, as a consequence of the inequalities

(266) $$|T(\omega)|+|P_{s,t}(\omega)| < e^{-\frac{1}{30\nu}qN\ln\frac{q}{\ln N}}$$
$$< (a\beta_0)^{-(2p+n+2)\nu}g^{-\nu n}t_0^{-2\nu(p+1)}(n+2(p+1))^{-\nu(n+2p+2)},$$

which follows directly from inequalities (263) and (265), the conditions of Lemma V are satisfied, from which it follows directly that any polynomial $P_{s,t}(x)$, where s and t are in the intervals given in (261), is divisible by the polynomial $T(x)$.

This means that

(267) $$P_{s,t}(x) = R(x)T(x),$$

where the height of $R(x)$ is, say, R and its degree does not exceed $2p-n+2$. We shall now find an upper bound for the quantity $\ln R$. Since the numbers t_{k_0,k_1} satisfy conditions (264) and $\omega_1, \omega_2, \ldots, \omega_\nu$ is a basis for the ring of integers in the field R_ν, then according to well-known properties of algebraic numbers we shall have that the height of $T(x)$ cannot be less than the quantity $\lambda_3 t_0^{-\nu}$ where $\lambda_3\geqslant 1$ depends only on the numbers $\omega_1, \ldots, \omega_\nu$. From this it follows, according to Lemma II and with the aid of inequality (262), that

(268) $$2p+2+2\lambda_1 N\sqrt{q\ln N}+\ln\lambda_3+\nu\ln t_0 > \ln R,$$

which immediately yields the inequality

(269) $$3(1+\lambda_1)N\sqrt{q\ln N} > \ln R.$$

Making use of inequality (269), from inequalities (264) we obtain a new inequality for $P_{s,t}(\omega)$, where s and t have bounds defined by (261), which inequality is stronger than (262), namely,

$$|P_{s,t}(\omega)| < e^{-Nq \ln \frac{q}{\ln N} \ln^{\delta} \ln q} < e^{-2Nq \ln q}, \tag{270}$$

since

$$2 \ln q < \ln \frac{q}{\ln N} \ln^{\delta} \ln q \tag{271}$$

as a consequence of the fact that $q > \ln^{1+\varepsilon} N$. But once inequalities (270) are satisfied for s and t in the bounds defined by (262), inequalities (258) must be valid, in other words,

$$|A_{k_0,k_1}(\omega)| = |\sum_{k=0}^{p} C_{k,k_0,k_1}\omega^k| < e^{-\frac{1}{4}Nq \ln q}, \tag{272}$$

where all the C_{k,k_0,k_1} are rational integers and satisfy conditions (254), and k_0, k_1 lie in the intervals $0 \leqslant k_0 \leqslant q$, $0 \leqslant k_1 \leqslant N$. But the polynomials $A_{k_0,k_1}(\omega)$ do not differ in any respect, as far as bounds for their degrees and coefficients are concerned, from the polynomials $P_{s,t}(\omega)$ and this means that every polynomial $A_{k_0,k_1}(x)$ must be divisible by $T(x)$. This is impossible by the very nature of the choice of the numbers $A_{k_0,k_1}(\omega)$, since we already got rid of the greatest common divisor of the $A_{k_0,k_1}(\omega)$. We have thus arrived at a contradiction which shows that the inequality

$$|Q(\omega)| < e^{-Nq \ln \frac{q}{\ln N} \ln^{\delta} \ln q} \tag{273}$$

is impossible for arbitrary $\delta > 0$, $\varepsilon > 0$ and arbitrary N, q such that $N \geqslant q \geqslant \ln^{1+\varepsilon} N$, if $Q(x)$ is a polynomial, irreducible over the rational field, with rational integral coefficients, of height $H \geqslant 3$ and degree n, where

$$\left\{\begin{aligned} n &< \sqrt{\frac{q}{\ln N}} \ln \frac{q}{\ln N} \ln^{-\delta} \ln q; \\ \ln H &< \frac{N}{\sqrt{q \ln N}} \ln \frac{q}{\ln N} \ln^{-\delta} \ln q, \end{aligned}\right. \tag{274}$$

for sufficiently large N. Now choosing $\sigma < 0$ and $\delta > 0$ arbitrarily small and solving the equations

$$\left\{\begin{aligned} n + \ln^{\sigma} \ln H &= \sqrt{\frac{x}{\ln z}} \ln \frac{x}{\ln z} \ln^{-\delta} \ln x, \\ n + \ln H &= \frac{z}{\sqrt{x \ln z}} \ln \frac{x}{\ln z} \ln^{-\delta} \ln x, \end{aligned}\right. \tag{275}$$

with $n+\ln H>A(\delta, \sigma)$ and setting $q=[x]$, $N=[z]$, we arrive finally at inequality (248).

In fact, suppose σ is a fixed arbitrarily small positive number.

We then have the equations

$$(276)\quad \begin{cases} \dfrac{z}{x} = \dfrac{n+\ln H}{n+\ln^{\sigma}\ln H} > 1, \\[2ex] (n+\ln^{\sigma}\ln H)(n+\ln H) = \dfrac{z}{\ln z}\ln^2\dfrac{x}{\ln z}\ln^{-2\delta}\ln x, \\[2ex] \dfrac{x}{\ln z} = (n+\ln^{\sigma}\ln H)^2\ln^{-2}\dfrac{x}{\ln z}\ln^{2\delta}\ln x, \end{cases}$$

from which it follows that

$$(277)\quad \begin{cases} \dfrac{z}{\ln z} = (n+\ln H)(n+\ln^{\sigma}\ln H)\times \\ \qquad\qquad \times\ln^{-2}[n+\ln^{\sigma}\ln H]\ln^{\delta_1}(n+\ln H), \\[2ex] \dfrac{x}{\ln z} = (n+\ln^{\sigma}\ln H)^2\times \\ \qquad\qquad \times\ln^{-2}(n+\ln^{\sigma}\ln H)\ln^{\delta_2}(n+\ln H), \\[1ex] \ln z = \ln(n+\ln H)\ln^{\delta_3}[n+\ln H], \\[1ex] \ln\dfrac{x}{\ln z} = 2\ln(n+\ln^{\sigma}\ln H)\ln^{\delta_4}(n+\ln H), \\[1ex] \lim\limits_{\sigma\to 0,\delta\to 0}[\delta_1{}^2+\delta_2{}^2+\delta_3{}^2+\delta_4{}^2] = 0. \end{cases}$$

It follows from these relations that

$$\frac{x}{\ln z} > n+\ln^{\sigma}\ln H > \ln^{\sigma} z$$

and

$$(278)\quad zx\ln\frac{x}{\ln z}\ln^{\delta}\ln x = \frac{z}{\ln z}\frac{x}{\ln z}\ln^2 z\ln\frac{x}{\ln z}\ln^{\delta}\ln x$$

$$= \frac{(n+\ln^{\sigma}\ln H)^3}{\ln^3(n+\ln^{\sigma}\ln H)}(n+\ln H)\ln^{2+\varepsilon_1}(n+\ln H),$$

$$(279)\quad \lim_{\substack{\delta\to 0\\ \sigma\to 0}}\varepsilon_1 = 0.$$

Since inequalities (273) and (274) cannot hold simultaneously for arbitrary $\varepsilon>0$ and $\delta>0$ if $N>N'(\varepsilon, \delta, \alpha, \beta)$, we have

$$(280)\quad |Q(\omega)| > \exp\left[-\frac{n^3}{1+\ln^3 n}(n+\ln H)\ln^{2+\varepsilon_1}(n+\ln H)\right]$$

for $n > \ln^\sigma \ln H$. But if $n < \ln^\sigma \ln H$ for arbitrary σ and $n+\ln H > A$, then

$$(281)\qquad |Q(\omega)| > \exp\left[-\ln H \ln^{2+\varepsilon_2}\ln H\right],$$

for arbitrary $\varepsilon_2 > 0$. This completes the proof of the first part of our Theorem III. Lemma VI′ enables us to omit the irreducibility of $Q(\omega)$.

We shall now prove that if α and β are algebraic, $\omega = \ln\alpha/\ln\beta$ is irrational, and $Q(x)$ is a polynomial, irreducible over the rational field, of degree n and height H, with rational integral coefficients, whose common divisor is unity, then

$$(282)\qquad \left|Q\left(\frac{\ln\alpha}{\ln\beta}\right)\right| > e^{-n^2(n+\ln H)^{2+\varepsilon}};\quad \varepsilon > 0,$$

where ε is arbitrarily small, but fixed, with

$$n+\ln H > C(\varepsilon; \ln\alpha, \ln\beta).$$

The transcendence of the number ω was proved earlier.

The proof of inequality (282) proceeds exactly as the proof of inequality (248) and hence we shall introduce certain simplifications in its proof.

Suppose the numbers α and β belong to the algebraic field K_ν; that $\omega_1, \ldots, \omega_\nu$ is a basis for the ring of integers in this field; and that a_0 is a rational integer such that $a_0\alpha$ and $a_0\beta$ are algebraic integers. Suppose also that ε ($\frac{1}{4} > \varepsilon > 0$) is arbitrarily small but fixed and that N is an arbitrarily large integer. We consider the function

$$(283)\qquad f_1(z) = \sum_{k_0=0}^{N}\sum_{k_1=0}^{N} A'_{k_0,k_1} e^{\ln\beta(k_0+k_1\omega)z},$$

$$A'_{k_0,k_1} = \sum_{k=0}^{q} C'_{k,k_0,k_1}\omega^k,\quad N^{1+\varepsilon} \leqslant q \leqslant N^{3/2}\ln^{-1/2} N,$$

where all the numbers C'_{k,k_0,k_1} are rational integers, all different from zero.

We now set $p = \left[\dfrac{4}{\varepsilon}\dfrac{N^2}{q}\right]$ and consider the $(p+1)(q+1)$ numbers

(284) $$f_{1,s,t} = \ln^{-s}\beta a_0^{2pN} f_1^{(s)}(t)$$
$$= \sum_{k_0=0}^{2q} \sum_{k_1=1}^{\nu} \omega_{k_1}\omega^{k_0} \sum_{n=0}^{q} \sum_{n_0=0}^{N} \sum_{n_1=0}^{N} C'_{n,n_0,n_1} B_{s,t,n,n_0,n_1,k_0,k_1},$$
$$0 \leqslant s \leqslant q,\ t = 0, 1, \ldots, p.$$

The numbers B, as can easily be seen, are rational integers and satisfy the inequalities

(285) $$|B_{s,t,n,n_0,n_1,k_0,k_1}| < e^{\lambda_0 \frac{N^3}{q}}.$$

The number of integers C' equals $(q+1)(N+1)^2$. A necessary condition that $f_{1,s,t}$ vanish for s, t in the intervals (284) is that $\nu(2q+1)$ conditions be satisfied. Therefore, a necessary condition that the equations

(286) $$f_{1,s,t} = 0;$$
$$0 \leqslant s \leqslant q; \quad t = 0, 1, \ldots, r; \quad r = \left[\frac{1}{8\nu}\frac{N^2}{q+1}\right] - 1,$$

be satisfied is that $\nu(r+1)(q+1)(2q+1)$ conditions hold. Since

(287) $$\tfrac{1}{4}(N+1)^2(q+1) > \nu(r+1)(q+1)(2q+1),$$

by Lemma I and inequalities (285), the numbers C'_{n,n_0,n_1} may be chosen to be rational integral, all different from zero and satisfying the inequalities

(288) $$|C'_{n,n_0,n_1}| < e^{\frac{\lambda_0}{2}\frac{N^3}{q}},$$

so that condition (286) is satisfied.

Having chosen the C'_{n,n_0,n_1} in this way, we keep them fixed and by the same token, we completely define our function $f_1(z)$.

We consider the numbers $A'_{k_0,k_1} = A'_{k_0,k_1}(\omega)$, fixed in this manner, which are polynomials with rational integral coefficients in the number ω. Suppose $R(\omega)$ is a polynomial with integral coefficients which is the greatest common divisor of the polynomials $A'_{k_0,k_1}(\omega)$. Then setting

(289) $$\frac{1}{R(\omega)} f_1(z) = f(z) = \sum_{k_0=0}^{N} \sum_{k_1=0}^{N} A_{k_0,k_1} e^{\ln\beta(k_0+k_1\omega)z},$$
$$A_{k_0,k_1} = \sum_{k=0}^{q} C_{k,k_0,k_1}\omega^k,$$

we can, by Lemma II, estimate the absolute values of the rational integers C_{k,k_0,k_1} and we obtain the inequalities

$$(290) \qquad |C_{k,k_0,k_1}| < e^{2\lambda_0 \frac{N^3}{q}}$$

for all numbers C_{k,k_0,k_1}. Again setting

$$(291) \qquad P_{s,t}(\omega) = \ln^{-s}\beta a_0^{2pN} f^{(s)}(t) = \sum_{k_0=0}^{2q} \sum_{k_1=1}^{\nu} g_{k_0,k_1} \omega^{k_0} \omega_{k_1},$$

$$0 \leqslant s \leqslant q; \quad 0 \leqslant t \leqslant p,$$

where, on the basis of inequalities (290), we have

$$(292) \qquad g_0 = \max_{\substack{0 \leqslant k_0 \leqslant 2q \\ 1 \leqslant k_1 \leqslant \nu}} |g_{k_0,k_1}| < e^{\lambda_1 \frac{N^3}{q}},$$

and all these numbers g_{k_0,k_1} are rational integers. Since conditions (286) are satisfied for $f(z)$, we have the integral representation

$$(293) \quad P_{s,t}(\omega) = \frac{s!\,a_0^{2pN}}{(2\pi i)^2 \ln^s \beta} \int_\Gamma \frac{dz}{(z-t)^{s+1}} \int_{\Gamma_1} \left[\frac{z(z-1)\ldots(z-r)}{\zeta(\zeta-1)\ldots(\zeta-r)}\right]^{q+1} \frac{f(\zeta)}{\zeta - z}\, d\zeta,$$

where the contour Γ is the circumference $|z| = \dfrac{8}{\varepsilon}\dfrac{N^2}{q}$ and the contour Γ_1 is the circumference $|\zeta| = N$, and s and t vary over the intervals (291).

Estimating the absolute values of the integrals in the right members of inequalities (293), we obtain the inequalities

$$(294) \qquad |P_{s,t}(\omega)| < e^{-\frac{1}{10\nu}N^2 \ln N},$$

$$0 \leqslant s \leqslant q, \quad 0 \leqslant t \leqslant p.$$

By Lemma IV, either the inequalities

$$(295) \qquad |A_{k_0,k_1}(\omega) < e^{-\frac{1}{4}N^2 \ln N}, \quad 0 \leqslant k_i \leqslant N,\ i = 0, 1$$

are valid for all $A_{k_0,k_1}(\omega)$ or the inequality

$$(296) \qquad |P_{s,t}(\omega)\,| > e^{-2N^2 \ln N}$$

holds for at least one pair (s, t), $0 \leqslant s \leqslant q$, $0 \leqslant t \leqslant p$.

Suppose $\frac{1}{4} > \delta > 0$ is fixed and that $Q(x)$ is a polynomial with rational integral coefficients which is irreducible over the rational field, where the common divisor of its coefficients is unity, of height H and degree n, satisfying the conditions

$$(297)\qquad \begin{cases} |Q(\omega)| < e^{-N^2 \ln^{1+2\delta} N}; \\ n \leqslant \dfrac{q}{N} \ln^{-\delta} N; \quad \ln H \leqslant \dfrac{N^2}{q} \ln^{-\delta} N. \end{cases}$$

Then, by Lemma VIII, there exists an irreducible polynomial $T(x)$ in the field K_ν whose coefficients are integers in the field K_ν, satisfying the conditions

$$(298)\qquad |T(\omega)| < e^{-N^2 \ln^{1+\delta} N}; \quad T(x) = \sum_{n_0=0}^{N} \sum_{n_1=1}^{\nu} t_{n_0,n_1} \omega_{n_1} x^{n_0},$$

$$t_0 = \max_{\substack{0 \leqslant n_0 \leqslant n \\ 1 \leqslant n_1 \leqslant \nu}} |t_{n_0,n_1}| < \exp\left[\frac{N^2}{q} \ln^{-\frac{\delta}{2}} N\right].$$

Suppose $P(x)$ is any one of the polynomials $P_{s,t}(x)$, where $0 \leqslant s \leqslant q$, $0 \leqslant t \leqslant p$. With the aid of inequalities (292) and (298), we obtain the inequality

$$(299)\qquad (a\beta)^{\nu(n+2q)} g_0^{\nu n} t_0^{2\nu q} (n+2q)^{\nu(n+2q)}$$
$$< \exp\left[\lambda_1 \nu \frac{N^3}{q} \cdot \frac{q}{N} + 2\nu \frac{N^2}{q} \cdot q + \lambda_2 \frac{N^3}{q}\right]$$
$$< \exp[\lambda_3 N^2] < \exp\left[\frac{1}{10\nu} N^2 \ln N\right],$$

where the numbers a and β are defined in Lemma V.

This means that the conditions of Lemma V are satisfied for the polynomials $P(x)$ and $T(x)$, and that $P(x)$ must be divisible by $T(x)$ in virtue of the fact that $T(x)$ is irreducible in the field K_ν. This means that

$$(300)\qquad P_{s,t}(x) = R(x)\, T(x),$$

where $R(x)$ is a polynomial of height R and of degree not greater than $2q-n$. With the aid of the same arguments which we used in the proof of the first part of our theorem, we immediately obtain the inequality

$$(301)\qquad N^2 > 2q + \lambda_1 \frac{N^3}{q} + \nu \frac{N^2}{q} > \ln R$$

for the height R. And this means that

$$(302)\qquad |P_{s,t}(\omega)| < e^{-\frac{1}{2} N^2 \ln^{1+\delta} N}, \quad 0 \leqslant s \leqslant q, \quad 0 \leqslant t \leqslant p$$

from which it follows that all the inequalities (295) are valid inasmuch as none of the inequalities (296) can hold. But since polynomials with rational integral coefficients $A_{k_0,k_1}(x)$ will now satisfy the same conditions as the polynomials $P_{s,t}(x)$, all the $A_{k_0,k_1}(x)$ must be divisible by $T(x)$, which is impossible in view of the fact that they do not have a common zero. Hence, suppose N is sufficiently large and that $N^{1+\varepsilon}\leqslant q\leqslant N^{3/2}\ln^{-1/2} N$, where $\varepsilon>0$ is arbitrarily small, but fixed. We also keep δ fixed, $\frac{1}{4}>\delta>0$. Suppose $Q(x)$ is a polynomial with rational integral coefficients, irreducible in the rational field, of degree n and height H

$$n \leqslant \frac{q}{N}\ln^{-1/4} N, \quad \ln H \leqslant \frac{N^2}{q}\ln^{-1/4} N. \tag{303}$$

We have proved that the inequality

$$|Q(\omega)| < e^{-N^2\ln^2 N} \tag{304}$$

is then impossible. We solve the equations

$$n+\ln^{\sigma} H = \frac{z}{x}\ln^{-1} x, \quad n+\ln H = \frac{x^2}{z}\ln^{-1} x \tag{305}$$

with $\frac{2\varepsilon}{1-\varepsilon}>\sigma>\frac{\varepsilon}{1-\varepsilon}$. We shall have:

$$x = \theta^2(n+\ln^{\sigma} H)(n+\ln H)\ln^2(n+\ln H), \quad 1 \leqslant \theta \leqslant 2 \tag{306}$$

and

$$z = \theta(n+\ln^{\sigma} H)^{z}(n+\ln H)\ln^3(n+\ln H). \tag{307}$$

Setting $N=[x]$, $q=[z\ln^{-1/2}(n+\ln H)]$, we see that for sufficiently large $n+\ln H$, inequalities (303) will be satisfied and q will be within the required bounds. This means that

$$|Q(\omega)| > e^{-N^2\ln^2 N} > e^{-n^2(n+\ln H)^{2+\varepsilon_1}}, \tag{308}$$

where $\varepsilon_1>0$ is arbitrarily small.

Now suppose $P_k(x)$ are reducible polynomials with integral coefficients, of degrees n_k and heights H_k such that the inequality

$$|P_k(\omega)| < e^{-n_k^2[n_k+\ln H_k]^{2+\varepsilon}} \tag{309}$$

holds for some $\varepsilon>0$, independently of k, and

$$\lim_{k\to\infty}(n_k+\ln H_k) = \infty.$$

Then, by Lemma VI, for arbitrary k there exists an irreducible polynomial $Q_k(x)$ of height h_k and degree q_k such that

$$(310)\qquad \begin{cases} A)\quad \ln h_k < \dfrac{2}{s}(n_k+\ln H_k), \quad q_k < \dfrac{1}{s}n_k, \\ B)\quad |Q_k(\omega)| < e^{-\frac{1}{2s}n_k^2[n_k+\ln H_k]^{2+\varepsilon}}. \end{cases}$$

Inequality B) shows that

$$\lim_{k\to\infty}(q_k+\ln h_k) = \infty,$$

and all these inequalities taken together lead to an inequality of type (308) for $Q(x)$ upon replacing ε by $\varepsilon/2$. This, by what was proved above, is impossible; this completes the proof of our theorem.

We note that we can also prove the transcendence of the numbers α^β and $\ln\alpha/\ln\beta$ in the same way that we proved Theorem III; the proof of the transcendence of these numbers requires a significantly simpler lemma than Lemma V. Besides its application to the proof of the above theorems, the method discussed above is applicable also to other transcendence problems.

One assertion, which may be formulated as a separate theorem, follows directly from inequalities (117) of Theorem III, §4, of the present chapter.

THEOREM IV. *The inequality*

$$(311)\qquad |x_1\ln\alpha+x_2\ln\beta| < e^{-\ln^{2+\varepsilon}x}, \quad |x_1|+|x_2| = x > 0,$$

where α *and* β *are algebraic numbers,* $\ln\alpha/\ln\beta$ *is irrational,* $\varepsilon>0$ *is an arbitrary fixed number, does not have a solution in rational integers* x_1, x_2 *with* $x>x_0$, $x_0=x_0(\alpha,\beta,\ln\alpha/\ln\beta,\varepsilon)$, *where* x_0 *is an effectively computable constant.*

Inequality (311) was used in an essential manner in the proof of a number of theorems in number theory. In order to make clear the range of applications of inequalities of this type, we introduce here the formulation of four theorems, which are proved with the aid of inequality (311).

The first theorem deals with the approximate representation of integers with the aid of the product of arbitrary powers of a finite number of fixed primes. Suppose $p_1,\dots,p_s$ are distinct primes, $s\geqslant 3$, $\frac{1}{3}>\varepsilon>0$, $\varepsilon>\delta>0$, N is any positive rational number. Then the above theorem will hold.

For the number of solutions $I_{N,s}$ of the equality

(312) $N = dp_1^{\alpha_1} \ldots p_s^{\alpha_s}$; $|\ln d| \leqslant \Delta$; $\Delta = e^{-[\ln \ln N]^{\frac{1}{\nu}-\varepsilon}}$, $\nu = 3$,

where $\alpha_1, \ldots, \alpha_s$ are positive integers, and d is a positive rational number, the asymptotic formula

$$I_{N,s} = \frac{2 \ln^{s-1} N\Delta}{(s-1)! \ln p_1 \ldots \ln p_s} + O\left[\ln^{s-1} N e^{-[\ln \ln N]^{\frac{1}{\nu}-\delta}}\right]$$

holds for large N. This theorem was proved by Segal [1] who made use of the preceding, less sharp inequality (311), where the power is 3 instead of 2. This theorem follows for $\nu = 2$ from inequality (311).

Another theorem deals with the properties of arithmetic functions of the type of characters. Suppose the function $\varphi(n)$ is defined for all rational integers, is multiplicative, i.e. $\varphi(n)\varphi(m) = \varphi(nm)$ and $\varphi(1) \neq 0$. Then the following theorem holds.

If $\varphi(p) = 0$ for all primes p with the exception of a finite number of them and there exists a constant C such that

$$\left| \sum_{k=1}^{N} \varphi(k) \right| < C$$

for arbitrary N, then $\varphi(p) = 0$ for all p except, perhaps, one $p = p_0$, with $\varphi(p_0) = e^{i\alpha} \neq 1$, where α is real.

This theorem was proved by Linnik and Chudakov [1]. Further, if α and β are real integers in the cubic field K with negative discriminant D, and the ratio α/β is irrational, then one can give an effective bound $x_0 = x_0(\alpha, \beta)$ for the magnitudes of the absolute values of the solutions of the equation

(313) $$N(\alpha x + \beta y) = 1$$

with

$$\frac{\alpha_3\beta_2 - \beta_3\alpha_2}{\alpha_1\beta_2 - \beta_1\alpha_2} = e^{2\pi i \frac{p}{q}}$$

if p, q, $(p, q) = 1$ are integers, and α_2, α_3 and β_2, β_3 are the conjugates of α and β, respectively.

This theorem was first proved by Delone [1] in a somewhat more general form, but it also follows directly from inequality (311).

We shall show briefly how this theorem is proved with the aid of inequality (311). Since in this case the field K has only one fundamental unit I, it follows from (313) that if x and y are a solution of (313), $\alpha x + \beta y$ is a unit and $\alpha x + \beta y = I^n$. Denoting by α_2, α_3, β_2, β_3

and I_2, I_3 numbers conjugate to $\alpha=\alpha_1$, $\beta=\beta_1$ and $I=I_1$, we obtain in this way the system of equations

$$\begin{aligned} \alpha_1 x+\beta_1 y &= I_1{}^n, \\ \alpha_2 x+\beta_2 y &= I_2{}^n, \\ \alpha_3 x+\beta_3 y &= I_3{}^n, \end{aligned}$$

where n is an integer. Eliminating x and y from this system, we obtain the equation

$$(\alpha_3\beta_2-\beta_3\alpha_2)I_1{}^n+(\alpha_1\beta_3-\beta_1\alpha_3)I_2{}^n+(\beta_1\alpha_2-\alpha_1\beta_2)I_3{}^n = 0 \tag{314}$$

for the number n. Assuming that I_1 is real, and that I_2, I_3 are complex conjugate, $|I_1|=\varrho$, we obtain that $|I_2|=|I_3|=|I_1|^{-1/2}$. It is easy to show that $n>0$ for sufficiently large $|x|$ or $|y|$. Setting $\eta=\ln I_3/I_2$ and dividing the left number of (314) by $(\alpha_1\beta_3-\beta_1\alpha_3)I_2{}^n$, we obtain the inequality

$$\left| e^{n\eta+2\pi i\left(m+\frac{p}{q}\right)}-1 \right| < C_0|I_1|^{-2n} = C_0 e^{-\lambda n} \tag{315}$$

in virtue of the condition of the theorem, where C_0 and λ are constants, and m is an arbitrary integer. The inequality

$$\left| n\ln\frac{I_3}{I_2}+m_1\ln e^{\frac{2k\pi i}{q}} \right| < C_1 e^{-\lambda n} \tag{316}$$

follows directly from (315), where n and m_1 are integers, C_1 is a constant, the numbers I_3/I_2, $e^{\frac{2k\pi i}{q}}$ are algebraic, and the values of their logarithms are arbitrary, but fixed. This same inequality, which is stronger than (311), cannot have solutions for $|n|$ and $|m_1|$ greater than some bound. If we could prove inequality (221), §5, to be effective like inequality (311) of this section, then we would obtain a computable bound for the magnitudes of the solutions of equation (313) in an arbitrary algebraic field K. As of this date, no realistic approach to this problem other than the one discussed above is apparent.

Finally, one can make use of inequalities (311) of the present section and (83) of §3 to prove very simply the following theorem due to Gelfond [9].

Suppose α, β and γ are real numbers in a finite algebraic field K which are not equal to 0, ± 1 and that at least one of them is not an algebraic unit. Then the equation

$$\alpha^x+\beta^y = \gamma^z, \quad |x|+|y|+|z| = t$$

does not have a solution in integers x, y, z when $t > t_0(\alpha, \beta, \gamma)$, except in the case

$$\alpha = \pm 2^{n_1}, \quad \beta = \pm 2^{n_2}, \quad \gamma = \pm 2^{n_3},$$

where n_1, n_2, n_3 are rational, and t_0 can be effectively computed.

The examples given above of using the estimates for approximating the ratio of logarithms of algebraic numbers by means of rational fractions show that the results and methods of transcendental number theory may be used not only in the theory of geometric construction, for example, to demonstrate the impossibility of squaring the circle, but in other connections as well.

Nontrivial lower bounds for linear forms, with integral coefficients, of an arbitrary number of logarithms of algebraic numbers, obtained effectively by methods of the theory of transcendental numbers, will be of extraordinarily great significance in the solution of very difficult problems of modern number theory. Therefore, one may assume, as was already mentioned above, that the most pressing problem in the theory of transcendental numbers is the investigation of the measures of transcendence of finite sets of logarithms of algebraic numbers. One must also recall that up to this time no way has been found to investigate the arithmetic nature of numbers of the type of the Euler constant or the values of $\zeta(z)$ for $z = 2n + 1$ where $n \geqslant 1$ is an integer.

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