MIT Real Analysis - Dr. Casey
Problem Set 4

Suroj Study

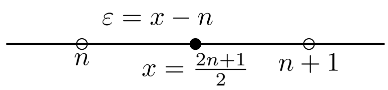
July 2025

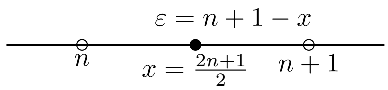
1 Exercise

1. We say a set $F \subset \mathbb{R}$ is *closed* if its complement $F^c := \mathbb{R} \setminus F$ is open (see Assignment 3 for a discussion of open sets). Since $\varnothing$ and $\mathbb{R}$ are open, it follows that $\varnothing$ and $\mathbb{R}$ are closed as well.

   (a) Let $a, b \in \mathbb{R}$ with $a < b$. Prove that $[a, b]$ is closed.
   Proof:
   Since $[a, b]^c = (-\infty, a) \cup (b, \infty)$, and both $(-\infty, a)$ and (b, ∞) are open sets (from Assignment 3, Q5.a), it follows that $[a, b]^c$ is open.
   Hence, $[a, b]$ is closed.

   (b) Is the set $\mathbb{Z} \subset \mathbb{R}$ closed? Provide a proof to substantiate your claim.
   Proof:
   Let $x \in \mathbb{R} \setminus \mathbb{Z}$. Then x lies between two consecutive integers, say $n < x < n + 1$ for some $n \in \mathbb{Z}$.
   Define
   $$\varepsilon = \min(x - n,\ n + 1 - x) > 0.$$
   Then the open interval $(x - \varepsilon, x + \varepsilon)$ contains x and does not intersect $\mathbb{Z}$, i.e.,
   $$(x - \varepsilon, x + \varepsilon) \subset \mathbb{R} \setminus \mathbb{Z}.$$
   Therefore, $\mathbb{R} \setminus \mathbb{Z}$ is open, and so $\mathbb{Z}$ is closed.





   (c) Is the set of rationals $\mathbb{Q} \subset \mathbb{R}$ closed? Provide a proof to substantiate your claim. ‘
   Proof:

(2) (a) Let Λ be a set (not necessarily a subset of $\mathbb{R}$), and for each $\lambda \in \Lambda$, let $F_\lambda \subset \mathbb{R}$. Prove that if F_λ is closed for all $\lambda \in \Lambda$, then the set

$$\bigcap_{\lambda \in \Lambda} F_\lambda = \{x \in \mathbb{R} : x \in F_\lambda \text{ for all } \lambda \in \Lambda\}$$

is closed.

Proof.

Let $x \in (\bigcap_{\lambda\in\Lambda} F_\lambda)^c \iff x \in \bigcup_{\lambda\in\Lambda} F_\lambda^c \implies x \in F_\lambda^c$, for some $\lambda \in \Lambda$

Since F_λ is closed set. $\iff F_\lambda^c$ is open set.

Thus $\exists \delta_\lambda > 0$ such that $(x-\delta_\lambda, x+\delta_\lambda) \subseteq F_\lambda^c \implies (x-\delta_\lambda, x+\delta_\lambda) \subseteq \bigcup_{\lambda\in\Lambda}^n F_\lambda^c$

Thus $\bigcap_{\lambda\in\Lambda} F_\lambda$ is open.

(b) Let $n \in \mathbb{N}$, and let $F_1, \ldots, F_n \subset \mathbb{R}$. Prove that if $F_1, \ldots, F_n$ are closed, then the set

$$\bigcup_{m=1}^{n} F_m$$

is closed.

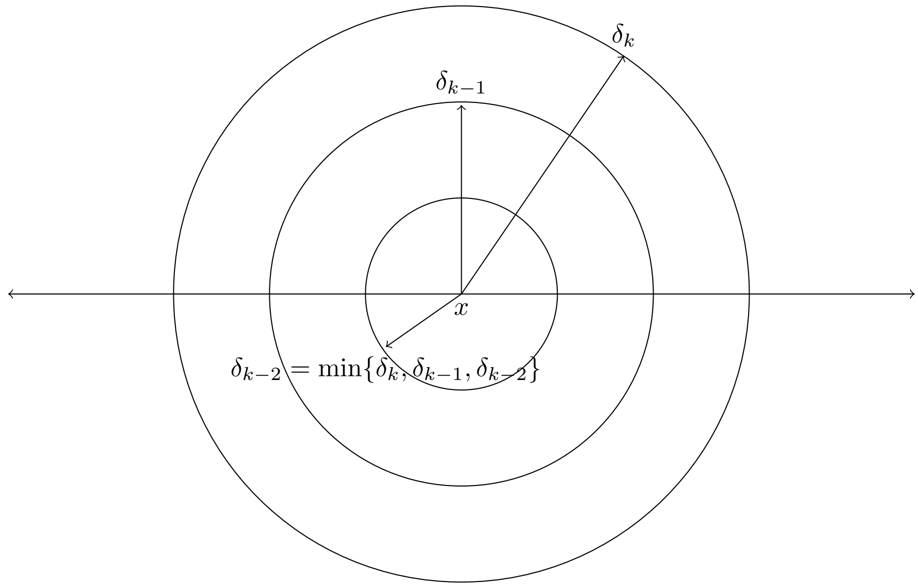
Proof.

Let $x \in (\bigcup_{m=1}^n F_n)^c$, $\iff x \in \bigcap_{m=1}^n F_m^c \implies x \in F_m^c, \forall m$

Since each F_m for all $m \in \mathbb{N}$ is closed set. $\implies F_m^c$ is open for all m.

Thus $\exists \delta_m > 0$ such that $(x-\delta_m, x+\delta_m) \subseteq F_m^c$, for each k

Define $\delta := \min\{\delta_1, \delta_2, ., \delta_k, ..)\,; k \in \mathbb{N}$



$\implies (x-\delta, x+\delta) \subseteq F_m^c$ for all $m \in \mathbb{N}$

$\implies (x-\delta, x+\delta) \subseteq \bigcap_{m=1}^n F_m^c$

Thus $\bigcup_{m=1}^n F_m$ is closed set.

(a) **Problem 3.** Let $F \subset \mathbb{R}$ be a closed set, and let $\{x_n\}$ be a sequence of elements of F converging to $x \in \mathbb{R}$. Prove that $x \in F$.

Hint: Assume that $x \in F^c$ and arrive at a contradiction.

Proof.

Let $F \subseteq \mathbb{R}$ be a closed set, and let $\{x_n\}$ be the sequence of elements of F convergences to $x \in \mathbb{R}$

So , Then $\forall \varepsilon > 0, \exists m \in \mathbb{N}$ for all $n \geq m$.

$$|x_n - x| < \varepsilon$$

$$\iff x - \varepsilon < x_n < x + \varepsilon (*)$$

Assume for contradiction $x \in F^c$
As F is closed set. $\iff$ F^c is open set.
So $\exists \delta > 0$ such that $(x - \delta, x + \delta) \subseteq F^c$

$$\iff (x - \delta < y < x + \delta \implies y \in F^c)\,(**)$$

By using $(*)$
We can choose $\varepsilon = \delta$, and for sufficiently large (n), We have

$$x - \delta < x_n < x + \delta \implies x_n \in F^c$$

This contradicts the fact that all $x_n \in F$, since $F \cap F^c = \emptyset$.

$$\therefore \quad \Rightarrow\Leftarrow \quad \text{Contradiction}$$

Therefore Our Assumption was wrong. So $x \in F$