Real Analysis — MIT MATH 18.100A

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Chapter 1

Lecture 9: Limsup , Liminf and the Bolzano-Weierstrass Theorem

1.1 Review: Binomial Theorem, Squeeze Theorem

Review

Theorem 1.1.

$$\lim_{n\to\infty} x_n = x \iff \lim_{n\to\infty} |x_n - x| = 0$$

Theorem 1.2. ***(Squeeze theorem)***. *If* $\forall n \in \mathbb{N}$ *such that* $\forall n \geq m$ $a_n \leq x_n \leq b_n$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L \implies \lim_{n\to\infty} x_n = L$$

Theorem 1.3. $\lim_{n\to\infty} c^n = 0$ *if* $|c| < 1$

Binomial Expansion

Theorem 1.4. $\forall n \in \mathbb{N},\ x, y \in \mathbb{R}$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Theorem

1. If $p > 0$ then $\lim_{n\to\infty} \frac{1}{n^p} = 0$
2. If $p > 0$ then $\lim_{n\to\infty} p^{1/n} = 1$
3. $\lim_{n\to\infty} n^{1/n} = 1$

Pf.

1. Let $a > 0$ choose $M \in \mathbb{N}$ such that $M > (\frac{1}{\varepsilon})^p$ then, if $n \geq M$

$$\left|\frac{1}{n^p} - 0\right| = \frac{1}{n^p} \leq \frac{1}{M^p} < \varepsilon$$

□

2. For $p = 1$ clear!

   $p-1$: if $p > 1$ $\left|p^{1/n} - 1\right|$

> if $x \geq -1$ then $(1+x)^n \geq a + nx$

then,

$$p = \left(1 + (p^{1/n} - 1)\right)^n \geq 1 + n(p^{1/n} - 1)$$
$$\iff \frac{p-1}{n} \geq p^{1/n} - 1 \geq 0; \quad (p \geq 1)$$

By S.T(1.2

$$\lim_{n\to\infty} \frac{p-1}{n} = 0$$

and

$$\lim_{n\to\infty} 0 = 0$$
$$\implies \lim_{n\to\infty} p^{1/n} - 1 = 0$$
$$\implies \lim_{n\to\infty} |p^{1/n} - 1| = 0, (p \geq 1)$$

By Theorem 1.1

$$\lim_{n\to\infty} p^{1/n} = 1$$

For $p < 1$, then

$$\lim_{n\to\infty} p^{1/n} = \lim_{n\to\infty} \frac{1}{(\frac{1}{p})^{1/n}}$$

By Previous case, $\frac{1}{p} > 1$ and limit respect fraction.So,

$$\lim_{n\to\infty} \frac{1}{(\frac{1}{p})^{1/n}} = 1$$

Thus, For any $p > 0$

$$\lim_{n\to\infty} p^{1/n} = 1$$

□

3. Let $x_n = n^{1/n} - 1 > 0$
   W.T.S $\lim\limits_{n\to\infty} x_n = 0$ Then

$$n = (1 + x_n)^n; \quad (1 + \underbrace{n^{1/n} - 1}_{x_n})^n = n$$

By Binomial expansion 1.4

$$n = (1 + x_n)^n = \sum_{k=0}^{n} \binom{n}{k} x_n^k \geq \binom{n}{2} x_n^2 = \frac{n!}{2!(n-2)!} x_n^2 = \frac{n(n-1)}{2} x_n^2$$

Thus, $\forall n > 1$

$$0 \leq x_n \leq \sqrt{\frac{2}{n-1}}$$

By S.T 1.2

$$\lim_{n\to\infty} x_n = 0$$
$$\lim_{n\to\infty} n^{1/n} - 1 = 0$$

By Theorem 1.1

$$\boxed{\lim_{n\to\infty} n^{1/n} = 1}$$

□

1.2 Limsup/Liminf

QUESTION

Q. Does every bounded Sequence have a convergent subsequence?
A. Yes, Bolzano-Weierstrass Theorem.

Definition: Limsup/Limlinf

Definition 1.1. *Let $\{x_n\}$ be the bounded Sequence. We define , if they exist,*

1. $\limsup x_n := \lim_{n\to\infty} (\sup\{x_k : k \geq n\})$
2. $\liminf x_n := \lim_{n\to\infty} (\inf\{x_k : k \geq n\})$

Theorem

Theorem 1.5. *Let $\{x_n\}$ be bounded sequence and Let*

$$a_n = \sup\{x_k : k \geq n\}$$

$$b_n = \inf\{x_k : k \geq n\}$$

Then,

1. $\{a_n\}$ is monotone decreasing bounded. $\{b_n\}$ is monotone increasing bounded. Thus

$$\lim_{n\to\infty} a_n \text{ and } \lim_{n\to\infty} b_n \text{ exists.}$$

2. $\liminf x_n \leq \limsup x_n$

Theorem

Theorem 1.6. *If $A, B \subseteq \mathbb{R}, A, b \neq \phi$, A and B are bounded and $A \subset B \implies \inf B \leq \inf A \leq \sup A \leq \sup B$.*

Pf. Since $\sup B$ is an upper bound for B and $A \subset B \implies \sup B$ is also an upper bound for A,

$$\implies \sup A \leq \sup B$$

Similarly, $\inf B$ is a lower bound for B and $A \subset B \implies \inf B$ is also a lower bound for A also,

$$\implies \inf B \leq \inf A$$

And it trivial that $\inf A \leq \sup A$

□

Pf.

1. Since

$$\{x_k : k \geq n+1\} \subset \{x_n : k \geq n\}$$

By theorem 1.6

$$\implies a_{n+1} = \sup\{x_k : k \geq n+1\} \leq a_n = \sup\{x_k : k \geq n\}$$

$$\forall n \in \mathbb{N}, \quad a_{n+1} \leq a_n$$

Thus, $a_n = \sup\{x_k : k \geq n\}$ is monotone decreasing sequence.

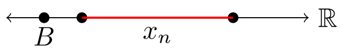
Similarly,

$$\inf\{x_k : k \geq n+1\} \geq \inf\{x_k : k \geq n\}$$

$$b_{n+1} \geq b_n$$

Thus, $b_n = \inf\{x_k : k \geq n\}$ is monotone increasing sequence. We have let $\{x_n\}$ is bounded so

$$\exists B > 0 : \forall n \in \mathbb{N}$$
$$-B \leq x_n \leq B$$



$$\implies -B \leq \inf\{x_k : k \geq n\} \leq \sup\{x_k : k \geq n\} \leq B$$

$$\implies -B \leq b_n \leq a_n \leq B$$

$$\implies |b_n| \leq B \text{ and } |a_n| \leq B$$

Since the sequence is *bounded* and *monotone* sequence, thus limit should exists.

2. By above,

$$\forall n, \quad b_n \leq a_n$$

$$\lim_{n\to\infty} b_n \leq \lim_{n\to\infty} a_n$$

Thus ,

$$\liminf x_n \leq \limsup x_n$$

□

Example

Example 1.1.1. $x_n = (-1)^n$.

$$\{x_k : k \geq n\} = \{(-1)^k : k \geq n\} = \{-1, 1\}$$

$$\sup\{(-1)^k : k \geq n\} = 1$$

$$\limsup\,(-1)^n = \lim_{n\to\infty} \sup\{(-1)^k : n \geq k\} = \lim_{n\to\infty} 1 = 1$$

Similarly, $\liminf(-1)^n = -1$

Example

Example 1.1.2. $x_n = \frac{1}{n}$

$$\left\{\frac{1}{k} : k \geq n\right\} = \left\{\frac{1}{n}, \frac{1}{n+1}, \frac{1}{n+2}, \dots\right\}$$

$$\sup\left\{\frac{1}{k} : k \geq n\right\} = \frac{1}{n}$$

$$\inf\left\{\frac{1}{k} : k \geq n\right\} = 0$$

$$\limsup\left\{\frac{1}{k} : k \geq n\right\} = \lim_{n\to\infty} \frac{1}{n} = 0$$

$$\limsup \frac{1}{n} = \lim_{n\to\infty} \sup\left\{\frac{1}{k} : k \geq n\right\} = \lim_{n\to\infty} \frac{1}{n} = 0$$

$$\liminf \frac{1}{n} = \lim_{n\to\infty} \inf\left\{\frac{1}{k} : k \geq n\right\} = \lim_{n\to\infty} 0 = 0$$

Remark

$\{x_n\}$ sequence convergence $\iff$ $\boxed{\liminf x_n = \limsup x_n = \lim_{n\to\infty}}$

Theorem

Theorem 1.7. *Let $\{x_n\}$ be bounded sequence then $\exists$ sub-sequence $\{x_{n_k}\}$ and $\{x_{m_k}\}$ such that*

$$\lim_{k\to\infty} x_{n_k} = \limsup x_n$$

$$\lim_{k\to\infty} x_{m_k} = \liminf x_n$$

1.3 Theorem: (Bolzano-Weierstrass)

1.3.1 Theorem: (Bolzano-Weierstrass) Every bounded sequence has convergence Subsequence

Pf.-

Let $a_n = \sup\{x_k, k \geq n\}$ By the least upper bound property of Reals , $\exists n_1 \geq 1$ such that

$$a_1 - 1 < x_{n_1} \leq a_1$$

a_1 is sup of set so , the above is true.

Since $a_{n_1+1} = \sup\{x_k, k \geq n_1 + 1\}$
$\exists n_1 > n_1$ such that

$$\underbrace{a_{n_1+1} - \frac{1}{2}}_{\text{not an upper bound}} < x_{n_2} \leq a_{n_1+1}$$

Since $a_{n_2+1} = \sup\{x_k : k \geq n_2 + 1\}$ $\exists n_3 > n_2$ s.t

$$a_{n_2+1} - \frac{1}{3} < x_{n_3} \leq a_{n_2+1}$$

Continuing in this manner, we obtain a sequence of natural numbers $n_1 < n_2 < n_3 < \cdots$ S.t $\forall k \in \mathbb{N}$

$$a_{n_k+1} - \frac{1}{k} \leq x_{n_k} \leq a_{n_k+1} \tag{1.1}$$

with $n_0 = 0$ So , Since $n_1 < n_2 < \cdots$

$$\implies n_1 + 1 < n_2 + 1 < n_3 + 1 < \cdots$$
$$\implies \{a_{n_{k-1}+1}$$

is a subsequence of $\{a_n\}$

Here $k - 1$ is mentions which is just to get sequence with being $k \in \mathbb{N} = 1, 2, 3, \ldots$ so we get subscript of a are $\mathbb{N}$

We know that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sup\{x_k : k \geq n\} = \limsup x_n$ and every subsequence of convergences sequence and that sequence converges to same thing. Thus, Using squeezed theorem 1.2 in equation 1.1
$\boxed{\lim_{k\to\infty} a_{n_{k-1}+1} = \lim_{n\to\infty} a_n = \limsup X_n}$

By S.T 1.2 on 1.1

$$\lim_{n\to\infty} x_{n_k} = \limsup x_n$$

□

Theorem

Theorem 1.8. *Let $\{x_n\}$ be bdd sequence. Then $\{x_n\}$ converges $\iff \limsup x_n = \liminf x_n$. If $\{x_n\}$converges, then, $\lim_{n\to\infty} x_n = \limsup x_n = \liminf x_n$*

Pf. $\impliedby$ Suppose $L = \limsup x_n = \liminf x_n$. Then $\forall n \in \mathbb{N}$

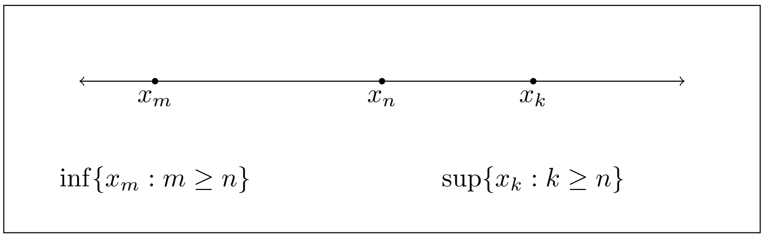
$$\inf\{x_k : k \geq n\} \leq x_n \leq \sup\{x_k : k \geq n\}$$

By S.T 1.2 This is true

$$\limsup x_n = \liminf x_n$$

So this must be true

$$\limsup x_n = \liminf x_n = \lim_{n\to\infty} x_n$$



($\implies$) Let $L = \lim_{n\to\infty} x_n$. By previous theorem that say every convergences sequence has convergence sub sequence so $\exists$ subsequence $\{x_{n_k}\}$ such that $\lim_{n\to\infty} x_{n_k} = \limsup x_n$
But $\{x_{n_k}\}$ is subsequence of convergent sequence $\{x_n\}$ thus it also converges to L. Thus,

$$\lim_{n\to\infty} x_{n_k} = \limsup x_n = L$$

Similarly, there exists subsequence $\{x_{m_k}\}$ such that $\lim_{n\to\infty} x_{m_k} = \liminf x_n$
But $\{x_{m_k}\}$ is subsequence of convergent sequence $\{x_n\}$ thus it also converges to L. Thus,

$$\lim_{n\to\infty} x_{n_k} = \liminf x_n = L$$

Thus,

$$\lim_{n\to\infty} x_{n_k} = \limsup x_n = \liminf x_n$$