1 Quiz 1

1. A dormitory has n students, all of whom like to gossip. One of the students hears a rumor, and tells it to one of the other $n-1$ students picked at random. Subsequently, each student who hears the rumor tells it to a student picked at random from the dormitory (excluding, of course, himself/herself and the person from whom he/she heard the rumor). Let p_r be the probability that the rumor is told r times without coming back to a student who has already heard it from a dormitory-mate. So $p_1 = p_2 = 1$, and $p_n = 0$. Find a formula for p_r for r between 3 and $n-1$.

2. A lie detector is known to be reliable 80% of the time when the person is guilty, and 90% reliable when the person is innocent. A suspect is chosen at random from a group of suspects of which only 1% have ever committed a crime. If the test indicates that the suspect is guilty, what is the probability that they are innocent?

3. In a raffle with 100 tickets, 10 people buy 10 tickets each. 3 winning tickets are drawn at random from this pool of 100 tickets. Find the probability that:
   (a) more than one person wins
   (b) there are 3 different winners

4. 8 rooks are placed at positions sampled randomly without replacement on an 8×8 chessboard. What is the probability that none of them attack each other? (Two rooks attack each other if they are in the same row, or in the same column)

5. A mail room has n empty mail slots, for some fixed positive integer n. The slots are labeled 1 through n. I have $2n$ letters. For each letter, I pick a mail slot uniformly at random and put the letter in it. Assume that my choices for the $2n$ letters are independent of each other.
   (a) Find the chance that Slot 1 has no letters in it after depositing all $2n$ letters.
   (b) Assuming that n is very large, find an exponential approximation for this probability

6. Let X, Y be independent, each uniformly distributed on $\{1 \dots n\}$. Find $P(\max X, Y = k)$

7. Box A contains 1 black and 3 white marbles, and box B contains 2 black and 4 white marbles. A box is selected at random, and a marble is drawn at random from the selected box. Given that the marble is black, find the probability that Box A was chosen.

8. In a hand of 13 cards drawn randomly from a pack of 52, find the chance that at most one **type** of court card appears (J,Q,K,A).

9. A hat contains n coins, f of which are fair, and b of which are biased to land heads with probability $2/3$. A coin is drawn at random from the hat and tossed twice. The first time it lands heads, and the second time it lands tails. Given this information, what is the probability that it is a fair coin?

10. A roulette wheel has 38 pockets. Of these, 18 are red, 18 black, and 2 green. A roulette wheel is spun 20 times. Assume that on each spin, each pocket is equally likely to appear; and assume that the spins are independent of each other. Find the probability that all the colors appear.

11. A deck of five cards contains two aces and three kings. The five cards are shuffled and dealt one by one. This problem is about the number of cards dealt till the aces appear. For example, if the order of the cards dealt was KAKKA, then the number of cards dealt until the first ace is 2, and until the second ace is 5.
    (a) Draw a table that displays the distribution of the number of cards dealt until the first ace.
    (b) Draw a table that displays the distribution of the number of cards dealt until the second ace. Hint: consider the symmetry!

12. Consider a 5 card hand from a standard card deck. Assume that all $\binom{52}{5}$ hands are equally likely. Find the probability of being dealt a
    (a) four of a kind (ranks a,a,a,a,b)
    (b) a pair (ranks a,a,b,c,d)

13. Cards are dealt from a well-shuffled standard deck until the first heart appears.
    (a) What is the probability that 5 or fewer deals are required?

(b) What is the probability that exactly 3 deals are required, given that 5 or fewer are required?

14. There are three boxes, each with two drawers. Box 1 has a gold coin in each drawer, and box 2 has a silver coin in each drawer. Box 3 has a silver coin in one drawer and a gold coin in the other. One box is chosen at random, and then a drawer is chosen at random from the box. Find the probability that box 1 is chosen, given that the chosen drawer yields a gold coin.

15. A classroom has n students, where 2 of the students are twins. Assume that each year consists of 365 days, each of the non-twins are equally likely to be born on any day independent of one another, and the twins are equally likely to be born on any day, independent of the rest. If r people are sampled from this room without replacement, find an *exact* expression for the probability that two students have the same birthday in this sample.

16. Seven dice are rolled. Write down an expression for P(three of one face, and four of another)

2 Quiz 2

17. A box contains n_R red balls, n_B blue balls, and n_G green balls. Balls are drawn one-by-one without replacement until all the red balls are drawn. Let D be the number of draws made. Calculate $E(D)$. Then, evaluate when $n_R = 5, n_B = 3, n_G = 3$.

18. Toss a p-coin until both faces have appeared. Let N be the number of tosses. Find $E(N)$. Then, evaluate when $p = \frac{1}{3}$.

19. Roll a standard fair die over and over. What is the expected number of rolls until the first pair of consecutive sixes appears?

20. Let X and Y be independent geometric random variables, where X has parameter p_1 and Y has parameter p_2. Find $E(X|X \leq Y)$. Then, evaluate your answer where $p_1 = \frac{1}{2}$ and $p_2 = \frac{2}{3}$.

21. Dibya and Jason are playing tennis. The first player to win s sets wins the match. Suppose the probability that Jason wins a set is p (where $0 < p < 1$), independent of the result of any other set. What is the probability that Jason wins the match.

22. A monkey types on a 26-letter keyboard that has lowercase letters only. Each letter is chosen independently and uniformly at random from the alphabet. If the monkey types n letters ($n > 11$), what is the expected number of times the sequence "probability" appears?

23. Suppose I have n balls where $n \sim \text{Poisson}(\lambda)$. Throw these n balls into m bins, where each ball is placed into a bin chosen uniformly at random, independent of all other balls. Find $P(ext there is an empty bin)$.

24. What is the expected number of black balls among n balls drawn at random from a box containing b_0 black balls, w_0 white balls, and d_0 balls drawn at random from another box of b black balls and w white balls? Assume all draws are made without replacement.

25. Throw n balls into m bins ($m \geq 2$). Each ball is thrown into a bin chosen uniformly at random, independent of all other balls. Let X_1 be the number of balls thrown into the first bin and X_2 in the second bin. Find $E(X_2|X_1)$.

26. A cereal company advertises a prize in every box of its cereal. In fact, only about 95% of their boxes have prizes in them. If a family buys one box of this cereal every week for a year, estimate the chance that they will collect more than 45 prizes.

27. The game of roulette involves rounds of betting on the colors red,black,or green, where red wins with probability $\frac{18}{38}$, black with probability $\frac{18}{38}$, and green with probability $\frac{2}{38}$. Each round is independent of all other rounds. A gambler decides to keep betting on red at roulette, and stop as soon as she has won a total of five bets. What is the probability that she has to make at least 9 bets?

28. Flip a p-coin repeatedly, where each flip is independent of the others. Let W_k denote the waiting time to see k heads in a row. Express $E[W_{k+1}]$ in terms of $E[W_k]$.

29. Suppose there are m distinguishable pairs of socks, but a prankster chooses d of these $2m$ socks at random, and pokes holes in them. Let X be the number of undamaged pairs of socks. Find $E(X)$.

30. Suppose X, Y are two independent RVs such that $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(\mu)$. Prove that $X + Y$ follows a Poisson distribution, and identify the parameter.

31. Suppose X, Y are two independent RVs such that $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(\mu)$. Prove that the conditional distribution of X given that $X + Y = k$ is binomial, and identify the parameters.

32. Roll a fair n-sided die m times. Assuming that each roll is independent of the other rolls, find the expected number of unique faces seen.

3 Quiz 3

33. A fair die is rolled n times. Let X be the number of ones and Y be the number of twos. Find $cov(X, Y)$. (Hint: Express X and Y as the sum of indicators.)

34. A random variable X has expectation 20 and standard deviation 5. Find the tightest upper and lower bounds you can on $P(15 < X < 30)$.

35. Dibya is taking a 50 question multiple choice test with 5 options each. He gets 5 points for writing his name and SID, and he earns 2 points for each question he answers correctly. Dibya did not study for the test, so he decides to select an answer choice for each question uniformly at random independently of all other questions. Assuming that he correctly writes his name and SID, find the variance of the number of points he will get on the test.

36. The new data science course has $2n$ students, paired into n project groups. The night that the first project was due, d students of the $2n$ students fell sick from the flu (you may assume the d people were selected uniformly at random from the $2n$ students). Let X be the number of project groups that couldn't finish because both members were sick. Find $Var(X)$.

37. Throw n balls into m bins ($m \geq 3$). Each ball is thrown into a bin chosen uniformly at random, independent of all other balls. Let X_1, X_2, X_3 be the number of balls in bins 1, 2, 3 respectively. Find $Cov(X_1, X_2 + X_3)$.

38. Suppose the IQ scores of a population have a mean of 100 and an SD of 10. Recalling that IQ scores are strictly positive,

    1. Find the best upper and lower bounds you can on $P(X \geq 120)$.
    2. The distribution on IQs is actually symmetric about 100. Mathematically, this means $P(X = 100+k) = P(X = 100-k)\ \forall k$. Now find the best upper and lower bounds you can on $P(X \geq 120)$.

39. Jason hits the streets of Las Vegas with \$1000 to play Roulette. Roulette is a game consisting of multiple rounds. On each round, Jason bets x dollars on black: with probability $\frac{18}{38}$, he will double that bet (get back $2x$ dollars), and with probability $\frac{20}{38}$, he will lose that money (get back 0 dollars). Since Jason is an experienced 140 TA, he decides to bet only \$50 on each round to minimize his risk. Let P be the amount of money Jason has after playing 20 rounds of roulette. Find $Var(P)$.

40. A building has $n+1$ floors: a ground floor G and floors $1 \ldots n$. One morning, m people enter the elevator on the ground floor, and each person gets off on one of the n floors above the ground floor, selected uniformly at random, independent of all other people. Let X be the number of floors which are stopped at. Find $Var(X)$.

41. A fair coin is tossed 300 times. Let H_{100} be the number of heads in the first 100 tosses, and H_{300} the total number of heads in the 300 tosses. Find $Cov(H_{100}, H_{300})$.

42. A random variable X has expectation 10 and standard deviation 5.

    1. Find the tightest upper bound you can on $P(X \geq 20)$.
    2. Can X be binomially distributed?

43. Dibya hits the streets of Las Vegas with \$1000 to play Roulette. Roulette is a game consisting of multiple rounds. On each round, Dibya bets x dollars on black: with probability $\frac{18}{38}$, he will double that bet (get back $2x$ dollars), and with probability $\frac{20}{38}$, he will lose that money (get back 0 dollars). Since Dibya is a high-roller, he bets \$500 on each round. Let P be the amount of money Dibya has after playing two rounds of roulette.Find $Var(P)$.

44. Suppose a sock drawer has m distinguishable pairs of socks, but a prankster chooses d of these $2m$ socks at random, and pokes holes in them. Let X be the number of undamaged pairs of socks. Find $Var(X)$.

45. Out of n individual voters at an election, r vote Republican and $n - r$ vote Democrat. At the next election the probability of Republican switching to vote Democrat is p_1, and of a Democrat switching is p_2. Suppose individuals behave independently. Find the variance of the number of Republican votes at the second election.

46. Heights in the male US population are at an average of 5 feet 9 inches, with standard deviation 3 inches. Two people are sampled randomly from the population (you may assume independence). Use Chebyshev's inequality to find upper and lower bounds for the probability that the heights of the men differ by at least 6 inches.

47. Jason is taking a 50 question multiple choice test with 4 options each. He earns 2 points for each question he answers correctly. Jason studied for only half an hour before the test, so he correctly answers the first 10 questions before getting thoroughly stumped on the rest of the questions. For each question thereafter, Jason decides to select an answer choice for each question uniformly at random independently of all other questions. Let J be the number of points that Jason receives on the test. Find $E(J)$ and $Var(J)$.

48. Students in a class are comparing their birthdays to one another. Assume that each student has a birthday which is equally likely to be any of 365 days, independent of all other students. Let X be the number of days on which at least one birthday will be celebrated. If there are n students in the class, find $Var(X)$.

4 Quiz 4

49. Let X be a random variable describing the relative change of Bitcoin in a year: a \$1 investment in bitcoin will be worth \$$X$ at the end of the year. Jason buys \$100 worth of bitcoin. Let Y be the profit made on this investment at the end of the year (this will be negative if $X < 1$). For example, if $X = 0.9$, then the profit is $-\$10$, and if $X = 1.1$, the profit is \$10.

$$f_X(x) = \frac{1}{x\sqrt{2\pi}} e^{-\frac{(\ln x)^2}{2}} \qquad E(X) = 1 \qquad Var(X) = e^2 - e$$

Assuming X follows the log-normal distribution, which has the above density and statistics, find

a) The density of Y: $f_Y(y)$

b) $E(Y)$ and $SD(Y)$

50. Let $X \sim$ Exponential(λ), and $Y = \log X$. Find the distribution of Y. Be sure to explicitly list the region on which the density is defined.

Hint: Recall that the Exponential(λ) distribution has density $f(x) = \lambda e^{-\lambda x}$

51. Suppose the distribution of height over a large population of individuals is approximately normal. Ten percent of individuals in the population are over 6 feet tall, while the average height is 5 feet 10 inches. What, approximately, is the probability that in a group of 100 people picked at random from this population there will be two or more individuals over 6 feet 2 inches tall?

52. Let X and Y have joint density f given by

$$f(x,y) \;=\; \begin{cases} \frac{3}{4}(x^2-y^2)/(x^5), & |y| < x,\ x \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

Write the each of the following in terms of f but do not simplify the expression. Integrals should not include regions where $f(x,y) = 0$.

a) $P(Y > 2)$

b) the conditional density of Y given $X = 3$ (please be clear about the values on which this density is positive)

c) $E(\frac{Y^2}{X^2})$

53. The distribution of the 3rd smallest element out of 10 independent Uniform(0,1) random variables is distributed Beta$(3, 8)$. Consider Y, the 3rd smallest element out of 10 independent Uniform(-1,1) random variables. We can write $Y = -1 + 2X$, where $X \sim$ Beta$(3, 8)$.

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \qquad E(X) = \frac{\alpha}{\alpha+\beta} \qquad Var(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Given the above density and statistics for the Beta distribution Beta(α, β), find

a) The density of Y: $f_Y(y)$

b) $E(Y)$ and $SD(Y)$

54. Let X be distributed with density $2x^{-3}e^{-x^{-2}}$ on the positive real numbers. Find the density of X^4. Be sure to explicitly list the region on which the density is defined.

55. Measurements on the weight of a lump of metal are believed to be independent and identically distributed; each measurement has mean 12 grams and SD 1.1 gram. Use the normal approximation to estimate the chance that the average of 100 measurements is between 11.8 and 12.2 grams.

56. Let X and Y have joint density f given by

$$f(x,y) \;=\; \begin{cases} \frac{63}{32}x(2-x)^5(x-y), & 0<y<x<2 \\ 0 & \text{otherwise} \end{cases}$$

Write the each of the following in terms of f but do not simplify the expression. Integrals should not include regions where $f(x,y)=0$.

a) Find $P(Y>1)$.

b) the conditional density of X given $Y=1$ (please be clear about the values on which the density is positive)

c) $E(XY^2)$

57. The temperature in Fahrenheit in Berkeley can be described by a random variable F which is distributed on the range $(0,100)$ according to the following density and statistics

$$f_F(f) = \frac{1}{100^{11}}x^5(100-x)^5 \qquad E(K)=50 \qquad Var(K)=\frac{2500}{11}$$

Let C denote the temperature in Celsius. Recall that Temperature in Celsius $=\frac{5}{9}$(Temperature in Farenheit $-\,32$). Find

a) The density of T: $f_T(t)$

b) $E(T)$ and $SD(T)$

58. Let $X\sim$ Normal$(0,1)$, and $Y=e^X$. Find the distribution of Y.

Hint: Recall that the Normal$(0,1)$ distribution has density $f(x)=\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$

59. Suppose $X_1\ldots X_{10}$ are independent uniform $(0,1)$ random variables, and define $S=\sum_{i=1}^{10}X_i$.
Recall that the uniform$(0,1)$ distribution has density $f(x)=1$ if $x\in(0,1)$, and has statistics $E(X)=\frac{1}{2}$ and $SD(X)=\frac{1}{\sqrt{12}}$.
Use the normal approximation to calculate $P(S\geq 8)$ approximately.

60. Let X and Y have joint density f given by

$$f(x,y) \;=\; \begin{cases} \frac{1}{4}(y^2-x^2)y^4e^{-y^2}, & |x|<y,\ y>0 \\ 0 & \text{otherwise} \end{cases}$$

Write the each of the following in terms of f but do not simplify the expression. Integrals should not include regions where $f(x,y)=0$.

a) Find $P(X>1)$

b) the conditional density of Y given $X=3$ (please be clear about the values on which the density is positive)

c) $E\left(\frac{X^2}{Y^2}\right)$

61. At any given moment of time at CERN, the temperature in Kelvin of the supercooling chamber can be described by a random variable K which is distributed according to the Maxwell–Boltzmann distribution, which has the following density and statistics

$$f_K(k)=\frac{2}{\pi}x^2e^{-x^2/2} \qquad E(K)=2\sqrt{\frac{2}{\pi}} \qquad Var(K)=\frac{3\pi-8}{\pi}$$

Let T denote the temperature in Fahrenheit. Recall that Temperature in Fahrenheit $=\frac{9}{5}$(Temperature in Kelvin)$\,-\,459.4$. Find

a) The density of T: $f_T(t)$

b) $E(T)$ and $SD(T)$

62. Let $X\sim$ Normal$(0,1)$, and $Y=X^5$. Find the distribution of Y.

Hint: Recall that the Normal$(0,1)$ distribution has density $f(x)=\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$

63. A large lot of marbles have diameters which are approximately normally distributed with a mean of 1cm. Of these, exactly one-third have diameters greater than 1.1cm. Find

a) The standard deviation of the distribution

b) The proportion whose diameters are within 0.2cm of the mean

64. Let X and Y have joint density f given by

$$f(x, y) = \begin{cases} \frac{10}{243} x(3-x)^4(x-y), & 0 < y < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

Write the each of the following in terms of f but do not simplify the expression. Integrals should not include regions where $f(x, y) = 0$.

a) Find $P(Y > 1)$

b) the conditional density of X given $Y = 1$ (please be clear about the values on which the density is positive)

c) $E(e^{XY})$