Transport System Bonus Task | Proof

$$T(E) = \mathrm{Tr}\left(G_R^{\dagger}(E)\,\Gamma^R(E)\,G_R(E)\,\Gamma^L(E)\right) \qquad (I)$$

with $\Gamma^L(E) = (\Delta_L(E) - \Delta_L^{\dagger}(E)) \wedge \Gamma^R(E) = (\Delta_R(E) - \Delta_R^{\dagger}(E))$

... assume that all matrix elements of $\Gamma^L(E) \wedge \Gamma^R(E)$ are zero except for $\Gamma^L_{\alpha\alpha}(E) = \Gamma^R_{\beta\beta}(E) = 1$

... show that $(I) \rightarrow T_{\alpha\beta}(E) = |G_{R,\alpha\beta}(E)|^2$

let
$$T(E) = \mathrm{Tr}\left[\begin{pmatrix} g_{11}^{\dagger} & \cdots & g_{1n}^{\dagger} \\ \vdots & & \vdots \\ g_{n1}^{\dagger} & \cdots & g_{nn}^{\dagger} \end{pmatrix} \cdot \begin{pmatrix} a_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} g_{11} & & g_{n1} \\ \vdots & & \vdots \\ g_{n1} & \cdots & g_{nn} \end{pmatrix} \begin{pmatrix} a_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn} \end{pmatrix}\right] =$$

with $a_{11} = a_{nn} = 1$ (Matrix-Multiplication is associative)

$$= \mathrm{Tr}\left[\begin{pmatrix} g_{11}^{\dagger} & \cdots & g_{1n}^{\dagger} \\ \vdots & & \vdots \\ g_{n1}^{\dagger} & \cdots & g_{nn}^{\dagger} \end{pmatrix} \begin{pmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & \ddots & \\ g_{n1} & & g_{nn} \end{pmatrix}\right] =$$

$$= \mathrm{Tr}\left[\begin{pmatrix} (g_{11}^{\dagger} \cdots g_{1n}^{\dagger})\cdot(g_{11} \cdots g_{n1}) & \cdots & (g_{11}^{\dagger} \cdots g_{1n}^{\dagger})\cdot(g_{1n} \cdots g_{nn}) \\ \vdots & \ddots & \\ (g_{n1}^{\dagger} \cdots g_{nn}^{\dagger})\cdot(g_{11} \cdots g_{n1}) & \cdots & (g_{n1}^{\dagger} \cdots g_{nn}^{\dagger})\cdot(g_{1n} \cdots g_{nn}) \end{pmatrix}\right]$$

with $\mathrm{TR}[A] = \sum_{j=1}^{n} a_{jj} \Rightarrow (g_{11}^{\dagger} \cdots g_{1n}^{\dagger})\cdot(g_{11} \cdots g_{n1}) + \cdots + (g_{n1}^{\dagger} \cdots g_{nn}^{\dagger})\cdot(g_{1n} \cdots g_{nn}) =$

$$= (g_{11} \cdot \ldots \cdot g_{1n})^{\dagger} \cdot (g_{11} \cdot \ldots \cdot g_{n1}) + \cdots + (g_{n1} \cdot \ldots \cdot g_{nn})^{\dagger} \cdot (g_{1n} \cdot \ldots \cdot g_{nn}) =$$

with $z^{\dagger} z = |z|^2$ and $g_{1n} = g_{n1}$ (values are the same on columns/rows)

$$= \left(|g_{11}|^2 + \cdots + |g_{1n}|^2 + \cdots + |g_{nn}|^2\right)$$

, so for one specific case the transmission prob. $T(E)_{\alpha\beta}$ is equal to $|g_{\alpha\beta}|^2$