2b)

I) Vaification that the velocity of the maximum of the wave packet is $v = \frac{q}{m}$

$$\Rightarrow \psi(x,t) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \tilde{\psi}(k,0) \cdot e^{i\cdot(kx-\omega t)}$$ ... Fourier Transform of distribution $\tilde{\psi}(k)$ in k-space

$$\Rightarrow \tilde{\psi}(k,0) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} \psi(x,0) \cdot e^{-ikx}$$ ... Inverse Fourier Transform at $t=0$

$\Rightarrow$ The dispersion relation is given by: $\omega(k) = \frac{\hbar \cdot k^2}{2m}$

$\Rightarrow$ The group velocity is given by $v_g = \frac{d\omega(k)}{dk}$

$$\Rightarrow v_g = \frac{2\hbar \cdot k}{2m} = \frac{\overbrace{\hbar \cdot k}^{=q}}{m} = \underline{\underline{\frac{q}{m}}}$$

II) Verification that the width of the packet is $d(t) = \sigma\sqrt{1+\Delta^2}$ with $\Delta = t/m\sigma^2$

$\Rightarrow$ general wave packet using momentum + Energie instead of wave number and frequency:

$$\psi(x,t) = \int_{-\infty}^{\infty} \frac{dq}{\sqrt{2\pi\hbar}} \cdot \tilde{\psi}(q,0) e^{\frac{i}{\hbar}\cdot(qx-Et)}$$

=> The inverse Fourier Transform at $t=0$ is given by:

$$\bar{\phi}(q,0) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi\hbar}} \cdot \psi(x,0) \cdot e^{-iqx}$$

$$= \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi\hbar}} \cdot \frac{1}{\sqrt{\sigma} \cdot \pi^{\frac{1}{4}}} \cdot e^{\left(-\frac{(x-x_0)^2}{2\sigma^2}\right)} \cdot e^{\frac{i}{\hbar} q_0 x} \cdot e^{-iqx}$$

$$= \frac{1}{\sqrt{2\pi\sigma\hbar}} \cdot \frac{1}{\pi^{\frac{1}{4}}} \cdot \underbrace{\int_{-\infty}^{\infty} dx \, e^{-\left(\frac{(x-x_0)^2}{2\sigma^2}\right)} \cdot e^{\frac{i}{\hbar} x \cdot (q_0 - q)}}_{\text{Integralrechner}}$$

$$= \cancel{\sqrt{2\pi}} \cdot \sigma \cdot e^{\frac{-\sigma^2 \cdot (q-q_0)^2}{2\hbar^2}} \cdot \frac{1}{\cancel{\sqrt{2\pi}}} \cdot \frac{1}{\sqrt{\sigma\hbar}} \cdot \frac{1}{\pi^{\frac{1}{4}}}$$

$$= \underbrace{\sqrt{\frac{\sigma}{\hbar \cdot \sqrt{\pi}}}}_{=N} \cdot e^{-\frac{\sigma^2 \cdot (q-q_0)^2}{2\hbar^2}}$$

$$\Rightarrow \psi(x,t) = \frac{N}{\sqrt{2\pi\hbar}} \cdot \underbrace{\int_{-\infty}^{\infty} dq \, e^{\left(\frac{-\sigma^2 (q-q_0)^2}{2\hbar^2}\right)} \cdot e^{\frac{i}{\hbar}(qx - Et)}}_{\text{simplify}} \quad \Big/ \, E = \frac{q^2}{2m}$$

$$* \quad a = \frac{\sigma^2}{2\hbar^2} + \frac{i \cdot t}{2m\hbar}, \quad b = \frac{\sigma^2 q_0}{2\hbar^2} + \frac{i \cdot x}{2\hbar}, \quad c = \frac{\sigma^2 q_0^2}{2\hbar^2}$$

$$\Rightarrow \psi(x,t) = \frac{N}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dq\, e^{\left(-a\left(q-\frac{b}{a}\right)^2 + \frac{b^2}{a} - c\right)} \qquad / \ u = p - \frac{b}{a}$$

$$\Rightarrow \int_{-\infty}^{\infty} du\, e^{-\alpha u^2} = \sqrt{\frac{\pi}{\alpha}} \quad \text{.. Formula of Gauss}$$

$$\Rightarrow \psi(x,t) = \frac{N}{\sqrt{2\pi\hbar}} \cdot \sqrt{\frac{\pi}{a}} \cdot e^{\frac{b^2}{a} - c} = \underline{\underline{\frac{N}{\sqrt{2a\hbar}} \cdot e^{\frac{b^2}{a} - c}}}$$

$\Rightarrow$ Probability Density:

$$|\psi(x,t)|^2 = \frac{N^2}{2\hbar|a|} \cdot e^{2\cdot \mathrm{Re}\left(\frac{b^2}{a} - c\right)}$$

$$* \ |a| = \sqrt{a \cdot a^*} = \sqrt{\mathrm{Re}(a)^2 + \mathrm{Im}(a)^2} = \sqrt{\left(\frac{\sigma^2}{2\hbar^2}\right)^2 + \left(\frac{t}{2m\hbar}\right)^2}$$

$$= \sqrt{\left(\frac{\sigma^2}{2\hbar^2}\right)^2 \cdot \left[1 + \underbrace{\frac{\hbar^2 \cdot t^2}{m^2\sigma^4}}_{=\Delta^2}\right]} = \frac{\sigma^2}{2\hbar^2} \cdot \sqrt{1 + \Delta^2} \qquad / \ \Delta = \frac{\hbar \cdot t}{m\sigma^2}$$

$$* \ 2 \cdot \mathrm{Re}\left(\frac{b^2}{a} - c\right) = 2 \cdot \mathrm{Re}\left(\frac{b^2 a^* - c \cdot |a|^2}{|a|^2}\right)$$

$$= 2 \cdot \mathrm{Re}\left(\frac{\left[\frac{\sigma^4 q_0^2}{4\hbar^4} + \frac{\sigma^2 q_0 i x}{2\hbar^3} - \frac{x^2}{4\hbar^2}\right]\left(\frac{\sigma^2}{2\hbar^2} - i \cdot \frac{t}{2m\hbar}\right) - \frac{\sigma^2 q_0^2}{2\hbar^2} \cdot \frac{\sigma^4}{4\hbar^4} \cdot (1 + \Delta^2)}{\frac{\sigma^4}{4\hbar^4} \cdot (1 + \Delta^2)}\right)$$

$$= 2 \cdot \frac{\left(\frac{\sigma^6 q_0^2}{8\hbar^6} - \frac{x^2\sigma^2}{8\hbar^4} + \frac{\sigma^2 q_0 x t}{4 m \hbar^4} - \frac{\sigma^6 q_0^2}{8\hbar^6} \cdot (1+\Delta^2) \right)}{\frac{\sigma^4}{4\hbar^4} \cdot (1+\Delta^2)}$$

$$= \frac{-\frac{\sigma^6 q_0^2}{\cancel{8\hbar^6}} \cdot \frac{\hbar^2 t^2}{m^2\sigma^4} - \frac{x^2\sigma^2}{\cancel{8\hbar^4}} + \frac{\sigma^2 q_0 x t \cdot 2}{\cancel{4 m\hbar^4}}}{\frac{\sigma^4}{\cancel{8\hbar^4}} \cdot (1+\Delta^2)}$$

$$= \frac{-\frac{q_0^2 t^2}{m^2} - x^2 + \frac{2 q_0 x t}{m}}{\sigma^2 \cdot (1+\Delta^2)} \qquad \Big| \; v = \frac{q_0}{m}$$

$$= \frac{-x^2 + 2vxt - v^2 t^2}{\sigma^2 \cdot (1+\Delta^2)} = -\frac{(x-vt)^2}{\sigma^2 \cdot (1+\Delta^2)}$$

$$\Rightarrow |\psi(x,t)|^2 = \frac{\sigma}{\cancel{\hbar} \cdot \sqrt{\pi} \cdot \cancel{2\hbar} \cdot \frac{\sigma^2}{\cancel{2\hbar^2}} \cdot \sqrt{1+\Delta^2}} \cdot e^{-\frac{(x-vt)^2}{\sigma^2 \cdot (1+\Delta^2)}}$$

$$= \frac{1}{\sqrt{\pi} \cdot \sigma \cdot \sqrt{1+\Delta^2}} \cdot e^{\left(\frac{-(x-vt)^2}{\sigma^2 \cdot (1+\Delta^2)} \right)}$$

$$\Rightarrow d(t) = \sigma \sqrt{(1+\Delta^2)}$$