CP1: Monte Carlo Methods WS 2023/24

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1. Exercise: Volume of a circle

Write a program to calculate π by Monte-Carlo estimation of the volume of a circle: pick a random point in the unit square ($0 < x, y < 1$) by generating two random numbers x and y with uniform probability between 0 and 1. This point has $x^2 + y^2 < 1$ if it is contained in the unit circle, and the probability for this is the area of the intersection of the unit circle and a unit square: $\pi/4$. We can thus approximate π by choosing a random point N times, and counting the number of points it is inside a unit circle: N_{in}. The estimator for π is thus $4N_{in}/N$.

Calculate the standard error of the estimator of π which is the square root of its variance. Use the standard random generator of your programming language. (In python, `random.random()` gives a random real number uniformly distributed in [0.0,1.0), `random.randint(a,b)` gives a random integer N such that $a \leq N \leq b$, uniformly distributed).

$$\text{Var}(x) = \langle (x - \langle x \rangle)^2 \rangle \tag{1}$$

An unbiased estimator of the variance using N samples is:

$$\frac{1}{N-1} \sum_{i=1}^{N} (x_i - \bar{x})^2 \tag{2}$$

with $\bar{x} = \sum_{i=1}^{N} x_i / N$.

2. Exercise: Volume of a 10-ball

Do the same for the 10-ball (Unit sphere in 10 dimensions) and use the following formula the volume of a sphere in 10 dimensional space

$$V_{10}(r) = r^{10} \frac{\pi^5}{120} \tag{3}$$

3. Exercise: Tic-Tac-Toe

Two players are playing Tic-Tac-Toe by always choosing at random from the available fields until one of them wins or the game ends with a draw. What is the probability that the first player wins? What is the probability of a draw?

This is doable using only a paper and a pencil, but it's much more convenient to use a Monte-Carlo simulation: write a program which uses the strategy described above to play, using the random generator to choose a field. After playing N times, give an estimation for the probabilities. Calculate also the errors. What's the mean and variance of the game length? (game length = number of symbols played)