Finite Approximation of Rational Numbers in Capacity-Limited Frameworks

Probabilistic Minds Consortium

December 15, 2024

Lemma 1 (Finite Approximation of Rational Targets)**.** *For any rational number $q \in \mathbb{Q}$ and any $\varepsilon > 0$, there exists a capacity state a in our finite framework such that we can represent a rational increment $\tilde{q}$ with increments at capacity a, where $|\tilde{q} - q| < \varepsilon$.*

Interpretation Given a rational number $q = \frac{r}{s} \in \mathbb{Q}$ (in lowest terms), we want to show that by choosing a capacity state a with sufficiently high resolution $N(a)$, we can produce increments of size $\frac{1}{N(a)}$ that allow a "representation" $\tilde{q} = \frac{k}{N(a)}$ for some integer k (with $0 \leq k \leq N(a)$) that differs from q by less than ε.

In other words, we can pick $N(a)$ large enough so that the granularity $\frac{1}{N(a)}$ is so fine that we can "zoom in" on q and choose a fraction that's within ε of it.

Proof

1. **Starting Point:**
   Let $q = \frac{r}{s}$ be our target rational number. We assume $r \in \mathbb{Z}$ and $s \in \mathbb{N}$. We also have a desired precision $\varepsilon > 0$.

2. **Capacity and Resolution:**
   By definition, each capacity state a comes with a resolution factor $N(a)$, where increments are multiples of $\frac{1}{N(a)}$. To approximate q within ε, we want:
   $$\left| \frac{k}{N(a)} - \frac{r}{s} \right| < \varepsilon$$
   for some integer k.

3. **Choosing a Large Enough $N(a)$:**
   Consider $N(a)$ as a variable we can make arbitrarily large by increasing the capacity. We know from the axioms of our framework (e.g., Axiom R1 and R2 for richness and incremental refinements) that for any required resolution, we can find a capacity state a with $N(a)$ at least as large as we need. If we want to get within ε of q, it suffices to choose $N(a) > \frac{1}{\varepsilon}$. With $N(a) > \frac{1}{\varepsilon}$, increments are smaller than ε.

4. **Construction of the Approximate Fraction:**
Since $q = \frac{r}{s}$, consider $k = \left\lfloor \frac{r}{s} N(a) \right\rfloor$ or $k = \left\lceil \frac{r}{s} N(a) \right\rceil$. Either choice ensures $\frac{k}{N(a)}$ is a fraction close to $\frac{r}{s}$. More concretely, as $N(a)$ grows large:

$$\left| \frac{r}{s} - \frac{k}{N(a)} \right| \leq \frac{1}{N(a)} < \varepsilon$$

Thus, by appropriate rounding, we find a k that gives $\tilde{q} = \frac{k}{N(a)}$ satisfying the desired inequality.

5. **Ensuring the Existence of Such a Capacity State a:**
By the axioms of our finite framework, we can always increase capacity $V(a)$ and thus increase $N(a)$. There is no upper bound on how large $N(a)$ can be (except for practical complexity limits, but theoretically we can assume capacity is arbitrarily large for this argument). Hence, for any $\varepsilon > 0$, we pick $N(a) > \max\left(s, \frac{1}{\varepsilon}\right)$ to ensure both representability of a fraction with denominator $N(a)$ finer than $1/\varepsilon$ and the ability to closely approximate $\frac{r}{s}$. Once $N(a)$ is fixed, we choose a capacity state a that realizes $N(a)$. Such a state a exists due to our capacity refinement axioms.

Conclusion We have shown that for any rational number q and any $\varepsilon > 0$, we can select a capacity state a with sufficiently large $N(a)$ to represent a fraction $\tilde{q} = \frac{k}{N(a)}$ such that $|\tilde{q} - q| < \varepsilon$.

This lemma establishes that the finite framework can approximate any given rational number to arbitrary precision, just by refining capacity and thereby reducing increments. This is a crucial stepping stone to simulating classical "infinite" processes (like taking limits) by approximating the desired rational approximations at finite, but increasingly fine, resolutions.

Comment on Extensions While the lemma focuses on a fixed rational target, it naturally extends to sequences of rationals and their limits. If we have a sequence (q_n) converging to L in the classical sense, we can pick an n large enough that q_n approximates L well, and then apply this lemma to approximate q_n within $\varepsilon/2$, ensuring the finite system gets arbitrarily close to L.

In this manner, this lemma supports the broader claim that we can emulate infinite-based rational arithmetic and limit approximations in a strictly finite and capacity-limited framework.