Appendix: Additional Theorems and Formal Results

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The following theorems and propositions are intended to supplement the main text by providing rigorous support for the underlying structure of the system. They illustrate how the finite sets, discrete arithmetic, algebraic structures, and geometric constructs are maintained and refined without invoking actual infinities. Moreover, they explain how errors are controlled and how the discrete framework approximates classical properties as closely as desired.

A.1 Closure and Consistency Theorems

Theorem A.1 (Closure of Finite Increments)

Statement:
Let

$$\mathrm{Increments}(a) = \left\{ \frac{k}{N(a)} \,\middle|\, k = 0, 1, \ldots, N(a) \right\}$$

for a given observer state a with capacity $N(a)$. Then:

1. **Addition:** For any $\frac{k}{N(a)}, \frac{m}{N(a)} \in \mathrm{Increments}(a)$, the sum

   $$\frac{k}{N(a)} + \frac{m}{N(a)}$$

   (when appropriately refined if necessary) is also representable as an element of a refined set $\mathrm{Increments}(a')$ (with a' such that $N(a')$ is a common multiple of $N(a)$).

2. **Subtraction and Multiplication:** Similar closure properties hold for subtraction and multiplication.

3. **Division (Partial):** Division is defined whenever the divisor is nonzero, with the result represented in an observer state with sufficient capacity.

Discussion:
This theorem guarantees that the basic arithmetic operations on discrete increments do not take us outside the finite framework. In practice, when different denominators are encountered, one "refines" the observer state to achieve a common denominator.

Theorem A.2 (Update Consistency Theorem)

Statement:
Let $\mathbf{U}_\mu$ be the update function that maps a probability measure from an observer state a to a refined state a'. Then, for any joint state (a, e) and any probability value $\alpha \in \{0, \frac{1}{N(a)}, \ldots, 1\}$, we have

$$\mathbf{U}_\mu(\alpha, a, e, a', e') \in \left\{0, \tfrac{1}{N(a')}, \ldots, 1\right\}$$

with the property that if $\alpha = \frac{k}{N(a)}$ then, under the refinement $N(a') = M \cdot N(a)$, we have

$$\mathbf{U}_\mu(\alpha, a, e, a', e') = \tfrac{k \cdot M}{N(a')}.$$

Discussion:
This theorem ensures that the update process preserves the finite structure of probability values. In essence, the information contained in the lower-capacity state is embedded without loss into the higher-capacity state.

A.2 Refinement and Embedding Theorems

Theorem A.3 (Capacity Refinement Theorem)

Statement:
Let a be an observer state with capacity $N(a)$ and a' a refined state with capacity $N(a') = M \cdot N(a)$ for some integer M. Then every structure defined over $\mathrm{Increments}(a)$—including probability measures, VOID Groups, and the discrete metric—is isomorphically embedded in the corresponding structure over $\mathrm{Increments}(a')$ via the mapping

$$\tfrac{k}{N(a)} \mapsto \tfrac{k \cdot M}{N(a')}.$$

Discussion:
This theorem formalizes the idea that increasing capacity (i.e., refining the system) is an information-preserving process. All algebraic and geometric structures defined in a lower-capacity observer state continue to exist and can be identified within a higher-capacity framework.

Corollary A.3.1 (Embedding of Classical Finite Structures)

Statement:
Any finite-dimensional vector space or finite field (as classically defined) that can be represented over a finite subfield of $\mathbb{Q}$ can be embedded into the capacity-based system by choosing an appropriate observer state with sufficiently high capacity.
Discussion:
This corollary connects the capacity-based system to classical finite algebraic structures, reinforcing the view that our model generalizes and refines familiar concepts from classical algebra.

A.3 Stability and Convergence Theorems

Theorem A.4 (Stability Preservation Under Refinement)

Statement:
Let x be a pattern that is stable over a region R at observer state a, i.e.,

$$\forall (a,e) \in R, \quad \left|\mu(a,e)(x) - \mu(a_0,e_0)(x)\right| \leq \varepsilon_\mu,$$

and similarly for its features. Then, if a' is a refinement of a (with $N(a') = M \cdot N(a)$), there exists a corresponding stability threshold ε'_μ (typically proportional to ε_μ) such that x remains stable in R when measured in a'.
Discussion:
This theorem indicates that the notion of stability is preserved—or even improved—under refinement. The pattern's behavior remains consistent when the system's resolution increases, supporting the robustness of the discrete "points" in the configuration space.

Theorem A.5 (Finite Approximation Theorem)

Statement:
For any classical property Prop (e.g., the value of a function or a geometric relation) and any desired error bound ε, there exists an observer state a' with capacity $N(a')$ such that the finite approximation of Prop in the system is within ε of the classical value, i.e.,

$$\left|\mathrm{Prop}_{\text{classical}} - \mathrm{Prop}_{\text{finite}}(a')\right| < \varepsilon.$$

Discussion:
While our system is entirely finite, this theorem guarantees that it can approximate classical (potentially continuous) properties as closely as desired. The approximation is achieved by choosing a sufficiently high capacity, demonstrating a kind of convergence through finite refinement.

A.4 Metric and Topological Theorems

Theorem A.6 (Discrete Metric Properties)

Statement:
Define the discrete distance between two patterns x and y at joint state (a,e) by

$$\mathrm{Dist}(a,e,x,y) = N(a) \sum_{g \in \mathrm{CommonFeats}(x,y)} \left| p_x(a,e)(g) - p_y(a,e)(g) \right|.$$

Then, for every fixed (a,e):

1. **Non-negativity:** $\mathrm{Dist}(a,e,x,y) \geq 0$,
2. **Identity:** $\mathrm{Dist}(a,e,x,y) = 0$ if and only if $x = y$,
3. **Symmetry:** $\mathrm{Dist}(a,e,x,y) = \mathrm{Dist}(a,e,y,x)$,
4. **Triangle Inequality:** For any three patterns x, y, and z,

$$\mathrm{Dist}(a,e,x,z) \leq \mathrm{Dist}(a,e,x,y) + \mathrm{Dist}(a,e,y,z).$$

Discussion:
This theorem verifies that the discrete distance function is indeed a metric. Although the metric space is finite, these properties mirror those of classical metric spaces, allowing the application of many standard techniques from metric topology in a fully combinatorial setting.

Theorem A.7 (Combinatorial Jordan Curve Theorem for Finite Graphs)

Statement:
Let L be a simple polygonal loop in the finite metric space of stable patterns (i.e., a closed cycle where each edge has a unit distance). Then L partitions the graph into two disjoint regions: an "inside" and an "outside," such that every pattern not on L belongs to exactly one of these regions.
Discussion:
While the classical Jordan Curve Theorem applies to continuous curves in the plane, this combinatorial version asserts an analogous partitioning property in the finite context. It lays a foundation for understanding topological concepts (such as connectedness and boundaries) within our discrete framework.

A.5 Error Control and Epsilon–Delta Theorems

Theorem A.8 (Finite Epsilon–Delta Theorem)

Statement:
For any property Prop and any discrete error bound $\varepsilon = \frac{\Delta}{N(a)}$ (with $\Delta \in \mathbb{N}$), there exists an observer state a' with capacity $N(a')$ (refining a) such that the error in approximating Prop satisfies

$$\left|\mathrm{Prop}_{\text{approx}}(a') - \mathrm{Prop}_{\text{ideal}}\right| < \frac{\Delta}{N(a')}.$$

Discussion:
This theorem recasts the classical epsilon–delta condition in finite terms. By choosing an observer state with sufficiently high capacity, one ensures that any desired level of approximation is achieved, and the error can be controlled in an explicit, integer-based manner.

Theorem A.9 (Error Composition Theorem)

Statement:
Assume that two operations yield errors ε_1 and ε_2 respectively. Then the composite operation (such as the sum or product) has an error bounded by a function $f(\varepsilon_1, \varepsilon_2)$, which is also of the form $\frac{\Delta}{N(a)}$ for some $\Delta \in \mathbb{N}$.
Discussion:
This theorem explains how errors propagate through composite operations. It ensures that the overall error remains within a finite bound and can be explicitly calculated, thereby reinforcing the robustness of our finite approximation techniques.

Concluding Remarks

The theorems presented in this appendix serve several purposes:

- They guarantee closure and consistency within the finite framework, ensuring that every operation is well-defined.
- They establish the preservation of structure under refinement, so that increasing capacity does not lead to loss of information.
- They formalize stability and approximation, showing that the system can emulate classical continuous behavior arbitrarily closely using only finite methods.
- They provide metric and topological underpinnings, linking discrete geometry to well-known results in classical topology.

- Finally, they offer rigorous error control that is essential both for theoretical clarity and practical computation.

Together, these results complete the mathematical foundation of the capacity-based, strictly finite system, bridging finitist philosophy with modern computational practices.