Chapter 9: Formal Axiomatic Summary

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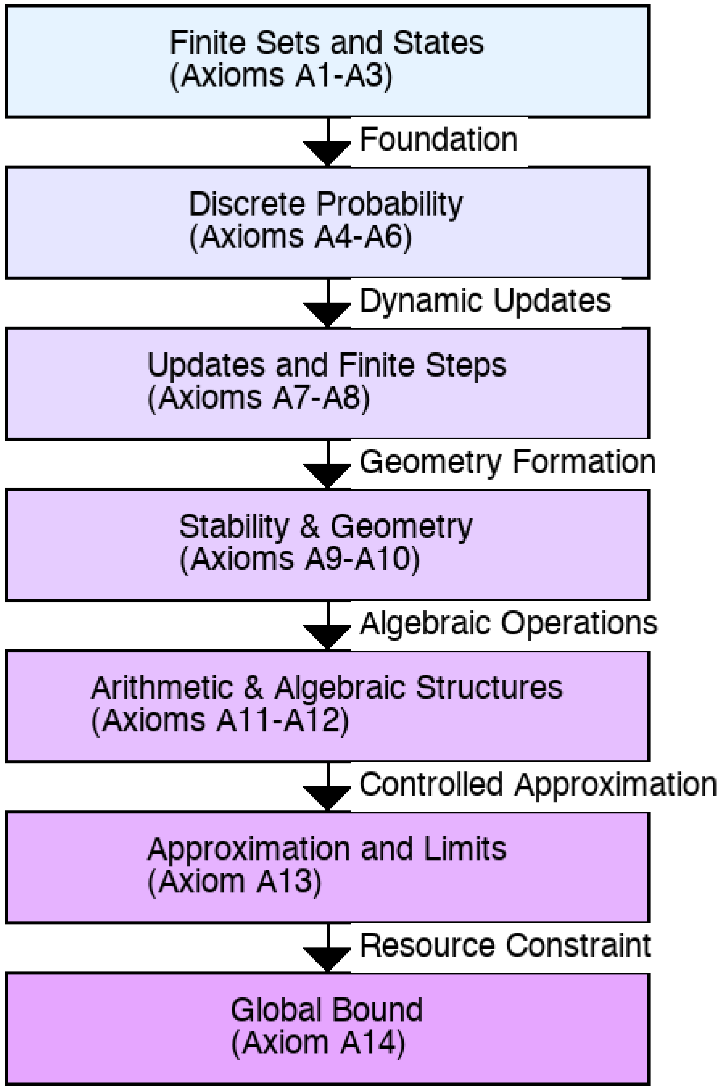
1 Introduction

Across the previous sections, we established a complete finite-capacity system for probabilities, geometry, algebra, and error control—without using infinite sets or continuous real numbers. In this chapter we summarize the core axioms (A1 through A14) that form the foundation of our grains-coded (finite) model. Each axiom is stated, explained, and linked to practical implementation aspects, thereby serving as a "contract" that guarantees our system remains logically consistent and fully finite.

The axioms ensure that:

- All sets (e.g., observer states, environment states, patterns, features) are finite.
- Numbers and probabilities are represented as discrete fractions $\frac{k}{N(a)}$, where the capacity $N(a)$ controls the resolution.
- Updates, refinements, and algebraic operations are carried out in exact, integer-based steps.
- Resource constraints (via a global bound Ω) are explicitly incorporated.

This summary also serves to highlight the theoretical soundness of our approach while bridging it to real-world, implementable code.



2 Sets and States

(Ax1) Finite Observer and Environment States:

- The set $\mathrm{ObsState}$ of observer states and the set $\mathrm{EnvState}$ of environment states are finite.
- This ensures that the system can be fully enumerated and that no infinite loops

occur.

(Ax2) Capacity Function:

$$V : \mathrm{ObsState} \to \mathbb{N}_{\geq 1}$$

- Each observer state a is assigned a capacity $V(a)$ (or resolution index).
- Higher capacity means more granular (finer) probability increments.

(Ax3) Joint States: The joint state is defined as

$$\mathrm{JointState} = \mathrm{ObsState} \times \mathrm{EnvState}.$$

- Since both factors are finite, the Cartesian product is finite.

3 Probability Assignments

(Ax4) Finite Patterns:

- The set $\mathrm{Pattern}$ is finite.
- Each pattern can be thought of as a discrete event label or shape.

(Ax5) Discrete Probability Measures: For every observer state a, environment state e, and pattern x:

$$\mu(a,e)(x) \in \left\{0, \tfrac{1}{N(a)}, \ldots, 1\right\}.$$

Similarly, feature distributions $p_x(a,e)(g)$ are in the same set.

(Ax6) Consistency Under Environment Changes: When the environment changes from e to e', all probability measures and feature distributions remain in their discrete form and update accordingly.

4 Updates and Finite Steps

(Ax7) Finite Update Functions: Define update functions (e.g., $\mathbf{U}_\mu$) that map grains-coded probabilities from one joint state (a,e) to another (a',e'):

$$\mathbf{U}_\mu : \{0, \tfrac{1}{N(a)}, \ldots, 1\} \times \mathrm{ObsState} \times \mathrm{EnvState} \times \mathrm{ObsState} \times \mathrm{EnvState} \to \{0, \tfrac{1}{N(a')}, \ldots, 1\}.$$

- Each update is a discrete, local step—there is no infinite recursion.

5 Capacity Refinement

(Ax8) Capacity Refinement: If an observer moves from state a to a higher capacity state a', then:

$$V(a') > V(a) \quad \Longrightarrow \quad N(a') \geq M \cdot N(a)$$

for some integer $M \geq 1$.

- Old grains-coded fractions embed exactly into the new scale:

$$\frac{k}{N(a)} \mapsto \frac{k \cdot M}{N(a')}.$$

6 Stability and Geometry

(Ax9) Stability: A pattern x is stable over a region R if:

$$\mathrm{Stable}(x, R, \varepsilon_\mu, \varepsilon_p) \iff \exists (a_0, e_0) \in R \text{ such that } \forall (a, e) \in R,$$

$$|\mu(a, e)(x) - \mu(a_0, e_0)(x)| \leq \varepsilon_\mu \quad \text{and} \quad |p_x(a, e)(g) - p_x(a_0, e_0)(g)| \leq \varepsilon_p \quad \forall\, g \in \mathrm{Feat}(x).$$

- This means the pattern and its feature distribution vary only minimally across R.

(Ax10) Discrete Distance: Define a distance function between patterns x and y at state (a, e):

$$\mathrm{Dist}(a, e, x, y) = N(a) \sum_{g \in \mathrm{CommonFeats}(x,y)} |p_x(a, e)(g) - p_y(a, e)(g)|\,.$$

- This discrete metric is finite, symmetric, and satisfies the triangle inequality.

7 Arithmetic and Algebraic Structures

(Ax11) Finite Arithmetic: The grains-coded fractions $\frac{k}{N(a)}$ form the basis for all arithmetic operations. By unifying denominators (through capacity refinement), addition, subtraction, multiplication, and division are performed exactly as with rational numbers.

(Ax12) Vector Spaces and Matrices: Tuples of grains-coded fractions form finite-dimensional vector spaces, and matrices constructed from these increments follow standard operations (addition, multiplication) in a finite, discrete setting.

8 Approximation and Limits

(Ax13) Epsilon-Style Approximation: For any desired precision $\varepsilon > 0$, there exists an observer state a' with a sufficiently large $N(a')$ such that for any property, the error satisfies:

$$\mathrm{Err}(\mathrm{Prop}, a, e) = \frac{\Delta}{N(a')} < \varepsilon.$$

- No actual limit process is needed; instead, precision is increased in discrete steps.

9 Finite Complexity

(Ax14) Global Capacity Bound: There exists a global bound Ω such that for every observer state a:

$$N(a) \leq \Omega.$$

- This ensures that no matter how much capacity is refined, all computations remain within finite, implementable limits.

10 Putting It All Together

The 14 axioms above define a complete finite, grains-coded mathematical system:

1. **Finite Sets (Ax1–Ax3):** Observer states, environment states, and their Cartesian product (joint states) are finite.
2. **Discrete Probability (Ax4–Ax6):** Patterns and features have probabilities expressed in discrete increments $\frac{k}{N(a)}$.
3. **Finite Updates (Ax7):** Transition functions update probabilities in exact, discrete steps.
4. **Capacity Refinement (Ax8):** Higher capacity allows finer increments, with old fractions embedding exactly into the new scale.
5. **Stability and Geometry (Ax9–Ax10):** Stable patterns and discrete distance metrics lead to well-defined geometric constructs.
6. **Algebraic Structures (Ax11–Ax12):** Classical algebraic operations (groups, fields, vector spaces, matrices) are recreated exactly within this finite framework.
7. **Approximation (Ax13):** For any desired error threshold, increasing capacity yields arbitrarily close approximations—without infinite processes.

8. **Finite Complexity (Ax14):** A global bound Ω ensures that all operations remain within physical and computational limits.

Final Remarks: By replacing infinite sets and floating-point arithmetic with exact, discrete grains-coded increments, our framework provides a resource-aware, fully finite approach to mathematics. Every procedure—from probability assignments to geometric constructions and algebraic operations—is exact and transparent. This axiomatic summary serves as both a theoretical guarantee and a practical blueprint for implementing robust, predictable systems in IT and beyond.