Markov Networks

Or undirected graphical models

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https://probabll.github.io

Outline and goals

Module 2 introduces *Markov networks* (MNs; Chapter 4).

ILOs After this module the student

- can map MNs to distributions and vice-versa;
- can reason using an MN;
- recognises the flow of probabilistic influence in MNs;
- recognises independence and separation.

Textbook for this course: Koller and Friedman [1].

Overview of Module 2

HC2b: MNs – semantics.

HC3a: MNs – reasoning.

LC3: MNs in code.

HC3b: MNs – influence.

WC3: exercises (semantics, reasoning and influence).

Table of contents

Semantics

Outline for this section

We will introduce *Markov Networks* (MNs; Chapter 4) via an example from the textbook (the *Misconception* example from Section 4.1).

After presenting the complete example, we will introduce MNs in full generality.

The *Misconception* example (example 3.8 and section 4.1)

Four students (Alice, Bob, Charles and Debbie) work together in pairs.

Alice and Charles are not on speaking terms, nor are Bob and Debbie. Only the following pairs study together:

- Alice and Bob
- Bob and Charles
- Charles and Debbie
- Debbie and Alice

In class, the professor accidentally misspoke, giving rise to a misconception. After class, the students may figure out the problem and share their newfound understanding with their study partners.

The rv A denotes whether Alice has the misconception a^1 or not a^0. We use B, C, D analogously for the other students.

Conditional Independences

Because Alice and Charles never speak, we have that A and C are conditionally independent given B and D. $A \perp C \mid B, D$

Similarly, B and D are conditionally independent given A and C.
$B \perp D \mid A, C$

Can we represent these two statements with a BN?

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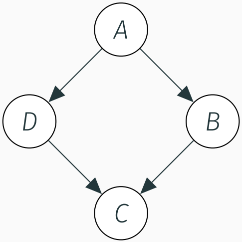

Attempt 1:

- $A \perp C \mid B, D$ ✓
- but D, B are independent given A alone ✗
- and if we are given A, C then D, B become dependent ✗

Conditional Independences

Because Alice and Charles never speak, we have that A and C are conditionally independent given B and D. $A \perp C \mid B, D$

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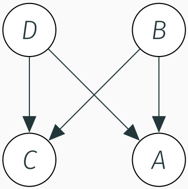

Attempt 2:

- $A \perp C \mid B, D$ ✓
- but now B and D are marginally independent ✗

Undirected Interaction

We want to represent two conditional independences $A \perp C \mid B, D$ and $B \perp D \mid A, C$ **and no other**.

It turns out that no BN structure can do precisely that. Instead,

- we need to represent the interaction between pairs
- and we have no reason to ascribe direction to any interaction

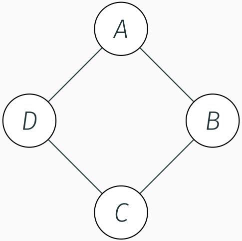

Figure 1: The MN graph for the *Misconception* example

We need an **undirected** representation of the relevant interactions!

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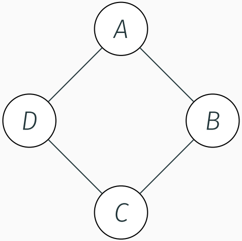
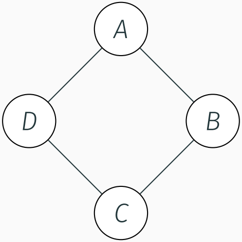


Figure 1: The MN graph for the *Misconception* example

We need an **undirected** representation of the relevant interactions! But how can we parameterise a joint distribution coherent with this view?

Factors for the *Misconception* example (Fig 4.1 of textbook)

A factor ϕ with scope X, Y represents the undirected *affinity* between possible outcomes of X and Y.



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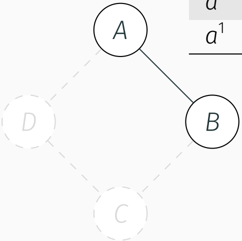

$\phi_1(A, B)$

a^0	b^0	30
a^0	b^1	5
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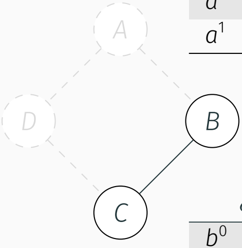

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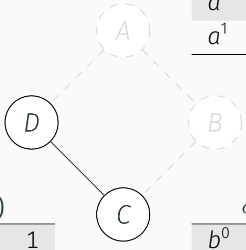

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Factors for the *Misconception* example (Fig 4.1 of textbook)

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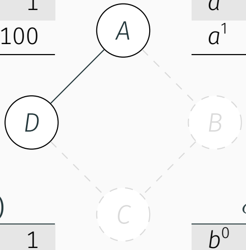

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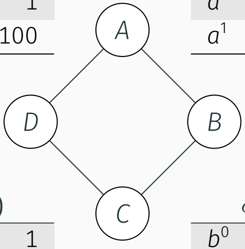

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Figure 2: Factors for the *Misconception* example

A factor ϕ is a function from the joint outcome space of a set D of rvs, which we call the factor's *scope*, to $\mathbb{R}_{\geq 0}$. Factors represent affinity amongst interacting variables.

Think of a factor ϕ as something like a python function with named arguments, the order of the arguments doesn't matter because they are 'named' with the names of the rvs in Scope[ϕ].

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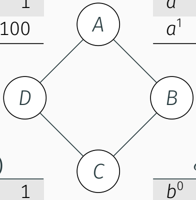

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- Bob is more strongly inclined to agree with Charles than with Alice.
- Charles and Debbie on the other hand are strongly inclined to disagreement.

Think of a factor ϕ as something like a python function with named arguments, the order of the arguments doesn't matter because they are 'named' with the names of the rvs in Scope[ϕ].

How can we define these joint probabilities?

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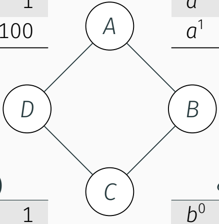

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1. $P(A = a^0, B = b^0, C = c^0, D = d^0) =$
2. $P(A = a^1, B = b^0, C = c^0, D = d^1) =$

A Representation of our Uncertainty over A, B, C, D (section 4.2.1)

A *joint probability distribution* is a means to represent our uncertainty over the *possible* assignments of the rvs of interest.

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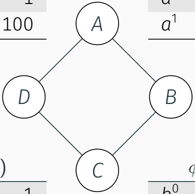

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With its undirected graph and factors, an MN identifies one such distribution.

The probability of any one of the possible outcomes of (A, B, C, D) in *Misconception* **is proportional to** $\phi_1(A, B) \times \phi_2(B, C) \times \phi_3(C, D) \times \phi_4(D, A)$.

Joint Distribution for the Misconception Example (fig 4.2)

In an MN, we define a distribution by taking the **product of the local factors** and then **normalising it** to define a valid joint distribution:

A	B	C	D	Unnormalised product of factors	$P(A, B, C, D)$
a^0	b^0	c^0	d^0	300 000	0.04
a^0	b^0	c^0	d^1	300 000	0.04
a^0	b^0	c^1	d^0	300 000	0.04
a^0	b^0	c^1	d^1	30	4.1×10^{-6}
a^0	b^1	c^0	d^0	500	6.9×10^{-5}
a^0	b^1	c^0	d^1	500	6.9×10^{-5}
a^0	b^1	c^1	d^0	5 000 000	0.69
a^0	b^1	c^1	d^1	500	6.9×10^{-5}
a^1	b^0	c^0	d^0	100	1.4×10^{-5}
a^1	b^0	c^0	d^1	1 000 000	0.14
a^1	b^0	c^1	d^0	100	1.4×10^{-5}
a^1	b^0	c^1	d^1	100	1.4×10^{-5}
a^1	b^1	c^0	d^0	10	1.4×10^{-6}
a^1	b^1	c^0	d^1	100 000	0.014
a^1	b^1	c^1	d^0	100 000	0.014
a^1	b^1	c^1	d^1	100 000	0.014
Sum				7 201 840	1.0

Table 1: Joint Distribution over $\mathrm{Val}(A) \times \mathrm{Val}(B) \times \mathrm{Val}(C) \times \mathrm{Val}(D)$

Compactness

The BN indeed *identifies* the joint distribution, but not by independently representing one probability value per joint outcome.

The probability value of any one of the joint outcomes is expressed via a product of local factors for directly interacting variables.

$\phi_4(D, A)$

d^0	a^0	100
d^0	a^1	1
d^1	a^0	1
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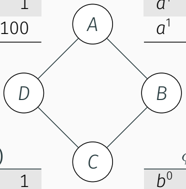

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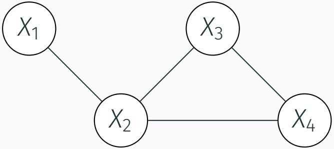
b^1	c^1	100

If each of N rvs takes on one of K values, we infer all K^N joint probabilities from the affinities of N K-by-K-dimensional factors.

While in this example (K=2, N=4), K^N happens to be equal to NK^2, for larger K and N the MN is very compact.

The notion of *directly interacting variables* corresponds to what is known as **complete subgraphs**. The factors in the MN, by definition, have complete subgraphs as their scope.



Some PGM jargon: we say that a factor 'covers' the rvs in its scope, and we also say that a factor 'covers' the edges that connect any two rvs in its scope.

Definition 2.13. In a complete subgraph, also known as *clique*, every two nodes are connected by some edge.

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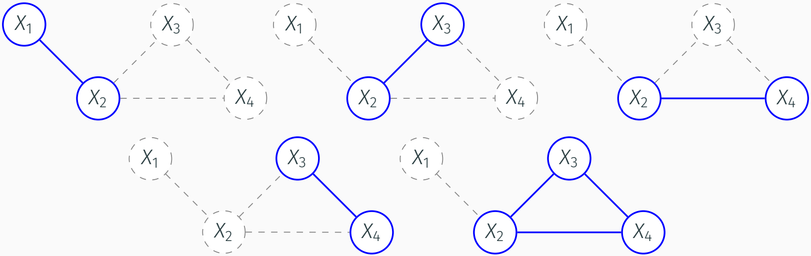

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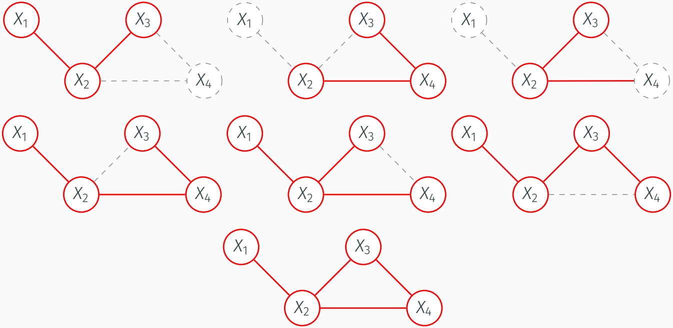

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Markov Network (MN) (section 4.2.2, def. 4.3 and 4.4)

An MN is a graph $\mathcal{H}$ and a collection Φ of factors such that:

- $\mathcal{H}$ is an undirected graph;
- nodes in $\mathcal{H}$ represent the rvs $X_1, \ldots, X_n$;
- the edges in $\mathcal{H}$ indicate direct interaction between rvs;
- the scope $\mathbf{D}_i$ of a factor $\phi_i \in \Phi$ is a complete subgraph of $\mathcal{H}$;
- Φ covers all nodes and all edges in the graph.

Z is also known as the *partition function* (a term that comes from statistical physics).

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- Φ covers all nodes and all edges in the graph.

The MN represents a joint distribution by normalisation of the product of factors Φ:

$$P_{\Phi}(X_1, \ldots, X_n) = \frac{1}{Z} \prod_{\phi_i \in \Phi} \phi_i(\mathbf{D}_i) \tag{1}$$

where $\tilde{P}_{\Phi}(X_1, \ldots, X_n) = \prod_{\phi_i \in \Phi} \phi_i(\mathbf{D}_i)$ is an unnormalised measure, and $Z = \sum_{X_1, \ldots, X_n} \tilde{P}_{\Phi}(X_1, \ldots, X_n)$ is the constant that normalises it.

Z is also known as the *partition function* (a term that comes from statistical physics).

Graphs and Distributions (1/4)

With its undirected graph $\mathcal{H}$ and a collection of factors Φ, an MN identifies a factorised distribution P_Φ.

In the *Misconception* example:

$\phi_4(D, A)$

d^0	a^0	100
d^0	a^1	1
d^1	a^0	1
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$\phi_1(A, B)$

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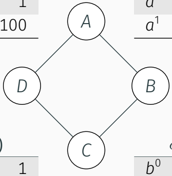

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Φ	$\mathbf{D}_i = \text{Scope}[\phi_i]$
ϕ_1	$\{A, B\}$
ϕ_2	$\{B, C\}$
ϕ_3	$\{C, D\}$
ϕ_4	$\{D, A\}$

Factors and their scope

$$P_\Phi(A, B, C, D) = \frac{1}{Z} \prod_{\phi_i \in \Phi} \phi_i(\mathbf{D}_i) = \frac{1}{Z} \phi_1(A, B)\phi_2(B, C)\phi_3(C, D)\phi_4(D, A)$$

This is similar for BNs, whose DAG and CPDs identify a factorised distribution.

Graphs and Distributions (2/4)

Since we defined factors as functions whose scopes are complete subgraphs (any two rvs in the scope must be connected by an edge), a collection of factors Φ identifies a graph $\mathcal{H}$ uniquely.

Can you identify the graph in each case?

Misconception	
Φ	$\mathbf{D}_i = \text{Scope}[\phi_i]$
ϕ_1	$\{A, B\}$
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Common Friend	
Φ	$\mathbf{D}_i = \text{Scope}[\phi_i]$
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ϕ_2	$\{A, C\}$
ϕ_3	$\{B, C\}$
ϕ_4	$\{C, D, E\}$

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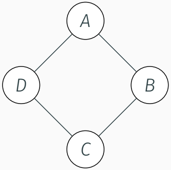

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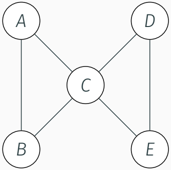

But if we only know the graph $\mathcal{H}$, then the factorisation is **not unique**!

Can you tell why?
(Hint: by definition, the scope of factor is a complete subgraph)

Graphs and Distributions (3/4)

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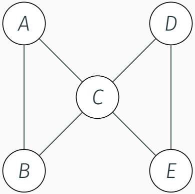
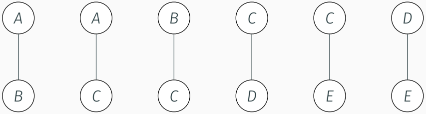
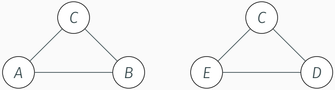
Consider this graph



One set of complete subgraphs for A, B, C, D:



another set:



and combinations of these, and others...

There are many sets of complete subgraphs (cliques) that cover all nodes and edges in $\mathcal{H}$. Each set corresponds to a factorisation.

There are differently-factorised distributions that are consistent with $\mathcal{H}$.

This is different in BNs, whose DAG implies a unique factorisation.

Then the question to ask is: if $\mathcal{H}$ does not imply a factorisation, what then do all distributions that are 'compatible' with $\mathcal{H}$ have in common?

There are differently-factorised distributions that are consistent with $\mathcal{H}$.

This is different in BNs, whose DAG implies a unique factorisation.

Then the question to ask is: if $\mathcal{H}$ does not imply a factorisation, what then do all distributions that are 'compatible' with $\mathcal{H}$ have in common?

$\mathcal{H}$ implies a set of conditional independence statements and *that* is what all distributions that factorise over $\mathcal{H}$ have in common.

Summary

- An MN uses an undirected graph $\mathcal{H}$ to encode statements of (conditional) independence without ascribing directionality to interactions amongst rvs.
- To parameterise the MN, we specify a collection of factors (such as tabular factors) whose scopes are complete subgraphs.
- Unlike we did for BNs, we cannot read a unique factorisation readily from the MN structure.
- But, having a collection of factors, we can identify the graph. With the graph and its factors Φ, an MN is a complete representation of a joint distribution P_Φ.

What's Next?

LC3: MNs in code.

HC3ab: MNs – reasoning and influence.

WC3: exercises (semantics, reasoning and influence).

Reasoning

Outline for this section

We begin by restating fundamental results of probability theory using general *factors* (as opposed to CPDs).

We then exploit those results and the MN representation to respond to queries about subsets of random variables and sets of observations.

We will start to see a connection between probabilistic inference and manipulation of the MN structure. This connection will be exploited more fully later in the course.

A Representation of our Uncertainty over A, B, C, D

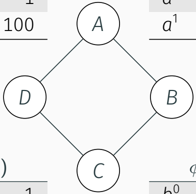
With its undirected graph and factors, an MN identifies a *joint probability distribution* over the rvs of interest.

$\phi_4(D, A)$

d^0	a^0	100
d^0	a^1	1
d^1	a^0	1
d^1	a^1	100

$\phi_1(A, B)$

a^0	b^0	30
a^0	b^1	5
a^1	b^0	1
a^1	b^1	10



$\phi_3(C, D)$

c^0	d^0	1
c^0	d^1	100
c^1	d^0	100
c^1	d^1	1

$\phi_2(B, C)$

b^0	c^0	100
b^0	c^1	1
b^1	c^0	1
b^1	c^1	100

The probability of any one of the possible outcomes of (A, B, C, D) in *Misconception* **is proportional to** $\phi_1(A, B) \times \phi_2(B, C) \times \phi_3(C, D) \times \phi_4(D, A)$.

Probability Calculus

Marginalisation

$$P(A) = \sum_B P(A, B)$$

These results extend to the cases where A and/or B are sets of rvs.

Probability Calculus

Marginalisation

$$P(A) = \sum_B P(A, B) = \sum_B \frac{1}{Z} \tilde{P}(A, B)$$

These results extend to the cases where A and/or B are sets of rvs.

Probability Calculus

Marginalisation

$$P(A) = \sum_B P(A, B) = \sum_B \frac{1}{Z} \tilde{P}(A, B) = \frac{1}{Z} \underbrace{\sum_B \tilde{P}(A, B)}_{\tilde{P}(A)} \tag{2}$$

These results extend to the cases where A and/or B are sets of rvs.

Probability Calculus

Marginalisation

$$P(A) = \sum_B P(A,B) = \sum_B \frac{1}{Z}\tilde{P}(A,B) = \frac{1}{Z}\underbrace{\sum_B \tilde{P}(A,B)}_{\tilde{P}(A)} \tag{2}$$

Conditioning

$$P(B|A) = \frac{P(A,B)}{P(A)}$$

These results extend to the cases where A and/or B are sets of rvs.

Probability Calculus

Marginalisation

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Conditioning

$$P(B|A) = \frac{P(A, B)}{P(A)} = \frac{\frac{1}{Z}\tilde{P}(A, B)}{\frac{1}{Z}\tilde{P}(A)}$$

These results extend to the cases where A and/or B are sets of rvs.

Probability Calculus

Marginalisation

$$P(A) = \sum_B P(A,B) = \sum_B \frac{1}{Z}\tilde{P}(A,B) = \frac{1}{Z}\underbrace{\sum_B \tilde{P}(A,B)}_{\tilde{P}(A)} \tag{2}$$

Conditioning

$$P(B|A) = \frac{P(A,B)}{P(A)} = \frac{\frac{1}{Z}\tilde{P}(A,B)}{\frac{1}{Z}\tilde{P}(A)} = \frac{\tilde{P}(A,B)}{\tilde{P}(A)} \tag{3}$$

These results extend to the cases where A and/or B are sets of rvs.

Let's compute some marginals for the *Misconception* example

A	B	C	D	$\tilde{P}(A, B, C, D)$
a^0	b^0	c^0	d^0	300 000
a^0	b^0	c^0	d^1	300 000
a^0	b^0	c^1	d^0	300 000
a^0	b^0	c^1	d^1	30
a^0	b^1	c^0	d^0	500
a^0	b^1	c^0	d^1	500
a^0	b^1	c^1	d^0	5 000 000
a^0	b^1	c^1	d^1	500
a^1	b^0	c^0	d^0	100
a^1	b^0	c^0	d^1	1 000 000
a^1	b^0	c^1	d^0	100
a^1	b^0	c^1	d^1	100
a^1	b^1	c^0	d^0	10
a^1	b^1	c^0	d^1	100 000
a^1	b^1	c^1	d^0	100 000
a^1	b^1	c^1	d^1	100 000
	Sum			7 201 840

1. $P(A = a^0) =$
2. $P(A = a^0, B = b^0) =$

Let's compute some marginals for the *Misconception* example

A	B	C	D	$\tilde{P}(A, B, C, D)$
a^0	b^0	c^0	d^0	300 000
a^0	b^0	c^0	d^1	300 000
a^0	b^0	c^1	d^0	300 000
a^0	b^0	c^1	d^1	30
a^0	b^1	c^0	d^0	500
a^0	b^1	c^0	d^1	500
a^0	b^1	c^1	d^0	5 000 000
a^0	b^1	c^1	d^1	500
a^1	b^0	c^0	d^0	100
a^1	b^0	c^0	d^1	1 000 000
a^1	b^0	c^1	d^0	100
a^1	b^0	c^1	d^1	100
a^1	b^1	c^0	d^0	10
a^1	b^1	c^0	d^1	100 000
a^1	b^1	c^1	d^0	100 000
a^1	b^1	c^1	d^1	100 000
	Sum			7 201 840

1. $P(A = a^0) = \frac{\text{yellow}}{\text{blue}} = \frac{5901530}{7201840} \approx 0.8194$
2. $P(A = a^0, B = b^0) =$

Let's compute some marginals for the *Misconception* example

A	B	C	D	$\tilde{P}(A, B, C, D)$
a^0	b^0	c^0	d^0	300 000
a^0	b^0	c^0	d^1	300 000
a^0	b^0	c^1	d^0	300 000
a^0	b^0	c^1	d^1	30
a^0	b^1	c^0	d^0	500
a^0	b^1	c^0	d^1	500
a^0	b^1	c^1	d^0	5 000 000
a^0	b^1	c^1	d^1	500
a^1	b^0	c^0	d^0	100
a^1	b^0	c^0	d^1	1 000 000
a^1	b^0	c^1	d^0	100
a^1	b^0	c^1	d^1	100
a^1	b^1	c^0	d^0	10
a^1	b^1	c^0	d^1	100 000
a^1	b^1	c^1	d^0	100 000
a^1	b^1	c^1	d^1	100 000
	Sum			7 201 840

1. $P(A = a^0) = \frac{\text{yellow}}{\text{blue}} = \frac{5901530}{7201840} \approx 0.8194$
2. $P(A = a^0, B = b^0) =$

Let's compute some marginals for the *Misconception* example

A	B	C	D	$\tilde{P}(A, B, C, D)$
a^0	b^0	c^0	d^0	300 000
a^0	b^0	c^0	d^1	300 000
a^0	b^0	c^1	d^0	300 000
a^0	b^0	c^1	d^1	30
a^0	b^1	c^0	d^0	500
a^0	b^1	c^0	d^1	500
a^0	b^1	c^1	d^0	5 000 000
a^0	b^1	c^1	d^1	500
a^1	b^0	c^0	d^0	100
a^1	b^0	c^0	d^1	1 000 000
a^1	b^0	c^1	d^0	100
a^1	b^0	c^1	d^1	100
a^1	b^1	c^0	d^0	10
a^1	b^1	c^0	d^1	100 000
a^1	b^1	c^1	d^0	100 000
a^1	b^1	c^1	d^1	100 000
	Sum			7 201 840

1. $P(A = a^0) = \frac{\text{yellow}}{\text{blue}} = \frac{5901530}{7201840} \approx 0.8194$
2. $P(A = a^0, B = b^0) = \frac{\text{dark yellow}}{\text{blue}} = \frac{900030}{7201840} \approx 0.1249$

Let's compute some conditionals for the *Misconception* example

A	B	C	D	$\tilde{P}(A, B, C, D)$
a^0	b^0	c^0	d^0	300 000
a^0	b^0	c^0	d^1	300 000
a^0	b^0	c^1	d^0	300 000
a^0	b^0	c^1	d^1	30
a^0	b^1	c^0	d^0	500
a^0	b^1	c^0	d^1	500
a^0	b^1	c^1	d^0	5 000 000
a^0	b^1	c^1	d^1	500
a^1	b^0	c^0	d^0	100
a^1	b^0	c^0	d^1	1 000 000
a^1	b^0	c^1	d^0	100
a^1	b^0	c^1	d^1	100
a^1	b^1	c^0	d^0	10
a^1	b^1	c^0	d^1	100 000
a^1	b^1	c^1	d^0	100 000
a^1	b^1	c^1	d^1	100 000
	Sum			7 201 840

1. $P(B = b^0 | A = a^0) =$
2. $P(B = b^1 | A = a^0) =$

Let's compute some conditionals for the *Misconception* example

A	B	C	D	$\tilde{P}(A, B, C, D)$
a^0	b^0	c^0	d^0	300 000
a^0	b^0	c^0	d^1	300 000
a^0	b^0	c^1	d^0	300 000
a^0	b^0	c^1	d^1	30
a^0	b^1	c^0	d^0	500
a^0	b^1	c^0	d^1	500
a^0	b^1	c^1	d^0	5 000 000
a^0	b^1	c^1	d^1	500
a^1	b^0	c^0	d^0	100
a^1	b^0	c^0	d^1	1 000 000
a^1	b^0	c^1	d^0	100
a^1	b^0	c^1	d^1	100
a^1	b^1	c^0	d^0	10
a^1	b^1	c^0	d^1	100 000
a^1	b^1	c^1	d^0	100 000
a^1	b^1	c^1	d^1	100 000
Sum				7 201 840

1. $P(B = b^0 | A = a^0) = \frac{\text{dark yellow}}{\text{light/dark yellow}} = \frac{900030}{5901530} \approx 0.1525$
2. $P(B = b^1 | A = a^0) = \frac{\text{light yellow}}{\text{light/dark yellow}} = \frac{900030}{5901530} \approx 0.8475$

Factor Operations

We can also reason using the MN representation, as opposed to the tabular view of the distribution.

For that, we need to learn more about how to manipulate factors:

- factor product
- factor normalisation
- factor reduction
- factor marginalisation (Chapter 9)

Let $\phi_1(X, Y)$ and $\phi_2(Y, Z)$ be two factors.

The **factor product** $\phi_1(X, Y) \times \phi_2(Y, Z)$ is the factor ψ defines as:

$$\psi(X, Y, Z) = \phi_1(X, Y) \times \phi_2(Y, Z) \tag{4}$$

The scope of the new factor is
$\text{Scope}[\psi] = \text{Scope}[\phi_1] \cup \text{Scope}[\phi_2] = \{X, Y, Z\}$.

This definition extends naturally to the case where X, Y, Z are replaced by disjoint sets of rvs.

Factor Product - Example 1/3

A	$\underline{B}$	ϕ_1
a^0	b^0	30
a^0	b^1	5
a^1	b^0	1
a^1	b^1	10

$\times$

$\underline{B}$	C	ϕ_2
b^0	c^0	100
b^0	c^1	1
b^1	c^0	1
b^1	c^1	100

Dark green (b^1), light green (b^0).

Factor Product - Example 1/3

A	$\underline{B}$	ϕ_1
a^0	b^0	30
a^0	b^1	5
a^1	b^0	1
a^1	b^1	10

$\times$

$\underline{B}$	C	ϕ_2
b^0	c^0	100
b^0	c^1	1
b^1	c^0	1
b^1	c^1	100

$=$

A	B	C	$\psi_{A,B,C}$
a^0	b^0	c^0	3000
a^0	b^0	c^1	30
a^0	b^1	c^0	5
a^0	b^1	c^1	500
a^1	b^0	c^0	100
a^1	b^0	c^1	1
a^1	b^1	c^0	10
a^1	b^1	c^1	1000

Dark green (b^1), light green (b^0).

Factor Product - Example 2/3

C	$\underline{D}$	ϕ_3
c^0	d^0	1
c^0	d^1	100
c^1	d^0	100
c^1	d^1	1

$\times$

$\underline{D}$	A	ϕ_4
d^0	a^0	100
d^0	a^1	1
d^1	a^0	1
d^1	a^1	100

Dark red (d^1), light red (d^0).

Factor Product - Example 2/3

C	$\underline{D}$	ϕ_3
c^0	d^0	1
c^0	d^1	100
c^1	d^0	100
c^1	d^1	1

$\times$

$\underline{D}$	A	ϕ_4
d^0	a^0	100
d^0	a^1	1
d^1	a^0	1
d^1	a^1	100

=

C	D	A	$\psi_{C,D,A}$
c^0	d^0	a^0	100
c^0	d^0	a^1	1
c^0	d^1	a^0	100
c^0	d^1	a^1	10000
c^1	d^0	a^0	10000
c^1	d^0	a^1	100
c^1	d^1	a^0	1
c^1	d^1	a^1	100

Dark red (d^1), light red (d^0).

Factor Product - Example 3/3

A	B	C	$\psi_{A,B,C}$
a^0	b^0	c^0	3000
a^0	b^0	c^1	30
a^0	b^1	c^0	5
a^0	b^1	c^1	500
a^1	b^0	c^0	100
a^1	b^0	c^1	1
a^1	b^1	c^0	10
a^1	b^1	c^1	1000

$\times$

C	D	A	$\psi_{C,D,A}$
c^0	d^0	a^0	100
c^0	d^0	a^1	1
c^0	d^1	a^0	100
c^0	d^1	a^1	10000
c^1	d^0	a^0	10000
c^1	d^0	a^1	100
c^1	d^1	a^0	1
c^1	d^1	a^1	100

Light yellow (a^0, c^0), dark yellow (a^0, c^1), light blue (a^1, c^0), dark blue (a^1, c^1).

Factor Product - Example 3/3

$\underline{A}$	B	$\underline{C}$	$\psi_{A,B,C}$
a^0	b^0	c^0	3000
a^0	b^0	c^1	30
a^0	b^1	c^0	5
a^0	b^1	c^1	500
a^1	b^0	c^0	100
a^1	b^0	c^1	1
a^1	b^1	c^0	10
a^1	b^1	c^1	1000

$\times$

$\underline{C}$	D	$\underline{A}$	$\psi_{C,D,A}$
c^0	d^0	a^0	100
c^0	d^0	a^1	1
c^0	d^1	a^0	100
c^0	d^1	a^1	10000
c^1	d^0	a^0	10000
c^1	d^0	a^1	100
c^1	d^1	a^0	1
c^1	d^1	a^1	100

$=$

$\underline{A}$	B	$\underline{C}$	D	$\psi_{A,B,C,D}$
a^0	b^0	c^0	d^0	300 000
a^0	b^0	c^0	d^1	300 000
a^0	b^0	c^1	d^0	300 000
a^0	b^0	c^1	d^1	30
a^0	b^1	c^0	d^0	500
a^0	b^1	c^0	d^1	500
a^0	b^1	c^1	d^0	5×10^6
a^0	b^1	c^1	d^1	500
a^1	b^0	c^0	d^0	100
a^1	b^0	c^0	d^1	1×10^6
a^1	b^0	c^1	d^0	100
a^1	b^0	c^1	d^1	100
a^1	b^1	c^0	d^0	10
a^1	b^1	c^0	d^1	100 000
a^1	b^1	c^1	d^0	100 000
a^1	b^1	c^1	d^1	100 000

Light yellow (a^0, c^0), dark yellow (a^0, c^1), light blue (a^1, c^0), dark blue (a^1, c^1).

Factor Normalisation

Let ϕ be a factor with scope X and $Z_\phi = \sum_{\mathsf{X}} \phi(\mathsf{X})$.

If $Z_\phi \in \mathbb{R}_{>0}$, we can obtain the normalised factor η:

$$\eta(\mathsf{X}) = \frac{1}{Z_\phi} \phi(\mathsf{X}) \tag{5}$$

A normalised factor may parameterise a joint probability distribution over X.

Factor Normalisation - *Misconception* Example

A	B	C	D	$\psi_{A,B,C,D} = \tilde{P}_{\Phi}$	$\eta_{A,B,C,D} = P_{\Phi}$
a^0	b^0	c^0	d^0	300 000	0.04
a^0	b^0	c^0	d^1	300 000	0.04
a^0	b^0	c^1	d^0	300 000	0.04
a^0	b^0	c^1	d^1	30	4.1×10^{-6}
a^0	b^1	c^0	d^0	500	6.9×10^{-5}
a^0	b^1	c^0	d^1	500	6.9×10^{-5}
a^0	b^1	c^1	d^0	5 000 000	0.69
a^0	b^1	c^1	d^1	500	6.9×10^{-5}
a^1	b^0	c^0	d^0	100	1.4×10^{-5}
a^1	b^0	c^0	d^1	1 000 000	0.14
a^1	b^0	c^1	d^0	100	1.4×10^{-5}
a^1	b^0	c^1	d^1	100	1.4×10^{-5}
a^1	b^1	c^0	d^0	10	1.4×10^{-6}
a^1	b^1	c^0	d^1	100 000	0.014
a^1	b^1	c^1	d^0	100 000	0.014
a^1	b^1	c^1	d^1	100 000	0.014
Sum				7 201 840	1.0

Factor Normalisation - Warning

While a normalised factor has the *form* of a joint distribution, not every normalised factor is coherent with a given Markov network!

For example, if we normalise $\psi_{A,B,C}$ the result 'looks' like a distribution over A, B, C:

A	B	C	$\psi_{A,B,C}$	$\eta_{A,B,C}$
a^0	b^0	c^0	3000	0.6458
a^0	b^0	c^1	30	0.0065
a^0	b^1	c^0	5	0.0011
a^0	b^1	c^1	500	0.1076
a^1	b^0	c^0	100	0.0215
a^1	b^0	c^1	1	0.0002
a^1	b^1	c^0	10	0.0021
a^1	b^1	c^1	1000	0.2152
	Sum		4646	1.0

While $\eta_{A,B,C}$ *can* parameterise *some distribution* over A and C, **it is not the marginal distribution** $P_\Phi(A, C)$ of the Misconception example.
Can you tell why?

Factor Normalisation - Warning

While a normalised factor has the *form* of a joint distribution, not every normalised factor is coherent with a given Markov network!

For example, if we normalise $\psi_{A,B,C}$ the result 'looks' like a distribution over A, B, C:

A	B	C	$\psi_{A,B,C}$	$\eta_{A,B,C}$
a^0	b^0	c^0	3000	0.6458
a^0	b^0	c^1	30	0.0065
a^0	b^1	c^0	5	0.0011
a^0	b^1	c^1	500	0.1076
a^1	b^0	c^0	100	0.0215
a^1	b^0	c^1	1	0.0002
a^1	b^1	c^0	10	0.0021
a^1	b^1	c^1	1000	0.2152
	Sum		4646	1.0

While $\eta_{A,B,C}$ *can* parameterise *some distribution* over A and C, **it is not the marginal distribution** $P_{\Phi}(A, C)$ of the Misconception example. Can you tell why?

In the Misconception MN, A and C depend on variables (D) and factors (ϕ_3 and ϕ_4) not yet accounted for in $\psi_{A,B,C}$.

Factor Reduction (definition 4.5)

Let $\phi(X, Y)$ be a factor, and $Y = y$ an assignment for $Y \in \text{Scope}[\phi]$. We define **the reduction of the factor ϕ to the context** $Y = y$, denoted $\phi[Y = y]$ and abbreviated to $\phi[y]$ when unambiguous, to be the factor:

$$\phi[y](x) = \phi(x, y)$$

The scope of the new factor is $\text{Scope}[\phi] \setminus \{Y\} = \{X\}$.

This definition extends naturally to the case where X and or Y are replaced by disjoint sets of variables.

Factor Reduction - Example 1/2

B	C	ϕ_2
b^0	c^0	100
b^0	c^1	1
b^1	c^0	1
b^1	c^1	100

$\xrightarrow{C=c^1}$

Dark pink (c^1), light pink (c^0).

Factor Reduction - Example 1/2

B	C	ϕ_2
b^0	c^0	100
b^0	c^1	1
b^1	c^0	1
b^1	c^1	100

$\xrightarrow{C=c^1}$

B	C	$\phi_2[c^1]$
b^0	c^1	1
b^1	c^1	100

Dark pink (c^1), light pink (c^0).

Factor Reduction - Example 2/2

Reduced factors are factors, they can take part in factor products:

A	$\underline{B}$	ϕ_1
a^0	b^0	30
a^0	b^1	5
a^1	b^0	1
a^1	b^1	10

$\times$

$\underline{B}$	C	$\phi_2[c^1]$
b^0	c^1	1
b^1	c^1	100

Dark green (b^1), light green (b^0).

Factor Reduction - Example 2/2

Reduced factors are factors, they can take part in factor products:

A	$\underline{B}$	ϕ_1
a^0	b^0	30
a^0	b^1	5
a^1	b^0	1
a^1	b^1	10

$\times$

$\underline{B}$	C	$\phi_2[c^1]$
b^0	c^1	1
b^1	c^1	100

=

A	B	C	$\psi_{A,B,C}[c^1]$
a^0	b^0	c^1	30
a^0	b^1	c^1	500
a^1	b^0	c^1	1
a^1	b^1	c^1	1000

Dark green (b^1), light green (b^0).

Factor Reduction - Example 2/2

Reduced factors are factors, they can take part in factor products:

A	$\underline{B}$	ϕ_1
a^0	b^0	30
a^0	b^1	5
a^1	b^0	1
a^1	b^1	10

$\times$

$\underline{B}$	C	$\phi_2[c^1]$
b^0	c^1	1
b^1	c^1	100

$=$

A	B	C	$\psi_{A,B,C}[c^1]$
a^0	b^0	c^1	30
a^0	b^1	c^1	500
a^1	b^0	c^1	1
a^1	b^1	c^1	1000

Do you think this is true?

$$(\phi_1(A,B) \times \phi_2(B,C))[c^1] = \phi_1(A,B) \times \phi_2[c^1](B)$$

Dark green (b^1), light green (b^0).

Factor Reduction - Example 2/2

Reduced factors are factors, they can take part in factor products:

A	$\underline{B}$	ϕ_1
a^0	b^0	30
a^0	b^1	5
a^1	b^0	1
a^1	b^1	10

$\times$

$\underline{B}$	C	$\phi_2[c^1]$
b^0	c^1	1
b^1	c^1	100

$=$

A	B	C	$\psi_{A,B,C}[c^1]$
a^0	b^0	c^1	30
a^0	b^1	c^1	500
a^1	b^0	c^1	1
a^1	b^1	c^1	1000

Do you think this is true?

$$(\phi_1(A, B) \times \phi_2(B, C))[c^1] = \phi_1(A, B) \times \phi_2[c^1](B)$$

Yes! And, if we are working with tabular factors, the second option is more efficient since we do not have to create the large table $\psi_{A,B,C}$.

Dark green (b^1), light green (b^0).

Factor Reduction and Conditioning

If we find that Debbie and Bob have the misconception, that is, $(D = d^1, B = b^1)$, then we can obtain a factor that is proportional to $P_\Phi(A, C|B = b^1, D = d^1)$ via:

$$(\phi_1(A, B) \times \phi_2(B, C) \times \phi_3(C, D) \times \phi_4(D, A))[b^1, d^1]$$

which is $\tilde{P}_\Phi(A, B = b^1, C, D = d^1)$.

And, of course, $P_\Phi(A, C|B = b^1, D = d^1) \propto \tilde{P}_\Phi(A, B = b^1, C, D = d^1)$

Factor Reduction and Conditioning

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which is $\tilde{P}_{\Phi}(A, B = b^1, C, D = d^1)$.

And, of course, $P_{\Phi}(A, C|B = b^1, D = d^1) \propto \tilde{P}_{\Phi}(A, B = b^1, C, D = d^1)$

Alternatively, **we can reduce the factors first**:

$$\psi_{b1d1}(A, C) = \phi_1[b^1](A) \times \phi_2[b^1](C) \times \phi_3[d^1](C) \times \phi_4[d^1](A)$$

Then, to obtain the distribution, we simply normalise ψ_{b1d1}.

Exercise

Express $P_\Phi(A, C | B = b^1, D = d^1)$ for the Misconception MN.

Exercise

Express $P_\Phi(A, C|B = b^1, D = d^1)$ for the Misconception MN.

A	B	$\phi_1[b_1]$
a^0	b^1	5
a^1	b^1	10

$\times$

B	C	$\phi_2[b^1]$
b^1	c^0	1
b^1	c^1	100

$\times$

C	D	$\phi_3[d^1]$
c^0	d^1	100
c^1	d^1	1

$\times$

D	A	$\phi_4[d^1]$
d^1	a^0	1
d^1	a^1	100

=

A	C	$\tilde{P}_\Phi[b^1, d^1]$	$P_\Phi(A, C\|B = b^1, D = d^1)$
a^0	c^0		
a^0	c^1		
a^1	c^0		
a^1	c^1		
Sum			

Exercise

Express $P_\Phi(A, C|B = b^1, D = d^1)$ for the Misconception MN.

A	B	$\phi_1[b_1]$
a^0	b^1	5
a^1	b^1	10

$\times$

B	C	$\phi_2[b^1]$
b^1	c^0	1
b^1	c^1	100

$\times$

C	D	$\phi_3[d^1]$
c^0	d^1	100
c^1	d^1	1

$\times$

D	A	$\phi_4[d^1]$
d^1	a^0	1
d^1	a^1	100

=

A	C	$\tilde{P}_\Phi[b^1, d^1]$	$P_\Phi(A, C\|B = b^1, D = d^1)$
a^0	c^0	500	0.0025
a^0	c^1	500	0.0025
a^1	c^0	100000	0.4975
a^1	c^1	100000	0.4975
Sum		201000	1.0

Exercise

Express $P_\Phi(A, C|B = b^1, D = d^1)$ for the Misconception MN.

A	B	$\phi_1[b_1]$
a^0	b^1	5
a^1	b^1	10

$\times$

B	C	$\phi_2[b^1]$
b^1	c^0	1
b^1	c^1	100

$\times$

C	D	$\phi_3[d^1]$
c^0	d^1	100
c^1	d^1	1

$\times$

D	A	$\phi_4[d^1]$
d^1	a^0	1
d^1	a^1	100

=

A	C	$\tilde{P}_\Phi[b^1, d^1]$	$P_\Phi(A, C\|B = b^1, D = d^1)$
a^0	c^0	500	0.0025
a^0	c^1	500	0.0025
a^1	c^0	100000	0.4975
a^1	c^1	100000	0.4975
Sum		201000	1.0

Can you now assess the marginal probability $P(B = b^1, D = d^1)$?

Exercise

Express $P_\Phi(A, C|B = b^1, D = d^1)$ for the Misconception MN.

A	B	$\phi_1[b_1]$
a^0	b^1	5
a^1	b^1	10

$\times$

B	C	$\phi_2[b^1]$
b^1	c^0	1
b^1	c^1	100

$\times$

C	D	$\phi_3[d^1]$
c^0	d^1	100
c^1	d^1	1

$\times$

D	A	$\phi_4[d^1]$
d^1	a^0	1
d^1	a^1	100

=

A	C	$\tilde{P}_\Phi[b^1, d^1]$	$P_\Phi(A, C\|B = b^1, D = d^1)$
a^0	c^0	500	0.0025
a^0	c^1	500	0.0025
a^1	c^0	100000	0.4975
a^1	c^1	100000	0.4975
Sum		201000	1.0

Can you now assess the marginal probability $P(B = b^1, D = d^1)$? It is $\frac{201000}{Z}$, we still need the normaliser Z of the the complete $\tilde{P}_\Phi$.

Summary

- An MN identifies a joint distribution, hence it is possible to reason marginally or conditionally.
- We can perform marginalisation and conditioning by manipulation of the unnormalised measure.
- We can use factor product, reduction and normalisation to condition more efficiently.
- Later in the course we will develop a general graph-based algorithm that can lead to more efficient inference (both conditioning and marginalisation).

What Next?

We will turn to influence and independence.

Influence

Outline for this section

We revisit independence.

Then trace influence in MNs to ascertain graphical separation.

We conclude with an overview of the complete set of conditional independences implied by the MN structure.

Independence (revisited)

A joint distribution P **satisfies independence between** the random variables X and Y, which we denote by $P \models A \perp B$, if:

- $P(X, Y) = P(X)P(Y)$
- $P(X|Y) = P(X)$
- $P(Y|X) = P(Y)$

A joint distribution P **satisfies independence** of the rvs X and Y *given* Z, which we denote by $P \models (X \perp Y \mid Z)$, if:

- $P(X, Y|Z) = P(X|Z)P(Y|Z)$
- $P(X|Y, Z) = P(X|Z)$
- $P(Y|X, Z) = P(Y|Z)$

Independence (revisited)

A joint distribution P **satisfies independence between** the random variables X and Y, which we denote by $P \models A \perp B$, if:

- $P(X, Y) = P(X)P(Y)$
- $P(X|Y) = P(X)$
- $P(Y|X) = P(Y)$
- $P(X, Y) \propto \phi_1(X)\phi_2(Y)$

A joint distribution P **satisfies independence** of the rvs X and Y *given* Z, which we denote by $P \models (X \perp Y \mid Z)$, if:

- $P(X, Y|Z) = P(X|Z)P(Y|Z)$
- $P(X|Y, Z) = P(X|Z)$
- $P(Y|X, Z) = P(Y|Z)$
- $P(X, Y, Z) \propto \phi_1(X, Z)\phi_2(Y, Z)$

where ϕ_1 and ϕ_2 are so-called **factors**.

Independence

Does $A \perp C | B, D$ hold?

B	D	A	C	$\tilde{P}(A, B, C, D)$
b^0	d^0	a^0	c^0	300 000
b^0	d^0	a^0	c^1	300 000
b^0	d^0	a^1	c^0	100
b^0	d^0	a^1	c^1	100
b^0	d^1	a^0	c^0	300 000
b^0	d^1	a^0	c^1	30
b^0	d^1	a^1	c^0	1×10^6
b^0	d^1	a^1	c^1	100
b^1	d^0	a^0	c^0	500
b^1	d^0	a^0	c^1	5×10^6
b^1	d^0	a^1	c^0	10
b^1	d^0	a^1	c^1	100 000
b^1	d^1	a^0	c^0	500
b^1	d^1	a^0	c^1	500
b^1	d^1	a^1	c^0	100 000
b^1	d^1	a^1	c^1	100 000

Colour: (B, D); dark: a^0; light: a^1.

Does $B \perp C | A, D$ hold?

A	D	B	C	$\tilde{P}(A, B, C, D)$
a^0	d^0	b^0	c^0	300 000
a^0	d^0	b^0	c^1	300 000
a^0	d^0	b^1	c^0	500
a^0	d^0	b^1	c^1	5×10^6
a^0	d^1	b^0	c^0	300 000
a^0	d^1	b^0	c^1	30
a^0	d^1	b^1	c^0	500
a^0	d^1	b^1	c^1	500
a^1	d^0	b^0	c^0	100
a^1	d^0	b^0	c^1	100
a^1	d^0	b^1	c^0	10
a^1	d^0	b^1	c^1	100 000
a^1	d^1	b^0	c^0	1×10^6
a^1	d^1	b^0	c^1	100
a^1	d^1	b^1	c^0	100 000
a^1	d^1	b^1	c^1	100 000

Colour: (A, D); dark: b^0; light: b^1.

Independence

Does $A \perp C | B, D$ hold? ✓

B	D	A	C	$\tilde{P}(A, B, C, D)$
b^0	d^0	a^0	c^0	300 000
b^0	d^0	a^0	c^1	300 000
b^0	d^0	a^1	c^0	100
b^0	d^0	a^1	c^1	100
b^0	d^1	a^0	c^0	300 000
b^0	d^1	a^0	c^1	30
b^0	d^1	a^1	c^0	1×10^6
b^0	d^1	a^1	c^1	100
b^1	d^0	a^0	c^0	500
b^1	d^0	a^0	c^1	5×10^6
b^1	d^0	a^1	c^0	10
b^1	d^0	a^1	c^1	100 000
b^1	d^1	a^0	c^0	500
b^1	d^1	a^0	c^1	500
b^1	d^1	a^1	c^0	100 000
b^1	d^1	a^1	c^1	100 000

Colour: (B, D); dark: a^0; light: a^1.

Does $B \perp C | A, D$ hold? ✗

A	D	B	C	$\tilde{P}(A, B, C, D)$
a^0	d^0	b^0	c^0	300 000
a^0	d^0	b^0	c^1	300 000
a^0	d^0	b^1	c^0	500
a^0	d^0	b^1	c^1	5×10^6
a^0	d^1	b^0	c^0	300 000
a^0	d^1	b^0	c^1	30
a^0	d^1	b^1	c^0	500
a^0	d^1	b^1	c^1	500
a^1	d^0	b^0	c^0	100
a^1	d^0	b^0	c^1	100
a^1	d^0	b^1	c^0	10
a^1	d^0	b^1	c^1	100 000
a^1	d^1	b^0	c^0	1×10^6
a^1	d^1	b^0	c^1	100
a^1	d^1	b^1	c^0	100 000
a^1	d^1	b^1	c^1	100 000

Colour: (A, D); dark: b^0; light: b^1.

Direct Influence

If there's an edge connecting X and Y in $\mathcal{H}$, then these variables will influence one another in any P factorising over $\mathcal{H}$.

Rationale:

- If there is an edge, for any P_Φ factorising over $\mathcal{H}$, there must be at least one factor ϕ covering this edge. The value of this factor will clearly depend on both X and Y.
  If it didn't, the factor would not 'cover' the edge.

This is similar to how it worked in BNs.

Indirect Influence without Observations

As edges are undirected, a node X can influence a node Y so long as there is a path connecting them.

Let's see this by characterising $P_{\Phi}(A, C)$ in the Misconception example:

$$P_{\Phi}(A, C) \propto \sum_{B} \sum_{D} \phi_1(A, B) \times \phi_2(B, C) \times \phi_3(C, D) \times \phi_4(D, A)$$

Reminder: $\tilde{P}_{\Phi}(A, B, C, D) = \phi_1(A, B) \times \phi_2(B, C) \times \phi_3(C, D) \times \phi_4(D, A)$

Indirect Influence without Observations

As edges are undirected, a node X can influence a node Y so long as there is a path connecting them.

Let's see this by characterising $P_\Phi(A, C)$ in the Misconception example:

$$\begin{aligned} P_\Phi(A, C) &\propto \sum_B \sum_D \phi_1(A, B) \times \phi_2(B, C) \times \phi_3(C, D) \times \phi_4(D, A) \\ &\propto \sum_B \phi_1(A, B) \times \phi_2(B, C) \times \sum_D \phi_3(C, D) \times \phi_4(D, A) \end{aligned}$$

See that because of the sums, we cannot re-express this as two factors $\psi_1(A) \times \psi_2(C)$, hence $P_\Phi \not\models A \perp C$.

Reminder: $\tilde{P}_\Phi(A, B, C, D) = \phi_1(A, B) \times \phi_2(B, C) \times \phi_3(C, D) \times \phi_4(D, A)$

Indirect Influence with Observations

If the paths connecting X and Y in $\mathcal{H}$ are 'blocked' by observations, then X and Y cannot influence one another.

To see this, let's use an example: can A influence D in the Misconception example when we observe $B = b$ and $D = d$?

$$P_\Phi(A, B = b, C, D = d) \propto \phi_1[b](A) \times \phi_2[b](C) \times \phi_3[d](C) \times \phi_4[d](A)$$

Reminder: $\tilde{P}_\Phi(A, B, C, D) = \phi_1(A, B) \times \phi_2(B, C) \times \phi_3(C, D) \times \phi_4(D, A)$

Indirect Influence with Observations

If the paths connecting X and Y in $\mathcal{H}$ are 'blocked' by observations, then X and Y cannot influence one another.

To see this, let's use an example: can A influence D in the Misconception example when we observe $B = b$ and $D = d$?

$$\begin{aligned} P_{\mathbf{\Phi}}(A, B = b, C, D = d) &\propto \phi_1[b](A) \times \phi_2[b](C) \times \phi_3[d](C) \times \phi_4[d](A) \\ &\propto \underbrace{\phi_1[b](A) \times \phi_4[d](A)}_{\psi_1(A)} \times \underbrace{\phi_2[b](C) \times \phi_3[d](C)}_{\psi_2(C)} \end{aligned}$$

Reminder: $\tilde{P}_{\mathbf{\Phi}}(A, B, C, D) = \phi_1(A, B) \times \phi_2(B, C) \times \phi_3(C, D) \times \phi_4(D, A)$

Indirect Influence with Observations

If the paths connecting X and Y in $\mathcal{H}$ are 'blocked' by observations, then X and Y cannot influence one another.

To see this, let's use an example: can A influence D in the Misconception example when we observe $B = b$ and $D = d$?

$$\begin{aligned} P_{\Phi}(A, B = b, C, D = d) &\propto \phi_1[b](A) \times \phi_2[b](C) \times \phi_3[d](C) \times \phi_4[d](A) \\ &\propto \underbrace{\phi_1[b](A) \times \phi_4[d](A)}_{\psi_1(A)} \times \underbrace{\phi_2[b](C) \times \phi_3[d](C)}_{\psi_2(C)} \end{aligned}$$

See that with factors reduced by observations, we can rearrange the terms as two independent factors $\psi_1(A) \times \psi_2(C)$, hence $P_{\Phi} \models A \perp C | B, D$.

Reminder: $\tilde{P}_{\Phi}(A, B, C, D) = \phi_1(A, B) \times \phi_2(B, C) \times \phi_3(C, D) \times \phi_4(D, A)$

Active Paths (definition 4.8)

A path $X_1, \ldots, X_k$ in $\mathcal{H}$ is **active** *given* $\mathbf{Z}$ if no X_i in the path is in $\mathbf{Z}$.

Separation (definition 4.9)

A set of nodes $\mathbf{Z}$ separates $\mathbf{X}$ and $\mathbf{Y}$ in $\mathcal{H}$, denoted $\mathsf{sep}(\mathbf{X}; \mathbf{Y} | \mathbf{Z})$, if there is no active path between any node $X \in \mathbf{X}$ and $Y \in \mathbf{Y}$ given $\mathbf{Z}$.

As in the case of BNs, graphical separation implies conditional independence and vice-versa. But now we have *separation* rather than directed-separation.

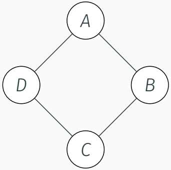
Global Independencies $\mathcal{I}(\mathcal{H})$ (section 4.3.1, definition 4.9)

Any distribution P that factorises over the MN structure $\mathcal{H}$ satisfies the **global independencies**:

$$\mathcal{I}(\mathcal{H}) = \{(\mathsf{X} \perp \mathsf{Y} \mid \mathsf{Z}) : \mathrm{sep}(\mathsf{X}; \mathsf{Y}|\mathsf{Z})\}$$

Local Independencies $\mathcal{I}_l(\mathcal{H})$ (definition 4.11)

A node is independent of all other nodes given its immediate neighbours. For example, $A \perp C \mid B, D$



The set of immediate neighbours of a node is called its *Markov Blanket*, denoted $\mathrm{MB}_{\mathcal{H}}(X_i)$.

$$\mathcal{I}_l(\mathcal{H}) = \{X_i \perp \mathrm{Rest}_{\mathcal{H}}(X_i) | \, \mathrm{MB}_{\mathcal{H}}(X_i) : \text{for all } X_i\}$$

When a distribution P that factorises over $\mathcal{H}$ is 'positive' (no zero probabilities allowed), the local independencies $\mathcal{I}_l(\mathcal{H})$ hold and imply the global ones.

The 'rest' of the nodes in $\mathcal{H}$ relative to X_i is $\mathcal{X} \setminus \{X_i\}$, we denote this by $\mathrm{Rest}_{\mathcal{H}}(X_i)$. The textbook does not use this (much cleaner) notation.

Summary

- Independence can be ascertain for unnormalised distributions.
- The MN structure (no need to know the factors) is sufficient to ascertain conditional independences, via graphical separation.
- While the MN structure does not identify a unique factorisation, it identifies a unique set of global independencies.
- Separation allows us to characterise this set.
- In the special (but common) case of positive distributions, we can also analyse the MN structure in terms of local independencies (Markov blankets). This will be useful when working through inference algorithms.

What Next?

WC3: exercises (semantics, reasoning and influence).

Friday: P3 deadline.

Midterm: BNs and MNs.

After midterm: Inference.

References

[1] Daphne Koller and Nir Friedman. *Probabilistic graphical models: principles and techniques.* MIT press, 2009.