

# Exact Inference

## Variable Elimination

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*<https://probabl1.github.io>*

**Module 3** introduces *a key algorithm for Exact Inference* known as **Variable Elimination** (VE; Chapter 9).

**ILOs** After this module the student

- can perform sum-product inference via VE;
- can predict the complexity of VE by constructing an induced graph;
- can optimise VE by ordering elimination steps opportunistically.

HC4a: naive sum-product inference and variable elimination.

LC4: VE in code.

HC4b: elimination ordering.

WC4: exercises.

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## Naive Exact Inference

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We have a joint distribution  $P(\mathbf{X})$  over a collection of rvs  $\mathbf{X}$ , this distribution is represented by a PGM such as a BN or MN.

Conditional probability queries:

- **Evidence:** an assignment  $\mathbf{E} = \mathbf{e}$  of a subset  $\mathbf{E} \subseteq \mathbf{X}$  of the rvs.
- **Query:** given the evidence, we are interest in reasoning about the possible assignments of the rvs in  $\mathbf{Y} \subseteq \mathbf{X}$ , where  $\mathbf{Y} \cap \mathbf{E} = \emptyset$ .
- **Task:** express  $P(\mathbf{Y}|\mathbf{E} = \mathbf{e})$

**Applications:** medical/fault diagnosis, image/text completion, speech analysis, ...

$$P(Y|E = e) = \frac{P(Y, E = e)}{P(E = e)} \propto P(Y, E = e)$$

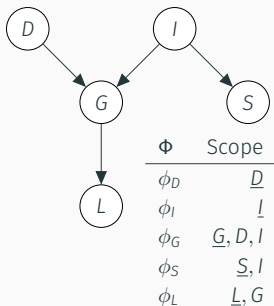
Let  $\mathbf{W}$  be the complement of  $\mathbf{E}$  and  $\mathbf{Y}$  in  $\mathbf{X}$ . That is, the ‘rest’ of the variables in the model.

Then, we can express  $P(Y, E = e)$  via marginalisation of  $\mathbf{W}$ . This is an instance of what we call **inference**. Specifically, *marginal inference*:

$$P(Y, E = e) = \sum_{\mathbf{W}} P(\mathbf{W}, Y, E = e)$$

To continue, we need to know more about how  $P(\underbrace{\mathbf{W}, Y, E}_{\mathbf{X}})$  factorises.

## A factor-view of the *Student* example



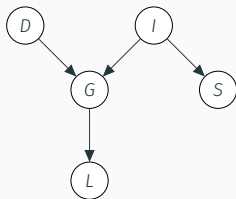
In factor form, we use  $\phi_X$  with scope  $\{X\} \cup \text{Pa}(X)$  for  $P(X|\text{Pa}(X))$ .

Joint distribution:

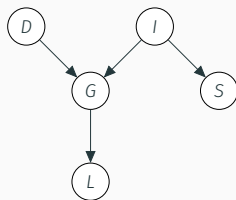
$$\begin{aligned} P(D, I, S, G, L) &= P(D)P(I)P(S|I)P(G|D, I)P(L|G) \\ &= \phi_D(D)\phi_I(I)\phi_S(S, I)\phi_G(G, D, I)\phi_L(L, G) \end{aligned}$$

In this view, we can develop a framework for inference that works both for BNs and MNs.





Evidence:  $D = d^1$ . Query:  $L$ .

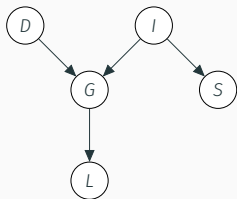


Evidence:  $D = d^1$ . Query:  $L$ .

To express  $P(L|D = d^1)$

1. assign  $D = d^1$ :

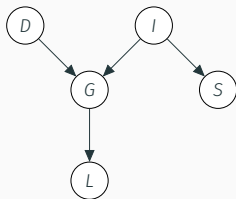
$$P(D = d^1, I, S, G, L)$$



Evidence:  $D = d^1$ . Query:  $L$ .

To express  $P(L|D = d^1)$

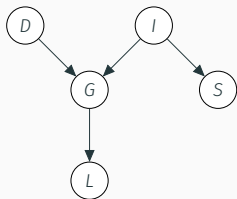
1. assign  $D = d^1$ :  
 $P(D = d^1, I, S, G, L)$
2. marginalise  $I, G, S$ :  
 $P(L, D = d^1);$



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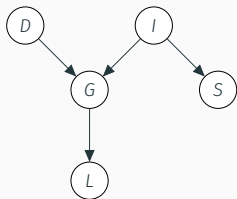
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Start with the joint distribution, but let's write it using *factors*:

$$\begin{aligned} P(D, I, G, S, L) \\ = \phi_D(D)\phi_I(I)\phi_G(G, D, I)\phi_S(S, I)\phi_L(L, G) \end{aligned}$$



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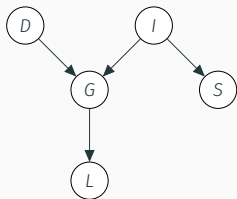
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1. Reduce the factors to the available evidence:

$$\begin{aligned}
 P(D = d^1, I, G, S, L) \\
 = \phi_D[d^1]\phi_I(I)\phi_G[d^1](G, I)\phi_S(S, I)\phi_L(L, G)
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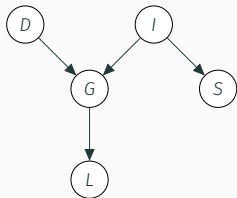
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2. Marginalise the variables that are neither assigned nor queried about:

$$\begin{aligned} P(L, D = d^1) \\ = \sum_{G, I, S} \phi_D[d^1]\phi_I(I)\phi_G[d^1](G, I)\phi_S(S, I)\phi_L(L, G) \end{aligned}$$



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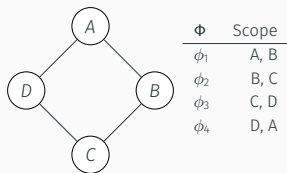
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3. Normaliser:  $\sum_{l \in \text{Val}(L)} P(L = l, D = d^1)$



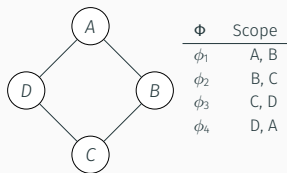


Evidence:  $A = a^1, C = c^1$ .

Query:  $B$ .

To express  $P_{\Phi}(B|A = a^1, C = c^1)$

1. assign  $A = a^1, C = c^1$ :  
 $\tilde{P}_{\Phi}(A = a^1, B, C = c^1, D)$

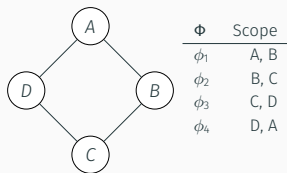


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Evidence:  $A = a^1, C = c^1$ .

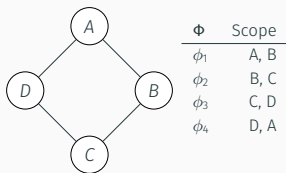
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To express  $P_{\Phi}(B|A = a^1, C = c^1)$

1. assign  $A = a^1, C = c^1$ :  
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2. marginalise  $D$ :  
 $\tilde{P}_{\Phi}(B, A = a^1, C = c^1);$
3. normalize over  $\text{Val}(B)$ .

Start with the unnormalised measure:

$$\begin{aligned}\tilde{P}_{\Phi}(A, B, C, D) \\ = \phi_1(A, B)\phi_2(B, C)\phi_3(C, D)\phi_4(D, A)\end{aligned}$$

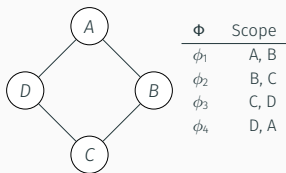


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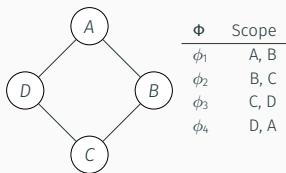
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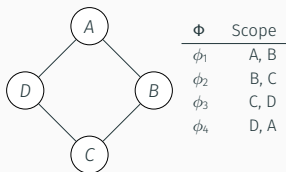
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2. Marginalise the variables that are neither assigned nor queried about:

$$\begin{aligned}\tilde{P}_{\Phi}(B, A = a^1, C = c^1) \\ = \sum_D \phi_1[a^1](B)\phi_2[c^1](B)\phi_3[c^1](D)\phi_4[a^1](D)\end{aligned}$$



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3. Normaliser:

$$\sum_{b \in \text{Val}(B)} \tilde{P}_{\Phi}(B = b, A = a^1, C = c^1)$$

This naive 'sum-product' algorithm is correct but requires creating the full joint (unnormalised) distribution.

Of course, we can reduce factors to the evidence before taking the large product, but you can see that this is not an efficient algorithm.

No matter the graphical structure, it will always involve a product of all factors, which, in the worst case, is a very large factor.

Next, we use the graphical structure to obtain better average performance via the so-called **variable elimination** algorithm.



# Variable Elimination

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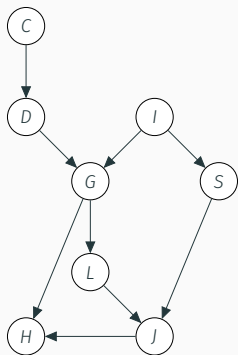
# Exploiting the Graph Structure

Naive sum-product does not use the graph structure at all. It starts from a full product of all factors.

The key challenge in inference is marginalisation of unassigned rvs. We build the entire product of factors because then we are sure that it is safe to marginalise rvs.

But consider marginalising one rv at a time, by looking around in the graphical structure, we might be able to see that not all factors are going to be relevant.

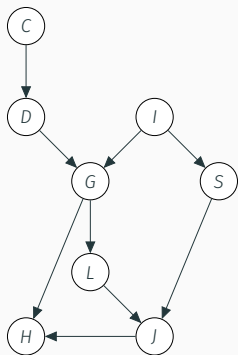
This intuition is the key to a more ‘incremental’ procedure known as **variable elimination** (VE; Section 9.3).



In factor form, we use  $\phi_X$  with scope  $\{X\} \cup \text{Pa}(X)$  for  $P(X|\text{Pa}(X))$ .

Joint distribution:

$$\begin{aligned} P(C, D, I, G, S, L, H, J) = & \phi_C(C) \phi_D(D, C) \phi_I(I) \phi_S(S, I) \\ & \times \phi_G(G, D, I) \phi_L(L, G) \phi_J(J, L) \phi_H(H, G, J) \end{aligned}$$

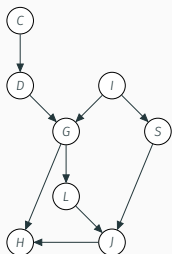


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Joint distribution:

$$\begin{aligned} P(C, D, I, G, S, L, H) &= \phi_C(C) \phi_D(D, C) \phi_I(I) \phi_S(S, I) \\ &\quad \times \phi_G(G, D, I) \phi_L(L, G) \phi_J(J, L) \phi_H(H, G, J) \end{aligned}$$

Example: let's express  $P(J)$  by eliminating  $C, D, I, H, G, S, L$  in this order [example 9.1]



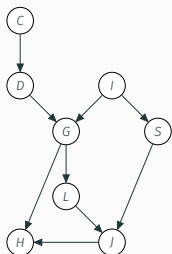
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Goal:  $P(J)$ .

Eliminate:  $\underline{C}, D, I, H, G, S, L$ .

$$\sum_C \phi_C(C) \phi_D(D, C) \phi_I(I) \phi_S(S, I) \phi_G(G, D, I) \phi_L(L, G) \phi_J(J, L) \phi_H(H, G, J)$$



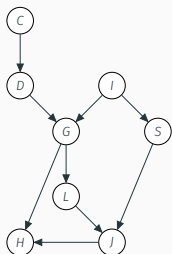
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Goal:  $P(J)$ .

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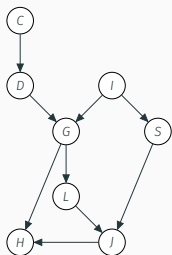
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$$P(C, D, I, G, S, L, J, H) = \phi_C(C) \phi_D(D, C) \phi_I(I) \phi_S(S, I) \\ \times \phi_G(G, D, I) \phi_L(L, G) \phi_J(J, L) \phi_H(H, G, J)$$

Goal:  $P(J)$ .

Eliminate:  $\underline{C}, D, I, H, G, S, L$ .

$$\begin{aligned} & \sum_C \phi_C(C) \phi_D(D, C) \phi_I(I) \phi_S(S, I) \phi_G(G, D, I) \phi_L(L, G) \phi_J(J, L) \phi_H(H, G, J) \\ &= \phi_I(I) \phi_S(S, I) \phi_G(G, D, I) \phi_L(L, G) \phi_J(J, L) \phi_H(H, G, J) \underbrace{\sum_C \phi_C(C) \phi_D(D, C)}_{\tau_1(D)} \\ &= \phi_I(I) \phi_S(S, I) \phi_G(G, D, I) \phi_L(L, G) \phi_J(J, L) \phi_H(H, G, J) \tau_1(D) \end{aligned}$$



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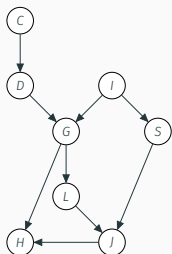
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Goal:  $P(J)$ .

Eliminate:  $\cancel{C}, \underline{D}, I, H, G, S, L$ .

$$\sum_D \phi_I(I) \phi_S(S, I) \phi_G(G, D, I) \phi_L(L, G) \phi_J(J, L) \phi_H(H, G, J) \tau_1(D)$$





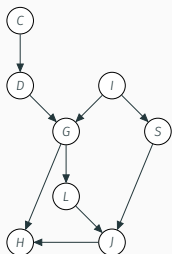
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Goal:  $P(J)$ .

Eliminate:  $\cancel{C}, \underline{D}, I, H, G, S, L$ .

$$\begin{aligned} & \sum_D \phi_I(I) \phi_S(S, I) \phi_G(G, D, I) \phi_L(L, G) \phi_J(J, L) \phi_H(H, G, J) \tau_1(D) \\ &= \phi_I(I) \phi_S(S, I) \phi_L(L, G) \phi_J(J, L) \phi_H(H, G, J) \underbrace{\sum_D \tau_1(D) \phi_G(G, D, I)}_{\tau_2(G, I)} \end{aligned}$$



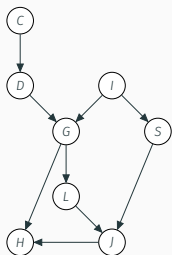
Joint distribution:

$$P(C, D, I, G, S, L, J, H) = \phi_C(C) \phi_D(D, C) \phi_I(I) \phi_S(S, I) \\ \times \phi_G(G, D, I) \phi_L(L, G) \phi_J(J, L) \phi_H(H, G, J)$$

Goal:  $P(J)$ .

Eliminate:  $\cancel{C}, \underline{D}, I, H, G, S, L$ .

$$\begin{aligned} & \sum_D \phi_I(I) \phi_S(S, I) \phi_G(G, D, I) \phi_L(L, G) \phi_J(J, L) \phi_H(H, G, J) \tau_1(D) \\ &= \phi_I(I) \phi_S(S, I) \phi_L(L, G) \phi_J(J, L) \phi_H(H, G, J) \underbrace{\sum_D \tau_1(D) \phi_G(G, D, I)}_{\tau_2(G, I)} \\ &= \phi_I(I) \phi_S(S, I) \phi_L(L, G) \phi_J(J, L) \phi_H(H, G, J) \tau_2(G, I) \end{aligned}$$



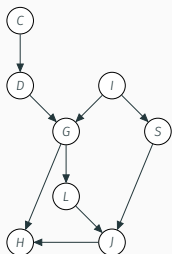
Joint distribution:

$$P(C, D, I, G, S, L, J, H) = \phi_C(C) \phi_D(D, C) \phi_I(I) \phi_S(S, I) \\ \times \phi_G(G, D, I) \phi_L(L, G) \phi_J(J, L) \phi_H(H, G, J)$$

Goal:  $P(J)$ .

Eliminate:  $\cancel{C}, \cancel{D}, \cancel{I}, H, G, S, L$ .

$$\sum_I \phi_I(I) \phi_S(S, I) \phi_L(L, G) \phi_J(J, L) \phi_H(H, G, J) \tau_2(G, I)$$



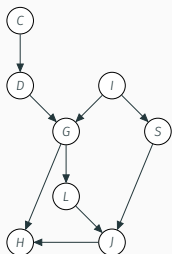
Joint distribution:

$$P(C, D, I, G, S, L, J, H) = \phi_C(C) \phi_D(D, C) \phi_I(I) \phi_S(S, I) \\ \times \phi_G(G, D, I) \phi_L(L, G) \phi_J(J, L) \phi_H(H, G, J)$$

Goal:  $P(J)$ .

Eliminate:  $C, D, I, H, G, S, L$ .

$$\sum_I \phi_I(I) \phi_S(S, I) \phi_L(L, G) \phi_J(J, L) \phi_H(H, G, J) \tau_2(G, I) \\ = \phi_L(L, G) \phi_J(J, L) \phi_H(H, G, J) \underbrace{\sum_I \overbrace{\phi_I(I) \phi_S(S, I) \tau_2(G, I)}^{\rho_3(I, G, S)}}_{\tau_3(G, S)}$$



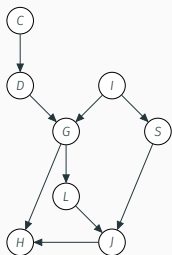
Joint distribution:

$$P(C, D, I, G, S, L, J, H) = \phi_C(C) \phi_D(D, C) \phi_I(I) \phi_S(S, I) \\ \times \phi_G(G, D, I) \phi_L(L, G) \phi_J(J, L) \phi_H(H, G, J)$$

Goal:  $P(J)$ .

Eliminate:  $C, D, I, H, G, S, L$ .

$$\begin{aligned} & \sum_I \phi_I(I) \phi_S(S, I) \phi_L(L, G) \phi_J(J, L) \phi_H(H, G, J) \tau_2(G, I) \\ &= \phi_L(L, G) \phi_J(J, L) \phi_H(H, G, J) \underbrace{\sum_I \phi_I(I) \phi_S(S, I) \tau_2(G, I)}_{\tau_3(G, S)} \\ &= \phi_L(L, G) \phi_J(J, L) \phi_H(H, G, J) \tau_3(G, S) \end{aligned}$$



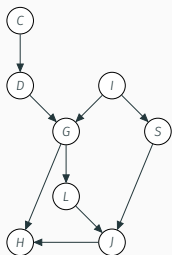
Joint distribution:

$$P(C, D, I, G, S, L, J, H) = \phi_C(C) \phi_D(D, C) \phi_I(I) \phi_S(S, I) \\ \times \phi_G(G, D, I) \phi_L(L, G) \phi_J(J, L) \phi_H(H, G, J)$$

Goal:  $P(J)$ .

Eliminate:  $\cancel{C}, \cancel{D}, \cancel{I}, \underline{H}, G, S, L$ .

$$\sum_H \phi_L(L, G) \phi_J(J, L) \phi_H(H, G, J) \tau_3(G, S)$$



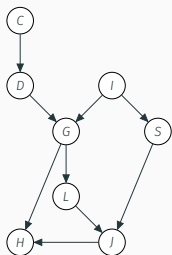
Joint distribution:

$$P(C, D, I, G, S, L, H, J) = \phi_C(C) \phi_D(D, C) \phi_I(I) \phi_S(S, I) \\ \times \phi_G(G, D, I) \phi_L(L, G) \phi_J(J, L) \phi_H(H, G, J)$$

Goal:  $P(J)$ .

Eliminate:  $\cancel{C}, \cancel{D}, \cancel{I}, \underline{H}, G, S, L$ .

$$\sum_H \phi_L(L, G) \phi_J(J, L) \phi_H(H, G, J) \tau_3(G, S) \\ = \phi_L(L, G) \phi_J(J, L) \tau_3(G, S) \underbrace{\sum_H \phi_H(H, G, J)}_{\tau_4(G, J)}$$



Joint distribution:

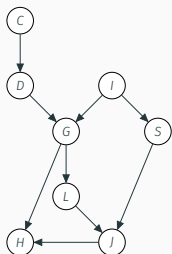
$$P(C, D, I, G, S, L, H, J) = \phi_C(C) \phi_D(D, C) \phi_I(I) \phi_S(S, I) \\ \times \phi_G(G, D, I) \phi_L(L, G) \phi_J(J, L) \phi_H(H, G, J)$$

Goal:  $P(J)$ .

Eliminate:  $\cancel{C}, \cancel{D}, \cancel{I}, \underline{H}, G, S, L$ .

$$\begin{aligned} & \sum_H \phi_L(L, G) \phi_J(J, L) \phi_H(H, G, J) \tau_3(G, S) \\ &= \phi_L(L, G) \phi_J(J, L) \tau_3(G, S) \underbrace{\sum_H \phi_H(H, G, J)}_{\tau_4(G, J)} \\ &= \phi_L(L, G) \phi_J(J, L) \tau_3(G, S) \tau_4(G, J) \end{aligned}$$





Joint distribution:

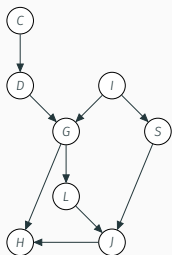
$$P(C, D, I, G, S, L, J, H) = \phi_C(C) \phi_D(D, C) \phi_I(I) \phi_S(S, I) \\ \times \phi_G(G, D, I) \phi_L(L, G) \phi_J(J, L) \phi_H(H, G, J)$$

Goal:  $P(J)$ .

Eliminate:  $\cancel{C}, \cancel{D}, \cancel{I}, \underline{H}, G, S, L$ .

$$\begin{aligned} & \sum_H \phi_L(L, G) \phi_J(J, L) \phi_H(H, G, J) \tau_3(G, S) \\ &= \phi_L(L, G) \phi_J(J, L) \tau_3(G, S) \underbrace{\sum_H \phi_H(H, G, J)}_{\tau_4(G, J)} \\ &= \phi_L(L, G) \phi_J(J, L) \tau_3(G, S) \tau_4(G, J) \end{aligned}$$

$\tau_4(G, J) = 1$  but, for generality, let's pretend we do not know that.



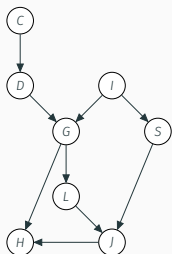
Joint distribution:

$$P(C, D, I, G, S, L, J, H) = \phi_C(C) \phi_D(D, C) \phi_I(I) \phi_S(S, I) \\ \times \phi_G(G, D, I) \phi_L(L, G) \phi_J(J, L) \phi_H(H, G, J)$$

Goal:  $P(J)$ .

Eliminate:  $\cancel{C}, \cancel{D}, \cancel{I}, \cancel{H}, \underline{G}, S, L$ .

$$\sum_G \phi_L(L, G) \phi_J(J, L) \tau_3(G, S) \tau_4(G, J)$$



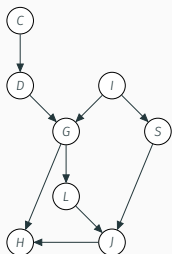
Joint distribution:

$$P(C, D, I, G, S, L, H) = \phi_C(C) \phi_D(D, C) \phi_I(I) \phi_S(S, I) \\ \times \phi_G(G, D, I) \phi_L(L, G) \phi_J(J, L) \phi_H(H, G, J)$$

Goal:  $P(J)$ .

Eliminate:  $\cancel{C}, \cancel{D}, \cancel{I}, \cancel{H}, \underline{G}, S, L$ .

$$\sum_G \phi_L(L, G) \phi_J(J, L) \tau_3(G, S) \tau_4(G, J) \\ = \phi_J(J, L) \sum_G \underbrace{\phi_L(L, G) \tau_3(G, S) \tau_4(G, J)}_{\tau_5(J, L, S)} \overbrace{\rho_5(G, J, L, S)}$$



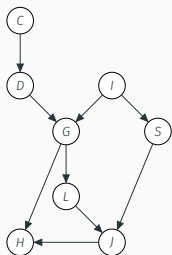
Joint distribution:

$$P(C, D, I, G, S, L, H) = \phi_C(C) \phi_D(D, C) \phi_I(I) \phi_S(S, I) \\ \times \phi_G(G, D, I) \phi_L(L, G) \phi_J(J, L) \phi_H(H, G, J)$$

Goal:  $P(J)$ .

Eliminate:  $C, D, I, H, G, S, L$ .

$$\begin{aligned} & \sum_G \phi_L(L, G) \phi_J(J, L) \tau_3(G, S) \tau_4(G, I) \\ &= \phi_J(J, L) \underbrace{\sum_G \overbrace{\phi_L(L, G) \tau_3(G, S) \tau_4(G, I)}^{\rho_5(G, J, L, S)}}_{\tau_5(J, L, S)} \\ &= \phi_J(J, L) \tau_5(J, L, S) \end{aligned}$$



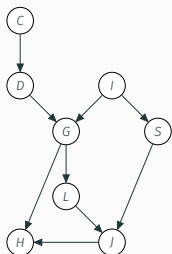
Joint distribution:

$$P(C, D, I, G, S, L, J, H) = \phi_C(C) \phi_D(D, C) \phi_I(I) \phi_S(S, I) \\ \times \phi_G(G, D, I) \phi_L(L, G) \phi_J(J, L) \phi_H(H, G, J)$$

Goal:  $P(J)$ .

Eliminate:  $\cancel{C}, \cancel{D}, \cancel{I}, \cancel{H}, \cancel{G}, \underline{S}, L$ .

$$\sum_S \phi_J(J, L) \tau_5(J, L, S)$$



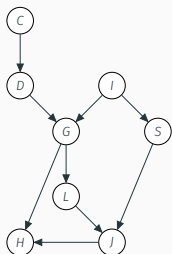
Eliminate:  $\cancel{C}, \cancel{D}, \cancel{I}, \cancel{H}, \cancel{G}, \underline{S}, L$ .

Joint distribution:

$$P(C, D, I, G, S, L, J, H) = \phi_C(C) \phi_D(D, C) \phi_I(I) \phi_S(S, I) \\ \times \phi_G(G, D, I) \phi_L(L, G) \phi_J(J, L) \phi_H(H, G, J)$$

Goal:  $P(J)$ .

$$\sum_S \phi_J(J, L) \tau_5(J, L, S) \\ = \phi_J(J, L) \underbrace{\sum_S \tau_5(J, L, S)}_{\tau_6(J, L)}$$



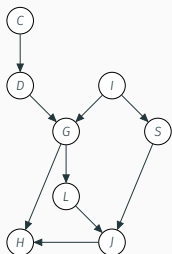
Eliminate:  $\cancel{C}, \cancel{D}, \cancel{I}, \cancel{H}, \cancel{G}, \underline{S}, L$ .

Joint distribution:

$$P(C, D, I, G, S, L, J, H) = \phi_C(C) \phi_D(D, C) \phi_I(I) \phi_S(S, I) \\ \times \phi_G(G, D, I) \phi_L(L, G) \phi_J(J, L) \phi_H(H, G, J)$$

Goal:  $P(J)$ .

$$\begin{aligned} & \sum_S \phi_J(J, L) \tau_5(J, L, S) \\ &= \phi_J(J, L) \underbrace{\sum_S \tau_5(J, L, S)}_{\tau_6(J, L)} \\ &= \phi_J(J, L) \tau_6(J, L) \end{aligned}$$



Eliminate:  $\cancel{C}, \cancel{D}, \cancel{I}, \cancel{H}, \cancel{G}, \cancel{S}, \underline{L}$ .

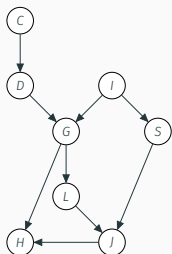
Joint distribution:

$$P(C, D, I, G, S, L, J, H) = \phi_C(C) \phi_D(D, C) \phi_I(I) \phi_S(S, I) \\ \times \phi_G(G, D, I) \phi_L(L, G) \phi_J(J, L) \phi_H(H, G, J)$$

Goal:  $P(J)$ .

$$\sum_L \overbrace{\phi_J(J, L) \tau_6(J, L)}^{p_7(J, L)}$$





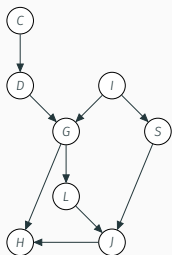
Eliminate:  $\cancel{C}, \cancel{D}, \cancel{I}, \cancel{H}, \cancel{G}, \cancel{S}, \underline{L}$ .

Joint distribution:

$$P(C, D, I, G, S, L, J, H) = \phi_C(C) \phi_D(D, C) \phi_I(I) \phi_S(S, I) \\ \times \phi_G(G, D, I) \phi_L(L, G) \phi_J(J, L) \phi_H(H, G, J)$$

Goal:  $P(J)$ .

$$\sum_L \overbrace{\phi_J(J, L) \tau_6(J, L)}^{p_7(J, L)} \\ = \tau_7(J)$$



Eliminate:  $\cancel{C}, \cancel{D}, \cancel{I}, \cancel{H}, \cancel{G}, \cancel{S}, \underline{L}$ .

Joint distribution:

$$P(C, D, I, G, S, L, J, H) = \phi_C(C) \phi_D(D, C) \phi_I(I) \phi_S(S, I) \\ \times \phi_G(G, D, I) \phi_L(L, G) \phi_J(J, L) \phi_H(H, G, J)$$

Goal:  $P(J)$ .

$$\sum_L \overbrace{\phi_J(J, L) \tau_6(J, L)}^{p_7(J, L)} \\ = \tau_7(J)$$

$\tau_7(J)$  is precisely  $P(J)$ .

## VE – Overview of Steps

We have a collection of factors  $\Phi$ .

(They may be reduced to account for some evidence.)

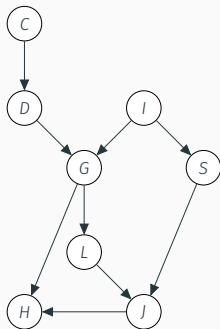
For each rv  $Y$  that we want to marginalise, we

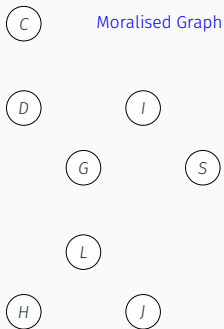
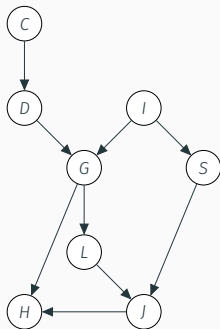
1. gather all relevant factors—those for which  $Y$  is in the scope—removing them from  $\Phi$ ;
2. take the product of the relevant factors;
3. marginalise  $Y$  out of the product;
4. add the resulting factor in the collection.

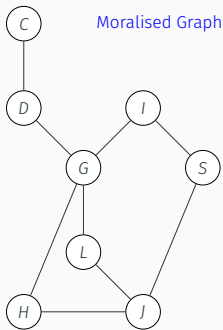
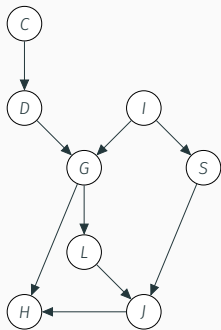
For example, marginalising  $G$  after having marginalised  $C, D, I, H$

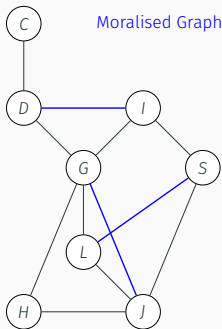
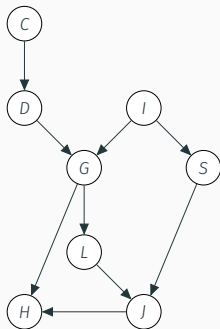
$$\sum_G \underbrace{\phi_L(L, G) \phi_J(J, L) \tau_3(G, S) \tau_4(G, J)}_{(1) \text{ relevant}} = \phi_J(J, L) \sum_G \underbrace{\phi_L(L, G) \tau_3(G, S) \tau_4(G, J)}_{(2) \text{ product}} = \phi_J(J, L) \underbrace{\tau_5(J, L, S)}_{(3) \text{ marginal: } \tau_5(J, L, S)} = \underbrace{\phi_J(J, L) \tau_5(J, L, S)}_{(4) \text{ new collection}}$$

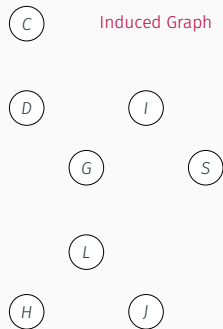
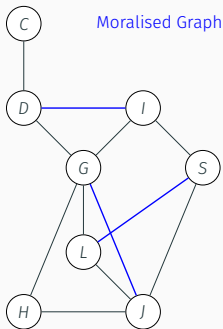
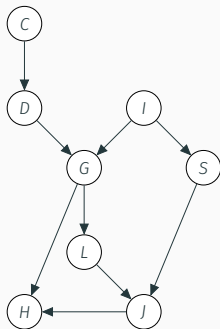
We repeat this until we are done marginalising.







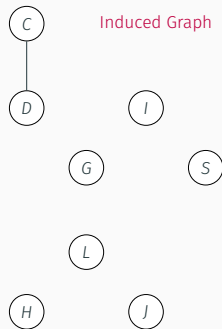
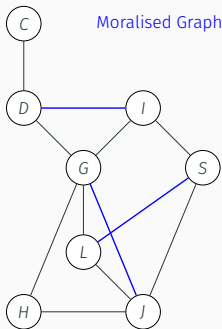
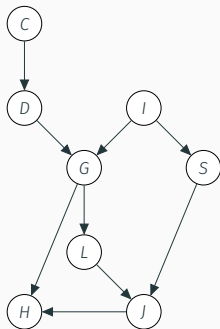




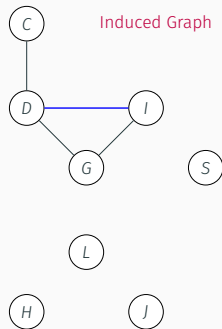
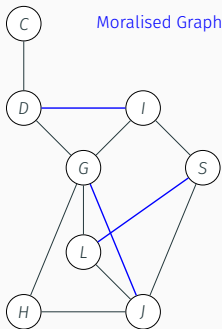
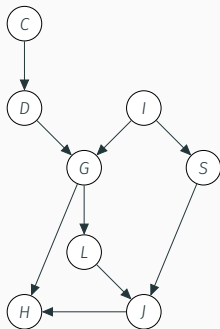
Eliminating Available factors

Relevant Intermediate factor New factor

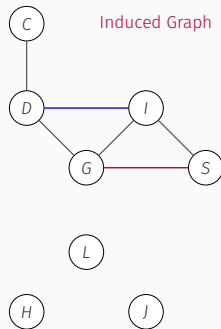
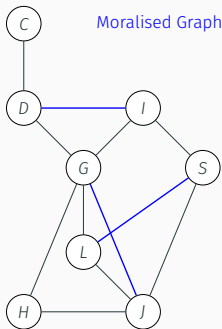
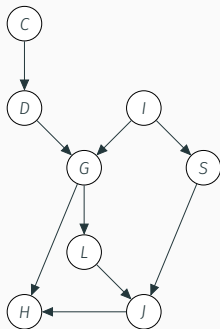




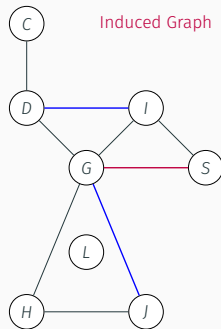
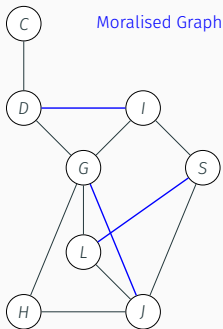
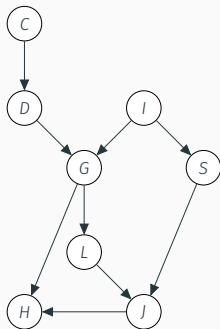
Eliminating	Available factors	Relevant	Intermediate factor	New factor
$C$	$\phi_C, \phi_D, \phi_I, \phi_S, \phi_G, \phi_L, \phi_H, \phi_J$	$\phi_C, \phi_D$	$\rho_1(C, D)$	$\tau_1(D)$



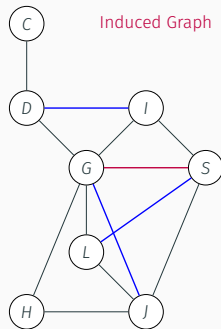
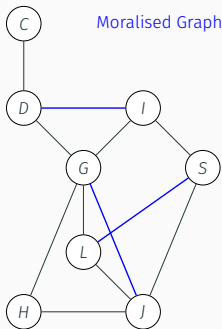
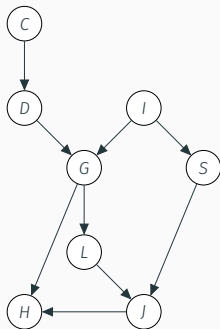
Eliminating	Available factors	Relevant	Intermediate factor	New factor
$C$	$\phi_C, \phi_D, \phi_I, \phi_S, \phi_G, \phi_L, \phi_H, \phi_J$	$\phi_C, \phi_D$	$\rho_1(C, D)$	$\tau_1(D)$
$D$	$\phi_I, \phi_S, \phi_G, \phi_L, \phi_H, \phi_J, \tau_1$	$\tau_1, \phi_G$	$\rho_2(G, D, I)$	$\tau_2(G, I)$



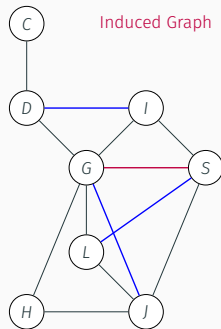
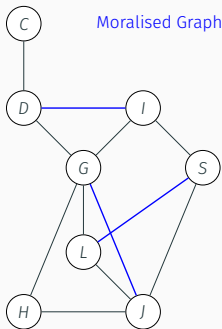
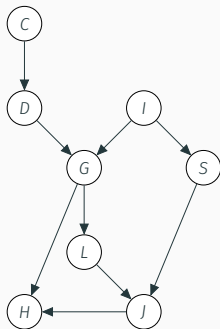
Eliminating	Available factors	Relevant	Intermediate factor	New factor
$C$	$\phi_C, \phi_D, \phi_I, \phi_S, \phi_G, \phi_L, \phi_H, \phi_J$	$\phi_C, \phi_D$	$\rho_1(C, D)$	$\tau_1(D)$
$D$	$\phi_I, \phi_S, \phi_G, \phi_L, \phi_H, \phi_J, \tau_1$	$\tau_1, \phi_G$	$\rho_2(G, D, I)$	$\tau_2(G, I)$
$I$	$\phi_I, \phi_S, \phi_L, \phi_H, \phi_J, \tau_2$	$\phi_I, \phi_S, \tau_2$	$\rho_3(I, G, S)$	$\tau_3(G, S)$



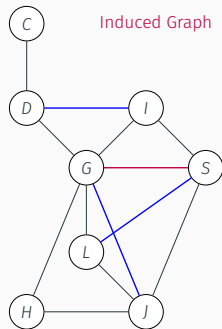
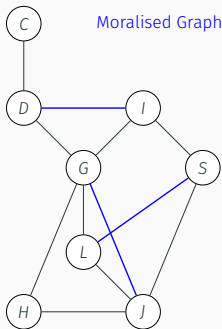
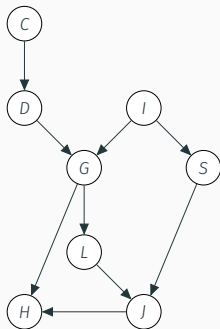
Eliminating	Available factors	Relevant	Intermediate factor	New factor
$C$	$\phi_C, \phi_D, \phi_I, \phi_S, \phi_G, \phi_L, \phi_H, \phi_J$	$\phi_C, \phi_D$	$\rho_1(C, D)$	$\tau_1(D)$
$D$	$\phi_I, \phi_S, \phi_G, \phi_L, \phi_H, \phi_J, \tau_1$	$\tau_1, \phi_G$	$\rho_2(G, D, I)$	$\tau_2(G, I)$
$I$	$\phi_I, \phi_S, \phi_L, \phi_H, \phi_J, \tau_2$	$\phi_I, \phi_S, \tau_2$	$\rho_3(I, G, S)$	$\tau_3(G, S)$
$H$	$\phi_L, \phi_H, \phi_J, \tau_3$	$\phi_H$	$\phi_H(H, G, J)$	$\tau_4(G, J)$



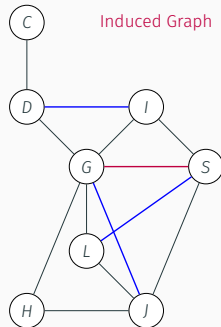
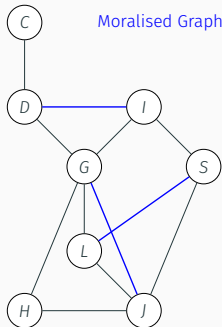
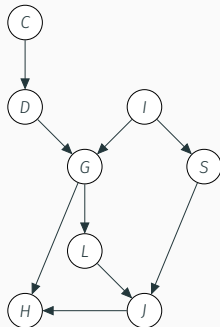
Eliminating	Available factors	Relevant	Intermediate factor	New factor
C	$\phi_C, \phi_D, \phi_I, \phi_S, \phi_G, \phi_L, \phi_H, \phi_J$	$\phi_C, \phi_D$	$\rho_1(C, D)$	$\tau_1(D)$
D	$\phi_I, \phi_S, \phi_G, \phi_L, \phi_H, \phi_J, \tau_1$	$\tau_1, \phi_G$	$\rho_2(G, D, I)$	$\tau_2(G, I)$
I	$\phi_I, \phi_S, \phi_L, \phi_H, \phi_J, \tau_2$	$\phi_I, \phi_S, \tau_2$	$\rho_3(I, G, S)$	$\tau_3(G, S)$
H	$\phi_L, \phi_H, \phi_J, \tau_3$	$\phi_H$	$\phi_H(H, G, J)$	$\tau_4(G, J)$
G	$\phi_L, \phi_J, \tau_3, \tau_4$	$\phi_L, \tau_3, \tau_4$	$\rho_5(G, J, L, S)$	$\tau_5(J, L, S)$



Eliminating	Available factors	Relevant	Intermediate factor	New factor
C	$\phi_C, \phi_D, \phi_I, \phi_S, \phi_G, \phi_L, \phi_H, \phi_J$	$\phi_C, \phi_D$	$\rho_1(C, D)$	$\tau_1(D)$
D	$\phi_I, \phi_S, \phi_G, \phi_L, \phi_H, \phi_J, \tau_1$	$\tau_1, \phi_G$	$\rho_2(G, D, I)$	$\tau_2(G, I)$
I	$\phi_I, \phi_S, \phi_L, \phi_H, \phi_J, \tau_2$	$\phi_I, \phi_S, \tau_2$	$\rho_3(I, G, S)$	$\tau_3(G, S)$
H	$\phi_L, \phi_H, \phi_J, \tau_3$	$\phi_H$	$\phi_H(H, G, J)$	$\tau_4(G, J)$
G	$\phi_L, \phi_J, \tau_3, \tau_4$	$\phi_L, \tau_3, \tau_4$	$\rho_5(G, J, L, S)$	$\tau_5(J, L, S)$
S	$\phi_J, \tau_5$	$\tau_5$	$\tau_5(J, L, S)$	$\tau_6(J, L)$



Eliminating	Available factors	Relevant	Intermediate factor	New factor
C	$\phi_C, \phi_D, \phi_I, \phi_S, \phi_G, \phi_L, \phi_H, \phi_J$	$\phi_C, \phi_D$	$\rho_1(C, D)$	$\tau_1(D)$
D	$\phi_I, \phi_S, \phi_G, \phi_L, \phi_H, \phi_J, \tau_1$	$\tau_1, \phi_G$	$\rho_2(G, D, I)$	$\tau_2(G, I)$
I	$\phi_I, \phi_S, \phi_L, \phi_H, \phi_J, \tau_2$	$\phi_I, \phi_S, \tau_2$	$\rho_3(I, G, S)$	$\tau_3(G, S)$
H	$\phi_L, \phi_H, \phi_J, \tau_3$	$\phi_H$	$\phi_H(H, G, I)$	$\tau_4(G, I)$
G	$\phi_L, \phi_J, \tau_3, \tau_4$	$\phi_L, \tau_3, \tau_4$	$\rho_5(G, J, L, S)$	$\tau_5(J, L, S)$
S	$\phi_J, \tau_5$	$\tau_5$	$\tau_5(J, L, S)$	$\tau_6(J, L)$
L	$\phi_J, \tau_6$	$\phi_J, \tau_6$	$\rho_7(J, L)$	$\tau_7(J)$

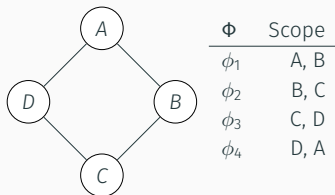


The **Moralised Graph** [definition 4.16] captures the factor structure of the BN: make edges undirected, ‘marry’ the parents of the colliders.

The **Induced Graph** [definition 9.5] captures the factor structure of VE. It differs by addition of ‘fill edges’: connect neighbours of the eliminated rv to one another. Which fill edges appear depends on the elimination ordering.

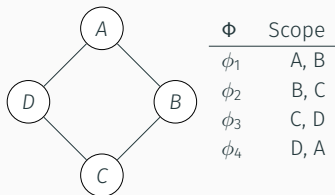
Exercise: elimination order  $G, I, S, L, H, C, D$  [solution in table 9.2].





Joint unnormalised distribution:

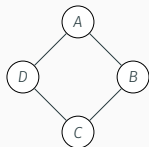
$$\tilde{P}_{\Phi}(A, B, C, D) = \phi_1(A, B)\phi_2(B, C)\phi_3(C, D)\phi_4(D, A)$$



Joint unnormalised distribution:

$$\tilde{P}_{\Phi}(A, B, C, D) = \phi_1(A, B)\phi_2(B, C)\phi_3(C, D)\phi_4(D, A)$$

Example: let's express  $P_{\Phi}(D)$



$\Phi$	Scope
$\phi_1$	A, B
$\phi_2$	B, C
$\phi_3$	C, D
$\phi_4$	D, A

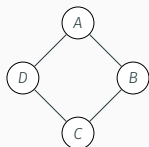
Joint unnormalised distribution:

$$\begin{aligned}\tilde{P}_{\Phi}(A, B, C, D) \\ = \phi_1(A, B)\phi_2(B, C)\phi_3(C, D)\phi_4(D, A)\end{aligned}$$

Goal:  $P_{\Phi}(D)$ .

Eliminate: A, B, C.

$$\sum_A \phi_1(A, B)\phi_2(B, C)\phi_3(C, D)\phi_4(D, A)$$



$\Phi$	Scope
$\phi_1$	A, B
$\phi_2$	B, C
$\phi_3$	C, D
$\phi_4$	D, A

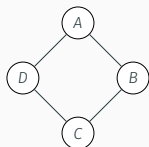
Joint unnormalised distribution:

$$\begin{aligned}\tilde{P}_{\Phi}(A, B, C, D) \\ = \phi_1(A, B)\phi_2(B, C)\phi_3(C, D)\phi_4(D, A)\end{aligned}$$

Goal:  $P_{\Phi}(D)$ .

Eliminate:  $\underline{A}, B, C$ .

$$\begin{aligned}\sum_A \phi_1(A, B)\phi_2(B, C)\phi_3(C, D)\phi_4(D, A) \\ = \phi_2(B, C)\phi_3(C, D) \sum_A \overbrace{\phi_1(A, B)\phi_4(D, A)}^{\rho_1(A, B, D)} \\ \underbrace{\hspace{10em}}_{\tau_1(B, D)}\end{aligned}$$



$\Phi$	Scope
$\phi_1$	A, B
$\phi_2$	B, C
$\phi_3$	C, D
$\phi_4$	D, A

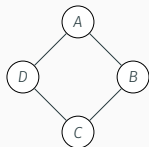
Joint unnormalised distribution:

$$\begin{aligned}\tilde{P}_{\Phi}(A, B, C, D) \\ = \phi_1(A, B)\phi_2(B, C)\phi_3(C, D)\phi_4(D, A)\end{aligned}$$

Goal:  $P_{\Phi}(D)$ .

Eliminate: A, B, C.

$$\begin{aligned}\sum_A \phi_1(A, B)\phi_2(B, C)\phi_3(C, D)\phi_4(D, A) \\ = \phi_2(B, C)\phi_3(C, D) \sum_A \underbrace{\phi_1(A, B)\phi_4(D, A)}_{\tau_1(B, D)} \\ = \phi_2(B, C)\phi_3(C, D)\tau_1(B, D)\end{aligned}$$



$\Phi$	Scope
$\phi_1$	A, B
$\phi_2$	B, C
$\phi_3$	C, D
$\phi_4$	D, A

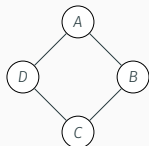
Joint unnormalised distribution:

$$\begin{aligned}\tilde{P}_{\Phi}(A, B, C, D) \\ = \phi_1(A, B)\phi_2(B, C)\phi_3(C, D)\phi_4(D, A)\end{aligned}$$

Goal:  $P_{\Phi}(D)$ .

Eliminate: ~~A~~, B, C.

$$\sum_B \phi_2(B, C)\phi_3(C, D)\tau_1(B, D)$$



$\Phi$	Scope
$\phi_1$	A, B
$\phi_2$	B, C
$\phi_3$	C, D
$\phi_4$	D, A

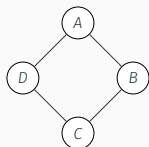
Joint unnormalised distribution:

$$\begin{aligned}\tilde{P}_{\Phi}(A, B, C, D) \\ = \phi_1(A, B)\phi_2(B, C)\phi_3(C, D)\phi_4(D, A)\end{aligned}$$

Goal:  $P_{\Phi}(D)$ .

Eliminate: ~~A~~, B, C.

$$\begin{aligned}\sum_B \phi_2(B, C)\phi_3(C, D)\tau_1(B, D) \\ = \phi_3(C, D) \underbrace{\sum_B \overbrace{\phi_2(B, C)\tau_1(B, D)}^{\rho_2(B, C, D)}}_{\tau_2(C, D)}\end{aligned}$$



$\Phi$	Scope
$\phi_1$	A, B
$\phi_2$	B, C
$\phi_3$	C, D
$\phi_4$	D, A

Joint unnormalised distribution:

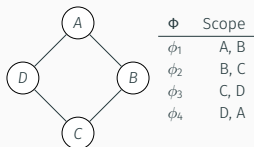
$$\begin{aligned}\tilde{P}_{\Phi}(A, B, C, D) \\ = \phi_1(A, B)\phi_2(B, C)\phi_3(C, D)\phi_4(D, A)\end{aligned}$$

Goal:  $P_{\Phi}(D)$ .

Eliminate: ~~A~~, B, C.

$$\begin{aligned}\sum_B \phi_2(B, C)\phi_3(C, D)\tau_1(B, D) \\ = \phi_3(C, D) \underbrace{\sum_B \overbrace{\phi_2(B, C)\tau_1(B, D)}^{\rho_2(B, C, D)}}_{\tau_2(C, D)} \\ = \phi_3(C, D)\tau_2(C, D)\end{aligned}$$





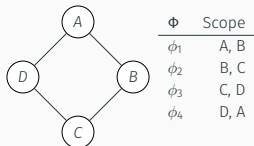
Eliminate: ~~A~~, ~~B~~, C.

Joint unnormalised distribution:

$$\begin{aligned}\tilde{P}_{\Phi}(A, B, C, D) \\ = \phi_1(A, B)\phi_2(B, C)\phi_3(C, D)\phi_4(D, A)\end{aligned}$$

Goal:  $P_{\Phi}(D)$ .

$$\sum_C \phi_3(C, D)\tau_2(C, D)$$



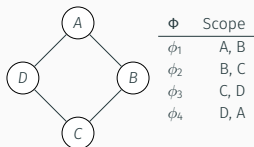
Eliminate: ~~A~~, ~~B~~, C.

Joint unnormalised distribution:

$$\begin{aligned}\tilde{P}_{\Phi}(A, B, C, D) \\ = \phi_1(A, B)\phi_2(B, C)\phi_3(C, D)\phi_4(D, A)\end{aligned}$$

Goal:  $P_{\Phi}(D)$ .

$$\begin{aligned}\sum_C \phi_3(C, D)\tau_2(C, D) \\ = \sum_C \overbrace{\phi_3(C, D)\tau_2(C, D)}^{\rho_3(C, D)}\end{aligned}$$



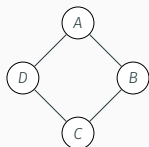
Eliminate: ~~A~~, ~~B~~, C.

Joint unnormalised distribution:

$$\begin{aligned}\tilde{P}_{\Phi}(A, B, C, D) \\ = \phi_1(A, B)\phi_2(B, C)\phi_3(C, D)\phi_4(D, A)\end{aligned}$$

Goal:  $P_{\Phi}(D)$ .

$$\begin{aligned}\sum_C \phi_3(C, D)\tau_2(C, D) \\ = \sum_C \overbrace{\phi_3(C, D)\tau_2(C, D)}^{\rho_3(C, D)} \\ = \tau_3(D)\end{aligned}$$



$\Phi$	Scope
$\phi_1$	A, B
$\phi_2$	B, C
$\phi_3$	C, D
$\phi_4$	D, A

Joint unnormalised distribution:

$$\begin{aligned}\tilde{P}_{\Phi}(A, B, C, D) \\ = \phi_1(A, B)\phi_2(B, C)\phi_3(C, D)\phi_4(D, A)\end{aligned}$$

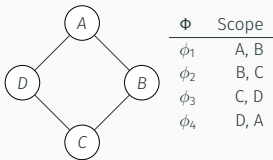
Goal:  $P_{\Phi}(D)$ .

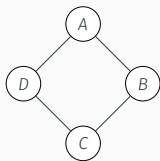
Eliminate: ~~A~~, ~~B~~, C.

$$\begin{aligned}\sum_C \phi_3(C, D)\tau_2(C, D) \\ = \sum_C \overbrace{\phi_3(C, D)\tau_2(C, D)}^{\rho_3(C, D)} \\ = \tau_3(D)\end{aligned}$$

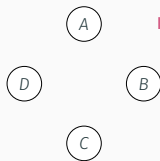
$\tau_3(D)$  is  $\tilde{P}_{\Phi}(D)$ .

To obtain  $P_{\Phi}(D)$  we normalise  $\tilde{P}(D)$  using the normaliser  $\sum_{d \in \text{Val}(D)} \tau_3(D = d)$ .

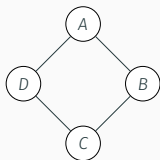




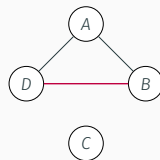
$\Phi$	Scope
$\phi_1$	A, B
$\phi_2$	B, C
$\phi_3$	C, D
$\phi_4$	D, A



Induced Graph

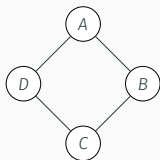


$\Phi$	Scope
$\phi_1$	A, B
$\phi_2$	B, C
$\phi_3$	C, D
$\phi_4$	D, A

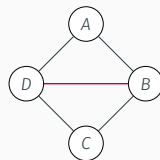


Induced Graph

Eliminating	Available factors	Relevant	Intermediate factor	New factor
A	$\phi_1, \phi_2, \phi_3, \phi_4$	$\phi_1, \phi_4$	$\rho_1(A, B, D)$	$\tau_1(B, D)$



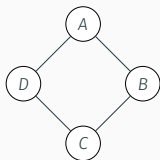
$\Phi$	Scope
$\phi_1$	A, B
$\phi_2$	B, C
$\phi_3$	C, D
$\phi_4$	D, A



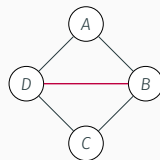
Induced Graph

Eliminating	Available factors	Relevant	Intermediate factor	New factor
A	$\phi_1, \phi_2, \phi_3, \phi_4$	$\phi_1, \phi_4$	$\rho_1(A, B, D)$	$\tau_1(B, D)$
B	$\phi_2, \phi_3, \tau_1$	$\phi_2, \tau_1$	$\rho_2(B, C, D)$	$\tau_2(C, D)$



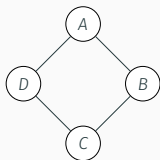


$\Phi$	Scope
$\phi_1$	A, B
$\phi_2$	B, C
$\phi_3$	C, D
$\phi_4$	D, A

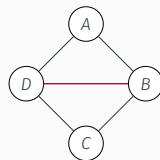


Induced Graph

Eliminating	Available factors	Relevant	Intermediate factor	New factor
A	$\phi_1, \phi_2, \phi_3, \phi_4$	$\phi_1, \phi_4$	$\rho_1(A, B, D)$	$\tau_1(B, D)$
B	$\phi_2, \phi_3, \tau_1$	$\phi_2, \tau_1$	$\rho_2(B, C, D)$	$\tau_2(C, D)$
C	$\phi_3, \tau_2$	$\phi_3, \tau_2$	$\rho_3(C, D)$	$\tau_3(D)$



$\Phi$	Scope
$\phi_1$	A, B
$\phi_2$	B, C
$\phi_3$	C, D
$\phi_4$	D, A



Induced Graph

Eliminating	Available factors	Relevant	Intermediate factor	New factor
A	$\phi_1, \phi_2, \phi_3, \phi_4$	$\phi_1, \phi_4$	$\rho_1(A, B, D)$	$\tau_1(B, D)$
B	$\phi_2, \phi_3, \tau_1$	$\phi_2, \tau_1$	$\rho_2(B, C, D)$	$\tau_2(C, D)$
C	$\phi_3, \tau_2$	$\phi_3, \tau_2$	$\rho_3(C, D)$	$\tau_3(D)$

There's no need for 'moralisation' because MNs do not have v-structures.

The **Induced Graph** [definition 9.5] captures the factor structure of VE. It differs by addition of '*fill edges*': connect neighbours of the eliminated rv to one another. Which fill edges appear depends on the elimination ordering.

Exercise: VE in order  $B, C, A$ .

For a BN with nodes  $X$ , start from the moralised graph with factors  $\Phi$ .

For an MN with nodes  $X$ , start from its factors  $\Phi$ .

If there's evidence  $E = e$ , reduce all factors in  $\Phi$ . [Section 9.3.2]

For a query  $Q$ , disjoint with  $E$ , we need to marginalise  $M = X \setminus (Q \cup E)$ .

In a given order  $\alpha$ , eliminate the variables in  $M$  one by one.

For each rv  $Y \in M$  that we want to marginalise, we

1. gather all relevant factors—those for which  $Y$  is in the scope—removing them from  $\Phi$ ;
2. take the product of the relevant factors;
3. marginalise  $Y$  out of the product;
4. add the resulting factor in the collection.

At the end, we take the product of whatever factors remain and normalise to obtain a distribution.

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The worst case of VE is intractable, much like naive sum-product, but because most models have *sparse graph structure*, **VE is typically much better than naive sum-product**.

*The largest factor in VE depends on the order of elimination.* But, there's no polynomial-time algorithm to find an optimal ordering.

# Summary

- Solving marginals is the key to complex probability queries (with or without evidence).
- Naive sum-product builds the largest possible factor—the (unnormalised) joint distribution—then factor marginalisation is safe to use. This algorithm is intractable for all but the smallest PGMs.
- VE marginalises ('eliminates') one rv at a time in a given order. By taking products involving only the necessary factors, VE achieves efficient inference (so long as the graph is sparse).

# What Next?

LC4: sum-product VE in code.

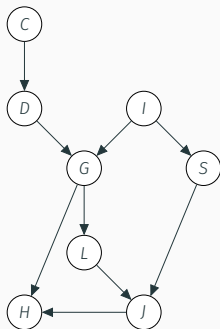
HC4b: elimination orderings.

WC4: exercises.

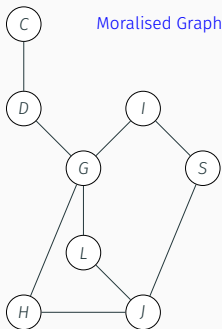
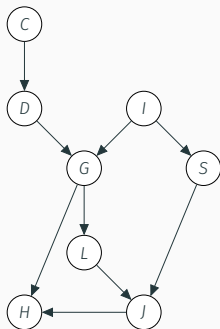
# Elimination Ordering

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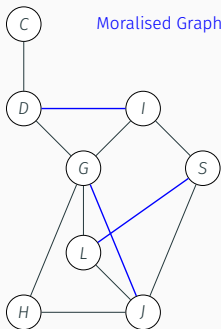
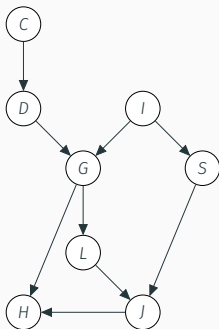
Let's run VE to express  $P(J)$  and let's start with eliminating  $G$



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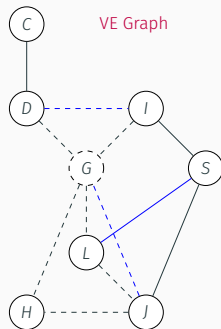
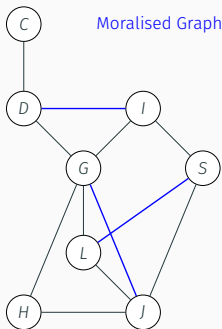
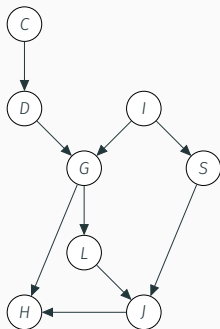


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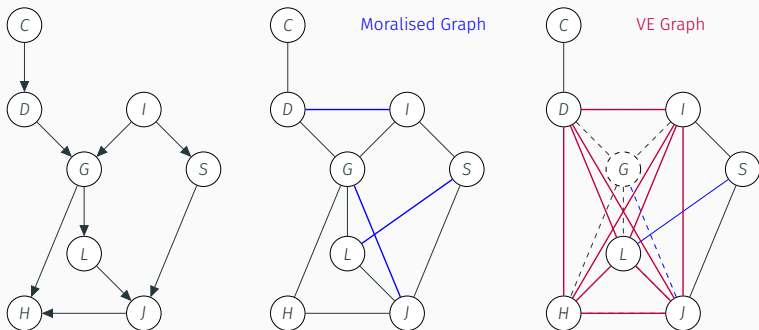


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All cliques involving  $G$  are dashed. Some or all of them will have factors in  $\Phi$ , we should identify those factors and take their product. This creates a **large intermediate factor** with scope  $D, G, H, I, J, L$ , which has cardinality  $3 \times 2^5$ . In example 9.1, the largest intermediate factor had scope  $G, J, L, S$ , which has cardinality  $3 \times 2^3$

# Optimal Ordering

An optimal ordering is one for which the largest intermediate factor is as small as possible.

Unfortunately, finding such an order is intractable. [Section 9.4.3]

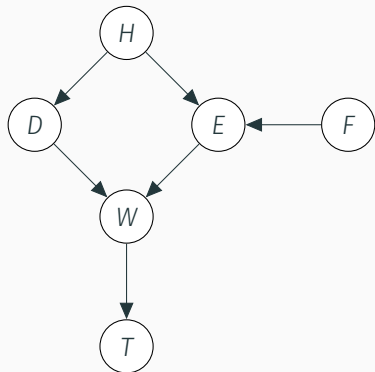
We resort to some useful heuristics: [Section 9.4.3.2]

- Min-neighbours
- Min-weight
- Min-fill
- Weighted-min-fill

These work surprisingly well in practice (min-fill and weighted-min-fill tend to work better on more problems).

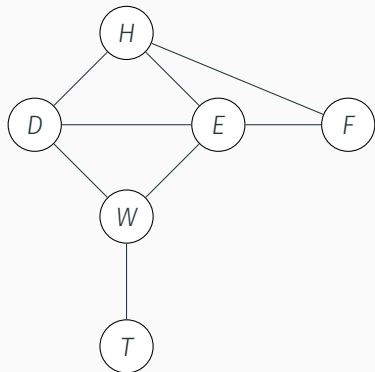
# Min-Neighbours

The cost of a node is the number of neighbours it has.



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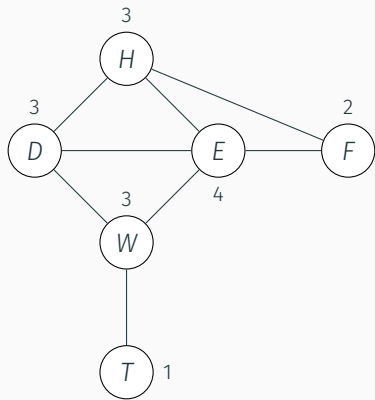
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Moralise the BN and count neighbours.

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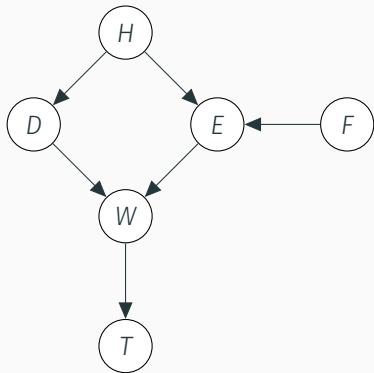


Moralise the BN and count neighbours.

Break ties lexicographically: T, F, D, H, W, E.

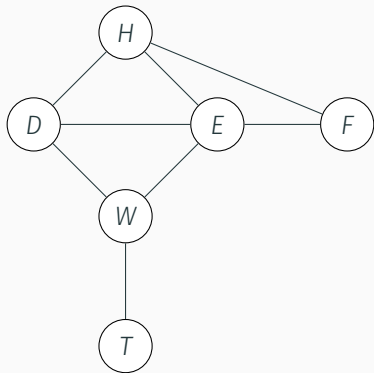
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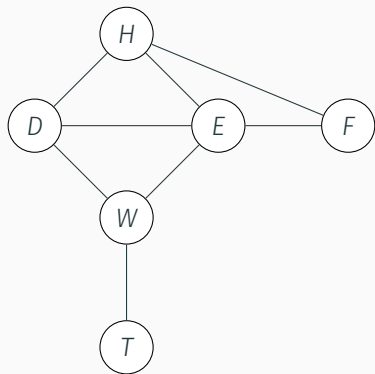


Moralise the BN and quantify weight.



# Min-Weight

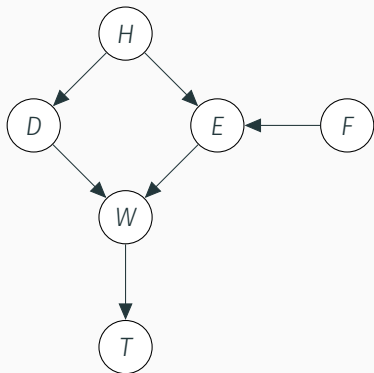
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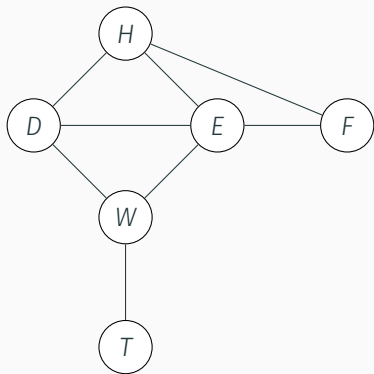
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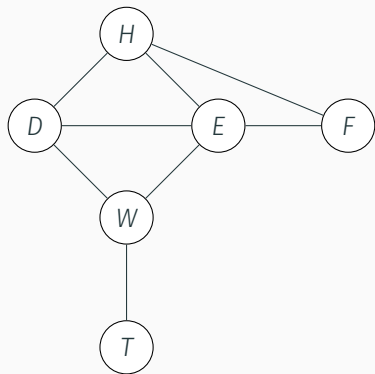
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Moralise the BN and count fill edges.

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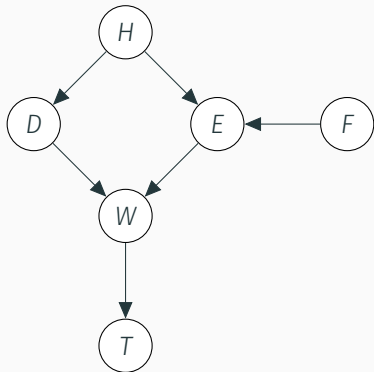
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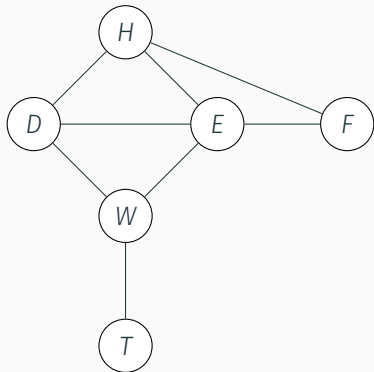
## Weighted-Min-Fill

The cost of a node is the sum of weights of the edges that need to be *added* to the graph due to its elimination, where weight of an edge is the product of weights of its vertices.



# Weighted-Min-Fill

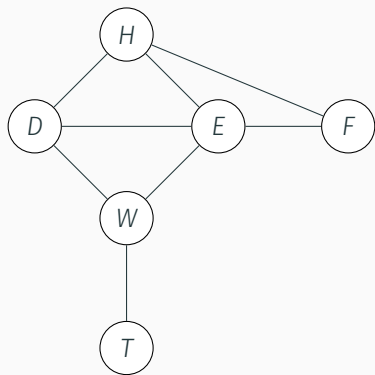
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Break ties lexicographically:

# Induced Graph and Search for Elimination Ordering

Suppose we have 4 heuristics (such as those just presented), we can simulate their corresponding induced graphs in VE (that is, we draw the induced graph without actually taking any factor product).

Then we can find out which heuristic leads to the most efficient VE run: the one whose induced graph requires the least prohibitive factor products.





- The most efficient VE run is the order that creates the smallest maximal clique.
- Optimal orderings are intractable, but we a number of heuristics work well in practice.
- We can even simulate the effect of different heuristics by working out their induced graphs.

# What Next?

WC4: exercises.

Next module: approximate inference.

## References

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- [1] Daphne Koller and Nir Friedman. *Probabilistic graphical models: principles and techniques*. MIT press, 2009.